2.4 The Hopf-Rinow theorem

In this section we state and prove the main result on *complete Riemannian manifolds* which links the geodesics of the manifold to its structure as a metric space. The technical core of this result is contained in the following lemma.

2.4.1 Lemma (Globally defined exp_p.). *Let M be a connected RMF and let $p \in M$ such that the exponential map $\exp_p$ at p is defined on all of T_pM (i.e., $\mathcal{D}_p = T_pM$). Then for each $q \in M$ there is a minimising geodesic from p to q.*

Proof. Let $U_\varepsilon(p)$ be a normal ε-neighborhood of p, cf. 2.3.7. If $q \in U_\varepsilon(p)$ then the claim follows from 2.3.6(ii). So let $q \notin U_\varepsilon(p)$ and denote by r the radius function at p, see (2.3.2). Now for $\delta > 0$ sufficiently small (i.e., $\delta < \varepsilon$) the 'sphere' $S_\delta := \{s \in M : \ r(m) = \delta\} = \exp_p(\{v \in T_pM : \ \|v\| = \delta\})$ lies within $U_\varepsilon(p)$. Since S_δ is compact the continuous function $S_\delta \ni s \mapsto d(s,q)$ attains its minimum in some point $m \in S_\delta$. We now show that

$$d(p,m) + d(m,q) = d(p,q). \tag{2.4.1}$$

Clearly we have $\geq$ in (2.4.1) by the triangle inequality. Conversely let $\alpha : [0,b] \to M$ be any curve from p to q and let $a \in (0,b)$ be any value of the parameter such that $\alpha(a) \in S_\delta$. (Clearly α initially lies inside S_δ and then has to leave it, Figure ♣ `insert figure` ♣). Write $\alpha_1 = \alpha|_{[0,a]}$ and $\alpha_2 = \alpha|_{[a,b]}$. Then by 2.3.6(ii) and the definition of m we have

$$L(\alpha) = L(\alpha_1) + L(\alpha_2) \geq \delta + L(\alpha_2) \geq \delta + d(m,q). \tag{2.4.2}$$

Again appealing to 2.3.6(ii) this implies

$$d(p,q) \geq \delta + d(m,q) = d(p,m) + d(m,q) \tag{2.4.3}$$

and we have proven (2.4.1).

Recalling that by assumption $\mathcal{D}_p = T_pM$ let now $c : [0,\infty) \to M$ be the unit speed geodesic whose initial piece is the radial geodesic form p to m. We show that c is the asserted minimising geodesic from p to q. To begin with set $d := d(p,q)$ and

$$T := \{t \in [0,d] : \ t + d(c(t),q) = d\}. \tag{2.4.4}$$

It suffices to show that $d \in T$ since in this case we have $d(c(d),q) = 0$ hence $c(d) = q$. Moreover we then have $L(c|_{[0,d]}) = d = d(p,q)$ and so c is minimising.

Now to show that $d \in T$ we first observe that $c_{[0,t]}$ is minimising for any $t \in T$. Clearly we have $t = L(c|_{[0,t]}) \geq d(p,c(t))$. Conversely by definition of T it holds that $d \leq d(p,c(t)) + d(c(t),q) = d(p,c(t)) + d - t$ and so $t = L(c|_{[0,t]}) \leq d(p,c(t))$. Hence in total we have $L(c|_{[0,t]}) = d(p,c(t))$.
Now let $\tilde{t}$ such that $c(\tilde{t}) = m$. Then by (2.4.1) we have $d = d(p,q) = d(c(0),c(\tilde{t})) + d(c(\tilde{d}),q) = \tilde{t} + d(c(\tilde{d}),q)$ and so $\tilde{t} \in T$.

So T is non-empty, closed and contained in $[0, d]$, hence compact. Writing $t_0 := \max T \leq d$ it remains to show that $t_0 = d$.

We assume to the contrary that $t_0 < d$. Then let $U_{\varepsilon'}(c(t_0))$ be a normal ε-neighbourhood of $c(t_0)$ which does not contain q, see Figure ♣ `insert figure` ♣. The same argument as in the beginning of the proof shows that there exists a radial unit speed geodesic $\sigma : [0, \delta'] \to U_{\varepsilon'}(c(t_0))$ that joins $c(t_0)$ with some point $m' \in S_{\delta'}$ with $d(m', q)$ minimal on $S_{\delta'}$. As in (2.4.1) we obtain

$$d(c(t_0), m') + d(m', q) = d(c(t_0), q). \tag{2.4.5}$$

Now observing that $d(c(t_0), m') = L(\sigma|_{[0,\delta']}) = \delta'$ we obtain from the fact that $t_0 \in T$ and (2.4.5)

