3.3 The Einstein equations

In this section we introduce the famous Einstein equations, i.e., the fundamental equations of General Relativity (GR), Albert Einstein's eminent theory of space, time and gravitation, which is the currently best available physical description of our universe at large. Just a year ago and only briefly after its centennial GR has seen a spectacular success by the direct observation of gravitational waves emitted from a binary black hole merger.
The Einstein equations are the so called field equations of GR and link the geometry and, in particular, the curvature of the spacetime manifold to its energy-matter content. Here we collect the mathematical prerequisites for their formulation.
We start by introducing the *Ricci tensor* and the *curvature scalar*, two 'curvature quantities' derived from the Riemann tensor.

3.3.1 Definition (Ricci tensor). *Let (M, g) be a SRMF with Riemann tensor R. The* Ricci tensor *Ric is defined as the contraction $C^1_3 R \in \mathcal{T}^0_2(M)$.*

The Ricci tensor's local coordinates are denoteby by R_{ij} and take the form (cf. page 22)

$$R_{ij} = R^m_{ijm}. \tag{3.3.1}$$

Moreover Ric is symmetric by pair symmetry of the Riemann tensor 3.1.2(iv) since

$$\begin{aligned}\mathrm{Ric}(X,Y) = (C^1_3 R)(X,Y) &= R(dx^i, X, Y, \partial_i)\\ &= dx^i\big(R(X,Y,\partial_i)\big) = \langle R_{Y\partial_i}X, \partial_i\rangle = \langle R_{X\partial_i}Y, \partial_i\rangle.\end{aligned} \tag{3.3.2}$$

Also we note the trace formula $\mathrm{Ric}(X,Y) = \mathrm{trace}(V \mapsto R_{XV}Y)$. A SRMF with $\mathrm{Ric} = 0$ is called *Ricci flat*. Clearly any flat manifold $R = 0$ is also Ricci flat but the converse is not true as we shall discuss below and which is essential for GR.
We proceed introducing the curvature scalar or scalar curvature of a SRMF.

3.3.2 Definition (Scalar curvature). *The* scalar curvature *S of the SRMF (M, g) is defined as the contractions of the Ricci tensor, $S = C(Ric) \in \mathcal{C}^\infty(M)$.*

Observe that since $\mathrm{Ric} \in \mathcal{T}^0_2(M)$ the contraction C unambiguously stands for C_{11}, cf. (3.2.10). In local coordinates we have

$$S = g^{ij} R_{ij} = g^{ij} R^m_{ijm} = R^{mj}{}_{jm}. \tag{3.3.3}$$

For our further considerations we rely on the following property of the curvature tensor.

3.3.3 Proposition (Second Bianchi identity). *For $x, y, z \in T_pM$ we have*

$$(\nabla_z R)(x,y) + (\nabla_x R)(y,z) + (\nabla_y R)(z,x) = 0. \tag{3.3.4}$$

Proof. As in the proof of 3.1.2 we may extend x, y, z arbirarily to vector fields X, Y, Z on a neighbourhood U of p. In this case we choose these extensions in such a way that their coefficients are constant w.r.t. some normal coordinates at p. Then again all Lie brackets vanish on U and moreover by 2.1.17(ii) all Christoffel symbols vanish at p and hence also all covariant derivatives of X, Y, Z w.r.t. each other vanish at p.
Using the notation used in the proof of 3.1.2(iii) we have to show that $S\nabla_Z R(X,Y) = 0$. Now by the product rule 1.3.18 we have for arbitrary V at p

$$\big((\nabla_Z R)(X,Y)\big)(V) = \nabla_Z\big(R(X,Y)V\big) - R(\underbrace{\nabla_Z X}_{=0}, Y)V - R(X, \underbrace{\nabla_Z Y}_{=0})V - R(X,Y)(\nabla_Z V)$$

and so again at p

$$(\nabla_Z R)(X,Y) = [\nabla_Z, R(X,Y)] = [\nabla_Z, [\nabla_Y, \nabla_X], \tag{3.3.5}$$

where we have used (3.1.4). Since the Jacobi identity holds also for ∇_X (cf. [10, 2.5.15(iii)]), the sum over all cyclic permutations of $(\nabla_Z R)(X,Y)$ vanishes as claimed. $\square$

