École polytechnique fédérale de Lausanne

Optionnal Semester Project

Verified double-hashing hash map

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1 Introduction

This report explain, in an informal way, the implementation and verification of a double-hash hash map. It does not aim to provide a complete and detailled explication of each line of code. Instead, the goal is to synthesize the main points needed to understand both the code and the verification.

The actual verification contains more than 100 new lemmas and fixpoints, and it would make no sense to foramlly describe each of them here, as a formal definition and proof are provided. Also, most of those are trivial proofs and the name is explicit enough to understand their behaviour.

Thus, this document is more intended to be an help when reading the actual proof.

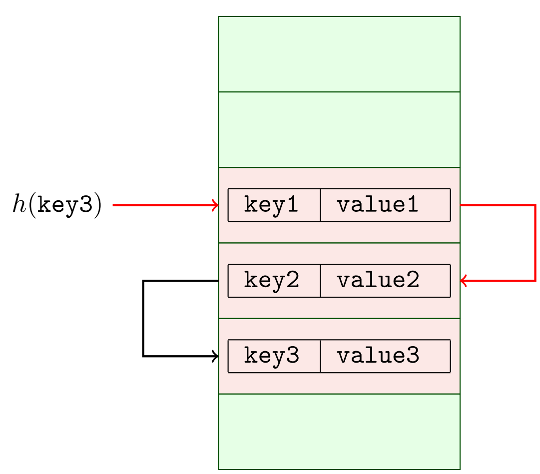
2 Implementation

2.1 Provided implementation

The implementation I was provided was a naive hash map, in which a $< key, value >$ tuple is inserted at the first free cell after $h(key)$, where h is a given hash function.

Thus, in case of multiple conflicts, the same cells will be tested. For instance, in Example 1, if $h(key1) = h(key2) = h(key3)$, then there is 2 unsuccessful accesses before finding an empty cell to insert $key3$ in.

Example 1: Multiple conflicts when inserting in a naive hash map.

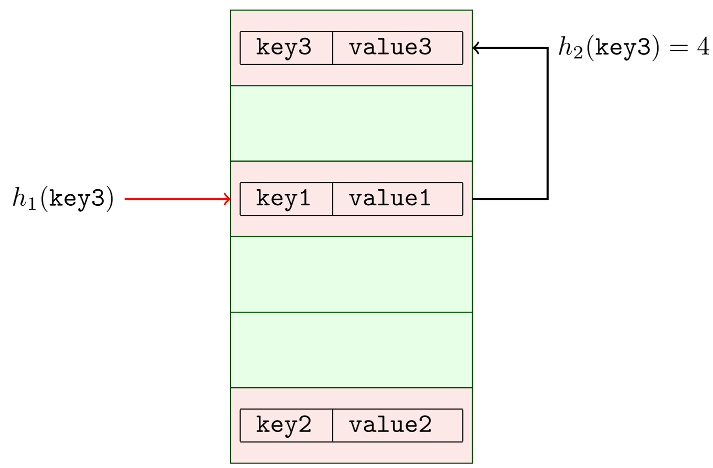


2.2 Double-hash implementation

The solution implemented during this project is *double-hashing*. In double-hashing, instead of searching in the following cell in case of conflict, the key is re-hashed using a second hash-function. This second hash-function determines an offset, and after each unsuccessful try, the $current\_index + offset$-th cell is looked-up.

For instance, Example 2 shows the same accesses as the previous example, but with double-hashing (the first hash-function being the same). As *key*2 and *key*3 have different second hashes, their second choice cell is not the same. Then inserting *key*3 only conflicts with *key*1.

Example 2: Multiple conflicts when inserting in a double-hash hash map.



2.3 Benchmark

3 Verification

3.1 Provided proof

3.1.1 Requirement (R)

Figure 1 present the relevant part of the proof of the original loop in `find_key`. The last statement (`no_key_found(ks, k)`) requires the property `not_my_key(k)` to be verified for all current keys in the mapping, ensured by the `up_to(nat_of_int(length(ks)), ...(not_my_key)(k)...)` statement.

