# update $\xi$

$$egin{aligned} oldsymbol{\xi}_1 &= \{\omega_1, \dots, \omega_L\} \ oldsymbol{\xi}_2 &= \{\Delta arphi_1, \dots, \Delta arphi_M\} \ oldsymbol{\xi}_3 &= \{\kappa_{1,1}, \dots, \kappa_{k,t}, \dots, \kappa_{K, au}\} \ oldsymbol{\xi}_4 &= \left\{\lambda^c, p_{01}^c, p_{10}^c, \mu_1^s, \sigma_1^s, \dots, \mu_k^s, \sigma_k^s 
ight\} \ oldsymbol{v} &\triangleq \{oldsymbol{x}, oldsymbol{\gamma}, oldsymbol{c}, oldsymbol{s} \} \end{aligned}$$

$$egin{aligned} oldsymbol{\xi}_{j}^{(i+1)} &= oldsymbol{\xi}_{j}^{(i)} + \gamma^{(i)} rac{\partial u \left(oldsymbol{\xi}_{j}, oldsymbol{\xi}_{-j}^{(i)}; oldsymbol{\xi}_{j}^{(i)}, oldsymbol{\xi}_{-j}^{(i)}
ight)}{\partial oldsymbol{\xi}_{j}} \Bigg|_{oldsymbol{\xi}_{j} = oldsymbol{\xi}_{j}^{(i)}} \ u(oldsymbol{\xi}; \dot{oldsymbol{\xi}}) &= u^{ ext{EM}}(oldsymbol{\xi}; \dot{oldsymbol{\xi}}) + \sum_{j \in \mathcal{J}_{c}^{1}} au_{j} \Big\| oldsymbol{\xi}_{j} - \dot{oldsymbol{\xi}}_{j} \Big\|^{2} \ u^{ ext{EM}}(oldsymbol{\xi}; \dot{oldsymbol{\xi}}) &= \int p(oldsymbol{v} \mid oldsymbol{p}, \dot{oldsymbol{\xi}}) \ln rac{p(oldsymbol{v}, oldsymbol{p}, oldsymbol{\xi})}{p(oldsymbol{v} \mid oldsymbol{p}, oldsymbol{\xi})} doldsymbol{v} \ &pprox \int q(oldsymbol{v}; \dot{oldsymbol{\xi}}) \ln rac{p(oldsymbol{v}, oldsymbol{p}, oldsymbol{\xi})}{q(oldsymbol{v}; \dot{oldsymbol{\xi}})} doldsymbol{v} \end{aligned}$$

首先求 $\frac{\partial}{\partial \boldsymbol{\xi}_{i}}\hat{u}^{EM}(\boldsymbol{\xi}_{j},\boldsymbol{\xi}_{-j}^{(i)};(\boldsymbol{\xi}_{j}^{(i)},\boldsymbol{\xi}_{-j}^{(i)}):$ 

$$\begin{split} \frac{\partial}{\partial \boldsymbol{\xi}_{j}} \hat{u}^{EM}(\boldsymbol{\xi}_{j}, \boldsymbol{\xi}_{-j}^{(i)}; \boldsymbol{\xi}_{j}^{(i)}, \boldsymbol{\xi}_{-j}^{(i)}) &= \frac{\partial}{\partial \boldsymbol{\xi}_{j}} \bigg\{ \bigg[ \int q(\boldsymbol{v}; \boldsymbol{\xi}_{j}^{(i)}, \boldsymbol{\xi}_{-j}^{(i)}) \ln p(\boldsymbol{v}, \boldsymbol{y}; \boldsymbol{\xi}_{j}, \boldsymbol{\xi}_{-j}^{(i)}) d\boldsymbol{v} - \int q\left(\boldsymbol{v}; \boldsymbol{\xi}_{j}^{(i)}, \boldsymbol{\xi}_{-j}^{(i)}\right) \ln q\left(\boldsymbol{v}; \boldsymbol{\xi}_{j}^{(i)}, \boldsymbol{\xi}_{-j}^{(i)}\right) d\boldsymbol{v} \bigg] \bigg\} \\ &= \frac{\partial}{\partial \boldsymbol{\xi}_{j}} \bigg[ \int q(\boldsymbol{v}; \boldsymbol{\xi}_{j}^{(i)}, \boldsymbol{\xi}_{-j}^{(i)}) \ln p(\boldsymbol{v}, \boldsymbol{y}; \boldsymbol{\xi}_{j}, \boldsymbol{\xi}_{-j}^{(i)}) d\boldsymbol{v} \bigg] \\ &= \int q(\boldsymbol{v}; \boldsymbol{\xi}_{j}^{(i)}, \boldsymbol{\xi}_{-j}^{(i)}) \frac{\partial}{\partial \boldsymbol{\xi}_{j}} \bigg[ \ln p(\boldsymbol{v}, \boldsymbol{y}; \boldsymbol{\xi}_{j}, \boldsymbol{\xi}_{-j}^{(i)}) \bigg] d\boldsymbol{v} \end{split}$$

注意到 $p(\boldsymbol{v},\boldsymbol{y};\boldsymbol{\xi}_{j},\boldsymbol{\xi}_{-j}^{(i)})$ 可以被进一步分解:

$$p(oldsymbol{v}, oldsymbol{y}; oldsymbol{\xi}_j, oldsymbol{\xi}_{-j}^{(i)}) = \underbrace{p(oldsymbol{x} \mid oldsymbol{\gamma}) p(oldsymbol{\kappa}) p(oldsymbol{y} \mid oldsymbol{s}, oldsymbol{\kappa}; oldsymbol{\xi}) p(oldsymbol{v}, oldsymbol{s}; oldsymbol{\xi}) p(oldsymbol{v} \mid oldsymbol{s}, oldsymbol{\kappa}; oldsymbol{\xi}) p(oldsymbol{v} \mid oldsymbol{s}, oldsymbol{s}; oldsymbol{s}, oldsymbol{s}; oldsymbol{s}) p(oldsymbol{v} \mid oldsymbol{s}, oldsymbol{s}; oldsymbol{s}) p(oldsymbol{s}, oldsymbol{s}; oldsymbol{s}, oldsymbol{s}; oldsymbol{s}; oldsymbol{s}, oldsymbol{s}; oldsymbol{s};$$

