2.2 Geodesic convexity

Geodesically convex sets, which sometimes are also called totally normal, are normal neighbourhoods of all of their points. In this section we are going to prove existence of such sets around each point in a SRMF. The arguments will rest on global properties of the exponential map. To formulate these we need to introduce product manifolds as a preparation. We skip the obvious proof of the following lemma.

2.2.1 Definition & Lemma (Product manifold). *Let M^m and N^n be smooth manifolds of dimension m and n respectively. Let $(p,q) \in M \times N$ and let $(\varphi = (x^1,\dots,x^m), U)$ and $(\psi = (y^1,\dots,y^n), V)$ be charts of M around p and of N around q, respectively. Then we call $(\varphi \times \psi, U \times V)$ a* product chart *of $M \times N$. The family of all product charts defines a $\mathcal{C}^\infty$-structure on $M \times N$ and we call the resulting smooth manifold the* product manifold *of M with N. The dimension of $M \times N$ is $m+n$.*

2.2.2 Remark (Properties of product manifolds). Let M^m and M^n be $\mathcal{C}^\infty$-manifolds.

(i) The natural manifold topology of $M \times N$ is precisely the product topology of M and N since the $\varphi \times \psi$ are homeomorphism by definition.

(ii) The projections $\pi_1 : M \times N \to M$, $\pi_1(p,q) = p$ and $\pi_2 : M \times N \to N$, $\pi_2(p,q) = q$ are smooth since $\varphi \circ \pi_1 \circ (\varphi \times \psi)^{-1} = \mathrm{pr}_1 : \varphi(U) \times \psi(V) \to \varphi(U)$ and similarly for π_2. From this local representation we even have that π_1, π_2 are surjective submersions. By ♣ `properly cite Cor. 1.1.23 in submanifold part of ANAMF` ♣ if follows that for $(p,q) \in M \times N$ the sets $M \times \{q\} = \pi_2^{-1}(\{q\})$ and $\{p\} \times N = \pi_1^{-1}(\{p\})$ are closed submanifolds of $M \times N$ of dimension m and n, respectively. Moreover the bijection $\pi_1|_{M\times\{q\}} : M \times \{q\} \to M$ is a diffeomorphism by ♣ `properly cite Thm. 1.1.5 in submanifold part of ANAMF` ♣ and analogously for $\pi_2|_{\{p\}\times N}$.

(iii) A mapping $f : P \to M \times N$ from a smooth manifold P into the product is smooth iff $\pi_1 \circ f$ and $\pi_2 \circ f$ are smooth. This fact is easily seen from the chart representations.

(iv) By (ii) we may identify $T_{(p,q)}(M \times \{q\})$ with T_pM and likewise for $T_{(p,q)}(\{q\} \times N)$ and T_qN. We now show that (using this identification)

$$T_{(p,q)}M \times N = T_pM \oplus T_qN. \tag{2.2.1}$$

Since $\dim T_pM = m$ and $\dim T_qN = n$ it suffices to show that $T_{(p,q)}(M \times \{q\}) \cap T_{(p,q)}(\{p\} \times N) = 0$. Suppose v is in this intersection. Now since $\pi_1|_{\{p\}\times N} \equiv p$ we have $T_{(p,q)}\pi_1|_{T_{(p,q)}(\{p\}\times N)} = 0$ and so $T_{(p,q)}\pi_1(v) = 0$. But by (ii) $T_{(p,q)}\pi_1|_{T_{(p,q)}(M\times\{q\})}$ is bijective and hence $v = 0$. In most cases one identifies $T_pM \oplus T_qn$ with $T_pM \times T_qN$ and hence writes $T_{(p,q)}(M \times N) \cong T_pM \times T_qN$.

