Esin-Refael Research

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Assume via a second quantization model of interacting system between plasmons and photons via a dipole moment. The dipole moment arises from the geometry of the wave function in the scenario.

$$\hat{H} = \sum_i \hbar\omega_{pl} b_i^\dagger b_i + \sum_k \hbar\omega_k \left(a_{R,k}^\dagger a_{R,k} + a_{L,k}^\dagger a_{L,k} \right) + d \sum_i \sum_k \hat{d}_i \cdot \mathbf{E}_k \left(b_i^\dagger + \text{h.c.} \right)$$

$$\mathbf{E}_k = i\sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}} \left[(\hat{\mathbf{x}} - i\hat{\mathbf{y}}) + (\hat{\mathbf{x}} + i\hat{\mathbf{y}}) - \text{h.c.} \right]$$

$$H_{int} = d \sum_i \sum_k \hat{d}_i \cdot \mathbf{E}_k \left(a_{R,k} b_i^\dagger + a_{L,k} b_i^\dagger + \text{h.c.} \right)$$

$$\propto i\sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}} \left[(\hat{\mathbf{x}} - i\hat{\mathbf{y}})\, a_{R,k} + (\hat{\mathbf{x}} + i\hat{\mathbf{y}})\, a_{L,k} - \text{h.c.} \right] \left(a_{R,k} b_i^\dagger + a_{L,k} b_i^\dagger + \text{h.c.} \right)$$

$$\hat{d}_i \cdot \mathbf{E}_k = i\sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}} \left[\hat{d}_{ix} \left(a_{R,k} + a_{L,k} - a_{R,k}^\dagger - a_{L,k}^\dagger \right) - i\hat{d}_{iy} \left(a_{R,k} - a_{L,k} - a_{R,k}^\dagger + a_{L,k}^\dagger \right) \right]$$

Assume transverse dipole moment:

$$H_{int} = d \sum_i \sum_k i\sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}} \left[\hat{d}_{ix} \left(a_{R,k} + a_{L,k} - a_{R,k}^\dagger - a_{L,k}^\dagger \right) - i\hat{d}_{iy} \left(a_{R,k} - a_{L,k} - a_{R,k}^\dagger + a_{L,k}^\dagger \right) \right] \times \left(a_{R,k} b_i^\dagger + a_{L,k} b_i^\dagger + a_{R,k}^\dagger b_i + a_{L,k}^\dagger b_i \right)$$

Let's apply the following structure:

$$H = \begin{pmatrix} \hbar\omega_{pl_1} & 0 & 0 & V_{11} & V_{12} \\ 0 & \hbar\omega_{pl_2} & 0 & V_{21} & V_{22} \\ 0 & 0 & \hbar\omega_{pl_3} & V_{31} & V_{32} \\ V_{11}^* & V_{21}^* & V_{31}^* & \hbar\omega_{k_1} & 0 \\ V_{12}^* & V_{22}^* & V_{32}^* & 0 & \hbar\omega_{k_2} \end{pmatrix}$$

Let's define the following to make this tractable:

$$A = \begin{pmatrix} \hbar\omega_{pl_1} & 0 & 0 \\ 0 & \hbar\omega_{pl_2} & 0 \\ 0 & 0 & \hbar\omega_{pl_3} \end{pmatrix}, B = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \\ V_{31} & V_{32} \end{pmatrix}, C = \begin{pmatrix} V_{11}^* & V_{21}^* & V_{31}^* \\ V_{12}^* & V_{22}^* & V_{32}^* \end{pmatrix}$$

and

$$\begin{pmatrix} \hbar\omega_{k_1} & 0 \\ 0 & \hbar\omega_{k_2} \end{pmatrix}$$

giving

$$H = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

Now modify A and D for diagonalization:

$$A = \begin{pmatrix} \hbar\omega_{pl_1} - \lambda & 0 & 0 \\ 0 & \hbar\omega_{pl_2} - \lambda & 0 \\ 0 & 0 & \hbar\omega_{pl_3} - \lambda \end{pmatrix},$$

