

# update $\xi$

$$\xi_1 = \{\omega_1, \dots, \omega_L\}$$

$$\xi_2 = \{\Delta\varphi_1, \dots, \Delta\varphi_M\}$$

$$\xi_3 = \{\kappa_{1,1}, \dots, \kappa_{k,t}, \dots, \kappa_{K,\tau}\}$$

$$\xi_4 = \{\lambda^c, p_{01}^c, p_{10}^c, \mu_1^s, \sigma_1^s, \dots, \mu_k^s, \sigma_k^s\}$$

$$\boldsymbol{v} \triangleq \{\boldsymbol{x}, \boldsymbol{\gamma}, \boldsymbol{c}, \boldsymbol{s}\}$$

$$\xi_j^{(i+1)} = \xi_j^{(i)} + \gamma^{(i)} \frac{\partial u \left( \xi_j, \xi_{-j}^{(i)}; \xi_j^{(i)}, \xi_{-j}^{(i)} \right)}{\partial \xi_j} \bigg|_{\xi_j = \xi_j^{(i)}}$$

$$u(\xi; \dot{\xi}) = u^{\text{EM}}(\xi; \dot{\xi}) + \sum_{j \in \mathcal{J}_c^1} \tau_j \|\xi_j - \dot{\xi}_j\|^2$$

$$\begin{aligned} u^{\text{EM}}(\xi; \dot{\xi}) &= \int p(\boldsymbol{v} \mid \boldsymbol{p}, \dot{\xi}) \ln \frac{p(\boldsymbol{v}, \boldsymbol{p}, \xi)}{p(\boldsymbol{v} \mid \boldsymbol{p}, \dot{\xi})} d\boldsymbol{v} \\ &\approx \int q(\boldsymbol{v}; \dot{\xi}) \ln \frac{p(\boldsymbol{v}, \boldsymbol{p}, \xi)}{q(\boldsymbol{v}; \dot{\xi})} d\boldsymbol{v} \end{aligned}$$

首先求  $\frac{\partial}{\partial \xi_j} \hat{u}^{EM}(\xi_j, \xi_{-j}^{(i)}; (\xi_j^{(i)}, \xi_{-j}^{(i)}))$ :

$$\begin{aligned} \frac{\partial}{\partial \xi_j} \hat{u}^{EM}(\xi_j, \xi_{-j}^{(i)}; \xi_j^{(i)}, \xi_{-j}^{(i)}) &= \frac{\partial}{\partial \xi_j} \left\{ \left[ \int q(\boldsymbol{v}; \xi_j^{(i)}, \xi_{-j}^{(i)}) \ln p(\boldsymbol{v}, \boldsymbol{y}; \xi_j, \xi_{-j}^{(i)}) d\boldsymbol{v} - \int q(\boldsymbol{v}; \xi_j^{(i)}, \xi_{-j}^{(i)}) \ln q(\boldsymbol{v}; \xi_j^{(i)}, \xi_{-j}^{(i)}) d\boldsymbol{v} \right] \right\} \\ &= \frac{\partial}{\partial \xi_j} \left[ \int q(\boldsymbol{v}; \xi_j^{(i)}, \xi_{-j}^{(i)}) \ln p(\boldsymbol{v}, \boldsymbol{y}; \xi_j, \xi_{-j}^{(i)}) d\boldsymbol{v} \right] \\ &= \int q(\boldsymbol{v}; \xi_j^{(i)}, \xi_{-j}^{(i)}) \frac{\partial}{\partial \xi_j} \left[ \ln p(\boldsymbol{v}, \boldsymbol{y}; \xi_j, \xi_{-j}^{(i)}) \right] d\boldsymbol{v} \end{aligned}$$

