Riemannian Geometry

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Preface

These lecture notes are based on the lecture course "Differentialgeometrie 2" taught by M.K. in the fall semesters of 2008 and 2012. The material has been slightly reorganised to serve as a script for the course "Riemannian geometry" by R.S. in the fall term 2016. It can be considered as a continuation of the lecture notes "Differential Geometry 1" of M.K. [9] and we will extensively refer to these notes.

Basically this is a standard introductory course on Riemannian geometry which is strongly influenced by the textbook "Semi-Riemannian Geometry (With Applications to Relativity)" by Barrett O'Neill [11]. The necessary prerequisites are a good knowledge of basic differential geometry and analysis on manifolds as is traditionally taught in a 3–4 hours course.

♣ `To be extended in due time` ♣

M.K., R.S., September 2016

Contents

Chapter 1

Semi-Riemannian Manifolds

In classical/elementary differential geometry of hypersurfaces in $\mathbb{R}^n$ and, in particular, of surfaces in $\mathbb{R}^3$ one finds that all intrinsic properties of the surface ultimately depend on the scalar product induced on the tangent spaces by the standard scalar product of the ambient Euclidean space. Our first goal is to generalise the respective notions of length, angle, curvature and the like to the setting of abstract manifolds. We will, however, allow for nondegenerate bilinear forms which are not necessarily positive definite to include central applications, in particular, general relativity. We will start with an account on such bilinear forms.

1.1 Scalar products

Contrary to basic linear algebra where one typically focusses on positive definite scalar products semi-Riemannian geometry uses the more general concept of nondegenerate bilinear forms. In this subsection we develop the necessary algebraic foundations.

1.1.1 Definition (Bilinear forms). *Let V be a finite dimensional vector space. A* bilinear form *on V is an $\mathbb{R}$-bilinear mapping $b:\ V \times V \to \mathbb{R}$. It is called* symmetric *if*

$$b(v,w) = b(w,v) \quad \textit{for all } v,w \in V. \tag{1.1.1}$$

A symmetric bilinear form is called

(i) Positive (negative) definite*, if $b(v,v) > 0$ (< 0) for all $0 \neq v \in V$,*

(ii) Positive (negative) semidefinite*, if $b(v,v) \geq 0$ (≤ 0) for all $v \in V$,*

(iii) nondegenerate*, if $b(v,w) = 0$ for all $w \in V$ implies $v = 0$.*

Finally we call b (semi)definite *if one of the alternatives in (i) (resp. (ii)) hold true. Otherwise we call b* indefinite*.*

In case b is definite it is semidefinite and nondegenerate and conversely if b is semidefinite and nondegenerate it is already definite. Indeed in the positive case suppose there is $0 \neq v \in V$ with $b(v, v) = 0$. Then for arbitrary $w \in V$ we find that

$$b(v + w, v + w) = \underbrace{b(v, v)}_{0} + 2b(v, w) + b(w, w) \geq 0 \tag{1.1.2}$$

$$b(v - w, v - w) = \underbrace{b(v, v)}_{0} - 2b(v, w) + b(w, w) \geq 0, \tag{1.1.3}$$

since b is positive semidefinite and so $2|b(v, w)| \leq b(w, w)$. But replacing w by λw with λ some positive number we obtain

$$2\ |b(v, w)| \leq \lambda\ b(w, w) \tag{1.1.4}$$

and since we may choose λ arbitrarily small we have $b(v, w) = 0$ for all w which by nondegeneracy implies $v = 0$, a contradiction.

If b is a symmetric bilinear form on V and if W is a subspace of V then clearly the restriction $b|_W$ (defined as $b|_{W \times W}$) of b to W is again a symmetric bilinear form. Obviously if b is (semi)definite then so is $b|_W$.

1.1.2 Definition (Index). *We define the* index r *of a symmetric bilinear form b on V by*

$$r := \max \{\dim W |\ W \textit{subspace of } V \textit{ with } b|_W \textit{ negative definite}\}. \tag{1.1.5}$$

By definition we have $0 \leq r \leq \dim V$ and $r = 0$ iff b is positive definite.

