Question # 1

(a) [3]

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 9 \end{bmatrix}$$

To find the inverse using inversion algorithm, we have

$$\left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 2 & 9 & 0 & 1 \end{array}\right]$$

$R_2 - 2R_1$

$$\left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right] \quad , \quad E_1 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

$R_1 - 4R_2$

$$\left[\begin{array}{cc|cc} 1 & 0 & 9 & -4 \\ 0 & 1 & -2 & 1 \end{array}\right] \quad E_2 = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad A^{-1} = \begin{bmatrix} 9 & -4 \\ -2 & 1 \end{bmatrix}.$$

$$A^{-1} = E_2 E_1$$

$$= \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

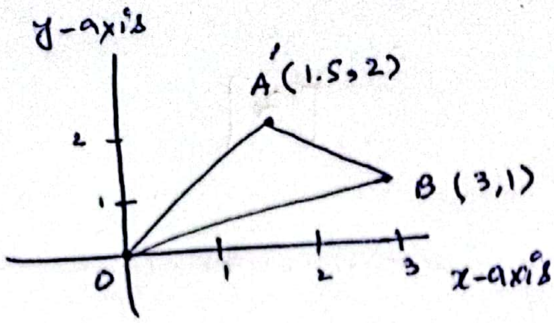
$$= \begin{bmatrix} 1+8 & 0-4 \\ 0-2 & 0+1 \end{bmatrix} = \begin{bmatrix} 9 & -4 \\ -2 & 1 \end{bmatrix}$$

(b) [2]

$$E_1^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \quad , \quad E_2^{-1} = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$$

$$A = E_1^{-1} E_2^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+0 & 4 \\ 2+0 & 8+0 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 9 \end{bmatrix}$$

(c) [3]

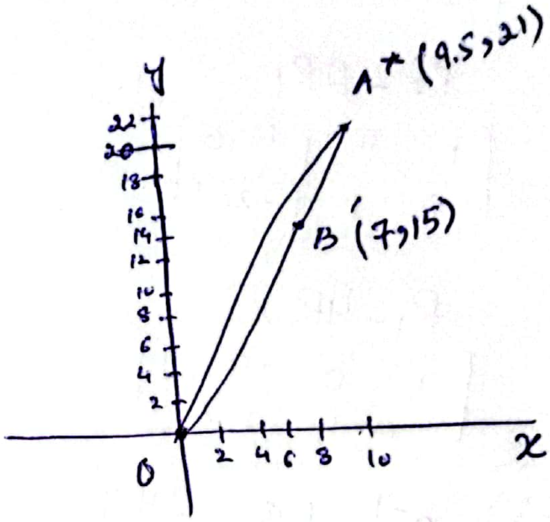


To find the image of triangle under multiplication,

$$\begin{bmatrix} 1 & 4 \\ 2 & 9 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 \\ 2 & 9 \end{bmatrix}\begin{bmatrix} 1.5 \\ 2 \end{bmatrix} = \begin{bmatrix} 9.5 \\ 21 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 \\ 2 & 9 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 15 \end{bmatrix}$$



[5]

(d)

(i) $E_2^{-1} = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$

Shear in the x-direction with a factor of 4

$E_1^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

Shear in the +ve y-direction by factor 2.

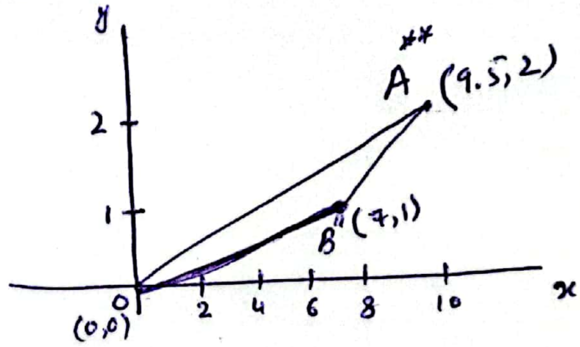
(ii)

$$E_2^{-1} = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$$

$$E_2^{-1}O = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$E_2^{-1}A' = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1.5 \\ 2 \end{bmatrix} = \begin{bmatrix} 1.5+8 \\ 0+2 \end{bmatrix} = \begin{bmatrix} 9.5 \\ 2 \end{bmatrix}$$

$$E_2^{-1}B = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3+4 \\ 0+1 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$$