注意到  $p(\boldsymbol{v}, \boldsymbol{y}; \xi_j, \xi_{-j}^{(i)})$  可以被进一步分解:

$$\begin{aligned} p(\boldsymbol{v}, \boldsymbol{y}; \xi_j, \xi_{-j}^{(i)}) &= \underbrace{p(\boldsymbol{x} \mid \boldsymbol{\gamma}) p(\boldsymbol{\kappa}) p(\boldsymbol{\gamma} \mid \boldsymbol{s})}_{\text{known distribution}} \underbrace{p(\boldsymbol{y} \mid \boldsymbol{x}, \boldsymbol{\kappa}; \xi) p(\boldsymbol{c}, \boldsymbol{s}; \xi)}_{\text{with unknown values}} \\ \ln p(\boldsymbol{v}, \boldsymbol{y}; \xi_j, \xi_{-j}^{(i)}) &= \ln p(\boldsymbol{x} \mid \boldsymbol{\gamma}) + \ln p(\boldsymbol{\kappa}) + \ln p(\boldsymbol{\gamma} \mid \boldsymbol{s}) \\ &\quad + \ln p(\boldsymbol{y} \mid \boldsymbol{x}, \boldsymbol{\kappa}; \xi_{1,2,3}) + \ln p(\boldsymbol{c}, \boldsymbol{s}; \xi_4) \end{aligned}$$

将其带入上式可得:

$$\begin{aligned} \frac{\partial}{\partial \xi_j} \hat{u}^{EM}(\xi_j, \xi_{-j}^{(i)}; \xi_j^{(i)}, \xi_{-j}^{(i)}) &= \int q(\boldsymbol{v}; \xi_j^{(i)}, \xi_{-j}^{(i)}) \frac{\partial}{\partial \xi_j} \left[ \ln p(\boldsymbol{v}, \boldsymbol{y}; \xi_j, \xi_{-j}^{(i)}) \right] d\boldsymbol{v} \\ &= \int q(\boldsymbol{v}; \xi_j^{(i)}, \xi_{-j}^{(i)}) \frac{\partial}{\partial \xi_j} [\ln p(\boldsymbol{y} \mid \boldsymbol{x}, \boldsymbol{\kappa}; \xi_{1,2,3}) + \ln p(\boldsymbol{c}, \boldsymbol{s}; \xi_4)] d\boldsymbol{v} \\ &= \begin{cases} \int q(\boldsymbol{v}; \xi_j^{(i)}, \xi_{-j}^{(i)}) \frac{\partial}{\partial \xi_j} \ln p(\boldsymbol{y} \mid \boldsymbol{x}, \boldsymbol{\kappa}; \xi_{1,2,3}) d\boldsymbol{v} & , j \in \{1, 2\} \\ \int q(\boldsymbol{v}; \xi_j^{(i)}, \xi_{-j}^{(i)}) \frac{\partial}{\partial \xi_j} \ln p(\boldsymbol{c}, \boldsymbol{s}; \xi_3) d\boldsymbol{v} & , j = 4 \end{cases} \\ &= \begin{cases} \int q(\boldsymbol{x}; \xi_j^{(i)}, \xi_{-j}^{(i)}) \frac{\partial}{\partial \xi_j} \ln p(\boldsymbol{y} \mid \boldsymbol{x}, \boldsymbol{\kappa}; \xi_{1,2,3}) d\boldsymbol{x} & , j \in \{1, 2\} \\ \int q(\boldsymbol{c}, \boldsymbol{s}; \xi_j^{(i)}, \xi_{-j}^{(i)}) \frac{\partial}{\partial \xi_j} \ln p(\boldsymbol{c}, \boldsymbol{s}; \xi_3) d\boldsymbol{c} d\boldsymbol{s} & , j = 4 \end{cases} \end{aligned}$$

## 首先以求解 $\omega_l$ 为例:

设置求导前等价函数  $f_l(\omega_l)$ :

$$f_l(\omega_l) \triangleq \int q(\boldsymbol{x}) \left\{ \sum_{k=1}^K \sum_{t=1}^{\tau} \ln \left[ \mathcal{CN}(y_{k,l,t}; \mathbf{F}_{k,l,t} \boldsymbol{x}_k, \boldsymbol{\kappa}_{k,t}^{-1}) \right] \right\} d\boldsymbol{x}$$

同时, 多维等价函数为  $f_L(\boldsymbol{\omega})$ :

$$f_L(\boldsymbol{\omega}) = \sum_{l=1}^L f_l(\omega_l)$$

$$\frac{\partial}{\partial \omega_l} \hat{u}^{EM} \left( \omega_l, \boldsymbol{\xi}_{-\omega_l}^{(i)}; \omega_l^{(i)}, \boldsymbol{\xi}_{-\omega_l}^{(i)} \right) = \int q(\mathbf{x}; \omega_l^{(i)}, \boldsymbol{\xi}_{-\omega_l}^{(i)}) \frac{\partial}{\partial \omega_l} \ln p(\mathbf{y} \mid \mathbf{x}, \boldsymbol{\kappa}; \omega_l, \boldsymbol{\xi}_{1, -\omega_l}^{(i)}) d\mathbf{x}$$