$$d = t_0 + d(c(t_0), q) = t_0 + \delta' + d(m', q). \tag{2.4.6}$$

Also $d = d(p, q) \leq d(p, m') + d(m', q)$ and so $d - d(m', q) = t_0 + \delta' \leq d(p, m')$. But this implies that the concatenation $\tilde{c}$ of $c|_{[0,t_0]}$ and σ is a curve joining p with m' which satisfies

$$d(p, m') \leq L(\tilde{c}) = t_0 + \delta' \leq d(p, m').$$

Hence $\tilde{c}$ is minimising and so by 2.3.11 an (unbroken) geodesic which implies that $\tilde{c} = c$. This immediately gives $m' = \sigma(\delta') = c(t_0 + \delta')$ and further by (2.4.6) we obtain

$$d - t_0 = \delta' + d(m', q) = \delta' + d(c(t_0 + \delta'), q).$$

But this means that $t_0 + \delta' \in T$ which contradicts the fact that $t_0 = \max T$. □

We may now head on to the main result of this section.

2.4.2 Theorem (Hopf-Rinow). *Let (M, g) be a connected RMF, then the following conditions are equivalent:*

(MC) The metric space (M, d) is complete (i.e., every Cauchy sequence converges).

(GC') There is $p \in M$ such that M is geodesically complete at p, i.e., $\exp_p$ is defined on all of T_pM.

(GC) (M, g) is geodesically complete.

(HB) M possesses the Heine-Borel property, i.e., every closed and bounded subset of M is compact.

The Hopf-Rinow theorem hence, in particular, guarantees that for connected Riemannian manifolds geodesic completeness coincides with completeness as a metric space. Therefore the term *complete Riemannian manifold* is unambiguous in the connected case and we will use it from now on. The theorem together with the previous Lemma 2.4.1 has the following immediate and mayor consequence.

2.4.3 Corollary (Geodesic connectedness). *In a connected complete Riemannian manifold any pair of points can be joined by a minimising geodesic.*

The converse of this result is obviously wrong; just consider the open unit disc in $\mathbb{R}^2$. Form the point of view of semi-Riemannian geometry the striking fact of the corollary is that in a complete RMF two arbitrary points can be connected by a geodesic *at all*. This property called *geodesic connectedness* fails to hold in complete connected Lorentzian manifolds. The great benefit of the corollary, of course, is that it allows to use geodesic constructions globally.
We now proceed to the proof of the Hopf-Rinow theorem.

Proof of 2.4.2.
(MC)$\Rightarrow$(GC): Let $c : [0, b) \to M$ be a unit speed geodesic. We have to show that c can be extended beyond b as a geodesic. By 2.2.8 is suffices to show that c can be extended continuously (as a curve) to b. To this end let $(t_n)_n$ be a sequence in $[0, b)$ with $t_n \to b$. Then $d(c(t_n), c(t_m)) \leq |t_n - t_m|$ and so $(c(t_n))_n$ is a Cauchy sequence in M , which by (MC) is convergent to a point called $c(b)$. If $(t'_n)_n$ is another such sequence then since $d(c(t_n), c(t'_n)) \leq |t_n - t'_n|$ we find that $(c(t'_n)_n$ also converges to $c(b)$. Hence we have c extended continuously to $[0, b]$.

(GC)$\Rightarrow$(GC') is clear.

(GC')$\Rightarrow$(HB): Let $A \subseteq M$ be closed and bounded. For any $q \in A$, by 2.4.1 there is a minimising geodesic $\sigma_q : [0, 1] \to M$ from p to q. As in (2.3.5) we have $\|\sigma'_q(0)\| = L(\sigma_q) = d(p, q)$.
Now since A is bounded there is $R > 0$ such that $d(p, q) \leq R$ for all $q \in A$. So $\sigma'_q(0) \in \overline{B_R(0)} = \{v \in T_pM : \|v\| \leq R\}$ which clearly is compact. But then $A \subseteq \exp_p(\overline{B_R(0)})$ hence is contained in a compact set and thus is compact itself.

(HB)$\Rightarrow$(MC)[3]: The point set of every Cauchy sequence is bounded hence its closure is compact by (HB). So the sequence possesses a convergent subsequence and being Chauchy it is convergent itself. $\Box$

Next we prove that any smooth manifold M (which, recall our convention, is assumed to be Hausdorff and second countable) can be equipped with a Riemannian metric g. Indeed g can simply be constructed by glueing the Euclidean metrics in the charts of an atlas as is done in the proof below.
In fact much more is true. The Theorem of Numizo and Ozeki (see e.g. [8, 62.12]) guarantees that on every smooth manifold there exists a *complete* Riemannian metric.

2.4.4 Theorem (Existence of Riemannian metrics). *Let M be a smooth manifold then there exists a Riemanninan metric g on M.*

[3]Observe that this part of the proof is purely topological.