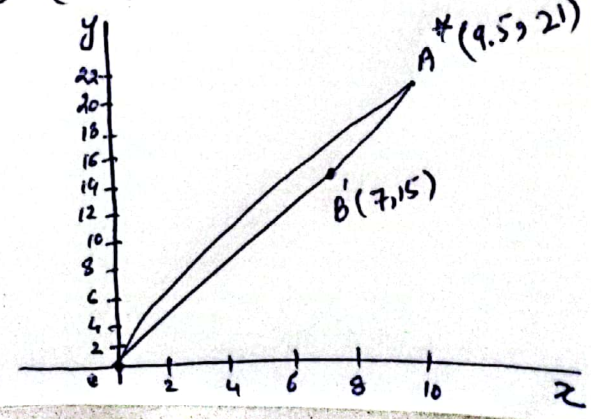


$$E_1^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$E_1^{-1}O = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$E_1^{-1}A^{**} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 9.5 \\ 2 \end{bmatrix} = \begin{bmatrix} 9.5+0 \\ 19+2 \end{bmatrix} = \begin{bmatrix} 9.5 \\ 21 \end{bmatrix}$$

$$E_1^{-1}B'' = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 7 \\ 1 \end{bmatrix} = \begin{bmatrix} 7+0 \\ 14+1 \end{bmatrix} = \begin{bmatrix} 7 \\ 15 \end{bmatrix}$$



[2]

(e)

We have seen that in part (d) the triangle is undergoing shear in x-direction and y-direction, i.e.,

$= E_1^{-1} E_2^{-1}$

$= A$ (from part b)

It can also be observed clearly that when matrix A is applied on the vertices of triangle, we get the image obtained in part (c).

(f) [5]

$y = -4x+3$

Let (x, y) be the point on $y=-4x+3$ and (x', y') be its image under multiplication by A

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 9 \end{bmatrix}^{-1} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$= \begin{bmatrix} 9 & -4 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 9x' - 4y' \\ -2x' + y' \end{bmatrix}$$

$\Rightarrow x = 9x' - 4y', \quad y = -2x' + y'$

Substitute these expressions in $y = -4x+3$

$-2x' + y' = -4(9x' - 4y') + 3 \Rightarrow -2x' + y' = -36x' + 16y' + 3$

$16y' - y' = 36x' - 2x' - 3 \Rightarrow 15y' = 34x' - 3$

$\boxed{y' = \frac{34}{15}x' - \frac{1}{5}}$ → Image of the given line.

Q No. 2 (a)

$u+v=(u_1+v_1-3,\ u_2+v_2-3)$

$Ku=(Ku_1,\ 0)$

Part (i)

$u+0=u$

Let $0=(a_1,\ a_2)$

$u+0=(u_1+a_1-3,\ u_2+a_2-3)=(u_1,u_2)$

$\Rightarrow u_1+a_1-3=u_1$ | $u_2+a_2-3=u_2$

$\Rightarrow a_1-3=0$ | $a_2-3=0$

$a_1=3$ | $a_2=3$

$\Rightarrow 0=(a_1,a_2)=(3,3)$

$\Rightarrow$ Identity element exist.

axiom 4 holds.

Q 2 (a)

Part (ii) For $u = (u_1, u_2)$

$$u + (-u) = 0$$

Here $0 = (3,3)$

Let $-u = (b_1, b_2)$ be the additive inverse of u, Then

$$u + (-u) = (u_1 + b_1 - 3, u_2 + b_2 - 3) = (3,3)$$

$\Rightarrow$

$u_1 + b_1 - 3 = 3$	$u_2 + b_2 - 3 = 3$
$b_1 = 6 - u_1$	$b_2 = 6 - u_2$

$$\Rightarrow -u = (6 - u_1, 6 - u_2)$$

Hence additive inverse $(-u)$ exist.

Q. No. 2(a)

Part (iii)

Axiom 8

$(K+m)u = Ku + mu.$

L.H.S $(K+m)u = (K+m)(u_1, u_2)$

$= ((K+m)u_1, 0)$

$= (Ku_1 + mu_1, 0)$ —— ①

R.H.S $Ku + mu = K(u_1, u_2) + m(u_1, u_2)$

$= (Ku_1, 0) + (mu_1, 0)$

$= Ku_1 + mu_1 - 3, 0+0-3)$

$= (Ku_1 + mu_1 - 3, -3)$ —— ②

From (i) and (2)

$(K+m)u \neq Ku + mu$

Axiom 8 fails.

Axiom 10

$1 \cdot u = u$.

$L.H.S = 1 \cdot u = 1 \cdot (u_1, u_2)$

$= (1 \cdot u_1, 0)$.