3.3.4 Corollary (Divergence of Ricci). *We have $dS = 2\mathrm{div}(Ric)$.*

Proof. By 3.1.3 and 1.3.25(v) we have in coordinates

$$(\nabla R)(\partial_j, \partial_k, \partial_l, \partial_r) = (\nabla_{\partial_r} R)_{\partial_k \partial_l}(\partial_j) = R^i_{jkl;r}\partial_i, \tag{3.3.6}$$

which upon using 3.3.3 gives $R^i_{jkl;r} + R^i_{jlr;k} + R^i_{jrk;l} = 0$. Now interchanging r and k in the final term (which by 3.1.2(i) causes a sign change) and contracting i with r gives

$$0 = R^r_{jkl;r} + R^r_{jlr;k} - R^r_{jkr;l} = R^r_{jkl;r} + R_{jl;k} - R_{jk;l} \tag{3.3.7}$$

and so

$$g^{jk}R^r_{jkl;r} + g^{jk}R_{jl;k} - S_{;l} = 0. \tag{3.3.8}$$

Next we note that $R_{mjkl} = R_{jmlk}$. Indeed by 3.1.2(i),(ii) we find $R_{jmlk} = g_{rj}R^r_{mlk} = \langle\partial_j, \partial_r\rangle R^r_{mlk} = \langle\partial_j, R^r_{mlk}\partial_r\rangle = \langle R_{\partial_l\partial_k}(\partial_m), \partial_j\rangle = -\langle R_{\partial_k\partial_l}(\partial_m), \partial_j\rangle = \langle R_{\partial_k\partial_l}(\partial_j), \partial_m\rangle = R_{mjkl}$. Moreover we have $R^r_{jkl} = g^{rm}R_{mjkl}$ and so

$$g^{jk}R^r_{jkl;r} = g^{jk}g^{rm}R_{mjkl;r} = g^{jk}g^{rm}R_{jmlk;r} = g^{rm}R^k_{mlk;r} = g^{rm}R_{ml;j} = R^r_{l;r}. \tag{3.3.9}$$

So by (3.3.8) we find

$$R^r_{l;r} + R^k_{l;k} = 2R^r_{l;r} = S_{;l}. \tag{3.3.10}$$

Finally since Ric is symmetric (cf. (3.3.2)) we have $C_{13}(\nabla\mathrm{Ric}) = C_{23}(\nabla\mathrm{Ric}) = \mathrm{div}(\mathrm{Ric})$, which in coordinates reads $g^{rs}R_{sl;r} = R^r_{l;r}$. So (3.3.10) gives $2\mathrm{div}(\mathrm{Ric}) = \nabla S = ds$. $\square$

We now very briefly discuss the basic principles of General Relativity. Naturally any discussion in the setting of this course has to be superficial and we refer e.g. to [13, Ch. 4] for a more appropriate account.

The stage of GR is *spacetime* which is the set of all events (t, x), labelled by a one-dimensional time coordinate and a three-dimensional space coordinate. Spacetime can be a model of e.g. the surroundings of a star, our solar system, or our universe as a whole. Mathematically spacetime is described by a 4-dimensional Lorentzian manifold (M, g), where the Lorentzian signature is chosen as to implement the causality structure already present in special relativity.

Now contrary to classical Newtonian physics gravity is *not* described as a force field on this manifold M but rather as the *curvature of spacetime*. This ground breaking idea which Einstein famously called his happiest thought, relies on taking the *principle of equivalence* to be the basic building block of the theory. Indeed, due to Galileo's principle of equivalence, all bodies fall the same in a gravitational field, so gravity can be thought of as being a 'property' of spacetime!

One can also argue why this 'property' has to be related to curvature. Generalising the Newtonian idea that bodies which move freely, i.e., without any force acting upon them move along straight paths, freely falling test bodies in GR should move along geodesics of spacetime. Now considering test bodies falling freely in a gravitational field of a point mass in the Newtonian picture one sees that they undergo a relative acceleration, do to so-called tidal forces. Translated into the spacetime perspective this means that geodesics focus—and the quantity that focusses geodesics clearly is curvature.

Consequently the curvature of spacetime has to be related to physical forces, or better to the all the mass and energy it contains. (Mass is equivalent to energy by the famous equation $E = mc^2$.) Already in classical mechanics and electrodynamics the matter variables (forces, strain, stress, etc.) are described by a single object, the so-called energy momentum tensor T which is a symmetric $(0, 2)$-tensor field. Moreover T is divergence free and this property implements *energy conservation*, another basic principle in all of physics.

So specifically Einstein in 1915 was looking for the correct equation that relates T to the curvature of spacetime. In a time where Riemannian resp. Lorentzian geometry has by far not been developed to its present state he first tried several variants of the Ricci curvature in his attempt to describe the perihelion precession of the planet mercury. However, the Ricci tensor is not divergence free and so he finally introduced the following quantity.

