Analytical Solution to Accumulation Model

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Introduction

This document describes the method for analytically solving the accumulation model from Brunton, 2013. The original formulation of this model included sticky bounds which absorb probability mass when the accumulated evidence reaches a certain threshold. The removal of the sticky bounds facilitates an analytical solution of the model. Below we will describe the solution to both the forward (predicting choice data) and posterior (utilizing choice data) distributions of the model.

Removing the sticky bounds can be motivated for several reasons. First, fits to rat behavior consistently show the bounds are large enough to have little effect on the rat's behavior. Second, in some situations the bounds are particularly harmful to behavior, and we would expect them to get extremely large. Finally, the bounds prevent an analytical solution, forcing the user to use a numerical evaluation which is slow and inaccurate.

The forward model solution describes the probability distribution of accumulated evidence at each time point, given some parameters, some trial data, and an initial starting distribution of evidence. The posterior distribution (previously called the backward pass) describes the probability distribution of the accumulated evidence at each time point with the same constraints as the forward model, but in addition the animal's choice. The forward model is required for fitting the accumulation model to animal choice behavior. The posterior distribution is not required for fitting the model to animal behavior, but is useful for evaluating neural data. This document describes the solution pursued in the accompanying code base to solve the model.

(1) Forward Model Solution

The model assumes an initial distribution of accumulation values $P(a|t=0) = \mathcal{N}\left(\mu_0 = 0, \sigma_i^2\right)$. At each moment in the trial, the forward model distribution of accumulation values $f(a|t, \delta_R, \delta_L) = P(a|t, \delta_R, \delta_L)$ is Gaussian distributed with mean (μ) and variance (σ^2) given by:

$$\mu(t) = \mu_0 e^{\lambda t} + \int_0^t \left(\delta_{R,s} \cdot C\left(R(s)\right) - \delta_{L,s} \cdot C\left(L(s)\right)\right) ds \tag{1}$$

$$\mu(t) = \mu_0 e^{\lambda t} + \sum_i^{\#R} e^{\lambda(t-R(i))} C(R(i)) - \sum_i^{\#L} e^{\lambda(t-L(i))} C(L(i)) \tag{2}$$

$$\sigma^2(t) = \sigma_i^2 e^{\lambda t} + \frac{\sigma_a^2}{2\lambda}\left(e^{2\lambda t} - 1\right) + \int_0^t \sigma_s^2 \left(\delta_{R,s} \cdot C\left(R(s)\right) - \delta_{L,s} \cdot C\left(L(s)\right)\right) e^{2\lambda t} ds \tag{3}$$

$$\sigma^2(t) = \sigma_i^2 e^{\lambda t} + \frac{\sigma_a^2}{2\lambda}\left(e^{2\lambda t} - 1\right) + \sum_i^{\#R} \sigma_s^2 C(R(i)) e^{2\lambda(t-R(i))} + \sum_i^{\#L} \sigma_s^2 C(L(i)) e^{2\lambda(t-L(i))} \tag{4}$$

Where $\#R$ is the number of right clicks on this trial up to time t, and $R(i)$ is the time of the i^{th} right click. $C(R(i))$ tells us the effective adaptation for that clicks. For a detailed discussion of a similar model, see Feng, 2009.

(2) Model Optimization

Given a forward model distribution of accumulation values $f(a|t, \delta_R, \delta_L) = \mathcal{N}\left(\mu(t), \sigma^2(t)\right)$, and the bias parameter B, we can compute the left and right choice probabilities by:

$$P(\text{go right}) = \frac{1}{2}\left(1 + \text{erf}\left(\frac{-\left(B - \mu(t)\right)}{\sigma\sqrt{2}}\right)\right), \tag{5}$$

$$P(\text{go left}) = 1 - P(\text{go right}). \tag{6}$$

These choice probabilities are then distorted by the lapse rate, which parameterizes how often a rat makes a random choice:. The model parameters θ are fit to each rat individually by maximizing the likelihood function:

$$L = \prod_{i}^{\#\text{trials}} P(\text{rat's choice on trial i}|\theta, \delta_R^i, \delta_L^i). \tag{7}$$

Here we note that other terms can be incorporated into the likelihood function, such as priors over specific parameters. To estimate the uncertainty on the parameter estimates, we use the inverse hessian matrix as a parameter covariance matrix (Daw, 2011). To compute the hessian of the model, we use automatic differentiation to exactly compute the local curvature.

