update $\boldsymbol{\xi}$

$\boldsymbol{\xi}_1 = \{\omega_1, \ldots, \omega_L\}$

$\boldsymbol{\xi}_2 = \{\Delta\varphi_1, \ldots, \Delta\varphi_M\}$

$\boldsymbol{\xi}_3 = \{\kappa_{1,1}, \ldots, \kappa_{k,t}, \ldots, \kappa_{K,\tau}\}$

$\boldsymbol{\xi}_4 = \left\{\lambda^c, p_{01}^c, p_{10}^c, \mu_1^s, \sigma_1^s, \ldots, \mu_k^s, \sigma_k^s\right\}$

$\boldsymbol{v} \triangleq \{\boldsymbol{x}, \boldsymbol{\gamma}, \boldsymbol{c}, \boldsymbol{s}\}$

$$\boldsymbol{\xi}_j^{(i+1)} = \boldsymbol{\xi}_j^{(i)} + \gamma^{(i)} \frac{\partial u\left(\boldsymbol{\xi}_j, \boldsymbol{\xi}_{-j}^{(i)}; \boldsymbol{\xi}_j^{(i)}, \boldsymbol{\xi}_{-j}^{(i)}\right)}{\partial \boldsymbol{\xi}_j}\Bigg|_{\boldsymbol{\xi}_j = \boldsymbol{\xi}_j^{(i)}}$$

$$u(\boldsymbol{\xi}; \dot{\boldsymbol{\xi}}) = u^{\mathrm{EM}}(\boldsymbol{\xi}; \dot{\boldsymbol{\xi}}) + \sum_{j \in \mathcal{J}_c^1} \tau_j \left\|\boldsymbol{\xi}_j - \dot{\boldsymbol{\xi}}_j\right\|^2$$

$$\begin{aligned} u^{\mathrm{EM}}(\boldsymbol{\xi}; \dot{\boldsymbol{\xi}}) &= \int p(\boldsymbol{v} \mid \boldsymbol{p}, \dot{\boldsymbol{\xi}}) \ln \frac{p(\boldsymbol{v}, \boldsymbol{p}, \boldsymbol{\xi})}{p(\boldsymbol{v} \mid \boldsymbol{p}, \dot{\boldsymbol{\xi}})} d\boldsymbol{v} \\ &\approx \int q(\boldsymbol{v}; \dot{\boldsymbol{\xi}}) \ln \frac{p(\boldsymbol{v}, \boldsymbol{p}, \boldsymbol{\xi})}{q(\boldsymbol{v}; \dot{\boldsymbol{\xi}})} d\boldsymbol{v} \end{aligned}$$

首先求$\frac{\partial}{\partial \boldsymbol{\xi}_j} \hat{u}^{EM}(\boldsymbol{\xi}_j, \boldsymbol{\xi}_{-j}^{(i)}; (\boldsymbol{\xi}_j^{(i)}, \boldsymbol{\xi}_{-j}^{(i)})$:

$$\begin{aligned} \frac{\partial}{\partial \boldsymbol{\xi}_j} \hat{u}^{EM}(\boldsymbol{\xi}_j, \boldsymbol{\xi}_{-j}^{(i)}; \boldsymbol{\xi}_j^{(i)}, \boldsymbol{\xi}_{-j}^{(i)}) &= \frac{\partial}{\partial \boldsymbol{\xi}_j}\left\{\left[\int q(\boldsymbol{v}; \boldsymbol{\xi}_j^{(i)}, \boldsymbol{\xi}_{-j}^{(i)}) \ln p(\boldsymbol{v}, \boldsymbol{y}; \boldsymbol{\xi}_j, \boldsymbol{\xi}_{-j}^{(i)}) d\boldsymbol{v} - \int q\left(\boldsymbol{v}; \boldsymbol{\xi}_j^{(i)}, \boldsymbol{\xi}_{-j}^{(i)}\right) \ln q\left(\boldsymbol{v}; \boldsymbol{\xi}_j^{(i)}, \boldsymbol{\xi}_{-j}^{(i)}\right) d\boldsymbol{v}\right]\right\} \\ &= \frac{\partial}{\partial \boldsymbol{\xi}_j}\left[\int q(\boldsymbol{v}; \boldsymbol{\xi}_j^{(i)}, \boldsymbol{\xi}_{-j}^{(i)}) \ln p(\boldsymbol{v}, \boldsymbol{y}; \boldsymbol{\xi}_j, \boldsymbol{\xi}_{-j}^{(i)}) d\boldsymbol{v}\right] \\ &= \int q(\boldsymbol{v}; \boldsymbol{\xi}_j^{(i)}, \boldsymbol{\xi}_{-j}^{(i)}) \frac{\partial}{\partial \boldsymbol{\xi}_j}\left[\ln p(\boldsymbol{v}, \boldsymbol{y}; \boldsymbol{\xi}_j, \boldsymbol{\xi}_{-j}^{(i)})\right] d\boldsymbol{v} \end{aligned}$$