$= (u_1, 0) \neq (u_1, u_2)$

$\neq u$.

Hence axiom ~~fails~~ 10 fails.

Q 2 (b)

$B = \{u_1, u_2\}$ $\quad u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; u_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$B' = \{u_1', u_2'\}$ $\quad u_1' = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \; u_2' = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

Find (i) $P_{B \to B'}$

(ii) $P_{B' \to B}$

Sol:~ For part (i)

$$[B' \mid B] = [\text{New basis} \mid \text{old basis}]$$

$$= \left[\begin{array}{cc|cc} 3 & 2 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right] \quad R_{12}$$

$$= \left[\begin{array}{cc|cc} 1 & -1 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{array}\right] \quad R_2 - 3R_1$$

$$= \left[\begin{array}{cc|cc} 1 & -1 & 0 & 1 \\ 0 & 5 & 1 & -3 \end{array}\right] \quad \frac{1}{5} R_2$$

$$= \left[\begin{array}{cc|cc} 1 & -1 & 0 & 1 \\ 0 & 1 & 1/5 & -3/5 \end{array}\right] \quad R_1 + R_2$$

$$= \left[\begin{array}{cc|cc} 1 & 0 & 1/5 & 2/5 \\ 0 & 1 & 1/5 & -3/5 \end{array}\right]$$

Hence

$$\boxed{P_{B \to B'} = \begin{bmatrix} 1/5 & 2/5 \\ 1/5 & -3/5 \end{bmatrix}}$$

Q 2C

$$A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -2 & 2 \\ 3 & 6 & -3 & 3 \\ 4 & 8 & -4 & 4 \end{bmatrix}$$

$R_2 - 2R_1$ $R_3 - 3R_1$ $R_4 - 4R_1$

$$\begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ — } 1$$

Now

Basis for row space $= (1, 2, -1, 1)$

Basis for the column space $= (1, 2, 3, 4)$

For Null space

$$x + 2y - z + w = 0$$

$$x = -2y + z - w$$

Let $y = t_1$

$z = t_2$

$w = t_3$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = t_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t_3 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Basis for the null space

$$\{(-2, 1, 0, 0), (1, 0, 1, 0), (-1, 0, 0, 1)\}$$

From (1)

Rank = 1

nullity = 3

Q#3(a) Find Eigenvalues

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & -1 \\ -2 & -4 & 4 \end{bmatrix}$$

Sol:- Here $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Multiplying by λ

$$\lambda I = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

We know that

$$\det(\lambda I - A) = 0$$

So, $\lambda I - A = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & -1 \\ -2 & -4 & 4 \end{bmatrix}$

$$= \begin{bmatrix} \lambda-2 & 0 & 0 \\ -1 & \lambda-4 & 1 \\ 2 & 4 & \lambda-4 \end{bmatrix}$$

$$\det(\lambda I - A) = \begin{vmatrix} \lambda-2 & 0 & 0 \\ -1 & \lambda-4 & 1 \\ 2 & 4 & \lambda-4 \end{vmatrix} = 0$$

Expanding by 1^{st} Row

$$\Rightarrow (\lambda-2)\left[(\lambda-4)^2 - 4\right] = 0$$

$$\Rightarrow (\lambda-2)\left[\lambda^2 - 2(\lambda)(4) + (4)^2 - 4\right] = 0$$

$$\Rightarrow (\lambda-2)\left[\lambda^2 - 8\lambda + 16 - 4\right] = 0$$

$$\Rightarrow (\lambda-2)\left[\lambda^2 - 8\lambda + 12\right] = 0$$

$\lambda - 2 = 0$, $\lambda^2 - 8\lambda + 12 = 0$

$\boxed{\lambda = 2}$, $\lambda^2 - 6\lambda - 2\lambda + 12 = 0$

$\lambda(\lambda - 6) - 2(\lambda - 6) = 0$

$(\lambda - 6)(\lambda - 2) = 0$

$\lambda - 6 = 0$, $\lambda - 2 = 0$

$\boxed{\lambda = 6}$, $\boxed{\lambda = 2}$

The Eigenvalue $\lambda = 2$ has algebraic Multiplicity $= 2$ & the eigenvalue $\lambda = 6$ has algebraic multiplicity $= 1$

<u>For Eigen Space :-</u>

Consider $(\lambda I - A) X = 0$

$$\begin{bmatrix} \lambda - 2 & 0 & 0 \\ -1 & \lambda - 4 & 1 \\ 2 & 4 & \lambda - 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

