3.1.8 Remark (Riemann tensor and parallel transport along closed curves). Let M be a SRMF, $p \in M$, $Z \in T_pM$ and c be a curve in M with $c(0) = p$. Further let $(x^1, \ldots, x^n)$ be coordinates around p and write as usual $c^i(t)$ for the coordinates of c w.r.t. this chart. Let now $Z(t) = Z^i(t)\partial_i|_{c(t)}$ be the vector field obtained from parallely transporting $Z = Z(0)$ along c. By (1.3.56) we then have

$$\frac{dZ^i(t)}{dt} + \Gamma^i_{\ kl}(x(c(t)))\,\frac{dc^k}{dt}\,Z^l(t) = 0. \tag{3.1.24}$$

Let now more generally $f : I \times J \to M$ a smooth two-parameter map with I and J intervals around 0. We denote by $x^i(u,v) = x^i \circ f(u,v)$ the local coordinates of f. For some fixed pair $(u,v) \in I \times J$ we define the 'corner points' (see Figure ♣ `insert figure` ♣) $P = f(0,0)$, $Q = f(u,0)$, $R = f(u,v)$ and $S = f(0,v)$. Now we transport $Z = Z_P \in T_PM$ parallel along f first to Q then to R and S and then back to P. We will denote the values of this vector field Z on f at the 'corner points' by Z_Q, Z_R, Z_S and finally $\bar{Z}_P = Z(u,v)$. In general the resulting vector $\bar{Z}_P$ will not equal the staring value Z_P but depend smoothly on u and v. Indeed the solutions of the ODE (3.1.24) depend smoothly on the data Q, R and S as well as the right hand side which themselves depend smoothly on (u,v).
Now we expand the components of these vectors w.r.t. the coordinates x^i in a Taylor series which gives

$$Z^i_Q = Z^i_P + \Big(\frac{\partial Z^i}{\partial u}\Big)_P u + \frac{1}{2}\Big(\frac{\partial^2 Z^i}{\partial u^2}\Big)_P u^2 + \mathcal{O}(u^3), \tag{3.1.25}$$

$$Z^i_R = Z^i_Q + \Big(\frac{\partial Z^i}{\partial v}\Big)_Q v + \frac{1}{2}\Big(\frac{\partial^2 Z^i}{\partial v^2}\Big)_Q v^2 + \mathcal{O}(v^3), \tag{3.1.26}$$

$$Z^i_S = Z^i_R + \Big(\frac{\partial Z^i}{\partial u}\Big)_R u - \frac{1}{2}\Big(\frac{\partial^2 Z^i}{\partial u^2}\Big)_R u^2 + \mathcal{O}(u^3), \tag{3.1.27}$$

$$\bar{Z}^i_P = Z^i_S + \Big(\frac{\partial Z^i}{\partial v}\Big)_S v - \frac{1}{2}\Big(\frac{\partial^2 Z^i}{\partial v^2}\Big)_S v^2 + \mathcal{O}(v^3). \tag{3.1.28}$$

Inserting (3.1.25) into (3.1.26), (3.1.26) into (3.1.27), and (3.1.27) into (3.1.28) we obtain for the difference between the starting an the final vector

$$\begin{aligned}\Delta Z^i_P := \bar{Z}^i_P - Z^i_P = &\left(\Big(\frac{\partial Z^i}{\partial u}\Big)_P - \Big(\frac{\partial Z^i}{\partial u}\Big)_R\right) u \;+\; \left(\Big(\frac{\partial Z^i}{\partial v}\Big)_Q - \Big(\frac{\partial Z^i}{\partial v}\Big)_S\right) v \\ &+ \left(\Big(\frac{\partial^2 Z^i}{\partial u^2}\Big)_P - \Big(\frac{\partial^2 Z^i}{\partial u^2}\Big)_R\right) \frac{u^2}{2} + \left(\Big(\frac{\partial^2 Z^i}{\partial v^2}\Big)_Q - \Big(\frac{\partial^2 Z^i}{\partial v^2}\Big)_S\right) \frac{v^2}{2} + \ldots\end{aligned} \tag{3.1.29}$$

Now we assume that the x^i are Riemannian normal coordinates at p which by 2.1.17(ii) leads to the vanishing of the first term in the Taylor expansion of the Christoffel symbols and we obtain

$$\Gamma^i_{\ jk}(x) = \sum_m \Big(\frac{\partial}{\partial x^m}\Gamma^i_{\ jk}\Big)_P x^m + \mathcal{O}(x^2). \tag{3.1.30}$$