$$D = \begin{pmatrix} \hbar\omega_{k_1} - \lambda & 0 \\ 0 & \hbar\omega_{k_2} - \lambda \end{pmatrix}$$

now $\det(H) = \det(A)\det(D - CA^{-1}B)$ if A^{-1} exists. Trivially, A^{-1} both exists and is easily calculated as:

$$A^{-1} = \begin{pmatrix} \frac{1}{\hbar\omega_{pl_1}-\lambda} & 0 & 0 \\ 0 & \frac{1}{\hbar\omega_{pl_2}-\lambda} & 0 \\ 0 & 0 & \frac{1}{\hbar\omega_{pl_3}-\lambda} \end{pmatrix}$$

It's clear to see that $\det(A) = (\hbar\omega_{pl_1} - \lambda)(\hbar\omega_{pl_2} - \lambda)(\hbar\omega_{pl_3} - \lambda)$ Now let's compute $CA^{-1}B$.

$$\begin{aligned} CA^{-1}B &= \begin{pmatrix} V_{11}^* & V_{21}^* & V_{31}^* \\ V_{12}^* & V_{22}^* & V_{32}^* \end{pmatrix} \begin{pmatrix} \frac{1}{\hbar\omega_{pl_1}-\lambda} & 0 & 0 \\ 0 & \frac{1}{\hbar\omega_{pl_2}-\lambda} & 0 \\ 0 & 0 & \frac{1}{\hbar\omega_{pl_3}-\lambda} \end{pmatrix} \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \\ V_{31} & V_{32} \end{pmatrix} \\ &= \begin{pmatrix} V_{11}^* & V_{21}^* & V_{31}^* \\ V_{12}^* & V_{22}^* & V_{32}^* \end{pmatrix} \begin{pmatrix} \frac{V_{11}}{\hbar\omega_{pl_1}-\lambda} & \frac{V_{12}}{\hbar\omega_{pl_1}-\lambda} \\ \frac{V_{21}}{\hbar\omega_{pl_2}-\lambda} & \frac{V_{22}}{\hbar\omega_{pl_2}-\lambda} \\ \frac{V_{31}}{\hbar\omega_{pl_2}-\lambda} & \frac{V_{32}}{\hbar\omega_{pl_2}-\lambda} \end{pmatrix} \\ &= \begin{pmatrix} \frac{|V_{11}|^2}{\hbar\omega_{pl_1}-\lambda} + \frac{|V_{21}|^2}{\hbar\omega_{pl_2}-\lambda} + \frac{|V_{31}|^2}{\hbar\omega_{pl_3}-\lambda} & \frac{V_{11}^* V_{12}}{\hbar\omega_{pl_1}-\lambda} + \frac{V_{21}^* V_{22}}{\hbar\omega_{pl_1}-\lambda} + \frac{V_{31}^* V_{32}}{\hbar\omega_{pl_1}-\lambda} \\ \frac{V_{11} V_{12}^*}{\hbar\omega_{pl_1}-\lambda} + \frac{V_{21} V_{22}^*}{\hbar\omega_{pl_1}-\lambda} + \frac{V_{31} V_{32}^*}{\hbar\omega_{pl_1}-\lambda} & \frac{|V_{12}|^2}{\hbar\omega_{pl_1}-\lambda} + \frac{|V_{22}|^2}{\hbar\omega_{pl_2}-\lambda} + \frac{|V_{32}|^2}{\hbar\omega_{pl_3}-\lambda} \end{pmatrix} \end{aligned}$$