(3) Posterior Distribution

Computing the posterior distribution is more complicated than the forward model. First, we fit the parameters to the model using choice data. Then, we compute the forward model for each trial, which tells us the probability distribution of observing an accumulated evidence value at each time point consistent with the stimulus and the initial evidence value. Next, we compute what I will refer to as the backward model $b(a|t, \delta_R, \delta_L, choice)$. Note this is not the 'backward pass distribution' commonly referenced by Brunton, 2013. The backward model here ignores the forward model, and instead computes the probability distribution of observing an accumulated evidence value at each time point consistent with the stimulus and the final evidence value. Importantly, the forward and backward distributions are conditionally independent, conditioned on the final value of the accumulated evidence. Given that they are independent, the posterior distribution that combines both observations is simply the product of the forward and backward distributions.

$$p(a|t, \delta_R, \delta_L, choice) = f\left(a|t, \delta_R, \delta_L, a(t_0) = 0\right) b\left(a|t, \delta_R, \delta_L, choice\right). \tag{8}$$

One technical wrinkle is that our solution for solving the model (forward or backward) relies on initial conditions that are gaussian. Our choice data only constrains the sign of a at the end of the trial, meaning our initial conditions for the backward model is a uniform distribution over $[0, \pm\infty]$. Since our solution method requires gaussian initial conditions, we appear to be stuck. A simple work-around is to discretize the a-value axis into small bins of width Δi, and solve the backward model for each bin assuming a delta function of initial probability mass at each bin. I'll refer to the backward distribution from each bin $a = i$ as the delta-backward solution $b_i(a)$. Our entire backward distribution is the mixture distribution over all the individual delta-backward solutions.

$$b(a) = \sum_{i=0}^{\pm\infty} w_i b_i(a) \tag{9}$$

The mixture weights w_i are all equal if the bin spacing is uniform. Note that it might be tempting to think that we need to weight each individual delta-backward solution by the Forward model's probability mass in each bin; however, this is not correct. Given that the backward model is independent of the forward model, we want the complete backward distribution to reflect all possible states consistent with the choice observation, which is the uniform distribution over the correct sign of a. With a set of $b_i(a)$ solutions, we can now combine them into the posterior distribution, "$p(a|t, \delta_R, \delta_L, choice)$". The exact solution as $\Delta i \to 0$ is given by:

$$p(a|t, \delta_R, \delta_L, choice) \propto f(a|t, \delta_R, \delta_L) \sum_{i=0}^{\pm\infty} w_i b_i(a|t, \delta_R, \delta_L, choice). \tag{10}$$

In practice we can truncate the infinite series at some reasonable value, with some finite bin spacing Δi. Give that $f(a)$, and $b_i(a)$ will be gaussian, let $p_i(a) = f(a)b_i(a)$, which is also gaussian. This lets us write the posterior distribution as the sum of many delta-posterior modes.

$$p(a|t, \delta_R, \delta_L, choice) \propto \sum_{i=0}^{\pm\infty} w_i p_i(a). \tag{11}$$

