(v) If in a local chart the tensor field $A \in \mathcal{T}^r_s(M)$ has components $A^{i_1\dots i_r}_{j_1\dots j_s}$ the components of its covariant differential $\nabla A \in \mathcal{T}^r_{s+1}(M)$ are denoted by $A^{i_1\dots i_r}_{j_1\dots j_s;k}$ and take the form

$$A^{i_1\dots i_r}_{j_1\dots j_s;k} = \frac{\partial A^{i_1\dots i_r}_{j_1\dots j_s}}{x^k} + \sum_{l=1}^{r} \Gamma^{i_l}_{km} A^{i_1\dots m\dots i_r}_{j_1\dots\dots\dots j_s} - \sum_{l=1}^{s} \Gamma^{m}_{kj_l} A^{i_1\dots\dots\dots i_r}_{j_1\dots m\dots j_s}\,. \tag{1.3.49}$$

Our next topic is the notion of a covariant derivative of vector fields which are not defined on all of M but just, say on (the image of) a curve. Of course then we can only expect to be able to define a derivative of the vector field in the direction of the curve. Intuitively such a notion corresponds to the rate of change of the vector field as we go along the curve. We begin by making precise the notion of such vector fields but do not restrict ourselves to the case of curves.

1.3.26 Definition (Vector field along a mapping). *Let N, M be smooth manifolds and let $f \in \mathcal{C}^\infty(N, M)$. A* vector field along f *is a smooth mapping*

$$Z:\ N \to TM \quad \textit{such that } \pi \circ Z = f, \tag{1.3.50}$$

where $\pi : TM \to M$ is the vector bundle projection. We denote the $\mathcal{C}^\infty(N)$-module of all vector fields along f by $\mathfrak{X}(f)$.

The definition hence says that $Z(p) \in T_{f(p)}M$ for all points $p \in N$. In the special case of $N = I \subseteq \mathbb{R}$ a real interval and $f = c : I \to M$ a $\mathcal{C}^\infty$-curve we call $\mathfrak{X}(c)$ the space of *vector fields along the curve* c. In particular, in this case $t \mapsto \dot c(t) \equiv c'(t) \in \mathfrak{X}(c)$. More precisely we have (cf. [9, below (2.5.3)]) $c'(t) = T_t c(1) = T_t c(\frac{\partial}{\partial t}|_t) \in T_{c(t)}M$. Also recall for later use that for any $f \in \mathcal{C}^\infty(M)$ we have $c'(t)(f) = T_t c(\frac{d}{dt}|_t)(f) = \frac{d}{dt}|_t(f \circ c)$ and consequently in coordinates $\varphi = (x^1, \dots, x^n)$ the local expression of the velocity vector takes the form $c'(t) = c'(t)(x^i)\partial_i|_{c(t)} = \frac{d}{dt}|_t(x^i \circ c)\partial_i|_{c(t)}$. (For more details see e.g. [11, 1.17 and below].)
In case M is a SRMF we may use the Levi-Civita covariant derivative to define the derivative Z' of $Z \in \mathfrak{X}(c)$ along the curve c.

1.3.27 Proposition (Induced covariant derivative). *Let $c : I \to M$ be a smooth curve into the SRMF M. Then there exists a unique mapping $\mathfrak{X}(c) \to \mathfrak{X}(c)$*

$$Z \mapsto Z' \equiv \frac{\nabla Z}{dt} \tag{1.3.51}$$

called the induced covariant derivative *such that*

(i) $(Z_1 + \lambda Z_2)' = Z_1' + \lambda Z_2' \qquad (\lambda \in \mathbb{R}),$

(ii) $(hZ)'(t) = \dfrac{dh}{dt} Z + h\, Z' \qquad (h \in \mathcal{C}^\infty(I, \mathbb{R})),$

(iii) $(X \circ c)'(t) = \nabla_{c'(t)} X \qquad (t \in I,\ X \in \mathfrak{X}(M)).$

In addition we have

(iv) $\frac{d}{dt}\langle Z_1, Z_2\rangle = \langle Z_1', Z_2\rangle + \langle Z_1, Z_2'\rangle$.