Therefore $D - CA^{-1}B$ is

$$= \begin{pmatrix} \hbar\omega_{k_1} - \lambda - \frac{|V_{11}|^2}{\hbar\omega_{pl_1}-\lambda} + \frac{|V_{21}|^2}{\hbar\omega_{pl_2}-\lambda} + \frac{|V_{31}|^2}{\hbar\omega_{pl_3}-\lambda} & \frac{V_{11}^* V_{12}}{\hbar\omega_{pl_1}-\lambda} + \frac{V_{21}^* V_{22}}{\hbar\omega_{pl_2}-\lambda} + \frac{V_{31}^* V_{32}}{\hbar\omega_{pl_3}-\lambda} \\ \frac{V_{11} V_{12}^*}{\hbar\omega_{pl_1}-\lambda} + \frac{V_{21} V_{22}^*}{\hbar\omega_{pl_2}-\lambda} + \frac{V_{31} V_{32}^*}{\hbar\omega_{pl_3}-\lambda} & \hbar\omega_{k_1} - \lambda - \frac{|V_{12}|^2}{\hbar\omega_{pl_1}-\lambda} + \frac{|V_{22}|^2}{\hbar\omega_{pl_2}-\lambda} + \frac{|V_{32}|^2}{\hbar\omega_{pl_3}-\lambda} \end{pmatrix}$$

For simplicity, let's define:

$$
\begin{aligned}
\alpha_1 &= \hbar\omega_{k_1} - \lambda - \left(\frac{|V_{11}|^2}{\hbar\omega_{pl_1} - \lambda} + \frac{|V_{21}|^2}{\hbar\omega_{pl_2} - \lambda} + \frac{|V_{31}|^2}{\hbar\omega_{pl_3} - \lambda} \right) \\
\alpha_2 &= \hbar\omega_{k_2} - \lambda - \left(\frac{|V_{12}|^2}{\hbar\omega_{pl_1} - \lambda} + \frac{|V_{22}|^2}{\hbar\omega_{pl_2} - \lambda} + \frac{|V_{32}|^2}{\hbar\omega_{pl_3} - \lambda} \right) \\
\beta &= -\left(\frac{V_{11}^* V_{12}}{\hbar\omega_{pl_1} - \lambda} + \frac{V_{21}^* V_{22}}{\hbar\omega_{pl_2} - \lambda} + \frac{V_{31}^* V_{32}}{\hbar\omega_{pl_3} - \lambda} \right) \\
\gamma &= -\left(\frac{V_{11} V_{12}^*}{\hbar\omega_{pl_1} - \lambda} + \frac{V_{21} V_{22}^*}{\hbar\omega_{pl_2} - \lambda} + \frac{V_{31} V_{32}^*}{\hbar\omega_{pl_3} - \lambda} \right)
\end{aligned}
$$

Therefore, the determinant is

$$
\det(D - CA^{-1}B) = \alpha_1\alpha_2 - \beta\gamma
$$

$$
\begin{aligned}
\det(H) = \det(A)\det(D - CA^{-1}B) = 0 \\
= (\hbar\omega_{pl_1} - \lambda)(\hbar\omega_{pl_2} - \lambda)(\hbar\omega_{pl_3} - \lambda)(\alpha_1\alpha_2 - \beta\gamma) = 0
\end{aligned}
$$

Which implies the only nontrivial eigenvalues are from $(\alpha_1\alpha_2 - \beta\gamma)$
Let's make the simplifying assumption that all these coupling are approximately the same...
Such that $V_{11} = V_{32} = ... = V$

$$
\begin{aligned}
\alpha_1 &= \hbar\omega_{k_1} - \lambda - |V|^2 \left(\frac{1}{\hbar\omega_{pl_1} - \lambda} + \frac{1}{\hbar\omega_{pl_2} - \lambda} + \frac{1}{\hbar\omega_{pl_3} - \lambda} \right) \\
\alpha_2 &= \hbar\omega_{k_2} - \lambda - |V|^2 \left(\frac{1}{\hbar\omega_{pl_1} - \lambda} + \frac{1}{\hbar\omega_{pl_2} - \lambda} + \frac{1}{\hbar\omega_{pl_3} - \lambda} \right) \\
\beta &= -|V|^2 \left(\frac{1}{\hbar\omega_{pl_1} - \lambda} + \frac{1}{\hbar\omega_{pl_2} - \lambda} + \frac{1}{\hbar\omega_{pl_3} - \lambda} \right) \\
\gamma &= -|V|^2 \left(\frac{1}{\hbar\omega_{pl_1} - \lambda} + \frac{1}{\hbar\omega_{pl_2} - \lambda} + \frac{1}{\hbar\omega_{pl_3} - \lambda} \right)
\end{aligned}
$$

