Generalizing the Penrose Cut-and-Paste Method: Null Shells with Pressure and Energy Flux

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1. Intro & Motivation
2. Impulsive gravitational waves & the classical Cut-and-Paste method
3. The (null) matching of spacetimes & the hypersurface data formalism
4. Matching two “Minkowski-halves” across a null hyperplane
5. Explicit forms of the metric
6. Conclusions & outlook

Outline

The Theme

- **Cut-and-Paste method:** construction method for *impulsive gravitational waves* [Roger Penrose, 1967–72]
- **impulsive grav. waves:** theoretical models of short but violent bursts of gravitational radiation
- **Why they are interesting:**
  1. exact solutions: interesting radiative (= type N) solutions
  2. physics: ultrarelativistic particles, quantum scattering, memory effect, entaglement harvesting
  3. maths: key examples of low regularity spacetimes

Goal: Generalise the method to construct more general such models

Approach: two pillars

1. understanding impulsive waves: Jiří Podolský, Robert Švarc, Clemens Sämann, S., . . .
2. formalism of hypersurface data: Marc Mars, Gabriel Sánchez-Pérez, Miguel Manzano

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- model short but strong pulses of gravitational radiation
- arise as shockwave generated by *ultrarelativistic particles*
- ultrarelativistic boost of Schwarzschild solution [Aichelburg-Sexl, 72]
  generalisations to Kerr-Newman family [Lousto-Sánchez, 89–98],
  [Balasin, 93–18], [Hotta-Tanaka, 93] $(\Lambda \neq 0)$, [S, 98]
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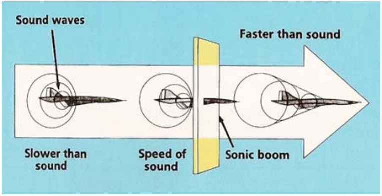

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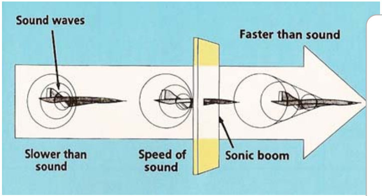

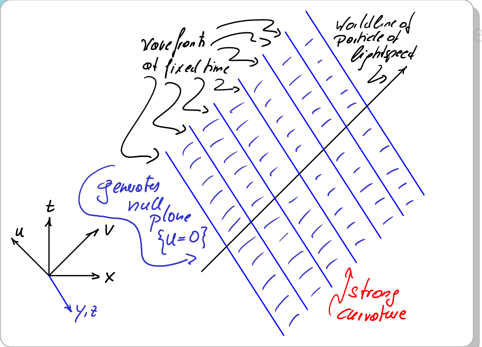

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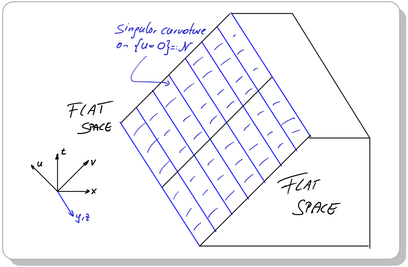

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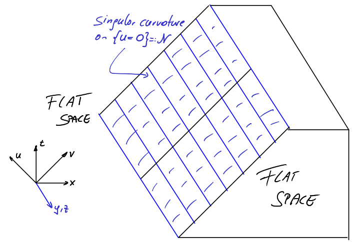

Aichelburg-Sexl metric

$$\boxed{\begin{aligned} ds^2 &= -2\,d\mathcal{U}\,d\mathcal{V} + dy^2 + dz^2 \quad (B) \\ &\quad {\color{red} +4\pi\,\log(\sqrt{y^2+z^2})\,\delta(\mathcal{U})\,d\mathcal{U}^2} \end{aligned}}$$

Brinkman-form of impulsive pp-wave

Impulsive gravitational waves: Exact solutions approach

- Brinkman form of pp-waves [Brinkmann, 1927]

$$\boxed{\mathrm{d}s^2 = -2\,\mathrm{d}\mathcal{U}\,\mathrm{d}\mathcal{V} + \mathrm{d}y^2 + \mathrm{d}z^2 + A(\mathcal{U},y,z)\,\mathrm{d}\mathcal{U}^2}$$

geometry: $\mathbf{k} = \partial_{\mathcal{V}}$ parallel null vector

curvature: $\mathbf{\Phi}_{2,2} = \triangle_{(y,z)} A = \rho$

$\mathbf{\Psi}_4 = (\partial_y^2 - \partial_z^2) A$ (type N, PND $\mathbf{k}$)

- no restriction the $\mathcal{U}$-dependence of A; so set $\boxed{A(\mathcal{U},y,z) = h(y,z)\delta(\mathcal{U})}$
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Outside GT-class $g \in H^1 \cap L^\infty$ [Geroch-Traschen, 87]

But work formally

or use nonlinear distributional geometry

[Grosser-Kunzinger-Oberguggenberger-S, 01]

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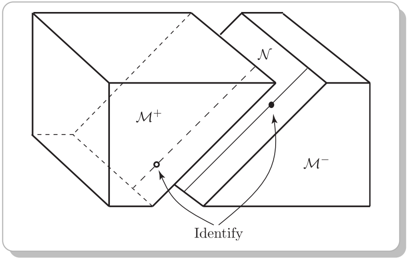
The Penrose cut-and-paste construction

