A new information dimension of complex network based on Rényi entropy

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HIGHLIGHTS

- Rényi dimension is proposed based on Rényi entropy and the generalized box-covering algorithm.
- SWF and PWF can explain the changing trend of Rényi dimension and Tsallis dimension.
- The concept of network attractors and absolute attraction network (AAN) are proposed in this paper.
- The physical meaning of the parameter α, SWF and PWF are discussed according to attractors in the network.
- Some experiments of the real networks and theoretical networks are applied in this paper.

ARTICLE INFO

ABSTRACT

With the development of high technology and artificial intelligence, it evolves into an open issue to calculate the dimension of the complex network. In this paper, a new dimension — Rényi dimension, combined with Rényi entropy and information dimension is proposed. A modified box-covering algorithm is introduced to calculate the minimum number and the length of the boxes needed to cover the whole network. Additionally, the self weight factor *(SWF)* and the positive weight factor *(PWF)* are defined to illustrate the change of the dimension value based on the perspective of both topology structure and dynamic property. The concept of attractors is proposed to illuminate the physical meaning of the weighted parameter in the formula of Rényi entropy — α, *PWF* and *SWF*. Finally, to demonstrate the efficiency of our method, it is applied to calculate the dimension of Sierpinski weighted fractal network, BA networks and many real-world networks. The results show that attractors exist in the network researched and α can access the attractiveness of attractors as a criterion. The *SWF* quantifies the total attractiveness of attractors. The comparison results with Tsallis dimension indicate the stability of the Rényi dimension.

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1. Introduction

Recently, complex networks have aroused great interests since they are applied to describe a wide range of systems in nature [1] and society [2]. The topology of complex network [3,4] can clearly illustrate many complicated systems in many subjects such as the social network of interpersonal relationships [5], the WWW [6] and the Food Web [7]. The small world [8,9] and the scale-free [10] have been acknowledged to be two fundamental properties of complex network. The small-world property of complex networks has many applications in the real world such as brain work mechanism [11,12]

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and information flow [13]. Similarly, the free-scaling of network is often used for testing the network performance under attacking [14,15], reaction–diffusion [16,17] and the targeted safeguard strategy [18]. BA network model is proposed by AL Barabási and R Albert to illustrate the scale-free property of the complex network [10,19]. In addition, the fractal [20] and self-similarity [21] firstly discovered by Song et al. [22] have also attracted many researchers. Many researches have been done about the properties [23,24]. In order to quantize the two properties above and demonstrate the structure of the complex network, entropy is introduced to calculate the uncertainty and the complexity of network [25,26]. With the further study of the nature of complex networks, many researches have been carried about the approach to estimate the importance of nodes [27] based on TOPSIS [28] and evidence theory [29,30] and lots of practical applications especially in ClusterRank [31], LeaderRank [32], the dynamic control strategies [33] and the construction of co-author networks [34].

The concept of dimension is developed after the discover of space-filling curves [35]. The fractal dimension value extends to the positive real number range as the theory of fractals is developed and improved by many mathematicians such as Mandelbrot [36] and Hutchinson [37]. The fractal dimension is introduced to many fields [38,39]. It is widely used in the field of complex network as an important network evaluation parameters [40,41]. To calculate the dimension of complex networks, the box-covering algorithm is proposed by Song et al. [22] and many improvement solutions have been designed to complete the algorithm including the way of finding the minimal [42] and maximal sizes of covering boxes [43] and the minimal partition coverings method [44].

The classical dimension, such as information dimension [45] and Hausdorff dimension [46], is designed to quantize the degree of fractal and complexity of the networks [47]. However, these dimensions are usually used for calculating the network dimension in planar area. It is segmentary to study a complex network in this way for some important information will be neglected unavoidably. So the modified box-covering algorithm proposed by Wei [48] is introduced to fill the requirements of comprehensiveness of network information and computational efficiency and some researches related to information dimension of weighted network [49] are studied. Additionally, compared with other dimensions combined with information entropy [50] or Tsallis entropy [51], the existence of parameter α makes the proposed method more flexible and expands its use in many different situations. Tsallis' entropy is another general form of Shannon entropy [52] controlled by q [53], which was firstly proposed by Tsallis in 1988. Now, with the development of information theory, Tsallis' entropy [54] is applied in many aspects, such as medical image analysis [55] stellar polytropes [56], cardiovascular physics [57] and communities detection [58]. Meanwhile, the generalized dimension attracts much attention [44,59,60]. Due to the mathematical qualities of Rényi entropy [61,62], the Rényi dimension values are more stable than that of Tsallis dimension.

Generalized dimension refers to the dimension which is controlled by certain parameter and degrades into fractal dimension, information dimension when setting the parameter as some fixed values. It can be divided into different types according to the various deductive methods and diverse application domains. Two of the most commonly adopted generalized dimension are Tsallis information dimension and Rényi dimension. In literature [63], Liu and Yu applied the generalized dimension to characterize the complexity of complex networks with the help of multi-fractal analysis and sandbox algorithm. In the recent work of Rosenberg [44], a generalized dimension proposed by Grassberger and by Hentschel and Procaccia in 1983, is adopted to calculate the densest part of complex network and solve the problem about minimal partition covering in box-covering algorithm. Wu [64] studied lossless analog compression in information-theoretic framework by strict theoretical derivation under the fundamental limit of the information dimension proposed by Rényi in 1959. Compared with the research results of similar study field mentioned above, we focus on the close integration of Rényi dimension of complex network and the modified box-covering algorithm. The discussion about the physical meaning of parameter and dimension takes both the topology structure and the dynamic property into account in order to illuminate the changing rule of the dimension from a practical perspective.

In the following part of the paper, the Rényi entropy and the classical dimension are introduced in Section 2. The modified box-covering algorithm is also illustrated in the same part as a preparation for the proposed method. The Rényi dimension of complex network inspired by Rényi entropy is proposed in Section 3. The *SWF* and *PWF* are also discussed in this part in order to figure out the changing rules of dimension. The definition of absolute attraction network (*AAN*) is clarified in the same part. Some applications in the theoretical network and real networks are listed in Section 4. Additionally, in Section 5, the Tsallis information dimension is also studied and explained by the *SWF* and *PWF* as a contrast. Proper interpretation about the physical meaning of parameter α are added as an analysis of the mass dimension results. Finally, Section 6 summarizes the conclusions of the whole paper.

2. Preliminaries

2.1. Rényi entropy and Tsallis' entropy

Shannon firstly defined the concept of Shannon entropy in order to calculate the uncertainty of a probability distribution [61]. The Shannon entropy has become one of the best-known measures of uncertainty, the headstone of the modern communication theory so far and it is commonly used for describing the uncertainty relations [65,66] and quantum mechanics predictions.

Definition 2.1. Shannon entropy is defined as follows [67]:

$$H(X) = \sum_k^n p_k \log_2 \frac{1}{p_k}$$

(1)

Where $X = (p_1, p_2, \ldots, p_k)$ is finite discrete probability distribution, which means $p_k \geq 0 (k = 1, 2, \ldots, n)$ and $\sum_{k=1}^n p_k = 1$

As information science advances, Shannon entropy may not meet all the requirements of diverse application domains. For further application of the entropy and more options to calculate the uncertainty of generalized distribution, Alfréd Rényi proposed a new entropy inspired by Shannon entropy. Rényi entropy is a family of entropies, which can be reduced to several important special cases by changing the parameter α [68]. This entropy conserves many convenient mathematical properties such as additivity.