将其带入上式可得:

$$\begin{split} \frac{\partial}{\partial \boldsymbol{\xi}_{j}} \hat{u}^{EM} \left(\boldsymbol{\xi}_{j}, \boldsymbol{\xi}_{-j}^{(i)}; \boldsymbol{\xi}_{j}^{(i)}, \boldsymbol{\xi}_{-j}^{(i)}\right) &= \int q(\boldsymbol{v}; \boldsymbol{\xi}_{j}^{(i)}, \boldsymbol{\xi}_{-j}^{(i)}) \frac{\partial}{\partial \boldsymbol{\xi}_{j}} \Big[ \ln p(\boldsymbol{v}, \boldsymbol{y}; \boldsymbol{\xi}_{j}, \boldsymbol{\xi}_{-j}^{(i)}) \Big] d\boldsymbol{v} \\ &= \int q(\boldsymbol{v}; \boldsymbol{\xi}_{j}^{(i)}, \boldsymbol{\xi}_{-j}^{(i)}) \frac{\partial}{\partial \boldsymbol{\xi}_{j}} \Big[ \ln p(\boldsymbol{y} \mid \boldsymbol{x}, \boldsymbol{\kappa}; \boldsymbol{\xi}_{1,2,3}) + \ln p(\boldsymbol{c}, \boldsymbol{s}; \boldsymbol{\xi}_{4}) \Big] d\boldsymbol{v} \\ &= \begin{cases} \int q(\boldsymbol{v}; \boldsymbol{\xi}_{j}^{(i)}, \boldsymbol{\xi}_{-j}^{(i)}) \frac{\partial}{\partial \boldsymbol{\xi}_{j}} \ln p(\boldsymbol{y} \mid \boldsymbol{x}, \boldsymbol{\kappa}; \boldsymbol{\xi}_{1,2,3}) d\boldsymbol{v} &, j \in \{1, 2\} \\ \int q(\boldsymbol{v}; \boldsymbol{\xi}_{j}^{(i)}, \boldsymbol{\xi}_{-j}^{(i)}) \frac{\partial}{\partial \boldsymbol{\xi}_{j}} \ln p(\boldsymbol{c}, \boldsymbol{s}; \boldsymbol{\xi}_{3}) d\boldsymbol{v} &, j = 4 \end{cases} \\ &= \begin{cases} \int q(\boldsymbol{x}; \boldsymbol{\xi}_{j}^{(i)}, \boldsymbol{\xi}_{-j}^{(i)}) \frac{\partial}{\partial \boldsymbol{\xi}_{j}} \ln p(\boldsymbol{y} \mid \boldsymbol{x}, \boldsymbol{\kappa}; \boldsymbol{\xi}_{1,2,3}) d\boldsymbol{x} &, j \in \{1, 2\} \\ \int q(\boldsymbol{c}, \boldsymbol{s}; \boldsymbol{\xi}_{j}^{(i)}, \boldsymbol{\xi}_{-j}^{(i)}) \frac{\partial}{\partial \boldsymbol{\xi}_{j}} \ln p(\boldsymbol{c}, \boldsymbol{s}; \boldsymbol{\xi}_{3}) d\boldsymbol{c} d\boldsymbol{s} &, j = 4 \end{cases} \end{split}$$

#### 首先以求解 $\omega_l$ 为例:

设置求导前等价函数 $f_l(\omega_l)$ :

$$f_l(\omega_l) riangleq \int q(oldsymbol{x}) igg\{ \sum_{k=1}^K \sum_{t=1}^ au \ln \left[ \mathcal{CN}\left(y_{k,l,t}; \mathbf{F}_{k,l,t} oldsymbol{x}_k, \kappa_{k,t}^{-1}
ight) 
ight] igg\} doldsymbol{x}$$

同时,多维等价函数为 $f_L(\omega)$ :

$$f_L(oldsymbol{\omega}) = \sum_{l=1}^L f_l(\omega_l)$$

$$\frac{\partial}{\partial \omega_l} \hat{u}^{EM} \left( \omega_l, \boldsymbol{\xi}_{-\omega_l}^{(i)}; \omega_l^{(i)}, \boldsymbol{\xi}_{-\omega_l}^{(i)} \right) = \int q(\boldsymbol{x}; \omega_l^{(i)}, \boldsymbol{\xi}_{-\omega_l}^{(i)}) \frac{\partial}{\partial \omega_l} \ln p(\boldsymbol{y} \mid \boldsymbol{x}, \boldsymbol{\kappa}; \omega_l, \boldsymbol{\xi}_{1, -\omega_l}^{(i)}) d\boldsymbol{x}$$

