# Reflected Ornstein–Uhlenbeck (ROU) Process

Brolin O'Connell

2025-06-17

## Contents



## Introduction

The Ornstein–Uhlenbeck (OU) process is a classical mean-reverting stochastic process used in a variety of applications including physics, finance, and queueing theory. In this document, we simulate and analyze both the standard OU process and its **reflected** form (ROU) in one and two dimensions, examining key properties such as stationarity, ergodicity, and convergence.

`{=latex} \vspace{10pt}`

---

## 1. Ornstein–Uhlenbeck Process (1D)

The 1D OU process is governed by the stochastic differential equation:

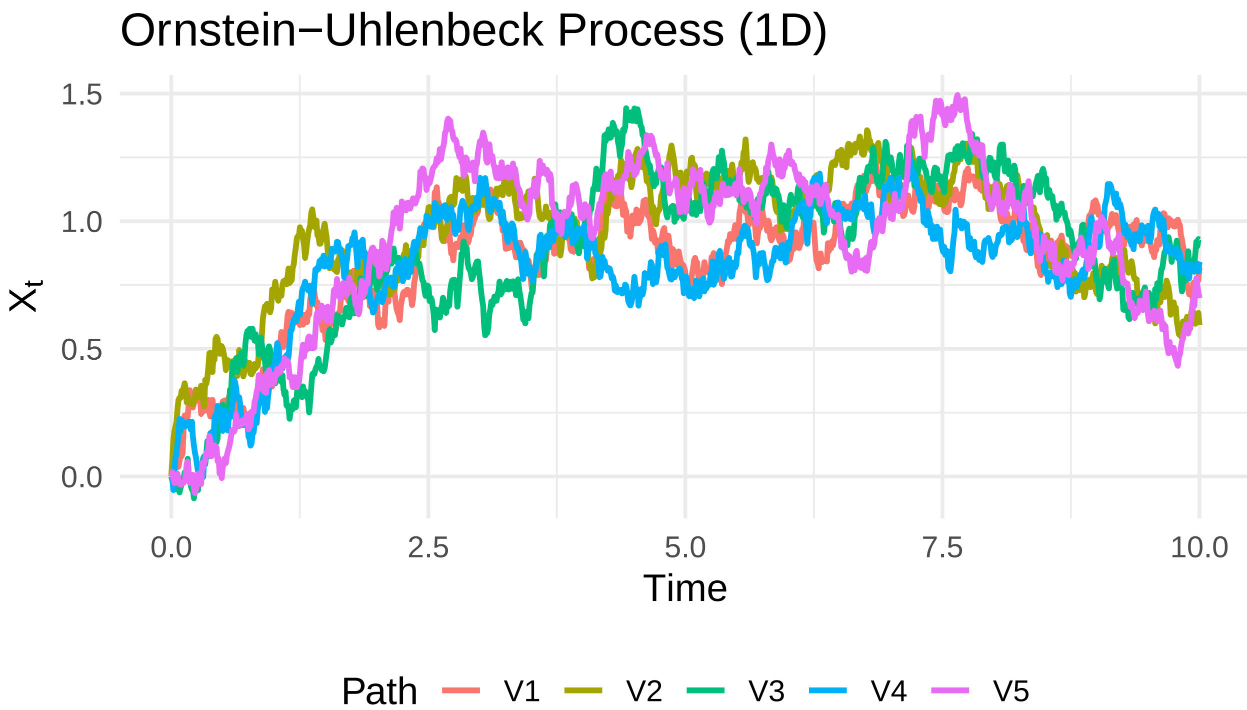
$$dX_t = -\theta(X_t - \mu)\,dt + \sigma\,dW_t \tag{1}$$

Where: - $X_t$: state at time $t$ - $\mu$: long-run mean - $\theta > 0$: mean reversion speed - $\sigma$: volatility - $dW_t$: standard Brownian motion increment

This process is stationary and has a well-defined limiting distribution:

$$\mathrm{Var}(X_t) = \frac{\sigma^2}{2\theta} \tag{2}$$

(see Lemma 7, Huang & Pang, 2022)