Clearly the last term from the expansion cancels giving the following:

$$
\begin{aligned}
&(\alpha_1\alpha_2 - \beta\gamma) \\
&= (\hbar\omega_{k_1} - \lambda)(\hbar\omega_{k_2} - \lambda) - (\hbar\omega_{k_1} + \hbar\omega_{k_2} - 2\lambda) \left(\frac{1}{\hbar\omega_{pl_1} - \lambda} + \frac{1}{\hbar\omega_{pl_2} - \lambda} + \frac{1}{\hbar\omega_{pl_3} - \lambda} \right) \\
&= (\hbar\omega_{k_1} - \lambda)(\hbar\omega_{k_2} - \lambda) - (\hbar\omega_{k_1} + \hbar\omega_{k_2} - 2\lambda) \left(\frac{\sum_{i=1}^{3} \hbar\omega_{pl} n_i - 3\lambda}{(\hbar\omega_{pl_1} - \lambda)(\hbar\omega_{pl_2} - \lambda)(\hbar\omega_{pl_3} - \lambda)} \right)
\end{aligned}
$$

$$
= \frac{(\hbar\omega_{k_1} - \lambda)(\hbar\omega_{k_2} - \lambda)(\hbar\omega_{pl_1} - \lambda)(\hbar\omega_{pl_2} - \lambda)(\hbar\omega_{pl_3} - \lambda) - (\hbar\omega_{k_1} + \hbar\omega_{k_2} - 2\lambda) \left(\sum_{i=1}^{3} \hbar\omega_{pl} n_i - 3\lambda \right)}{(\hbar\omega_{pl_1} - \lambda)(\hbar\omega_{pl_2} - \lambda)(\hbar\omega_{pl_3} - \lambda)})
$$

Which then cancels with the det[A]

$$(\hbar\omega_{k_1}-\lambda)(\hbar\omega_{k_2}-\lambda)(\hbar\omega_{pl_1}-\lambda)(\hbar\omega_{pl_2}-\lambda)(\hbar\omega_{pl_3}-\lambda)-(\hbar\omega_{k_1}+\hbar\omega_{k_2}-2\lambda)\left(\sum_{i=1}^{3}\hbar\omega_{pl_i}-3\lambda\right) = 0$$

Which gives

$$\begin{aligned}
&-2\lambda^5 \\
&+\left[2\left(\omega_{pl_1}+\omega_{pl_2}+\omega_{pl_3}\right)+\hbar\left(\omega_{k_1}+\omega_{k_2}\right)\right]\lambda^4 \\
&-\Big[2\left(\omega_{pl_1}\omega_{pl_2}+\omega_{pl_1}\omega_{pl_3}+\omega_{pl_2}\omega_{pl_3}\right)+\hbar^2\left(\omega_{k_1}+\omega_{k_2}\right)\left(\omega_{pl_1}+\omega_{pl_2}+\omega_{pl_3}\right) \\
&\qquad -\hbar^3\left(\omega_{k_1}+\omega_{k_2}\right)\left(\omega_{pl_1}\omega_{pl_2}+\omega_{pl_1}\omega_{pl_3}+\omega_{pl_2}\omega_{pl_3}\right)+6\Big]\lambda^3 \\
&+\Big[\hbar^3\omega_{k_1}\omega_{k_2}\left(\omega_{pl_1}+\omega_{pl_2}+\omega_{pl_3}\right)+\hbar^4\left(\omega_{k_1}+\omega_{k_2}\right)\left(\omega_{pl_1}\omega_{pl_2}\omega_{pl_3}\right) \\
&\qquad -2\hbar^3\left(\omega_{pl_1}\omega_{pl_2}\omega_{pl_3}\right)-3\hbar\left(\omega_{k_1}+\omega_{k_2}\right)-2\hbar\left(\omega_{pl_1}+\omega_{pl_2}+\omega_{pl_3}\right)\Big]\lambda^2 \\
&-\Big[\hbar^4\omega_{k_1}\omega_{k_2}\left(\omega_{pl_1}\omega_{pl_2}+\omega_{pl_1}\omega_{pl_3}+\omega_{pl_2}\omega_{pl_3}\right)-\hbar^4\left(\omega_{k_1}+\omega_{k_2}\right)\left(\omega_{pl_1}\omega_{pl_2}\omega_{pl_3}\right)\Big]\lambda \\
&+\hbar^5\omega_{k_1}\omega_{k_2}\omega_{pl_1}\omega_{pl_2}\omega_{pl_3}-\hbar^2\left(\omega_{k_1}+\omega_{k_2}\right)\left(\omega_{pl_1}+\omega_{pl_2}+\omega_{pl_3}\right)=0
\end{aligned}$$