注意到$p(\boldsymbol{v}, \boldsymbol{y}; \boldsymbol{\xi}_j, \boldsymbol{\xi}_{-j}^{(i)})$ 可以被进一步分解:

$$\begin{aligned} p(\boldsymbol{v}, \boldsymbol{y}; \boldsymbol{\xi}_j, \boldsymbol{\xi}_{-j}^{(i)}) &= \underbrace{p(\boldsymbol{x} \mid \boldsymbol{\gamma}) p(\boldsymbol{\kappa}) p(\boldsymbol{\gamma} \mid \boldsymbol{s})}_{\text{known distribution}} \underbrace{p(\boldsymbol{y} \mid \boldsymbol{x}, \boldsymbol{\kappa}; \boldsymbol{\xi}) p(\boldsymbol{c}, \boldsymbol{s}; \boldsymbol{\xi})}_{\text{with unknown valuables}} \\ \ln p(\boldsymbol{v}, \boldsymbol{y}; \boldsymbol{\xi}_j, \boldsymbol{\xi}_{-j}^{(i)}) &= \ln p(\boldsymbol{x} \mid \boldsymbol{\gamma}) + \ln p(\boldsymbol{\kappa}) + \ln p(\boldsymbol{\gamma} \mid \boldsymbol{s}) \\ &\quad + \ln p(\boldsymbol{y} \mid \boldsymbol{x}, \boldsymbol{\kappa}; \boldsymbol{\xi}_{1,2,3}) + \ln p(\boldsymbol{c}, \boldsymbol{s}; \boldsymbol{\xi}_4) \end{aligned}$$

将其带入上式可得:

$$\begin{aligned} \frac{\partial}{\partial \boldsymbol{\xi}_j} \hat{u}^{EM}\left(\boldsymbol{\xi}_j, \boldsymbol{\xi}_{-j}^{(i)}; \boldsymbol{\xi}_j^{(i)}, \boldsymbol{\xi}_{-j}^{(i)}\right) &= \int q(\boldsymbol{v}; \boldsymbol{\xi}_j^{(i)}, \boldsymbol{\xi}_{-j}^{(i)}) \frac{\partial}{\partial \boldsymbol{\xi}_j}\left[\ln p(\boldsymbol{v}, \boldsymbol{y}; \boldsymbol{\xi}_j, \boldsymbol{\xi}_{-j}^{(i)})\right] d\boldsymbol{v} \\ &= \int q(\boldsymbol{v}; \boldsymbol{\xi}_j^{(i)}, \boldsymbol{\xi}_{-j}^{(i)}) \frac{\partial}{\partial \boldsymbol{\xi}_j}[\ln p(\boldsymbol{y} \mid \boldsymbol{x}, \boldsymbol{\kappa}; \boldsymbol{\xi}_{1,2,3}) + \ln p(\boldsymbol{c}, \boldsymbol{s}; \boldsymbol{\xi}_4)] d\boldsymbol{v} \\ &= \begin{cases} \int q(\boldsymbol{v}; \boldsymbol{\xi}_j^{(i)}, \boldsymbol{\xi}_{-j}^{(i)}) \frac{\partial}{\partial \boldsymbol{\xi}_j} \ln p(\boldsymbol{y} \mid \boldsymbol{x}, \boldsymbol{\kappa}; \boldsymbol{\xi}_{1,2,3}) d\boldsymbol{v} & , j \in \{1, 2\} \\ \int q(\boldsymbol{v}; \boldsymbol{\xi}_j^{(i)}, \boldsymbol{\xi}_{-j}^{(i)}) \frac{\partial}{\partial \boldsymbol{\xi}_j} \ln p(\boldsymbol{c}, \boldsymbol{s}; \boldsymbol{\xi}_3) d\boldsymbol{v} & , j = 4 \end{cases} \\ &= \begin{cases} \int q(\boldsymbol{x}; \boldsymbol{\xi}_j^{(i)}, \boldsymbol{\xi}_{-j}^{(i)}) \frac{\partial}{\partial \boldsymbol{\xi}_j} \ln p(\boldsymbol{y} \mid \boldsymbol{x}, \boldsymbol{\kappa}; \boldsymbol{\xi}_{1,2,3}) d\boldsymbol{x} & , j \in \{1, 2\} \\ \int q(\boldsymbol{c}, \boldsymbol{s}; \boldsymbol{\xi}_j^{(i)}, \boldsymbol{\xi}_{-j}^{(i)}) \frac{\partial}{\partial \boldsymbol{\xi}_j} \ln p(\boldsymbol{c}, \boldsymbol{s}; \boldsymbol{\xi}_3) d\boldsymbol{c} d\boldsymbol{s} & , j = 4 \end{cases} \end{aligned}$$

首先以求解ω_l为例:

设置求导前等价函数$f_l(\omega_l)$:

$$f_l(\omega_l) \triangleq \int q(\boldsymbol{x}) \left\{\sum_{k=1}^{K} \sum_{t=1}^{\tau} \ln\left[\mathcal{CN}\left(y_{k,l,t}; \mathbf{F}_{k,l,t} \boldsymbol{x}_k, \kappa_{k,t}^{-1}\right)\right]\right\} d\boldsymbol{x}$$

同时，多维等价函数为$f_L(\boldsymbol{\omega})$:

$$f_L(\boldsymbol{\omega}) = \sum_{l=1}^{L} f_l(\omega_l)$$

$$\frac{\partial}{\partial \omega_l}\hat{u}^{EM}\left(\omega_l, \boldsymbol{\xi}_{-\omega_l}^{(i)}; \omega_l^{(i)}, \boldsymbol{\xi}_{-\omega_l}^{(i)}\right) = \int q(\boldsymbol{x}; \omega_l^{(i)}, \boldsymbol{\xi}_{-\omega_l}^{(i)})\frac{\partial}{\partial \omega_l}\ln p(\boldsymbol{y} \mid \boldsymbol{x}, \boldsymbol{\kappa}; \omega_l, \boldsymbol{\xi}_{1,-\omega_l}^{(i)})d\boldsymbol{x}$$