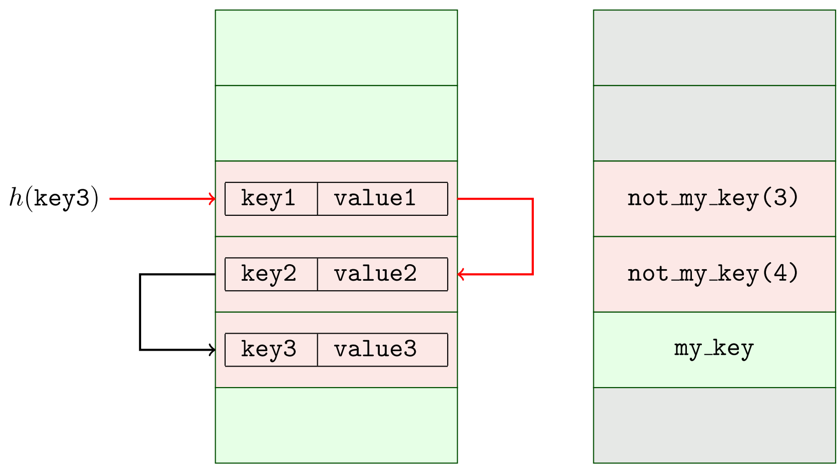
`up_to(nat, prop)` verifies that `prop` is ensured for all `i` below `nat`:
`up_to(0, prop) = true`
`up_to(n, prop) = prop(n-1) && up_to(n-1, prop)`

This `up_to` statement is proved by the *for*-loop invariant: at each round, `not_my_key(k, nth(index, ks))` is ensured, either because the cell is empty (`no_busy_no_key` lemma), either because the hash does not match (`no_hash_no_key` lemma), either because the key does not match (hence inferred by Verifast).

Finally, the *for*-loop only proves that the `up_to` statement holds when starting from `index = start` and looping. The lemma `by_loop_for_all` prove that this loop access is equivalent to a continuous access from `0` to `length`.

Example 3 represent a successful search in the map. The search starts at index $h(key3)$. As long as the key is not found at index i, `not_my_key(i)` is asserted. Finally, when *key*3 is found, it is ensured to be the right key and returned.

Example 3: Successful search

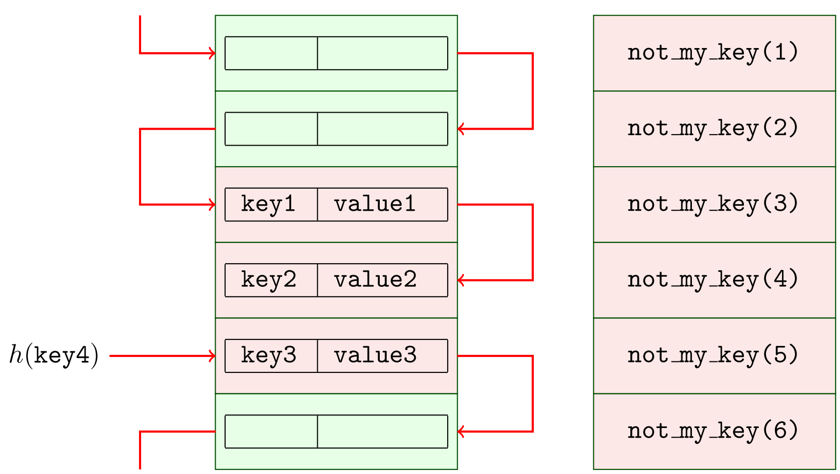


In case of unsuccessful search, as in Example 4, `not_my_key(i)` is asserted for all indexes, ensuring that the key is not present in the map.

Hence, an invariant of the *for*-loop is that `not_my_key` is asserted *for all indexes from start up to i*, by ring accesses, i being the loop iterator.

Example 4

`not_my_key` is ensured both when the cell is empty (green cell in this example), or when the cell is busy, but occupied by an other key (a busy cell is in red in this example).