For $\lambda = 2$

$$\begin{bmatrix} 0 & 0 & 0 \\ -1 & -2 & 1 \\ 2 & 4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now, convert it into Echleon Form

Take,
$$\begin{bmatrix} 0 & 0 & 0 \\ -1 & -2 & 1 \\ 2 & 4 & -2 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -2 & 1 \\ 2 & 4 & -2 \\ 0 & 0 & 0 \end{bmatrix} \quad R_1 \leftrightarrow R_2, \; R_2 \leftrightarrow R_3$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ 0 & 0 & 0 \end{bmatrix} \quad (-1)R_1$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad R_2 - 2R_1$$

So, $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$x_1 + 2x_2 - x_3 = 0 \longrightarrow (i)$$

x_1 is leading variable

$\boxed{x_2 = s}$, $\boxed{x_3 = t}$

Put in (i)

$$x_1 + 2s - t = 0$$

$$\boxed{x_1 = -2s + t}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2s+t \\ s \\ t \end{bmatrix}$$

$$= s \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Here are two eigen vectors corresponding to eigen value $\boxed{\lambda = 2}$

The geometric multiplicity of eigenvalue is defined as dimension of eigen space associated with that eigen value.

Eigen vectors are

$\boxed{\lambda_1 = 2}$ $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

The dimension is 2.

The Geometric multiplicity of $\lambda = 2$ is 2.

For $\lambda = 6$

$$\begin{bmatrix} \lambda-2 & 0 & 0 \\ -1 & \lambda-4 & 1 \\ 2 & 4 & \lambda-4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 0 & 0 \\ -1 & 2 & 1 \\ 2 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Take,

$$\begin{bmatrix} -1 & 2 & 1 \\ 4 & 0 & 0 \\ 2 & 4 & 2 \end{bmatrix} \quad R_1 \approx R_2$$

$$\begin{bmatrix} 1 & -2 & -1 \\ 4 & 0 & 0 \\ 2 & 4 & 2 \end{bmatrix} \quad (-1) R_1$$

$$\begin{bmatrix} 1 & -2 & -1 \\ 0 & 8 & 4 \\ 0 & 8 & 4 \end{bmatrix} \quad \begin{matrix} R_2 - 4R_1 \\ R_3 - 2R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & -2 & -1 \\ 0 & 8 & 4 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 - R_2$$

$$\begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix} \quad \frac{1}{8} R_2$$

$$x_1 - 2x_2 - x_3 = 0 \longrightarrow (ii)$$

$$x_2 + \frac{1}{2} x_3 = 0 \longrightarrow (iii)$$

$$\boxed{x_3 = t}$$

$$\boxed{x_2 = -\frac{1}{2} t}$$

Put values in (ii)

$$x_1 - 2\left(\frac{-1}{2}t\right) - t = 0$$

$$x_1 + \cancel{t} - \cancel{t} = 0$$

$$\boxed{x_1 = 0}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{2}t \\ t \end{bmatrix} = t\begin{bmatrix} 0 \\ -1/2 \\ 1 \end{bmatrix}$$

The dimension is 1.

The Geometric multiplicity of $\lambda = 6$ is 1.

(b) Determine whether A is diagonalizable?

Matrix A is diagonalizable if the geometric multiplicity of each eigen value is equal to algebraic multiplicity.

In our case, it is hold.

To find matrix P, write basis vectors in column form.

$$P_1 = \begin{bmatrix} 0 \\ -1/2 \\ 1 \end{bmatrix}, \quad P_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \quad P_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} P_1 & P_2 & P_3 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & -2 & 1 \\ -1/2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

(c) Prove that $P^{-1}AP = D$

Now, we find P^{-1}

$$\left[\begin{array}{ccc|ccc} 0 & -2 & 1 & 1 & 0 & 0 \\ -1/2 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ -1/2 & 1 & 0 & 0 & 1 & 0 \\ 0 & -2 & 1 & 1 & 0 & 0 \end{array}\right] \quad R_3 \approx R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1/2 & 0 & 1 & 1/2 \\ 0 & -2 & 1 & 1 & 0 & 0 \end{array}\right] \quad R_2 + \frac{1}{2} R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1/2 & 0 & 1 & 1/2 \\ 0 & 0 & 2 & 1 & 2 & 1 \end{array}\right] \quad R_3 + 2R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1/2 & 0 & 1 & 1/2 \\ 0 & 0 & 1 & 1/2 & 1 & 1/2 \end{array}\right] \quad \frac{1}{2} R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1/4 & 1/2 & 1/4 \\ 0 & 0 & 1 & 1/2 & 1 & 1/2 \end{array}\right] \quad R_2 - \frac{1}{2} R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/2 & -1 & 1/2 \\ 0 & 1 & 0 & -1/4 & 1/2 & 1/4 \\ 0 & 0 & 1 & 1/2 & 1 & 1/2 \end{array}\right] \quad R_1 - R_3$$