`{=latex} \vspace{10pt}`

---

## 2. Stationarity and Ergodicity

The OU process is **second-order stationary**: its mean and variance are time-invariant, and autocovariance depends only on lag. It is also **ergodic**, meaning:

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T X_t \, dt = \mathbb{E}[X_t] = \mu \tag{3}$$

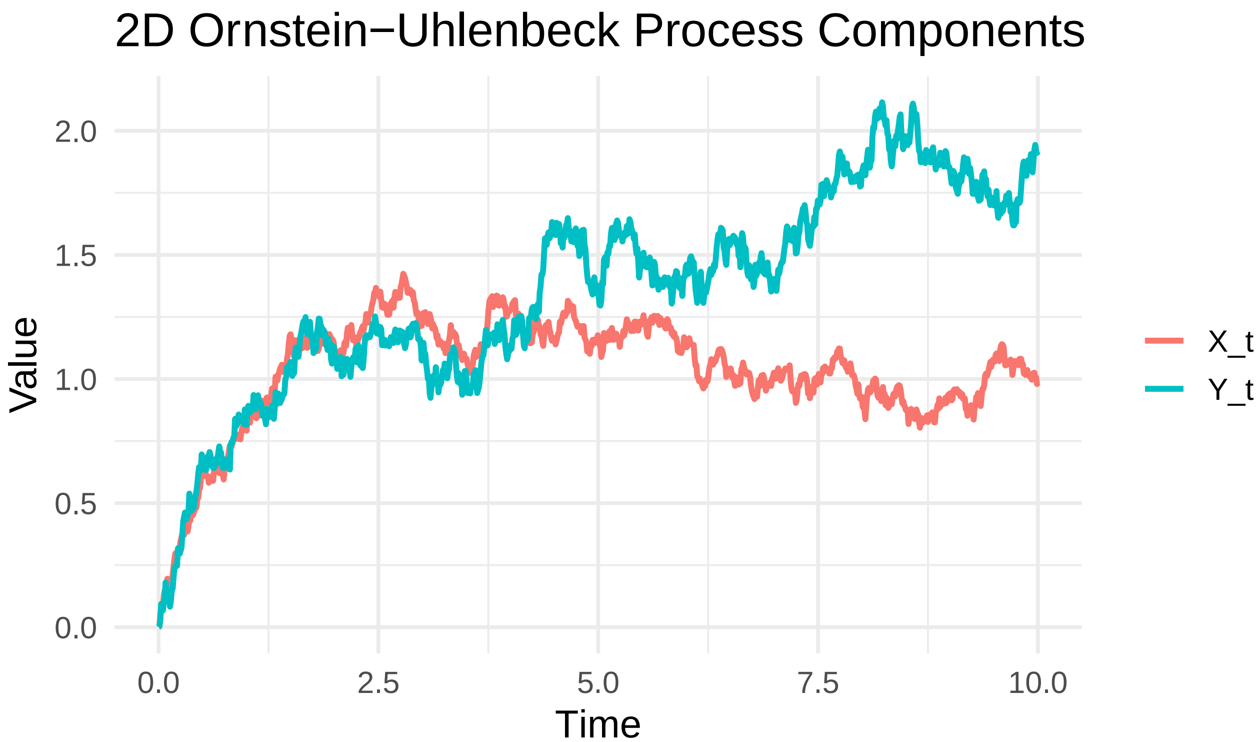
(see Theorem 2.4, Huang & Pang, 2022) `{=latex} \vspace{10pt}` —

## 3. Multivariate OU Process (2D)

We now extend the OU process to two dimensions, where the state vector $\mathbf{X}_t = (X_t, Y_t) \in \mathbb{R}^2$ evolves as:

$$d\mathbf{X}_t = -\Theta(\mathbf{X}_t - \boldsymbol{\mu})\, dt + \Sigma\, d\mathbf{W}_t \tag{4}$$