Assume to linear $\hbar$ order, which is similar to the QED expansion technique, we get:

$$\begin{aligned}
&-2\lambda^5 \\
&+\left[2\left(\omega_{pl_1}+\omega_{pl_2}+\omega_{pl_3}\right)+\hbar\left(\omega_{k_1}+\omega_{k_2}\right)\right]\lambda^4 \\
&-\left[2\left(\omega_{pl_1}\omega_{pl_2}+\omega_{pl_1}\omega_{pl_3}+\omega_{pl_2}\omega_{pl_3}\right)+6\right]\lambda^3 \\
&+\left[-3\hbar\left(\omega_{k_1}+\omega_{k_2}\right)-2\hbar\left(\omega_{pl_1}+\omega_{pl_2}+\omega_{pl_3}\right)\right]\lambda^2 \\
=\;& 0
\end{aligned}$$

which we can then solve, or

We may redefine the Hamiltonian in terms of momentum coordinates, q_x, q_y for the plasmons and k_x, k_y for the photons:

$$H=\begin{pmatrix} 0 & q_x-iq_y & 0 & V_{11} & V_{12} \\ q_x-iq_y & 0 & q_x+iq_y & V_{21} & V_{22} \\ 0 & q_x+iq_y & 0 & V_{31} & V_{32} \\ V_{11}^* & V_{21}^* & V_{31}^* & c\sqrt{q_x^2+q_y^2} & 0 \\ V_{12}^* & V_{22}^* & V_{32}^* & 0 & c\sqrt{q_x^2+q_y^2} \end{pmatrix}$$

Now define

$$H-\lambda I=\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with

$$A = \begin{pmatrix} -\lambda & q_x - iq_y & 0 \\ q_x + iq_y & -\lambda & q_x - iq_y \\ 0 & q_x + iq_y & -\lambda \end{pmatrix},$$
$$B = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \\ V_{31} & V_{32} \end{pmatrix},$$
$$C = \begin{pmatrix} V_{11}^* & V_{21}^* & V_{31}^* \\ V_{12}^* & V_{22}^* & V_{32}^* \end{pmatrix},$$
$$D = \begin{pmatrix} c\sqrt{q_x^2 + q_y^2} - \lambda & 0 \\ 0 & c\sqrt{q_x^2 + q_y^2} - \lambda \end{pmatrix}.$$

$$BD^{-1}C = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \\ V_{31} & V_{32} \end{pmatrix} \begin{pmatrix} \frac{1}{c\sqrt{q_x^2+q_y^2}-\lambda} & 0 \\ 0 & \frac{1}{c\sqrt{q_x^2+q_y^2}-\lambda} \end{pmatrix} \begin{pmatrix} V_{11}^* & V_{21}^* & V_{31}^* \\ V_{12}^* & V_{22}^* & V_{32}^* \end{pmatrix}$$

$$BD^{-1}C = K \begin{pmatrix} |V_{11}|^2 + |V_{12}|^2 & V_{11}V_{21}^* + V_{12}V_{22}^* & V_{11}V_{31}^* + V_{12}V_{32}^* \\ V_{21}V_{11}^* + V_{22}V_{12}^* & |V_{21}|^2 + |V_{22}|^2 & V_{21}V_{31}^* + V_{22}V_{32}^* \\ V_{31}V_{11}^* + V_{32}V_{12}^* & V_{31}V_{21}^* + V_{32}V_{22}^* & |V_{31}|^2 + |V_{32}|^2 \end{pmatrix}$$