Observe that in 1.3.27(iii) we have $X \circ c \in \mathfrak{X}(c)$ and since by 1.3.2(i) we have $\nabla_{c'(t)}X \in T_{c(t)}M$ also the right hand side makes sense.

Proof. (Local) uniqueness: Let $Z \mapsto Z'$ be a mapping as above that satisfies (i)–(iii) and let $(\phi = (x^1, \dots, x^n), U)$ be a chart and let $c : I \to U$ be a smooth curve. For $Z \in \mathfrak{X}(c)$ we then have

$$Z(t) = \sum_i Z(t)(x^i)\partial_i|_{c(t)} =: \sum_i Z^i(t)\partial_i|_{c(t)} \equiv Z^i(t)\partial_i|_{c(t)}. \tag{1.3.52}$$

By (i)–(iii) we then obtain

$$Z'(t) = \frac{dZ^i}{dt}\,\partial_i|_{c(t)} + Z^i(t)(\partial_i \circ c)' = \frac{dZ^i}{dt}\,\partial_i|_{c(t)} + Z^i(t)\nabla_{c'(t)}\partial_i. \tag{1.3.53}$$

So Z' is completely determined by the Levi-Civita connection ∇ and hence unique.

Existence: For any $J \subseteq I$ such that $c(J)$ is contained in a chart domain U we define the mapping $Z \mapsto Z'$ by equation (1.3.53). Then properties (i)–(iii) hold. Indeed (i) is obvious. Property (ii) follows from the following straight forward calculation

$$(hZ)'(t) = \frac{d(hZ^i)}{dt}\,\partial_i|_{c(t)} + h(t)Z^i(t)\nabla_{c'(t)}\partial_i = h(t)Z'(t) + \frac{dh}{dt}Z^i(t)\partial_i|_{c(t)}.$$

Finally to prove (iii) let $X \in \mathfrak{X}(M)$, $X|_U = X^i\partial_i$. Then (as explained prior to the proposition) $\frac{d}{dt}(X^i \circ c) = c'(t)(X^i)$ and so by (∇_3)

$$\begin{aligned}\nabla_{c'(t)}X = \nabla_{c'(t)}(X^i\partial_i) &= c'(t)(X^i)\partial_i|_{c(t)} + X^i(c(t))\nabla_{c'(t)}\partial_i \\ &= \frac{d(X^i \circ c)}{dt}\,\partial_i|_{c(t)} + X^i \circ c(t)\nabla_{c'(t)}\partial_i = (X \circ c)'(t).\end{aligned} \tag{1.3.54}$$

Now suppose J_1, J_2 are two subintervals of I with corresponding maps $F_i : Z \mapsto Z'$. Then on $J_1 \cap J_2$ both F_i satisfy properties (i)–(iii) hence coincide by the uniqueness argument from above and we obtain a well-defined map on the whole of I.

Finally we obtain (iv) since we have using the chart (φ, U)

$$\langle Z_1', Z_2\rangle + \langle Z_1, Z_2'\rangle = \frac{dZ_1^i}{dt}\,Z_2^j\langle\partial_i, \partial_j\rangle + Z_1^iZ_2^j\langle\nabla_{c'}\partial_i, \partial_j\rangle + Z_1^i\frac{dZ_2^j}{dt}\langle\partial_i, \partial_j\rangle + Z_1^iZ_2^j\langle\partial_i, \nabla_{c'}\partial_j\rangle$$

and on the other hand

$$\frac{d}{dt}\langle Z_1, Z_2\rangle = \frac{d}{dt}\big(Z_1^iZ_2^j\langle\partial_i, \partial_j\rangle\big) = \frac{dZ_1^i}{dt}\,Z_2^j\langle\partial_i, \partial_j\rangle + Z_1^i\,\frac{dZ_2^j}{dt}\,\langle\partial_i, \partial_j\rangle + Z_1^iZ_2^j\,\frac{d}{dt}\,\langle\partial_i, \partial_j\rangle.$$

