Digital Communications – Assignment 1

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1 Deriving Theoretical Error Rates

To find the probability of error in a single dimension we use

$$p_e = Q\left(\frac{d}{2\sigma}\right)$$

Where the Q function is

$$Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-z^2}{2}\right) dx$$

The probability of error in two dimensions depends on the decision boundaries around a symbol. We can see that there are two sets, of four symbols each, for which there are matching symbols. The probability of error for these sets is given in Equations 1 and 2

$$p(Error \mid s_i) = 2 \cdot Q\left(\sqrt{\frac{E_s}{3N_o}}\right) - Q^2\left(\sqrt{\frac{E_s}{3N_o}}\right), i = \{1, 2, 7, 8\} \tag{1}$$

$$p(Error \mid s_i) = 3 \cdot Q\left(\sqrt{\frac{E_s}{3N_o}}\right) - 2 \cdot Q^2\left(\sqrt{\frac{E_s}{3N_o}}\right), i = \{3, 4, 5, 6\} \tag{2}$$

1.1 Symbol Error Rate (SER)

For SER we can see that the average SER is the weighted average of Equations 1 and 2. Following this we can see that the theoretical SER is given by Equation 4.

$$\begin{aligned} p_e = \frac{1}{8}\Bigg[4\left[2 \cdot Q\left(\sqrt{\frac{E_s}{3N_o}}\right) - Q^2\left(\sqrt{\frac{E_s}{3N_o}}\right)\right] \\ +4\left[3 \cdot Q\left(\sqrt{\frac{E_s}{3N_o}}\right) - 2 \cdot Q^2\left(\sqrt{\frac{E_s}{3N_o}}\right)\right]\Bigg] \end{aligned}$$

$$p_e = \frac{1}{2}\left[5 \cdot Q\left(\sqrt{\frac{E_s}{3N_o}}\right) - 3 \cdot Q^2\left(\sqrt{\frac{E_s}{3N_o}}\right)\right] \tag{3}$$

$$p_e = \left[\frac{5}{2} \cdot Q\left(\sqrt{\frac{E_s}{3N_o}}\right) - \frac{3}{2} \cdot Q^2\left(\sqrt{\frac{E_s}{3N_o}}\right)\right] \tag{4}$$

1.2 Bit Error Rate (BER)

As $E_b = \frac{E_s}{3}$ then $Q\left(\sqrt{\frac{E_s}{3N_o}}\right)$ becomes $Q\left(\sqrt{\frac{E_b}{3N_o}}\right)$. Where we previously averaged across 8 signals we now have 24 bits, 3 bits per symbol. Noting these differences, Equation 5 is the BER equivalent of Equation 3 and the theoretical BER is given by Equation 6.

$$p_e = \frac{1}{2 \cdot 3}\left[5 \cdot Q\left(\sqrt{\frac{E_b}{N_o}}\right) - 3 \cdot Q^2\left(\sqrt{\frac{E_b}{N_o}}\right)\right] \tag{5}$$

$$p_e = \frac{5}{6} \cdot Q\left(\sqrt{\frac{E_b}{N_o}}\right) - \frac{1}{2} \cdot Q^2\left(\sqrt{\frac{E_b}{N_o}}\right) \tag{6}$$

2 Transmitter Encoding Scheme

The mapping of bits to symbols was done with a Gray Code, seen in Table 1. This serves several purposes:

- If a symbol is in error with a neighbouring symbol there is only one bit error
- It is possible to map each bit in the three bit word to a direction decision

Table 1: Gray-Coding Scheme

000	100
001	101
011	111
010	110

It is clear from Table 1 that any neighbouring errors only result in a bit error.

Table 2 and Figure 1 show how the code maps words to symbols. It is used to generate the correct signal and for the transmitter to know what was sent. This can be seen also by comparing the descriptions in Table 2 with the positions of the words in Table 1.