其中：

$$\begin{aligned} \frac{\partial}{\partial \omega_l} \ln p(\mathbf{y} \mid \mathbf{x}, \boldsymbol{\kappa}; \omega_l, \boldsymbol{\xi}_{1, -\omega_l}^{(i)}) &= \frac{\partial}{\partial \omega_l} \sum_{k=1}^K \sum_{l=1}^L \sum_{t=1}^{\tau} \ln [\mathcal{CN}(y_{k,l,t}; \mathbf{F}_{k,l,t} \mathbf{x}_k, \boldsymbol{\kappa}_{k,t}^{-1})] \\ &= \frac{\partial}{\partial \omega_l} \sum_{k=1}^K \sum_{t=1}^{\tau} \ln [\mathcal{CN}(y_{k,l,t}; \mathbf{F}_{k,l,t} \mathbf{x}_k, \boldsymbol{\kappa}_{k,t}^{-1})] \\ &= \sum_{k=1}^K \sum_{t=1}^{\tau} \frac{\partial}{\partial \omega_l} \ln [\mathcal{CN}(y_{k,l,t}; \mathbf{F}_{k,l,t} \mathbf{x}_k, \boldsymbol{\kappa}_{k,t}^{-1})] \\ &= \sum_{k=1}^K \sum_{t=1}^{\tau} \underbrace{\frac{\partial}{\partial (\mathbf{F}_{k,l,t} \mathbf{x}_k)} \ln [\mathcal{CN}(y_{k,l,t}; \mathbf{F}_{k,l,t} \mathbf{x}_k, \boldsymbol{\kappa}_{k,t}^{-1})]}_{\text{part 1}} \underbrace{\frac{\partial}{\partial \omega_l} (\mathbf{F}_{k,l,t} \mathbf{x}_k)}_{\text{part 2}} \end{aligned}$$

接下来将被求和项目展开，其中

$$\begin{aligned} \mathbf{F}_{k,l,t} \mathbf{x}_k &= \left[ \boldsymbol{\Phi}^H \mathbf{V}(\omega_l) \mathbf{D}_M(\Delta \varphi) \right]_{t,:} \mathbf{x}_k \\ &= \left[ \boldsymbol{\Phi}^H \right]_{t,:} \mathbf{V}(\omega_l) \mathbf{D}_M(\Delta \varphi) \mathbf{x}_k \\ &= \begin{bmatrix} e^{j\vartheta_{t,1}} & \dots & e^{j\vartheta_{t,M}} \end{bmatrix} \begin{bmatrix} \mathbf{a}_M(\omega_l)_1 [\mathbf{U}_M]_{1,:} \\ \vdots \\ \mathbf{a}_M(\omega_l)_M [\mathbf{U}_M]_{M,:} \end{bmatrix} \mathbf{D}_M(\Delta \varphi) \mathbf{x}_k \\ &= \sum_{m=1}^M e^{j\vartheta_{t,m}} \mathbf{a}_M(\omega_l)_m [\mathbf{U}_M]_{m,:} \mathbf{D}_M(\Delta \varphi) \mathbf{x}_k \\ &= \sum_{m=1}^M e^{j\vartheta_{t,m}} e^{-j2\pi(m-1)\omega_l} [\mathbf{U}_M]_{m,:} \mathbf{D}_M(\Delta \varphi) \mathbf{x}_k \\ \frac{\partial}{\partial \omega_l} \mathbf{F}_{k,l,t} \mathbf{x}_k &= \sum_{m=1}^M e^{j\vartheta_{t,m}} \frac{\partial}{\partial \omega_l} (e^{-j2\pi(m-1)\omega_l}) [\mathbf{U}_M]_{m,:} \mathbf{D}_M(\Delta \varphi) \mathbf{x}_k \\ &= \underbrace{\sum_{m=1}^M e^{j\vartheta_{t,m}} (-j2\pi(m-1)) e^{-j2\pi(m-1)\omega_l} [\mathbf{U}_M]_{m,:} \mathbf{D}_M(\Delta \varphi) \mathbf{x}_k}_{\text{part 2}} \end{aligned}$$