Given a symmetric bilinear form b we call the function

$$q :\ V \to \mathbb{R}, \quad q(v) = b(v, v) \tag{1.1.6}$$

the *quadratic form associated with b*. Frequently it is more convenient to work with q than with b. Recall that by polarisation $b(v, w) = 1/2(q(v + w) - q(v) - q(w))$ we can recover b from q and so all the information of b is also encoded in q.

Let $\mathcal{B} = \{e_1, \ldots, e_n\}$ be a basis of V, then

$$(b_{ij}) := (b(e_i, e_j))_{i,j=1}^n \tag{1.1.7}$$

is called the *matrix of* b with respect to $\mathcal{B}$. It is clearly symmetric and entirely determines b since $b(\sum v_i\, e_i, \sum w_j\, e_j) = \sum b_{ij} v_i w_j$. Moreover nondegeneracy of b is characterised by its matrix (w.r.t. any basis):

1.1.3 Lemma. *A symmetric bilinear form is nondegenerate iff its matrix w.r.t. one (and hence any) basis is invertible.*

Proof. Let $\mathcal{B} = \{e_1, \dots, e_n\}$ be a basis of V. Given $v \in V$ we have $b(v, w) = 0$ for all w iff $0 = b(v, e_i) = b(\sum_j v_j e_j, e_i) = \sum_j b_{ij} v_j$ for all $1 \leq i \leq n$. So b is degenerate iff there are $(v_1, \dots, v_n) \neq (0, \dots, 0)$ with $\sum_j b_{ij} v_j = 0$ for all i. But this means that the kernel of (b_{ij}) is non trivial and (b_{ij}) is singular. □

We now introduce a terminology slightly at odds with linear algebra standards but which reflects our interest in non positive definite symmetric bilinear forms.

1.1.4 Definition (scalar product, inner product). *A* scalar product g *on a vector space* V *is a nondegenerate symmetric bilinear form. An* inner product *is a positive definite scalar product.*

1.1.5 Example.

(i) *The* example of an inner product is the *standard scalar product* of Euclidean space $\mathbb{R}^n$: $v \cdot w = \sum_i v_i w_i$.

(ii) The most simple example of a vector space with indefinite scalar product is *two-dimensional Minkowski space* $\mathbb{R}^2$ with

$$g = \eta : \ \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}, \qquad g(v, w) = -v_1 w_1 + v_2 w_2. \tag{1.1.8}$$

Obviously g is bilinear and symmetric. To see that it is nondegenerate suppose that $g(v, w) = 0$ for all $w \in \mathbb{R}^2$. Setting $w = (1, 0)$ and $w = (0, 1)$ gives $v_1 = 0$ and $v_2 = 0$, respectively and so $v = 0$. Hence η is a scalar product but it is not an inner product since it is indefinite:

$$g\big((1, 0), (1, 0)\big) = -1 < 0, \quad \text{but} \quad g\big((0, 1), (0, 1)\big) = 1 > 0. \tag{1.1.9}$$

The corresponding quadratic form is $q(v) = -v_1^2 + v_2^2$.

In the following V will always be a (finite dimensional, real) vector space with a scalar product g in the sense of 1.1.4. A vector $0 \neq v \in V$ with $q(v) = 0$ will be called a *null vector*. Such vectors exists iff g is indefinite. Note that the zero vector 0 is *not* a null vector.

1.1.6 Example. We consider the lines $q = c$ and $q = -c$ $(c > 0)$ in two-dimensional Minkowski space of Example 1.1.5(ii). They are either hyperbolas or straight lines in case $c = 0$, see Figure 1.1.

A pair of vectors $u, w \in V$ is called *orthogonal*, $u \perp w$, if $g(u, w) = 0$. Analogously we call subspaces U, W of V orthogonal, if $g(u, w) = 0$ for all $u \in U$ and all $w \in W$. **Warning:** In case of indefinite scalar products vectors that are orthogonal need not to be at right angels to one another as the following example shows.