The next and laborious step consist in calculating the coefficients $(\partial Z^i/\partial u)_P$, $(\partial^2 Z^i/\partial u^2)_P$ etc. from the ODE (3.1.24). We obtain

(i) $$\left.\frac{\partial Z^i}{\partial u}\right|_P = 0$$

Indeed we have by parallel transport (cf. (3.1.24)) along the curve $u \mapsto x(u,0)$ that $\partial_u Z^i(u,0) = -\Gamma^i_{\ kl}(x(u,0))\,\partial_u Z^k(u,0)\,Z^l(u,0)$ and so $\partial_u Z^i(0,0) = -\Gamma^i_{\ kl}(P)\cdot\ldots = 0$, since we have assumed x^i to be Riemannian normal coordinates around P.

(ii) $$\left.\frac{\partial Z^i}{\partial v}\right|_Q = -\Big(\frac{\partial}{\partial x^m}\,\Gamma^i_{\ jk}\,\frac{\partial x^m}{\partial u}\,\frac{\partial x^j}{\partial v}\,Z^k\Big)_P\,u + \ldots$$

Parallel transport along $v \mapsto x(u,v)$ from Q to R gives $(\partial_u Z^i)_Q = -(\Gamma^i_{\ jk}\,\partial_v x^j\,Z^k)_Q$. Now by Taylor expansion at P (cf. (3.1.30)) we obtain $\Gamma^i_{\ jk}(Q) = \partial_m\Gamma^i_{\ jk}(P)\,x^m + \ldots$ and also $x^m(Q) = 0 + \partial_u x^m(P)\,u + \partial_v x^m(P)\,0 + \ldots$, since $Q = (u,0)$. Similarly $\partial_v x^j(Q) = \partial_v x^j(P) + u\cdot\ldots + 0\cdot\ldots + \ldots$, and $Z^k(Q) = Z^k(P) + \partial_u Z^k(P)\,u + \partial_v Z^k(P)\,0 + \ldots$. Collecting terms together we finally arrive at the asserted result.

(iii) $$\left.\frac{\partial Z^i}{\partial u}\right|_R = -\Big(\frac{\partial}{\partial x^m}\,\Gamma^i_{\ jk}\,\frac{\partial x^m}{\partial u}\,\frac{\partial x^j}{\partial u}\,Z^k\Big)_P\,u - \Big(\frac{\partial}{\partial x^m}\,\Gamma^i_{\ jk}\,\frac{\partial x^m}{\partial v}\,\frac{\partial x^j}{\partial u}\,Z^k\Big)_P\,v + \ldots$$

Now we use parallel transport from R to S along $s \mapsto x(u-s,v)$, $s \in [0,u]$ to obtain $\partial_s Z^i(R) = -\big(\Gamma^i_{\ jk}(x(u-s,v))\,\partial_s(x^j(u-s,v))\,Z^k\big)_R$. Next note that $\partial_s(Z^i(u-s,v)) = -\partial_u Z(u-s,v)$ and $\partial_s(x^j(u,s,v)) = -\partial_u x^j(u-s,v)$ and so we have $\partial_u Z^i(R) = -\big(\Gamma^i_{\ jk}(x(u-.,v))\,\partial_u x^j(u-.,v)\,Z^k\big)_R = -(\Gamma^i_{\ jk}\,\partial_u x^j\,Z^k)_R$. Again Taylor expansion now at P gives $\Gamma^i_{\ jk}(R) = \partial_m\Gamma^i_{\ jk}(P)x^m(R) + \ldots = \partial_m\Gamma^i_{\ jk}(P)\partial_u x^m(P)\,u + \partial_v x^m(P)\,v + \ldots$, and $\partial_j(R) = \partial_u x^j(P) + u\cdot\ldots + v\cdot\ldots$. Moreover by (3.1.25), (3.1.26) $Z^k(R) = Z^k(P) + \partial_u Z^k(P)u + \partial_v Z^k(Q)\,v + \ldots$. Again collecting together the respective terms we obtain the asserted formula.