现在考虑  $\frac{\partial}{\partial (\mathbf{F}_{k,l,t} \mathbf{x}_k)} \ln [\mathcal{CN}(y_{k,l,t}; \mathbf{F}_{k,l,t} \mathbf{x}_k, \boldsymbol{\kappa}_{k,l,t}^{-1})]$

化简模型：  $\frac{\partial}{\partial \mu} \ln [\mathcal{CN}(y; \mu, \kappa^{-1})]$

$$\begin{aligned} p_{xy} &= \frac{1}{2\pi\sigma^2} e^{-\frac{(x-\mu_x)^2 + (y-\mu_y)^2}{2\sigma^2}} \\ p_z &= \frac{1}{2\pi\sigma^2} e^{-\frac{(z-\mu_z)^2}{2\sigma^2}} \\ p_z &= \frac{1}{\pi\sigma_z^2} e^{-\frac{(z-\mu_z)^2}{\sigma_z^2}} \end{aligned}$$

复数求偏微分的方法如下：

$$\begin{aligned}
z &= x + iy \\
\frac{df}{dz} &= \frac{1}{2} \left( \frac{df}{dx} - i \frac{df}{dy} \right) \\
\frac{dz}{dz} &= 1, \frac{dz^*}{dz} = 0
\end{aligned}$$

复高斯分布可以写为：

$$\mathcal{CN}(y; \mu, \sigma^2) = \frac{1}{2\pi\sigma^2} e^{-\frac{(y_x - \mu_x)^2 + (y_y - \mu_y)^2}{2\sigma^2}}$$

对 $\mu_x$ 和 $\mu_y$ 分开求导：

对于 $\mu_x$ ：

$$\frac{\partial}{\partial \mu_x} \mathcal{CN}(y; \mu, \sigma^2) = \frac{1}{2\pi\sigma^2} \left( -\frac{1}{2\sigma^2} \right) 2(y_x - \mu_x)(-1) e^{-\frac{(y_x - \mu_x)^2 + (y_y - \mu_y)^2}{2\sigma^2}}$$

对于 $\mu_y$ ：

$$\frac{\partial}{\partial \mu_y} \mathcal{CN}(y; \mu, \sigma^2) = \frac{1}{2\pi\sigma^2} \left( -\frac{1}{2\sigma^2} \right) 2(y_y - \mu_y)(-1) e^{-\frac{(y_x - \mu_x)^2 + (y_y - \mu_y)^2}{2\sigma^2}}$$

合并：

$$\begin{aligned}
\frac{\partial}{\partial \mu} \mathcal{CN}(y; \mu, \sigma^2) &= \frac{1}{2} \frac{1}{2\pi\sigma^2} \left( -\frac{1}{2\sigma^2} \right) 2(y_x - \mu_x)(-1) e^{-\frac{(y_x - \mu_x)^2 + (y_y - \mu_y)^2}{2\sigma^2}} \\
&+ \left( -\frac{1}{2} i \right) \frac{1}{2\pi\sigma^2} \left( -\frac{1}{2\sigma^2} \right) 2(y_y - \mu_y)(-1) e^{-\frac{(y_x - \mu_x)^2 + (y_y - \mu_y)^2}{2\sigma^2}} \\
&= \frac{1}{4\pi\sigma^4} (y_x - \mu_x) e^{-\frac{(y_x - \mu_x)^2 + (y_y - \mu_y)^2}{2\sigma^2}} + \\
&\quad (-i) \frac{1}{4\pi\sigma^4} (y_y - \mu_y) e^{-\frac{(y_x - \mu_x)^2 + (y_y - \mu_y)^2}{2\sigma^2}} \\
&= \frac{1}{4\pi\sigma^4} e^{-\frac{(y - \mu)^2}{2\sigma^2}} (y^* - \mu^*)
\end{aligned}$$