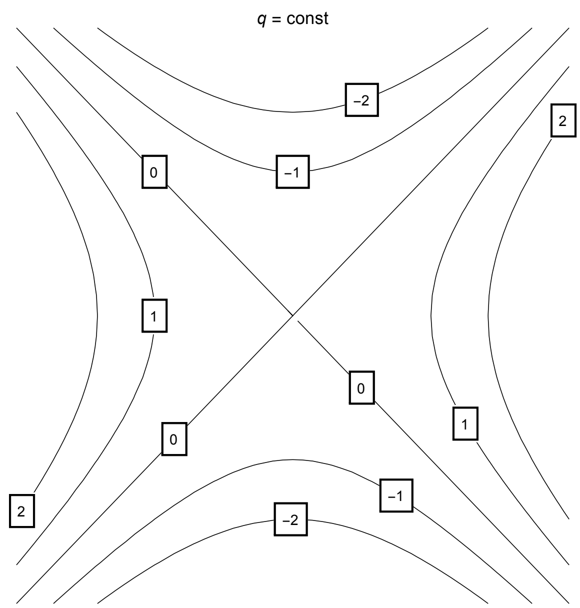

Figure 1.1: Contours of q in 2-dimensional Minkowski space

1.1.7 Example. The following pairs of vectors v, v' are orthogonal in two-dimensional Minkowski space, see Figure 1.2: $w = (1,0)$ and $w' = (0,1)$, $u = (1,b)$ and $u' = (b,1)$ for some $b > 0$, $z = (1,1) = z'$.

The example of the vectors z, z' above hints at the fact that null vectors are precisely those vectors that are orthogonal to themselves.
If W is a subspace of V let

$$W^\perp := \{v \in V : \ v \perp w \quad \text{for all } w \in W\}. \tag{1.1.10}$$

Clearly $W^\perp$ is a subspace of V which we call W *perp*. **Warning:** We cannot call $W^\perp$ the orthogonal complement of W since in general $W + W^\perp \neq V$, e.g. if $W = \text{span}(z)$ in Example 1.1.7 we even have $W^\perp = W$. However $W^\perp$ has two familiar properties.

1.1.8 Lemma. *Let W be a subspace of V. Then we have*

(i) $\dim W + \dim W^\perp = \dim V$,

(ii) $(W^\perp)^\perp = W$.

Proof.

(i) Let $\{e_1, \ldots, e_k\}$ be a basis of W which we extend to a basis $\{e_1, \ldots, e_k, e_{k+1}, \ldots, e_n\}$ of V. Then we have

$$v \in W^\perp \Leftrightarrow g(v, e_i) = 0 \quad \text{for } 1 \leq i \leq k \ \Leftrightarrow \ \sum_{j=1}^{n} g_{ij} v_j = 0 \quad \text{for } 1 \leq i \leq k. \tag{1.1.11}$$

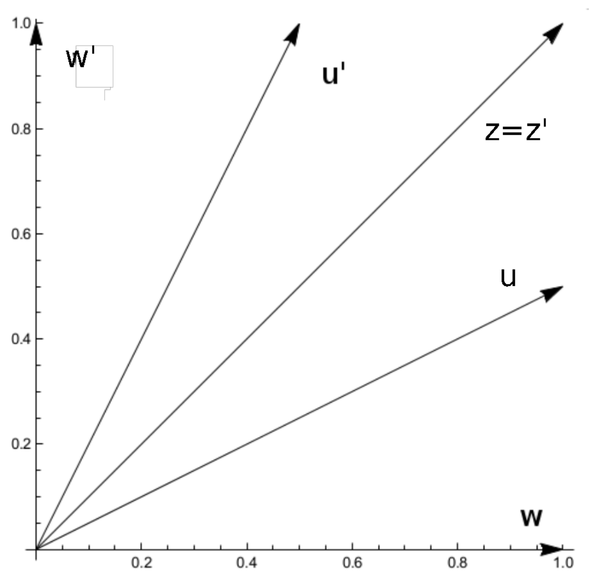

Figure 1.2: Pairs of orthogonal vectors in 2-dimensional Minkowski space

Now by Lemma 1.1.3 (g_{ij}) is invertible and hence the rows in the above linear system of equations are linearly independent and its space of solutions has dimension $n-k$. So $\dim W^\perp = n-k$.

