Calculations for the Accelerating Photon BEC Experiment

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1 Calculations

https://fsl.npre.illinois.edu/2010%20NPRE%20421/421-MHD%20Eq.%20&%20Plasma%20Stab..pdf

For a cylindrical plasma microcavity, the following condition needs to hold in order to produce a stable plasma:

$$\frac{r}{\pi}\left(\frac{1}{\mu}\cdot\frac{\partial\mu}{\partial r}\right)+\frac{8\pi}{B_z^2}(\frac{\partial\rho}{\partial r})>0$$

Under the condition that the keep pitch constant, this condition simplifies to:

$$\frac{8\pi}{B_z^2}(\frac{\partial\rho}{\partial r})>0$$

which implies given $\frac{\partial\rho}{\partial r}=k_BT_e\frac{\partial n_e}{\partial r}$ that:

$$\frac{8\pi k_BT}{B_z^2}\frac{\partial n_e}{\partial r}>0$$

Which gives us the very useful information that $\frac{\partial n_e}{\partial r}$ must point in the positive z-dir, assuming a traditional reference frame.
Now,

$$n_e=n_e(0)e^{\frac{e\phi(r)}{k_BT_e}}\implies\frac{\partial n_e}{\partial r}=n_e(0)(\frac{e}{k_BT_e})(\frac{\partial\phi(r)}{\partial r})e^{\frac{e\phi(r)}{k_BT_e}}$$

Now, n_e is physically constrained https://journals.aps.org/pre/supplemental/10.1103/PhysRevE.108.L013201/SM1.pdf
from 10^{12} to 10^{16}. This implies a maximal graident of $\frac{\partial\phi(r)}{\partial r}\approx 10^4$, and under the initial starting density of 10^{12}, we have the following:

$$10^{-8}=(\frac{e}{k_BT_e})(\frac{\partial\phi(r)}{\partial r})e^{\frac{e\phi(r)}{k_BT_e}}\implies 10^{-8}\frac{k_BT_e}{e}e^{-\frac{e\phi(r)}{k_BT_e}}=\frac{\partial\phi(r)}{\partial r}$$

Now T_e and e are on the same order and $\frac{1}{k_B}$ so the exponential is quite small and a taylor series to a first order, giving us a linear solution of $\phi(r)\approx\alpha r+C$ where $\alpha\approx 1.5\cdot 10^7$ and $C\approx -50$ for $T_e=3eV$, implying I can get a stable change in potential of around $1.5kV$!

2 Theory

1. Hamiltonian in Second Quantization

$$H = \int d^2r\, \hat{a}^\dagger(\mathbf{r}) \left[\frac{1}{2m^*}(\mathbf{p} - q\mathbf{A}(\mathbf{r},t))^2 \right] \hat{a}(\mathbf{r})$$

$$H = \int d^2r\, \hat{a}^\dagger(\mathbf{r}) \left[\frac{-\hbar^2}{2m^*}\nabla^2 + \frac{iq\hbar}{m^*}\mathbf{A}(\mathbf{r},t)\cdot\nabla + \frac{q^2}{2m^*}|\mathbf{A}(\mathbf{r},t)|^2 \right] \hat{a}(\mathbf{r})$$

2. Gross-Pitaevskii Equation (GPE)

$$i\hbar\frac{\partial\psi(r,\theta,t)}{\partial t} = \left[g|\psi(r,\theta,t)|^2 - \frac{\hbar^2}{2m^*}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right) + \frac{iq\hbar}{m^*}\mathbf{A}(r,\theta,t)\cdot\nabla + \frac{q^2}{2m^*}|\mathbf{A}(r,\theta,t)|^2 \right]\psi(r,\theta,t)$$

therefore

$$E[\psi] = \int rdrd\theta \left[\frac{\hbar^2}{2m^*}\left(|\frac{\partial\psi}{\partial r}|^2 + \frac{1}{r^2}|\frac{\partial\psi}{\partial\theta}|^2\right) + \frac{g}{2}|\psi|^4 + V_{ext}|\psi|^2 + \frac{iq\hbar}{m^*}\mathbf{A}\cdot\psi^*\nabla\psi + \frac{q^2}{2m^*}|\mathbf{A}|^2|\psi|^2 \right]$$