Definition 2.2. The Rényi entropy of order α is given as [61]:

$$H_\alpha(X) = \frac{1}{1 - \alpha} \log \sum_{k=1}^n p_k{}^\alpha$$

(2)

For $\alpha \in (0, 1) \cup (1, \infty)$, and the corresponding limit for $\alpha \in \{0, 1, \infty\}$. For $\alpha = 1$, the limit of Rényi entropy can degenerate to the Shannon entropy, so the Rényi entropy is the generalization of Shannon entropy.

Definition 2.3. The Tsallis' entropy of order q is shown as follows [51]:

$$S_q \equiv k \frac{1 - \sum_{i=1}^N p_i^q}{q - 1}$$

(3)

Where N is the total number of the elements of the probability set and p_i represents the corresponding probabilities. When the order q equals to 1, the Tsallis' entropy degenerates to the Shannon entropy.

The order α in the Rényi entropy is usually compared with the order q in the Tsallis' entropy to analyze the stability of them in the rapidly and slightly changing situation [69,70]. In this paper, the two entropies are applied to calculate the dimension of complex network and evaluated according to the value of dimension in Section 4.

2.2. Box-covering algorithm

Given a complex network, $G = (N, V)$ and $N = (1, 2, \ldots, n)$, $V = (1, 2, \ldots, m)$ where n is the number of nodes and m is the number of edges. G is a matrix which can express all the necessary information of a complex network.

Definition 2.4. The strength of a weighted network is presented as follows:

$$S_i = \sum_{j=1}^n w_{ij}$$

(4)

where i is the focal node and j represents all other nodes, w_{ij} is the weight between i and j directly.

Definition 2.5. The shortest path between two nodes i and j, d_{ij} is defined as follows:

$$d_{ij} = \min(w_{ij_1} + w_{j_1j_2} + \cdots + w_{j_kj})$$

(5)

where $j_k(k = 1, 2, \ldots)$ represents the IDs of the other nodes except i and j. The maximum of d_{ij} is defined as the diameter of a complex network.

Song et al. [22] have proposed a box-covering algorithm to calculate the fractal dimension of complex network by choosing a beginning node randomly and averaging the results of huge amount of experiments which takes rather long time to get the final result. Usually to get a representative value of dimension, we must repeat the algorithm over 1000 times and get the average value. So here we adopt Wei's method [48], which is modified to be easily calculated the weighted network. The total running time and space complexity of Wei's algorithm are nearly 5 times shorter than that of the box coverage algorithm which takes 0.023 s to calculate the dimension of a BA network with 1000 nodes. What is more, the result of the theoretic network — Sierpinski network in the fourth section is 1.4663 which is close to the theoretic value 1.5850. As a result, the algorithm can not only save computing time but can also represent the average result. The procedure is illustrated as follows:

1. Calculate the size of box l_B by adding up the weights from small to large until the sum is greater than the diameter of the network.
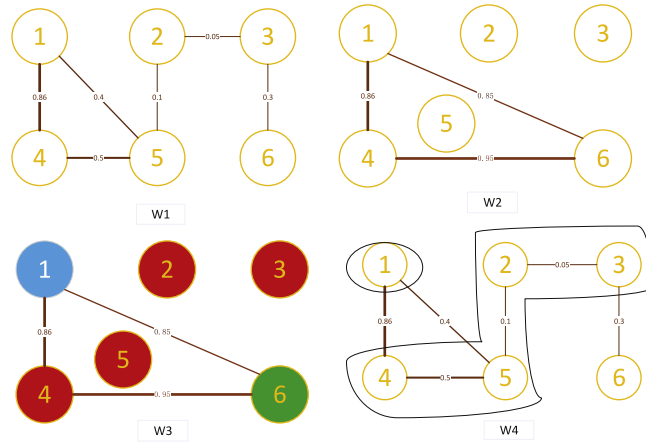
Fig. 1. The process of coloring the network. $W1$ is a weighted network with six nodes and six edges. The diameter of the network is 0.95. So we choose $0.85(0.05 + 0.1 + 0.3 + 0.4 = 0.85)$ as the box size and connect the nodes i, j with $d_{ij} \leq 0.85$ and obtain $W2$. Color the nodes connected in $W3$. The nodes 1, 4, 6 have different colors and the other nodes share the same color with node 4 because it has the maximum strength.

2. Connect those pairs of nodes in the original network whose distance is equal or greater than l_B and a new network G' will be obtained .

3. Color the nodes which are connected directly in network G' with different colors which means they belong to different boxes. The left isolated nodes will be arranged to the box with the node which holds the maximum strength.

4. Count the number of colors used to cover the boxes and then yield the number of boxes needed to cover the whole network. Calculate the information content I according to the color scheme.

5. Add the edge weights again until the sum is less than the first l_B and get the refreshed box size. Then repeat the step 2 to 4 in order to get a series of information content and box size respectively.

6. Perform a linear regression of $ln(I)$ and $ln(l_B)$. Choose the slope as the dimension of the complex network.

The detailed algorithm to calculate the box number is given in Fig. 1.

After calculating the minimum number of the boxes N_B of a complex network, the Hausdorff dimension, also known as the box dimension, can be obtained through the formula written as follows:

$$d_H = -\lim_{l_B \to 0} \frac{\log N_B}{\log l_B} \tag{6}$$

2.3. Information dimension and Tsallis information dimension

Information dimension is firstly used for estimating the information load on decision making [71] and measure strange attractors. Wei et al. [72] defined an information dimension based on information entropy and box-covering algorithm.

Definition 2.6. The information content concluded in a complex network can be calculated as follows:

$$I = -\sum_{i=1}^{N_B} p_i \ln p_i \tag{7}$$

Where p_i represents the probability of nodes in the ith box, which can be written as follows:

$$p_i = \frac{n_i}{n} \tag{8}$$

Where n_i is the number of nodes in the ith box, and n is the total number of nodes in the network. So the information dimension of the network is defined as follows:

$$d_I = -\lim_{l \to 0} \frac{I}{\log l_B} = \lim_{l \to 0} \frac{\sum_{i=1}^{N_B} \frac{n_i(l_B)}{n} \log\left(\frac{n_i(l_B)}{n}\right)}{\log l_B} \tag{9}$$

Where l_B represents the length of the boxes needed to cover the network, $\frac{n_i(l_B)}{n}$ is the probability of nodes in the ith box when the box edge is l_B.

The Tsallis information dimension is proposed by Zhang et al. [73], inspired by the Tsallis' entropy, with the aim of explaining the structure complexity of the complex network and reflecting the degree of self-similarity and fractal properties.

Definition 2.7. The Tsallis information dimension is defined as follows [73]:

$$d_T = \lim_{l_B \to 0} \frac{\frac{1 - \sum_{i=1}^{N_B} p_i(l_B)^q}{q - 1}}{\ln l_B} \tag{10}$$

According to the Eq. (10), the information content of the Tsallis' entropy is rewritten as follows:

$$I_T = \frac{\sum_{i=1}^{N_B} p_i(l_B)^q - 1}{1 - q} \tag{11}$$

Where l_B represents the length of the covering boxes and the p_i is the probability associated with the box-covering results. q is the restraining parameter of the generalized entropy.

3. Rényi dimension

There are many dimensions which can be used for illustrating the complexity and the uncertainty of the complex network. However the existing dimensions generally have two drawbacks − one is that they lack of flexibility because of the fixed formula (For instance, the information dimension is a certain value when it is applied on one complex network while the Rényi dimension can change as the variation of α), while another is that the topology structure and the dynamic feature of complex network cannot be balanced and the computing time is usually too long because of the average box-covering algorithm.