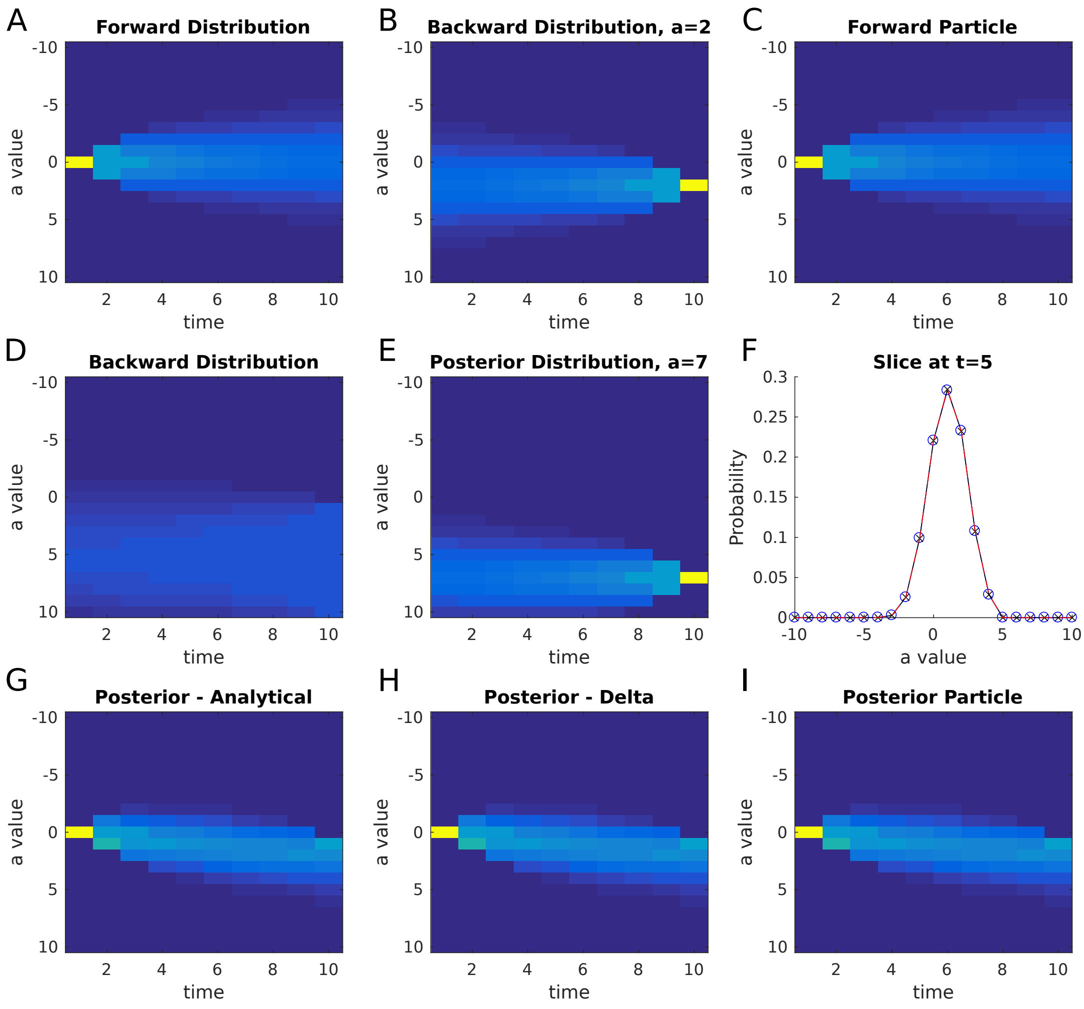
(4) Backward Model Solution

To solve each delta-backward solution $b_i(a)$, we use the time-reversed solution to the forward model.

(5) Toy Model example

To illustrate and confirm this method, I developed a toy model. The model is a random walk, where on each time step there is a 1/3 probability of staying in place, 1/3 probability of taking a step of size +1, and a 1/3 probability of taking a step of size −1. We start at $a(t = 0) = 0$, and then observe the process some time later $a(t = 10) > 0$. We now want to compute the forward model, and the posterior distribution for this data.

I can compute analytically the distribution of probability mass at any time forward, using a binomial distribution. In panel A of the figure below, we see the forward model. Which tell us the probability of observing the random walk at each accumulated value, at each point in time. Then, I can compute the backward model, which tell us the probability of the random walk, given the future observation $a(t = 10) > 0$, panel D. Combining these two independent distributions, I can predict the posterior distribution, which is all possible states the random walk could be in given these two observations. This is shown in panel G. Note here that we do not need to weight the initial distribution used in the backward model by the forward model's prediction for $t = 10$. Because the forward and backward model are independent, the backward model only needs to consider equal probability of all $a > 0$ states.



This simple model allows a direct computation of the entire backward distribution directly, but our accumulation model requires the composition of delta-backward solutions. To check this, I computed delta-backward solutions starting at all values of a for $a > 0$. Panels B and E show two sample solutions starting at $a = 2$ and $a = 7$. Panel H shows the posterior distribution composed of the mixture of each individual delta solution and the product with the forward model. All individual solutions are weighted equally.

To confirm the model solution, I simulated $2,000,000$ sample random walks starting at $a(t=0) = 0$. The distribution of these random walks is shown in panel C. This sampled distribution matches the analytical solution on panel A. To check the posterior distribution, I then filtered the sampled random walks to select for only the random walks which ended at $a(t=10) > 0$. Panel I shows the distribution of these filtered samples, which matches the analytically predicted distribution. To confirm that individual time-point distributions match closely, Panel F shows the predicted (red), and sampled (blue) distributions for $t = 5$.

We can now move on to the accumulation model, which has the same basic random walk structure, but with a few more bells and whistles.