(ii) Let $w \in W$, then $w \perp W^\perp$ and $w \in (W^\perp)^\perp$, which implies $W \subseteq (W^\perp)^\perp$. Moreover, by (i) we have $\dim W = \dim (W^\perp)^\perp = k$ and so $W = (W^\perp)^\perp$. □

A scalar product g on V is nondegenerate, iff $V^\perp = \{0\}$. A subspace W of V is called *nondegenerate*, if $g|_W$ is nondegenerate. If g is an inner product, then any subspace W is again an inner product space, hence nondegenerate. If g is indefinite, however, there always exists degenerate subspaces, e.g. $W = \mathrm{span}(w)$ for any null vector w. Hence a subspace W of a vector space with scalar product in general is *not* a vector space with scalar product. Indeed W could be degenerate. We now give a simple characterisation of nondegeneracy for subspaces.

1.1.9 Lemma (Nondegenerate subspaces). *A subspace W of a vectors space V with scalar product is nondegenerate iff*

$$V = W \oplus W^\perp. \tag{1.1.12}$$

Proof. By linear algebra we know that

$$\dim(W + W^\perp) + \dim(W \cap W^\perp) = \dim W + \dim W^\perp = \dim V. \tag{1.1.13}$$

So equation (1.1.12) holds iff

$$\dim(W \cap W^\perp) = 0 \;\Leftrightarrow\; \{0\} = W \cap W^\perp = \{w \in W : w \perp W\}$$

which is equivalent to the nondegeneracy of W. □

As a simple consequence we obtain by using 1.1.8(ii), i.e., $(W^\perp)^\perp = W$.

1.1.10 Corollary. *W is nondegenerate iff $W^\perp$ is nondegenerate.*

Our next objective is to deal with *orthonormal bases* in a vector space with scalar product. To begin with we define unit vectors. However, due to the fact that q takes negative values we have to be a little careful.

1.1.11 Definition (Norm). *We define the* norm *of a vector $v \in V$ by*

$$|v| := |g(v,v)|^{\frac{1}{2}}. \tag{1.1.14}$$

A vector $v \in V$ is called a unit vector *if $|v| = 1$, i.e., if $g(v,v) = \pm 1$. A family of pairwise orthogonal unit vectors is called* orthonormal.

Observe that an orthonormal system of $n = \dim V$ elements automatically is a basis. The existence of orthonormal bases (ONB) is guaranteed by the following statement.

1.1.12 Lemma (Existence of orthonormal bases). *Every vector space $V \neq \{0\}$ with scalar product possesses an orthonormal basis.*

Proof. There exists $v \neq 0$ with $g(v,v) \neq 0$, since otherwise by polarisation we would have $g(v,w) = 0$ for all pairs of vectors v, w, which implies that g is degenerate. Now $v/|v|$ is a unit vector and it suffices to show that any orthonormal system $\{e_1, \ldots, e_k\}$ can be extended by one vector.
So let $W = \mathrm{span}\{e_1, \ldots, e_k\}$. Then by lemma 1.1.3 W is nondegenerate and so is $W^\perp$ by corollary 1.1.10. Hence by the argument given above $W^\perp$ contains a unit vector e_{k+1} which extends $\{e_1, \ldots, e_k\}$. □

The matrix of g w.r.t. any ONB is diagonal, more precisely

$$g(e_i, e_j) = \delta_{ij}\varepsilon_j, \quad \text{where } \varepsilon_j := g(e_j, e_j) = \pm 1. \tag{1.1.15}$$

In the following we will always order any ONB $\{e_i, \ldots, e_n\}$ in such a way that in the so-called *signature* $(\varepsilon_1, \ldots, \varepsilon_n)$ the negative signs come first. Next we give the representation of a vector w.r.t an ONB. Once again we have to be careful about the signs.

