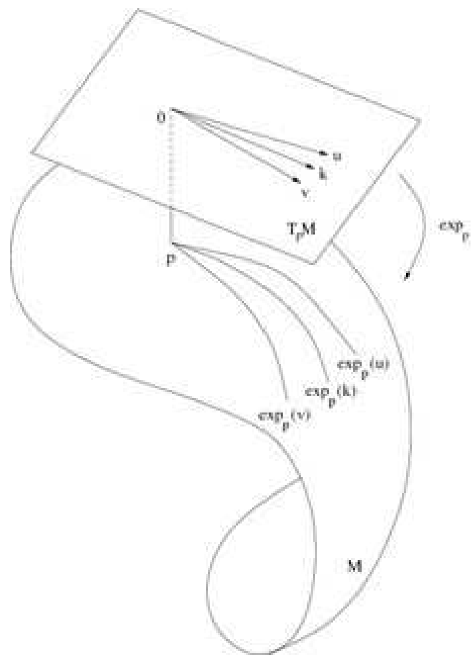

Figure 2.2: $\exp_p$ maps straight lines through $0 \in T_pM$ to geodesics through p.

2.1.14 Theorem (Exponential map). *Let p be a point in a SRMF M. Then there exist neighbourhoods $\tilde{U}$ of 0 in T_pM and U of p in M such that the exponential map $\exp_p : \tilde{U} \to U$ is a diffeomorphism.*

Proof. The plan of the proof is to first show that $\exp_p$ is smooth on a suitable neighbourhood and then to apply the inverse function theorem.
To begin with we recall that by 2.1.11 the mapping $(w, s) \mapsto c_w(s)$ is smooth on $\mathcal{N} \times I$, where $\mathcal{N}$ is an open neighbourhood in TM and I an interval around 0 which we assume w.l.o.g. to be $I = (-a, a)$. Now set $\mathcal{N}_p := \{\frac{v}{a'} : v \in \mathcal{N} \cap T_pM\}$, where a' is a fixed number with $a' > 1/a$. By (2.1.12) the map $(w, s) \mapsto c_w(s)$ is then defined on $\mathcal{N}_p \times J$ where $J \supseteq I$. Indeed for $w = v/a' \in \mathcal{N}_p$ we have $c_w(s) = c_{v/a'}(s) = c_v(s/a')$ with $s/a' \in I = (-a, a)$ and so $s \in a'I \supseteq [0, 1]$. So in total $\exp_p : \mathcal{N}_p \to M$ is smooth.
We next show that $T_0(\exp_p) : T_0(T_pM) \to T_pM$ is a linear isomorphism. To this end let $v \in T_0(T_pM)$ which we may identify with T_pM (cf. [9, 2.4.10]). Set $\rho(t) = tv$. Then $v = \rho'(0) = T_0\rho(\frac{d}{dt}|_0)$ and so

$$T_0(\exp_p)(v) = T_0(\exp_p)(\rho'(0)) = T_0(\exp_p \circ \rho)(\frac{d}{dt}|_0) = T_0(c_v)(\frac{d}{dt}|_0) = c_v'(0) = v. \quad (2.1.14)$$

Hence $T_0 \exp_p = \mathrm{id}_{T_pM}$ and the claim now follows from the inverse function theorem, see e.g. [9, 1.1.5]. $\square$

A subset S of a vector space is called *star shaped* (around 0) if $v \in S$ and $t \in [0,1]$ implies that $tv \in S$. If U and $\tilde{U}$ are as in 2.1.14 and $\tilde{U}$ is star shaped then we call U a *normal neighbourhood* of p. In this case U is star shaped in the following sense.

2.1.15 Proposition (Radial geodesics). *Let U be a normal neighbourhood of p. Then for each $q \in U$ there exists a unique geodesic $\sigma : [0,1] \to U$ from p to q called the* radial geodesic *from p to q. Moreover we have $\sigma'(0) = \exp_p^{-1}(q) \in \tilde{U}$.*

Proof. By assumption $\tilde{U} \subseteq T_pM$ is star shaped and $\exp_p : \tilde{U} \to U$ is a diffeomorphisms. Let $q \in U$ and set $v := \exp_p^{-1}(q) \in \tilde{U}$. Since $\tilde{U}$ is star shaped we have that $\rho(t) = tv \in \tilde{U}$ for $t \in [0,1]$. Hence $\sigma = \exp_p \circ \rho$ which by (2.1.13) is a geodesic and connects p with q is contained in U. Moreover we have by (2.1.14)

$$\sigma'(0) = T_0(\exp_p)(\rho'(0)) = T_0 \exp_p(v) = v = \exp_p^{-1}(q). \tag{2.1.15}$$