(v) If (M, g_M) and (N, g_N) are SRMFs then $M \times N$ is again a SRMF with metric

$$g := \pi_1^*(g_M) + \pi_2^*(g_n), \quad \text{i.e., cf. [9, 2.7.24]}), \tag{2.2.2}$$
$$g_{(p,q)}(v, w) = g_M(p)\big(T_{(p,q)}\pi_1(v), T_{(p,q)}\pi_1(w)\big) + g_N(q)\big(T_{(p,q)}\pi_2(v), T_{(p,q)}\pi_2(v)\big).$$

Indeed g is obviously symmetric and it is nondegenerate since suppose $g(v, w) = 0$ for all $w \in T_{(p,q)}(M \times N)$. Now choosing $w \in T_{(p,q)}(M \times \{q\}) \cong T_pM$ we obtain by $T_{(p,q)}\pi_2(w) = 0$ that

$$g_m(p)\big(T_{(p,q)}\pi_1(v), \underbrace{T_{(p,q)}\pi_1(w)}_{(*)}\big) = 0. \tag{2.2.3}$$

Now since $T_{(p,q)}\pi_1$ is bijective the term $(*)$ attains all values in T_pM hence by nondegeneracy of g_M, $T_{(p,q)}\pi_1(v) = 0$ and analogously $T_{(p,q)}\pi_2(v) = 0$. Since $v = v_M + v_N$ where $v_M \in T_pM$ and $v_N \in T_qN$ we finally obtain $v = 0$.

The SRMF $(M \times N, g)$ is called the *semi-Riemannian product* of M with N.

Now we are in a position to collect together the maps $\exp_p : T_pM \supseteq \mathcal{D}_p \to M$ $(p \in M)$ to a single mapping. To begin with we assume M to be complete, i.e., we assume that $\exp_p$ is defined on all of T_pM for all $p \in M$. Let $\pi : TM \to M$ the projection and define the mapping

$$E : TM \to M \times M, \quad E(v) = (\pi(v), \exp_{\pi(v)} v), \tag{2.2.4}$$

that is for $v \in T_pM \subseteq TM$ we have $E(v) = (p, \exp_p(v))$.
In case M is not complete the maximal domain of E is given by $\mathcal{D} := \{v \in TM : c_v \text{ exists at least on } [0,1]\}$. Now for each p we have that the maximal domain $\mathcal{D}_p$ of $\exp_p$ is $\mathcal{D}_p = \mathcal{D} \cap T_pM$. Observe that $\mathcal{D}$ also is the maximal domain of $\exp := \Pi_2 \circ E : v_p \mapsto \exp_p(v_p) = c_{v_p}(1)$. We now have the following result which we have already used in the proof of the Gauss lemma 2.1.21.

2.2.3 Proposition (The domain of exp**).** *The domain $\mathcal{D}$ of E is open in TM and the domain $\mathcal{D}_p$ of $\exp_p$ is open and star shaped around 0 in T_pM.*

Proof. Let $G \in \mathfrak{X}(TM)$ be the geodesic spray as in 2.1.12. Then as proven there the flow lines of G are the derivatives of geodesics, i.e., $Fl_t^G(v) = c_v'(t)$. The maximal domain $\tilde{\mathcal{D}}$ of Fl^G is open in $\mathbb{R} \times TM$ by [9, 2.5.17(iii)].
Clearly $\tilde{\mathcal{D}}$ also is the maximal domain of $\pi \circ \mathrm{Fl}^G : (t, v) \mapsto c_v(t)$. Now let $\Phi : TM \to TM \times \mathbb{R}$, $\Phi(v) = (v, 1)$. Then Φ is smooth and we have $\mathcal{D} = \{v \in TM : \exists c_v(1)\} = \{v \in TM : (v, 1) \in \tilde{\mathcal{D}}\} = \Phi^{-1}(\tilde{\mathcal{D}})$, hence open. But then also $\mathcal{D}_p = \mathcal{D} \cap T_pM$ is open in T_pM.
Let finally $v \in \mathcal{D}_p$ then c_v is defined on $[0, 1]$ and by (2.1.12) we have $c_{tv}(1) = c_v(t)$. So $tv \in \mathcal{D}_p$ for all $t \in [0, 1]$ and hence $\mathcal{D}_p$ is star shaped. □

We now introduce sets which generalise the notion of convex subsets of Euclidean space.

2.2.4 Definition (Convex sets). *An open subset $C \subseteq M$ of a SRMF is called* (geodesically) convex *if C is a normal neighbourhood of all of its points.*

By 2.1.15 for any pair p, q of points in a convex set C there hence exists a unique geodesic called $c_{pq} : [0,1] \to M$ which connects p to q and stays entirely in C. Observe that there also might be other geodesics in M that connect p and q. They, however, have to leave C, as is depicted in the case of the sphere in Figure 2.2.