Make the simplifying assumption that the coupling strengths are approximately equal:

$$A - BD^{-1}C = \begin{pmatrix} -\lambda - 2K|V|^2 & q_x - iq_y - 2K|V|^2 & -2K|V|^2 \\ q_x + iq_y - 2K|V|^2 & -\lambda - 2K|V|^2 & q_x - iq_y - 2K|V|^2 \\ -2K|V|^2 & q_x + iq_y - 2K|V|^2 & -\lambda - 2K|V|^2 \end{pmatrix}$$
$$A - BD^{-1}C = \begin{pmatrix} a & b - 2K|V|^2 & -2K|V|^2 \\ b^* - 2K|V|^2 & a & b - 2K|V|^2 \\ -2K|V|^2 & b^* - 2K|V|^2 & a \end{pmatrix}$$
$$\text{with } a = -\lambda - 2K|V|^2, b = q_x - iq_y$$
$$\det(A - BD^{-1}C) = a(a^2 - (b - 2K|V|^2)(b^* - 2K|V|^2))$$
$$-(b - 2K|V|^2)((b^* - 2K|V|^2)a + 2K|V|^2(b - 2K|V|^2))$$
$$-2K|V|^2((b^* - 2K|V|^2)^2 - a(-2K|V|^2))$$

$$\begin{aligned}
\det(A - BD^{-1}C) &= a(a^2 - (b - 2K|V|^2)(b^* - 2K|V|^2)) \\
&\quad - (b - 2K|V|^2)((b^* - 2K|V|^2)a + 2K|V|^2(b - 2K|V|^2)) \\
&\quad - 2K|V|^2((b^* - 2K|V|^2)^2 - a(-2K|V|^2)) \\
&= a^3 - a(bb^* - 2K|V|^2 b - 2K|V|^2 b^* + 4K^2|V|^4) \\
&\quad - bb^* a + 2K|V|^2 ba + 2K|V|^2 b^* a - 4K^2|V|^4 a \\
&\quad - 2K|V|^2 bb^* + 4K^2|V|^4 b + 4K^2|V|^4 b^* - 8K^3|V|^6 \\
&\quad + 2K|V|^2 a^2 + 4K^2|V|^4 a
\end{aligned}$$

$$\begin{aligned}
\det(A - BD^{-1}C) = -\lambda^3 &+ \frac{6\lambda^2|V|^2}{c\sqrt{q_x^2+q_y^2} - \lambda} - \frac{12\lambda|V|^4}{(c\sqrt{q_x^2+q_y^2} - \lambda)^2} + \frac{8|V|^6}{(c\sqrt{q_x^2+q_y^2} - \lambda)^3} \\
&+ \lambda(q_x^2+q_y^2) - \frac{2|V|^2(q_x^2+q_y^2)}{c\sqrt{q_x^2+q_y^2} - \lambda} + \frac{4\lambda|V|^2(q_x^2+q_y^2)}{(c\sqrt{q_x^2+q_y^2} - \lambda)^2} \\
&- \frac{8|V|^4(q_x^2+q_y^2)}{(c\sqrt{q_x^2+q_y^2} - \lambda)^3} + \lambda(q_x^2+q_y^2) - \frac{2|V|^2(q_x^2+q_y^2)}{c\sqrt{q_x^2+q_y^2} - \lambda} \\
&+ \frac{4\lambda|V|^2(q_x^2+q_y^2)}{(c\sqrt{q_x^2+q_y^2} - \lambda)^2} - \frac{8|V|^4(q_x^2+q_y^2)}{(c\sqrt{q_x^2+q_y^2} - \lambda)^3} - \frac{2|V|^2(q_x^2+q_y^2)}{c\sqrt{q_x^2+q_y^2} - \lambda} \\
&+ \frac{4|V|^4(q_x - iq_y)}{c\sqrt{q_x^2+q_y^2} - \lambda} + \frac{4|V|^4(q_x + iq_y)}{c\sqrt{q_x^2+q_y^2} - \lambda} - \frac{8|V|^6}{(c\sqrt{q_x^2+q_y^2} - \lambda)^3} \\
&+ \frac{2\lambda|V|^2}{c\sqrt{q_x^2+q_y^2} - \lambda} - \frac{8|V|^4}{(c\sqrt{q_x^2+q_y^2} - \lambda)^2} + \frac{16|V|^6}{(c\sqrt{q_x^2+q_y^2} - \lambda)^3} \\
&- \frac{4\lambda|V|^4}{(c\sqrt{q_x^2+q_y^2} - \lambda)^2} + \frac{16|V|^6}{(c\sqrt{q_x^2+q_y^2} - \lambda)^3}
\end{aligned}$$