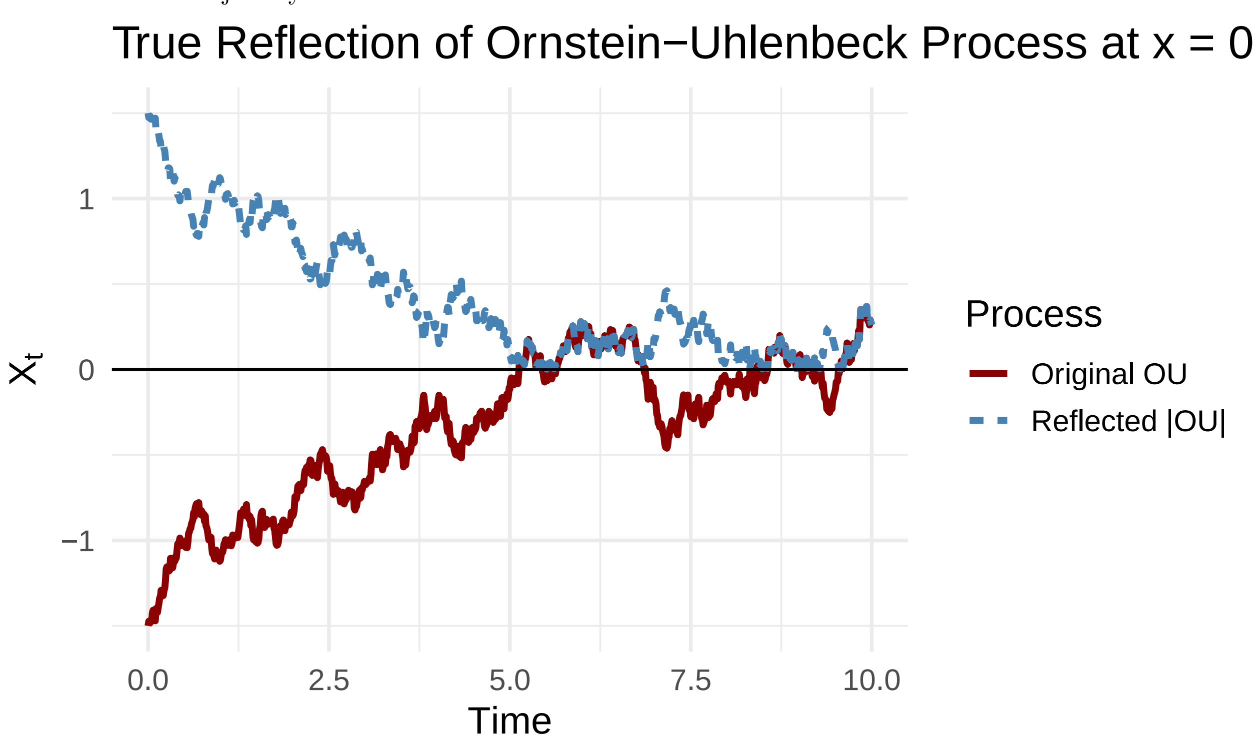
Where: - $\Theta$: mean reversion matrix - $\Sigma$: diffusion matrix - $\boldsymbol{\mu}$: long-run mean vector - $d\mathbf{W}_t$: vector of independent Brownian motions




{=latex} \vspace{10pt}

**4. Reflected Ornstein–Uhlenbeck Process (1D)**

We now simulate a **reflected OU process** in 1D, where the process is constrained to remain in $\mathbb{R}_+$. This imposes a Skorokhod reflection at the boundary $X_t = 0$, approximated here by taking the absolute value of a standard OU trajectory.