其中:

$$\begin{aligned}
\frac{\partial}{\partial \omega_l}\ln p(\boldsymbol{y} \mid \boldsymbol{x}, \boldsymbol{\kappa}; \omega_l, \boldsymbol{\xi}_{1,-\omega_l}^{(i)}) &= \frac{\partial}{\partial \omega_l}\sum_{k=1}^{K}\sum_{l=1}^{L}\sum_{t=1}^{\tau}\ln\left[\mathcal{CN}(y_{k,l,t}; \mathbf{F}_{k,l,t}\boldsymbol{x}_k, \kappa_{k,t}^{-1})\right] \\
&= \frac{\partial}{\partial \omega_l}\sum_{k=1}^{K}\sum_{t=1}^{\tau}\ln\left[\mathcal{CN}(y_{k,l,t}; \mathbf{F}_{k,l,t}\boldsymbol{x}_k, \kappa_{k,t}^{-1})\right] \\
&= \sum_{k=1}^{K}\sum_{t=1}^{\tau}\frac{\partial}{\partial \omega_l}\ln\left[\mathcal{CN}(y_{k,l,t}; \mathbf{F}_{k,l,t}\boldsymbol{x}_k, \kappa_{k,t}^{-1})\right] \\
&= \sum_{k=1}^{K}\sum_{t=1}^{\tau}\underbrace{\frac{\partial}{\partial(\mathbf{F}_{k,l,t}\boldsymbol{x}_k)}\ln\left[\mathcal{CN}(y_{k,l,t}; \mathbf{F}_{k,l,t}\boldsymbol{x}_k, \kappa_{k,t}^{-1})\right]}_{\text{part 1}}\underbrace{\frac{\partial}{\partial \omega_l}(\mathbf{F}_{k,l,t}\boldsymbol{x}_k)}_{\text{part 2}}
\end{aligned}$$

接下来将被求和项目展开，其中

$$\begin{aligned}
\mathbf{F}_{k,l,t}\boldsymbol{x}_k &= \left[\boldsymbol{\Phi}^H\mathbf{V}(\omega_l)\mathbf{D}_M(\Delta\boldsymbol{\varphi})\right]_{t,:}\boldsymbol{x}_k \\
&= \left[\boldsymbol{\Phi}^H\right]_{t,:}\mathbf{V}(\omega_l)\mathbf{D}_M(\Delta\boldsymbol{\varphi})\boldsymbol{x}_k \\
&= \begin{bmatrix} e^{j\vartheta_{t,1}} & \cdots & e^{j\vartheta_{t,M}} \end{bmatrix}\begin{bmatrix} \mathbf{a}_M(\omega_l)_1[\mathbf{U}_M]_{1,:} \\ \vdots \\ \mathbf{a}_M(\omega_l)_M[\mathbf{U}_M]_{M,:} \end{bmatrix}\mathbf{D}_M(\Delta\boldsymbol{\varphi})\boldsymbol{x}_k \\
&= \sum_{m=1}^{M} e^{j\vartheta_{t,m}}\mathbf{a}_M(\omega_l)_m[\mathbf{U}_M]_{m,:}\mathbf{D}_M(\Delta\boldsymbol{\varphi})\boldsymbol{x}_k \\
&= \sum_{m=1}^{M} e^{j\vartheta_{t,m}}e^{-j2\pi(m-1)\omega_l}[\mathbf{U}_M]_{m,:}\mathbf{D}_M(\Delta\boldsymbol{\varphi})\boldsymbol{x}_k \\
\frac{\partial}{\partial \omega_l}\mathbf{F}_{k,l,t}\boldsymbol{x}_k &= \sum_{m=1}^{M} e^{j\vartheta_{t,m}}\frac{\partial}{\partial \omega_l}(e^{-j2\pi(m-1)\omega_l})[\mathbf{U}_M]_{m,:}\mathbf{D}_M(\Delta\boldsymbol{\varphi})\boldsymbol{x}_k \\
&= \underbrace{\sum_{m=1}^{M} e^{j\vartheta_{t,m}}\big(-j2\pi(m-1)\big)e^{-j2\pi(m-1)\omega_l}[\mathbf{U}_M]_{m,:}\mathbf{D}_M(\Delta\boldsymbol{\varphi})\boldsymbol{x}_k}_{\text{part 2}}
\end{aligned}$$

现在考虑$\frac{\partial}{\partial(\mathbf{F}_{k,l,t}\boldsymbol{x}_k)}\ln\left[\mathcal{CN}\left(y_{k,l,t}; \mathbf{F}_{k,l,t}\boldsymbol{x}_k, \kappa_{k,l,t}^{-1}\right)\right]$

化简模型：$\frac{\partial}{\partial \mu}\ln\left[\mathcal{CN}\left(y; \mu, \kappa^{-1}\right)\right]$

$$\begin{aligned}
p_{xy} &= \frac{1}{2\pi\sigma^2}e^{-\frac{(x-\mu_x)^2+(y-\mu_y)^2}{2\sigma^2}} \\
p_z &= \frac{1}{2\pi\sigma^2}e^{-\frac{(z-\mu_z)^2}{2\sigma^2}} \\
p_z &= \frac{1}{\pi\sigma_z^2}e^{-\frac{(z-\mu_z)^2}{\sigma_z^2}}
\end{aligned}$$

复数求偏微分的方法如下;

$$\begin{aligned}
z &= x + iy \\
\frac{df}{dz} &= \frac{1}{2}\left(\frac{df}{dx} - i\frac{df}{dy}\right) \\
\frac{dz}{dz} &= 1, \frac{dz^*}{dz} = 0
\end{aligned}$$

复高斯分布可以写为:

$$\mathcal{CN}(y;\mu,\sigma^2)=\frac{1}{2\pi\sigma^2}e^{-\frac{(y_x-\mu_x)^2+(y_y-\mu_y)^2}{2\sigma^2}}$$