- **cut** Minkowski space $(\mathbb{M}, \eta = -2d\mathcal{U}\, d\mathcal{V} + dy^2 + dz^2)$ along null plane $\mathcal{N} = \{\mathcal{U} = 0\}$
- **shift** resulting half-spaces $\mathcal{M}^-$, $\mathcal{M}^+$
- **paste** by identifying boundary points in $\mathcal{N}$ according to the *Penrose junction conditions*

$$\boxed{\mathcal{V} \in \mathcal{M}^- \mapsto \mathcal{V} - h(y,z) \in \mathcal{M}^+}$$



Leads (again) to an impulsive pp-wave in (B)-form

$$\begin{aligned} \mathrm{d}s^2 &= -2\,\mathrm{d}\mathcal{U}\,\mathrm{d}\mathcal{V} + \mathrm{d}y^2 + \mathrm{d}z^2 + h(y,z)\,\delta(\mathcal{U})\,\mathrm{d}\mathcal{U}^2 \\ &= -2\,\mathrm{d}\mathcal{U}\,\mathrm{d}\mathcal{V} + \mathrm{d}\zeta\,\mathrm{d}\bar{\zeta} + \hat{h}(\zeta,\bar{\zeta})\,\delta(\mathcal{U})\,\mathrm{d}\mathcal{U}^2 \qquad \text{(B)} \end{aligned}$$

$\zeta \overset{\text{def}}{=} y + \mathrm{i}z \in \mathbb{Z}$, complex coordinates simplify matters, $\ldots h \rightsquigarrow \widehat{h}$

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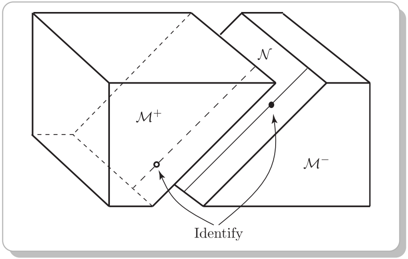

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The Rosen form of impulsive pp-waves

- [Penrose, 67–72], [Griffiths-Podolský, 1990-ies]
- start with Minkowski $\boxed{\eta = \mathrm{d}\zeta\,\mathrm{d}\bar{\zeta} - 2\,\mathrm{d}\mathcal{U}\,\mathrm{d}\mathcal{V} \quad \text{(M)}}$
- consider (formal) coordinate transform

$$\begin{aligned} \mathcal{U} &= u \\ \mathcal{V} &= v + \Theta(u)\,\widehat{h} + u_+\,\widehat{h}_{,Z}\,\widehat{h}_{,\bar{Z}} \quad (T) \\ \zeta &= Z + u_+\,\widehat{h}_{,\bar{Z}} \end{aligned}$$

- applying (T) to (M) separately for $u < 0$ and $u > 0$ and then writing a combined metric gives

$$\boxed{\mathrm{d}s^2 = 2\left|\mathrm{d}Z + u_+(\widehat{h}_{,\bar{Z}Z}\,\mathrm{d}Z + \widehat{h}_{,\bar{Z}\bar{Z}}\,\mathrm{d}\bar{Z})\right|^2 - 2\,\mathrm{d}u\,\mathrm{d}v \quad (R)}$$

the (Lipschitz) continuous *Rosen form* of impulsive pp-waves

The “discontinuous coordinate transformation”

- (B) and (R) forms of the metric are physically equivalent
- Both have distributional curvature concentrated on $\mathcal{N}$

$$\mathbf{\Psi}_4 = \widehat{h}_{,ZZ}\,\delta(u) \text{ (type } N\text{, PND } \partial_v), \quad \mathbf{\Phi}_{2,2} = \widehat{h}_{,Z\bar{Z}}\,\delta(u) = \tau^{vv} = \rho$$

- related by “discontinuous transformation” [Penrose, 72]
- Made rigorous in nonlinear generalized functions [Kunzinger-S, 99] including Λ [Podolský-Sämann-Švarc-Schinnerl-S, 2016–24].
  (A) Geometric insight: transf. given by special family of null geodesics
  (B) Analytic insight: nonlinear distributional analysis of geodesics of (B)

Goal: Construct impulsive waves w. more *general energy-momentum tensor*
Find both the *distributional* and *Lipschitz* continuous metrics

Method: *General matching* of “Minkowski-halves” along null hyperplanes using Marc Mars' *hypersurface data formalism*

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Now keeping the distributional terms (T) (formally) takes

$$\mathrm{d}s^2 = -2\,\mathrm{d}\mathcal{U}\,\mathrm{d}\mathcal{V} + 2\mathrm{d}\zeta\,\mathrm{d}\bar{\zeta} + 2\,\widehat{h}(\zeta,\bar{\zeta})\,\delta(\mathcal{U})\,\mathrm{d}\mathcal{U}^2 \qquad (B)$$

to

$$\mathrm{d}s^2 = 2\left|\mathrm{d}Z + u_+(\widehat{h}_{,\bar{Z}Z}\,\mathrm{d}Z + \widehat{h}_{,\bar{Z}\bar{Z}}\,\mathrm{d}\bar{Z})\right|^2 - 2\,\mathrm{d}u\,\mathrm{d}v \qquad (R)$$