$$\left[\begin{array}{c|c} I & P^{-1} \end{array}\right]$$

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$$P^{-1} = \begin{bmatrix} -1/2 & -1 & 1/2 \\ -1/4 & 1/2 & 1/4 \\ 1/2 & 1 & 1/2 \end{bmatrix}$$

Now, $P^{-1}AP = D$

$$P^{-1}AP = \begin{bmatrix} -1/2 & -1 & 1/2 \\ -1/4 & 1/2 & 1/4 \\ 1/2 & 1 & 1/2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & -1 \\ -2 & -4 & 4 \end{bmatrix} \begin{bmatrix} 0 & -2 & 1 \\ -1/2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & -6 & 3 \\ -1/2 & 1 & 1/2 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & -2 & 1 \\ -1/2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = D$$

where diagonal enteries are the eigen values of matrix A.

(d) Check that Matrix A & $P^{-1}AP$ have same trace.

Trace $(P^{-1}AP)$ = sum of diagonal enteries

$$= \text{trace} \begin{bmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= 6 + 2 + 2$$

$$= 10$$

$$\text{Trace } (A) = \text{tr.} \begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & -1 \\ -2 & -4 & 4 \end{bmatrix}$$

$= 2 + 4 + 4$

$= 10$

Both A & $P^{-1}AP$ have same trace.

Q) 1 Gram-Schmidt Process

$u_1 = (1,1,1)$, $u_2 = (-1,1,0)$, $u_3 = (1,2,1)$

$V_1 = u_1 = (1,1,1) \Rightarrow \boxed{V_1 = (1,1,1)}$

$$V_2 = u_2 - \frac{\langle u_2, V_1 \rangle}{\|V_1\|^2} V_1$$

$$= (-1,1,0) - \frac{(\cancel{-1} + \cancel{1} + 0)}{1^2+1^2+1^2}(1,1,1)$$

$$= (-1,1,0) - 0$$

$$\boxed{V_2 = (-1,1,0)}$$

$$V_3 = u_3 - \frac{\langle u_3, V_1 \rangle}{\|V_1\|^2} \overline{V_1} - \frac{\langle u_3, V_2 \rangle}{\|V_2\|^2} \overline{V_2}$$

$$= (1,2,1) - \frac{(1+2+1)}{1+1+1}(1,1,1) - \frac{(-1+2+0)}{1+1+0}(-1,1,0)$$

$$= (1,2,1) - \frac{4}{3}(1,1,1) - \frac{1}{2}(-1,1,0)$$

$$\boxed{V_3 = \left(\frac{1}{6}, \frac{1}{6}, -\frac{1}{3}\right)}$$

$$q_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{3}}(1,1,1) = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$q_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{2}}(-1,1,0) = \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

$$q_3 = \frac{v_3}{\|v_3\|} = \frac{1}{(\cancel{1}\sqrt{6})}\left(\frac{1}{6}, \frac{1}{6}, \frac{-1}{3}\right)$$

$$= \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}\right)$$

b)

$$Q = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{bmatrix}$$

$$R = \begin{bmatrix} \langle u_1, q_1 \rangle & \langle u_2, q_1 \rangle & \langle u_3, q_1 \rangle \\ 0 & \langle u_2, q_2 \rangle & \langle u_3, q_2 \rangle \\ 0 & 0 & \langle u_3, q_3 \rangle \end{bmatrix}$$

$$\langle u_1, q_1 \rangle = (1,1,1)\cdot\left(\frac{1}{\sqrt3},\frac{1}{\sqrt3},\frac{1}{\sqrt3}\right)$$

$$= (1)\left(\frac{1}{\sqrt3}\right) + (1)\frac{1}{\sqrt3} + (1)\left(\frac{1}{\sqrt3}\right)$$

$$= \frac{1}{\sqrt3} + \frac{1}{\sqrt3} + \frac{1}{\sqrt3} = \frac{3}{\sqrt3}$$

$$\langle u_2, q_1 \rangle = (-1,1,0)\cdot\left(\frac{1}{\sqrt3},\frac{1}{\sqrt3},\frac{1}{\sqrt3}\right)$$