Proof. Cover M by charts $((x^1_\alpha, \dots, x^n_\alpha), U_\alpha)$ and let $(\chi_\alpha)_\alpha$ be a partition of unity subordinate to this cover with $\mathrm{supp}(\chi_\alpha) \subseteq U_\alpha$, cf. [10, 2.3.14]. Then on U_α set

$$g_\alpha := \sum_i dx^i_\alpha \otimes dx^i_\alpha \quad \text{and finally define} \quad g := \sum_\alpha \chi_\alpha g_\alpha \quad \text{on } M. \tag{2.4.7}$$

Since a linear combination of positive definite scalar products with positive coefficients again is positive definite, g indeed is a Riemannian metric on M. □

This result has the following immediate topological consequence.

2.4.5 Corollary (Metrisability). *Every smooth manifold is metrisable.*

Proof. In case M is connected this is immediate from 2.4.4 and 2.3.9. In the general case we have $M = \mathring{\bigcup} M_i$ with all (countably many) M_i connected. On each M_i we have a metric d_i for which we may assume w.l.o.g. $d_i < 1$ (otherwise replace d_i by $\frac{d_i}{1+d_i}$). Then

$$d(x,y) := \begin{cases} d_i(x,y) & \text{if } x, y \in M_i \\ 1 & \text{else} \end{cases} \tag{2.4.8}$$

clearly is a metric on M. □

The construction used in the proof can also be employed to draw the following conclusion from general topology.

2.4.6 Corollary. *Every compact Riemannian manifold is complete.*

Proof. We first construct a metric on M from the distance functions in each of the (countably many) connected components as in (2.4.8). Then we just use the fact that any compact metric space is complete. In fact this follows from the proof of (HB)⇒(MC) in 2.4.2 by noting that the Heine-Borel property (HB) holds trivially in any compact metric space. □

To finish this section we remark that the Lorentzian situation is much more complicated, somewhat more precisely we have

2.4.7 Remark (Lorentzian analogs). On Lorentzian manifolds the above results are wrong in general! In some more detail we have the following.

(i) There does not exist a Lorentzian metric on any smooth manifold M. Observe that the above proof fails since linear combinations of nondegenerate scalar products need not be nondegenerate. In fact, there exist topological obstructions to the existence of Lorentzian metrics: M can be equipped with a Lorentzian metric iff there exists a nowhere vanishing vector field hence iff M is non compact or compact with Euler characteristic 0.

(ii) The Lorentzian analog to 2.3.4 is the following: Let U be a normal neighborhood of p. If there is a timelike curve form p to some point $q \in U$ then the radial geodesic from p to q is the *longest* curve from p to q.

(iii) There is no Lorentzian analog of the Hopf Rinow theorem 2.4.2. If a Lorentzian manifold is connected and geodesically complete it need not even be geodesically connected. For a counterexample see e.g. [12, p. 150].

Chapter 3

Curvature

♣ `Intro to be done` ♣

3.1 The curvature tensor

To motivate the definition of the curvature tensor on a SRMF we consider the parallel transport of a vector along a curve. If we parallel transport a vector along a closed curve in the plane then upon returning to the starting point we end up with the same vector as we have started with. This, however, is not the case in the sphere. If we parallel transport a vector say along a geodesic triangle on the sphere we do not end up with the vector we have started with, see Figure ♣ `Figure` ♣.
Now since parallel transport is defined via the covariant derivative, see page 30, the difference between the starting vector and the final vector can be expressed in terms of the *non commutativity* of covariant derivatives. This idea lies behind the following definition to which we will return also at the end of this section.

3.1.1 Definition & Lemma (Riemanninan curvature tensor). *Let M be a SRMF with Levi-Civita connection ∇. Then the mapping $R : \mathfrak{X}(M)^3 \to \mathfrak{X}(M)$ given by*

$$R_{XY}Z := \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z \tag{3.1.1}$$

is a $(1,3)$ tensor field called the Riemanninan curvature tensor *of M.*

Proof. By (1.3.24) we only have to show that R is $\mathcal{C}^\infty(M)$-multi linear. So let $f \in \mathcal{C}^\infty(M)$. We then have by [10, 2.5.15(iv)] that $[X, fY] = XfY + f[X,Y]$ and so

$$\begin{aligned} R_{X,fY}Z &= \nabla_{[X,fY]}Z - \nabla_X \nabla_{fY} Z - \nabla_{fY}\nabla_X Z \\ &= X(f)\nabla_Y Z + f\nabla_{[X,Y]}Z - \nabla_X(f\nabla_Y Z) + f\nabla_Y\nabla_X Z \\ &= Xf\nabla_Y Z - X(f)\nabla_Y Z + fR_{XY}Z = fR_{XY}Z. \end{aligned} \tag{3.1.2}$$