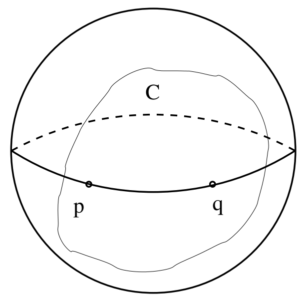

Figure 2.5: The unique geodesic connecting p and q within the convex set C is the shorter great circle arc between p and q.

To study further properties of the mapping E we need some further preparations.

2.2.5 Definition (Diagonal, null section). *Let M be a smooth manifold. The* diagonal *$\triangle_M \subseteq M \times M$ is defined by $\triangle_M := \{(p,p) : p \in M\}$. The* zero section *$TM_0$ of TM is defined as $TM_0 := \{0_p : p \in M\}$.*

Observe that in a chart φ around p we have that the map $f : M \times M \ni (p, p') \mapsto \varphi(p) - \varphi(p')$ is a submersion and we locally have that $\triangle_M = f^{-1}(0)$ and so $\triangle_M$ is a submanifold of M, cf. ♣ `properly cite 1.1.22 of submf.pdf` ♣. Moreover the mapping $p \mapsto (p,p)$ is a diffeomorphism from M to $\triangle_M$. Finally by [9, 2.5.6] the zero section is precisely the base of the vector bundle $TM \to M$ and so it is also diffeomorphic to M. We now have.

2.2.6 Theorem (Properties of E). *The mapping E is a diffeomorphism from a neighbourhood of the zero section TM_0 of TM to a neighbourhood of the diagonal $\triangle_M \subseteq M \times M$.*

Proof. We first show that every $x \in TM_0$ possesses a neighbourhood where E is a diffeomorphism. To begin with observe that we have $E(x) = (\pi(x), \exp_p(0)) = (p,p)$. We aim at applying the inverse function theorem (see eg. ♣ `properly cite 1.1.5 of submf.pdf` ♣) and so we have to show that $T_x E : T_x(TM) \to T_{E(x)}(M \times M)$ is bijective. But since $\dim TM = \dim(M \times M) = 2 \dim M$ it suffices to establishes injectivity at every point $x \in TM_0$. So let $x = 0_p \in TM_0$ and suppose $T_x E(v) = 0$ for $v \in T_x(TM)$. We have to show that $v = 0$.
Denoting by $\pi : TM \to M$ the bundle projection and by $\pi_1 : M \times M \to M$ the projection onto the first factor we have $\pi = \pi_1 \circ E$. So $T_x\pi(v) = T_{E(x)}\pi_1(T_x E(v)) = 0$ hence $v \in$

$T_x(T_pM)$ since $\ker(T_x\pi) = T_x(T_pM)$ (cf. ♣ `properly cite 1.1.26 of submf.pdf` ♣). Now $E|_{T_pM} = \exp_p$ (identifying $\{p\} \times M$ with M) and so by (2.1.14)

$$0 = T_xE(v) = T_x(E|_{T_pM})(v) = T_{0_p}\exp_p(v) = v \tag{2.2.5}$$

as claimed.

In a second step we show that E is a diffeomorphism on a suitable domain. First we observe that for U in a basis of neighbourhoods of p and $\varepsilon > 0$ the sets of the form

$$W_{U,\varepsilon} := (T\psi)^{-1}\big(\psi(U) \times B_\varepsilon(0)\big) \tag{2.2.6}$$

form a basis of neighbourhoods of $v \in TM_0$ with ψ a chart of M around $\pi(v) = p$.
Now cover TM_0 by such neighbourhoods W_{U_i,ε_i} such that $E|_{W_{U_i,\varepsilon_i}}$ is a diffeomorphism for all i. Then $W = \cup_i W_{U_i,\varepsilon_i}$ is a neighbourhood as claimed. Indeed E is a local diffeomorphism on W and we only have to show that is is also injective. To this end let $w_1, w_2 \in W$, $w_i \in W_{U_i,\varepsilon_i}$ $(i = 1, 2)$ with $E(w_1) = E(w_2)$. But then $\pi(w_1) = \pi(w_2) =: p \in U_1 \cap U_2$ and supposing w.l.o.g. that $\varepsilon_1 \le \varepsilon_2$ we have $w_1, w_2 \in W_{U_2,\varepsilon_2}$. But then $w_1 = w_2$ by the fact that $E|_{W_{U_2,\varepsilon_2}}$ is a diffeomorphism. □