3.3.5 Definition (Einstein tensor). *Let (M, g) be a Lorentzian manifold. We define the* Einstein tensor *as*

$$G := Ric - \frac{1}{2} S\, g. \tag{3.3.11}$$

The essential properties of G are now:

3.3.6 Lemma (Properties of G). *The Einstein tensor of a spacetime has the following properties:*

(i) G is a symmetric and divergence free $(0,2)$-tensor field.

(ii) $Ric = G - \frac{1}{2}\,C(G)g$.

Proof. (i) Symmetry of G follows immediately from symmetry of Ric and g. We calculate the divergence $\mathrm{div}(Sg) = C_{13}(\nabla(Sg))$ by the product rule to be

$$\begin{aligned}\nabla(Sg)(X,Y,Z) = \nabla_Z(Sg)(X,Y) &= \nabla_Z(Sg(X,y)) - Sg(\nabla_Z X, Y) - Sg(X, \nabla_Z Y)\\ &= (\nabla_Z S)\, g(X,Y) + S\nabla_Z(g(X,Y)) - Sg(\nabla_Z X, Y) - Sg(X, \nabla_Z Y).\end{aligned} \tag{3.3.12}$$

Moreover by ($\nabla 5$) we have $0 = (\nabla_Z g)(X,Y) = \nabla g(X,Y,Z) = \nabla_Z(g(X,Y)) - g(\nabla_Z X, Y) - g(X, \nabla_Z Y)$ and so

$$\nabla(Sg)(X,Y,Z) = (\nabla_Z S)\, g(X,Y) = dS(Z)\, g(X,Y) = g \otimes dS(X,Y,Z). \tag{3.3.13}$$

Now for convenience proceeding in coordinates we find $(\nabla(Sg))_{ijk} = g_{ij}(dS)_k = g_{ij}\partial_k S$ and hence

$$\mathrm{div}(Sg)_j = C_{13}(\nabla(Sg))_j = g^{ki} g_{ij} \frac{\partial S}{\partial x^k} = \frac{\partial S}{\partial x^j} = (dS)_j. \tag{3.3.14}$$

So we finally arrive at $\mathrm{div}(Sg) = dS$ and so by 3.3.4

$$\mathrm{div}(G) = \mathrm{div}(\mathrm{Ric} - \frac{1}{2}\, Sg) = \frac{1}{2}(dS - dS) = 0. \tag{3.3.15}$$

(ii) We have $C(g) = g^{ij} g_{ij} = \delta^i_i = \dim M = 4$, hence by definition 3.3.2 $C(G) = C(\mathrm{Ric}) - 1/2\, SC(g) = S - 2S = -S$ and finally

$$\mathrm{Ric} = G + \frac{1}{2} Sg = G - \frac{1}{2}\, C(G) g. \tag{3.3.16}$$

□

The significance of the previous results lie in the fact that (i) says that in the light of the above discussion G is a formally qualified candidate for the curvature quantity to be equated with T, while (ii) guarantees that it is also a sensible one, since it encodes the same information as Ric. So we finally arrive at:

The Einstein equations. If (M,g) is a spacetime with energy momentum tensor T then

$$G \;=\; \frac{8\pi N}{c^4}\, T. \tag{3.3.17}$$

Here $N = 6.67 \cdot 10^{-11} m^3/(kg \cdot s^2)$ is Newton's gravitational constant and $c = 2.99 \cdot 10^8 m/s$ is the speed of light in vacuum. Usually one sets $N/c^4 = 1$ which amounts to using so-called geometric units. In the very important special case of vacuum, i.e., in the absence of matter, the equations reduce to

$$\mathrm{Ric} \; = \; 0, \tag{3.3.18}$$

since taking the trace of (3.3.17) gives $C(G) = S = 8\pi C(T)$. So vacuum solutions to Einstein equations are Ricci flat but far from (locally) flat, i.e. $R = 0$, as is exemplified e.g. by the notorious *Schwarzschild metric* which is the (unique) spherically symmetric solution of (3.3.18) and provides the simplest model of a *black hole*.

Now in a sense General Relativity is the study of solutions of the Einstein equations. From the coordinate formulae one sees that they form a highly complicated system of (by symmetries of G) 10 coupled nonlinear (quasilinear, to be precise) partial differential equations for g. Although there are literally thousands of known exact solutions to (3.3.17) accompanied by a big wealth of deep results in Lorentzian geometry and also recently the global existence theory of Einstein's equations has made great advances it is still fair to say that one is far from reaching a comprehensive understanding of their full content. So General Relativity is a very active field of research today, combining many fascinating aspects of (Lorentzian) geometry and analysis.