With the aim of improving the situation mentioned above, a new dimension is proposed to calculate the complexity of complex network which is more flexible to use than some dimensions of the same kind because of its weighted parameter α in this section. The new dimension is inspired by the Rényi entropy and the information dimension.

3.1. Basic method

Definition 3.1. The Rényi dimension is defined as follows:

$$d_R = \lim_{l_B \to 0} \frac{\frac{1}{1 - \alpha} \log \sum_{k=1}^{n} p_k{}^\alpha}{\log l_B} \tag{12}$$

where l_B is the box size of the box-covering algorithm. The numerator is the Rényi entropy defined in Eq. (2).

When $\alpha = 1$, the Rényi dimension degenerates to information dimension which can be easily proved using L'Hospital's rule. When $\alpha = 0$, the dimension is exactly the classical Hausdorff dimension. Just as the Eq. (12) shows, the change trend function of dimension cannot be judged directly. In other words, it is confusing to view the whole formula as an entire part. As a result, it is a simple and intuitional approach to separate the Rényi entropy into two impact factors according to the connection between the network structure and the mathematical property which are defined as positive weight factor and self weight factor. Here we dismantle the Rényi entropy in order to illustrate the meaning of α in the process of dimension calculation.

Definition 3.2 (*Positive Weight Factor (PWF)*)**.** In order to distinguish the influence of the weight parameter α, a positive weight factor *(PWF)* w_p is defined as

$$w_p = \frac{1}{1 - \alpha} \tag{13}$$

where α is the parameter of Rényi entropy.

This factor is completely decided by the value of α, which can be given by different researchers according to different analysis measures they apply to complex network. According to the formula of the function of *PWF*, the value of *PWF* increases with the increasing of α.
Case 1 When $\alpha \in (0, 1)$, the value of *PWF* is positive, which contributes to the increase of the final result of Rényi entropy. The more belief the expert gives to a certain measure, the more important *PWF* will be;
Case 2 When $\alpha \in (1, \infty)$, the value of *PWF* becomes negative. So the factor influences the entropy in an opposite way, though the function is still incremental. And the importance of α begins to descend, since too much weight will be taken as invalid measure.

Definition 3.3 (*Self Weight Factor(SWF)*)**.** Considering the structural features of complex network, a self weight factor w_s is proposed, indicated by

$$w_s = \sum_{k=1}^{N} p_k{}^\alpha \tag{14}$$

Table 1
The dimension of different network with different complexity.

α	Probability distribution	SWF
0.5	(1,0)	1.000
0.5	(0.6,0.4)	1.407
0.5	(0.5,0.2,0.3)	1.702
0.5	(0.4,0.3,0.2,0.1)	1.943
0.5	(0.3,0.3,0.2,0.1,0.1)	2.175
2.0	(1,0)	1.000
2.0	(0.6,0.4)	0.520
2.0	(0.5,0.2,0.3)	0.380
2.0	(0.4,0.3,0.2,0.1)	0.300
2.0	(0.3,0.3,0.2,0.1,0.1)	0.240

The *SWF* is influenced by the complexity of the network construction and the given parameter α. $X = (p_1, p_2, \ldots, p_k)$ represents the original feature of a network, while α indicates the weight given by the users. For each p_k belongs to $(0, 1)$, the value of *SWF* increases when α pertains to $(0, 1)$ and decreases when α belongs to $(1, \infty)$. Also the value of *SWF* is greater than 1 when α belongs to $(0, 1)$, while the value is less than 1 when α changes in the domain of definition greater than 1.

For the Tsallis information dimension, Eq. (11) can be regarded as the combination of *SWF* and *PWF* when the parameter α is replaced by the parameter q in the Tsallis' entropy. α and q play quite similar roles in the two dimensions. The two information content based on the Rényi entropy and Tsallis' entropy can be rewritten as:

$$I_R = w_p \log w_s \tag{15}$$

$$I_T = w_p(w_s - 1) \tag{16}$$

Where w_p and w_s are the positive weight factor and the self weight factor.

3.2. Example explanation

A succinct example is taken to interpret the effect of *SWF*. It is assumed that there are four different networks, the edge of the boxes needed to cover the whole network is a fixed value. When setting the length of the covering box as a fixed value, the metric is certain, the original characteristic of complex network is the only factor that influences the *SWF*. As a result, the increment of the number of the elements means the complexity of the networks is increasing. According to Table 1, two values of α are chosen to show the changing rule dominated by *SWF*. It is revealed by the data in Table 1 that when setting α as 0.5, the increase of the boxes leads to the amplification of the complexity of network and the value of *SWF*. The result is exactly the reverse when $\alpha = 2.0$.

Case 1 When $\alpha \in (0, 1)$, the value of *SWF* can directly reflect the complexity of network. However, restricted by the formula of *SWF*, the increment is not quite large. It remains rather steady when the weight parameter changes.

Case 2 When $\alpha \in (1, \infty)$, the value of *SWF* is degressive so the *SWF* must be taken as a negative feedback of the structure of network.

For a network, the degree of complexity is decided by the nodes and the edges it has. *SWF*, a factor connects closely with the structure, can reflects the property of the structure cell and the similarity of the cells. As is demonstrated in Table 1, the more complex the networks become, the less increment the *SWF* has. This can demonstrate that the complex networks will have more similar parts when the number of nodes and edges grow, which is a proof of the existence of the complex networks' fractal property. The feature is evident especially when calculating the *SWF* of some typical fractal networks. Some detailed explanations are listed in the Section 4.

Taking both of the factors into consideration, the final trend of Rényi entropy can be educed. Nevertheless, it remains to be proved when one of factors plays a decisive role with the increase of weight parameter α. Therefore there emerges a milestone in the series of Rényi dimension.

3.3. The physical meaning of the weight parameter α and SWF

Inspired by strange attractors that cause nonlinear effects of the dynamical system, we try to find attractors in a growing network. The nodes which maintain the larger degree (in unweighted network) or strength (in weighted network) during the process of growth are defined as the attractors of the network, because having greater degrees is more likely to have an impact on the growth of the network and the connection of other nodes, which symbolizes greater appeal of the attractors. The attractors are usually the nodes which are colored firstly in the box-covering algorithm. In this way, the ordinary parameter α gets the new physical meaning and represents the parameters for assessing the attractiveness of the attractor. According to the definition of *SWF* in Eq. (5), the meaning of α is illustrated as follows.

1. When $\alpha > 1$, with the evaluation criteria α increasing, the boxes with less attractors are being phased out because it does not reach the attractiveness required by α, it means that $p_k{}^\alpha$ will be small enough to be ignored for some weak attractors. And the *SWF* value gradually becomes smaller, so α exhibits strong screening ability.

2. When $1 > \alpha > 0$, α also reflects weak screening ability, but the requirement of the value of $p_k{}^\alpha$ is not very demanding. Take a pellucid example, $0.1^{0.1} = 0.79432$ is not quite different from $0.9^{0.1} = 0.98952$ while $0.1^{10} = 10^{-10}$ has ten orders of magnitude difference with $0.9^{10} = 0.34867$.

As a result, the physical meaning of *SWF* refers to the total attractiveness of the attractors in the network under the evaluation of a certain attracting ability index α.

Case 1: $\alpha = 0$, *SWF* is equal to the number of boxes required by the box-covering algorithm, indicating the number of potential attractors.

Case 2: $\alpha = 1$, *SWF* equals to 1, the attractor is in equilibrium and does not exhibit attractive qualities, which only reflects the sum of p_k.

Case 3: $\alpha > 1$, *SWF* embodies attractiveness under strong screening conditions.