对μ_x和μ_y分开求导:

对于μ_x:

$$\frac{\partial}{\partial\mu_x}\mathcal{CN}(y;\mu,\sigma^2)=\frac{1}{2\pi\sigma^2}(-\frac{1}{2\sigma^2})2(y_x-\mu_x)(-1)e^{-\frac{(y_x-\mu_x)^2+(y_y-\mu_y)^2}{2\sigma^2}}$$

对于μ_y:

$$\frac{\partial}{\partial\mu_y}\mathcal{CN}(y;\mu,\sigma^2)=\frac{1}{2\pi\sigma^2}(-\frac{1}{2\sigma^2})2(y_y-\mu_y)(-1)e^{-\frac{(y_x-\mu_x)^2+(y_y-\mu_y)^2}{2\sigma^2}}$$

合并:

$$\begin{aligned}\frac{\partial}{\partial\mu}\mathcal{CN}(y;\mu,\sigma^2)&=\frac{1}{2}\frac{1}{2\pi\sigma^2}(-\frac{1}{2\sigma^2})2(y_x-\mu_x)(-1)e^{-\frac{(y_x-\mu_x)^2+(y_y-\mu_y)^2}{2\sigma^2}}\\&+\left(-\frac{1}{2}i\right)\frac{1}{2\pi\sigma^2}(-\frac{1}{2\sigma^2})2(y_y-\mu_y)(-1)e^{-\frac{(y_x-\mu_x)^2+(y_y-\mu_y)^2}{2\sigma^2}}\\&=\frac{1}{4\pi\sigma^4}(y_x-\mu_x)e^{-\frac{(y_x-\mu_x)^2+(y_y-\mu_y)^2}{2\sigma^2}}+\\&\quad(-i)\frac{1}{4\pi\sigma^4}(y_y-\mu_y)e^{-\frac{(y_x-\mu_x)^2+(y_y-\mu_y)^2}{2\sigma^2}}\\&=\frac{1}{4\pi\sigma^4}e^{-\frac{(y-\mu)^2}{2\sigma^2}}(y^*-\mu^*)\end{aligned}$$

考虑$\ln()$带来对影响:

$$\begin{aligned}\frac{\partial}{\partial\mu}\ln\mathcal{CN}(y;\mu,\sigma^2)&=\frac{1}{4\pi\sigma^4}e^{-\frac{(y-\mu)^2}{2\sigma^2}}(y^*-\mu^*)\cdot\mathcal{CN}(y;\mu,\sigma^2)^{-1}\\&=\frac{1}{4\pi\sigma^4}e^{-\frac{(y-\mu)^2}{2\sigma^2}}(y^*-\mu^*)\cdot2\pi\sigma^2e^{\frac{(y-\mu)^2}{2\sigma^2}}\\&=\frac{1}{2\sigma^2}(y^*-\mu^*)\end{aligned}$$

则$\sigma^2=\kappa_{k,l,t}^{-1},\mu=\mathbf{F}_{k,l,t}\boldsymbol{x}_k$:

$$\frac{\partial}{\partial\left(\mathbf{F}_{k,l,t}\boldsymbol{x}_k\right)}\ln\left[\mathcal{CN}\left(y_{k,l,t};\mathbf{F}_{k,l,t}\boldsymbol{x}_k,\kappa_{k,l,t}^{-1}\right)\right]=\underbrace{\frac{\kappa_{k,l,t}}{2}(y_{k,l,t}^*-(\mathbf{F}_{k,l,t}\boldsymbol{x}_k)^*)}_{\text{part 1}}$$

将二级结论带入:

$$\begin{aligned}\frac{\partial}{\partial\omega_l}\ln p(\boldsymbol{y}\mid\boldsymbol{x},\boldsymbol{\kappa};\omega_l,\boldsymbol{\xi}_{1,-\omega_l}^{(i)})&=\sum_{k=1}^{K}\sum_{t=1}^{\tau}\frac{\partial}{\partial(\mathbf{F}_{k,l,t}\boldsymbol{x}_k)}\ln\left[\mathcal{CN}(y_{k,l,t};\mathbf{F}_{k,l,t}\boldsymbol{x}_k,\kappa_{k,t}^{-1})\right]\frac{\partial}{\partial\omega_l}(\mathbf{F}_{k,l,t}\boldsymbol{x}_k)\\&=\sum_{k=1}^{K}\sum_{t=1}^{\tau}\text{part 2}\cdot\text{part 1}\\&=\sum_{k=1}^{K}\sum_{t=1}^{\tau}\frac{\kappa_{k,t}}{2\pi}(y_{k,l,t}^*-(\mathbf{F}_{k,l,t}\boldsymbol{x}_k)^*)\cdot\\&\quad\sum_{m=1}^{M}e^{j\vartheta_{t,m}}\big(-j2\pi(M-1)\big)e^{-j2\pi(m-1)\omega_l}[\mathbf{U}_M]_{m,:}\mathbf{D}_M(\Delta\boldsymbol{\varphi})\boldsymbol{x}_k\end{aligned}$$

化简表示：$\mathbf{F}_1\triangleq\mathbf{F}_{k,l,t}$,

$\mathbf{F}_2\triangleq\sum_{m=1}^{M}e^{j\vartheta_{t,m}}\big(-j2\pi(M-1)\big)e^{-j2\pi(m-1)\omega_l}[\mathbf{U}_M]_{m,:}\mathbf{D}_M(\Delta\boldsymbol{\varphi})$