3.1.2 Impact of the modifications

The modifications have two main impacts: first, the accesses are not performed in the same order. The scond impact is that the specification is not true anymore in the general case.

Access order: As explain above, with double hashing, cells of the map are not accessed by loop anymore. Hence, the `by_loop_for_all` lemma doesn't apply anymore. Let *stripe* be the function which, given a loop iteration, returns the index of the cell looked-up at this iteration (parametrized by *start*, *step* and *capacity*). This problem is solved by computing the antecedant of each cell.

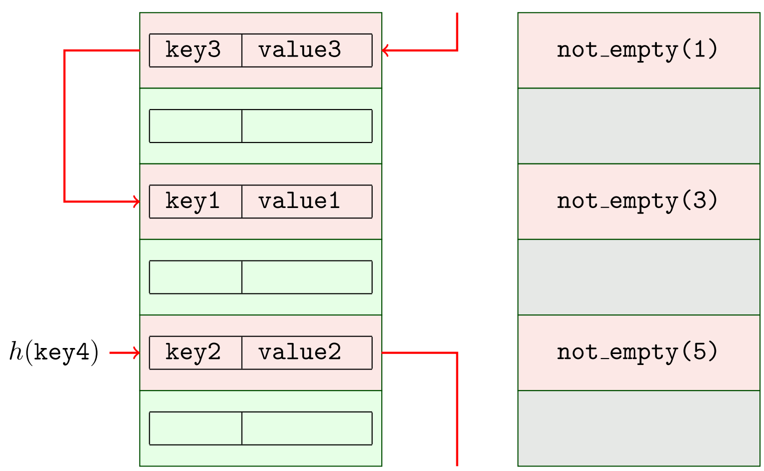
Hence, at iteration i, for any cell *map*[`index`], if the antecedent of *map*[`index`] is less than i, then `not_my_key(index)` is ensured.

The new way to ensure `not_my_key` for all indexes is then to ensures that all index has an antecedant w.r.t. the *stripe* function.

New requirements: However, it is not always the case that every cell is reached. An example is provided in Example 5. Actually, after the Chinese remainder theorem, the *step* and the *capacity* must be coprime in order to ensure that every cell is eventually tested.

Hence, this coprimeness is a new requirement of the specification. Technically, it is sufficient to have a capacity being a power of 2 and to have only odd $offset$ hashes.

Example 5



3.2 The `stripe_l_fp` fixpoint

3.2.1 Definition

First, a fixpoint is defined, which returns the index to be updated after n iterations with an offset of *step*, starting from *start* with capacity *capa*.

A lemma ensuring that `stripe` $= start + n * step \% capa$ is proved.

Definition 1: stripe(int start, int step, nat n, int capa)

```
fixpoint int stripe(int start, int step, nat n, int capa) {
  switch(n) {
    case zero: return start;
    case succ(m): return
      (stripe(start, step, m, capa) + step) % capa;
  }
}
```

The `stripe_l_fp` fixpoint builds a `list<option<nat>>` given a starting point, an offset, a number of accesses and a capacity. The base case of this fixpoint is to generate a `list` containing only `nones` (fixpoint `gen_none`), if `zero` accesses are performed. The recursive case is to update the $start + n * offset \% capa$ cell, using the above `stripe` fixpoint.

Definition 2: stripe_l_fp(int start, int step, nat bound, int capa)

```
fixpoint list<option<nat> > stripe_l_fp(int start,
  int step, nat n, int capa)
{
  switch(n) {
    case zero: return gen_none(nat_of_int(capa));
    case succ(m): return
      update(stripe(start, step, n, capa), some(n),
        stripe_l_fp(start, step, m, capa));
  }
}
```

The `update (index, elem, list)` fixpoint returns `list` with the `index`-th element updated to `elem`.

Example 6: `stripe_l_fp(0, 2, 5, 7)`

Calling `stripe_l_fp(0, 2, 5, 7)` produces the following list. Notice that the base case returns a list containing only `nones`, not a list containing a `some(0)`.

none
some(4)
some(1)
some(5)
some(2)
none
some(3)

3.2.2 Properties

The main property required is that the number of cell containing `some(i)` (for any `i`) is equal to the number of steps done. This property will later be used to ensures that all cells are eventually reached (see Subsection 3.2.4). The function that count the number of such cells is named `count_some(list<option<nat>> list)`.