1.1.13 Lemma. *Let $\{e_1, \ldots, e_n\}$ be an ONB for V. Then any $v \in V$ can be uniquely written as*

$$v = \sum_{i=1}^{n} \varepsilon_i \, g(v, e_i) \, e_i. \tag{1.1.16}$$

Proof. We have that

$$\langle v - \sum_i \varepsilon_i g(v, e_i) e_i, e_j \rangle = \langle v, e_j \rangle - \sum_i \varepsilon_i \, \langle v, e_i \rangle \, \underbrace{\langle e_i, e_j \rangle}_{\varepsilon_i \delta_{ij}} = 0 \tag{1.1.17}$$

for all j and so by nondegeneracy $v = \sum_i \varepsilon_i g(v, e_i) e_i$. Uniqueness now simply follows since $\{e_1, \ldots, e_n\}$ is a basis. □

If a subspace W is nondegenerate we have by lemma 1.1.9 that $V = W \oplus W^\perp$. Let now π be the orthogonal projection of V onto W. Since any ONB $\{e_1, \ldots, e_k\}$ of W can be extended to an ONB of V (cf. the proof of 1.1.12) we have for any $v \in V$

$$\pi(v) = \sum_{j=1}^{k} \varepsilon_j \, g(v, e_j) \, e_j. \tag{1.1.18}$$

Next we give a more vivid description of the index r of g (see definition 1.1.2), which we will also call the index of V and denote it by ind V

1.1.14 Proposition (Index and signature). *Let $\{e_1, \ldots, e_n\}$ be any ONB of V. Then the index of V equals the number of negative signs in the signature $(\varepsilon_1, \ldots, \varepsilon_n)$.*

Proof. Let exactly the first m of the ε_i be negative. In case g is definite we have $m = r = 0$ or $m = r = n = \dim V$ and we are done.
So suppose $0 < m < n$. Obviously g is negative definite on $S = \mathrm{span}\{e_1, \ldots, e_m\}$ and so $m \leq r$.
To show the converse let W be a subspace with $g|_W$ negative definite and define

$$\pi : W \to S, \quad \pi(w) := -\sum_{i=1}^{m} g(w, e_i) e_i. \tag{1.1.19}$$

Then π is obviously linear and we will show below that it is injective. Then clearly $\dim W \leq \dim S$ and since W was arbitrary $r \leq \dim S = m$.
Finally π is injective since if $\pi(w) = 0$ then by lemma 1.1.13 $w = \sum_{i=m+1}^{n} g(w, e_i) e_i$. Since $w \in W$ we also have $0 \geq g(w, w) = \sum_{i=m+1}^{n} g(w, e_i)^2$ which implies $g(w, e_j) = 0$ for all $j > m$. But then $w = 0$. □

The index of a nondegenerate subspace can now easily be related to the index of V.

1.1.15 Corollary. *Let W be a nondegenerate subspace of V. Then*

$$\mathit{ind}\, V = \mathit{ind}\, W + \mathit{ind}\, W^{\perp}. \tag{1.1.20}$$

Proof. Let $\{e_1, \dots, e_k\}$ be an ONB of W and $\{e_{k+1}, \dots, e_n\}$ be an ONB of $W^{\perp}$ such that $\{e_1, \dots, e_n\}$ is an ONB of V, cf. the proof of 1.1.12. Now the assertion follows from proposition 1.1.14. □

To end this section we will introduce *linear isometries*. Let (V_1, g_1) and (V_2, g_2) be vector spaces with scalar product.

1.1.16 Definition (Linear isometry). *A linear map $T : V_1 \to V_2$ is said to* preserve the scalar product *if*

$$g_2(Tv, Tw) = g_1(v, w). \tag{1.1.21}$$

A linear isometry *is a linear bijective map that preserves the scalar product.*

In case there is no danger of missunderstanding we will write equation (1.1.21) also as

$$\langle Tv, Tw \rangle = \langle v, w \rangle. \tag{1.1.22}$$

If equation (1.1.21) holds true then T automatically preserves the associated quadratic forms, i.e., $q_2(Tv) = q_1(v)$ for all $v \in V$. The converse assertions clearly holds by polarisation.
A map that preserves the scalar product is automatically *injective* since $Tv = 0$ by virtue of (1.1.21) implies $g_1(v, w) = 0$ for all w and so $v = 0$ by nondegeneracy. Hence a linear mapping is an isometry iff $\dim V_1 = \dim V_2$ and equation (1.1.21) holds. Moreover we have the following characterisation.