Now we are ready to state and prove the main result of this section. ♣ `insert figure` ♣

2.2.7 Theorem (Existence of convex sets). *Every point p in a SRMF M possesses a basis of neighbourhoods consisting of convex sets.*

Proof. Let V be a normal neighbourhood of p with Riemannian normal coordinates $\psi = (x^1, \dots, x^n)$ and define on V the function $N(q) = \sum_i (x^i(q))^2$. Then the sets $V(\delta) := \{q \in V : N(q) < \delta\}$ are diffeomorphic via ψ to the open balls $B_{\sqrt{\delta}}(0)$ in $\mathbb{R}^n$, hence they form a basis of neighbourhoods of p in M.
By 2.2.6 choosing δ small enough the map E is a diffeomorphism of an open neighbourhood W of $0_p \in TM$ to $V(\delta) \times V(\delta)$. We may choose W in such a way that $[0,1]W \subseteq W$ (e.g. by setting $W = T\varphi^{-1}(\varphi(U) \times B(0))$ with φ a chart of M and $B(0)$ is a suitable Euclidean ball).
Let now $B \in \mathcal{T}^0_2(V)$ be the symmetric tensor field with

$$B_{ij}(q) = \delta_{ij} - \sum_l \Gamma^l_{ij}(q)\, x^l(q). \tag{2.2.7}$$

B by 2.1.17 is positive definite at p and hence if we further shrink δ and W it is positive definite on $V(\delta)$. We now claim that $V(\delta)$ is a normal neighbourhood of each of its points. Let $q \in V(\delta)$ and set $W_q = W \cap T_qM$ then by construction $E|_{W_q}$ is a diffeomorphism onto $\{q\} \times V(\delta)$ and so is $\exp_q|_{W_q}$ onto $V(\delta)$.
It remains to show that W_q is star shaped. For $q \neq \tilde{q} \in V(\delta)$ we set $v := E^{-1}(q, \tilde{q}) = \exp_q^{-1}(\tilde{q}) \in W_q$. Then $\sigma : [0,1] \to M$, $\sigma(t) = c_v(t)$ is a geodesic joining q with $\tilde{q}$. If we can

show that σ lies in $V(\delta)$ then the proof of 2.1.15 (see the footnote) shows that $tv \in W_q$ for all $t \in [0,1]$ and we are done since any $v \in W_q$ is of the form $E^{-1}(q,\tilde{q})$.
Hence we assume by contradiction that σ leaves U. Then there is a $t \in (0,1)$ with $N(\sigma(t)) \geq \delta$. Since $N(q), N(\tilde{q}) < \delta$ there is $t_0 \in (0,1)$ such that $t \mapsto N \circ \sigma$ takes a maximum in t_0. Then for $\sigma^i = x^i \circ \sigma$ we find by the geodesic equation

$$\frac{d^2(N \circ \sigma)}{dt^2} = 2\sum_i \big((\frac{d\sigma^i}{dt})^2 + \sigma^i \frac{d^2(\sigma^i)}{dt^2}\big) = 2\sum_{ij} \big(\delta_{ij} - \sum_k \Gamma^k_{ij}\sigma^k\big)\frac{d\sigma^i}{dt}\frac{d\sigma^j}{dt} \tag{2.2.8}$$

and so

$$\frac{d^2(N \circ \sigma)}{dt^2}(t_0) = 2B(\sigma'(t_0), \sigma'(t_0)) > 0, \tag{2.2.9}$$

since $\sigma' \neq 0$. But this contradicts the fact that $N \circ \sigma(t_0)$ is maximal. □

Convex neighbourhoods are of great technical significance as we can see e.g. in the following statement.

2.2.8 Corollary (Extendability of geodesics). *Let $c : [0,b) \to M$ be a geodesic. Then c is continuously extendible (as a curve) to $[0,b]$ iff it is extendible to $[0,b]$ as a geodesic.*

Proof. The 'only if' part of the assertion is clear. Let now $\tilde{c} : [0,b] \to M$ be a continuous extension of c. By 2.2.7, $\tilde{c}(b)$ has a convex neighbourhood C. Let now $a \in [0,b)$ be such that $\tilde{c}([a,b]) \subseteq C$. Then C is also a normal neighbourhood of $p := c(a)$ and $c|_{[a,b)}$ is a radial geodesic by 2.1.15 and can hence be extended until it reaches ∂C or to $[a,\infty)$. But since $\tilde{c}(b) \in C$ (and hence $\notin \partial C$) c can be extended as a geodesic beyond b. □