其中:

$$\begin{split} \frac{\partial}{\partial \omega_{l}} \ln p(\boldsymbol{y} \mid \boldsymbol{x}, \boldsymbol{\kappa}; \omega_{l}, \boldsymbol{\xi}_{1,-\omega_{l}}^{(i)}) &= \frac{\partial}{\partial \omega_{l}} \sum_{k=1}^{K} \sum_{l=1}^{T} \sum_{t=1}^{\tau} \ln \left[ \mathcal{CN}(y_{k,l,t}; \mathbf{F}_{k,l,t} \boldsymbol{x}_{k}, \kappa_{k,t}^{-1}) \right] \\ &= \frac{\partial}{\partial \omega_{l}} \sum_{k=1}^{K} \sum_{t=1}^{\tau} \ln \left[ \mathcal{CN}(y_{k,l,t}; \mathbf{F}_{k,l,t} \boldsymbol{x}_{k}, \kappa_{k,t}^{-1}) \right] \\ &= \sum_{k=1}^{K} \sum_{t=1}^{\tau} \frac{\partial}{\partial \omega_{l}} \ln \left[ \mathcal{CN}(y_{k,l,t}; \mathbf{F}_{k,l,t} \boldsymbol{x}_{k}, \kappa_{k,t}^{-1}) \right] \\ &= \sum_{k=1}^{K} \sum_{t=1}^{\tau} \frac{\partial}{\partial (\mathbf{F}_{k,l,t} \boldsymbol{x}_{k})} \ln \left[ \mathcal{CN}(y_{k,l,t}; \mathbf{F}_{k,l,t} \boldsymbol{x}_{k}, \kappa_{k,t}^{-1}) \right] \underbrace{\frac{\partial}{\partial \omega_{l}} (\mathbf{F}_{k,l,t} \boldsymbol{x}_{k})}_{\text{part 1}} \end{split}$$

接下来将被求和项目展开, 其中

$$\begin{split} \mathbf{F}_{k,l,t} \boldsymbol{x}_k &= \left[ \boldsymbol{\Phi}^H \mathbf{V}(\omega_l) \mathbf{D}_M(\Delta \boldsymbol{\varphi}) \right]_{t,:} \boldsymbol{x}_k \\ &= \left[ \boldsymbol{\Phi}^H \right]_{t,:} \mathbf{V}(\omega_l) \mathbf{D}_M(\Delta \boldsymbol{\varphi}) \boldsymbol{x}_k \\ &= \left[ e^{j\vartheta_{t,1}} \quad \cdots \quad e^{j\vartheta_{t,M}} \right] \begin{bmatrix} \mathbf{a}_M(\omega_l)_1 [\mathbf{U}_M]_{1,:} \\ \vdots \\ \mathbf{a}_M(\omega_l)_M [\mathbf{U}_M]_{M,:} \end{bmatrix} \mathbf{D}_M(\Delta \boldsymbol{\varphi}) \boldsymbol{x}_k \\ &= \sum_{m=1}^M e^{j\vartheta_{t,m}} \mathbf{a}_M(\omega_l)_m [\mathbf{U}_M]_{m,:} \mathbf{D}_M(\Delta \boldsymbol{\varphi}) \boldsymbol{x}_k \\ &= \sum_{m=1}^M e^{j\vartheta_{t,m}} e^{-j2\pi(m-1)\omega_l} [\mathbf{U}_M]_{m,:} \mathbf{D}_M(\Delta \boldsymbol{\varphi}) \boldsymbol{x}_k \\ &= \sum_{m=1}^M e^{j\vartheta_{t,m}} \frac{\partial}{\partial \omega_l} (e^{-j2\pi(m-1)\omega_l}) [\mathbf{U}_M]_{m,:} \mathbf{D}_M(\Delta \boldsymbol{\varphi}) \boldsymbol{x}_k \\ &= \sum_{m=1}^M e^{j\vartheta_{t,m}} \left( -j2\pi(m-1) \right) e^{-j2\pi(m-1)\omega_l} [\mathbf{U}_M]_{m,:} \mathbf{D}_M(\Delta \boldsymbol{\varphi}) \boldsymbol{x}_k \end{split}$$

现在考虑 $\frac{\partial}{\partial (\mathbf{F}_{k,l}, \boldsymbol{x}_k)} \ln \left[ \mathcal{CN} \left( y_{k,l,t}; \mathbf{F}_{k,l,t} \boldsymbol{x}_k, \kappa_{k,l,t}^{-1} \right) \right]$ 

化简模型:  $\frac{\partial}{\partial \mu} \ln \left[ \mathcal{CN} \left( y; \mu, \kappa^{-1} \right) \right]$ 

$$p_{xy} = rac{1}{2\pi\sigma^2} e^{-rac{(x-\mu_x)^2+(y-\mu_y)^2}{2\sigma^2}} \ p_z = rac{1}{2\pi\sigma^2} e^{-rac{(z-\mu_z)^2}{2\sigma^2}} \ p_z = rac{1}{\pi\sigma_z^2} e^{-rac{(z-\mu_z)^2}{\sigma_z^2}}$$

复数求偏微分的方法如下;

$$z = x + iy$$

$$\frac{df}{dz} = \frac{1}{2} \left( \frac{df}{dx} - i \frac{df}{dy} \right)$$

$$\frac{dz}{dz} = 1, \frac{dz^*}{dz} = 0$$

复高斯分布可以写为:

$$\mathcal{CN}(y;\mu,\sigma^2) = rac{1}{2\pi\sigma^2}e^{-rac{(yx-\mu_x)^2+(yy-\mu_y)^2}{2\sigma^2}}$$

对 $\mu_x$ 和 $\mu_y$ 分开求导:

对于 $\mu_x$ ;

$$\frac{\partial}{\partial \mu_x} \mathcal{CN}(y;\mu,\sigma^2) = \frac{1}{2\pi\sigma^2} (-\frac{1}{2\sigma^2}) 2(y_x - \mu_x) (-1) e^{-\frac{(y_x - \mu_x)^2 + (y_y - \mu_y)^2}{2\sigma^2}}$$

对于 $\mu_y$ :

$$\frac{\partial}{\partial \mu_y} \mathcal{CN}(y;\mu,\sigma^2) = \frac{1}{2\pi\sigma^2} (-\frac{1}{2\sigma^2}) 2(y_y - \mu_y) (-1) e^{-\frac{(y_x - \mu_x)^2 + (y_y - \mu_y)^2}{2\sigma^2}}$$