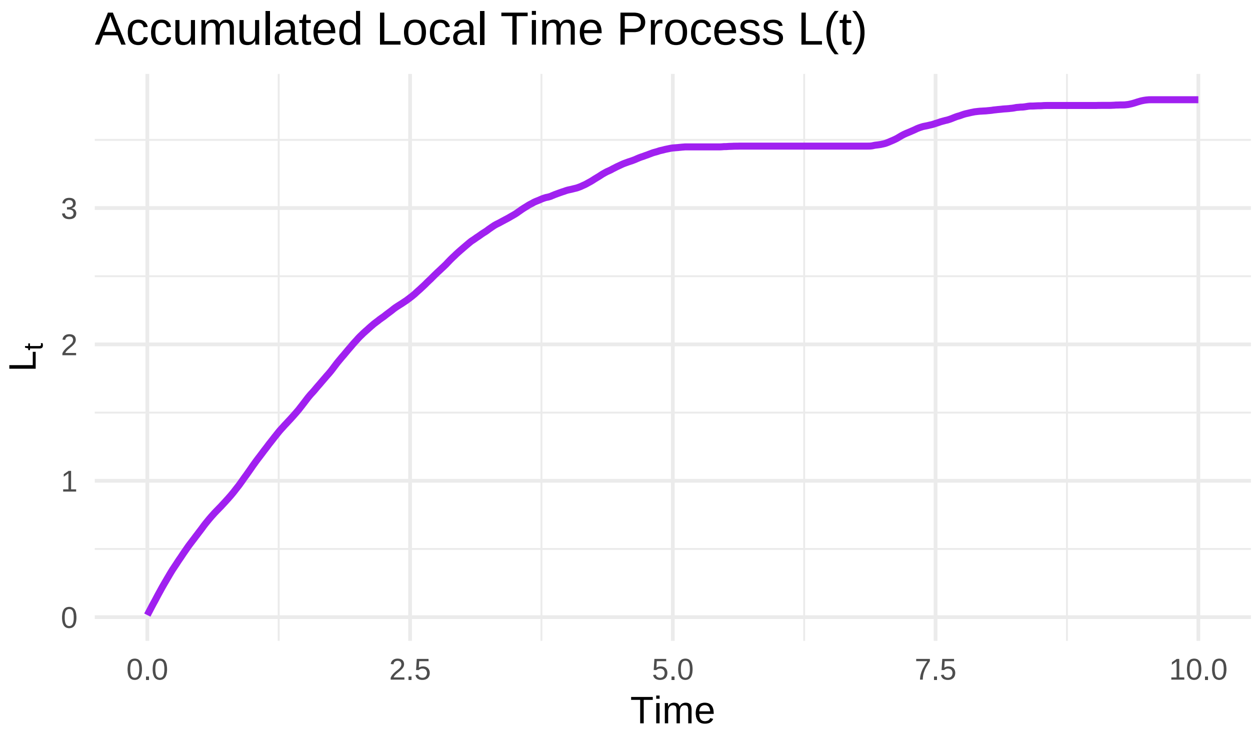


{=latex} \vspace{10pt}

---

#### 5. Local Time Approximation

Following Huang & Pang (2022), the reflected process introduces a local time $L_t$, which increases only when $X_t = 0$. To approximate this numerically, we accumulate the negative excursions of the unreflected path:

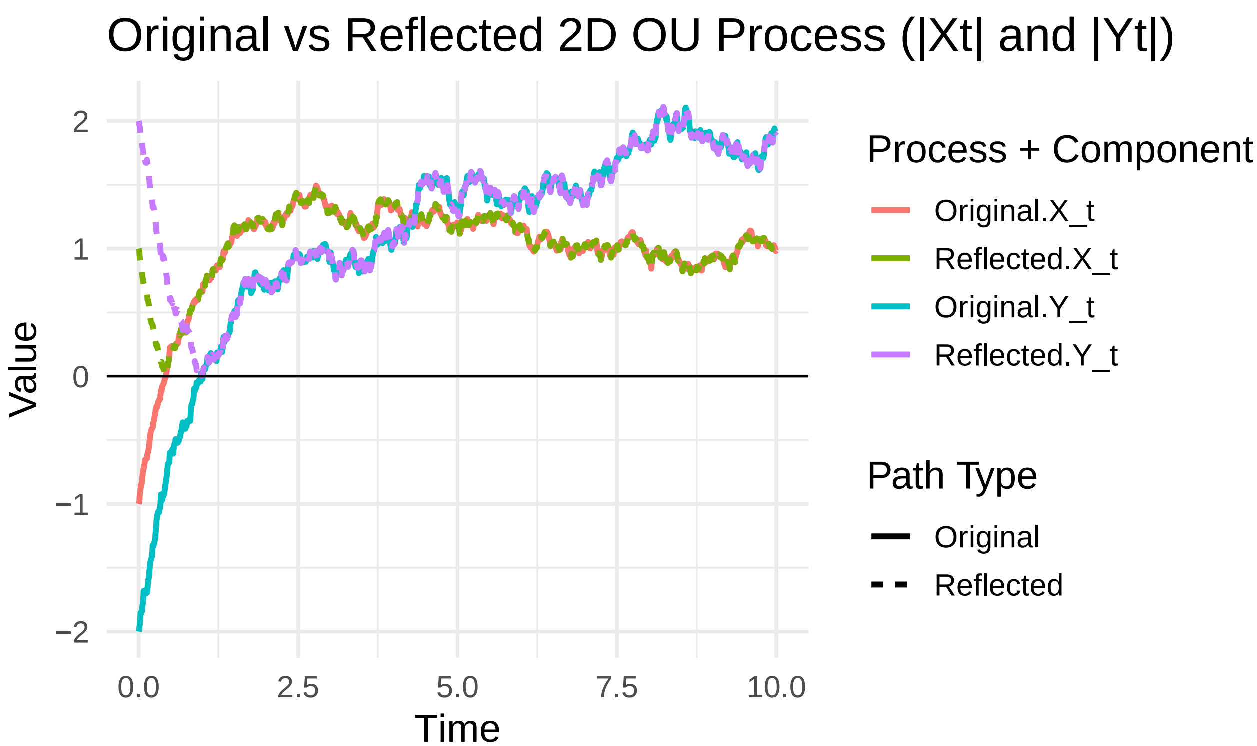



{=latex} \vspace{10pt}

---

#### 6. Reflected OU Process in $\mathbb{R}^2_+$

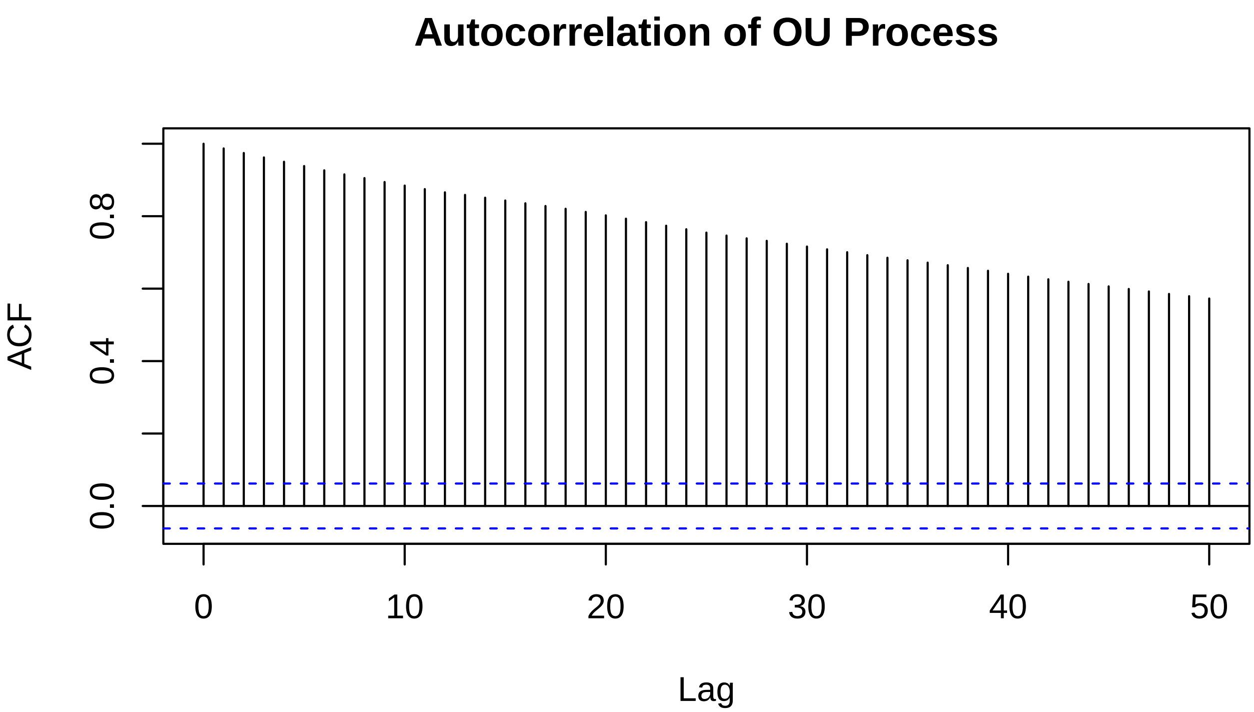
We extend the OU system to 2D and impose reflection into the nonnegative orthant by projecting any negative value to zero.