Table 2: The Roll of Each Bit in Symbol Mapping

	Bit is 0	**Bit is 1**
1st Bit	Negative half plane of x axis	Positive half plane of x axis
2nd Bit	Positive half plane of y axis	Negative half plane of y axis
3rd Bit	Most extreme y position	Least extreme y position

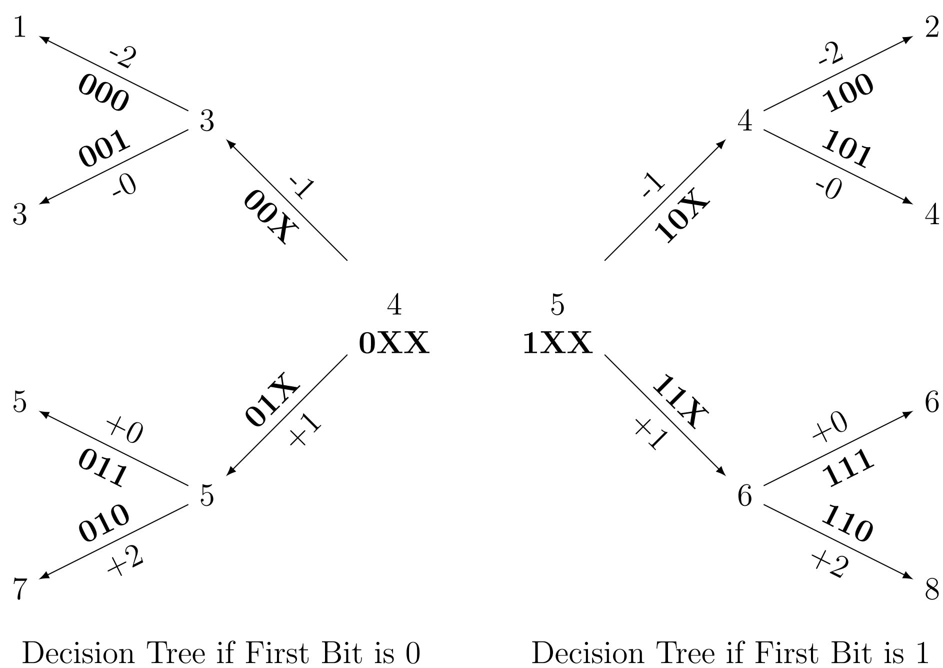

Figure 1: Decision Tree for Tracking Signal Enumeration During Modulation

Figure 2 shows how the bits in the words, following the descriptions in Table 2, generates the signal piecewise, reading one bit at a time.