$\mathbf{F}_3\triangleq\mathbf{F}_1^H\mathbf{F}_2$

$c_1=\frac{\kappa_{k,l,t}}{2},c_2=y_{k,l,t}^*$

则，偏导可以表示为:

$$\begin{aligned}\frac{\partial}{\partial \omega_l}\ln p(\boldsymbol{y} \mid \boldsymbol{x}, \boldsymbol{\kappa}; \omega_l, \boldsymbol{\xi}^{(i)}_{1,-\omega_l}) &= \sum_{k=1}^{K}\sum_{t=1}^{\tau} c_1\left(c_2 - (\mathbf{F}_1\boldsymbol{x}_k)^*\right)\cdot \mathbf{F}_2\boldsymbol{x}_k \\ &= \sum_{k=1}^{K}\sum_{t=1}^{\tau}\left(c_1c_2\mathbf{F}_2\boldsymbol{x}_k - c_1\boldsymbol{x}_k^H\mathbf{F}_1^H\mathbf{F}_2\boldsymbol{x}_k\right)\end{aligned}$$

接下来对其进行针对$q(\boldsymbol{x})$的积分：

$$\begin{aligned}&\int q(\boldsymbol{x})\frac{\partial}{\partial \omega_l}\ln p(\boldsymbol{y} \mid \boldsymbol{x}, \boldsymbol{\kappa}; \omega_l, \boldsymbol{\xi}^{(i)}_{1,-\omega_l})d\boldsymbol{x} \\ &= \sum_{k=1}^{K}\int q(\boldsymbol{x}_k)\sum_{t=1}^{\tau}\left(c_1c_2\mathbf{F}_2\boldsymbol{x}_k - c_1\boldsymbol{x}_k^H\mathbf{F}_1^H\mathbf{F}_2\boldsymbol{x}_k\right)d\boldsymbol{x}_k \\ &= \sum_{k=1}^{K}\sum_{t=1}^{\tau}\left\{c_1c_2\mathbf{F}_2\underbrace{\int q(\boldsymbol{x}_k)\boldsymbol{x}_k\cdot d\boldsymbol{x}_k}_{\text{integration 1}} - c_1\underbrace{\int q(\boldsymbol{x}_k)\boldsymbol{x}_k^H\mathbf{F}_3\boldsymbol{x}_k\cdot d\boldsymbol{x}_k}_{\text{integration 2}}\right\}\end{aligned}$$

Integration 1:

$$\begin{aligned}\mathbf{F}_2\int q(\boldsymbol{x}_k)\boldsymbol{x}_k\cdot d\boldsymbol{x}_k &= \mathbf{F}_2\cdot \mathrm{E}_q[\boldsymbol{x}_k] \\ &= \mathbf{F}_2\cdot \underbrace{\boldsymbol{\mu}_k}_{\text{the parameter in E-step}}\end{aligned}$$

Integration 2:

首先展开$q(\boldsymbol{x}_k)$:

$$q(\boldsymbol{x}_k) = \mathcal{CN}(\boldsymbol{x}_k; \underbrace{\boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k}_{\text{the parameters in E-step}})$$

根据 The Matrix Cookbook [http://matrixcookbook.com] Kaare Brandt Petersen Michael Syskind Pedersen Version: November 15, 2012：

$$\begin{aligned}\int q(\boldsymbol{x}_k)\boldsymbol{x}_k^H\mathbf{F}_3\boldsymbol{x}_k\cdot d\boldsymbol{x}_k &= E_q[\boldsymbol{x}_k^H\mathbf{F}_3\boldsymbol{x}_k] \\ &= \mathrm{Tr}(\mathbf{F}_3\boldsymbol{\Sigma}_k) + \boldsymbol{\mu}_k^H\mathbf{F}_3\boldsymbol{\mu}_k\end{aligned}$$

最终，将所有变量代换：

$$\begin{aligned}&\frac{\partial}{\partial \omega_l}\hat{u}^{EM}\left(\omega_l, \boldsymbol{\xi}^{(i)}_{-\omega_l}; \omega_l^{(i)}, \boldsymbol{\xi}^{(i)}_{-\omega_l}\right) = \\ &\sum_{k=1}^{K}\sum_{t=1}^{\tau}\left(c_1c_2\mathbf{F}_2\boldsymbol{\mu}_k - c_1\left(\mathrm{Tr}(\mathbf{F}_3\boldsymbol{\Sigma}_k) + \boldsymbol{\mu}_k^H\mathbf{F}_3\boldsymbol{\mu}_k\right)\right)\end{aligned}$$

接下来求解$\Delta\varphi_m$：

原函数$f_m(\varphi_m)$

$$f_m(\varphi_m) = \int q(\boldsymbol{x})\left\{\sum_k^K\sum_l^L\sum_t^\tau \ln\left[\mathcal{CN}\left(y_{k,l,t}; \mathbf{F}_{k,l,t}\boldsymbol{x}_k, \kappa_{k,t}^{-1}\right)\right]\right\}d\boldsymbol{x}$$

首先：

$$\frac{\partial}{\partial \Delta\varphi_m}\hat{u}^{EM}\left(\omega_l, \boldsymbol{\xi}^{(i)}_{-\omega_l}; \omega_l^{(i)}, \boldsymbol{\xi}^{(i)}_{-\omega_l}\right) = \int q(\boldsymbol{x}; \omega_l^{(i)}, \boldsymbol{\xi}^{(i)}_{-\omega_l})\frac{\partial}{\partial \Delta\varphi_m}\ln p(\boldsymbol{y} \mid \boldsymbol{x}, \boldsymbol{\kappa}; \omega_l, \boldsymbol{\xi}^{(i)}_{1,-\omega_l})d\boldsymbol{x}$$