Since by definition $R_{XY}Z = -R_{YX}Z$ we also find that $R_{fXY}Z = fR_{XY}Z$. Finally by an analogous calculation one finds that $R_{XY}fZ = fR_{X,Y}Z$. □

We will follow the widespread convention to also write $R(X,Y)Z$ for $R_{XY}Z$. Moreover we will also call R the Riemann tensor of curvature tensor for short. Since R is a tensor field one may also insert individual tangent vectors into its slots. In particular for $x, y \in T_pM$ the mapping

$$R_{xy} : T_pM \to T_pM, \qquad z \mapsto R_{xy}z \tag{3.1.3}$$

is called the *curvature operator*. We next study the symmetry properties of the curvature tensor.

3.1.2 Proposition (Symmetries of the Riemann tensor). *Let $x, y, z, v, x \in T_pM$ then for the curvature operator we have the following identities*

(i) $R_{xy} = -R_{yx}$ *(skew-symmetry)*

(ii) $\langle R_{xy}v, w\rangle = -\langle R_{xy}w, v\rangle$ *(skew-adjointness)*

(iii) $R_{xy}z + R_{yz}x + R_{zx}y = 0$ *(first Bianchi identity)*

(iv) $\langle R_{xy}v, w\rangle = \langle R_{vw}x, y\rangle$ *(symmetry by pairs)*

Proof. Since ∇_X and $[\,,\,]$ are local operations (see 1.3.6 and [10, 2.5.15(v)] respectively) it suffices to work on any neighbourhood of p. Moreover all identities are tensorial and we may extend $x, y, \ldots$ in any convenient way to vector fields $X, Y, \ldots$ on that neighbourhood. In the present case it is beneficial to do so in such a way that all Lie-brackets vanish which is achieved by taking the vector fields to have constant components w.r.t. a coordinate basis (Recall that $[\partial_i, \partial_j] = 0$, cf. [10, 2.5.15(vi)] ♣ `will be (vii) in the new version` ♣.) We then have

$$R_{XY}Z = \nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z \tag{3.1.4}$$

and we go on proving the individual items of the proposition.

(i) follows directly form the anti symmetry of $[\,,\,]$. (Observe that (i) also is easy to see from the definition and we have already used it in the proof of 3.1.1.)

(ii) It suffices to show that $\langle R_{xy}v, v\rangle = 0$ since the assertion then follows by replacing v by $v + w$. We have using 1.3.4(∇5)

$$\begin{aligned}\langle R_{XY}V, V\rangle &= \langle \nabla_Y\nabla_X V, V\rangle - \langle \nabla_X\nabla_Y V, V\rangle \\ &= Y\langle \nabla_X V, V\rangle - \cancel{\langle \nabla_X V, \nabla_Y V\rangle} - X\langle \nabla_Y V, V\rangle + \cancel{\langle \nabla_Y V, \nabla_X V\rangle} \\ &= \frac{1}{2}YX\langle V, V\rangle - \frac{1}{2}XY\langle V, V\rangle = -\frac{1}{2}\underbrace{[X,Y]}_{=0}(\langle V, V\rangle) = 0.\end{aligned} \tag{3.1.5}$$

(iii) follows from the following more general reasoning. Let $F : \mathfrak{X}(M)^3 \to \mathfrak{X}(M)$ be an $\mathbb{R}$-multi linear map and define the mapping $S(F) : \mathfrak{X}(M)^3 \to \mathfrak{X}(M)$ as the sum of the cyclic permutations of F, i.e.,

$$S(F)(X, Y, Z) = F(X, Y, Z) + F(Y, Z, X) + F(Z, X, Y). \tag{3.1.6}$$

Then a cyclic permutation of X, Y, Z obviously leaves $S(F)(X, Y, Z)$ unchanged. Consequently we find using 1.3.4(∇4)

$$\begin{aligned} S(R)_{XY} Z &= S\nabla_Y \nabla_X Z - S\nabla_X \nabla_Y Z \\ &= S\nabla_X \nabla_Z Y - S\nabla_X \nabla_Y Z = -S\nabla_X [Y, Z] = 0. \end{aligned} \tag{3.1.7}$$

(iv) is a combinatorial exercise. By (iii) $\langle S(R)_{YV} X, W\rangle = 0$. Now summing over the four cyclic permutations of Y, V, X, W and writing out $S(R)$ one obtains 12 terms, 8 of which cancel in pairs by (i) and (ii) leaving

$$2\langle R_{XY} V, W\rangle + 2\langle R_{WV} X, Y\rangle = 0, \tag{3.1.8}$$

which by another appeal to (i) gives the asserted identity.