1.1.17 Proposition (Linear isometries). *Let (V_1, g_1) and (V_2, g_2) be vector spaces with scalar product. Then the following are equivalent:*

(i) $\dim V_1 = \dim V_2$ *and* $\mathit{ind}\, V_1 = \mathit{ind}\, V_2$

(ii) There exists a linear isometry $T : V_1 \to V_2$

Proof. (i)⇒(ii): Choose ONBs $\{e_1, \dots, e_n\}$ of V_1 and $\{e'_1, \dots, e'_n\}$ of V_2. By proposition 1.1.14 we may assume that $\langle e_i, e_i \rangle = \langle e'_i, e'_i \rangle$ for all i. Now we define a linear map T via $Te_i = e'_i$. Then clearly $\langle Te_i, Te_j \rangle = \langle e_i, e_j \rangle$ for all i, j and T is an isometry.
(ii)⇒(i): If T is an isometry then $\dim V_1 = \dim V_2$ and T maps any ONB of V_1 to an ONB of V_2. But then equation (1.1.21) and proposition 1.1.14 imply that $\mathrm{ind}\, V_1 = \mathrm{ind}\, V_2$. □

1.2 Semi-Riemannian metrics

In this section we start our program to transfer the setting of elementary differential geometry to abstract manifolds. The key element is to equip each tangent space with a scalar product that varies smoothly on the manifold. We start right out with the central definition.

1.2.1 Definition (Metric). *A semi-Riemannian* metric tensor *(or* metric*, for short) on a smooth manifold*[1] *M is a smooth, symmetric and nondegenerate $(0,2)$-tensor field g on M of constant index.*

In other words g smoothly assigns to each point $p \in M$ a symmetric nondegenerate bilinear form $g(p) \equiv g_p : T_pM \times T_pM \to \mathbb{R}$ such that the index r_p of g_p is the same for all p. We call this common value r_p the *index* r of the metric g. We clearly have $0 \le r \le n = \dim M$. In case $r = 0$ all g_p are inner products on T_pM and we call g a *Riemannian metric*, cf. [9, 3.1.14]. In case $r = 1$ and $n \ge 2$ we call g a *Lorentzian metric.*

1.2.2 Definition. *A* semi-Riemannian manifold (SRMF) *is a pair (M, g), where g is a metric on M. In case g is Riemannian or Lorentzian we call (M, g) a* Riemannian manifold (RMF) *or* Lorentzian manifold (LMF)*, respectively.*

We will often sloppily call just M a (S)RMF or LMF and write $\langle\ ,\ \rangle$ instead of g and use the following convention

- $g_p(v, w) = \langle v, w\rangle \in \mathbb{R}$ for vectors $v, w \in T_pM$ and $p \in M$, and
- $g(X, Y) = \langle X, Y\rangle \in \mathcal{C}^\infty(M)$ for vector fields $X, Y \in \mathfrak{X}(M)$.

If (V, φ) is a chart of M with coordinates $\varphi = (x^1, \dots, x^n)$ and natural basis vector fields $\partial_i \equiv \frac{\partial}{\partial x^i}$ we write

$$g_{ij} = \langle \partial_i, \partial_j\rangle \qquad (1 \le i, j \le n) \tag{1.2.1}$$

for the local components of g on V. Denoting the dual basis covector fields of ∂_i by dx^i we have

$$g|_U = g_{ij}\, dx^i \otimes dx^j, \tag{1.2.2}$$

where we have used the summation convention (see [9, p. 54]) which will be in effect from now on.

Since g_p is nondegenerate for all p the matrix $(g_{ij}(p))$ is invertible by lemma 1.1.3 and we write $(g^{ij}(p))$ for its inverse. By the inversion formula for matrices the g^{ij} are smooth functions on V and by symmetry of g we have $g^{ij} = g^{ji}$ for all i and j.

[1] In accordance with [9] we assume all smooth manifolds to be second countable and Hausdorff. For background material on topological properties of manifolds see [9, Section 2.3].

1.2.3 Example.

(i) We consider $M = \mathbb{R}^n$. Clearly $T_pM \cong \mathbb{R}^n$ for all points p and the standard scalar product induces a Riemannian metric on $\mathbb{R}^n$ which we denote by

$$\langle v_p, w_p \rangle = v \cdot w = \sum_i v^i\, w^i. \tag{1.2.3}$$

We will always consider $\mathbb{R}^n$ equipped with this Riemannian metric.