Taking the limit of $q = \sqrt{q_x^2+q_y^2} \to 0$

$$\begin{aligned}
\det(A - BD^{-1}C) = -\lambda^3 - 6\lambda|V|^2 &+ \frac{12|V|^4}{\lambda} - \frac{8|V|^6}{\lambda^3} \\
&- \frac{4|V|^4(q_x - iq_y)}{\lambda} - \frac{4|V|^4(q_x + iq_y)}{\lambda} + \frac{8|V|^6}{\lambda^3} \\
&- 2|V|^2 + \frac{8|V|^4}{\lambda^2} - \frac{16|V|^6}{\lambda^3} \\
&+ \frac{4|V|^4}{\lambda} - \frac{16|V|^6}{\lambda^3}
\end{aligned}$$

$$\det(A - BD^{-1}C) = -\lambda^3 - 6\lambda|V|^2 + \frac{16|V|^4}{\lambda} - \frac{8|V|^4 q_x}{\lambda}$$
$$- 2|V|^2 + \frac{8|V|^4}{\lambda^2} - \frac{32|V|^6}{\lambda^3}$$

We may redefine the Hamiltonian in terms of momentum coordinates, q_x, q_y for the plasmons and k_x, k_y for the photons, considering one mode:

$$H = \begin{pmatrix} 0 & q_x - iq_y & 0 & V_1 \\ q_x - iq_y & 0 & q_x + iq_y & V_2 \\ 0 & q_x + iq_y & 0 & V_3 \\ V_1^* & V_2^* & V_3^* & c\sqrt{q_x^2 + q_y^2} \end{pmatrix}$$

The dispersion relation for the photon uses the same momentum coordinates since the photon is assumed to be near resonance with the plasmons.

Now define

$$H - \lambda I = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with

$$A = \begin{pmatrix} -\lambda & q_x - iq_y & 0 \\ q_x + iq_y & -\lambda & q_x - iq_y \\ 0 & q_x + iq_y & -\lambda \end{pmatrix},$$
$$B = \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix},$$
$$C = \begin{pmatrix} V_1^* & V_2^* & V_3^* \end{pmatrix},$$
$$D = \left(c\sqrt{q_x^2 + q_y^2} - \lambda \right).$$

$$BD^{-1}C = \begin{pmatrix} V_1^* \\ V_2^* \\ V_3^* \end{pmatrix} \left(\frac{1}{c\sqrt{q_x^2+q_y^2}-\lambda} \right) \begin{pmatrix} V_1 & V_2 & V_3 \end{pmatrix}$$

$$BD^{-1}C = \frac{1}{c\sqrt{q_x^2 + q_y^2} - \lambda} \begin{pmatrix} |V_1|^2 & V_1^* V_2 & V_1^* V_3 \\ V_2^* V_1 & |V_2|^2 & V_2^* V_3 \\ V_3^* V_1 & V_3^* V_2 & |V_3|^2 \end{pmatrix}$$