The result now follows from from differentiating $\langle \partial_i, \partial_j \rangle$ (actually $\langle \partial_i, \partial_j \rangle \circ c$):

$$\frac{d}{dt} \langle \partial_i, \partial_j \rangle = c'(t)\langle \partial_i, \partial_j \rangle = \langle \nabla_{c'} \partial_i, \partial_j \rangle + \langle \partial_i, \nabla_{c'} \partial_j \rangle,$$

where we have used (∇5). □

We now write Z' in terms of the Christoffel symbols. In a chart $(\varphi = (x^1, \dots x^n), V)$ we have

$$\nabla_{c'(t)} \partial_i = \nabla_{\frac{d(x^j \circ c)}{dt} \partial_j} \partial_i = \frac{d(x^j \circ c)}{dt} \nabla_{\partial_j} \partial_i = \frac{d(x^j \circ c)}{dt} \Gamma^k_{ij} \partial_k \tag{1.3.55}$$

and hence

$$Z'(t) = \left(\frac{dZ^k}{dt}(t) + \Gamma^k_{ij}(c(t)) \frac{d(x^j \circ c)}{dt}(t)\, Z^i(t) \right) \partial_k|_{c(t)}. \tag{1.3.56}$$

In the special case that $Z = c'$ we call $Z' = c''$ the *acceleration* of c. Also we call a vector field $Z \in \mathfrak{X}(c)$ *parallel* if $Z' = 0$. The above formula (1.3.56) shows that this condition actually is expressed by a system of linear ODEs of first order implying the following result.

1.3.28 Proposition (Parallel vector fields). *Let $c : I \to M$ be a smooth curve into a SRMF M. Let $a \in I$ and $z \in T_{c(a)}M$. Then there exists an unique parallel vector field $Z \in \mathfrak{X}(c)$ with $Z(a) = z$.*

Proof. As noted above locally Z obeys (1.3.56), which is a system of linear first order ODEs. Given an initial condition such an equation possesses a unique solution defined on the whole interval where the coefficient functions are given. Hence the claim follows from covering $c(I)$ by chart neighbourhoods. □

This result gives rise to the following notion.

1.3.29 Definition (Parallel transport). *Let $c : T \to M$ be a smooth curve into a SRMF M. Let $a, b \in I$ and write $c(a) = p$ and $c(b) = q$. For $z \in T_pM$ let Z_z be as in 1.3.28 with $Z(a) = z$. Then we call the mapping*

$$P = P^b_a(c) : T_pM \ni z \mapsto Z_z(b) \in T_qM \tag{1.3.57}$$

the parallel transport *(or* parallel translation*) along c from p to q.*

Finally we have the following crucial property of parallel translation.

1.3.30 Proposition. *Parallel transport is a linear isometry.*

Proof. Let $z, y \in T_pM$ with parallel vector fields Z_z, Z_y. Since then also $Z_y + Z_y$ and λZ_z (for $\lambda \in \mathbb{R}$) are parallel we have $P(z + y) = Z_z(b) + Z_y(b) = P(z) + P(y)$ and $P(\lambda z) = \lambda Z_z(b) = \lambda P(z)$ so that P is linear.
Let $P(z) = 0$ then by uniqueness $Z_z = 0$ hence also $z = 0$. So P is injective hence bijective.
Finally $\langle Z_z, Z_y \rangle$ is constant along c since

$$\frac{d}{dt} \langle Z_z, Z_y \rangle = \langle Z_z', Z_y \rangle + \langle Z_z, Z_y' \rangle = 0 \tag{1.3.58}$$

and so

$$\langle P(z), P(y) \rangle = \langle Z_z(b), Z_y(b) \rangle = \langle Z_z(a), Z_y(a) \rangle = \langle z, y \rangle. \tag{1.3.59}$$