$$E[\psi] = \int rdrd\theta \left[\frac{\hbar^2}{2m^*}|\frac{\partial\psi}{\partial r}|^2 + \frac{g}{2}|\psi|^4 + V_{ext}|\psi|^2 + \frac{iq\hbar}{m^*}\mathbf{A}\cdot\psi^*\nabla\psi + \frac{q^2}{2m^*}|\mathbf{A}|^2|\psi|^2 \right]$$

The proceeding derivations apply to a 2d case, done in radial coordinates with a corresponding Jacobian of $J(r) = r$

$$E[\psi] = \int rdrd\theta \left[\frac{\hbar^2}{2m^*}|\frac{\partial\psi}{\partial r}|^2 + \frac{g}{2}|\psi|^4 + V_{ext}|\psi|^2 + \frac{iq\hbar}{m^*}\frac{B_0 r}{2}\cdot\psi^*\nabla\psi + \frac{q^2B_0^2r^2}{8m^*}|\psi|^2 \right]$$

with $A = (A_x, A_y) = (-\frac{B_0 y}{2}, \frac{B_0 x}{2})$ and therefore $A = (A_r, A_\theta) = (\frac{B_0 r}{2}, 0)$
let

$$\varphi(r) = (\frac{N}{w^2a^2\pi})^{\frac{1}{2}}\exp[\frac{-r^2}{2wa}]$$

Thus we have

$$\nabla\varphi = \nabla\varphi^* = \frac{\partial}{\partial r}[(\frac{N}{w^2a^2\pi^{3/2}})^{\frac{1}{2}}\exp[\frac{-r^2}{2wa}]] = (\frac{N}{w^2a^2\pi})^{\frac{1}{2}}(\frac{-r}{wa})\exp[\frac{-r^2}{2wa}]$$

$$\implies |\nabla\varphi|^2 = \frac{N}{w^3a^3\pi}\frac{r^2}{w^2a^2}\exp[-\frac{r^2}{wa}]$$

$$\frac{\hbar^2}{2m^*}\int drd\phi(|\nabla\varphi|^2)r = \frac{\hbar^2}{2m^*}\int drd\phi\frac{N}{w^2a^2\pi}\frac{r^3}{w^2a^2}\exp[-\frac{r^2}{wa}]$$

$$= \frac{\hbar^2}{2m^*}\frac{N}{w^2a^2\pi}\frac{(aw)^2}{2(aw)^2}\int d\phi$$

$$= \frac{\hbar^2}{2m^*}\frac{N}{(aw)^2}$$

Now we are defining $V_{ext}(x,y) = \frac{1}{2}m\omega^2(x^2+y^2) = V_{ext}(r) = \frac{1}{2}m\omega^2 r^2$ therefore:

$$\begin{aligned}\int drd\phi d\theta(V_{ext}(r)|\varphi|^2)dr = \int drd\phi\frac{1}{2}m\omega^2 r^3\exp[\frac{-r^2}{wa}]\frac{N}{w^2a^2\pi} \\ = \frac{Nm\omega^2}{w^2a^2\pi}\frac{(aw)^2}{2}\int d\phi \\ = Nm\omega^2\end{aligned}$$

$$\begin{aligned}\int drd\phi(\frac{g}{2}|\varphi|^4)r &= \frac{g}{2}(\frac{N}{w^2a^2\pi})^2(\int drd\phi[r\exp[\frac{-2r}{wa}]]) \\ &= \frac{g}{2}\frac{N^2}{(aw)^4\pi^2}\frac{(aw)}{4}\int d\phi = -\frac{gN^2}{4(aw)^3\pi}\end{aligned}$$