□

Next we derive a local formula for the Riemann tensor.

3.1.3 Lemma (Coordinate expression for R). *Let $(x^1, \ldots, x^n)$ be local coordinates. Then we have $R_{\partial_k \partial_l} \partial_j = R^i_{jkl} \partial_i$, where*

$$R^i_{jkl} = \frac{\partial}{\partial x^l} \Gamma^i_{\ kj} - \frac{\partial}{\partial x^k} \Gamma^i_{\ lj} + \Gamma^i_{\ lm} \Gamma^m_{\ kj} - \Gamma^i_{\ km} \Gamma^m_{\ lj}. \tag{3.1.9}$$

Proof. Since $[\partial_i, \partial_j] = 0$ for all i, j we have by (3.1.4) that

$$R_{\partial_k \partial_l} \partial_j = \nabla_{\partial_l} \nabla_{\partial_k} \partial_j - \nabla_{\partial_k} \nabla_{\partial_l} \partial_j. \tag{3.1.10}$$

Now by 1.3.8 we have

$$\nabla_{\partial_l}(\nabla_{\partial_k} \partial_j) = \nabla_{\partial_l}(\Gamma^m_{\ kj} \partial_m) = \frac{\partial}{\partial x^l} \Gamma^m_{\ kj} \partial_m + \Gamma^m_{\ kj} \Gamma^r_{\ lm} \partial_r = \left(\frac{\partial}{\partial x^l} \Gamma^i_{\ kj} + \Gamma^i_{\ lm} \Gamma^m_{\ kj}\right) \partial_i. \tag{3.1.11}$$

Now exchanging k and l and subtracting the respective terms gives the assertion. □

In the remainder of this section we want to give two interpretations of the Riemann tensor to aid also an intuitive understanding of this pretty complicated geometric object.

(1) We show that the Riemann tensor is an obstruction to the manifold being locally flat, i.e., the manifold being covered by charts in which the metric is flat, see Theorem 3.1.7, below.

(2) We make precise the idea which we already discussed prior to definition 3.1.1. We will establish that the the curvature tensor is a measure for the failure of a vector to return to ist starting value when parallely transported along a closed curve.

To arrive at (1) we need some preparations. Our proof of the result alluded to above depends on the construction of an especially simple coordinate system. Recall that the natural basis vector fields ∂_i in any coordinate system commute, i.e., $[\partial_i, \partial_j] = 0$ for all i, j ([10, 2.5.15(vi)]). We now establish the converse of this result: any n-tuple of commuting and linearly independent local vector fields is the natural basis for some chart. We begin with the following characterisation of commuting flows resp. vector fields, which is of clear independent interest.

3.1.4 Lemma (Commuting flows). *Let M be a smooth manifold and let $X, Y \in \mathfrak{X}(M)$. Then the following conditions are equivalent:*

(i) $[X, Y] = 0$,

(ii) $(\mathrm{Fl}^X_t)^* Y = Y$, *wherever the l.h.s. is defined,*

(iii) $\mathrm{Fl}^X_t \circ \mathrm{Fl}^Y_s = \mathrm{Fl}^Y_s \circ \mathrm{Fl}^X_t$, *wherever one (hence both) side(s) are defined.*

Proof. (i)⇒(ii): We have

$$\begin{aligned}
\frac{d}{dt}(\mathrm{Fl}^X_t)^* Y &= \frac{d}{ds}\Big|_0 (\mathrm{Fl}^X_{t+s})^* Y = \frac{\partial}{\partial s}\Big|_0 (T\mathrm{Fl}^X_{-(s+t)} \circ Y \circ \mathrm{Fl}^X_{s+t}) \\
&= \frac{\partial}{\partial s}\Big|_0 (T\mathrm{Fl}^X_{-t)} \circ \mathrm{Fl}^X_{-s} Y \circ \mathrm{Fl}^X_s \circ \mathrm{Fl}^X_t) = T\mathrm{Fl}^X_{-t} \circ \underbrace{\Big(\frac{\partial}{\partial s}\Big|_0 \big(\mathrm{Fl}^X_s\big)^* Y \Big)}_{=L_X Y} \circ \mathrm{Fl}^X_t \\
&= (\mathrm{Fl}^X_t)^*(L_X Y) = 0,
\end{aligned} \tag{3.1.12}$$

and so $(\mathrm{Fl}^X_T)^* Y = (\mathrm{Fl}^X_0)^* Y = Y$.

(ii)⇒(i): follows directly from the definition of the Lie derivative.