(iv) $$\left.\frac{\partial Z^i}{\partial v}\right|_S = -\Big(\frac{\partial}{\partial x^m}\,\Gamma^i_{\ jk}\,\frac{\partial x^m}{\partial v}\,\frac{\partial x^j}{\partial v}\,Z^k\Big)_P\,v + \ldots$$

We parallely transport Z along $t \mapsto (0, v-t)$ from S to P to obtain $\partial_t\big(Z^i(0,v-t)\big) = -\big(\Gamma^i_{\ jk}(x(0,v-t))\,\partial_t(x^j(0,v-t))Z^k\big)_S$ which by analogous reasoning as in (iii) gives $\partial_v Z^i(S) = -(\Gamma^i_{\ jk}\,\partial_v\,x^j\,Z^k)_S$, where by Taylor expansion in P, $\Gamma^i_{\ jk}(S) = \partial_m\Gamma^i_{\ jk}(P)x^m(S) + \ldots = \partial_m\Gamma^i_{\ jk}(P)\,(\partial_u x^m\cdot 0 + \partial_v x^m\cdot v)_P + \ldots$ as well as $\partial_v x^j(S) = \partial_v x^j(P) + u\cdot\ldots + v\cdot\ldots + \ldots$ and using (3.1.25)–(3.1.27) $Z^k(S) = Z^k(P) + \ldots$ Once more collecting the terms gives the result.

(v) $$\left.\frac{\partial^2 Z^i}{\partial u^2}\right|_P = -\Big(\frac{\partial\Gamma^i_{\ jk}}{\partial x^m}\,\frac{\partial x^m}{\partial u}\,\frac{\partial x^j}{\partial u}\,Z^k\Big)_P + \ldots$$ ♣ `insert calculation` ♣

(vi) $$\left.\frac{\partial^2 Z^i}{\partial u^2}\right|_R = -\Big(\frac{\partial\Gamma^i_{\ jk}}{\partial x^m}\,\frac{\partial x^m}{\partial u}\,\frac{\partial x^j}{\partial u}\,Z^k\Big)_P + \ldots$$ ♣ `insert calculation` ♣

(vii) $$\left.\frac{\partial^2 Z^i}{\partial v^2}\right|_Q = -\Big(\frac{\partial\Gamma^i_{\ jk}}{\partial x^m}\,\frac{\partial x^m}{\partial v}\,\frac{\partial x^j}{\partial v}\,Z^k\Big)_P + \ldots$$ ♣ `insert calculation` ♣

(viii) $\displaystyle \frac{\partial^2 Z^i}{\partial v^2}\Big|_S = -\Big(\frac{\partial \Gamma^i_{\ jk}}{\partial x^m}\frac{\partial x^m}{\partial v}\frac{\partial x^j}{\partial v} Z^k\Big)_P + \ldots$ ♣ `insert calculation` ♣

Now we may plug (i)–(viii) into (3.1.29) to arrive at

$$\Delta Z^i_P = \left(\Big(\frac{\partial \Gamma^i_{\ jk}}{\partial x^m} - \frac{\partial \Gamma^i_{\ mk}}{\partial x^j}\Big)\frac{\partial x^j}{\partial u}\frac{\partial x^m}{\partial v} Z^k\right)_P uv + \ldots \tag{3.1.31}$$

Further by 3.1.3 and 2.1.17(ii) we obtain

$$R^i_{kmj}\,|_P = \left(\frac{\partial \Gamma^i_{\ mk}}{\partial x^j} - \frac{\partial \Gamma^i_{\ jk}}{\partial x^m} + \Gamma^i_{\ jr}\Gamma^r_{\ mk} - \Gamma^i_{\ ms}\Gamma^s_{\ jk}\right)_P = -\left(\frac{\partial \Gamma^i_{\ jk}}{\partial x^m} - \frac{\partial \Gamma^i_{\ mk}}{\partial x^j}\right)_P \tag{3.1.32}$$

and so we finally obtain

$$\Delta Z^i_P = -\Big(R^i_{kmj}\frac{\partial x^j}{\partial u}\frac{\partial x^m}{\partial v} Z^k\Big)_P uv + \cdots = -R^i\Big(\frac{\partial x}{\partial v}\frac{\partial x}{\partial u}\Big)(Z)\,|_P\, uv + \ldots . \tag{3.1.33}$$

From this result we may now immediately draw the following conclusions:

(1) If $R = 0$ by 3.1.7 M is locally isometric to $\mathbb{R}^n_r$ and since parallel transport then is trivial we obtain $\Delta Z^i_P = 0$.