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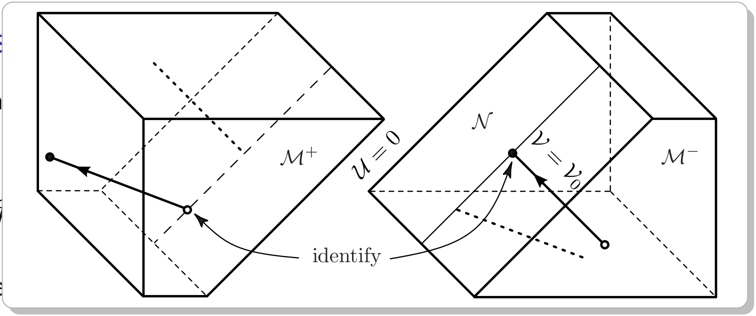

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Abstract matching of Semi-Riemannian Manifolds

Setting: (M_1, g_1), (M_2, g_2) C^∞-SR-manifolds of same dim. & signature with diffeomorphic boundaries $\phi : \partial M_1 \to \partial M_2$

Manifold matching: The adjunction space $M := M_1 \cup_\phi M_2$ $(\partial M_1 \ni x \approx \phi(x) \in \partial M_2)$ is a smooth manifold with $M_i \hookrightarrow M_i'$ proper,

$$M_1' \cup M_2' = M_1 \cup_\phi M_2, \quad M_1' \cap M_2' = \partial M_1' = \partial M_2' =: \mathcal{N}.$$

Adding the metric needs ϕ isometry **and** some care:
ϕ needs to be ξ-aligning for a(ny) transversal vector ξ on $\mathcal{N}$:

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If ϕ is ξ-aligning then there is a unique locally Lipschitz continous metric g on M agreeing with g_i on M_i (hence it is smooth off $\mathcal{N}$).

Straightens a result by [Clarke-Dray, 87].

Abstract matching of Semi-Riemannian Manifolds

Setting: (M_1, g_1), (M_2, g_2) C^∞-SR-manifolds of same dim. & signature with diffeomorphic boundaries $\phi : \partial M_1 \to \partial M_2$

Manifold matching: The adjunction space $M := M_1 \cup_\phi M_2$ ($\partial M_1 \ni x \approx \phi(x) \in \partial M_2$) is a smooth manifold with $M_i \hookrightarrow M_i'$ proper,

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Matching & the Hypersurface Data Formalism

Matching spacetimes has long history: [Darmois, 1927], [Israel, 66], [Barrabés-Israel, 91], [Mars-Senovilla, 93], ...

HSD-Formalism: fresh perspective [Mars, 13–]

Works with a "detached boundary manifold" $\mathcal{N}$ to

- clearly separates intrinsic from extrinsic geometry of $\mathcal{N}$, and
- still enable very explicit calculations (of energy momentum tensor)

- $\mathcal{N}$ smooth n-dim. manifold
- γ symmetric $(0,2)$-tensor w. signature $(0,+,\ldots,+)$ on $\mathcal{N}$
- ℓ one-form & $\ell^{(2)}$ scalar function on $\mathcal{N}$
- such that the matrix $\mathcal{A} = \begin{pmatrix} \gamma_{ab} & \ell_a \\ \ell_b & \ell^{(2)} \end{pmatrix}$ is non-degenerate on $\mathcal{N}$
- $\mathbf{Y}$ symmetric $(0,2)$-tensor on $\mathcal{N}$

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Matching & the junction conditions

Embedded null hypersurface data

$\{\mathcal{N},\gamma,\ell,\ell^{(2)},Y\}$ is $\{\phi,\zeta\}$*-embedded* in $(n+1)$-dim. (M,g) if there is

- a smooth embedding $\phi:\mathcal{N}\hookrightarrow\mathcal{M}$,
- a *rigging* vector field ζ along $\phi(\mathcal{N})$, everywhere transversal:

$$\phi^*(g)=\gamma,\quad \phi^*(g(\zeta,\cdot))=\boldsymbol{\ell},\quad \phi^*(g(\zeta,\zeta))=\ell^{(2)},\quad \frac{1}{2}\phi^*(\pounds_\zeta g)=\mathbf{Y}.$$

Matching two spacetimes $(M^\pm,g^\pm)$ with boundaries $\widetilde{\mathcal{N}}^\pm$ requires

NMHD $\{\mathcal{N},\gamma,\boldsymbol{\ell},\ell^{(2)}\}$, 2 embeddings $\phi^\pm:\mathcal{N}\hookrightarrow\mathcal{M}^\pm$, 2 riggings $\zeta^\pm$:
$\{\mathcal{N},\gamma,\boldsymbol{\ell},\ell^{(2)}\}$ can be $\{\phi^\pm,\zeta^\pm\}$-embedded in both $(\mathcal{M}^\pm,g^\pm)$, and
(i) $\phi^\pm(\mathcal{N})=\widetilde{\mathcal{N}}^\pm$ (ii) $\zeta^\pm$ point inwards/outwards.

Matching & the junction conditions

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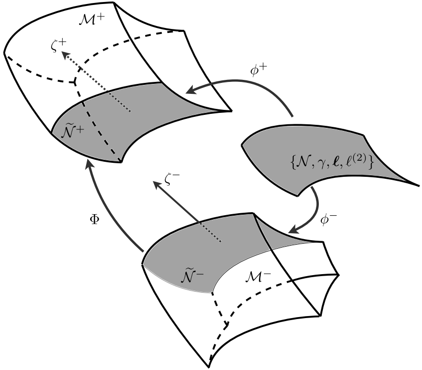

Hence

$\gamma = (\phi^\pm)^\star(g^\pm)$

$\boldsymbol{\ell} = (\phi^\pm)^\star(g^\pm(\zeta^\pm, \cdot))$ (JC)

$\ell^{(2)} = (\phi^\pm)^\star(g^\pm(\zeta^\pm, \zeta^\pm))$

matter on shell encoded in jump of extrinsic curvature

$$\boxed{[\mathbf{Y}] \stackrel{\text{def}}{=} \mathbf{Y}^+ - \mathbf{Y}^-}$$

where $\mathbf{Y}^\pm \stackrel{\text{def}}{=} \frac{1}{2}(\phi^\pm)^\star(\pounds_{\zeta^\pm} g^\pm)$.