(ii)⇒(iii): Observe that condition (ii) precisely says that $(\mathrm{Fl}^X_t)^* Y$ and Y are Fl^X_t-related. Recall that given a smooth function between two manifolds $f : M_1 \to M_2$ one says that $X_1 \in \mathfrak{X}(M_1)$ is f-related to $X_2 \in \mathfrak{X}(M_2)$, $X_1 \sim_f X_2$, if $Tf \circ X_1 = X_2 \circ f$.
Now we have in general that

$$X_1 \sim_f X_2 \ \Rightarrow\ f \circ \mathrm{Fl}^{X_1}_t = \mathrm{Fl}^{X_2}_t \circ f, \tag{3.1.13}$$

wherever one (hence both) side(s) are defined. Indeed we have

$$\frac{d}{dt}\Big|_0 (f \circ \mathrm{Fl}_t^{X_1}(p)) = T_{\mathrm{Fl}_t^{X_1}(p)} f\big(\frac{d}{dt}\big|_0 \mathrm{Fl}_t^{X_1}(p)\big) = T_{\mathrm{Fl}_t^{X_1}(p)} f \circ X \circ \mathrm{Fl}_t^{X_1}(p) = X_2(f(\mathrm{Fl}_t^{X_1}(p)),$$

and $f \circ \mathrm{Fl}_0^{X_1}(p) = f(p)$. So by the definition of the flow we obtain $f \circ \mathrm{Fl}_t^{X_1}(p) = \mathrm{Fl}_t^{X_2}(f(p))$.

Now we obtain for X, Y resp. its flows

$$\begin{aligned} \mathrm{Fl}_t^X \circ \mathrm{Fl}_s^Y = \mathrm{Fl}_s^Y \circ \mathrm{Fl}_t^X \;&\Leftrightarrow\; \mathrm{Fl}_s^Y = \mathrm{Fl}_{-t}^X \circ \mathrm{Fl}_s^Y \circ \mathrm{Fl}_t^X = \mathrm{Fl}_{-t}^X \circ \mathrm{Fl}_t^X \circ \mathrm{Fl}_s^{(\mathrm{Fl}_t^X)^*Y} = \mathrm{Fl}_s^{(\mathrm{Fl}_t^X)^*Y} \\ &\Leftrightarrow\; Y = (\mathrm{Fl}_t^X)^*Y \end{aligned}$$

□

3.1.5 Lemma (Coordinates adapted to given vector fields). *Let V be an open subset of a smooth manifold M. Given vector fields $X_1, \dots, X_n \in \mathfrak{X}(M)$ such that $[X_i, X_j] = 0$ for all i, j and $\{X_1(p), \dots, X_n(p)\}$ is a basis of T_pM for all $p \in V$ then around each p there is a chart $(\varphi = (x^1, \dots, x^n), U)$ with $U \subseteq V$ and*

$$X_i\big|_V = \frac{\partial}{\partial x^i} \qquad \textit{for all } i = 1, \dots, n. \tag{3.1.14}$$

Proof. Fix some $p \in V$ and set $F : t = (t^1, \dots, t^n) \mapsto (\mathrm{Fl}_{t^1}^{X_1} \circ \dots \circ \mathrm{Fl}_{t^n}^{X_n})(p)$. Then F is smooth on an open neighbourhood W of $0 \in \mathbb{R}^n$ and we may assume that $F(W) \subseteq V$. Since $[X_i, X_j] = 0$ by 3.1.4 the flows of the X_i commute and we have

$$\frac{\partial}{\partial t^i} F(t) = \frac{\partial}{\partial t^i} \mathrm{Fl}_{t^i}^{X_i} \circ \mathrm{Fl}_{t^1}^{X_1} \circ \dots \circ \mathrm{Fl}_{t^n}^{X_n}(p) = X_i(F(t)). \tag{3.1.15}$$

Since the $X_i(F(t))$ are a basis it follows that F is a local diffeomorphism hence w.l.o.g. $F : W \to U \subseteq V$ is a diffeomorphism.
Now define $\varphi = F^{-1} : U \to W$. Then we have for $q = \varphi^{-1}(t)$ using (3.1.15)

$$\frac{\partial}{\partial x^i}\Big|_q = \big(T_q\varphi\big)^{-1}(e_i) = T_tF(e_i) = \frac{\partial}{\partial t^i} F(t) = X_i(q). \tag{3.1.16}$$

□

Next we establish an explicit expression for the commutator of the induced covariant derivative in terms of the curvature tensor, which obviously is of independent interest.