□

Chapter 2

Geodesics

In Euclidean space $\mathbb{R}^n$ the shortest path between to arbitrary points is uniquely given by the straight line connecting these two points. That is, straight lines have two decisive properties: they are parallel, i.e., their velocity vector is parallely transported and they globally minimise length. Already in spherical geometry matters become more involved. The curves which possess a parallel velocity vector are the great circles (e.g. the meridians). Intuitively they are the 'straightest possible' lines on the sphere in the sense that their curvature equals the curvature of the sphere and so is the smallest possible. Also the minimize length but only locally. Indeed a great circle starting in a point p is no longer minimising after it passes through the antipodal point $-p$. Also between antipodal points (e.g. the north and the south pole) there are infinitely many great circles which all have the same length.
In this section we study geodesics on SRMFs that is curves with parallel velocity vector and their respective properties. We introduce the exponential map which maps straight lines in the tangent space to geodesics of the manifold and in turn use it introduce normal coordinates which provide a chart which is adapted to the geometry of the manifold. We introduce the length functional and the Riemannian distance function and prove that locally in a RMF (radial) geodesics minimise length. Finally we prove the Hopf-Rinow theorem which says that the Riemannian distance function encodes the topology of the manifold. ♣ `polish formulation` ♣

2.1 Geodesics and the exponential map

In this subsection we generalise the notion of a straight line in Euclidean space. As in [9, Sec. 3.3] we define a *geodesic* to be a curve c such that its tangent vector c' is parallel along c. Equivalently we have for the acceleration $c'' = 0$. Locally this condition translates into a system of nonlinear ODEs of second order. More precisely we have.

2.1.1 Proposition (Geodesic equation). *Let $(\varphi = (x^1, \dots, x^n), U)$ be a chart of the SRMF M and let $c : I \to U$ be a smooth curve. Then c is a geodesic iff the local coordinate* *expressions* $x^k \circ c$ *of* c *obey the* geodesic equation,

$$\frac{d^2(x^k \circ c)}{dt} + \Gamma^k_{ij} \circ c \, \frac{d(x^i \circ c)}{dt} \, \frac{d(x^j \circ c)}{dt} = 0 \qquad (1 \leq k \leq n). \tag{2.1.1}$$

Proof. The curve c is a geodesic iff $(c')' = 0$. We have $c'(t) = \frac{d(x^k \circ c)}{dt} \partial_k|_{c(t)}$ and inserting $\frac{d(x^k \circ c)}{dt}$ for Z^k in (1.3.56) we obtain (2.1.1). □

It is most common to abbreviate the local expressions $x^k \circ c$ of c by c^k. Using this notation the geodesic equation (2.1.1) takes the form

$$\frac{d^2 c^k}{dt} + \Gamma^k_{ij} \frac{dc^i}{dt} \frac{dc^j}{dt} = 0 \qquad (1 \leq k \leq n). \tag{2.1.2}$$

Obviously the geodesic equation is a system of nonlinear ODEs of second order and so by basic ODE-theory we obtain the following result on existence and uniqueness of geodesics.

2.1.2 Lemma (Existence of geodesics). *Let* $p \in (M, g)$ *and let* $v \in T_pM$. *Then there exists a real interval* I *around* 0 *and a unique geodesic* $c : I \to M$ *with* $c(0) = p$ *and* $c'(0) = v$.

We call c as in 2.1.2 the geodesic starting a p with initial velocity v.

2.1.3 Examples (Geodesics of flat space). The geodesic equations in $\mathbb{R}^n_r$ are trivial, i.e., they take the form $\frac{d^2 c^k}{dt^2} = 0$ since all Christoffel symbols Γ^i_{jk} vanish. Hence the geodesics are the straight lines $c(t) = p + tv$.