考虑 $\ln()$ 带来对影响：

$$\begin{aligned}
\frac{\partial}{\partial \mu} \ln \mathcal{CN}(y; \mu, \sigma^2) &= \frac{1}{4\pi\sigma^4} e^{-\frac{(y - \mu)^2}{2\sigma^2}} (y^* - \mu^*) \cdot \mathcal{CN}(y; \mu, \sigma^2)^{-1} \\
&= \frac{1}{4\pi\sigma^4} e^{-\frac{(y - \mu)^2}{2\sigma^2}} (y^* - \mu^*) \cdot 2\pi\sigma^2 e^{\frac{(y - \mu)^2}{2\sigma^2}} \\
&= \frac{1}{2\sigma^2} (y^* - \mu^*)
\end{aligned}$$

则 $\sigma^2 = \kappa_{k,l,t}^{-1}$ ,  $\mu = \mathbf{F}_{k,l,t} \mathbf{x}_k$ ：

$$\frac{\partial}{\partial (\mathbf{F}_{k,l,t} \mathbf{x}_k)} \ln \left[ \mathcal{CN}(y_{k,l,t}; \mathbf{F}_{k,l,t} \mathbf{x}_k, \kappa_{k,l,t}^{-1}) \right] = \underbrace{\frac{\kappa_{k,l,t}}{2} (y_{k,l,t}^* - (\mathbf{F}_{k,l,t} \mathbf{x}_k)^*)}_{\text{part 1}}$$

将二级结论带入：

$$\begin{aligned}
\frac{\partial}{\partial \omega_l} \ln p(\mathbf{y} \mid \mathbf{x}, \boldsymbol{\kappa}; \omega_l, \boldsymbol{\xi}_{1,-\omega_l}^{(i)}) &= \sum_{k=1}^K \sum_{t=1}^{\tau} \frac{\partial}{\partial (\mathbf{F}_{k,l,t} \mathbf{x}_k)} \ln [\mathcal{CN}(y_{k,l,t}; \mathbf{F}_{k,l,t} \mathbf{x}_k, \kappa_{k,t}^{-1})] \frac{\partial}{\partial \omega_l} (\mathbf{F}_{k,l,t} \mathbf{x}_k) \\
&= \sum_{k=1}^K \sum_{t=1}^{\tau} \text{part 2} \cdot \text{part 1} \\
&= \sum_{k=1}^K \sum_{t=1}^{\tau} \frac{\kappa_{k,t}}{2\pi} (y_{k,l,t}^* - (\mathbf{F}_{k,l,t} \mathbf{x}_k)^*) \cdot \\
&\quad \sum_{m=1}^M e^{j\vartheta_{l,m}} (-j2\pi(M-1)) e^{-j2\pi(m-1)\omega_l} [\mathbf{U}_M]_{m,:} \mathbf{D}_M(\Delta\varphi) \mathbf{x}_k
\end{aligned}$$

化简表示： $\mathbf{F}_1 \triangleq \mathbf{F}_{k,l,t}$ ,

$\mathbf{F}_2 \triangleq \sum_{m=1}^M e^{j\vartheta_{l,m}} (-j2\pi(M-1)) e^{-j2\pi(m-1)\omega_l} [\mathbf{U}_M]_{m,:} \mathbf{D}_M(\Delta\varphi)$

$$\mathbf{F}_3 \triangleq \mathbf{F}_1^H \mathbf{F}_2$$

$$c_1 = \frac{\kappa_{k,l,t}}{2}, c_2 = y_{k,l,t}^*$$

则，偏导可以表示为：

$$\begin{aligned} \frac{\partial}{\partial \omega_l} \ln p(\mathbf{y} | \mathbf{x}, \boldsymbol{\kappa}; \omega_l, \boldsymbol{\xi}_{1,-\omega_l}^{(i)}) &= \sum_{k=1}^K \sum_{t=1}^{\tau} c_1 (c_2 - (\mathbf{F}_1 \mathbf{x}_k)^*) \cdot \mathbf{F}_2 \mathbf{x}_k \\ &= \sum_{k=1}^K \sum_{t=1}^{\tau} \left( c_1 c_2 \mathbf{F}_2 \mathbf{x}_k - c_1 \mathbf{x}_k^H \mathbf{F}_1^H \mathbf{F}_2 \mathbf{x}_k \right) \end{aligned}$$