其中：

$$\begin{aligned}&\frac{\partial}{\partial \Delta\varphi_m}\ln p(\boldsymbol{y} \mid \boldsymbol{x}, \boldsymbol{\kappa}; \Delta\varphi_m, \boldsymbol{\xi}^{(i)}_{2,-\Delta\varphi_m}) \\ &= \sum_{k=1}^{K}\sum_{l=1}^{L}\sum_{t=1}^{\tau}\underbrace{\frac{\partial}{\partial(\mathbf{F}_{k,l,t}\boldsymbol{x}_k)}\ln\left[\mathcal{CN}(y_{k,l,t}; \mathbf{F}_{k,l,t}\boldsymbol{x}_k, \kappa_{k,t}^{-1})\right]}_{\text{Partial derivative 1}}\underbrace{\frac{\partial}{\partial \Delta\varphi_m}(\mathbf{F}_{k,l,t}\boldsymbol{x}_k)}_{\text{Partial derivative 2}}\end{aligned}$$

Partial derivative 1和上一部分一样:

$$\frac{\partial}{\partial\left(\mathbf{F}_{k,l,t}\boldsymbol{x}_k\right)}\ln\left[\mathcal{CN}\left(y_{k,l,t};\mathbf{F}_{k,l,t}\boldsymbol{x}_k,\kappa_{k,t}^{-1}\right)\right]=\underbrace{\frac{\kappa_{k,t}}{2\pi}\left(y_{k,l,t}^{*}-\left(\mathbf{F}_{k,l,t}\boldsymbol{x}_k\right)^{*}\right)}_{\text{Partial derivative 1}}$$
$$=c_1\left(c_2-\left(\mathbf{F}_1\boldsymbol{x}_k\right)^{*}\right)$$

Partial derivative 2 写为:

$$\begin{aligned}
\mathbf{F}_{k,l,t}\boldsymbol{x}_k &= \left[\mathbf{\Phi}^H\mathbf{V}(\omega_l)\mathbf{D}_M(\Delta\boldsymbol{\varphi})\right]_{t,:}\boldsymbol{x}_k \\
&= \underbrace{\left[\mathbf{\Phi}^H\right]_{t,:}}_{(1\times M)}\mathbf{V}(\omega_l)\mathbf{D}_M(\Delta\boldsymbol{\varphi})\boldsymbol{x}_k \\
&= \underbrace{\left[\mathbf{\Phi}^H\right]_{t,:}\mathbf{V}(\omega_l)}_{(1\times M)}\begin{bmatrix} D_M(\Delta\varphi_1,1) & \cdots & D_M(\Delta\varphi_m,1) & \cdots & D_M(\Delta\varphi_M,1) \\ D_M(\Delta\varphi_1,2) & \cdots & D_M(\Delta\varphi_m,2) & \cdots & D_M(\Delta\varphi_M,2) \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ D_M(\Delta\varphi_1,m') & \cdots & D_M(\Delta\varphi_m,m') & \cdots & D_M(\Delta\varphi_M,m') \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ D_M(\Delta\varphi_1,M) & \cdots & D_M(\Delta\varphi_m,M) & \cdots & D_M(\Delta\varphi_M,M) \end{bmatrix}\boldsymbol{x}_k
\end{aligned}$$

对其做偏导:

$$\frac{\partial}{\partial\Delta\varphi_m}\mathbf{F}_{k,l,t}\boldsymbol{x}_k=\underbrace{\left[\mathbf{\Phi}^H\right]_{t,:}\mathbf{V}(\omega_l)}_{(1\times M)}\begin{bmatrix} 0 & \cdots & \frac{\partial}{\partial\Delta\varphi_m}D_M(\Delta\varphi_m,1) & \cdots & 0 \\ 0 & \cdots & \frac{\partial}{\partial\Delta\varphi_m}D_M(\Delta\varphi_m,2) & \cdots & 0 \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & \cdots & \frac{\partial}{\partial\Delta\varphi_m}D_M(\Delta\varphi_m,m') & \cdots & 0 \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & \cdots & \frac{\partial}{\partial\Delta\varphi_m}D_M(\Delta\varphi_m,M) & \cdots & 0 \end{bmatrix}\boldsymbol{x}_k$$

接下来求解$\frac{\partial}{\partial\Delta\varphi_m}D_M\left(\Delta\varphi_m,m'\right)$:

$$D_M(\Delta\varphi_m,m')=\begin{cases} f_M(2\pi(\frac{m'-m}{M}-1+\Delta\varphi_m))\,, & \frac{m'-1}{M}<0.5 \\ f_M(2\pi(\frac{m'-m}{M}+\Delta\varphi_m)), & \frac{m'-1}{M}\geq 0.5 \end{cases}$$

其中:

$$f_M(x)=\frac{1}{\sqrt{M}}e^{jx(M-1)/2}\frac{\sin(Mx/2)}{\sin(x/2)}$$

对其求偏导: $((u\pm v)'=u'\pm v'$, $\left(\frac{u}{v}\right)'=\frac{u'v-uv'}{v^2}$) , $(uv)'=u'v+v'u$

$$\begin{aligned}
f_M'(x)\triangleq\frac{\partial}{\partial x}f_M(x) &= \frac{1}{\sqrt{M}}\frac{j(M-1)}{2}e^{jx(M-1)/2}\frac{\sin(Mx/2)}{\sin(x/2)} \\
&+\frac{1}{\sqrt{M}}e^{jx(M-1)/2}\frac{\frac{M}{2}\cos(Mx/2)\sin(x/2)-\frac{1}{2}\cos(x/2)\sin(Mx/2)}{\sin^2(x/2)}
\end{aligned}$$

$$\frac{\partial}{\partial\Delta\varphi_m}D_M(\Delta\varphi_m,m')=\begin{cases} 2\pi f_M'(x)|_{x=2\pi(\frac{m'-m}{M}-1+\Delta\varphi_m)} & ,\frac{m'-1}{M}<0.5 \\ 2\pi f_M'(x)|_{x=2\pi(\frac{m'-m}{M}+\Delta\varphi_m)} & ,\frac{m'-1}{M}\geq 0.5 \end{cases}$$