Let now $\tau : [0,1] \to U$ be an arbitrary geodesic in U connecting p with q. Set $w := \tau'(0)$. Then the geodesics τ and $t \mapsto \exp_p(tw)$ both have the same velocity vector at p hence coincide by 2.1.5.
We show that $w \in \tilde{U}$. Suppose to the contrary that $w \notin \tilde{U}$ and set $\tilde{t} := \sup\{t \in [0,1] : tw \in \tilde{U}\}$. Then $\tilde{t}w \in \partial\tilde{U}$. Now $\tau([0,1]) \subseteq U$ is compact and so is $(\exp_p|\tilde{U})^{-1}(\tau([0,1]))$ in $\tilde{U}$ and hence it has a positive distance to $\partial\tilde{U}$ and hence to $\tilde{t}w$. So by the definition of the supremum there is $t_0 < \tilde{t}$ arbitrarily close to $\tilde{t}$ such that $\tilde{U} \ni t_0 w \notin (\exp_p|\tilde{U})^{-1}(\tau([0,1]))$. But then $\exp_p(t_0 w) \notin \tau([0,1])$, which contradicts the fact that $\tau = t \mapsto \exp_p(tw)$ and so $w \in \tilde{U}$[1]
Finally we have $\exp_p(w) = \tau(1) = q = \exp_p(v)$ and by injectivity of the exponential map $v = w$. But this implies that $\sigma = \tau$. □

By a *broken geodesic* we mean a piecewise smooth curve whose smooth parts are geodesics. In $\mathbb{R}^n_r$ these are just the polygons. We now can prove the following criterion for connectedness.

2.1.16 Lemma (Connectedness via broken geodesics). *A SRMF is connected iff every pair of points may be joined by a broken geodesic.*

Proof. The condition is obviously sufficient. Let now M be connected and choose $p \in M$. Set $C := \{q \in M :\ q \text{ can be connected to } p \text{ by a broken geodesic}\}$. Let now $q \in M$ and U be a normal neighbourhood of q. If $q \in C$ then by 2.1.15 $U \subseteq C$ hence C is open. Also in $q \notin M$ then $U \subseteq M \setminus C$ and so C is also closed. Hence $C = M$. □

[1]Observe that this argument even shows that the entire segment $\{tw | t \in [0,1]\}$ lies in $\tilde{U}$. Just replace w by some $\tilde{w} = sw$ for $s \in [0,1]$.

Normal neighbourhoods are particularly useful in constructing special coordinate systems, called *Riemannian normal coordinates (RNC)* which are of great importance for explicit calculations. Let $p \in M$ and let U, $\tilde{U}$ be as above and let $\mathcal{B} = \{e_1, \dots, e_n\}$ be an orthonormal basis of T_pM. The Riemannian normal coordinate system $(\varphi = (x^1, \dots, x^n), U)$ around p defined by $\mathcal{B}$ assigns to any point $q \in U$ the coordinates of $\exp_p^{-1}(q) \in \tilde{U} \subseteq T_pM$ w.r.t. $\mathcal{B}$, i.e.,

$$\exp_p^{-1}(q) = x^i(q)e_i. \tag{2.1.16}$$

If $\mathcal{B}' = \{f^1, \dots, f^n\}$ is the dual basis of $\mathcal{B}$ then we have

$$x^i \circ \exp_p = f^i \quad \text{on } \tilde{U} \tag{2.1.17}$$

Indeed set $q = \exp_p(w)$ in (2.1.16) then $w = x^i(\exp_p(w))e_i$.
The most important properties of RNCs around p are that in this coordinates the metric at p is precisely the flat metric and also at p all Christoffel symbols vanish. It is essential to point out that these properties only hold at the point in which the coordinates are centered and in general fail already arbitrarily near to p. However, tensor fields are defined pointwise and so in many situations it is very beneficial to check certain tensorial identities in (the center point of) RNCs. More precisely we have.

2.1.17 Proposition (Normal coordinates). *Let $x^1, \dots, x^n$ be RNC around p. Then we have for all i, j, k*

(i) $g_{ij}(p) = \delta_{ij}\varepsilon_j$, *and*

(ii) $\Gamma^i_{\ jk} = 0$.