3.1.6 Proposition (Exchanging 2nd order derivatives.). *Let $f : \mathcal{D} \to M$ be a two-parameter map into a SRMF M and let $Z \in vf(M)$. Then we have*

$$\frac{\nabla}{du}\frac{\nabla}{dv} - \frac{\nabla}{du}\frac{\nabla}{dv}\, Z \;=\; Z_{uv} - Z_{vu} \;=\; R(f_u, f_v)Z. \tag{3.1.17}$$

Proof. We work in a chart and write $Z = Z^k \partial_k$. Then by (1.3.56) we have $Z_u = Z^k_u \partial_k$ with $Z^k_u = \partial Z^k/\partial u + \Gamma^k_{lm} Z^l \partial f^m/\partial u$ and so

$$Z_{uv} = \Big(\frac{\partial^2 Z^k}{\partial u \partial v} + \frac{\partial \Gamma^k_{lm}}{\partial v} Z^l \frac{\partial f^m}{\partial u} + \Gamma^k_{lm} \frac{\partial Z^l}{\partial v} \frac{\partial f^m}{\partial u} + \Gamma^k_{lm} Z^l \frac{\partial^2 f^m}{\partial u \partial v} + \Gamma^k_{ij} \frac{\partial Z^i}{\partial u} \frac{\partial f^j}{\partial v} + \Gamma^k_{ij} \Gamma^i_{lm} Z^l \frac{\partial f^m}{\partial u} \frac{\partial f^j}{\partial v}\Big) \partial_k \tag{3.1.18}$$

and analogously for Z_{vu}. Upon inserting the symmetric terms cancel and we obtain

$$\begin{aligned} &Z_{uv} - Z_{vu} \\ &= \Big(Z^l \Big(\frac{\partial \Gamma^k_{lm}}{\partial v} \frac{\partial f^m}{\partial u} - \frac{\partial \Gamma^k_{lm}}{\partial u} \frac{\partial f^m}{\partial v}\Big) + \Gamma^k_{ij} \Gamma^i_{lm} Z^l \Big(\frac{\partial f^m}{\partial u} \frac{\partial f^j}{v} - \frac{\partial f^m}{\partial v} \frac{\partial f^j}{u}\Big)\Big) \partial k \end{aligned} \tag{3.1.19}$$

On the other hand we have

$$\begin{aligned} R(f_u, f_v) Z &= R\Big(\frac{\partial f^i}{\partial u} \partial_i, \frac{\partial f^j}{\partial} \partial_j\Big)\big(Z^l \partial_l\big) \\ &= Z^l \frac{\partial f^i}{\partial u} \frac{\partial f^j}{\partial v} R(\partial_i, \partial_j) \partial_k = Z^l \frac{\partial f^i}{\partial u} \frac{\partial f^j}{\partial v} R^k_{lij} \partial_k \\ &= \Big(Z^l \frac{\partial f^i}{\partial u} \frac{\partial f^j}{\partial v} \Big(\frac{\partial \Gamma^k_{il}}{\partial x^j} - \frac{\partial \Gamma^k_{jl}}{\partial x^i} + \Gamma^k_{jm} \Gamma^m_{il} - \Gamma^k_{im} \Gamma^m_{jl}\Big)\Big) \partial_k \qquad (3.1.20) \\ &= Z^l \Big(\frac{\partial f^i}{\partial u} \underbrace{\frac{\partial \Gamma^k_{il}}{\partial x^j} \frac{\partial f^j}{\partial v}}_{\frac{\partial \Gamma^k_{il}}{\partial v}} - \frac{\partial f^j}{\partial v} \underbrace{\frac{\partial \Gamma^k_{jl}}{\partial x^i} \frac{\partial f^i}{\partial u}}_{\frac{\partial \Gamma^k_{jl}}{\partial u}}\Big) \partial_k + Z^l \Big(\Gamma^k_{jm} \Gamma^m_{il} \tfrac{\partial f^i}{\partial u} \tfrac{\partial f^j}{\partial v} - \Gamma^k_{im} \Gamma^m_{jl} \tfrac{\partial f^i}{\partial u} \tfrac{\partial f^j}{\partial v} \Big) \partial_k \\ &= Z^l \Big(\frac{\partial \Gamma^k_{il}}{\partial v} \frac{\partial f^m}{\partial u} - \frac{\partial \Gamma^k_{ml}}{\partial u} \frac{\partial f^m}{\partial v}\Big) \partial_k + Z^l \Gamma^k_{jm} \Gamma^m_{il} \Big(\frac{\partial f^i}{\partial u} \frac{\partial f^j}{\partial v} - \frac{\partial f^i}{\partial v} \frac{\partial f^j}{\partial u}\Big) \partial_k, \end{aligned}$$

where for the last equality we have interchanged the summation index i by m in the first term and j by m in the second. Now the r.h.s. of (3.1.20) equals the r.h.s. of (3.1.19) upon exchanging the summation index i by m in the second term and we are done. □

With this we can now prove the following characterisation for the vanishing of the curvature tensor.