Let now C be a convex open set in a SRMF M and let $p, q \in M$. Denoting by σ_{pq} the unique geodesic in C such that $\sigma_{pq}(0) = p$ and $\sigma_{pq}(1) = q$, cf. 2.1.15. Then we denote by $\vec{pq} := \sigma'_{pq}(0) \in T_pM$ the *displacement vector* of p and q. We then have

2.2.9 Lemma (Displacement vector). *Let $C \subseteq M$ be convex. Then the mapping*

$$\Phi : C \times C \to TM, \quad (p,q) \mapsto \vec{pq} \tag{2.2.10}$$

is smooth and a diffeomorphism onto its image $\Phi(C \times C)$ in TM.

Proof. Let $(p_0, q_0) \in C \times C$, then $T_{q_0}\exp_{p_0}$ is regular. Also we have $\Phi(p,q) = \exp_p^{-1}(q)$ is the solution to the equation $\exp_p(f(p,q)) = q$ and so Φ is smooth by the implicit function theorem (see e.g. [9, 2.1.2]).
Moreover $E(\Phi(p,q)) = (p,q)$ and E is invertible at $\Phi(p,q)$ for all $(p,q) \in C \times C$ since $T_{\Phi(p,q)}E = \begin{pmatrix} \mathrm{id} & 0 \\ * & T_q\exp_p \end{pmatrix}$ is nonsingular. So locally $\Phi = E^{-1}$ and so Φ is a local diffeomorphism and, in particular, $\Phi(C \times C)$ is open.

Finally Φ is injective and hence it is a diffeomorphism with inverse $E^{-1}|_{\Phi(C\times C)}$. $\square$

In Euclidean space the intersection of convex sets is convex. This statement fails to hold on SRMFs already in simple situations as the following conounterexample shows.

2.2.10 Examples (Convex sets on S^1). Let $M = S^1$. Then the exponential function is a diffeomorphism from $(-\pi,\pi) \in T_pS^1$ to $S^1 \setminus \{\bar{p}\}$, where $\bar{p}$ is the antipodal point of p, see ♣ `insert figure` ♣. The sets $C = S^1 \setminus \{p\}$ and $C' = S^1 \setminus \{\bar{p}\}$ are convex but their intersection $C \cap C'$ is not even connected hence, in particular, not convex.

However, we have the following result in a special case.

2.2.11 Lemma (Intersection of convex sets). *Let $C_1, C_2 \subseteq M$ be convex and suppose C_1 and C_2 are contained in a convex set $D \subseteq M$. Then the intersection $C_1 \cap C_2$ is convex*

Proof. Let $p \in C_1 \cap C_2$. We show that $C_1 \cap C_2$ is a normal neighbourhood of p. To begin with $\exp_p : \tilde{D} \to D$ is a diffeomorphism. Set $\tilde{C}_i := \exp_p^{-1}(C_i)$ $(i = 1,2)$, then $\exp_p : \tilde{C}_1 \cap \tilde{C}_2 \to C_1 \cap C_2$ is a diffeomorphism. Hence it only remains to show that $\tilde{C}_1 \cap \tilde{C}_2$ is star shaped.
Indeed let $v \in \tilde{C}_1 \cap \tilde{C}_2$ then $q := \exp_p(v) \in C_1 \cap C_2$ and so $\sigma_{pq}(t) = \exp_p(tv)$ is the unique geodesic in D that joins p and q. By convexity of the C_i we have that $\sigma_{pq}(t) = \exp_p(tv) \in C_1 \cap C_2$ for all $t \in [0,1]$ and so $tv \in \tilde{C}_1 \cap \tilde{C}_2$ and we are done. $\square$

The above lemma allows to prove the existence of a convex refinement of any open cover of a SRMF M. To be more precise we call a covering $\mathcal{C}$ of M by open and convex sets a *convex covering* if all non trivial intersections $C \cap C'$ of sets in $\mathcal{C}$ are convex. Then one can show that for any given open covering $\mathcal{O}$ of M there exists a convex covering $\mathcal{C}$ such that any set of $\mathcal{C}$ is contained in some element of $\mathcal{O}$, see [11, Lemma 5.10].