Notice the negative for the potential
and of course:

$$\int drd\phi[\frac{q^2B_0^2r^3}{8m^*}\frac{N}{w^2a^2\pi}\exp[\frac{-r^2}{wa}]] = \frac{Nq^2B_0^2}{8m^*(aw)^2\pi}\frac{(aw)^2}{2}\int d\phi = \frac{q^2B_0^2N}{8m^*}$$

and

$$\int drd\phi[\frac{iq\hbar B_0r^2}{2m^*}\frac{N}{w^2a^2\pi}\exp[\frac{-r^2}{wa}]] = \frac{-iq\hbar B_0N}{4\pi m^*}$$

Which gives us a total functional of:

$$E[w] = \frac{\hbar^2}{2m^*}\frac{N}{(aw)^2} + Nm\omega^2 - \frac{gN^2}{4(aw)^3\pi} + \frac{q^2B_0^2N}{8m^*} - \frac{iq\hbar B_0N}{4\pi m^*}$$

The following is the welll known plasma effect

$$\begin{aligned}\Delta n \approx \frac{-n_0\Delta\epsilon}{2\epsilon_0} \implies \frac{n_0}{2\epsilon_0}\frac{\omega_p^2}{\omega^2+i\gamma\omega} \\ \Delta\epsilon = \frac{\omega_p^2}{\omega^2+i\gamma\omega} \\ \implies \Delta n = \frac{n_0}{2}\frac{1}{\omega^2+i\gamma\omega}\frac{e^2}{\epsilon_0 m^*}\frac{N_q}{A}\end{aligned}$$

Now I want to parameterize $\frac{N_q}{A}$, lets define $n(r) = \frac{N_q}{A}r$ where R is the total radius within the VCSEL.
Now we have:

$$\Delta n = \frac{n_0}{2\epsilon_0}\frac{1}{\omega^2+i\gamma\omega}\frac{e^2}{\epsilon_0 m^*}n(r)$$

My claim is that in a cavity in which a photon BEC forms, the intensity $|\psi|^2$ takes form is directly proportional the refractive index. Therefore we have the following:

$$\Delta n \propto d|\psi^2| = \frac{\partial}{\partial r}|\psi|^2 = \frac{\partial \psi^*}{\partial r}\psi + \psi^* \frac{\partial \psi}{\partial r} = \frac{n_0}{2\epsilon_0}\frac{1}{\omega^2 + i\gamma\omega}\frac{e^2}{\epsilon_0 \mu}n(r)$$

Now what this implies is that if one induces a gradient in the electron like I have proposed:

$$\frac{\partial}{\partial r}[\frac{\partial}{\partial r}|\psi|^2] = \frac{\partial^2 \psi^*}{\partial r^2}\psi + 2|\frac{\partial \psi}{\partial r}|^2 + \psi^* \frac{\partial^2 \psi}{\partial r^2} = \frac{n_0}{2\epsilon_0}\frac{1}{\omega^2 + i\gamma\omega}\frac{e^2}{\epsilon_0 \mu}(\frac{dN_q r}{A} + \frac{N_q}{A})$$

Now $|\psi|^2$ has units of photons per area for the 2d photon bec system, implying we need to add this term to the energy derived from the electrons in this system. Let's solve for dN_q, so that we may solve for energy via density of states

$$([\frac{\partial^2 \psi^*}{\partial r^2}\psi + 2|\frac{\partial \psi}{\partial r}|^2 + \psi^* \frac{\partial^2 \psi}{\partial r^2}]\frac{2(\omega^2 + i\gamma\omega)\mu\epsilon_0(A)}{n_0 e^2} - N_q)\frac{1}{r} = dN_q$$

$$dN_q = \frac{\mu^*}{\pi\hbar^2}dE$$

$$\implies ([\frac{\partial^2 \psi^*}{\partial r^2}\psi + 2|\frac{\partial \psi}{\partial r}|^2 + \psi^* \frac{\partial^2 \psi}{\partial r^2}]\frac{2(\omega^2 + i\gamma\omega)\mu\epsilon_0(A)N_q}{n_0 e^2} - \frac{N_q^2}{2})\frac{1}{r} = \frac{\mu E^*}{\pi\hbar^2}$$