2.1.4 Examples (Geodesic on the cylinder). Let $M \in \mathbb{R}^3$ be the cylinder of radius 1 and ψ the chart $(\cos\varphi, \sin\varphi, z) \mapsto (\varphi, z)$ $(\varphi \in (0, 2\pi))$. The natural basis of T_pM w.r.t. ψ is then given by (cf. [9, 2.4.11])

$$\partial_\varphi = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}, \quad \partial_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad (p = (\cos\varphi, \sin\varphi)). \tag{2.1.3}$$

Also we have $g_{11} = \langle \partial_\varphi, \partial_\varphi \rangle = 1$, $g_{12} = g_{21} = \langle \partial_\varphi, \partial_z \rangle = 0$, $g_{22} = \langle \partial_z, \partial_z \rangle = 1$ and so

$$(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad (g^{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{2.1.4}$$

which immediately implies that $\Gamma^i_{jk} = 0$ for all i, j, k. Hence the geodesic equations for a curve $c(t) = (\cos\varphi(t), \sin\varphi(t), z(t))$ using the notation $c^1(t) = x^1 \circ c(t) = \varphi(t)$, $c^2(t) = x^2 \circ c(t) = z(t)$ take the form

$$\ddot{\varphi}(t) = 0, \qquad \ddot{z}(t) = 0, \tag{2.1.5}$$

which are readily solved to obtain $\varphi(t) = a_1 t + a_0$, $z(t) = b_1 t + b_0$. So we find

$$c(t) = \Big(\cos(a_1 t + a_0), \sin(a_1 t + a_0), b_1 t + b_0\big) \tag{2.1.6}$$

revealing that the geodesics of the cylinder are helices with initial point and speed given by the a_i, b_i, see figure 2.1, left. This also includes the extreme cases of circles of latitude $z = c$ ($b_1 = 0$, $b_0 = c$) and generators ($a_1 = 0$). Another way to see that these are the

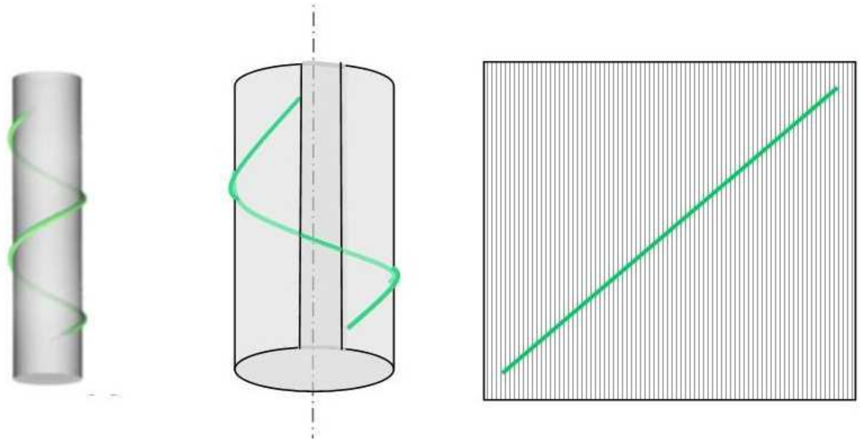

Figure 2.1: Geodesics on the cylinder

geodesics of the cylinder is by 'unwrapping' the cylinder to the plain, see Figure 2.1, right. This of course amounts to applying the chart ψ, which in this case is an isometry since $g_{ij} = \delta_{ij}$. Now the geodesics of $\mathbb{R}^2$ are straight lines which via ψ^{-1} are wrapped as helices onto the cylinder.

We now return to the general study of geodesics. Being solutions of a nonlinear ODE geodesics will in general not be defined for all values of the parameter. However, ODE theory provides us with unique maximal(ly extended) solutions. To begin with we have.