合并:

$$\begin{split} \frac{\partial}{\partial \mu} \mathcal{CN}(y;\mu,\sigma^2) &= \frac{1}{2} \frac{1}{2\pi\sigma^2} (-\frac{1}{2\sigma^2}) 2(y_x - \mu_x) (-1) e^{-\frac{(y_x - \mu_x)^2 + (y_y - \mu_y)^2}{2\sigma^2}} \\ &\quad + \left( -\frac{1}{2}i \right) \frac{1}{2\pi\sigma^2} (-\frac{1}{2\sigma^2}) 2(y_y - \mu_y) (-1) e^{-\frac{(y_x - \mu_x)^2 + (y_y - \mu_y)^2}{2\sigma^2}} \\ &= \frac{1}{4\pi\sigma^4} (y_x - \mu_x) e^{-\frac{(y_x - \mu_x)^2 + (y_y - \mu_y)^2}{2\sigma^2}} + \\ &\quad (-i) \frac{1}{4\pi\sigma^4} (y_y - \mu_y) e^{-\frac{(y_x - \mu_x)^2 + (y_y - \mu_y)^2}{2\sigma^2}} \\ &= \frac{1}{4\pi\sigma^4} e^{-\frac{(y - \mu)^2}{2\sigma^2}} (y^* - \mu^*) \end{split}$$

考虑ln()带来对影响:

$$egin{aligned} rac{\partial}{\partial \mu} \ln \mathcal{CN}(y;\mu,\sigma^2) &= rac{1}{4\pi\sigma^4} e^{-rac{(y-\mu)^2}{2\sigma^2}} (y^*-\mu^*) \cdot \mathcal{CN}(y;\mu,\sigma^2)^{-1} \ &= rac{1}{4\pi\sigma^4} e^{-rac{(y-\mu)^2}{2\sigma^2}} (y^*-\mu^*) \cdot 2\pi\sigma^2 e^{rac{(y-\mu)^2}{2\sigma^2}} \ &= rac{1}{2\sigma^2} (y^*-\mu^*) \end{aligned}$$

则 $\sigma^2 = \kappa_{k,l,t}^{-1}, \mu = \mathbf{F}_{k,l,t}oldsymbol{x}_k$ :

$$\frac{\partial}{\partial \left(\mathbf{F}_{k,l,t} \boldsymbol{x}_{k}\right)} \mathrm{ln} \left[ \mathcal{CN} \left(y_{k,l,t}; \mathbf{F}_{k,l,t} \boldsymbol{x}_{k}, \kappa_{k,l,t}^{-1} \right) \right] = \underbrace{\frac{\kappa_{k,l,t}}{2} (y_{k,l,t}^{*} - (\mathbf{F}_{k,l,t} \boldsymbol{x}_{k})^{*})}_{\mathrm{part 1}}$$

将二级结论带入:

$$\begin{split} \frac{\partial}{\partial \omega_{l}} \ln p(\boldsymbol{y} \mid \boldsymbol{x}, \boldsymbol{\kappa}; \omega_{l}, \boldsymbol{\xi}_{1,-\omega_{l}}^{(i)}) &= \sum_{k=1}^{K} \sum_{t=1}^{\tau} \frac{\partial}{\partial (\mathbf{F}_{k,l,t} \boldsymbol{x}_{k})} \ln \left[ \mathcal{CN}(y_{k,l,t}; \mathbf{F}_{k,l,t} \boldsymbol{x}_{k}, \kappa_{k,t}^{-1}) \right] \frac{\partial}{\partial \omega_{l}} (\mathbf{F}_{k,l,t} \boldsymbol{x}_{k}) \\ &= \sum_{k=1}^{K} \sum_{t=1}^{\tau} \operatorname{part} 2 \cdot \operatorname{part} 1 \\ &= \sum_{k=1}^{K} \sum_{t=1}^{\tau} \frac{\kappa_{k,t}}{2\pi} (y_{k,l,t}^{*} - (\mathbf{F}_{k,l,t} \boldsymbol{x}_{k})^{*}) \cdot \\ &\sum_{m=1}^{M} e^{j\vartheta_{t,m}} \left( -j2\pi(M-1) \right) e^{-j2\pi(m-1)\omega_{l}} [\mathbf{U}_{M}]_{m,:} \mathbf{D}_{M}(\Delta \boldsymbol{\varphi}) \boldsymbol{x}_{k} \end{split}$$

化简表示:  $\mathbf{F}_1 \triangleq \mathbf{F}_{k,l,t}$ ,

$$\mathbf{F}_2 riangleq \sum_{m=1}^M e^{jartheta_{t,m}} ig( -j2\pi(M-1) ig) e^{-j2\pi(m-1)\omega_l} [\mathbf{U}_M]_{m,:} \mathbf{D}_M(\Deltaoldsymbol{arphi})$$

$$\mathbf{F}_3 \triangleq \mathbf{F}_1^H \mathbf{F}_2$$

$$c_1=rac{\kappa_{k,l,t}}{2}$$
 ,  $c_2=y_{k,l,t}^*$ 

则,偏导可以表示为:

$$egin{aligned} rac{\partial}{\partial \omega_l} & \ln p(oldsymbol{y} \mid oldsymbol{x}, oldsymbol{\kappa}; \omega_l, oldsymbol{\xi}_{1,-\omega_l}^{(i)}) = \sum_{k=1}^K \sum_{t=1}^ au c_1 ig( c_2 - (\mathbf{F}_1 oldsymbol{x}_k)^* ig) \cdot \mathbf{F}_2 oldsymbol{x}_k \ & = \sum_{k=1}^K \sum_{t=1}^ au \left( c_1 c_2 \mathbf{F}_2 oldsymbol{x}_k - c_1 oldsymbol{x}_k^H \mathbf{F}_1^H \mathbf{F}_2 oldsymbol{x}_k 
ight) \end{aligned}$$