There are also other properties which are used internally to prove the `count_some` property. The main lemma is named `stripe_l`.

Lemma 1: Prototype of `stripe_l`

```
lemma list<option<nat> > stripe_l(int start, int step, nat n,
   int capa)
requires 0 <= start &*& start < capa &*& step > 0
   &*& n <= capa &*& coprime(step, capa) &*& step < capa;
ensures count_some(result) == n
   &*& length(result) == capa
   &*& true == up_to(nat_of_int(capa),
      (list_contains_stripes)(result, start, step))
   &*& true == up_to(nat_of_int(capa),
      (lst_opt_less_than_n)(result, n))
   &*& true == forall(result, opt_not_zero)
   &*& result == stripe_l_fp(start, step, n, capa)
   &*& coprime(step, capa);
```

`list_contains_stripe` ensures that if a cell contains `some(i)`, then `i` is the antecedant of the cell index w.r.t. stripe.

3.2.3 Proof of `stripe_l`

The proof of these properies relies on the fact that the same cell is not updated twice. Once this is ensured, the construction of the fixpoint ensures the validity of the properties.

Algorithm 1 shows the main steps of the proof. In the base case, all trivially holds. In the inductive case, if the `stripe(start, step, n, capa)`-th cell (i.e. the one hit at n-th iteration) already contains `some(i)`, the `list_contains_stripes` property ensures that `stripe(start, step, i, capa)`-th cell is the one we are hitting. Hence, $start + step \times i\%capa = start + step \times n\%capa$, with $n - i < capa$. Then the Chinese remainder theorem leads to a contradiction.

```
Input: int start, int step, nat n, int capa
switch n do
  case zero
    // All hold by construction
  end
  case succ(m)
    // Recursive call, the termination is ensured by n > m
    list lst ⟵ stripe_l(start, step, m, capa)
    // Now, we want to update the stripe(start, step, n, capa)-th to
       some(n)
    // Proof by contradiction that the cell contains none
    switch nth(stripe(start, step, n, capa), lst do
      case some(i)
        assert start + i × step%capa = start + n × step%capa;
        assert (n − i) × step%capa = 0;
        assert n − i < capa;
        // The chinese remainder theorem applies and shows a
           contradiction
        chinese_remainder_theorem(step, capa, (n − i) × step);
      end
      case none
      end
    endsw
    // We now that the stripe(start, step, n, capa)-th cell contains
       a none, which we update to some(n), so the properties hold
       for the updated list.
    return update(stripe(start, step, n, capa), some(n), lst)
  end
endsw
```

$n - i$ is noted $diff$ in the following parts

Algorithm 1: Proof of `stripe_l`

3.2.4 From `stripe` fixpoint to `R`

3.3 Proof of the *Chinese remainder theorem*

3.3.1 Properties

The goal of the *Chinese remainder theorem* is to highlight a contradiction in the `stripe_l` proof. We have that $diff \times step\%capa = 0$. The contradiction we want to highlight is that in the given environment, $diff$ can only be 0, i.e. the supposed previous value is the same that the one we want to write, that is the n-th iteration is supposed to be already written to the list.

This is reduced to the following lemma: if $x\%n1 = 0$, $x\%n2 = 0$, $n1$ and $n2$ are coprime, and $x < n1 \times n2$, then $x = 0$. It is also required that $n1 > 0$, $n2 > 0$ and $x \geq 0$.

In `stripe_l`, this is applied to $x = diff \times$ `step`, $n1 =$ `step` and $n2 =$ `capa`.

This lemma is a direct consequence of the *uniqueness* property of the *Chinese remainder theorem*. Although it is simple to show informally, Verifast first requires to build *gcd* which is quite long. All the proof is done in a separate file `chinese_remainder_th.gh`. The proof takes around 1000 lines of code (which less than 200 are *assert*-s or comments and could be remove).