Assume simplifying assumption of $V_1 = V_2 = V_3 = V$ and take the $q =$

$\sqrt{q_x^2+q_y^2} \to 0$ limit:

$$-\frac{|V|^2}{c\sqrt{q_x^2+q_y^2}-\lambda}\begin{pmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{pmatrix}$$

leaving us with $A - BD^{-1}C$ as

$$= \begin{pmatrix} -\lambda - \frac{|V|^2}{c\sqrt{q_x^2+q_y^2}-\lambda} & q_x - iq_y - \frac{|V|^2}{c\sqrt{q_x^2+q_y^2}-\lambda} & -\frac{|V|^2}{c\sqrt{q_x^2+q_y^2}-\lambda} \\ q_x + iq_y - \frac{|V|^2}{c\sqrt{q_x^2+q_y^2}-\lambda} & -\lambda - \frac{|V|^2}{c\sqrt{q_x^2+q_y^2}-\lambda} & q_x - iq_y - \frac{|V|^2}{c\sqrt{q_x^2+q_y^2}-\lambda} \\ -\frac{|V|^2}{c\sqrt{q_x^2+q_y^2}-\lambda} & q_x + iq_y - \frac{|V|^2}{c\sqrt{q_x^2+q_y^2}-\lambda} & -\lambda - \frac{|V|^2}{c\sqrt{q_x^2+q_y^2}-\lambda} \end{pmatrix}$$

Whose determinant simplifies to (via algebraic calculator assuming real values for all variables except V):

$$\begin{aligned} &\frac{4Vq_y^2\overline{V}}{c\sqrt{q_x^2+q_y^2}-\lambda} - \frac{c\lambda^3\sqrt{q_x^2+q_y^2}}{c\sqrt{q_x^2+q_y^2}-\lambda} \\ &+\frac{\lambda^4}{c\sqrt{q_x^2+q_y^2}-\lambda} + \lambda^2[-\frac{3V\overline{V}}{c\sqrt{q_x^2+q_y^2}-\lambda} - \frac{2q_x^2}{c\sqrt{q_x^2+q_y^2}-\lambda} \\ &-\frac{2q_y^2}{c\sqrt{q_x^2+q_y^2}-\lambda}] \\ &+\lambda[-\frac{4Vq_x\overline{V}}{c\sqrt{q_x^2+q_y^2}-\lambda} \\ &+\frac{2cq_x^2\sqrt{q_x^2+q_y^2}}{c\sqrt{q_x^2+q_y^2}-\lambda} + \frac{2cq_y^2\sqrt{q_x^2+q_y^2}}{c\sqrt{q_x^2+q_y^2}-\lambda}] \end{aligned}$$

Mutliplying by the determinant of D gives:

$$\begin{aligned} 4q_y^2|V|^2 - c\lambda^3\sqrt{q_x^2+q_y^2} + \lambda^4 + \lambda^2[-3|V|^2 - 2(q_x^2+q_y^2)]+ \\ \lambda[-4|V|^2q_x - 2c(q_x^2+q_y^2)\sqrt{q_x^2+q_y^2}] \end{aligned}$$

Now we take the $q = \sqrt{q_x^2+q_y^2} \to 0$ limit which necessarily implies $q_y^2 \to 0$ and $q^2 \to 0$ giving us:

$$\lambda^4 - 3|V|^2\lambda^2 - 4|V|^2q_x\lambda$$
$$\lambda(\lambda^3 - 3|V|^2\lambda - 4|V|^2q_x) = 0$$

This has the form of a depressed cubic:

$$\lambda(\lambda^3 + p\lambda + q) = 0$$

with $p = -3|V|^2$ and $q = -4|V|^2 q_x$ Purely real expression may be obtained for this cubic since $4p^3 + 27q^2 = -108|V|^6 + 108|V|^4 q_x^2 < 0$ for this limit, giving us three purely real eigenvalues of

$$\lambda_k = 2\sqrt{|V|^2}\cos(\frac{1}{3}\arccos(2q_x\sqrt{\frac{1}{|V|^2}} - \frac{2\pi k}{3}))$$

with $k \in 0, 1, 2$ and an eigenvalue of 0