2.1.5 Lemma (Uniqueness of geodesics). *Let $c_1, c_2 : I \to M$ geodesics. If there is $a \in I$ such that $c_1(a) = c_2(a)$ and $\dot c_1(a) = \dot c_2(a)$ then we already have $c_1 = c_2$.*

Proof. Suppose under the hypothesis of the lemma there is $t_0 \in I$ such that $c_1(t_0) \neq c_2(t_0)$. Without loss of generality we may assume $t_0 > a$ and set $b = \inf\{t \in I :\ t > a \text{ and } c_1(t) \neq c_2(t)\}$. We now argue that $c_1(b) = c_2(b)$ and $\dot c_1(b) = \dot c_2(b)$. Indeed if $b = a$ this follows by assumption. Also in case $b > a$ we have that $c_1 = c_2$ on (a, b) and the claim follows by continuity.
Now since also $t \mapsto c_i(b + t)$ are geodesics ($i = 1, 2$) lemma 2.1.2 implies that c_1 and c_2 agree on a neighbourhood of b which contradicts the definition of b. $\square$

2.1.6 Proposition (Maximal geodesics). *Let $v \in T_pM$. Then there exists a unique geodesic c_v such that*

(i) $c_v(0) = p$ *and* $c_v'(t) = v$

(ii) the domain of c_v is maximal, that is if $c : J \to M$ is any geodesic with $c(0) = p$ and $c'(0) = v$ then $J \subseteq I$ and $c_v|_J = c$.

Proof. Let $G := \{c : I_c \to M :\ 0 \in I_c,\ c(0) = p,\ c'(0) = v\}$, then by 2.1.2 $G \neq \emptyset$ and by 2.1.5 we have for any pair c_1, $c_2 \in G$ that $c_1|_{I_{c_1} \cap I_{c_2}} = c_2|_{I_{c_1} \cap I_{c_2}}$. So the geodesics in G define a unique geodesic satisfying the assertions of the statement. □

2.1.7 Definition (Completeness). *We call a geodesic c_v as in 2.1.5* maximal. *If in a SRMF M all maximal geodesics are defined on the whole of $\mathbb{R}$ we call M* geodesically complete.

2.1.8 Examples (Complete manifolds).

(i) $\mathbb{R}^n_r$ is geodesically complete as well as the cylinder of 2.1.4.

(ii) $\mathbb{R}^n_r \setminus \{0\}$ is not geodesically complete since all geodesics of the form $t \mapsto tv$ are defined either on $\mathbb{R}^+$ or $\mathbb{R}^-$ only.

We next turn to the study of the causal character of geodesics. We begin with the following definition.

2.1.9 Definition (Causal character of curves). *A curve c into a SRMF M is called* spaceliek, timelike *or* null *if for all t its velocity vector $c'(t)$ is spacelike, timelike or null, respectively. We call c* causal *if it is timelike or null. These properties of c are commonly referred to as its* causal character.

In general a curve need not have a causal character, i.e., its velocity vector could change its causal character along the curve. However, geodesics do have a causal character: Indeed if c is a geodesic then by definition c' is parallel along c. But parallel transport by 1.3.28 is an isometry so that $\langle c'(t), c'(t)\rangle = \langle c'(t_0),,\rangle c'(t_0)$ for all t. This fact can also be seen directly by the following simple calculation

$$\frac{d}{dt}\,\langle c'(t), c'(t)\rangle = 2\langle c''(t), c'(t)\rangle = 0. \tag{2.1.7}$$

Moreover we also have that the speed of a geodesic is constant, i.e., $\|c'(t)\| = \|c(t_0)\|$ for all t. The following technical result is significant.

2.1.10 Lemma (Geodesic parametrisation). *Let $c : I \to M$ be a nonconstant geodesic. A reparametrisation $c \circ h : J \to M$ of c is a geodesic iff h is of the form $h(t) = at + b$ with $a, b \in \mathbb{R}$.*