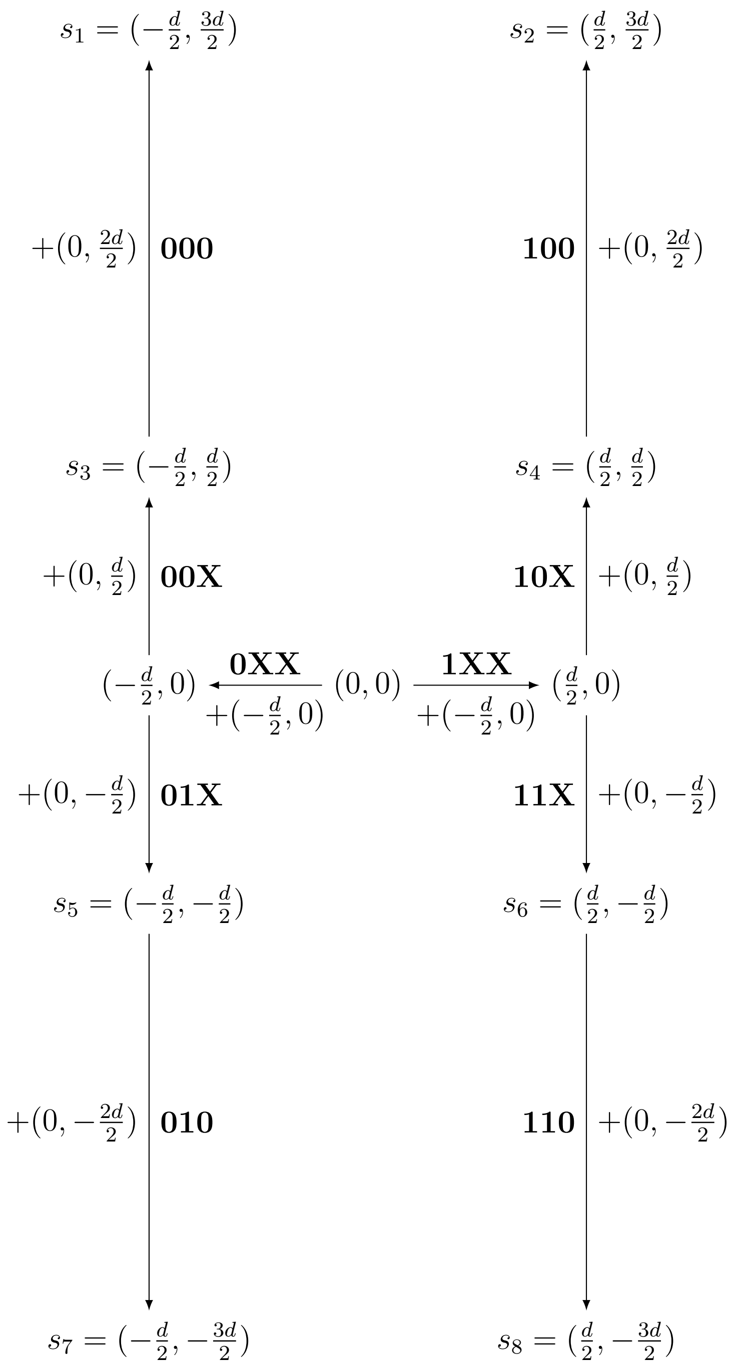

Figure 2: Decision Tree for Modulating 3-Bit Words to Signals

The MATLAB script uses the decision trees shown in both Figure 1 and Figure 2 to generate the signal to transmit and to know what that signal is, for checking error rate once it is received.

3 Results

3.1 SER Below 10^{-4}

The SER lies below 10^{-4} for E_s/N_o of ∼16.7 dB.

3.2 BER Below 10^{-4}

The BER lies below 10^{-4} for E_b/N_o of ∼11.2 dB.

3.3 Graphs

Figures 3 and 5 graph the SER and BER respectively while Figures 4 and 6 show the percentage error between the theoretical and actual values. With both SER and BER show the percentage error increases towards the end while BER has more errors at the start. There are greater actual errors with the BER than the theoretical errors as the model only accounts for single-bit errors. These errors are more common at high SNR which is why the actual errors are greater for low SNR, where a symbol can be in error but cause more than a single-bit error.

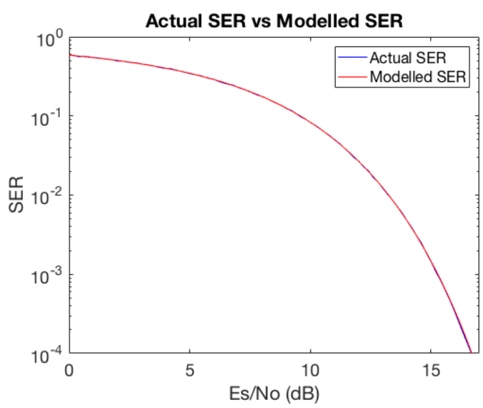

Figure 3: SER

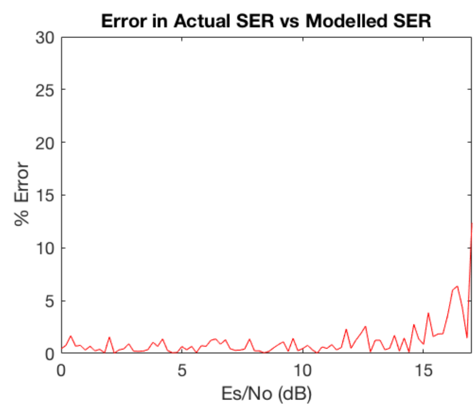

Figure 4: SER Error

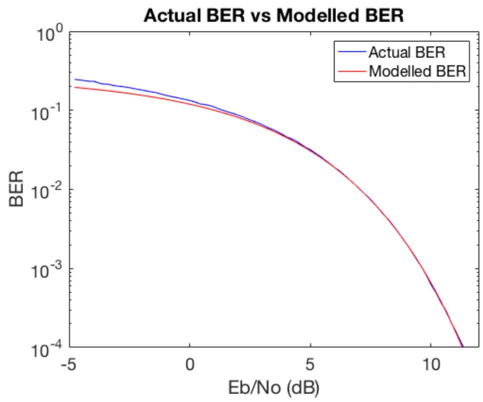

Figure 5: BER

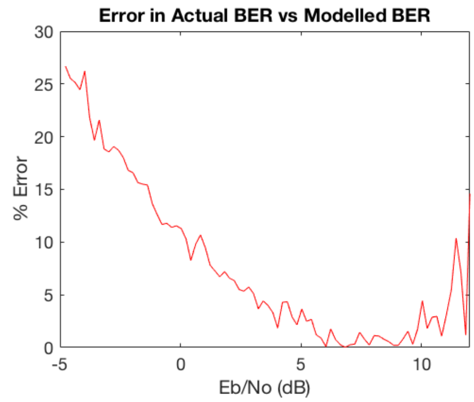

Figure 6: BER Error