3.1.7 Theorem (Locally flat SRMF). *For any SRMF M the following are equivalent:*

(i) M is locally flat, *that is, for all points $p \in M$ there is a chart (U, φ) around p where the metric is flat, i.e., $(\varphi_* g)_{ij} = \varepsilon_j \delta_{ij}$ on $\varphi(U)$.*

(ii) The Riemann tensor vanishes.

Proof. (i)⇒(ii) follows simply from 3.1.3 and the fact that in the chart φ the Christoffel symbols all vanish.

(ii)⇒(i): The statement is local, so we may assume that $M = \mathbb{R}^n$ and $p = 0$. Let $e_1, \dots, e_n$ be an ONB at 0 and choose the e_i as coordinate axes for coordinates $x^1, \dots, x^n$ on $\mathbb{R}^n$. Then we have $\partial_i = e_i$ for $1 \le i \le n$.
Now for each i we first parallel transport e_i along the x^1-axis ($t \mapsto (t, 0, \dots, 0)$) and then from each point t_0 on the x^1-axis along the x^2-axis ($t \mapsto (t_0, t, 0, \dots, 0)$) and so on for the x^3, x^4,...-axes.
In this way we obtain vector fields $E_1, \dots, E_n \in \mathfrak{X}(\mathbb{R}^n)$ which are smooth since parallel transport is governed by an ODE whose solutions by ODE-theory depend smoothly on the initial data. Moreover since parallel transport preserves scalar products (1.3.28) the E_i form an ONB at every $q \in \mathbb{R}^n$.
Now for $1 \le k \le n$ set $M_k := \mathbb{R}^k \times \{0\} \subseteq \mathbb{R}^n$. By construction for all $1 \le j \le n$ we have that $E_j|_{M_k}$ is a vector field along the mapping $f_k : (t_1, \dots, t_k) \mapsto \sum_{i=1}^k t_i e_i = (t_1, \dots, t_k, 0, \dots, 0)$ and we may consider $\frac{\nabla}{\partial_{t_i}}(E_j|_{M_k})$ for all $1 \le i \le k$, see 41 but now for all k parameters. Let now $j \in \{1, \dots, n\}$

We claim $\frac{\nabla}{dt_i}(E_j|_{M_k}) = 0$ for all $1 \le i \le k$.
We proceed by induction. For $k = 1$ the equality follows by definition of parallel transport.
$k \mapsto k + 1$: Let $\frac{\nabla}{dt_i}(E_j|_{M_k}) = 0$ for all $1 \le i \le k$. By construction we also have $\frac{\nabla}{dt_{k+1}}(E_j|_{M_{k+1}}) = 0$. Now by 3.1.6 and our assumption that $R = 0$ we have

$$\frac{\nabla}{dt_{k+1}} \frac{\nabla}{dt_i} E_j(t_1, \dots, t_{k+1}, 0, \dots, 0) = \frac{\nabla}{dt_i} \frac{\nabla}{dt_{k+1}} E_j(t_1, \dots, t_{k+1}, 0, \dots, 0) = 0. \tag{3.1.21}$$

So $0 = \frac{\nabla}{dt_i} E_j(t_1, \dots, t_k, 0, \dots, 0)$ is parallely transported along the straight line $t_{k+1} \mapsto (t_1, \dots, t_k, t_{k+1}, 0, \dots, 0)$ hence vanishes on all of M_{k+1}.

Next we claim that $E_1, \dots, E_n \in \mathfrak{X}(\mathbb{R}^n)$ are parallel.
It suffices to show that $\nabla_{\partial_i} E_j = 0$ for all i, j. For fixed $x^1, \dots, x^n \in \mathbb{R}^n$ let $c_i : t \mapsto (x^1, \dots, x^{i-1}, t, x^{i+1}, \dots, x^n)$. Then we have by 1.3.27(iii)

$$\nabla_{\partial_i} E_j(x^1, \dots, x^n) = \nabla_{c_i'(x^i)} E_j = \frac{\nabla}{dt_i}(E_j \circ c)|_{t_i = x^i} = 0, \tag{3.1.22}$$

where for the last equality we have used the previous claim in case $k = n$.

Finally from the latter claim we have $[E_i, E_j] = \nabla_{E_i} E_j - \nabla_{E_i} E_j = 0$ for all i, j. So there are coordinates $((y^1, \dots, y^n), V)$ locally around p such that $E_j|_V = \partial_{y^i}$ for all $1 \le j \le n$. But in this coordinates we have

$$g_{ij} = g(\partial_{y^i}, \partial_{y^j}) = g(E_i, E_j) = \varepsilon_i \delta_{ij}. \tag{3.1.23}$$

□

Finally we come to item (2) of our list on page 64 which we make precise in the following remark. Here we closely follow [7, II.22 (p. 65)].