接下来对其进行针对q(x)的积分:

$$egin{aligned} \int q(oldsymbol{x}) rac{\partial}{\partial \omega_l} & \ln p(oldsymbol{y} \mid oldsymbol{x}, oldsymbol{\kappa}; \omega_l, oldsymbol{\xi}_{1,-\omega_l}^{(i)}) doldsymbol{x} \ &= \sum_{k=1}^K \int q(oldsymbol{x}_k) \sum_{t=1}^ au \left( c_1 c_2 \mathbf{F}_2 oldsymbol{x}_k - c_1 oldsymbol{x}_k^H \mathbf{F}_1^H \mathbf{F}_2 oldsymbol{x}_k 
ight) doldsymbol{x}_k \ &= \sum_{k=1}^K \sum_{t=1}^ au \left\{ c_1 c_2 \mathbf{F}_2 \int q(oldsymbol{x}_k) oldsymbol{x}_k \cdot doldsymbol{x}_k - c_1 \int q(oldsymbol{x}_k) oldsymbol{x}_k^H \mathbf{F}_3 oldsymbol{x}_k \cdot doldsymbol{x}_k 
ight. \end{aligned}$$

Integration 1:

$$egin{aligned} \mathbf{F}_2 \int q(m{x}_k) m{x}_k \cdot dm{x}_k &= \mathbf{F}_2 \cdot \mathrm{E}_qig[m{x}_kig] \ &= \mathbf{F}_2 \cdot oldsymbol{\mu}_k \ & ext{the parameter in E-step} \end{aligned}$$

Integration 2:

首先展开 $q(\boldsymbol{x}_k)$ :

$$q(oldsymbol{x}_k) = \mathcal{CN}(oldsymbol{x}_k; \underbrace{oldsymbol{\mu}_k, oldsymbol{\Sigma}_k}_{ ext{the parameters in E-step}})$$

根据 The Matrix Cookbook [ <a href="http://matrixcookbook.com">http://matrixcookbook.com</a> ] Kaare Brandt Petersen Michael Syskind Pedersen Version: November 15, 2012:

$$egin{aligned} \int q(oldsymbol{x}_k) oldsymbol{x}_k^H \mathbf{F}_3 oldsymbol{x}_k \cdot doldsymbol{x}_k &= E_q[oldsymbol{x}_k^H \mathbf{F}_3 oldsymbol{x}_k] \ &= \operatorname{Tr}(\mathbf{F}_3 oldsymbol{\Sigma}_k) + oldsymbol{\mu}_k^H \mathbf{F}_3 oldsymbol{\mu}_k \end{aligned}$$

最终,将所有变量代换:

$$egin{aligned} rac{\partial}{\partial \omega_l} \hat{u}^{EM} \left( \omega_l, oldsymbol{\xi}_{-\omega_l}^{(i)}; \omega_l^{(i)}, oldsymbol{\xi}_{-\omega_l}^{(i)} 
ight) = \ & \sum_{k=1}^K \sum_{t=1}^ au \left( c_1 c_2 \mathbf{F}_2 oldsymbol{\mu}_k - c_1 ig( \operatorname{Tr}(\mathbf{F}_3 oldsymbol{\Sigma}_k) + oldsymbol{\mu}_k^H \mathbf{F}_3 oldsymbol{\mu}_k ig) 
ight) \end{aligned}$$

## 接下来求解 $\Delta \varphi_m$ :

原函数 $f_m(\varphi_m)$ 

$$f_m(arphi_m) = \int q(oldsymbol{x}) \Bigg\{ \sum_k^K \sum_l^L \sum_t^ au \ln \left[ \mathcal{CN} \left( y_{k,l,t}; \mathbf{F}_{k,l,t} oldsymbol{x}_k, \kappa_{k,t}^{-1} 
ight) 
ight] \Bigg\} doldsymbol{x}$$

首先:

$$rac{\partial}{\partial \Delta arphi_m} \hat{u}^{EM} \left( \omega_l, oldsymbol{\xi}_{-\omega_l}^{(i)}; \omega_l^{(i)}, oldsymbol{\xi}_{-\omega_l}^{(i)} 
ight) = \int q(oldsymbol{x}; \omega_l^{(i)}, oldsymbol{\xi}_{-\omega_l}^{(i)}) rac{\partial}{\partial \Delta arphi_m} \ln p(oldsymbol{y} \mid oldsymbol{x}, oldsymbol{\kappa}; \omega_l, oldsymbol{\xi}_{1,-\omega_l}^{(i)}) doldsymbol{x}$$

其中:

$$\begin{split} \frac{\partial}{\partial \Delta \varphi_m} & \ln p(\boldsymbol{y} \mid \boldsymbol{x}, \boldsymbol{\kappa}; \Delta \varphi_m, \boldsymbol{\xi}_{2, -\Delta \varphi_m}^{(i)}) \\ &= \sum_{k=1}^K \sum_{l=1}^L \sum_{t=1}^\tau \underbrace{\frac{\partial}{\partial (\mathbf{F}_{k,l,t} \boldsymbol{x}_k)} \ln \left[\mathcal{CN}(y_{k,l,t}; \mathbf{F}_{k,l,t} \boldsymbol{x}_k, \kappa_{k,t}^{-1})\right]}_{\text{Partial derivative 1}} \underbrace{\frac{\partial}{\partial \Delta \varphi_m} (\mathbf{F}_{k,l,t} \boldsymbol{x}_k)}_{\text{Partial derivative 2}} \end{split}$$