接下来对其进行针对 $q(\mathbf{x})$ 的积分：

$$\begin{aligned} &\int q(\mathbf{x}) \frac{\partial}{\partial \omega_l} \ln p(\mathbf{y} | \mathbf{x}, \boldsymbol{\kappa}; \omega_l, \boldsymbol{\xi}_{1,-\omega_l}^{(i)}) d\mathbf{x} \\ &= \sum_{k=1}^K \int q(\mathbf{x}_k) \sum_{t=1}^{\tau} \left( c_1 c_2 \mathbf{F}_2 \mathbf{x}_k - c_1 \mathbf{x}_k^H \mathbf{F}_1^H \mathbf{F}_2 \mathbf{x}_k \right) d\mathbf{x}_k \\ &= \sum_{k=1}^K \sum_{t=1}^{\tau} \left\{ \underbrace{c_1 c_2 \mathbf{F}_2 \int q(\mathbf{x}_k) \mathbf{x}_k \cdot d\mathbf{x}_k}_{\text{integration 1}} - c_1 \underbrace{\int q(\mathbf{x}_k) \mathbf{x}_k^H \mathbf{F}_3 \mathbf{x}_k \cdot d\mathbf{x}_k}_{\text{integration 2}} \right\} \end{aligned}$$

Integration 1:

$$\begin{aligned} \mathbf{F}_2 \int q(\mathbf{x}_k) \mathbf{x}_k \cdot d\mathbf{x}_k &= \mathbf{F}_2 \cdot \mathbb{E}_q[\mathbf{x}_k] \\ &= \mathbf{F}_2 \cdot \underbrace{\boldsymbol{\mu}_k}_{\text{the parameter in E-step}} \end{aligned}$$

Integration 2:

首先展开 $q(\mathbf{x}_k)$ :

$$q(\mathbf{x}_k) = \mathcal{CN}(\mathbf{x}_k; \underbrace{\boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k}_{\text{the parameters in E-step}})$$

根据 The Matrix Cookbook [ <http://matrixcookbook.com> ] Kaare Brandt Petersen Michael Syskind Pedersen Version: November 15, 2012:

$$\begin{aligned} \int q(\mathbf{x}_k) \mathbf{x}_k^H \mathbf{F}_3 \mathbf{x}_k \cdot d\mathbf{x}_k &= \mathbb{E}_q[\mathbf{x}_k^H \mathbf{F}_3 \mathbf{x}_k] \\ &= \text{Tr}(\mathbf{F}_3 \boldsymbol{\Sigma}_k) + \boldsymbol{\mu}_k^H \mathbf{F}_3 \boldsymbol{\mu}_k \end{aligned}$$

最终，将所有变量代换：

$$\begin{aligned} \frac{\partial}{\partial \omega_l} \hat{u}^{EM}(\omega_l, \boldsymbol{\xi}_{-\omega_l}^{(i)}; \omega_l^{(i)}, \boldsymbol{\xi}_{-\omega_l}^{(i)}) &= \\ &\sum_{k=1}^K \sum_{t=1}^{\tau} \left( c_1 c_2 \mathbf{F}_2 \boldsymbol{\mu}_k - c_1 (\text{Tr}(\mathbf{F}_3 \boldsymbol{\Sigma}_k) + \boldsymbol{\mu}_k^H \mathbf{F}_3 \boldsymbol{\mu}_k) \right) \end{aligned}$$

## 接下来求解 $\Delta \varphi_m$ ：

首先：

$$\frac{\partial}{\partial \Delta \varphi_m} \hat{u}^{EM}(\omega_l, \boldsymbol{\xi}_{-\omega_l}^{(i)}; \omega_l^{(i)}, \boldsymbol{\xi}_{-\omega_l}^{(i)}) = \int q(\mathbf{x}; \omega_l^{(i)}, \boldsymbol{\xi}_{-\omega_l}^{(i)}) \frac{\partial}{\partial \Delta \varphi_m} \ln p(\mathbf{y} | \mathbf{x}, \boldsymbol{\kappa}; \omega_l, \boldsymbol{\xi}_{1,-\omega_l}^{(i)}) d\mathbf{x}$$