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Explicit calculations (1) — Setup of [Manzano-Mars, 21]

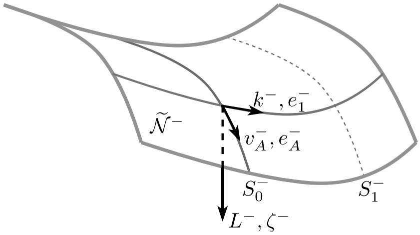
Bases $\{L^{\pm}, k^{\pm}, v_I^{\pm}\}$ of $\Gamma\left(T\mathcal{M}^{\pm}\right)|_{\widetilde{\mathcal{N}}^{\pm}}$

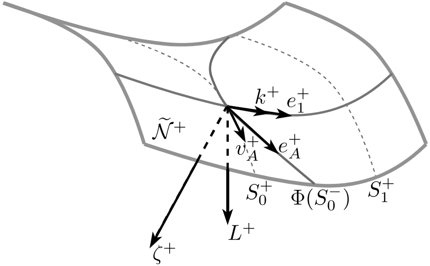
Assume $\widetilde{\mathcal{N}}^{\pm} = S^{\pm} \times \mathbb{R}$, with $S^{\pm}$ spacelike cross-sections

(A) $k^{\pm}$ are future affine null generators of $\widetilde{\mathcal{N}}^{\pm}$

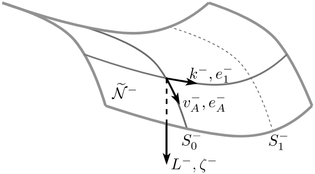
(B) $\{v_I^{\pm}\} \in \Gamma(T\widetilde{\mathcal{N}}^{\pm}): \ v_I^{\pm}|_{S^{\pm}} \in \Gamma(TS^{\pm}), \ [k^{\pm}, v_I^{\pm}] = 0, \ [v_I^{\pm}, v_J^{\pm}] = 0$

(C) $L^{\pm}$ past riggings: $g^{\pm}(L^{\pm}, k^{\pm}) = 1, \quad g^{\pm}(L^{\pm}, v_I^{\pm}) = 0$



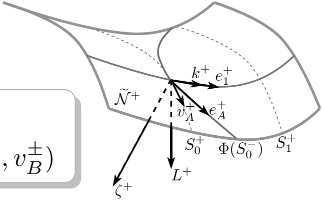


Explicit calculations (2)



metric on $S^\pm$

$$h^\pm_{AB} \overset{\textbf{def}}{=} g^\pm(v^\pm_A, v^\pm_B)$$



Coordinates $\{z^a\} = \{z^1 = v, z^A\}$ on $\mathcal{N}$ to construct matched bases

- $\Gamma(T\widetilde{\mathcal{N}}^-)$: $\{e_a^- \overset{\textbf{def}}{=} \phi_\star^-(\partial_{z^a})\}$; enforce $\boxed{e_1^- = k^-,\ e_I^- = v_I^-,\ \zeta^- = L^-}$
- $\Gamma(T\widetilde{\mathcal{N}}^+)$: $\{e_a^+ \overset{\textbf{def}}{=} \phi_\star^+(\partial_{z^a})\}$; (JC) $\Rightarrow$ $\exists$ fcts. $H(v, z^B)$, $h^A(z^B)$ on $\mathcal{N}$:

$$\boxed{e_1^+ = \frac{\partial H(v,z^A)}{\partial v} k^+, \quad e_I^+ = \frac{\partial H(v,z^A)}{\partial z^I} k^+ + \frac{\partial h^J(z^A)}{\partial z^I} v_J^+}$$ where

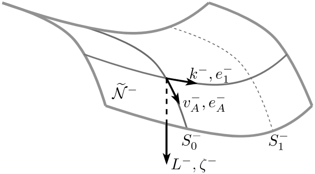
(a) $\partial_v H > 0$

(b) $\dfrac{\partial(h^2, \ldots, h^{n+1})}{\partial(z^2, \ldots, z^{n+1})}$ non-singular

(c) $h^-_{AB}|_{\phi^-(p)} = \dfrac{\partial h^C}{\partial z^A} \dfrac{\partial h^D}{\partial z^B} h^+_{CD}|_{\phi^+(p)}$

(core solvability issue)

Explicit calculations (2)



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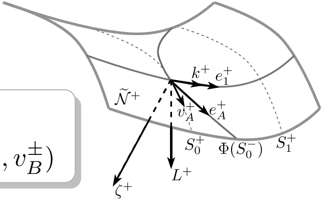

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(core solvability issue)

Step function & energy momentum tensor of the shell

$\{H, h^A\}$ fully encode the matching

Given coordinates $\{v_+, u_+^I\}$ on $\widetilde{\mathcal{N}}^+$, $\{u_I^+\}$ constant along generators

$$\phi^+(v, z^A) = (v_+ = H(v, z^B), u_+^A = h^A(z^B)).$$

- *step function* H measures the jump across the null direction
- $\{h^A\}$ rule identification of null generators of $\widetilde{\mathcal{N}}^\pm$