Case 4: $1 > \alpha > 0$, *SWF* reflects attractiveness while the screening condition is weak.

In order to clearly give an index that quantifies attractors and attractive forces, we give the following definition according to α and *SWF*.

Definition 3.4 (*Absolute Attraction Network (AAN)*).

$$w_{sA} = \sum_{k=1}^{N} p_k{}^{10} \tag{17}$$

A absolute attraction network *(AAN)* is a network whose w_{sA} is greater or equal to 0.10737 ($0.8^{10} + 0.2^{10} = 0.10737$). When $w_{sA} < 0.10737$, the network is defined as non-absolute attraction network *(NAAN)*.

The definition shows that the complex network which has attractors with rather great degree or strength is more likely to become *AAN*. This can be proved by the box-covering algorithm described in Section 2. The isolated nodes are arranged to the boxes which contains the nodes with largest degree or strength, then the p_k of these boxes will be large enough to influence the final result — *SWF* and make the network become *AAN* eventually.

4. Application

In this section, the proposed Rényi dimension is applied to calculate the series of dimension of a theoretical network — Sierpinski weighted fractal network, BA networks of different types and five real weighted networks, namely, the 500 busiest commercial airports in the United States (*top*500) [74], the Caenorhabditis elegans worm's neural network (*celeg*) [75], the collaboration scientists working on network theory weighted network (*Netscience*) [76], the collaboration scientists in computation geometry weighted networks (*goem*) [76] and the US Air-line network [76]. For comparison, the Tsallis' dimension is also applied in the real networks to show how the two factors mentioned effects the value of dimension. Also we discuss the physical meaning of two parameter α and *SWF* according to the dimensions of different networks. The PWF in Table 3 is fit for all the network and is only related to the weight parameter α.

4.1. Applications on Sierpinski network

Sierpinski network is a theoretical weighted network, which is featured by its regular fractal structure referring to Fig. 2. The length of the edge halves and the number of nodes increases threefold when the number of iterations increases by one at a time. As a result, the Hausdorff dimension can be precisely calculated according to its iterative rule, as follows.

$$d_H = -\lim_{l_H \to 0} \frac{\log N_B}{\log l_H} = \frac{\log 3}{\log 2} \doteq 1.585 \tag{18}$$

In Table 4, the Rényi dimension of Sierpinski weighted fractal network is listed. Furthermore, according to the result of Table 5, though the iteration is increasing from $G5$ to $G9$, the gap between two *SWF* is lower than 0.0035. This phenomenon also reveals that for a certain theoretical fractal network, the degree of self-similarity is quite high, and the self structure is merely decided by the most fundamental cell of the iteration no matter how larger the number of iteration is. For a network which has strong fractal property, *SWF* works as both a measure to testify the character and a reference to ascertain the primary geometry.

Influenced by both the *SWF* and *PWF*, the dimension of Sierpinski network progressively increases as the weight factor α increases. From $G5$ (which is contained by 40 nodes and 121 edges) to $G9$ (which consists of 3280 nodes and 9841 edges), the larger the number of iterations is, the lower the theoretical value and the actual value gap is for the ideal network is infinite. In order to reflect the average changing trend and the closest dimension to the theoretical network, the dimension of $G9$ is shown by Fig. 4.
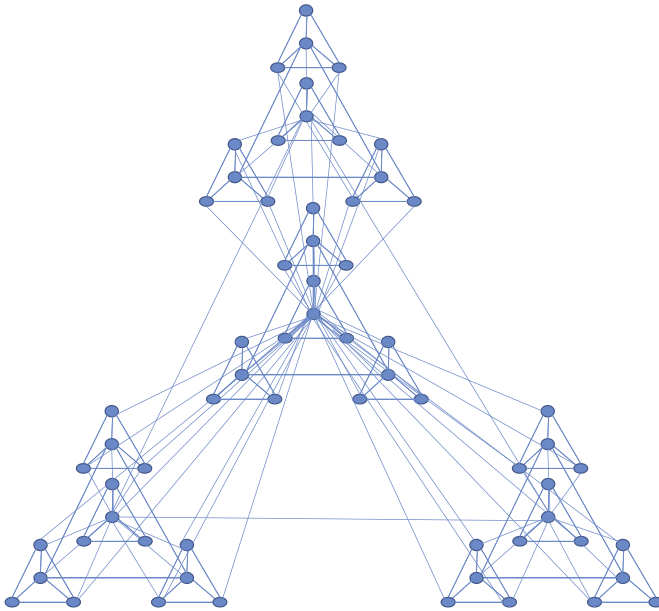
Fig. 2. Sierpinski weighted fractal network.

Table 2
The SWF of BA network as α changes.

α	0	0.5	0.8	1	1.5	2	3	5	10
SWF(BA66361)	6	2.0295	1.2733	1	0.6478	0.4799	0.3013	0.1318	0.0173
SWF(BA66362)	5	1.8636	1.2362	1	0.6801	0.5185	0.3401	0.1616	0.0261

4.2. Applications on BA networks of different types

AL Barabási and R Albert proposed a scale-free network model (BA model) to explain the power law generation mechanism [19]. The BA model has two characteristics. One is growth which refers to the continuous increase of the network size. In the research network, the nodes of the network are continuously increasing; The second is the preferential connection mechanism which means that the new nodes that are continuously generated in the network are more likely to be connected to those nodes that are more connected. It can be seen in Fig. 3. Rényi dimension increases as the parameter α increases expect when α equals to 1. The information dimension is much smaller than other dimension.

This can be explained by the generating rules of the BA network. The BA network is generated according to the power law, so it will obey the rules called "Rich get richer". It means that the node with the greater degree at the beginning is more likely to connect the new node during the growth of the network scale. As a result, the average path length of the BA network is rather short, and the cluster coefficient is also quite small. From the perspective of information entropy, the source information is mastered in a few important nodes which can be seen as attractors in the network, the uncertainty in the propagation process is reduced, so the entropy is smaller, and the dimension is smaller.

The result of *SWF* is a reflection of how many attractors the network has. The network with the name ended by 1 like *BA*66361, is the network whose initial network nodes are isolated nodes. The networks with the name ended by 2 like *BA*66362 begin with complete networks. According to Table 2, the *SWF* of *BA*66362 is larger than that of *BA*66361 when α is greater than 1. This shows that when α performs strong screening ability, the more complicated the initial structure of the BA network is, the larger the *SWF* is.