定义辅助矩阵$\mathbf{F}_4$:

$$
\mathbf{F}_4=\underbrace{\left[\mathbf{\Phi}^H\right]_{t,:}\mathbf{V}(\omega_l)}_{(1\times M)}\begin{bmatrix} 0 & \cdots & \frac{\partial}{\partial\Delta\varphi_m}D_M(\Delta\varphi_m,1) & \cdots & 0 \\ 0 & \cdots & \frac{\partial}{\partial\Delta\varphi_m}D_M(\Delta\varphi_m,2) & \cdots & 0 \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & \cdots & \frac{\partial}{\partial\Delta\varphi_m}D_M(\Delta\varphi_m,m') & \cdots & 0 \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & \cdots & \frac{\partial}{\partial\Delta\varphi_m}D_M(\Delta\varphi_m,M) & \cdots & 0 \end{bmatrix}
$$

Partial derivative 2 最终写为：

$$
\frac{\partial}{\partial\Delta\varphi_m}\mathbf{F}_{k,l,t}\boldsymbol{x}_k=\mathbf{F}_4\boldsymbol{x}_k
$$

定义$\mathbf{F}_5\triangleq\mathbf{F}_1^H\mathbf{F}_4$，则：，偏导可以表示为：

$$
\begin{aligned}
\frac{\partial}{\partial\Delta\varphi_m}\ln p(\boldsymbol{y}\mid\boldsymbol{x},\boldsymbol{\kappa};\Delta\varphi_m,\boldsymbol{\xi}^{(i)}_{1,-\Delta\varphi_m}) &= \sum_{k=1}^{K}\sum_{t=1}^{\tau}c_1\left(c_2-(\mathbf{F}_1\boldsymbol{x}_k)^*\right)\cdot\mathbf{F}_4\boldsymbol{x}_k \\
&= \sum_{k=1}^{K}\sum_{t=1}^{\tau}\left(c_1c_2\mathbf{F}_4\boldsymbol{x}_k-c_1\boldsymbol{x}_k^H\mathbf{F}_5\boldsymbol{x}_k\right)
\end{aligned}
$$

对其进行积分有：

$$
\begin{aligned}
&\int q(\boldsymbol{x})\frac{\partial}{\partial\Delta\varphi_m}\ln p(\boldsymbol{y}\mid\boldsymbol{x},\boldsymbol{\kappa};\Delta\varphi_m,\boldsymbol{\xi}^{(i)}_{1,-\Delta\varphi_m})d\boldsymbol{x} \\
&=\sum_{k=1}^{K}\sum_{l=1}^{L}\sum_{t=1}^{\tau}\int q(\boldsymbol{x}_k)\left(c_1c_2\mathbf{F}_4\boldsymbol{x}_k-c_1\boldsymbol{x}_k^H\mathbf{F}_1^H\mathbf{F}_4\boldsymbol{x}_k\right)d\boldsymbol{x}_k \\
&=\sum_{k=1}^{K}\sum_{l=1}^{L}\sum_{t=1}^{\tau}\left\{c_1c_2\underbrace{\mathbf{F}_4\int q(\boldsymbol{x}_k)\boldsymbol{x}_k\cdot d\boldsymbol{x}_k}_{\text{integration 1}}-c_1\underbrace{\int q(\boldsymbol{x}_k)\boldsymbol{x}_k^H\mathbf{F}_5\boldsymbol{x}_k\cdot d\boldsymbol{x}_k}_{\text{integration 2}}\right\}
\end{aligned}
$$

对于integration 1来说：

$$
\begin{aligned}
\mathbf{F}_4\int q(\boldsymbol{x}_k)\boldsymbol{x}_k\cdot d\boldsymbol{x}_k &= \mathbf{F}_4\cdot\mathrm{E}_q\left[\boldsymbol{x}_k\right] \\
&= \mathbf{F}_4\cdot\underbrace{\boldsymbol{\mu}_k}_{\text{the parameter in E-step}}
\end{aligned}
$$

对于integration 2来说：

$$
\begin{aligned}
\int q(\boldsymbol{x}_k)\boldsymbol{x}_k^H\mathbf{F}_5\boldsymbol{x}_k\cdot d\boldsymbol{x}_k &= E_q[\boldsymbol{x}_k^H\mathbf{F}_5\boldsymbol{x}_k] \\
&= \mathrm{Tr}(\mathbf{F}_5\mathbf{\Sigma}_k)+\boldsymbol{\mu}_k^H\mathbf{F}_5\boldsymbol{\mu}_k
\end{aligned}
$$