Proof. We first show that $\partial_i|_p = e_i$. Let $v \in \tilde{U} \subseteq T_pM$, $v = a^ie_i$. By (2.1.13), (2.1.17) we have

$$x^i(c_v(t)) = x^i(\exp_p(tv)) = f^i(tv) = ta^i \tag{2.1.18}$$

and so by (2.1.16), $\exp_p^{-1} \circ c_v = (a^1t, \dots, a^nt)$. Hence we have $T_p \exp_p^{-1}(c_v'(0)) = (a^1, \dots a^n)$ which gives $v = c_v'(0) = a^i\partial_i|_p$. Since $v = a^ie_i$ was arbitrary we obtain $\partial_i|_p = e_i$ which immediately gives (i).
Now since $x^i \circ c_v(t) = ta^i$ the geodesic equation for c_v reduces to $\Gamma^k_{ij}(c_v(t))a^ia^j = 0$ for all k. Inserting $t = 0$ we have $\Gamma^k_{ij}(p)a^ja^i = 0$ for all $a = (a^1, \dots a^n) \in \mathbb{R}^n$. So for fixed k the quadratic form $a \mapsto \Gamma^k_{ij}(p)a^ia^j$ vanishes and by polarisation we find $(a, b) \mapsto \Gamma^k_{ij}(p)a^ib^j = 0$ and so $\Gamma^k_{ij}(p) = 0$, hence (ii) holds. □

2.1.18 Examples (Exponential map of $\mathbb{R}^n_r$). Let $v \in T_p(\mathbb{R}^n_r)$, then the geodesic c_v starting at p is just $t \mapsto p + tv$. Hence we have $\exp_p : v \mapsto c_v(1) = p + v$. This is a *global* diffeomorphism and even an isometry.

Our next goal is to prove an essential result that goes by the name of Gauss lemma and states that the exponential map is a 'radial isometry'. This means that the orthogonality to radial directiosn is preserved. We first need some preparations.

2.1.19 Definition (Two-parameter mappings). *Let $\mathcal{D} \subseteq \mathbb{R}^2$ be open and such that vertical and horizontal straight lines intersect $\mathcal{D}$ in intervals. A* two-parameter mapping *on $\mathcal{D}$ is a smooth map $f : \mathcal{D} \to M$.*

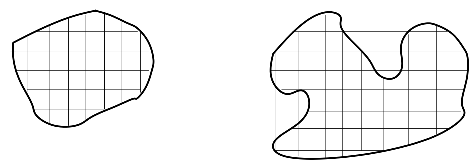

Figure 2.3: Sets $\mathcal{D}$ that (fail) to have the property of 2.1.19

For examples of sets that obey respectively lack the above property see Figure 2.1. Two-parameter maps are also often called *singular surfaces* since there is no condition on the rank of f.
Denoting the coordinates in $\mathbb{R}^2$ by (t, s) then a two-parameter map f defines two families of smooth curves: the t-parameter curves $s = s_0 : \ t \mapsto f(t, s_0)$ and the s-parameter curves $t = t_0 : \ s \mapsto f(t_0, s)$. By definition all such curves are defined on intervals. The corresponding partial derivatives

$$f_t(t, s) := T_{(t,s)} f(\partial_t) \quad \text{and} \quad f_s(t, s) := T_{(t,s)} f(\partial_s) \tag{2.1.19}$$

are then vector fields along f in the sense of definition 1.3.26. Observe that $f_t(t_0, s_0)$ is the velocity of the t-parameter curve $s = s_0$ at t_0 and analogous for $f_s(t_0, s_0)$.
If the image of f is contained in a chart $((x^1, \dots, x^n), V)$ of M then we denote the coordinate functions $x^1 \circ f$ of f by f^i $(1 \leq i \leq n)$. We then have $f^i = x^i \circ f : \mathcal{D} \to \mathbb{R}$ and by [9, 2.4.14]

$$f_t = \frac{\partial f^i}{\partial t}\, \partial_i \quad \text{and} \quad f_s = \frac{\partial f^i}{\partial s}\, \partial_i. \tag{2.1.20}$$

Now let M be a SRMF and $Z \in \mathfrak{X}(f)$. Then $t \mapsto Z(t, s_0)$ and $s \mapsto Z(t_0, s)$ are vector fields along the curves $t \mapsto f(t, s_0)$ and $s \mapsto f(t_0, s)$, respectively. We denote the corresponding induced covariant derivatives by

$$Z_t = \frac{\nabla Z}{\nabla t} = \nabla_{\partial_t} Z \quad \text{and} \quad Z_s = \frac{\nabla Z}{\nabla s} = \nabla_{\partial_s} Z, \tag{2.1.21}$$

respectively. By (1.3.53) we have

$$\nabla \partial_t Z(t, s) = Z_t(t, s) = \left(\frac{\partial Z^k}{\partial t}(t, s) + \Gamma^k_{ij}(f(t, s))\, Z^i(t, s)\, \frac{\partial f^j}{\partial t}(t, s) \right) \partial_k \tag{2.1.22}$$

and analogously for Z_s. In particular, for $Z = f_t$ we call $Z_t = f_{tt}$ the acceleration of the t-parameter curve and analogously for f_{ss}. We now note the following essential fact.