4.3. Applications on real networks

The result of real network dimension is presented in Table 7. In Table 7, d_I represents the information dimension. The value of d_I is the same as the Rényi dimension when the weighted parameter α equals 1, which suggests that in this case, the Rényi dimension degenerates to the box dimension. And it is evident that when setting $\alpha = 0$, the dimension is exactly the box dimension. The results show that there will emerge a inflection point with the increase of the parameter α. In summary, the trend of the dimension will increase at first and then decrease. The trend is caused by the irregular variation of the distance between two nodes and the uneven growth of the length of the boxes used to cover the whole network. Thus the *SWF* and the *PWF* cannot have a certain change rule and for different networks, the rule of the emergence of inflection
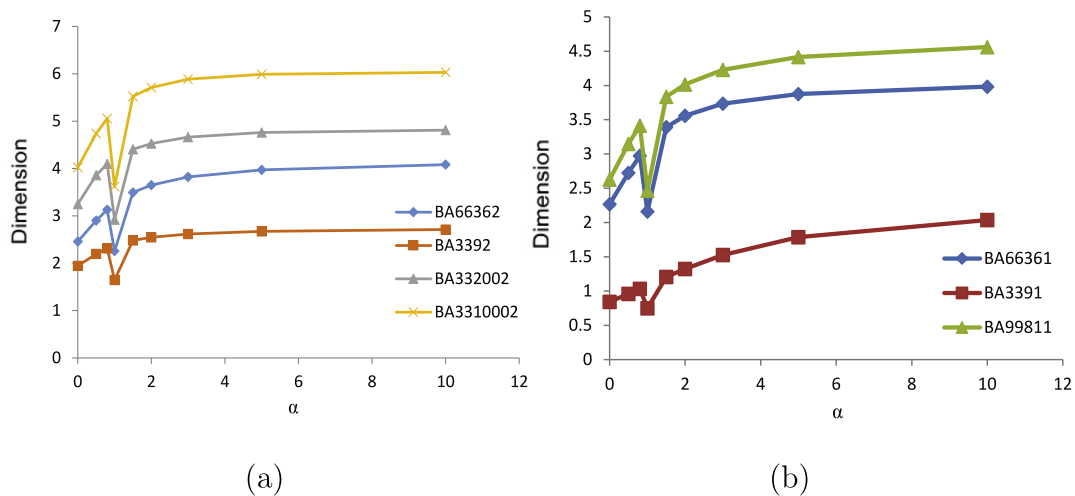
Fig. 3. The Rényi dimension of BA network belongs to two different types when the parameter α changes. (a) shows the dimension of BA networks whose initial nodes constitutes a complete graph while (b) refers to the dimension of BA networks with isolated nodes at the very beginning.

Table 3
The value of PWF.

α	0	0.5	0.8	1.0	1.5	2.0	3.0	5.0	10.0	100.0
PWF	1.0	2.0	5.0	–	−2.0	−1.0	−0.5	−0.25	−0.11	−0.01

Table 4
Rényi dimension of Sierpinski weighted fractal networks with different iterations.

	G5	G6	G7	G8	G9
d_I	1.2814	1.3378	1.3822	1.4170	1.4443
$d_{R(\alpha=0)}$	1.2425	1.3077	1.3558	1.3940	1.4245
$d_{R(\alpha=0.5)}$	1.2673	1.3272	1.3749	1.4122	1.4414
$d_{R(\alpha=0.8)}$	1.2763	1.3341	1.3798	1.4154	1.4434
$d_{R(\alpha=1.0)}$	**1.2814**	**1.3378**	**1.3822**	**1.4170**	**1.4443**
$d_{R(\alpha=1.5)}$	1.2920	1.3451	1.3871	1.4202	1.4465
$d_{R(\alpha=2.0)}$	1.3003	1.3509	1.3912	1.4233	1.4489
$d_{R(\alpha=3.0)}$	1.3124	1.3599	1.3981	1.4287	1.4532
$d_{R(\alpha=5.0)}$	1.3270	1.3713	1.4037	1.4362	1.4593
$d_{R(\alpha=10.0)}$	1.3424	1.3838	1.4174	1.4445	1.4663
$d_{R(\alpha=100.0)}$	1.3589	1.3968	1.4277	–	–

point is ambiguous. What is more, the *PWF* and *SWF* effect the final dimension in different degrees for different structure of networks. This two main reasons make the prediction of the inflection point remain an unsolved problem.

The Tsallis information dimension is combined with *SWF* and *PWF*. The *SWF* of Tsallis information dimension is exactly the *SWF* of Rényi dimension, and the value of *SWF* in the same real network decreases as the parameter increases according to Table 6. The *PWF* is fixed when given the weighted parameter and increases when the parameter increases. The final changing rule of Tsallis information dimension is discussed into two cases as follows:

Case 1 When $q \in (0, 1), (w_s - 1)$ attenuates with q grows while w_p changes inversely. The changing rate of $(w_s - 1)$ is greater than w_p. So the *SWF* plays a conclusive role in the attenuation of the Tsallis information content and the Tsallis information dimension.

Case 2 When $q \in (1, \infty), (1 - w_s)$ aggrandizes but $-w_p$ decreases more drastically when q expands gradually. In this way, the Tsallis information dimension is decided by the *PWF* and reduces accordingly.

Taking both the results in Table 8 and analysis above into consideration, the Tsallis information dimension monotonically decreases when the weight parameter q increases. Comparing the dimension proposed in this paper with Tsallis information dimension, the Rényi dimension is more stable because the logarithmic structure of Rényi information content imposes restrictions on the variation of the dimension.

5. Discussion

Through the comparison of the Sierpinski weighted fractal network and the real networks, the law of change about Rényi dimension is decided by the *SWF* and the *PWF*. The *PWF* is only decided by the parameter α which can be considered to be
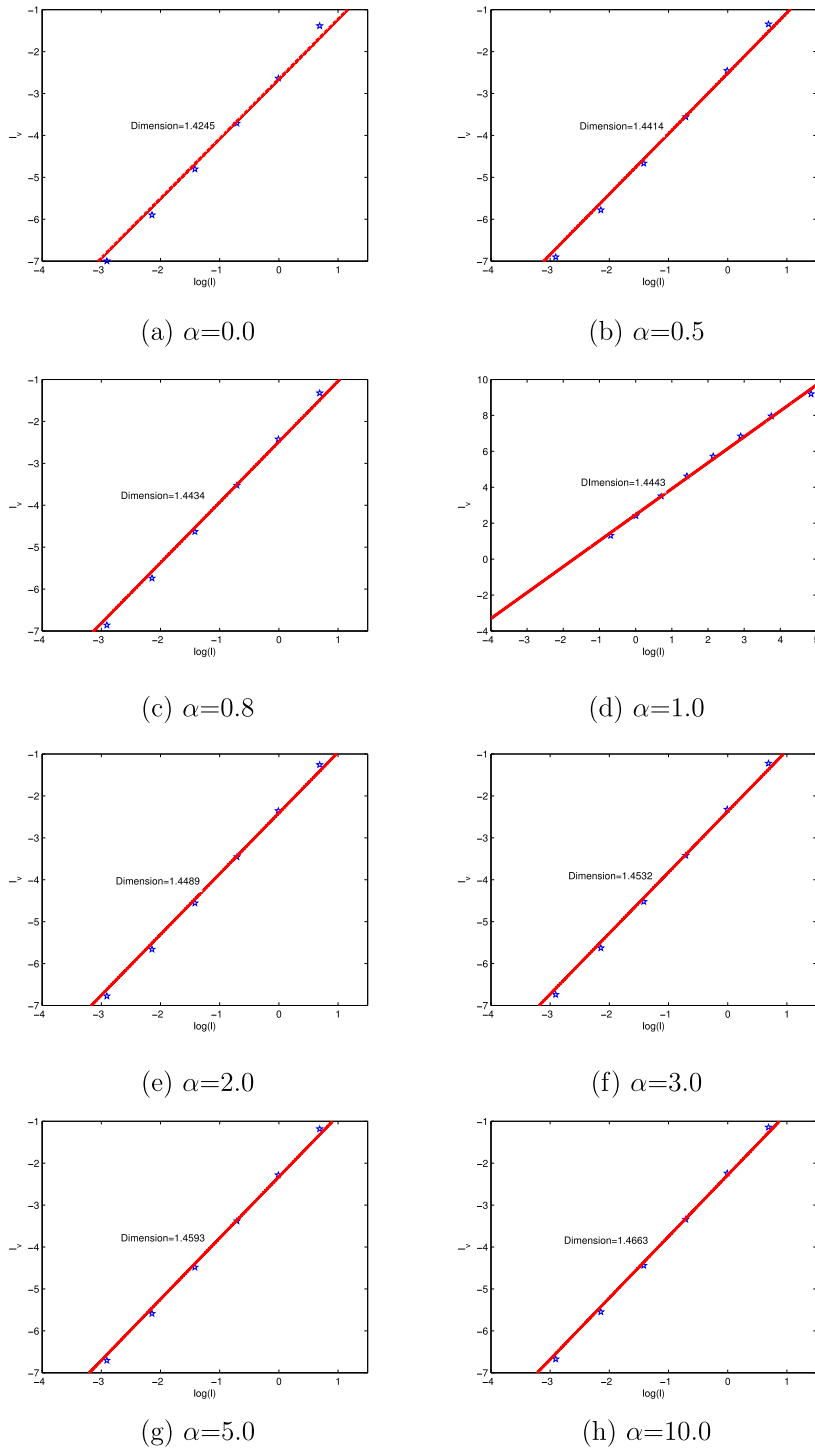
Fig. 4. The Rényi dimension of Sierpinski weighted fractal network. The x-axis − log(l) represents logarithm of the box side length and the y-axis − I-v is the Rényi information content. According to the box-covering algorithm, the slope is the value of Rényi dimension.