Partial derivative 1和上一部分一样:

$$\frac{\partial}{\partial \left(\mathbf{F}_{k,l,t} \boldsymbol{x}_{k}\right)} \ln \left[ \mathcal{CN}\left(y_{k,l,t}; \mathbf{F}_{k,l,t} \boldsymbol{x}_{k}, \kappa_{k,t}^{-1}\right) \right] = \underbrace{\frac{\kappa_{k,t}}{2\pi} (y_{k,l,t}^{*} - (\mathbf{F}_{k,l,t} \boldsymbol{x}_{k})^{*})}_{\text{Partial derivative 1}} \\ = c_{1} \left(c_{2} - (\mathbf{F}_{1} \boldsymbol{x}_{k})^{*}\right)$$

Partial derivative 2 写为:

$$\begin{split} \mathbf{F}_{k,l,t} \boldsymbol{x}_k &= \left[\mathbf{\Phi}^H \mathbf{V}(\omega_l) \mathbf{D}_M(\Delta \boldsymbol{\varphi})\right]_{t,:} \boldsymbol{x}_k \\ &= \underbrace{\left[\mathbf{\Phi}^H\right]_{t,:} \mathbf{V}(\omega_l)}_{(1 \times M)} \mathbf{D}_M(\Delta \boldsymbol{\varphi}) \boldsymbol{x}_k \\ &= \underbrace{\left[\mathbf{\Phi}^H\right]_{t,:} \mathbf{V}(\omega_l)}_{(1 \times M)} \begin{bmatrix} D_M(\Delta \varphi_1, 1) & \cdots & D_M(\Delta \varphi_m, 1) & \cdots & D_M(\Delta \varphi_M, 1) \\ D_M(\Delta \varphi_1, 2) & \cdots & D_M(\Delta \varphi_m, 2) & \cdots & D_M(\Delta \varphi_M, 2) \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ D_M(\Delta \varphi_1, m') & \cdots & D_M(\Delta \varphi_m, m') & \cdots & D_M(\Delta \varphi_M, m') \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ D_M(\Delta \varphi_1, M) & \cdots & D_M(\Delta \varphi_m, M) & \cdots & D_M(\Delta \varphi_M, M) \end{bmatrix} \boldsymbol{x}_k \end{split}$$

对其做偏导:

$$egin{aligned} rac{\partial}{\partial \Delta arphi_m} \mathbf{F}_{k,l,t} oldsymbol{x}_k = \underbrace{\left[ oldsymbol{\Phi}^H 
ight]_{t,:} \mathbf{V}(\omega_l)}_{(1 imes M)} egin{aligned} egin{aligned} 0 & \cdots & rac{\partial}{\partial \Delta arphi_m} D_M(\Delta arphi_m, 1) & \cdots & 0 \ 0 & \cdots & rac{\partial}{\partial \Delta arphi_m} D_M(\Delta arphi_m, 2) & \cdots & 0 \ dots & \cdots & dots & \cdots & dots \ 0 & \cdots & rac{\partial}{\partial \Delta arphi_m} D_M(\Delta arphi_m, m') & \cdots & 0 \ dots & \cdots & dots & \cdots & dots \ 0 & \cdots & rac{\partial}{\partial \Delta arphi_m} D_M(\Delta arphi_m, M) & \cdots & 0 \end{aligned} egin{aligned} oldsymbol{x}_k \end{aligned}$$

接下来求解 $\frac{\partial}{\partial \Delta \varphi_m} D_M (\Delta \varphi_m, m')$ :

$$D_M(\Deltaarphi_m,m') = egin{cases} f_M(2\pi(rac{m'-m}{M}-1+\Deltaarphi_m))\,,rac{m'-1}{M} < 0.5 \ f_M(2\pi(rac{m'-m}{M}+\Deltaarphi_m)),rac{m'-1}{M} \geq 0.5 \end{cases}$$

其中:

$$f_M(x) = \frac{1}{\sqrt{M}} e^{jx(M-1)/2} \frac{\sin(Mx/2)}{\sin(x/2)}$$
对其求偏导: 
$$((u \pm v)' = u' \pm v', \ \left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}) \ , \ (uv)' = u'v + v'u$$

$$f'_M(x) \triangleq \frac{\partial}{\partial x} f_M(x) = \frac{1}{\sqrt{M}} \frac{j(M-1)}{2} e^{jx(M-1)/2} \frac{\sin(Mx/2)}{\sin(x/2)}$$

$$+ \frac{1}{\sqrt{M}} e^{jx(M-1)/2} \frac{\frac{M}{2} \cos(Mx/2) \sin(x/2) - \frac{1}{2} \cos(x/2) \sin(Mx/2)}{\sin^2(x/2)}$$

$$\frac{\partial}{\partial \Delta \varphi_m} D_M(\Delta \varphi_m, m') = \begin{cases} 2\pi f'_M(x)|_{x=2\pi(\frac{m'-m}{M}-1+\Delta \varphi_m)} &, \frac{m'-1}{M} < 0.5 \\ 2\pi f'_M(x)|_{x=2\pi(\frac{m'-m}{M}+\Delta \varphi_m)} &, \frac{m'-1}{M} \geq 0.5 \end{cases}$$

定义辅助矩阵 $\mathbf{F}_4$ :

$$\mathbf{F}_4 = \underbrace{\left[ \mathbf{\Phi}^H 
ight]_{t,:}}^{\mathbf{V}(\omega_l)} egin{bmatrix} 0 & \cdots & rac{\partial}{\partial \Delta arphi_m} D_M(\Delta arphi_m, 1) & \cdots & 0 \ 0 & \cdots & rac{\partial}{\partial \Delta arphi_m} D_M(\Delta arphi_m, 2) & \cdots & 0 \ dots & \cdots & dots & \cdots & dots \ 0 & \cdots & rac{\partial}{\partial \Delta arphi_m} D_M(\Delta arphi_m, m') & \cdots & 0 \ dots & \cdots & dots & \cdots & dots \ 0 & \cdots & rac{\partial}{\partial \Delta arphi_m} D_M(\Delta arphi_m, m') & \cdots & 0 \ \end{bmatrix}$$