$$= \cancel{-\frac{1}{\sqrt3}} + \cancel{\frac{1}{\sqrt3}} + 0 = 0$$

$$\langle u_3, q_1 \rangle = (1,2,1)\cdot\left(\frac{1}{\sqrt3},\frac{1}{\sqrt3},\frac{1}{\sqrt3}\right)$$

$$= \frac{1}{\sqrt3} + \frac{2}{\sqrt3} + \frac{1}{\sqrt3} = \frac{4}{\sqrt3}$$

$$\langle u_2, q_2 \rangle = (-1,1,0)\cdot\left(-\frac{1}{\sqrt2},\frac{1}{\sqrt2},0\right)$$

$$= \frac{1}{\sqrt2} + \frac{1}{\sqrt2} = \frac{2}{\sqrt2}$$

$$\langle u_3, q_2 \rangle = (1,2,1)\cdot\left(-\frac{1}{\sqrt2},\frac{1}{\sqrt2},0\right)$$

$$= -\frac{1}{\sqrt2} + \frac{2}{\sqrt2} + 0 = \frac{1}{\sqrt2}$$

$$\langle u_3, q_3 \rangle = (1,2,1)\cdot\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}\right)$$

$$= \frac{1}{\sqrt{6}} + \cancel{\frac{2}{\sqrt{6}}} - \cancel{\frac{2}{\sqrt{6}}}$$

$$= \frac{1}{\sqrt{6}}$$

So

$$R = \begin{bmatrix} \frac{3}{\sqrt{3}} & 0 & \frac{4}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{6}} \end{bmatrix}$$

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Question No 5

(a) Consider basis $S = \{v_1, v_2\}$ for $\mathbb{R}^2$, where $v_1 = (-2, 1)$ and $v_2 = (1, 3)$ and let : $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation such that $T(v_1) = (-1, 2, 0)$ and $T(v_2) = (0, -3, 5)$. Find a formula for $T(x_1, x_2)$ and use formula to find $T(2, -3)$.

Sol.

Firstly, we have to find $T(x_1, x_2) = A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, where,

$$A = \left[T\begin{pmatrix} 1 \\ 0 \end{pmatrix} \middle| T\begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]$$

Now,

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = c_1 v_1 + c_2 v_2 = c_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2c_1 + c_2 \\ c_1 + 3c_2 \end{pmatrix}$$

$\Rightarrow$ $-2c_1 + c_2 = 1$, $c_1 + 3c_2 = 0$

$-2c_1 - \frac{1}{3}c_1 = 1$ $\quad$ $c_1 = -3c_2$

$\frac{-7c_1}{3} = 1$

$\Rightarrow c_1 = \frac{-3}{7}$

$\Rightarrow c_2 = -\frac{1}{3}\left(\frac{-3}{7}\right) = \frac{1}{7}$

Thus,

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = T(c_1 v_1 + c_2 v_2) = c_1 T(v_1) + c_2 T(v_2)$$

$$= \frac{-3}{7}\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + \frac{1}{7}\begin{pmatrix} 0 \\ -3 \\ 5 \end{pmatrix}$$

$$= \left(3/7 \,;\, -9/7 ,\, 5/7 \right)$$

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$$\begin{pmatrix}0\\1\end{pmatrix} = a_1 v_1 + a_2 v_2 = a_1(-2,1) + a_2(1,3)$$

$$\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-2a_1 + a_2\\ a_1 + 3a_2\end{pmatrix}$$

$\Rightarrow \quad -2a_1 + a_2 = 0, \qquad a_1 + 3a_2 = 1$

$\Rightarrow \quad a_2 = 2a_1 \qquad a_1 + 3(2a_1) = 1$

$\Rightarrow \quad a_2 = 2/7 \qquad 7a_1 = 1$

$a_1 = 1/7$

Thus,

$$T\begin{pmatrix}0\\1\end{pmatrix} = T(a_1 v_1 + a_2 v_2) = a_1 T(v_1) + a_2 T(v_2)$$

$$= \frac{1}{7}\begin{pmatrix}-1\\2\\0\end{pmatrix} + \frac{2}{7}\begin{pmatrix}0\\-3\\5\end{pmatrix}$$

$$= \left(-1/7, -4/7, 10/7\right)$$

Now,

$$T(x_1, x_2) = \begin{bmatrix}\frac{3}{7} & -1/7\\ -9/7 & -4/7\\ 5/7 & 10/7\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}(3x_1 - x_2)/7\\ (-9x_1 - 4x_2)/7\\ (5x_1 + 10x_2)/7\end{bmatrix} \text{ Answer.}$$