Proof. To begin with note that $(c\circ h)'(t) = c'(h(t))h'(t)$ since $(c\circ h)'(t) = T_t(c\circ h)(\frac{d}{dt}|_t) = T_{h(t)}c(T_t h(\frac{d}{dt}|_t)) = T_{h(t)}c(h'(t)\frac{d}{dt}|_t)) = h'(t)T_{h(t)}c(\frac{d}{dt}|_t) = h'(t)c'(h(t))$.
Now, if $Z \in \mathfrak{X}(c)$ then $Z\circ h \in \mathfrak{X}(c\circ h)$ and from (1.3.56) we have $(Z\circ h)'(t) = Z'(h(t))h'(t)$. Applying this to $Z = c'$, we obtain with 1.3.27(ii)

$$(c\circ h)''(t) = \Big(c'(h(t))h'(t)\Big)' = h''(t)c'(h(t)) + h'(t)^2 c''(h(t)). \tag{2.1.8}$$

Now since c is nonconstant we have $c'(t) \neq 0$ and we obtain

$c\circ h$ is a geodesic $\Leftrightarrow$ $(c\circ h)'' = 0$ $\Leftrightarrow$ $h'' = 0$ $\Leftrightarrow$ $h(t) = at + b$. □

This result shows that the parametrisation of a geodesic has a geometric significance. More generally a curve that has a reparametrisation as a geodesic is called a *pregeodesic*.

Next we turn to a deeper analysis of the geodesic equations as a system of second order ODEs. The first result is concerned with the dependence of a geodesic on its initial speed and is basically a consequence of smooth dependence of solutions of ODEs on the data.

2.1.11 Lemma (Dependence on the initial speed). *Let $v \in T_pM$ then there exists a neighbourhood $\mathcal{N}$ of v in TM and an interval I around 0 such that the mapping*

$$\mathcal{N} \times I \ni (w,s) \mapsto c_w(s) \in M \tag{2.1.9}$$

is smooth.

Proof. c_w is the solution of the second order ODE (2.1.1) which depends smoothly on t as well as on $c_w(0) =: p$ and $c_w'(0) = w$. This follows from ODE theory e.g. by rewriting (2.1.1) as a first order system as in [9, 2.5.16]. □

Our next aim is to make the rewriting of the geodesic equations as a first order system explicit. To this end let $(\psi = (x^1,\dots x^n), V)$ be a chart. Then $T\psi : TV \to \psi(V)\times\mathbb{R}^n$, $T\psi = (x^1,\dots,x^n,y^1,\dots,y^n)$ is a chart of TM. Now $c : I \to M$ is a geodesic iff $t \mapsto (c^1(t),\dots,c^n(t),y^1(t),\dots,y^n(t))$ solves the following first order system

$$\begin{aligned}\frac{dc^k}{dt} &= y^k(t)\\ \frac{dy^k}{dt} &= -\Gamma^k_{ij}(x(t))\, y^i(t)\, y^j(t) \qquad (1 \le k \le n).\end{aligned} \tag{2.1.10}$$

Locally (2.1.10) is an ODE on TM since its right hand side is a vector field on TV, i.e., an element of $\mathfrak{X}(TV)$. The geodesics hence correspond to the flow lines of such a vector field. More precisely we have.

2.1.12 Theorem (Geodesic flow). *There exists a uniquely defined vector field $G \in \mathfrak{X}(TM)$, the so-called* geodesic field *or* geodesic spray *with the following properties: The projection $\pi : TM \to M$ establishes a one-to-one correspondence between (maximal) integral curves of G and (maximal) geodesics of M.*

Proof. Given $v \in TM$ the mapping $s \mapsto c_v'(s)$ is a smooth curve in TM. Let $G_v := G(v)$ be the initial speed of this curve, i.e., $G_v := \frac{d}{dt}|_0(c_v'(s)) \in T_v(TM)$ (since $c_v'(0) = v$). Now by 2.1.11 $G \in \mathfrak{X}(TM)$. We now prove the following two statements:

(i) If c is a geodesic of M then c' is an integral curve of G.
Indeed let $\alpha(s) := c'(s)$ and for any fixed t set $w := c'(t)$ and $\beta(s) := c_w'(s)$. By **??** we have $c(t+s) = c_w(s)$ (since $c_w(0) = c(t)$ and $c_w'(0) = w = c'(t)$). Differentiation w.r.t. s yields $\alpha(s+t) = c_w'(s) = \beta(s)$. So we have in $T(TM)$ that $\alpha'(t+s) = \beta'(s)$. In particular, $\alpha'(t) = \beta'(0) = G_w = G_{\alpha(t)}$ and hence α is an integral curve of G.