Prop. (Energy momentum tensor of the shell) [Manzano-Mars, 21]

For a null matching across totally geodesic boundaries we have

$$\tau^{vv} = -\epsilon h^{IJ}[\mathsf{Y}](\partial_{z^I}, \partial_{z^J}), \quad \tau^{vz^I} = \epsilon h^{IJ}[\mathsf{Y}](\partial_v, \partial_{z^J}), \quad \tau^{z^I z^J} = -\epsilon h^{IJ}[\mathsf{Y}](\partial_v, \partial_v),$$

where $[\mathsf{Y}_{vv}] = -\dfrac{\partial_v \partial_v H}{\partial_v H}$, $\quad [\mathsf{Y}_{vz^J}] = \boldsymbol{\sigma}_L^+(W_J) - \boldsymbol{\sigma}_L^-(v_J^-) - \dfrac{\partial_v \partial_{z^J} H}{\partial_v H}$

$$[\mathsf{Y}_{z^I z^J}] = \tfrac{1}{\partial_v H}\Big(2(\nabla^h_{(I} H)\boldsymbol{\sigma}_L^+(W_{J)}) + \boldsymbol{\Theta}_+^L(W_{(I}, W_{J)}) - (\partial_v H)\boldsymbol{\Theta}_-^L(v^-_{(I}, v^-_{J)}) - \nabla^h_I \nabla^h_J H\Big)$$

Step function & energy momentum tensor of the shell

$\{H, h^A\}$ fully encode the matching

Recall: $\boxed{[\mathbf{Y}] \overset{\text{def}}{=} \mathbf{Y}^+ - \mathbf{Y}^-}$ where $\mathbf{Y}^\pm \overset{\text{def}}{=} \frac{1}{2}(\phi^\pm)^\star(\pounds_{\zeta^\pm} g^\pm)$

And moreover we have defined

$$\mathbf{\Theta}^L_\pm(X_\pm, Y_\pm) \overset{\text{def}}{=} g^\pm\big(\nabla_{X_\pm} L, Y_\pm\big), \quad \boldsymbol{\sigma}^\pm_L(X_\pm) \overset{\text{def}}{=} -g^\pm\big(\nabla_{X_\pm} k, L\big)$$

with the vector fields $\{W_I \overset{\text{def}}{=} (\partial_{z^I} h^B) v^+_B\}$ on $\widetilde{\mathcal{N}}^+$ assumed to be totally geodesic

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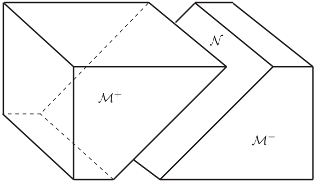
Outline

Null matching of Minkowski

Setup: $(\mathbb{M}^{\pm}, \eta^{\pm})$, with

$$\eta^{\pm} = -2d\mathcal{U}_{\pm}d\mathcal{V}_{\pm} + \delta_{AB}dx_{\pm}^{A}dx_{\pm}^{B}$$

$$\mathcal{U}_{+} \geq 0,\ \mathcal{U}_{-} \leq 0,\ \widetilde{\mathcal{N}}^{\pm} \overset{\text{def}}{=} \{\mathcal{U}_{\pm} = 0\}$$



Then the boundaries $\widetilde{\mathcal{N}}_{\pm}$ are

- null, foliated by spacelike sections $S^{\pm} \overset{\text{def}}{=} \{\mathcal{U}_{\pm} = 0, \mathcal{V}_{\pm} = 0\}$
- totally geodesic $\Rightarrow$ Proposition applies $\quad \boldsymbol{\sigma}_{L}^{\pm} = 0 = \boldsymbol{\Theta}_{\pm}^{L}$

Bases $\{L^{\pm}, k^{\pm}, v_{I}^{\pm}\}$ of $\Gamma(T\mathbb{M}^{\pm})|_{\widetilde{\mathcal{N}}^{\pm}}$: $\boxed{L^{\pm} = -\partial_{\mathcal{U}_{\pm}},\ k^{\pm} = \partial_{\mathcal{V}_{\pm}},\ v_{I}^{\pm} = \partial_{x_{\pm}^{I}}}$

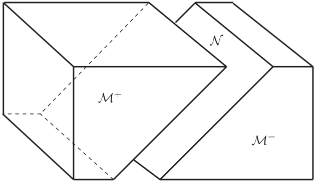
Construct null MHD embedded in $\mathcal{M}^{\pm}$: Consider $\phi^{\pm} : \mathcal{N} \hookrightarrow \widetilde{\mathcal{N}}^{\pm} \subset \mathcal{M}^{\pm}$

$$\phi^{-}(v, z^{I}) = \left(\mathcal{U}_{-} = 0, \mathcal{V}_{-} = v, x_{-}^{I} = z^{I}\right) \rightsquigarrow h_{AB} = \delta_{AB} \rightsquigarrow h^{I} = \mathrm{id}$$

$$\phi^{+}(v, z^{I}) = \left(\mathcal{U}_{+} = 0, \mathcal{V}_{+} = H(v, z^{I}), x_{+}^{I} = z^{J}\right)$$

Null MHD: $\left\{\mathcal{N},\ \gamma = \delta_{AB}dz^{A} \otimes dz^{B},\ \boldsymbol{\ell} = dv,\ \ell^{(2)} = 0\right\}$