$$\implies E^* = ([\frac{\partial^2 \psi^*}{\partial r^2}\psi + 2|\frac{\partial \psi}{\partial r}|^2 + \psi^* \frac{\partial^2 \psi}{\partial r^2}]\frac{2(\omega^2 + i\gamma\omega)\mu\epsilon_0(A)N_q}{n_0 e^2} - \frac{N_q^2}{2})\frac{\pi\hbar^2}{\mu r}$$

$$E^* = ([\frac{\partial^2 \psi^*}{\partial r^2}\psi + 2|\frac{\partial \psi}{\partial r}|^2 + \psi^* \frac{\partial^2 \psi}{\partial r^2}]\frac{2(\omega^2 + i\gamma\omega)\pi\hbar^2\epsilon_0(A)N_q}{n_0 e^2} - \frac{N_q^2\pi\hbar^2}{2\mu r})$$

`https://pubs.aip.org/aip/apl/article/54/20/1989/55152/Influence-of-band-gap-shrinkage-on-th`
This link provides the finding of refractive index change in GaAs based semiconductors, from there we can derives the next effect for the energy funcion that will be produced via variation in electron density.

$$\Delta n_0 = \frac{-r_0\lambda^2}{2\pi n_0}(\frac{N_q^+}{m_e} + \frac{N_q^-}{m_p})$$

Through the same rigamarole as last time:

$$\Delta n_0 \propto d|\psi|^2 = \frac{\partial \psi^*}{\partial r}\psi + \psi^* \frac{\partial \psi}{\partial r} = \frac{-r_0\lambda^2}{2\pi n_0}(\frac{N_q^-}{m_e} + \frac{N_q^+}{m_p})$$

Now define $\frac{N_q}{\mu} = \frac{N_q^+ m_p + N_q m_e}{m_e + m_p}$. And we parameter N_q again, and apply differentials again

$$\frac{\partial \psi^*}{\partial r}\psi + \psi^* \frac{\partial \psi}{\partial r} = \frac{-r_0\lambda^2}{2\pi n_0}(\frac{N_q r}{\mu R})$$

$$\frac{\partial^2 \psi^*}{\partial r^2}\psi + 2|\frac{\partial \psi}{\partial r}|^2 + \psi^* \frac{\partial^2 \psi}{\partial r^2} = \frac{-r_0\lambda^2}{2\pi n_0}(\frac{dN_q r}{\mu R} + \frac{N_q}{\mu R})$$

$$dN_q = \frac{\mu}{\pi\hbar^2}dE$$

$$([\frac{\partial^2 \psi^*}{\partial r^2}\psi + 2|\frac{\partial \psi}{\partial r}|^2 + \psi^* \frac{\partial^2 \psi}{\partial r^2}]\frac{-2\pi n_0}{r_0\lambda^2}N_q\mu R - \frac{N_q^2}{2})\frac{\pi\hbar^2}{\mu r} = E^*$$

$$E^* = [\frac{\partial^2 \psi^*}{\partial r^2}\psi + 2|\frac{\partial \psi}{\partial r}|^2 + \psi^* \frac{\partial^2 \psi}{\partial r^2}]\frac{-2\pi n_0}{r_0\lambda^2 r}N_q\pi\hbar^2 - \frac{N_q^2\pi\hbar^2}{2\mu r} \text{ similar...}$$

$$E^* = ([\frac{\partial^2 \psi^*}{\partial r^2}\psi + 2|\frac{\partial \psi}{\partial r}|^2 + \psi^* \frac{\partial^2 \psi}{\partial r^2}]\frac{2(\omega^2 + i\gamma\omega)\pi\hbar^2\epsilon_0^2(AR)N_q}{n_0 e^2 r} - \frac{N_q^2\pi\hbar^2}{2\mu r})$$

In short we have an effective energy functional modification of

$$E^* = \frac{\partial^2|\psi|^2}{\partial r^2}(\frac{2(\omega^2 + i\gamma\omega)\epsilon_0^2(AR)}{n_0 e^2 r} - \frac{2\pi n_0}{r_0\lambda^2 r})N_q\pi\hbar^2 - \frac{\pi\hbar^2 N_q^2}{\mu r}$$