a parameter depended on expert's decision. The *SWF* is a part which can reflect the parameter α and the structural feature of the network.

Table 5
SWF of Sierpinski weighted fractal networks with different iterations.

SWF	G5	G6	G7	G8	G9
$\alpha = 0$	4.0000	4.0000	4.0000	4.0000	4.0000
$\alpha = 0.5$	1.9624	1.9605	1.9598	1.9596	1.9595
$\alpha = 0.8$	1.3043	1.3035	1.3033	1.3032	1.3031
$\alpha = 1.0$	1.0000	1.0000	1.0000	1.0000	1.0000
$\alpha = 1.5$	0.5249	0.5261	0.5265	0.5266	0.5267
$\alpha = 2.0$	0.2817	0.2832	0.2837	0.2839	0.2839
$\alpha = 3.0$	0.0849	0.0859	0.0863	0.0864	0.0864
$\alpha = 5.0$	0.0085	0.0087	0.0088	0.0088	0.0088
$\alpha = 10.0$	3.15E−05	3.32E−05	3.39E−05	3.40E−05	3.41E−05

Table 6
SWF of different real networks.

	top500	USAir	Netscience	geom	celeg
$\alpha = 0$	2.0000	2.0000	397.0000	2061.0000	2.0000
$\alpha = 0.5$	1.0437	1.0534	16.3878	31.4410	1.0564
$\alpha = 0.8$	1.0053	1.0072	2.8217	3.1506	1.0078
$\alpha = 1.0$	1.0000	1.0000	1.0000	1.0000	1.0000
$\alpha = 1.5$	0.9971	0.9957	0.1627	0.3557	0.9951
$\alpha = 2.0$	0.9960	0.9940	0.0603	0.2432	0.9933
$\alpha = 3.0$	0.9940	0.9910	0.0134	0.1198	0.9899
$\alpha = 5.0$	0.9900	0.9850	7.52E−04	0.0291	0.9832
$\alpha = 10.0$	0.9802	0.9703	5.65E−07	8.48E−04	0.9667

Table 7
Rényi dimension of different real complex networks.

	top500	USAir	Netscience	geom	celeg
d_I	0.9220	1.1906	0.5964	1.5055	2.0712
$d_{R(\alpha=0)}$	0.6303	0.9600	0.3276	0.3730	1.7315
$d_{R(\alpha=0.5)}$	0.8259	1.1309	0.4170	0.6248	2.0273
$d_{R(\alpha=0.8)}$	0.9014	1.1767	0.5095	1.0378	2.0661
$d_{R(\alpha=1.0)}$	**0.9220**	**1.1906**	**0.5964**	**1.5055**	**2.0712**
$d_{R(\alpha=1.5)}$	0.8989	1.1972	0.8642	2.3799	2.0642
$d_{R(\alpha=2.0)}$	0.8476	1.1888	1.0558	2.5913	2.0516
$d_{R(\alpha=3.0)}$	0.7812	1.1646	1.1891	2.6624	2.0284
$d_{R(\alpha=5.0)}$	0.7288	1.1245	1.2240	2.6583	1.9975
$d_{R(\alpha=10.0)}$	0.6932	1.0674	1.2021	2.6320	1.9661

Table 8
Tsallis information dimension of different real complex networks.

	top500	USAir	Netscience	geom	celeg
d_I	0.9220	1.1906	0.5964	1.5055	2.0712
$d_{T(\alpha=0)}$	**68.4895**	**53.6739**	**258.0225**	**1.46E+03**	**92.3206**
$d_{T(\alpha=0.5)}$	6.0820	6.2191	10.4605	32.2256	10.8002
$d_{T(\alpha=0.8)}$	1.8017	2.1435	1.7663	4.4508	3.7252
$d_{T(\alpha=1.0)}$	**0.9220**	**1.1906**	**0.5964**	**1.5055**	**2.0712**
$d_{T(\alpha=1.5)}$	0.2870	0.4135	0.0641	0.2499	0.7284
$d_{T(\alpha=2.0)}$	0.1471	0.2172	0.0131	0.0884	0.3926
$d_{T(\alpha=3.0)}$	0.0691	0.1025	0.0031	0.0213	0.1969
$d_{T(\alpha=5.0)}$	0.0303	0.0452	3.29E−05	0.0025	0.0972
$d_{T(\alpha=10.0)}$	0.0104	0.0157	9.33E−09	3.04E−05	0.0417

5.1. Connection between parameter α and Rényi dimension

The Rényi dimension does not only symbolize the fractal geometry of the network but also expresses the dynamic characteristics of the growth of the network.

So when the concept of attractors is introduced in the study of complex network, for theoretical networks and unweighted networks like BA networks, with the increase of the assessing parameter α, the fractal feature is weakened while the dynamic characteristics become the main factors affecting the complexity of the network. The node with the largest degree in the beginning continuously strengthens the control of the new generation nodes in the process of network growth (the rich get richer thoughts). Attraction continues to increase, and dimensions increase, but the speed of growth slows down (referring to the dimension gap between $d_{R(\alpha=1)}$, $d_{R(\alpha=10)}$ and $d_{R(\alpha=1000)}$) because the number of attractors with absolute attraction will get less and less in the screening process. As for the weighted network in the real world, the SWF holds the same

Table 9
The judgment of AAN.

	Network	The number of covered box	w_{sA}	Type
Sierpinski network	G5	4	3.15E−05	NAAN
	G6	4	3.32E−05	NAAN
	G7	4	3.39E−05	NAAN
BA networks with complete graph	BA3310002	4	**0.7374**	**AAN**
	BA3392	2	**0.3079**	**AAN**
	BA332002	4	**0.56796**	**AAN**
BA networks with isolated nodes	BA66361	6	0.0173	NAAN
	BA3391	6	1.72E−05	NAAN
	BA99811	10	0.0011	NAAN

influence as unweighted network, but the growth of the edge does not have definite pattern, the strength of attractors will be changed dramatically after a generation of nodes growth, the dynamic feature is erratic, so the changing rule of dimension is unpredictable.

According to the experiments of Sierpinski network and BA networks in Table 9, the theoretical network whose nodes grow evenly and steadily from a single source and the BA networks with isolated nodes initially are tend to be NAAN. The reason is that the existence of isolated nodes cuts down the degrees of the ordinary network and makes the number of boxes increase. Then the network will be more likely to become NAAN in the expansion of the nodes. On the contrary, the networks which need less boxes in the first round of coloring in the box-covering algorithm are usually AAN. So the type of network mainly depends on the number of the isolated nodes and the growth law of the nodes and edges.