Null matching of Minkowski



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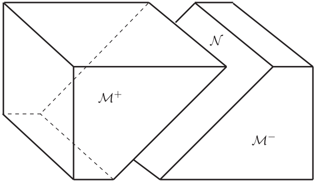

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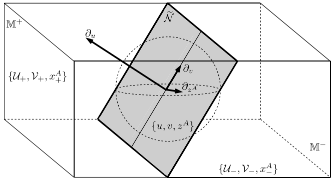
The energy-momentum tensor

For the general null matching of Minkowski we have

The jump is $[\mathrm{Y}_{ab}] = -\dfrac{\partial_{z^a}\partial_{z^b}H}{\partial_v H}$ and so we have

- energy density: $\tau^{vv} = \rho = -\delta^{AB}\dfrac{\partial_{z^A}\partial_{z^B}H}{\partial_v H}$
- energy flux: $\tau^{vz^A} = j^A = \delta^{AB}\dfrac{\partial_v\partial_{z^B}H}{\partial_v H}$
- pressure: $\tau^{z^A z^B} = \delta^{AB}p = -\dfrac{\partial_v\partial_v H}{\partial_v H}$



Special cases

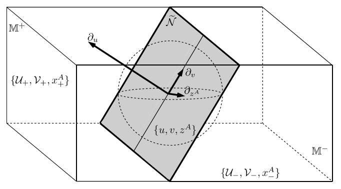
- No-shell: $[\mathbf{Y}] = 0$, $H = av + b_J z^J + c$, $(a > 0)$ and $H = v$ after isometry
- Gravitational / null-dust: $[\mathbf{Y}] \neq 0$ and $\tau = 0$ / $\rho \neq 0$, $j^A = p = 0$,
  in both cases, $H = av + \mathcal{H}(z^A)$ $(a > 0) \rightsquigarrow$ *Penrose cut & paste*
- Generic shell: $\boxed{H(v, z^A) = \beta(z^A)\int \exp\left(-\int p(v, z^A)dv\right)dv + \mathcal{H}(z^A)\ \text{(GS)}}$
  $\beta(z^A), \mathcal{H}(z^A)$ Lie-constant along generators, $(\partial_v H > 0)$

The energy-momentum tensor

For the general null matching of Minkowski we have

The jump is $[\mathsf{Y}_{ab}] = -\dfrac{\partial_{z^a}\partial_{z^b}H}{\partial_v H}$ and so we have

- energy density: $\tau^{vv} = \rho = -\delta^{AB}\dfrac{\partial_{z^A}\partial_{z^B}H}{\partial_v H}$
- energy flux: $\tau^{vz^A} = j^A = \delta^{AB}\dfrac{\partial_v\partial_{z^B}H}{\partial_v H}$
- pressure: $\tau^{z^A z^B} = \delta^{AB}p = -\dfrac{\partial_v\partial_v H}{\partial_v H}$



Special cases

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Outline

The Lipschitz metric (1): Issues & strategies

So far: $\mathcal{M} = (\mathbb{M}^+ \cup \mathbb{M}^-)/\widetilde{\mathcal{N}} = \mathbb{M}^+ \cup_\Phi \mathbb{M}^-$ with stepfunction (GS)

Questions & Issues:

(Q1) What is the regularity of g?

(Q2) Find good explicit form of g?

Strategy: new coordinates $\{u, \hat{v}, \hat{z}^A\}$ in neighb. $\mathcal{O} \subseteq \mathcal{M}$ of $\widetilde{\mathcal{N}}$

- enforce trivial identification $\left\{\mathcal{U}_- = u,\ \mathcal{V}_- = \hat{v},\ x_-^A = \hat{z}^A\right\}\Big|_{\mathbb{M}^-}$
- relate vector fields $\{\zeta^-, e_a^-\}$ to basis $\{\partial_u, \partial_{\hat{v}}, \partial_{\hat{z}^A}\}$ of $\Gamma\big(T\mathcal{M}\big)|_{\widetilde{\mathcal{N}}}$

$$e_1^- = \partial_{\hat{v}}, \quad e_I^- = \partial_{\hat{z}^I}, \quad \zeta^- = -\partial_u \ \text{ on } \mathcal{N}$$

- since $\{e_a^\pm, \zeta^\pm\}$ are identified in the matching

$$e_1^+ = (\partial_v H)\partial_{\mathcal{V}_+} = \partial, \qquad e_I^+ = (\partial_{z^I} H)\partial_{\mathcal{V}_+} + \partial_{x_+^I} = \partial_I,$$

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Only tangential derivatives of H appear and by our choice of ϕ^-

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The Lipschitz metric (2): Technicalities

Next 'integrate'

$$e_1^+ = (\partial_v H)\partial_{\mathcal{V}_+} = \partial_v, \qquad e_I^+ = (\partial_{z^I} H)\partial_{\mathcal{V}_+} + \partial_{x_+^I} = \partial_{z^I},$$
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to express $\{\mathcal{U}_+, \mathcal{V}_+, x_+^A\}$ in terms of $\{u, v, z^A\}$. Using

- $d\mathcal{U}_+$, $d\mathcal{V}_+$ and dx_+^A are covariantly constant on $\mathbb{M}^+ \subset \mathcal{M}$
- $\partial_u\partial_u(d\mathcal{U}_+) = 0$, $\partial_u\partial_u(d\mathcal{V}_+) = 0$, $\partial_u\partial_u(dx_+^A) = 0$
- previous choices & calculating one obtains

$$\mathcal{U}_+ = u\,\frac{1}{\partial_v H}, \quad x_+^A = z^A + u\delta^{AB}\partial_{z^B} H\,\frac{1}{\partial_v H}$$
$$\mathcal{V}_+ = H + u\,\frac{1}{2\partial_v H}\delta^{AB}\partial_{z^A} H \partial_{z^B} H$$

From here calculate $\eta^\pm$ ...