2.1.20 Lemma (Mixed secon derivatives of 2-parameter maps commute). *Let M be a SRMF and $f : \mathcal{D} \to M$ a 2-parameter map. Then we have $\nabla_{\partial_t}(\partial_s f) = \nabla_{\partial_s}(\partial_t f)$ or for short $f_{ts} = f_{st}$.*

Proof. By (2.1.22) we have

$$f_{ts} = \left(\frac{\partial^2 f^k}{\partial t\, \partial s} + \Gamma^k_{ij} \circ f \; \frac{\partial f^i}{\partial t} \frac{\partial f^j}{\partial s} \right) \partial_k, \tag{2.1.23}$$

which by the symmetry of the Christoffel symbols is symmetric in i and j. □

As a final preparation we consider $x \in T_pM$. Since T_p is a finite dimensional vector space we may identify $T_x(T_pM)$ with T_p itself, cf. [9, 2.4.10]. Hence if $v_x \in T_x(T_pM)$ we will view v_x also as an element of T_pM. We call v_x *radial* if is is a multiple of x.
Now we finally may state and prove the following result.

2.1.21 Theorem (Gauss lemma). *Let M be a SRMF and let $p \in M$, $0 \neq x \in \mathcal{D}_p \subseteq T_pM$. Then for any v_x, $w_x \in T_x(T_pM)$ with v_x radial we have*

$$\langle (T_x \exp_p)(v_x), (T_x \exp_p)(w_x) \rangle = \langle v_x, w_x \rangle. \tag{2.1.24}$$

Proof. Since v_x is radial and (2.1.24) is linear we may suppose w.l.o.g. that $v_x = x$ and to simplify notations we choose to denote this vector by v. Also we write w instead of w_x. Let now

$$f(t, s) := \exp_p(t(v + sw)). \tag{2.1.25}$$

The mapping $(t, s) \mapsto t(v + sw)$ is continuous and maps $[0, 1] \times \{0\}$ into the set $\mathcal{D}_p$. By [], below $\mathcal{D}_p$ is open and so there is $\varepsilon, \delta > 0$ such that f is defined on the square $\mathcal{D} := \{(t, s) :\ -\delta \leq t \leq 1 + \delta,\ -\varepsilon \leq s \leq \varepsilon\}$ (see figure 2.4) and so f is a 2-parameter mapping.

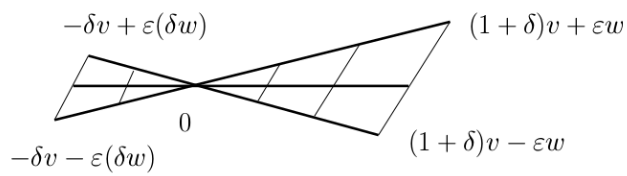

Figure 2.4: The shape of $g(\mathcal{D})$ (where $g(t, s) = t(v + sw)$).

We now have $f_t(1, 0) = T_v \exp_p(v)$ and $f_s(1, 0) = T_x \exp_p(w)$ and so we have to show that $\langle f_t(1, 0), f_s(1, 0) \rangle = \langle v, w \rangle$.
The curves $t \mapsto f(t, s)$ are geodesics with initial speed $v + sw$. Hence $f_{tt} = 0$ and so $\langle f_t, f_t \rangle = \text{const} = \langle v + sw, v + sw \rangle$ since $f_t(0, s) = T_0 \exp_p(v + sw) = v + sw$.

Moreover by 2.1.20 $f_{st} = f_{ts}$ and so we have

$$\begin{aligned}\frac{\partial}{\partial t}\langle f_s, f_t\rangle &= \langle f_{st}, f_t\rangle + \langle f_s, \underbrace{f_{tt}}_{=0}\rangle = \langle f_{ts}, f_t\rangle \\ &= \frac{1}{2}\frac{\partial}{\partial s}\langle f_t, f_t\rangle = \frac{1}{2}\frac{\partial}{\partial s}\langle v+sw, v+sw\rangle = \langle v, w\rangle + s\langle w, w\rangle \qquad (2.1.26)\end{aligned}$$

which implies

$$\left(\frac{\partial}{\partial t}\langle f_s, f_t\rangle\right)(t,0) = \langle v, w\rangle \quad \text{for all } t. \qquad (2.1.27)$$

Now $f(0,s) = \exp_p(0) = p$ for all s and so $f_s(0,0) = 0$ which gives $\langle f_s, f_t\rangle(0,0) = 0$. Now integrating (2.1.27) yields $\langle f_s, f_t\rangle(t,0) = t\langle v, w\rangle$. Finally setting $t = 1$ we obtain $\langle f_t(1,0), f_s(1,0)\rangle = \langle v, w\rangle$. □