(2) In the general case ΔZ^i_P is of second order in (u, v) and depends on the curvature tensor R at P. Hence we may view R as an obstruction to the vanishing of ΔZ^i_P. In particular, (3.1.33) gives the following alternative characterisation of the Riemann tensor

$$R^i\Big(\frac{\partial x}{\partial v}\frac{\partial x}{\partial u}\Big)(Z)\,|_P = -\lim_{u,v\to 0}\frac{1}{uv}\,\Delta Z^i_P. \tag{3.1.34}$$

3.2 Some differential operators

The aim of this section is to introduce on a SRMF the generalisations of the classical differential operators of *gradient, divergence*and *Laplacian*. To deal with these in an appropriate way we need some preparations, namely we need to introduce the operations of *type-changing* of higher order tensor fields and *metric contraction*.

The former operation is nothing but the generalisation of the musical isomorphism of 1.3.3 of vector fields and one-forms to higher order tensors. To achieve this goal we proceed as follows: Given a tensor field $A \in \mathcal{T}^r_s(M)$ on a SRMF (M, g) we define for any $1 \le a \le r$, $1 \le b \le s$ the tensor field $\downarrow^a_b A \in \mathcal{T}^{r-1}_{s+1}(M)$ via

$$\begin{aligned}(\downarrow^a_b A)(\omega^1, \ldots, \omega^{r-1}, X_1 \ldots, X_{s+1})&\\ := A(\omega^1, \ldots, \underset{\text{slot } a}{X^\flat_b}, \ldots, \omega^{r-1}, X_1 \ldots, X_{b-1}, X_{b+1}, \ldots, X_{s+1}),&\end{aligned} \tag{3.2.1}$$

where $X^\flat_b$ is the metric equivalent one-form of the vector field X_b, cf. 1.3.3. So on the r.h.s. we extract the bth vector field and insert its metrically equivalent one-form in the ath slot among the one-forms. It is instructive to consider an example.

3.2.1 Example (Index lowering). Let A be a $(2,2)$-tensor field on M, then $B :=\downarrow^1_2 A$ is the $(1,3)$-tensor field given by

$$B(\omega, X, Y, Z) = A(Y^\flat, \omega, X, Z). \tag{3.2.2}$$

Now let $(x^1, \ldots, x^n)$ be local coordinates on M. Then first observe that $\partial_k^\flat = g_{km} dx^m$ since $\partial_k^\flat(V^l \partial_l) = \langle \partial_k, V^l \partial_l \rangle = g_{kl} V^l$. And so we have

$$B^i_{jkl} = B(dx^i, \partial_j, \partial_k, \partial_l) = A(g_{km} dx^m, dx^i, \partial_j, \partial_l) = g_{km} A^{mi}_{jl}. \tag{3.2.3}$$

We now see that the operation $\downarrow^1_2$ changes the first upper index of A into the second lower index of B.

The operator $\downarrow^a_b\colon\ \mathcal{T}^r_s(M) \to \mathcal{T}^{r-1}_{s+1}(M)$ is classically also called the *lowering* of the respective indices. It is obviously $\mathcal{C}^\infty$-linear and moreover it is an isomorphism with inverse $\uparrow^a_b$: $\mathcal{T}^{r'}_{s'}(M) \to \mathcal{T}^{r'+1}_{s'-1}(M)$ given by

$$\begin{aligned}(\uparrow^a_b A)(\omega^1, \ldots, \omega^{r'+1}, X_1, \ldots, X_{s'-1})& \\ := A(\omega^1, \ldots, \omega^{a-1}, \omega^{a+1}, \ldots \omega^{r'+1}, X_1, \ldots, &\underbrace{(\omega^a)^\sharp}_{\text{slot } b}, \ldots, X_{s'-1}),\end{aligned} \tag{3.2.4}$$

where now $(\omega^b)^\sharp$ is the vector field metrically equivalent to the one-form ω^b. This operation extracts the the ath one-form and inserts its metrically equivalent vector field in the bth slot among the vector fields and is classically called the *raising* of the respective index. Generally we call all tensors which are derived from a given tensor by raising or lowering an index *metrically equivalent*.
We again look at an example where we also demonstrate that the lowering and raising of indices are inverse operations.