(ii) Let $0 \leq r \leq n$. Then

$$\langle v_p, w_p \rangle = -\sum_{i=1}^{r} v^i\, w^i + \sum_{j=r+1}^{n} v^j\, w^j \tag{1.2.4}$$

defines a metric on $\mathbb{R}^n$ of index r. We will denote $\mathbb{R}^n$ with this metric tensor by $\mathbb{R}^n_r$. Clearly $\mathbb{R}^n_0$ is $\mathbb{R}^n$ in the sense of (i). For $n \geq 2$ the space $\mathbb{R}^n_1$ is called n-dimensional *Minkowski space*. In case $n = 4$ this is the simplest spacetime in the sense of Einstein's general relativity. In fact it is the flat spacetime of special relativity.

Setting $\varepsilon_i = -1$ for $1 \leq i \leq r$ and $\varepsilon_i = 1$ for $r + 1 \leq i \leq n$ the metric of $\mathbb{R}^n_r$ takes the form

$$g = \varepsilon_i\; dx^i \otimes dx^i = \varepsilon_i\; e^i \otimes e^i. \tag{1.2.5}$$

As is clear from section 1.1 a nonvanishing index allows for the existence of null vectors. Here we further pursue this line of ideas.

1.2.4 Definition (Causal character). *Let M be a SRMF, $p \in M$. We call $v \in T_pM$*

(i) spacelike *if $\langle v, v \rangle > 0$ or if $v = 0$,*

(ii) null *if $\langle v, v \rangle = 0$ and $v \neq 0$,*

(iii) timelike *if $\langle v, v \rangle < 0$.*

The above notions define the so-called *causal character* of v. The set of null vectors in T_pM is called the *null cone* at p respectively *light cone* at p in the Lorentzian case. In this case we also refer to null vectors as *lightlike* and call a vector *causal* if it is either timelike or lightlike.

1.2.5 Example. Let v be a vector in 2-dimensional Minkowski space $\mathbb{R}^2_1$. Then v is null iff $v_1^2 = v_2^2$, i.e., iff $v_1 = \pm v_2$, see also figure 1.3.

The above terminology is of course motivated by physics and relativity in particular. Setting the speed of light $c = 1$ then a flash of light emitted at the origin of Minkowski space travels along the light cone. A point is timelike if a signal with speed $v < 1$ can reach it and it is spacelike if it needs superluminal speed to reach it.

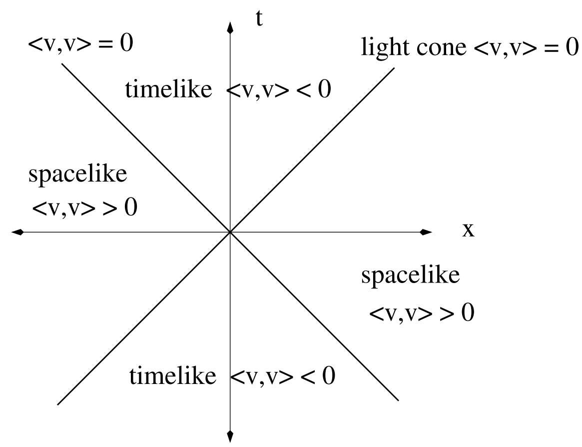

Figure 1.3: The lightcone in 2-dimensional Minkowski space

Let q be the quadratic form associated with g, i.e., $q(v) = \langle v, v\rangle$ for all $v \in T_pM$. Then by polarisation q determines the metric but it is *not* a tensor field since for $X \in \mathfrak{X}(M)$ and $f \in \mathcal{C}^\infty(M)$ we clearly have $q(fX) = f^2 q(X)$ (cf. [9, 2.6.19]). It is nevertheless frequently used in 'classical terminology' where it is called the *line element* and denoted by ds^2. Locally one writes

$$q = ds^2 = g_{ij}\, dx^i\, dx^j, \tag{1.2.6}$$

where juxtaposition of differentials means multiplication in each tangent space, that is

$$q(X) = g_{ij}\, dx^i(X)\, dx^j(X) = g_{ij} X^i X^j \tag{1.2.7}$$

for a vector field locally given by $X = X^i \partial_i$. Finally we write for the *norm* of a tangent vector