——— (A)

For $T(2,-3)$ put $x_1 = 2$, $x_2 = -3$ in eqn (A),

$$T(2,-3) = \begin{bmatrix}(3(2)-3)/7\\ (-9(2)-4(-3))/7\\ (5(2)+10(-3))/7\end{bmatrix} = \begin{bmatrix}3/7\\ 6/7\\ -20/7\end{bmatrix}$$

$$T(2,-3) = \left(3/7, 6/7, -20/7\right) \text{ Answer.}$$

5(b)

let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation defined by $T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = \begin{bmatrix}-4x_1\\ -x_1 + 2x_2\\ -2x_1 + 5x_2\end{bmatrix}$

(i) Find the matrix for the transformation T i.e. $[T]_{B',B} = [\,[T(u_1)]_{B'} \mid [T(u_2)]_{B'}\,]$ relative to the basis $B = \{\vec{u}_1, \vec{u}_2\}$ and $B' = \{v_1, v_2, v_3\}$ where

$$u_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad v_1 = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

$$v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Sol. $T(u_1) = c_1 v_1 + c_2 v_2 + c_3 v_3$

From the given formula

$$T(u_1) = T\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} -4(1) \\ -1+2(3) \\ -2(1)+5(3) \end{bmatrix} = \begin{bmatrix} -4 \\ 5 \\ 13 \end{bmatrix}$$

Now

$(-4, 5, 13) = c_1(2,-2,0) + c_2(1,0,1) + c_3(0,1,0)$

$\Rightarrow \quad 2c_1 + c_2 = -4, \quad -2c_1 + c_3 = 5, \quad \boxed{c_2 = 13}$

$\Rightarrow \quad c_1 = \frac{1}{2}(-4-13) \qquad c_3 = 5 + 2\left(\frac{-17}{2}\right)$

$\boxed{c_1 = -17/2} \qquad \boxed{c_3 = -12}$

$$[T(u_1)]_{B'} = \begin{bmatrix} -17/2 \\ 13 \\ -12 \end{bmatrix}$$

Now, $T(u_2) = a_1 v_1 + a_2 v_2 + a_3 v_3$

From the given formula

$$T(u_2) = T\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} -4(2) \\ -(2)+2(-1) \\ -2(2)+5(-1) \end{bmatrix} = \begin{bmatrix} -8 \\ -4 \\ -9 \end{bmatrix}$$

$T(u_2) = (-8, -4, -9)$

$(-8,-4,-9) = a_1(2,-2,0) + a_2(1,0,1) + a_3(0,1,0)$

$\Rightarrow 2a_1 + a_2 = -8, \quad -2a_1 + a_3 = -4, \quad a_2 = -9$

$a_1 = \frac{1}{2}(-8+9), \quad a_3 = -4 + 2(\frac{1}{2})$

$\boxed{a_1 = \frac{1}{2}}, \quad \boxed{a_3 = 3}$

Thus, $[T(u_2)]_{B'} = \begin{bmatrix} 1/2 \\ -9 \\ 3 \end{bmatrix}$

$$[T]_{B',B} = \begin{bmatrix} -\frac{17}{2} & \frac{1}{2} \\ 13 & -9 \\ -12 & 3 \end{bmatrix} = T$$

$\boxed{(ii)}$ Find the Kernal of T i.e. Ker(T).

$Ker(T) = \{ (x_1, x_2) \mid T(x_1, x_2) = (0,0,0) \}$

Consider,

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{17}{2} & \frac{1}{2} \\ 13 & -9 \\ -12 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} (-17x_1 + x_2)/2 \\ (13x_1 - 9x_2) \\ -12x_1 + 3x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow -17x_1 + x_2 = 0, \quad 13x_1 - 9x_2 = 0, \quad -12x_1 + 3x_2 = 0$