(6) Accumulation Model Solution

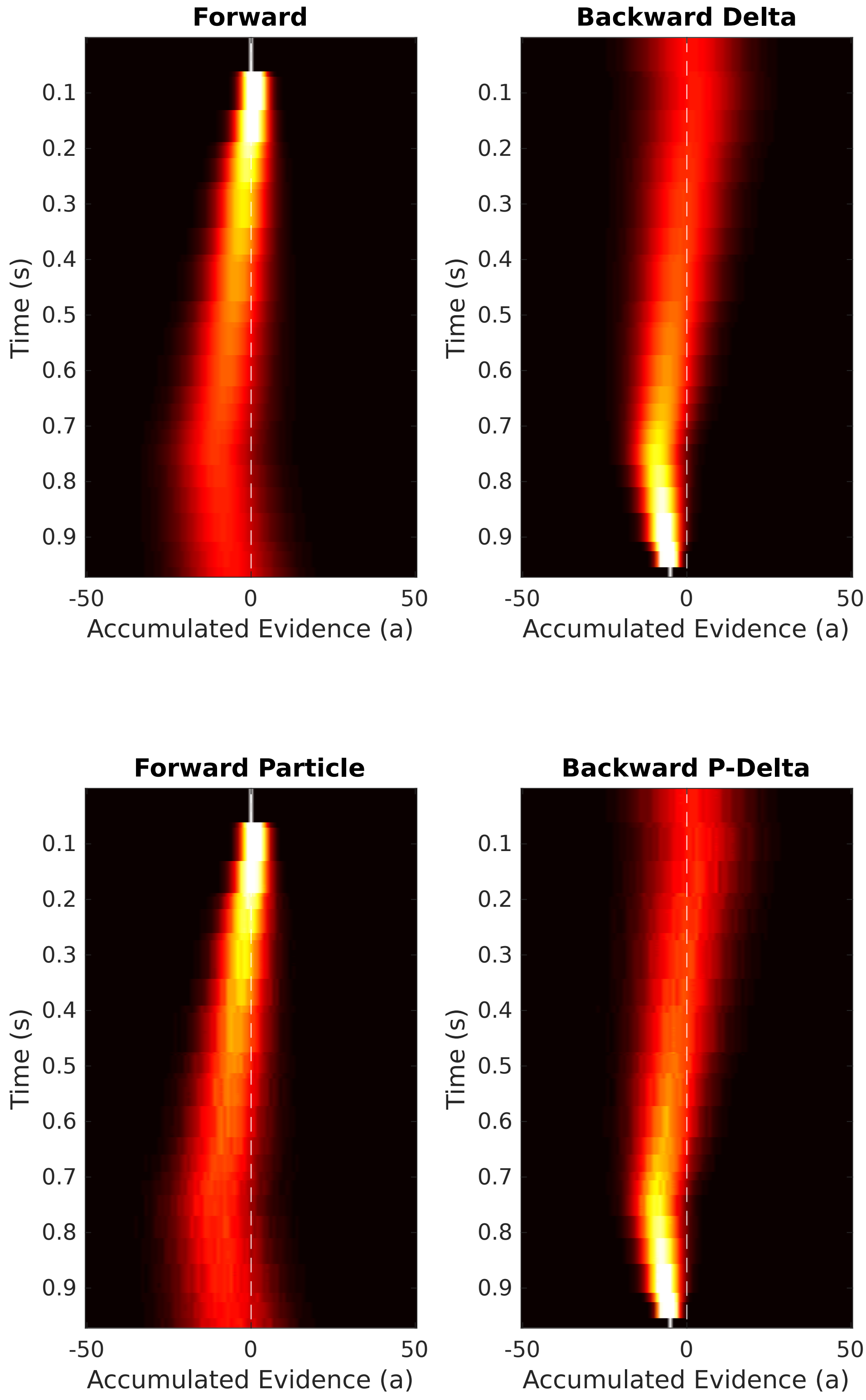
To evaluate my implementation of the accumulation model, I present a series of plots comparing elements of my analytical solution with the simulation of 10,000 sample trajectories of the model. Each trajectory has a unique noise realization. For both the model and sample trajectories, I will show the forward distribution, an example delta-backward distribution and its corresponding delta-posterior distribution, and the entire posterior distribution. Matlab's 'imagesc()' function can be deceptive with its color scaling, so I will also show verification plots with the