5.2. Comparison of Rényi dimension to Tsallis information dimension

In general, Rényi dimension of complex network is monotonously increasing when the nodes and length change at a regular rule. The physical significance of α is debatable by now. As a result, in the part of application, through the change of the dimension of the theoretical network and the real networks, α is given the meaning of weighted parameter and can be chosen to be different values with different applications when calculating the complexity of a weighted network.

The values of Tsallis information dimension decrease as parameter q increases. So the physical meaning of the weight parameter in Tsallis information dimension is credibility that is given by the surveyors and would influence the standard of real networks' complexity. The changing trend is quite distinct, because the *SWF* and *PWF* control the tendency of dimension alternatively with 1 as the limit. As a result, when calculating the dimension of complex networks, the parameter q must be chosen meticulously based on the applied range and stability requirements. Though the Tsallis' entropy dominated by q shares some common properties with Rényi entropy controlled by α, the changing rule of Tsallis information dimension is fixed and the range ability of the former is greater than the latter.

The attractors exist in all kinds of network. Whether the parameter form is α or q, *SWF* can represent the attractiveness of the network because it is a crucial factor in both of the dimension. To some extent, *SWF* is a bridge that can connect all the generalized dimensions with the same structure as $\sum_{k=1}^{N} p_k{}^{\alpha}$. The explanation of the physical meaning in the article can be followed by other generalized dimension.

6. Conclusion

In this paper, a generalized information dimension, which can be calculated by the modified box-covering algorithm, is proposed inspired by the Rényi entropy. The existing information dimension, Hausdorff dimension and collision dimension (often used in quantum mechanics) are the special cases of Rényi dimension when α equals to *1, 0* and *2*. The *PWF* and *SWF* is defined on the basis of Rényi entropy and analyzed from the perspective of both the topology structure and the dynamic feature. Additionally, the concept of attractors is proposed to expound the physical meaning of α — the assessment parameter of attractiveness. A point that should be stressed is that SWF — the attractiveness of the entire network changes under different attraction conditions as α increases. The results of BA networks of different types prove the existence of network attractors. Finally, the comparison between the Rényi dimension and the Tsallis dimension indicates that the proposed dimension fluctuates within a smaller range when the parameter α changes which shows the robustness of Rényi dimension.

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Conflict of interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] Y. Xiao, M.T. Angulo, J. Friedman, M.K. Waldor, S.T. Weiss, Y.-Y. Liu, Mapping the ecological networks of microbial communities, Nature Commun. 8 (2017) 2042.
[2] A. Bandyopadhyay, S. Kar, Coevolution of cooperation and network structure in social dilemmas in evolutionary dynamic complex network, Appl. Math. Comput. 320 (2018) 710–730.
[3] J. Wang, H. Mo, F. Wang, F. Jin, Exploring the network structure and nodal centrality of Chinaś air transport network: A complex network approach, J. Transp. Geogr. 19 (2011) 712–721.
[4] R. Albert, A.-L. Barabási, Statistical mechanics of complex networks, Rev. Modern Phys. 74 (2002) 47.
[5] C. Song, Z. Qu, N. Blumm, A.-L. Barabási, Limits of predictability in human mobility, Science 327 (2010) 1018–1021.
[6] M. Faloutsos, P. Faloutsos, C. Faloutsos, On power-law relationships of the internet topology, in: ACM SIGCOMM Computer Communication Review, volume 29, ACM, pp. 251–262..
[7] J.M. Montoya, R.V. Solé, Small world patterns in food webs, J. Theoret. Biol. 214 (2002) 405–412.
[8] D.J. Watts, S.H. Strogatz, Collective dynamics of small-world networks, Nature 393 (1998) 440.
[9] J. Travers, S. Milgram, The small world problem, Psychol. Today 1 (1967) 61–67.
[10] A.-L. Barabási, R. Albert, H. Jeong, Mean-field theory for scale-free random networks, Physica A 272 (1999) 173–187.
[11] L. Gallos, M. Sigman, H. Makse, The conundrum of functional brain networks: Small-world efficiency or fractal modularity, Front. Physiol. 3 (2012) 123.
[12] L.K. Gallos, H.A. Makse, M. Sigman, A small world of weak ties provides optimal global integration of self-similar modules in functional brain networks, Proc. Natl. Acad. Sci. 109 (2012) 2825–2830.
[13] H.D. Rozenfeld, C. Song, H.A. Makse, Small-world to fractal transition in complex networks: A renormalization group approach, Phys. Rev. Lett. 104 (2010) 025701.
[14] L.K. Gallos, R. Cohen, P. Argyrakis, A. Bunde, S. Havlin, Stability and topology of scale-free networks under attack and defense strategies, Phys. Rev. Lett. 94 (2005) 188701.
[15] L.K. Gallos, P. Argyrakis, Scale-free networks resistant to intentional attacks, Europhys. Lett. 80 (2007) 58002.
[16] L.K. Gallos, C. Song, H.A. Makse, Scaling of degree correlations and its influence on diffusion in scale-free networks, Phys. Rev. Lett. 100 (2008) 248701.
[17] L.K. Gallos, P. Argyrakis, Reaction–diffusion processes on correlated and uncorrelated scale-free networks, Phys. Rev. E 72 (2005) 017101.
[18] T. Zhou, B. Wang, Catastrophes in scale-free networks, Chin. Phys. Lett. 22 (2005) 1072.
[19] R. Albert, H. Jeong, A.-L. Barabási, Internet: Diameter of the world-wide web, Nature 401 (1999) 130.
[20] C. Song, S. Havlin, H.A. Makse, Origins of fractality in the growth of complex networks, Nat. Phys. 2 (2006) 275–281.
[21] C. Song, S. Havlin, H.A. Makse, Self-similarity of complex networks, Nature 433 (2005) 392–395.
[22] C. Song, L.K. Gallos, S. Havlin, H.A. Makse, How to calculate the fractal dimension of a complex network: The box covering algorithm, J. Stat. Mech. Theory Exp. 2007 (2007) P03006.
[23] A.A. Koronovskii, M.K. Kurovskaya, O.I. Moskalenko, A. Hramov, S. Boccaletti, Self-similarity in explosive synchronization of complex networks, Phys. Rev. E 96 (2017) 062312.
[24] Q. Zhang, M. Li, Y. Deng, Measure the structure similarity of nodes in complex networks based on relative entropy, Physica A 491 (2018) 749–763.
[25] W. Jiang, A correlation coefficient for belief functions, International Journal of Approximate Reasoning 2018 Published on line. http://dx.doi.org/10.1016/j.ijar.2018.09.001.
[26] Z. He, W. Jiang, An evidential dynamical model to predict the interference effect of categorization on decision making, Knowl.-Based Syst. 150 (2018) 139–149.
[27] L. Fei, Q. Zhang, Y. Deng, Identifying influential nodes in complex networks based on the inverse-square law, Physica A (2018).
[28] Z. He, W. Jiang, An evidential markov decision making model, Inform. Sci. 467 (2018) 357–372.
[29] L. Yin, Y. Deng, Toward uncertainty of weighted networks: An entropy-based model, Physica A 508 (2018) 176–186.
[30] Y. Han, Y. Deng, An Evidential Fractal AHP target recognition method, Def. Sci. J. 68 (2018) 367–373.
[31] W. Jiang, W. Hu, An improved soft likelihood function for Dempster-Shafer belief structures, Int. J. Intell. Syst. 33 (2018) 1264–1282.
[32] X. Deng, Analyzing the monotonicity of belief interval based uncertainty measures in belief function theory, Int. J. Intell. Syst. 33 (2018) 1869–1879.
[33] X. Deng, D. Han, J. Dezert, Y. Deng, Y. Shyr, Evidence combination from an evolutionary game theory perspective, IEEE Trans. Cybern. 46 (2016) 2070–2082.
[34] J. Liu, Y. Li, Z. Ruan, G. Fu, X. Chen, R. Sadiq, Y. Deng, A new method to construct co-author networks, Physica A 419 (2015) 29–39.
[35] J. Theiler, Estimating fractal dimension, J. Opt. Soc. Amer. A 7 (1990) 1055–1073.
[36] B.B. Mandelbrot, R. Pignoni, The fractal geometry of nature, volume 173, WH freeman, New York, 1983.
[37] J.E. Hutchinson, Fractals and self similarity, Indiana Univ. Math. J. 30 (1981) 713–747.
[38] A. Mashiah, O. Wolach, J. Sandbank, O. Uziel, P. Raanani, M. Lahav, Lymphoma and leukemia cells possess fractal dimensions that correlate with their biological features, Acta Haematol 119 (2008) 142–150.
[39] Y. Li, Y. Deng, Generalized Ordered Propositions Fusion Based on Belief Entropy, Int. J. Comput. Commun. Control 13 (2018) 771–786.
[40] E. Rosenberg, The correlation dimension of a rectilinear grid, J. Interconnect. Netw. 16 (2016) 1550010.
[41] T. Wen, M. Song, W. Jiang, Evaluating topological vulnerability based on fuzzy fractal dimension, Int. J. Fuzzy Syst. 20 (2018) 1956–1967.
[42] E. Rosenberg, Lower bounds on box counting for complex networks, J. Interconnect. Netw. 14 (2013) 1350019.
[43] E. Rosenberg, Maximal entropy coverings and the information dimension of a complex network, Phys. Lett. A 381 (2017) 574–580.
[44] E. Rosenberg, Minimal box size for fractal dimension estimation, Community Ecol. 17 (2016) 24–27.
[45] J.D. Farmer, Information dimension and the probabilistic structure of chaos, Z. Nat.forsch A 37 (1982) 1304–1326.
[46] C. McMullen, The Hausdorff dimension of general Sierpiński carpets, Nagoya Math. J. 96 (1984) 1–9.
[47] S.M. Pincus, Approximate entropy as a measure of system complexity, Proc. Natl. Acad. Sci. 88 (1991) 2297–2301.
[48] D.-J. Wei, Q. Liu, H.-X. Zhang, Y. Hu, Y. Deng, S. Mahadevan, Box-covering algorithm for fractal dimension of weighted networks, Sci. Rep. 3 (2013) 3049.
[49] T. Wen, W. Jiang, An information dimension of weighted complex networks, Physica A 501 (2018) 388–399.
[50] R.M. Gray, Entropy and Information Theory, Springer, 2011.
[51] C. Tsallis, Possible generalization of Boltzmann–Gibbs statistics, J. Stat. Phys. 52 (1988) 479–487.
[52] R.J. dos Santos, Generalization of Shannonś theorem for Tsallis entropy, J. Math. Phys. 38 (1997) 4104–4107.
[53] C. Tsallis, A.R. Plastino, W.M. Zheng, Power-law sensitivity to initial conditions New entropic representation, Chaos Solitons Fractals 8 (1997) 885–891.
[54] C. Tsallis, L.J.L. Cirto, Thermodynamics is more powerful than the role to it reserved by Boltzmann–Gibbs statistical mechanics, Eur. Phys. J. Spec. Top. 223 (2014) 2161–2175.
[55] V. Rajinikanth, S.C. Satapathy, Segmentation of ischemic stroke lesion in brain MRI based on social group optimization and fuzzy-tsallis entropy, Arab. J. Sci. Eng. (2018) 1–14.
[56] A. Plastino, A. Plastino, Stellar polytropes and Tsallis' entropy, Phys. Lett. A 174 (1993) 384–386.