3.3.2 Computation of *gcd*

3.3.3 *gcd* properties

3.3.4 Proof by contradiction

In Verifast, the proof of the `bin_chinese_remainder_theorem` lemma is quite long (approx. 300 lines). However, most of it is only arithmetic statements. Hence, informally, the proof is much shorter. Algorithm 2 sketches the main cases. The main part (*if* $x > 1$ branch) decompose x into $n1 \times k1 = n2 \times k2$. After justifying why $k1\%n2 \neq 0$, it considers $gcd(k1, n2) = a$, which can not be 1. Then remaining case ($a \neq 1$, $k1\%n2 \neq 0$, $x > 1$) calls recursively the theorem, on $\frac{n2}{a} = b$.

3.3.5 Assumed lemma

One lemma remains assumed:

Lemma 2: `gcd_mul`

```
lemma void gcd_mul(int n1, int n2, int n3)
requires coprime(n1, n3) &*& coprime(n2, n3);
ensures coprime(n1*n2, n3) &*& coprime(n1, n3)
   &*& coprime(n2, n3);
```

```
if x = 1 then
    x%n1 = 0 ⇒ n1 = 1;
    x%n2 = 0 ⇒ n2 = 1;
    assert n1 × n2 = 1;
    assert x = n1 × n2;
    contradiction;
else if x > 1 then
    x%n1 = 0 ⇒ ∃k1|n1 × k1 = x ;
    x%n2 = 0 ⇒ ∃k2|n2 × k2 = x ;
    assert k1 ≠ 0;
    if k1%n2 = 0 then
        β ⟵ k1/n2;
        assert β × n2 = k1;
        assert β ≥ 1;
        assert β × n2 ≤ k1;
        assert x = n1 × k1 ≥ n1 × n2;
        contradiction;
    else
        a ⟵ gcd(k1, n2);
        b ⟵ n2/a, assert b ≠ 0;
        γ ⟵ k2/a;
        assert gcd(b, γ) = 1;
        if gcd(n1, b) ≠ 1 then
            assert gcd(n1, a × b) ≠ 1;
            assert gcd(n1, n2) ≠ 1;
        end
        if a = 1 then
            assert γ = k1 ∧ b = n2;
            gcd(b, γ) = 1 ∧ gcd(n1, b) = 1 ⇒ gcd(n1 × γ, b) = 1;
            contradiction gcd(x, n2) = 1;
        else
            // The termination is ensured by b < n2
            bin_chinese_remainder_theorem(n1, b, k2 × b);
            assert k2 × b = 0;
            assert k2 = 0;
            contradiction n2 × k2 = x = 0;
        end
    end
else
    assert x = 0;
end
```

Algorithm 2: Proof of `bin_chinese_remainder_theorem`

4 Conclusion

4.1 Validity of benchmark

4.2 Forthcoming work

```
int i = 0;
for (; i < capacity; ++i)
/*@ invariant ... &*&
    true == up_to(nat_of_int(i),
        (byLoopNthProp)(ks, (not_my_key)(k),
            capacity, start));
@*/
//@ decreases capacity - i;
{
    int index = loop(start + i, capacity);
    int bb = busybits[index];
    int kh = k_hashes[index];
    void* kp = keyps[index];
    if (bb != 0 && kh == key_hash) {
        if (eq(kp, keyp)) {
            //@ hmap_find_this_key(hm, index, k);
            return index;
        }
    } else {
        //@ if (bb != 0) no_hash_no_key(ks, khs, k, index, hsh);
        //@ if (bb == 0) no_bb_no_key(ks, bbs, index);
    }
    //@ assert(true == not_my_key(k, nth(index, ks)));
}
/*@ by_loop_for_all(ks, (not_my_key)(k),
    start, capacity, nat_of_int(capacity));
@*/
/*@ assert true == up_to(nat_of_int(length(ks)),
    (nthProp)(ks, (not_my_key)(k)));
@*/
//@ no_key_found(ks, k);
```

Figure 1: Original *for*-loop for searching a key