mean and variance of each of the model components across time, as well as a few example "slices" at different points in time. The analytical solution is in blue, and the particle trajectories are in red.

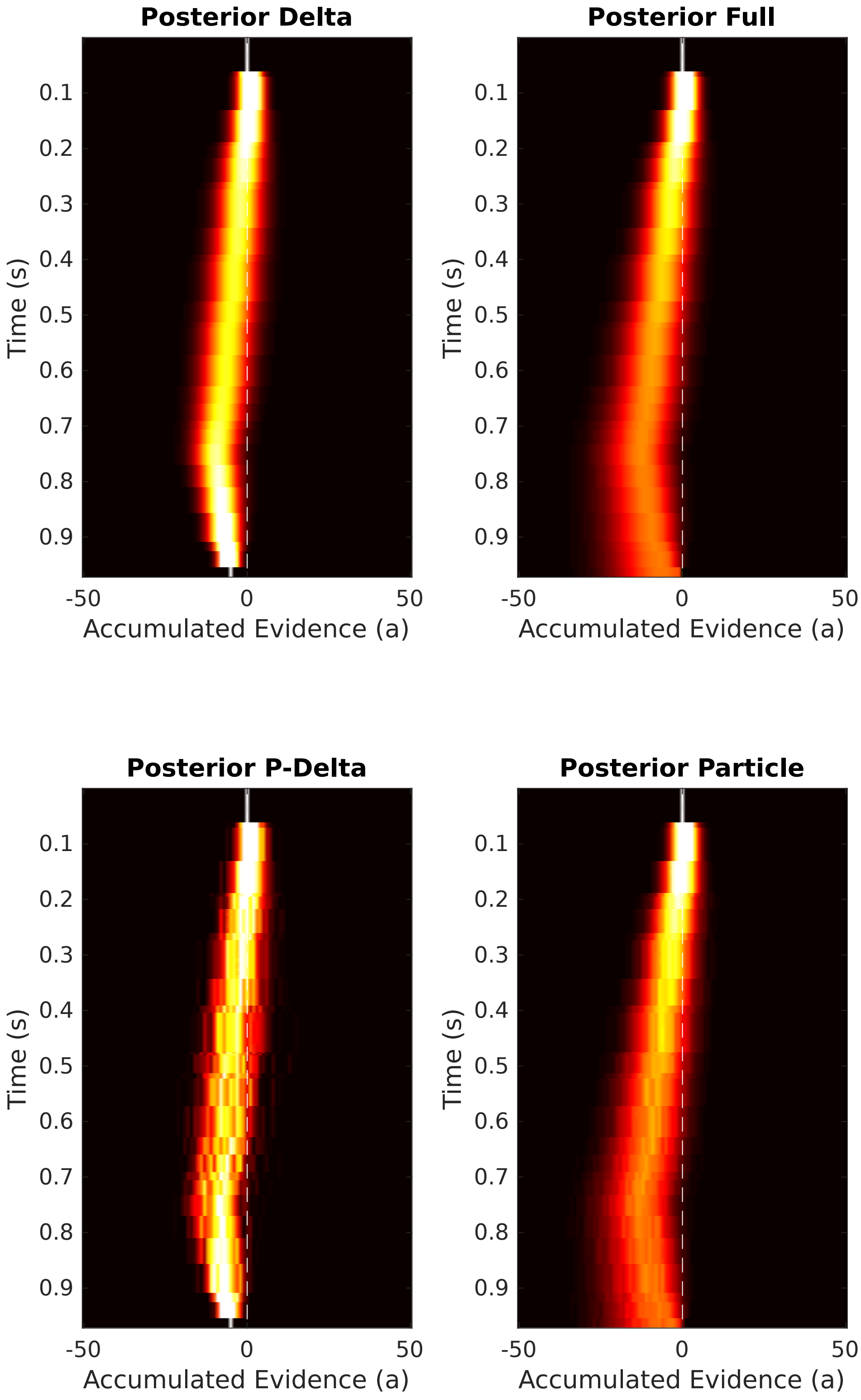
Note that the example delta-posterior solution of the sample trajectories suffers from low-n, because if I simulate 10,000 particles, only 500 or so end up in the right bin. That is why the variance deviates a bit on these plots. Luckily the entire posterior distribution does not suffer from this problem. The entire solution was computed using 51 delta-backward solutions on a grid from $-50 : 1 : 0$.