最终，将所有变量代换：

$$
\begin{aligned}
&\frac{\partial}{\partial\Delta\varphi_m}\hat{u}^{EM}\left(\Delta\varphi_m,\boldsymbol{\xi}^{(i)}_{-\Delta\varphi_m};\Delta\varphi_m^{(i)},\boldsymbol{\xi}^{(i)}_{-\Delta\varphi_m}\right)= \\
&\sum_{k=1}^{K}\sum_{l=1}^{L}\sum_{t=1}^{\tau}\left\{c_1c_2\mathbf{F}_4\boldsymbol{\mu}_k-c_1\left(\mathrm{Tr}(\mathbf{F}_5\mathbf{\Sigma}_k)+\boldsymbol{\mu}_k^H\mathbf{F}_5\boldsymbol{\mu}_k\right)\right\}
\end{aligned}
$$

结论：

对于ω_l

$$
\begin{aligned}
&\frac{\partial}{\partial\omega_l}\hat{u}^{EM}\left(\omega_l,\boldsymbol{\xi}^{(i)}_{-\omega_l};\omega_l^{(i)},\boldsymbol{\xi}^{(i)}_{-\omega_l}\right)= \\
&\sum_{k=1}^{K}\sum_{t=1}^{\tau}\left(c_1c_2\mathbf{F}_2\boldsymbol{\mu}_k-c_1\left(\mathrm{Tr}(\mathbf{F}_3\mathbf{\Sigma}_k)+\boldsymbol{\mu}_k^H\mathbf{F}_3\boldsymbol{\mu}_k\right)\right) \\
&\propto\sum_{k=1}^{K}\sum_{t=1}^{\tau}\left(c_2\mathbf{F}_2\boldsymbol{\mu}_k-\mathrm{Tr}(\mathbf{F}_3\mathbf{\Sigma}_k)-\boldsymbol{\mu}_k^H\mathbf{F}_3\boldsymbol{\mu}_k\right)
\end{aligned}
$$

其中:

$c_1 = \frac{\kappa_{k,l,t}}{2}, c_2 = y^*_{k,l,t}$

$\mathbf{F}_1 \triangleq \mathbf{F}_{k,l,t},$

$\mathbf{F}_2 \triangleq \sum_{m=1}^{M} e^{j\vartheta_{t,m}} \big(-j2\pi(M-1) \big) e^{-j2\pi(m-1)\omega_l} [\mathbf{U}_M]_{m,:} \mathbf{D}_M(\Delta\boldsymbol{\varphi})$

$\mathbf{F}_3 \triangleq \mathbf{F}_1^H \mathbf{F}_2$

对于$\Delta\varphi_m$

$$\begin{aligned}\frac{\partial}{\partial \Delta\varphi_m} \hat{u}^{EM} \left(\Delta\varphi_m, \boldsymbol{\xi}^{(i)}_{-\Delta\varphi_m}; \Delta\varphi^{(i)}_m, \boldsymbol{\xi}^{(i)}_{-\Delta\varphi_m} \right) &= \\ \sum_{k=1}^{K} \sum_{l=1}^{L} \sum_{t=1}^{\tau} &\left\{ c_1 c_2 \mathbf{F}_4 \boldsymbol{\mu}_k - c_1 \big(\mathrm{Tr}(\mathbf{F}_5 \boldsymbol{\Sigma}_k) + \boldsymbol{\mu}_k^H \mathbf{F}_5 \boldsymbol{\mu}_k \big) \right\} \\ \propto \sum_{k=1}^{K} \sum_{l=1}^{L} \sum_{t=1}^{\tau} &\left\{ c_2 \mathbf{F}_4 \boldsymbol{\mu}_k - \mathrm{Tr}(\mathbf{F}_5 \boldsymbol{\Sigma}_k) - \boldsymbol{\mu}_k^H \mathbf{F}_5 \boldsymbol{\mu}_k \right\}\end{aligned}$$

其中:

$c_1 = \frac{\kappa_{k,l,t}}{2}, c_2 = y^*_{k,l,t}$

$$\mathbf{F}_4 = \underbrace{\left[\boldsymbol{\Phi}^H \right]_{t,:} \mathbf{V}(\omega_l)}_{(1\times M)} \begin{bmatrix} 0 & \cdots & \frac{\partial}{\partial \Delta\varphi_m} D_M(\Delta\varphi_m, 1) & \cdots & 0 \\ 0 & \cdots & \frac{\partial}{\partial \Delta\varphi_m} D_M(\Delta\varphi_m, 2) & \cdots & 0 \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & \cdots & \frac{\partial}{\partial \Delta\varphi_m} D_M(\Delta\varphi_m, m') & \cdots & 0 \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & \cdots & \frac{\partial}{\partial \Delta\varphi_m} D_M(\Delta\varphi_m, M) & \cdots & 0 \end{bmatrix}$$

$\mathbf{F}_5 \triangleq \mathbf{F}_1^H \mathbf{F}_4$

where:

$$\frac{\partial}{\partial \Delta\varphi_m} D_M(\Delta\varphi_m, m') = \begin{cases} 2\pi f'_M(x)|_{x=2\pi(\frac{m'-m}{M}-1+\Delta\varphi_m)} & , \frac{m'-1}{M} < 0.5 \\ 2\pi f'_M(x)|_{x=2\pi(\frac{m'-m}{M}+\Delta\varphi_m)} & , \frac{m'-1}{M} \geq 0.5 \end{cases}$$

and

$$\begin{aligned} f'_M(x) \triangleq \frac{\partial}{\partial x} f_M(x) = &\frac{1}{\sqrt{M}} \frac{j(M-1)}{2} e^{jx(M-1)/2} \frac{\sin(Mx/2)}{\sin(x/2)} \\ &+ \frac{1}{\sqrt{M}} e^{jx(M-1)/2} \frac{\frac{M}{2}\cos(Mx/2)\sin(x/2) - \frac{1}{2}\cos(x/2)\sin(Mx/2)}{\sin^2(x/2)} \end{aligned}$$