3.2.2 Example (Index lowering). First observe that in local coordinates we have $(dx^i)^\sharp = g^{ij}\partial_j$ since $\langle g^{ij}\partial_j, v^k \partial_k \rangle = \delta^i_k v^k = v^i = dx^i(v^k \partial_k)$. Now for a $(1,3)$-tensor field B we have

$$(\uparrow^1_2 B)^{ij}_{kl} = (\uparrow^1_2 B)(dx^i, dx^j, \partial_k, \partial_l) = B(dx^j, \partial_k, g^{im}\partial_m, \partial_l) = g^{im}\, B^j_{kml}. \tag{3.2.5}$$

So as expected $\uparrow^1_2$ turns the second lower index into the first upper index using the inverse metric. Now to check that it is the inverse of $\downarrow^1_2$ we write using equations (3.2.3) and (3.2.5)

$$(\uparrow^1_2 \downarrow^1_2 A)^{ij}_{kl} = g^{im} (\downarrow^1_2 A)^j_{kml} = g^{im}\, g_{mn}\, A^{nj}_{kl} = \delta^i_n\, A^{nj}_{kl} = A^{ij}_{kl}. \tag{3.2.6}$$

We give another example to emphasise how natural actually type changing is; in fact it often occurs in calculations without even being noticed.

3.2.3 Example (Type changing). As in (1.3.24) we consider a $(1,s)$-tensor field A given as a $\mathcal{C}^\infty$-multilinear map $A : \mathfrak{X}(M)^s \to \mathfrak{X}(M)$. The we have (using $\bar{A}$ as in (1.3.24))

$$\begin{aligned}(\downarrow^1_1 \bar{A})(V, X_1, \ldots, X_s) &= \bar{A}(V^\flat, X_1, \ldots, X_s) \\ &= V^\flat\big(A(X_1, \ldots, X_s)\big) = \langle V, A(X_1, \ldots, X_s)\rangle.\end{aligned} \tag{3.2.7}$$

Finally we point at one peculiar issue in dealing with the coordinate expression for the Riemann tensor which arises due to historic reasons. Actually the coordinate version of differential geometry was developed long before the invariant approach and in harmonising these two this issue requires some care.

3.2.4 Remark (Coordinate expression of the Riemann tensor). We have written the coordinate expression of the curvature tensor in 3.1.3 (according to the classical pattern) as

$$R_{\partial_k \partial_l}(\partial_j) = R^i_{jkl}\partial_i, \text{ hence the order of arguments is } R_{XY}Z = R(Z, X, Y). \tag{3.2.8}$$

Indeed using the convention of (1.3.24) we obtain $R^i_{jkl} = \bar{R}(dx^i, \partial_j, \partial_k, \partial_l) = dx^i\big(R(\partial_j, \partial_k, \partial_l)\big)$. The components of the $(0,4)$-tensor $\downarrow^1_1 R$ are then given by

$$\begin{aligned} R_{ijkl} = (\downarrow^1_1 \bar{R})(\partial_i, \partial_j, \partial_k, \partial_l) &= \langle \partial_i, R(\partial_j, \partial_k, \partial_l)\rangle \\ &= \langle \partial_i, R_{\partial_k\partial_l}(\partial_j)\rangle = \langle \partial_i, R^m_{jkl}\partial_m\rangle = g_{im}R^m_{jkl}, \end{aligned} \tag{3.2.9}$$

where we have used (3.2.7).

Next we turn to the operation of *metric contraction*. On smooth manifolds we have introduced the contraction $C^i_j : \mathcal{T}^r_s(M) \to \mathcal{T}^{r-1}_{s-1}$ on page 22. There the ith contravariant index or slot is contracted with the jth covariant one, i.e., in coordinates $(C^i_j A)^{i_1\dots i_{r-1}}_{j_1\dots j_{s-1}} = A^{i_1\dots \overset{i}{m}\dots i_r}_{j_1\dots \underset{j}{m}\dots j_s}$. On a SRMF we may use the metric to also contract two covariant or two contravariant slots by first raising respectively lowering the respective index, that is we combine the metric type changing with the contraction. More precisely let $1 \le a < b \le s$, then for arbitrary r we define $C_{ab} : \mathcal{T}^r_s(M) \to \mathcal{T}^r_{s-2}(M)$ locally by

$$(C_{ab}A)^{i_1\dots\dots i_r}_{j_1\dots j_{s-2}} := g^{lm}\, A^{i_1\dots\dots\dots\dots i_r}_{j_1\dots \underset{a}{l}\dots \underset{b}{m}\dots j_{s-2}}. \tag{3.2.10}$$

Analogously we define for $1 \le a < b \le r$ and for s arbitrary $C^{ab} : \mathcal{T}^r_s(M) \to \mathcal{T}^{r-2}_s(M)$ locally by

$$(C^{ab}A)^{i_1\dots\dots i_{r-2}}_{j_1\dots\dots j_s} := g_{lm}\, A^{i_1\dots \overset{a}{l}\dots \overset{b}{m}\dots j_{s-2}}_{j_1\dots\dots\dots\dots j_r}. \tag{3.2.11}$$

We now have the following compatibility result.