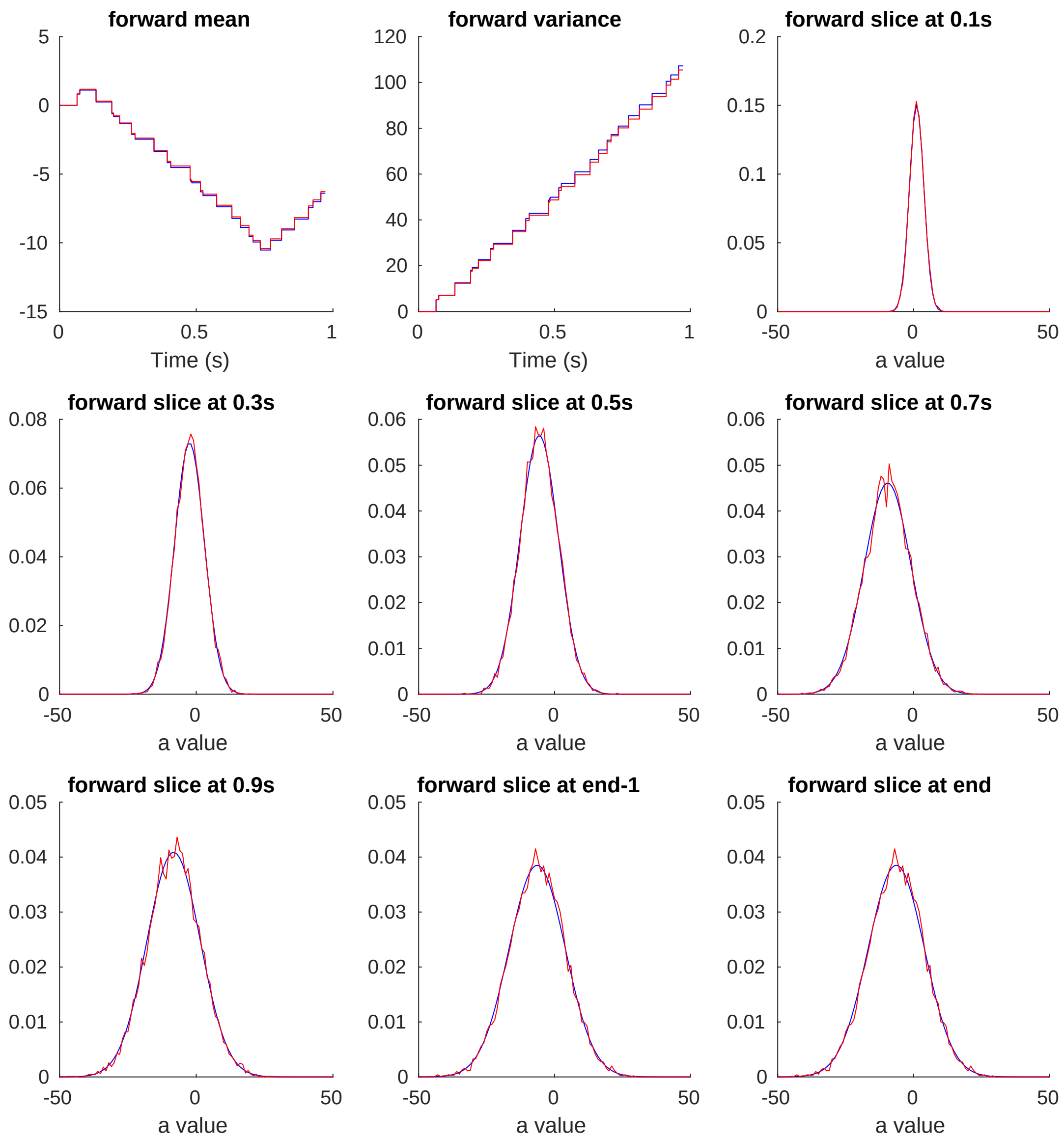
A few notes on the advantages of the analytical solution. First, a large increase in accuracy of the model. Second, I can compute the posterior distribution for only a subset of all time points without computing the solution for all time points. This fact allows for very rapid computation of the posterior distribution.