The Lipschitz metric (3): The result

Theorem [Manzano-Ohanyan-S, 25]

Let $(\mathcal{M}, g)$ be the general null matching of Minkowski with step function

$$H(v, z^A) = \beta(z^A) \int \exp\left(-\int p(v, z^A) dv\right) dv + \mathcal{H}(z^A).$$

(i) There are Gaussian null coordinates $\{u, v, z^A\}$ on both sides of $\widetilde{\mathcal{N}}$:
$\partial_v|_{\widetilde{\mathcal{N}}}$ null generator, $g(\partial_u, \partial_v)|_{\widetilde{\mathcal{N}}} = -1$,
$\partial_u|_{\widetilde{\mathcal{N}}}$ future-directed, null rigging of $\widetilde{\mathcal{N}}$ orthogonal to the spacelike planes

(ii) In these coordinates the metric takes the Lipschitz continuous form

$$\begin{aligned} g = &- 2dudv + \delta_{AB} dz^A dz^B + u_+ dv^2 \left(u\delta^{AB}[\mathsf{Y}_{vz^A}][\mathsf{Y}_{vz^B}] - 2p\right) \\ &+ 2u_+[\mathsf{Y}_{vz^I}]) dvdz^A \left(u\delta^{BI}[\mathsf{Y}_{z^A z^B}] - 2\delta^I_A\right) \qquad (L) \\ &- 2u_+ dz^A dz^B \left([\mathsf{Y}_{z^A z^B}] - \frac{u}{2}\delta^{IJ}[\mathsf{Y}_{z^I z^A}][\mathsf{Y}_{z^J z^B}]\right) \end{aligned}$$

(iii) For a purely gravitational or a null-dust shell ($H = av + \mathcal{H}(z^A)$) we recover the Rosen form of impulsive pp-waves (R).

The distributional metric (1): Issues & strategies

So far: $\mathcal{M} = (\mathbb{M}^+ \cup \mathbb{M}^-)/\widetilde{\mathcal{N}}$ w. stepfunct. (GS); Lip.-metric (L)

Issue: Find distributional form of g

in terms of δ, in coordinates $\{\mathcal{U}, \mathcal{V}, \mathcal{X}, \mathcal{Y}\} = \{\mathcal{U}_\pm, \mathcal{V}_\pm, x^2_\pm, x^3_\pm\}$ on $\mathbb{M}^\pm \setminus \widetilde{\mathcal{N}}$.

(+) manifestly flat off shell & information encoded in impulsive term

(−) at face value, metric is only formal

Strategy: Ansatz for

- metric $\boxed{g = -2d\mathcal{U}d\mathcal{V} + \delta_{AB}d\mathcal{X}^A\mathcal{X}^B + 2\mathscr{H}(v, z^C)\delta(u)d\mathcal{U}^2 \quad \text{(D)}}$, with
- "discontinuous transformation" $\qquad \mathscr{H}(v, z^C) \stackrel{\text{def}}{=} (\partial_v H)\frac{1+(\partial_v H)}{1+(\partial_v H)^2}(H - v)$

$$\mathcal{U} \stackrel{\text{def}}{=} u + u_+ (1/\partial_v H - 1), \qquad \mathcal{X}^A \stackrel{\text{def}}{=} z^A + u_+ x^A_1$$

$$\mathcal{V} \stackrel{\text{def}}{=} v + \theta(u)(H - v) + u_+ \frac{1}{2\partial_v H}\delta^{AB}\partial_{z^A} H \partial_{z^B} H,$$

and transform (D) to obtain (L).

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The distributional metric (2): Technicalities

Careful calculations:

- $\mathcal{U}, \mathcal{X}^A$ Lip. $\rightsquigarrow d\mathcal{U}, d\mathcal{X}^A \in L^\infty_{\text{loc}}$
  $\rightsquigarrow d\mathcal{U}$-part L^∞_{loc} : $d\mathcal{U}^2 = \big(1-\theta(u)\big)d\mathcal{U}_-^2 + \theta(u)d\mathcal{U}_+^2$
  $\rightsquigarrow$ spatial part L^∞_{loc} : $\delta_{AB}\Big(\big(1-\theta(u)\big)dx_-^A dx_-^B + \theta(u)dx_+^A dx_+^B\Big)$
- $\mathcal{V}$ only $L^\infty_{\text{loc}} \subseteq L^1_{\text{loc}} \subseteq \mathcal{D}'$; product rule in $C^\infty \cdot \mathcal{D}'$
  $$d\mathcal{V} = \big(1-\theta(u)\big)d\mathcal{V}_- + \theta(u)d\mathcal{V}_+ + \delta(u)(H-v)du$$
- problem: $-2d\mathcal{U}d\mathcal{V} \sim \theta \cdot \delta \in L^\infty_{\text{loc}} \cdot \mathcal{D}'$ $\rightsquigarrow$ regularisation product
- with this: $-2d\mathcal{U}d\mathcal{V} =$
  $-2\big(1-\theta(u)\big)d\mathcal{U}_-d\mathcal{V}_- - 2\theta(u)d\mathcal{U}_+d\mathcal{V}_+ - \delta(u)(H-v)\big(1+1/\partial_v H\big)du^2$
- similarly: $2\mathscr{H}(v,z^C)\delta(u)d\mathcal{U}^2 = \delta(u)(H-v)\left(1+1/\partial_v H\right)du^2$
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Model product (intrinsic distributional product [Oberguggenberger, 92])