Therefore we have a modification to our GPE Hamiltonian via the following:

$$H_{eff} = H_0 + H^*$$

$$H^* = \int r dr d\theta [\frac{d^2|\psi|^2}{dr^2}\frac{2(\omega^2 + i\gamma\omega)\epsilon_0^2(AR)}{n_0 e^2 r} - \frac{2n_0}{r_0\lambda^2 r}N_q\pi\hbar^2 - \frac{\pi\hbar^2 N_q^2}{\mu r}]$$

Therefore let's evaluate the energy functional using this modification: $H^* \cong E[\varphi]^*$ using $\varphi(r) = (\frac{N}{w^2 a^2 \pi})^{\frac{1}{2}}\exp[\frac{-r^2}{2wa}]$ Here we go:

$$\frac{\partial^2|\psi|^2}{\partial r^2} = (\frac{\partial^2 \psi^*}{\partial r^2}\psi + 2|\frac{\partial \psi}{\partial r}|^2 + \psi^* \frac{\partial^2 \psi}{\partial r^2})$$

$$\frac{\partial \psi}{\partial r} = (\frac{N}{(aw)^2\pi})^{1/2}(\frac{-r}{aw})\exp[\frac{-r^2}{2aw}]$$

$$\frac{\partial^2 \psi}{\partial r^2} = (\frac{N}{(aw)^2\pi})(\exp[\frac{-r^2}{aw}])(\frac{-1}{aw} + \frac{r^2}{(aw)^2})$$

letting $C = \frac{2(\omega^2+i\gamma\omega)\epsilon_0 A}{n_0 e^2}$ therefore

$$\int r dr d\theta [\frac{2C}{r}\frac{N}{(aw)^2\pi}\exp[\frac{-r^2}{aw}]][\frac{-1}{aw} + \frac{r^2}{(aw)^2}]$$

$$= \int r dr d\theta [\frac{4CN}{(}\frac{-\sqrt{\pi}}{(aw)^{5/2}} + \frac{\sqrt{\pi}}{(aw)^{5/2}}L)$$

$$= \frac{4CN\sqrt{\pi}}{(aw)^{5/2}} = \frac{8(\omega^2 + i\gamma\omega)\epsilon_0 A\sqrt{\pi}N_{ph}}{n_0 e^2 L(aw)^{5/2}}$$

Unit check:

$$\frac{\frac{1}{s^2}\frac{C^2}{Nm^2}m^2(Js)^2}{C^2m} = \frac{1}{Nm}J^2 = J$$

Now our true total function is

$$E[w] = \frac{\hbar^2}{2m^*}\frac{N}{(aw)^2} + Nm\omega^2 - \frac{gN^2}{4(aw)^3\pi} + \frac{q^2B_0^2N}{8m^*} - \frac{iq\hbar B_0N}{4\pi m^*} + \frac{8(\omega^2+i\gamma\omega)\epsilon_0A\sqrt{\pi}N_{ph}}{n_0e^2L(aw)^{5/2}}$$

Given electron density modulation and

$$E[w] = \frac{\hbar^2}{2m^*}\frac{N}{(aw)^2} + Nm\omega^2 - \frac{gN^2}{4(aw)^3\pi} + \frac{q^2B_0^2N}{8m^*} - \frac{iq\hbar B_0N}{4\pi m^*}$$

for without

However, this is all done neglecting finite size effects, unfortunately, we cannot do this whatsoever due to the following:

$$\begin{aligned}
&\text{Assumptions:}\\
&\text{VCSEL cavity length: } L \sim 10\,\mu m\\
&\text{photon wavelength: } \lambda \approx 850\,nm\\
&\text{condensate temperature: } T \sim 298
\end{aligned}$$