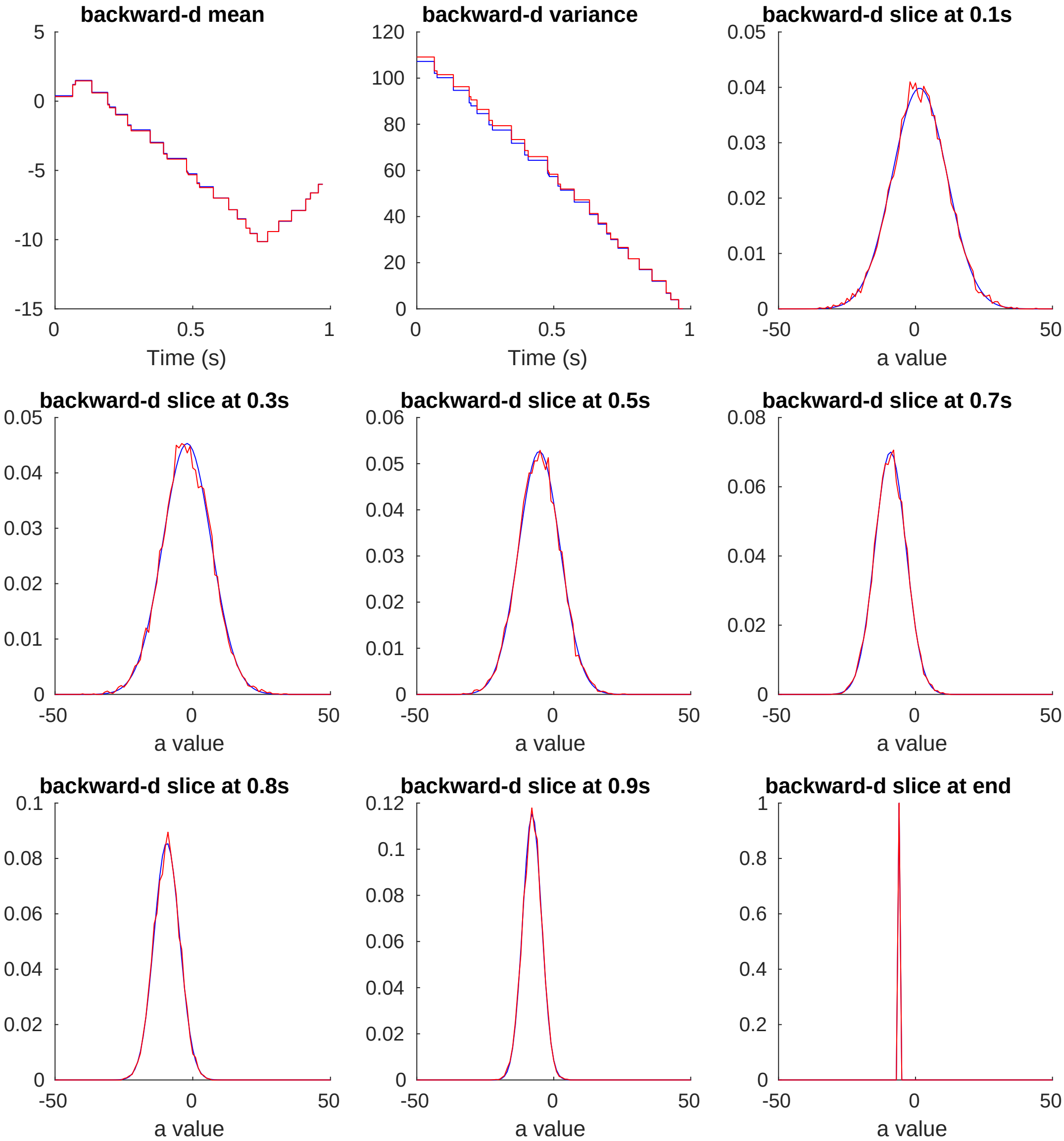
(7) In progress

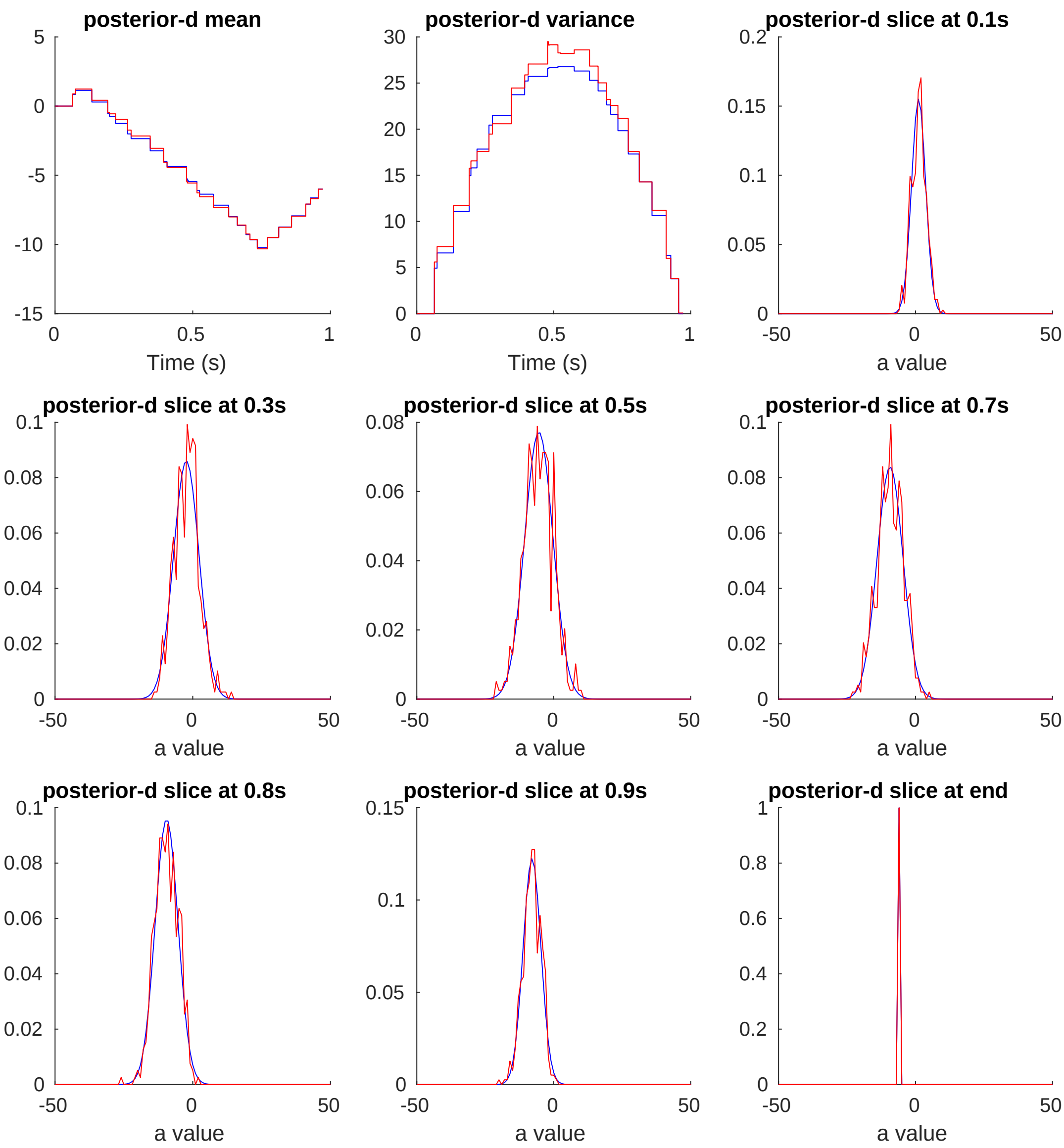
I am working on a few things right now. (1) Speeding up my implementation. (2) Optimizing how many, and which, delta-backward solutions I need to compute using an adaptive mesh of solutions. (3) Integrating my code with Hanks, 2015 style tuning curve analysis. I should be able to only compute the posterior at the time points with spikes, meaning the posterior distribution probably doesn't have to be computed ahead of time. (4) When the total variance for the trial is low, there are some numerical instabilities, and I should have a separate function that handles that case separately.