[57] M.M. Ozimek, Analysis of Selected Phenomena in Cardiovascular Physics using Nonextensive Statistical Physics (Ph.D. thesis), Zakład Fizyki Układów Złożonych, 2017.
[58] K.A. Dawson, J.O. Indekeu, H.E. Stanley, C. Tsallis, Communities detection as a tool to assess a reform of the Italian interlocking directorship network, Physica A 466 (2017) 91–104.
[59] E. Rosenberg, Non-monotonicity of the generalized dimensions of a complex network, Phys. Lett. A 381 (2017) 2222–2229.
[60] X. Deng, W. Jiang, Z. Wang, Zero-sum polymatrix games with link uncertainty: A Dempster-Shafer theory solution, Appl. Math. Comput. 340 (2019) 101–112.
[61] A. Rényi, On Measures of Entropy and Information, Technical Report, Hungarian Academy Of Sciences Budapest Hungary, 1961.
[62] P. Erdos, A. Rényi, On the evolution of random graphs, Publ. Math. Inst. Hung. Acad. Sci 5 (1960) 17–60.
[63] J.-L. Liu, Z.-G. Yu, V. Anh, Determination of multifractal dimensions of complex networks by means of the sandbox algorithm, Chaos 25 (2015) 023103.
[64] Y. Wu, S. Verdú, Rényi information dimension: Fundamental limits of almost lossless analog compression, IEEE Trans. Inform. Theory 56 (2010) 3721–3748.
[65] C.E. Shannon, Communication theory of secrecy systems, Bell Labs Tech. J. 28 (1949) 656–715.
[66] X. Deng, W. Jiang, Dependence assessment in human reliability analysis using an evidential network approach extended by belief rules and uncertainty measures, Ann. Nucl. Energy 117 (2018) 183–193.
[67] C.E. Shannon, A mathematical theory of communication, ACM SIGMOBILE Mob. Comput. Commun. Rev. 5 (2001) 3–55.
[68] K. Zyczkowski, Rényi extrapolation of Shannon entropy, Open Syst. Inf. Dyn. 10 (2003) 297–310.
[69] S. Abe, Stability of Tsallis entropy and instabilities of Rényi and normalized Tsallis entropies: A basis for q-exponential distributions, Phys. Rev. E 66 (2002) 046134.
[70] A. Bărar, G. Mocanu, I. Raşa, Heun functions related to entropies, R. Acad. Cienc. Exactas Fıś. Nat. Ser. A Mat. (2018) 1–12.
[71] M.I. Hwang, J.W. Lin, Information. dimension, Information dimension information overload and decision quality, J. Inf. Sci. 25 (1999) 213–218.
[72] D. Wei, B. Wei, Y. Hu, H. Zhang, Y. Deng, A new information dimension of complex networks, Phys. Lett. A 378 (2014) 1091–1094.
[73] Q. Zhang, C. Luo, M. Li, Y. Deng, S. Mahadevan, Tsallis information dimension of complex networks, Physica A 419 (2015) 707–717.
[74] V.A. Colizza V., R. Pastor-Satorras, Reaction–diffusion processes and metapopulation models in heterogeneous networks, Nat. Phys. (2007) 266–282.
[75] D.J. S. S. H. Watts, Collective dynamics of ° small-world networks, Nature 393 (1998) 440–442.
[76] A.M. Vladimi. Batagelj, Pajek datasets, 2006. http://vlado.fmf.uni-lj.si/pub/networks/data/.