$x_2 = 17x_1 \qquad 13x_1 - 9(17x_1) = 0$

$\Rightarrow \boxed{x_2 = 0} \qquad -140x_1 = 0$

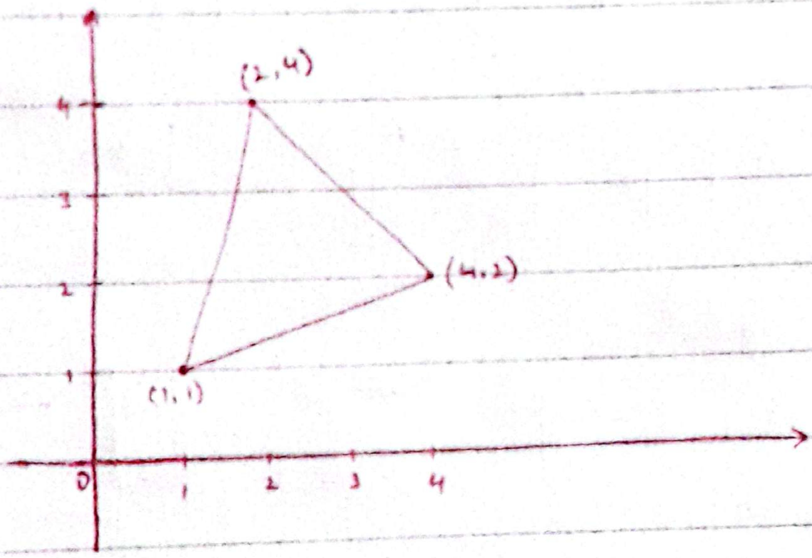
$\boxed{x_1 = 0}$

$Ker(T) = \{(0,0)\}$ Ans.

(iii) Since, by Rank-Nullity theorem:

$$\dim \text{Range}(T) + \dim(\text{Ker}(T)) = \dim(\mathbb{R}^3)$$
$$\text{Rank}(T) + 0 = 3$$
$$\Rightarrow \text{Range}(T) = \mathbb{R}^3.$$

(c) If the following is the image of a triangle ΔMON



Find the standard matrices for the mentioned below parts and plot the transformed image of a given triangle.

i) Expand by a factor of 2 in the x-direction

let T be Transformation that is an expansion by a factor of 2 in the x-direction, then

$$T(x,y) = (2x, y)$$

Now, for standarad matrix A

$$T(1,0) = (2,0), \quad T(0,1) = (0,1)$$

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$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$ is the standard matrix.

<u>For plotting:</u>

Given points of ΔMON are

$(1, 1)$, $(4, 2)$, $(2, 4)$

Now,

$$T(1,1) = A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$T(1,1) = (2, 1)$

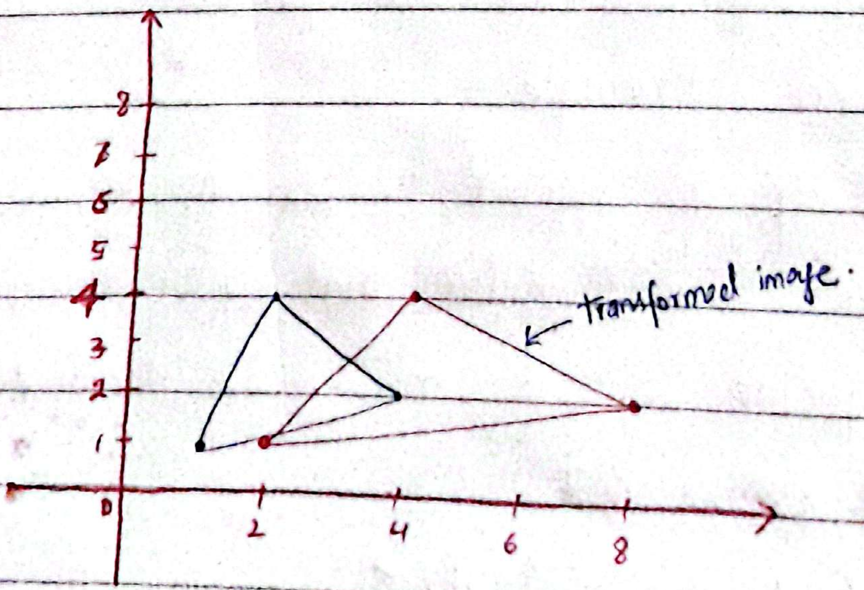
$$T(4,2) = A\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$$

$T(4,2) = (8, 2)$

$$T(2,4) = A\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

$T(2,4) = (4, 4)$

The Transformed image of given triangle is



(ii) Reflect the given triangle about $y = x$.

If T is reflection about line $y = x$ then

$$T(x, y) = (y, x)$$

For Standarad Matric,

$T(1, 0) = (0, 1)$, $T(0, 1) = (1, 0)$

Thus, $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is the standard matrix

Now, Reflection of the given triangle $\triangle MON$.

$$T(1, 1) = A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$T(4, 2) = A\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$T(2, 4) = A\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

Now, the transformed image is :