(ii) If α is an integral curve of G then $\pi \circ \alpha$ is a geodesic of M.
Let $\alpha : I \to TM$ and $t \in I$. Since $\alpha(t) \in TM$ we have by (i) that the map $s \mapsto G_{\alpha(t)}'(s)$ an integral curve of G. For $s = 0$ both integral curves $G_{\alpha(t)}'(s)$ and $s \mapsto \alpha(s+t)$ attain the value $\alpha(t)$. Hence by [9, 2.5.17] they coincide on their entire domain. So we obatin for all s that

$$\alpha(s+t) = c_{\alpha(t)}'(s) \ \Rightarrow\ \pi(\alpha(s+t)) = c_{\alpha(t)}(s) \ \Rightarrow \pi \circ \alpha \quad \text{is a geodesis of } M.$$

Finally we have $\pi \circ c' = c$ and $(\pi\alpha)'(t) = \frac{d}{ds}|_0 \pi(\alpha(t+s)) = \frac{d}{ds}|_0 c_{\alpha(t)}(s) = c_{\alpha(t)}'(0) = \alpha(t)$ and so the maps $\alpha \mapsto \pi \circ \alpha$ and $c \mapsto c'$ are inverse to each other. Also G is unique since its integral curves are prescribed. □

The flow of the vector field G in the above Theorem is called *the geodesic flow* of M. Next we will introduce the *exponential* map which is one of the essential tools of semi-Riemannian geometry.

2.1.13 Definition (Exponenatial map). *Let p be a point in the SRMF M and set $\mathcal{D}_p := \{v \in T_pM : c_v$ is at least defined on $[0,1]\}$. The* exponential map *of M at p is defined as*

$$\exp_p : \mathcal{D}_p \to M, \qquad \exp_p(v) := c_v(1). \tag{2.1.11}$$

Observe that $\mathcal{D}_p$ is the maximal domain of $\exp_p$. In case M is geodesically complete we have $\mathcal{D}_p = T_pM$ for all p.
Let now $v \in T_pM$ and fix $t \in \mathbb{R}$. The geodesic $s \mapsto c_v(ts)$ (cf. 2.1.10) has initial speed $tc_v'(0) = tv$ and so we have

$$c_{tv}(s) = c_v(ts) \tag{2.1.12}$$

for all t, s for which one and hence both sides of (2.1.12) are defined. This implies for the exponential map that

$$\exp_p(tv) = c_{tv}(1) = c_v(t) \tag{2.1.13}$$

which has the following immediate geometric interpretation: the exponential map $\exp_p$ maps straight lines $t \mapsto tv$ through the origin in T_pM to geodesics $c_v(t)$ through p in M, see Figure 2.2. This is actually done in a diffeomorphic way.

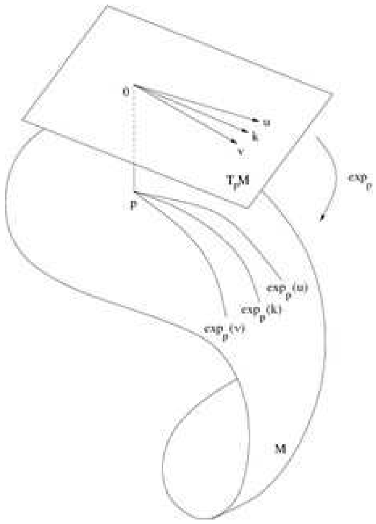

Figure 2.2: The exponential map at a point p.

2.1.14 Theorem (Exponential map). *Let p be a point in a SRMF M. Then there exist neighbourhoods $\tilde{U}$ of 0 in T_pM and U of p in M such that the exponential map $\exp_p : \tilde{U} \to U$ is a diffeomorphism.*