3.2.5 Lemma (Metric contraction and ∇). *On a SRMF (M, g) the covariant derivative as well as the covariant differential commute with type changing and metric contraction.*

Proof. For the case of the covariant derivative and type changing it suffices to consider the case $\downarrow^a_1$ since the assertion then follows by permutation for $\downarrow^a_b$ and by the following argument for $\uparrow^a_b$: Let $B = \downarrow^a_b A$ then we have

$$\uparrow^a_b \nabla_V B = \uparrow^a_b \nabla_V(\downarrow^a_b A) = \uparrow^a_b \downarrow^a_b \nabla_V A = \nabla_V A = \nabla_V \uparrow^a_b B. \tag{3.2.12}$$

Now to consider $\downarrow_1^a A$ first note that $\downarrow_1^a A = C_1^a(g \otimes A)$. Indeed in coordinates we have (cf. (3.2.3)) $C_1^a(g \otimes A)^{i_1 \cdots r-1}_{m j_1 \dots j_s} = g_{m_1 l} A^{i_1 \dots l \dots i_r}_{j_1 \dots j_s}$. Now by definition 1.3.22 ∇_V is a tensor derivation which by definition 1.3.16(ii) commutes with contractions and moreover satisfies the metric property ($\nabla 5$) (cf. 1.3.25(iv))so we obtain

$$\nabla_V(\downarrow_1^a A) = \nabla_V\big(C_1^a(g \otimes A)\big) = C_1^a \nabla_V(g \otimes A) = C_1^a(g \otimes \nabla_V A) = \downarrow_1^a \nabla_V A. \tag{3.2.13}$$

By equations 3.2.10 resp. (3.2.11) metric contraction is just the composition of type changing and contraction, so ∇_V also commutes with this operation.

The analogous assertions for the covariant differential ∇ follow easily from those of ∇_V. We demonstrate this for type changing just in a special case which makes clear how to proceed in the general case. Let $A \in \mathcal{T}_1^2(M)$, then $\nabla A \in \mathcal{T}_2^2(M)$, $\downarrow_1^1 \nabla A \in \mathcal{T}_3^1(M)$ and we have

$$\begin{aligned}(\downarrow_1^1 \nabla A)(\omega, X, Y, Z) &= \nabla A(X^\flat, \omega, Y, Z) = \nabla_Z A(X^\flat, \omega, Y) \qquad (3.2.14)\\ &= (\downarrow_1^1 \nabla_Z A)(\omega, X, Y) = \big(\nabla_Z(\downarrow_1^1 A)\big)(\omega, X, Y) = \big(\nabla(\downarrow_1^1 A)\big)(\omega, X, Y, Z).\end{aligned}$$

Finally one easily verifies that ∇ commutes with tensor products and contractions and hence with metric contraction. $\square$

Now we are finally in a position to introduce the above mentioned differential operators on SRMFs.

3.2.6 Definition (Gradient). *For a function $f \in \mathcal{C}^\infty(M)$ we define its* gradient $\mathrm{grad}(f)$ *(or* $\mathrm{grad} f$ *for short) as the vector field metrically equivalent to $df \in \Omega^1(M)$, i.e.,*

$$\langle \mathrm{grad}(f), X \rangle = df(X) = X(f) \qquad \textit{for all } X \in \mathfrak{X}(M). \tag{3.2.15}$$

We clearly see that while the differential df of a function is defined on any smooth manifold it needs a metric to define the gradient. In local coordinates we have $df = \partial_i f dx^i$ and so

$$\mathrm{grad} f = g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j} = \uparrow_1^1 df \tag{3.2.16}$$

since $\langle g^{ij} \partial_i f \partial_j, v^k \partial_k \rangle = \partial_i f v^k g^{ij} \delta_{jk} = \partial_i f v^i = \partial_i f dx^i(v^k \partial_k)$.
As a simple example we note that on flat space $\mathbb{R}^n_r$ we have $\mathrm{grad} f = \varepsilon_i \partial_i f dx^i$ which on $\mathbb{R}^n$ reduces to the well known formula $\mathrm{grad} f = \partial_i f \partial_i$.