Partial derivative 2 最终写为:

$$rac{\partial}{\partial \Delta arphi_m} \mathbf{F}_{k,l,t} oldsymbol{x}_k = \mathbf{F}_4 oldsymbol{x}_k$$

定义 $\mathbf{F}_5 \triangleq \mathbf{F}_1^H \mathbf{F}_4$ ,则:,偏导可以表示为:

$$egin{aligned} rac{\partial}{\partial \Delta arphi_m} & \ln p(oldsymbol{y} \mid oldsymbol{x}, oldsymbol{\kappa}; \Delta arphi_m, oldsymbol{\xi}_{1, -\Delta arphi_m}^{(i)}) = \sum_{k=1}^K \sum_{t=1}^ au c_1 ig( c_2 - (\mathbf{F}_1 oldsymbol{x}_k)^* ig) \cdot \mathbf{F}_4 oldsymbol{x}_k \ & = \sum_{k=1}^K \sum_{t=1}^ au ig( c_1 c_2 \mathbf{F}_4 oldsymbol{x}_k - c_1 oldsymbol{x}_k^H \mathbf{F}_5 oldsymbol{x}_k ig) \end{aligned}$$

对其进行积分有:

$$egin{aligned} \int &q(oldsymbol{x})rac{\partial}{\partial\Deltaarphi_m} \ln p(oldsymbol{y}\midoldsymbol{x},oldsymbol{\kappa};\Deltaarphi_m,oldsymbol{\xi}_{1,-\Deltaarphi_m}^{(i)})doldsymbol{x} \ &=\sum_{k=1}^K\sum_{l=1}^L\sum_{t=1}^ au\int q(oldsymbol{x}_k)\Big(c_1c_2oldsymbol{F}_4oldsymbol{x}_k-c_1oldsymbol{x}_k^Holdsymbol{F}_1^Holdsymbol{F}_4oldsymbol{x}_k\Big)doldsymbol{x}_k \ &=\sum_{k=1}^K\sum_{l=1}^L\sum_{t=1}^ au\Big\{c_1c_2oldsymbol{F}_4\int q(oldsymbol{x}_k)oldsymbol{x}_k\cdot doldsymbol{x}_k - c_1oldsymbol{\int}q(oldsymbol{x}_k)oldsymbol{x}_k^Holdsymbol{F}_5oldsymbol{x}_k\cdot doldsymbol{x}_k\Big\} \ &=\sum_{k=1}^K\sum_{l=1}^L\sum_{t=1}^ au\Big\{c_1c_2oldsymbol{F}_4\int q(oldsymbol{x}_k)oldsymbol{x}_k\cdot doldsymbol{x}_k - c_1oldsymbol{\int}q(oldsymbol{x}_k)oldsymbol{x}_k^Holdsymbol{F}_5oldsymbol{x}_k\cdot doldsymbol{x}_k\Big\} \ &=\sum_{k=1}^K\sum_{l=1}^L\sum_{t=1}^ au\Big\{c_1c_2oldsymbol{F}_4\int q(oldsymbol{x}_k)oldsymbol{x}_k\cdot doldsymbol{x}_k - c_1oldsymbol{x}_koldsymbol{x}_k^Holdsymbol{F}_5oldsymbol{x}_k\cdot doldsymbol{x}_k\Big\} \ &=\sum_{k=1}^K\sum_{l=1}^L\sum_{t=1}^ au\Big\{c_1c_2oldsymbol{F}_4\int q(oldsymbol{x}_k)oldsymbol{x}_k\cdot doldsymbol{x}_k - c_1oldsymbol{x}_koldsymbol{x}_k^Holdsymbol{F}_5oldsymbol{x}_k\cdot doldsymbol{x}_k\Big\} \ &=\sum_{k=1}^K\sum_{l=1}^L\sum_{t=1}^ au\Big\{c_1c_2oldsymbol{x}_k^Holdsymbol{x}_k\cdot doldsymbol{x}_k - c_1oldsymbol{x}_k^Holdsymbol{x}_k^Holdsymbol{x}_k\cdot doldsymbol{x}_k\Big\} \ &=\sum_{k=1}^K\sum_{l=1}^L\sum_{t=1}^ au\Big\{c_1c_2oldsymbol{x}_k^Holdsymbol{x}_k^Holdsymbol{x}_k\cdot doldsymbol{x}_k^Holdsymbol{x}_k^Holdsymbol{x}_k^Holdsymbol{x}_k^Holdsymbol{x}_k^Holdsymbol{x}_k\Big\} \ &=\sum_{l=1}^K\sum_{l=1}^ au\Big\{c_1c_2oldsymbol{x}_k^Holdsymbol{$$

对于integration 1来说:

$$\mathbf{F}_4 \int q(m{x}_k) m{x}_k \cdot dm{x}_k = \mathbf{F}_4 \cdot \mathrm{E}_qig[m{x}_kig] \ = \mathbf{F}_4 \cdot oldsymbol{\mu}_k$$
 the parameter in E-step

对于integration 2来说:

$$egin{aligned} \int q(oldsymbol{x}_k) oldsymbol{x}_k^H \mathbf{F}_5 oldsymbol{x}_k \cdot doldsymbol{x}_k &= E_q[oldsymbol{x}_k^H \mathbf{F}_5 oldsymbol{x}_k] \ &= \mathrm{Tr}(\mathbf{F}_5 oldsymbol{\Sigma}_k) + oldsymbol{\mu}_k^H \mathbf{F}_5 oldsymbol{\mu}_k \end{aligned}$$