$$\|v\| := |q(v)|^{1/2} = |\langle v, v\rangle|^{1/2}. \tag{1.2.8}$$

1.2.6 Remark. The origin of the somewhat strange notation ds^2 is the following heuristic consideration: Consider two 'neighbouring points' p and p' with coordinates $(x^1, \dots, x^n)$ and $(x^1 + \triangle x^1, \dots, x^n + \triangle x^n)$. Then the 'tangent vector' $\triangle p = \sum \triangle x^i \partial_i$ at p points approximately to p' and so the 'distance' $\triangle s$ from p to p' should approximately be given by

$$\triangle s^2 = \|\triangle p\|^2 = \langle \triangle p, \triangle p\rangle = \sum_{i,j} g_{ij}(p) \triangle x^i \triangle x^j. \tag{1.2.9}$$

Let N be a submanifold of a RMF M with embedding $j : N \hookrightarrow M$. Then the pull back j^*g of the metric g to the submanifold N is given by (see [9, 2.7.24])

$$(j^*g)(p)(v, w) = g(j(p))(T_p j v, T_p j w) = g(p)(v, w), \tag{1.2.10}$$

where in the final equality we have identified $T_p j(T_p N)$ with $T_p N$, see [9, 3.4.11].

Hence j^*g is just the restriction of g_p to the subspace T_pN of T_pM. Since g is Riemannian this restriction is positive definite and so j^*g turns N into a RMF.
However, if M is only a SRMF then the $(0,2)$-tensor field j^*g on N need not be a metric. Indeed (cf. section 1.1) j^*g is a metric and hence (N, j^*g) a SRMF iff every T_pN is nondegenerate in T_pM and the index of T_pN is the same for all $p \in N$. Of course this index can be different from the index of g.
These considertaions lead to the following definition.

1.2.7 Definition (Semi-Riemannian submanifold). *A submanifold N of a SRMF M is called a* semi-Riemannian submanifold (SRSMF) *if j^*g is a metric on N.*

If N is Riemannian or Lorentzian these terms replace semi-Riemannian in the above definition. However, note that while every SRSMF of a RMF is again Riemannian, a LMF can have Lorentzian as well as Riemannian submanifolds.

Finally we turn to isometries, i.e., diffeomorphisms that preserve the metric.

1.2.8 Definition. *Let (M, g_M) and (N, g_N) be SRMF and $\phi : M \to N$ be a diffeomorphism. We call ϕ an* isometry *if $\phi^* g_N = g_M$.*

Recall that the defining property of an isometry by [9, 2.7.24] in some more detail reads

$$\langle T_p\phi(v), T_p\phi(w)\rangle = g_N(\phi(p))\big(T_p\phi(v), T_p\phi(w)\big) = g_M(p)(v, w) = \langle v, w\rangle \qquad (1.2.11)$$

for all $v, w \in T_pM$ and all $p \in M$. Since ϕ is a diffeomorphism, every tangent map $T_p\phi : T_pM \to T_{\phi(p)}N$ is a linear isometry (cf. 1.1.16). Also note that $\phi^* g_N = g_M$ is equivalent to $\phi^* q_N = q_M$ since g is always uniquely determined by q. If there is an isometry between the SRMFs M and N we call them *isometric*.

1.2.9 Remark.

(i) Clearly id_M is an isometry. Moreover the inverse of an isometry is an isometry again and if ϕ_1 and ϕ_2 are isometries so is $\phi_1 \circ \phi_2$. Hence the isometries of M form a group, called the *isometry group* of M.

(ii) Given two vector spaces V and W with scalar product and a local isometry $\phi : V \to W$. If we consider V and W as SRMF. Then ϕ is also an isometry of SRMF.

(iii) If V is a vector space with scalar product and $\dim V = n$, ind $V = r$ then V as a SRMF is isometric to $\mathbb{R}^n_r$. Just choose an ONB $\{e_1, \dots, e_n\}$ of V. Then the coordinate mapping $V \ni v = \sum v^i e_i \mapsto (v^1, \dots, v^n) \in \mathbb{R}^n$ is a linear isometry and (ii) proves the claim.