{=latex} \vspace{10pt}

---

**7. Autocorrelation Function (ACF)**

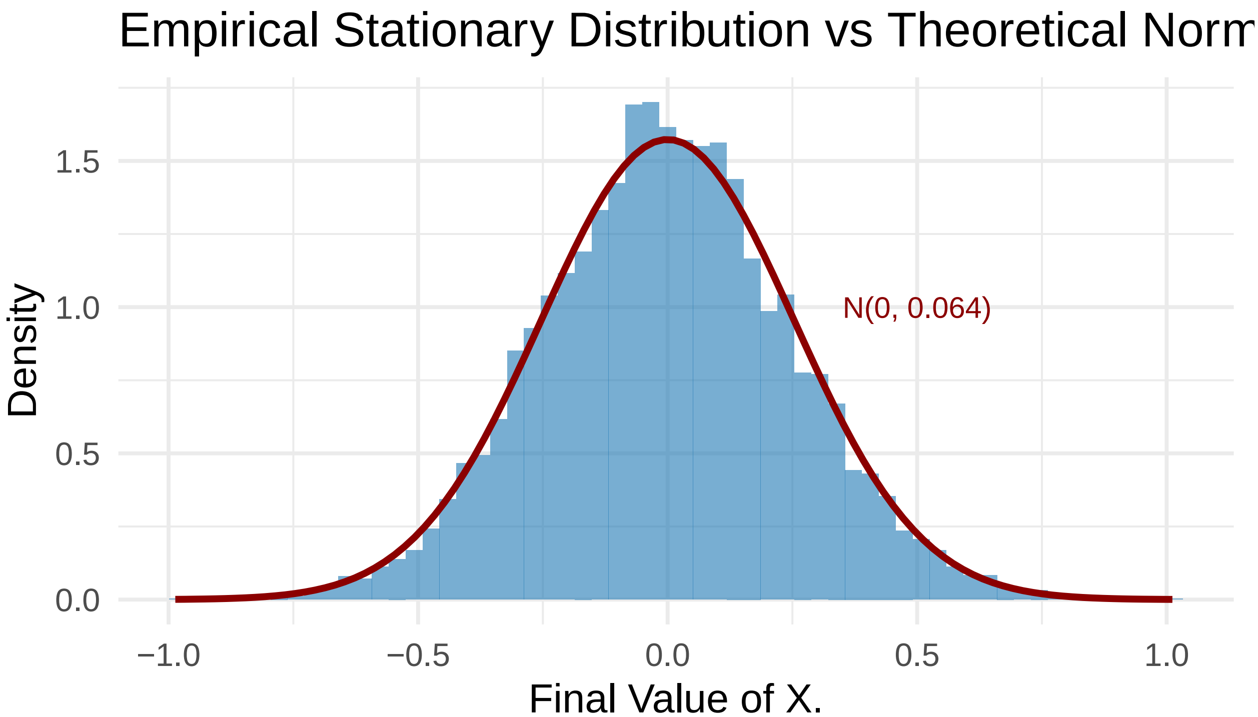



OU processes exhibit exponentially decaying autocorrelations: $\rho(\tau) = e^{-\theta\tau}$.

{=latex} \vspace{10pt}

**8. Stationary Distribution Convergence**

We compare empirical distribution at time $T$ to the theoretical stationary normal:




Huang & Pang (2022) confirm the convergence of the ROU process toward a stationary distribution under bounded local time and ergodicity (Theorem 2.4).