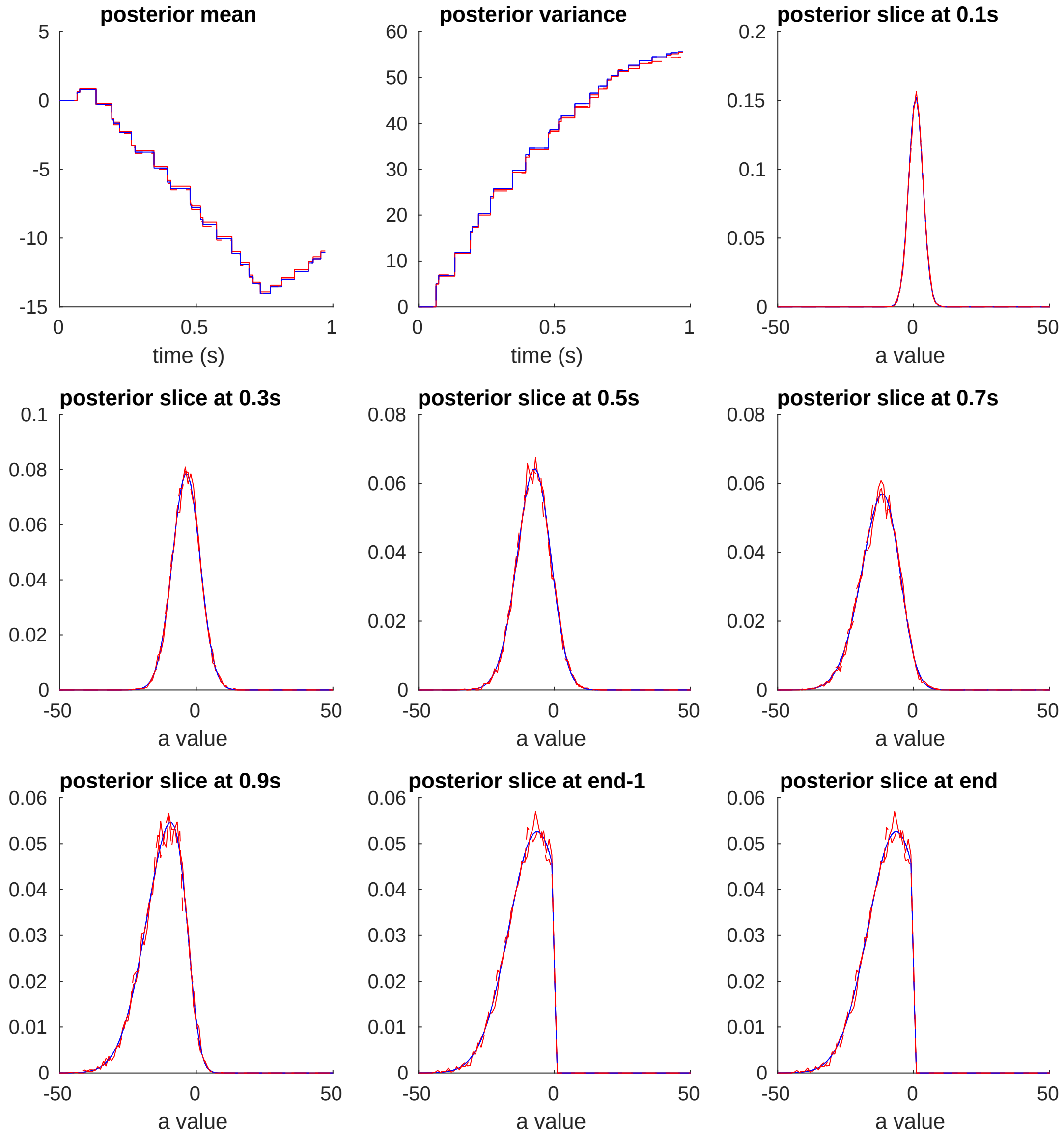

Forward
Backward Delta
Forward Particle
Backward P-Delta
Time (s)
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
-50
0
50
Accumulated Evidence (a)


Posterior Delta
Posterior Full
Posterior P-Delta
Posterior Particle
Time (s)
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
-50
0
50
Accumulated Evidence (a)


forward mean
Time (s)
forward variance
Time (s)
forward slice at 0.1s
a value
forward slice at 0.3s
a value
forward slice at 0.5s
a value
forward slice at 0.7s
a value
forward slice at 0.9s
a value
forward slice at end-1
a value
forward slice at end
a value


backward-d mean
Time (s)
backward-d variance
Time (s)
backward-d slice at 0.1s
a value
backward-d slice at 0.3s
a value
backward-d slice at 0.5s
a value
backward-d slice at 0.7s
a value
backward-d slice at 0.8s
a value
backward-d slice at 0.9s
a value
backward-d slice at end
a value


posterior-d mean
Time (s)
posterior-d variance
Time (s)
posterior-d slice at 0.1s
a value
posterior-d slice at 0.3s
a value
posterior-d slice at 0.5s
a value
posterior-d slice at 0.7s
a value
posterior-d slice at 0.8s
a value
posterior-d slice at 0.9s
a value
posterior-d slice at end
a value


posterior mean
time (s)
posterior variance
time (s)
posterior slice at 0.1s
a value
posterior slice at 0.3s
a value
posterior slice at 0.5s
a value
posterior slice at 0.7s
a value
posterior slice at 0.9s
a value
posterior slice at end-1
a value
posterior slice at end
a value