$$\begin{aligned}
\lambda_{dB} &= \frac{h}{\sqrt{2\pi m_{\text{photon}}k_BT}}\\
&\text{effective mass of photon:}\\
m_{\text{photon}} &= \frac{h\nu}{c^2} = \frac{h}{\lambda c}\\
m_{\text{photon}} &\approx \frac{6.626\times10^{-34}J\cdot s}{(0.85\times10^{-6}m)(3\times10^8m/s)} \approx 2.6\times10^{-36}kg\\
&\text{de Broglie wavelength at a room temp:}\\
\lambda_{dB} &\approx \frac{6.626\times10^{-34}J\cdot s}{\sqrt{2\pi(2.6\times10^{-36}kg)(1.381\times10^{-23}J/K)(298K)}}\\
\lambda_{dB} &\approx 4.5\times10^{-6}m = 4.5\,\mu m
\end{aligned}$$

Clearly we are working on very comparable characteristic length scales here.

$$\int_0^R rdrd\theta[\frac{g|\varphi|^4}{2}] = \frac{g}{2}(\frac{N}{(aw)^2\pi})^2\int_0^R\int_0^{2\pi} drd\theta r\exp[\frac{-2r^2}{aw}] = \frac{g}{2}(\frac{N}{(aw)^2\pi})^2(\frac{aw}{4} - \frac{(aw)\exp[\frac{-2R^2}{aw}]}{4})$$
$$= \frac{gN^2}{8\pi(aw)}(1-\exp[\frac{-2R^2}{aw}])$$

$$\int_0^R rdrd\theta[\frac{\hbar^2}{2m^*}|\frac{\partial\varphi}{\partial r}|^2] = \int_0^R [\frac{\hbar^2}{2m^*}\frac{r^2}{(aw)^2}\frac{N}{(aw)^3\pi}\exp[-\frac{r^2}{2aw}]]$$

$$= \frac{N\hbar^2}{2m^*(aw)^3}\frac{1}{(aw)^2}(\frac{(aw)^2}{2}(1-\exp[-\frac{R^2}{aw}])] + \frac{awR^2}{2})$$

$$= \frac{N\hbar^2}{2m^*}[\frac{1-\exp[-\frac{R^2}{aw}]}{(aw)^3} + \frac{R^2}{2(aw)^4}]$$

$$\int_0^R rdrd\theta[(\frac{m\omega^2 r^2}{2})\exp[\frac{-r^2}{aw}](\frac{N}{(aw)^2\pi})] = \frac{N}{(aw)^2\pi}\frac{m^*\omega^2}{2}$$

$$= \frac{N}{(aw)^2\pi}\frac{m^*\omega^2}{2}(2\pi)(\frac{(aw)^2}{2}(1-\exp[-\frac{R^2}{aw}])] + \frac{awR^2}{2})$$

$$= Nm^*\omega^2[\frac{(1-\exp[-\frac{R^2}{aw}])}{2} + \frac{R^2}{2(aw)}]$$

$$\Delta n = \frac{\alpha}{2\epsilon_0}n$$

$$\Delta n = n_2 I = \frac{3\chi^{(3)}}{4\epsilon_0 c n_0^2}I$$

We estirmate g as:

$$g = \frac{3\hbar\omega\chi^{(3)}}{2\epsilon_0^2 c n_0^2 \alpha}$$

$$\alpha \sim 4\pi\epsilon_0\epsilon_b a_0^3$$

$$g \approx \frac{3}{4}\frac{10^{-34}10^{12}10^{-10}}{2(10^{-11})^2(3\cdot 10^8)10^{-2}} \approx 1.25\cdot 10^{-15}$$

$$\frac{(cn^{-1})}{L} \implies \omega \approx 10^{13}$$

and estimate mass with

$$\approx \frac{\hbar c}{\lambda} \approx 10^{-36}$$

Now I solved for the only critical point w.r.t N of our function and I got $N \approx 3.75\cdot 10^4$ which roughly agrees with what has been verified experimentally thus far.