3.2.7 Definition (Divergence). *For a tensor field A we call a* divergence *of A every contraction of the new covariant slot of ∇A with any of its original contravariant slots.*

We discuss some special cases. For a vector field $V \in \mathfrak{X}(M)$ the only possibility is $\mathrm{div} V = C(\nabla V)$ which in coordinates reads

$$\mathrm{div} V = C(\nabla V) = dx^i(\nabla_{\partial_i} V) = dx^i \left(\Big(\frac{\partial V^m}{\partial x^i} + \Gamma^m_{ik} V^k \Big) \partial_m \right) = \sum_i \Big(\frac{\partial V^i}{\partial x^i} + \Gamma^i_{ik} V^k \Big). \tag{3.2.17}$$

In the special case of flat space $\mathbb{R}^n_r$ we obtain the well-know formula from analysis $\mathrm{div}V = \sum_i \partial_i V^i$.

3.2.8 Definition (Hessian). *The* Hesse tensor H^f, *or* Hessian *for short, of a function $f \in \mathcal{C}^\infty(M)$ is defined as the second covariant differential of f, i.e.,*

$$H^f = \nabla(\nabla f). \tag{3.2.18}$$

3.2.9 Lemma (Hessian explicitly). *The Hessian H^f of $f \in \mathcal{C}^\infty(M)$ is a symmetric $(0,2)$-tensor field and we have*

$$H^f(X,Y) = XYf - (\nabla_X Y)f = \langle \nabla_X(\mathrm{grad}f), Y\rangle. \tag{3.2.19}$$

Proof. Since $\nabla f = df$ we have

$$\begin{aligned} H^f(X,Y) = \nabla(df)(X,Y) &= \big(\nabla_Y(df)\big)(X) \\ &= Y\big(df(X)\big) - df(\nabla_Y X) = Y(Xf) - (\nabla_Y X)(f). \end{aligned} \tag{3.2.20}$$

Symmetry now follows from the torsion free condition ($\nabla 4$), i.e., by $XY - YX = [X,Y] = \nabla_X Y - \nabla_Y X$. Finally we have by the metric condition ($\nabla 5$)

$$\langle \nabla_X(\mathrm{grad}f), Y\rangle = X\langle \mathrm{grad}f, Y\rangle - \langle \mathrm{grad}f, \nabla_X Y\rangle = XYf - \nabla_X Y(f) = H^f(X,Y).$$

□

In local coordinates we hence have for the Hessian

$$H^f_{ij} = H^f(\partial_i, \partial_j) = \partial_i\partial_j f - (\nabla_{\partial_i}\partial_j)f = \partial_i\partial_j f - \Gamma^k_{ij}\partial_k f. \tag{3.2.21}$$

3.2.10 Definition (Laplace). *The Laplace-Beltrami operator on a SRMF (M,g) is the mapping*

$$\triangle : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M), \qquad \triangle f = \mathrm{div}(\mathrm{grad}f). \tag{3.2.22}$$

More explicitly we have

$$\begin{aligned} \triangle f = \mathrm{div}(\mathrm{grad}f) &= C\big(\nabla(\mathrm{grad}f)\big) = C\big(\nabla(\uparrow^1_1 df)\big) \\ &= C\big(\uparrow^1_1 \nabla(df)\big) = (C\uparrow^1_1)H^f = C_{12}(H^f), \end{aligned} \tag{3.2.23}$$

and we see that the Laplace-Beltrami operator is the metric contraction of the Hessian tensor. In local coordinates we hence obtain

$$\triangle f = g^{ij}H_{ij} = g^{ij}\nabla_{\partial_i}\nabla_{\partial_j} f = g^{ij}(\partial_i\partial_j f - \Gamma^k_{ij}\partial_k f), \tag{3.2.24}$$

which on flat $\mathbb{R}^n_r$ gives $\triangle f = \sum \varepsilon_i \partial^2 f/\partial x_i^2$. Obviously this gives the Laplace operator on $\mathbb{R}^n$ and the wave operator on Minkowski space $\mathbb{R}^n_1$. This is the reason why on Riemanninan and Lorentzian manifolds $\triangle$ sometimes is called the Laplace and the wave operator, respectively.