最终,将所有变量代换:

$$egin{aligned} rac{\partial}{\partial \Delta arphi_m} \hat{u}^{EM} \left( \Delta arphi_m, oldsymbol{\xi}_{-\Delta arphi_m}^{(i)}; \Delta arphi_m^{(i)}, oldsymbol{\xi}_{-\Delta arphi_m}^{(i)} 
ight) = \ & \sum_{k=1}^K \sum_{l=1}^L \sum_{t=1}^{ au} \left\{ c_1 c_2 \mathbf{F}_4 oldsymbol{\mu}_k - c_1 ig( \operatorname{Tr}(\mathbf{F}_5 oldsymbol{\Sigma}_k) + oldsymbol{\mu}_k^H \mathbf{F}_5 oldsymbol{\mu}_k ig) 
ight\} \end{aligned}$$

## 结论:

### 对于 $\omega_l$

$$egin{aligned} rac{\partial}{\partial \omega_l} \hat{u}^{EM} \left( \omega_l, oldsymbol{\xi}_{-\omega_l}^{(i)}; oldsymbol{\omega}_l^{(i)}, oldsymbol{\xi}_{-\omega_l}^{(i)} 
ight) = \ & \sum_{k=1}^K \sum_{t=1}^ au \left( c_1 c_2 \mathbf{F}_2 oldsymbol{\mu}_k - c_1 ig( \operatorname{Tr}(\mathbf{F}_3 oldsymbol{\Sigma}_k) + oldsymbol{\mu}_k^H \mathbf{F}_3 oldsymbol{\mu}_k ig) 
ight) \ & \propto \sum_{k=1}^K \sum_{t=1}^ au \left( c_2 \mathbf{F}_2 oldsymbol{\mu}_k - \operatorname{Tr}(\mathbf{F}_3 oldsymbol{\Sigma}_k) - oldsymbol{\mu}_k^H \mathbf{F}_3 oldsymbol{\mu}_k ig) \end{aligned}$$

其中:

$$c_1=rac{\kappa_{k,l,t}}{2}$$
 ,  $c_2=y_{k,l,t}^*$ 

$$\mathbf{F}_1 \triangleq \mathbf{F}_{k,l,t}$$

$$\mathbf{F}_2 riangleq \sum_{m=1}^M e^{jartheta_{t,m}} ig( -j2\pi(M-1) ig) e^{-j2\pi(m-1)\omega_l} [\mathbf{U}_M]_{m,:} \mathbf{D}_M(\Deltaoldsymbol{arphi})$$

$$\mathbf{F}_3 \triangleq \mathbf{F}_1^H \mathbf{F}_2$$

## 对于 $\Delta \varphi_m$

$$egin{aligned} rac{\partial}{\partial \Delta arphi_m} \hat{u}^{EM} \left( \Delta arphi_m, oldsymbol{\xi}_{-\Delta arphi_m}^{(i)}; \Delta arphi_m^{(i)}, oldsymbol{\xi}_{-\Delta arphi_m}^{(i)} 
ight) = \ & \sum_{k=1}^K \sum_{l=1}^L \sum_{t=1}^ au \left\{ c_1 c_2 \mathbf{F}_4 oldsymbol{\mu}_k - c_1 ig( \operatorname{Tr}(\mathbf{F}_5 oldsymbol{\Sigma}_k) + oldsymbol{\mu}_k^H \mathbf{F}_5 oldsymbol{\mu}_k ig) 
ight\} \ & \propto \sum_{k=1}^K \sum_{l=1}^L \sum_{t=1}^ au \left\{ c_2 \mathbf{F}_4 oldsymbol{\mu}_k - \operatorname{Tr}(\mathbf{F}_5 oldsymbol{\Sigma}_k) - oldsymbol{\mu}_k^H \mathbf{F}_5 oldsymbol{\mu}_k 
ight\} \end{aligned}$$

其中:

$$c_1=rac{\kappa_{k,l,t}}{2}$$
 ,  $c_2=y_{k,l,t}^*$ 

$$\mathbf{F}_4 = \underbrace{\left[ \mathbf{\Phi}^H 
ight]_{t,:} \mathbf{V}(\omega_l)}_{(1 imes M)} egin{bmatrix} 0 & \cdots & rac{\partial}{\partial \Delta arphi_m} D_M(\Delta arphi_m, 1) & \cdots & 0 \ 0 & \cdots & rac{\partial}{\partial \Delta arphi_m} D_M(\Delta arphi_m, 2) & \cdots & 0 \ dots & \cdots & dots & \cdots & dots \ 0 & \cdots & rac{\partial}{\partial \Delta arphi_m} D_M(\Delta arphi_m, m') & \cdots & 0 \ dots & \cdots & dots & \cdots & dots \ 0 & \cdots & rac{\partial}{\partial \Delta arphi_m} D_M(\Delta arphi_m, m') & \cdots & 0 \ \end{bmatrix}$$

$$\mathbf{F}_5 \triangleq \mathbf{F}_1^H \mathbf{F}_4$$

where:

$$rac{\partial}{\partial \Delta arphi_m} D_M(\Delta arphi_m, m') = egin{cases} 2\pi f_M'(x)|_{x=2\pi(rac{m'-m}{M}-1+\Delta arphi_m)} &, rac{m'-1}{M} < 0.5 \ 2\pi f_M'(x)|_{x=2\pi(rac{m'-m}{M}+\Delta arphi_m)} &, rac{m'-1}{M} \geq 0.5 \end{cases}$$

and

$$f_M'(x) riangleq rac{\partial}{\partial x} f_M(x) = rac{1}{\sqrt{M}} rac{j(M-1)}{2} e^{jx(M-1)/2} rac{\sin(Mx/2)}{\sin(x/2)} \ + rac{1}{\sqrt{M}} e^{jx(M-1)/2} rac{rac{M}{2} \cos(Mx/2) \sin(x/2) - rac{1}{2} \cos(x/2) \sin(Mx/2)}{\sin^2(x/2)}$$