Now let's see what happens to N critical after the application of the linear gradient in electron density.

letting $C = \frac{2(\omega^2+i\gamma\omega)\epsilon_0 A}{n_0 e^2}$ therefore

$$\int rdrd\theta[\frac{2C}{r}\frac{N}{(aw)^2\pi}\exp[\frac{-r^2}{aw}]][\frac{-1}{aw} + \frac{r^2}{(aw)^2}]$$

$$= \int_0^R dr[\frac{-4CN}{(aw)^3 r}\exp[\frac{-r^2}{aw}]]] + \int_0^R dr[\frac{4CNr}{(aw)^4}\exp[\frac{-r^2}{aw}]]]$$

Let's evaluate the first, let $B = \frac{-4NC}{(aw)^3}$, and $c = \frac{1}{aw}$

$$\begin{aligned}
\int_0^R [B\frac{\exp[-cr^2]}{r}] &= B\int_0^R \frac{1}{r}\sum_{n=1}^{\infty}\frac{(-cr^2)^n}{n!}dx = B\int_0^R \frac{1}{r}\sum_{n=0}^{\infty}\frac{(-1)^n c^n r^{2n}}{n!}dx \\
&= B\sum_{n=0}^{\infty}\frac{(-1)^n c^n}{n!}\int_0^R r^{2n-1}dr = B\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{1}{(aw)^n}\int_0^R r^{2n-1}dr \\
&= B\int_0^R \frac{1}{r}dr + B\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\frac{1}{(aw)^n}\int_0^R r^{2n-1}dr
\end{aligned}$$

Use the typical trick to set our potential to 0 near that singularity.

$$\begin{aligned}
& B\ln[R] + B\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\frac{1}{(aw)^n}\int_0^R r^{2n-1}dr \\
&= B\ln[R] + B\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\frac{1}{(aw)^n}\int_0^R r^{2n-1}dr \\
&= B\ln[R] + B\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\frac{1}{(aw)^n}\frac{R^2}{2n} \\
&= B\ln[R] + B\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\frac{1}{(aw)^n}\frac{R^2}{2n} = B\ln[R] + \frac{B}{2}\sum_{n=1}^{\infty}\frac{(-\frac{R^2}{aw})^n}{n\cdot n!} \\
&= B\ln[R] + \frac{B}{2}(E_1[\frac{R^2}{aw}] + \gamma + \ln[\frac{R^2}{aw}]) = -\frac{4NC}{(aw)^3}[\ln[R] + \frac{1}{2}(E_1[\frac{R^2}{aw}] + \gamma + \ln[\frac{R^2}{aw}])] \\
&= -\frac{4NC}{(aw)^3}[\ln[\frac{R^2}{\sqrt{aw}}] + \frac{1}{2}(E_1[\frac{R^2}{aw}] + \gamma)]
\end{aligned}$$

Where γ is the euler-mascheroni constant. Now for the next integral:

$$\begin{aligned}
\int_0^R dr[\frac{4CNr}{(aw)^4}\exp[\frac{-r^2}{aw}]] &= \frac{4NC}{(aw)^4}(\frac{aw}{2}(1-\exp[-\frac{R^2}{aw}])) \\
&= \frac{2NC}{(aw)^3}(1-\exp[-\frac{R^2}{aw}])
\end{aligned}$$

Now we can formulate a full energy functional with finite size effects accounted for:

$$\begin{aligned}
E[w] = \frac{gN^2}{8\pi(aw)}(1-\exp[\frac{-2R^2}{aw}]) &+ \frac{N\hbar^2}{2m^*}[\frac{1-\exp[-\frac{R^2}{aw}]}{(aw)^3} + \frac{R^2}{2(aw)^4}] \\
&+ Nm^*\omega^2[\frac{(1-\exp[-\frac{R^2}{aw}])}{2} + \frac{R^2}{2(aw)}] \\
-\frac{4NC}{(aw)^3}[\ln[\frac{R^2}{\sqrt{aw}}] + \frac{1}{2}(E_1[\frac{R^2}{aw}] + \gamma)] &+ \frac{2NC}{(aw)^3}(1-\exp[-\frac{R^2}{aw}])
\end{aligned}$$

After yet again finding critical values of N for some fixed w, the estimated critical photon number explodes to 10^{78} photons needed for condensation (unphysically large), which implies a collapse occured after the modification of the potential in this way.