- mollifier: $\rho \in C^\infty$, $\mathrm{supp}(\rho) \subseteq B_1(0)$, $\int \rho = 1$
- model δ-net: $\rho_\varepsilon(x) \stackrel{\textbf{def}}{=} \frac{1}{\varepsilon^n}\rho\left(\frac{x}{\varepsilon}\right)$ $(\varepsilon \in (0,1])$
- regularisation:

$$\mathcal{D}' \ni u \mapsto \boxed{u_\varepsilon \stackrel{\textbf{def}}{=} u * \rho_\varepsilon(x) \stackrel{\textbf{def}}{=} \langle u(x-y), \rho_\varepsilon(y)\rangle} \in C^\infty \to u \in \mathcal{D}'$$

- note: $\rho_\varepsilon \to \delta$; very general regularisation of δ.
- model product: provided limit exists and coincides for all ρ_ε

$$[u\,v] \stackrel{\textbf{def}}{=} \lim_{\varepsilon\to 0}(u * \rho_\varepsilon)\,(v * \rho_\varepsilon)$$

- calculate: $[\theta\,\delta] = \frac{1}{2}\,\delta$.

But why choose twice the same ρ_ε? Physical modelling!

- view thin shell/imp. wave as limiting case of thick shell/sandwich wave

$\rightsquigarrow$ δ and θ come from the same "source", i.e. θ_ε should be prime fct. of ρ_ε

- But this is compatible with the above:

$$\theta_\varepsilon(x) \stackrel{\textbf{def}}{=} \theta * \rho_\varepsilon(x) = \int_{-1}^{x/\varepsilon} \rho\,(y)\,dy.$$

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Outline

Final results and outlook

We have found for the most general null matching of Minkowski

- step function $H(v, z^A) = \beta(z^A) \int \exp\left(-\int p(v, z^A) dv\right) dv + \mathcal{H}(z^A)$ (GS)
- Lipschitz metric

$$\begin{aligned} g = &- 2dudv + \delta_{AB} dz^A dz^B + u_+ dv^2 \left(u\delta^{AB}[\mathsf{Y}_{vz^A}][\mathsf{Y}_{vz^B}] - 2p\right) \\ &+ 2u_+[\mathsf{Y}_{vz^I}]) dvdz^A \left(u\delta^{BI}[\mathsf{Y}_{z^A z^B}] - 2\delta^I_A\right) \qquad (L) \\ &- 2u_+ dz^A dz^B \left([\mathsf{Y}_{z^A z^B}] - \frac{u}{2}\delta^{IJ}[\mathsf{Y}_{z^I z^A}][\mathsf{Y}_{z^J z^B}]\right) \end{aligned}$$

- distr. metric $g = -2d\mathcal{U}d\mathcal{V} + \delta_{AB} d\mathcal{X}^A \mathcal{X}^B + 2\mathscr{H}(v, z^C)\delta(u) d\mathcal{U}^2$ (D)
- “discontinuous transformation”

$$\mathcal{U} = u + u_+ \left(1/\partial_v H - 1\right), \quad \mathcal{X}^A \overset{\mathsf{def}}{=} z^A + u_+ x_1^A$$

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- generalised Penrose junction conditions $\mathcal{V} \in \mathbb{M}^- \mapsto H\left(\mathcal{V}_-, x^A_-\right) \in \mathbb{M}^+$

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& - 2u_+ dz^A dz^B \left([\mathsf{Y}_{z^A z^B}] - \frac{u}{2}\delta^{IJ}[\mathsf{Y}_{z^I z^A}][\mathsf{Y}_{z^J z^B}]\right)
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Caveats

- $\delta = \delta(u)$ and not $\delta(\mathcal{U})$
  generically $\mathcal{U}$ only Lipschitz at shell $\rightsquigarrow$ delicate to deal with $\delta\,(\mathcal{U})$.
- $\mathscr{H} = \mathscr{H}(v, z^A)$ and not $\mathscr{H}(\mathcal{V}, \mathcal{X}^A)$
  similar; $\mathcal{V}$ only L^∞ and not defined on shell

Future work:

- Consider geodesics in (D) with smooth, generic $\mathscr{H}(\mathcal{V}, \mathcal{X}^A))$ and $\delta(\mathcal{U})$
- Find geometric meaning & regularisation of "discontinuous transformation"
- ...

- distr. metric $g = -2d\mathcal{U}d\mathcal{V} + \delta_{AB}d\mathcal{X}^A\mathcal{X}^B + 2\mathscr{H}(v, z^C)\delta(u)d\mathcal{U}^2$ (D)
- "discontinuous transformation"

$$\mathcal{U} = u + u_+\,(1/\partial_v H - 1), \quad \mathcal{X}^A \stackrel{\text{def}}{=} z^A + u_+ x_1^A$$
$$\mathcal{V} = v + \theta(u)\,(H - v) + u_+ \frac{1}{2\partial_v H}\delta^{AB}\partial_{z^A} H \partial_{z^B} H,$$

- generalised Penrose junction conditions $\mathcal{V} \in \mathbb{M}^- \mapsto H\left(\mathcal{V}_-, x_-^A\right) \in \mathbb{M}^+$

Thank you for your attention

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