其中：

$$\begin{aligned} &\frac{\partial}{\partial \Delta \varphi_m} \ln p(\mathbf{y} | \mathbf{x}, \boldsymbol{\kappa}; \Delta \varphi_m, \boldsymbol{\xi}_{2,-\Delta \varphi_m}^{(i)}) \\ &= \sum_{k=1}^K \sum_{l=1}^L \sum_{t=1}^{\tau} \underbrace{\frac{\partial}{\partial (\mathbf{F}_{k,l,t} \mathbf{x}_k)} \ln [\mathcal{CN}(y_{k,l,t}; \mathbf{F}_{k,l,t} \mathbf{x}_k, \boldsymbol{\kappa}_{k,t}^{-1})]}_{\text{Partial derivative 1}} \underbrace{\frac{\partial}{\partial \Delta \varphi_m} (\mathbf{F}_{k,l,t} \mathbf{x}_k)}_{\text{Partial derivative 2}} \end{aligned}$$

Partial derivative 1和上一部分一样:

$$\begin{aligned}\frac{\partial}{\partial (\mathbf{F}_{k,l,t} \mathbf{x}_k)} \ln \left[ \mathcal{CN} \left( y_{k,l,t}; \mathbf{F}_{k,l,t} \mathbf{x}_k, \kappa_{k,t}^{-1} \right) \right] &= \underbrace{\frac{\kappa_{k,t}}{2\pi} (y_{k,l,t}^* - (\mathbf{F}_{k,l,t} \mathbf{x}_k)^*)}_{\text{Partial derivative 1}} \\ &= c_1 (c_2 - (\mathbf{F}_1 \mathbf{x}_k)^*)\end{aligned}$$

Partial derivative 2 写为:

$$\begin{aligned}\mathbf{F}_{k,l,t} \mathbf{x}_k &= \left[ \Phi^H \mathbf{V}(\omega_l) \mathbf{D}_M(\Delta \varphi) \right]_{t,:} \mathbf{x}_k \\ &= \underbrace{\left[ \Phi^H \right]_{t,:}}_{(1 \times M)} \mathbf{V}(\omega_l) \mathbf{D}_M(\Delta \varphi) \mathbf{x}_k \\ &= \underbrace{\left[ \Phi^H \right]_{t,:}}_{(1 \times M)} \mathbf{V}(\omega_l) \begin{bmatrix} D_M(\Delta \varphi_1, 1) & \cdots & D_M(\Delta \varphi_m, 1) & \cdots & D_M(\Delta \varphi_M, 1) \\ D_M(\Delta \varphi_1, 2) & \cdots & D_M(\Delta \varphi_m, 2) & \cdots & D_M(\Delta \varphi_M, 2) \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ D_M(\Delta \varphi_1, m') & \cdots & D_M(\Delta \varphi_m, m') & \cdots & D_M(\Delta \varphi_M, m') \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ D_M(\Delta \varphi_1, M) & \cdots & D_M(\Delta \varphi_m, M) & \cdots & D_M(\Delta \varphi_M, M) \end{bmatrix} \mathbf{x}_k\end{aligned}$$

对其做偏导:

$$\frac{\partial}{\partial \Delta \varphi_m} \mathbf{F}_{k,l,t} \mathbf{x}_k = \underbrace{\left[ \Phi^H \right]_{t,:}}_{(1 \times M)} \mathbf{V}(\omega_l) \begin{bmatrix} D_M(\Delta \varphi_1, 1) & \cdots & \frac{\partial}{\partial \Delta \varphi_m} D_M(\Delta \varphi_m, 1) & \cdots & D_M(\Delta \varphi_M, 1) \\ D_M(\Delta \varphi_1, 2) & \cdots & \frac{\partial}{\partial \Delta \varphi_m} D_M(\Delta \varphi_m, 2) & \cdots & D_M(\Delta \varphi_M, 2) \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ D_M(\Delta \varphi_1, m') & \cdots & \frac{\partial}{\partial \Delta \varphi_m} D_M(\Delta \varphi_m, m') & \cdots & D_M(\Delta \varphi_M, m') \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ D_M(\Delta \varphi_1, M) & \cdots & \frac{\partial}{\partial \Delta \varphi_m} D_M(\Delta \varphi_m, M) & \cdots & D_M(\Delta \varphi_M, M) \end{bmatrix} \mathbf{x}_k$$

接下来求解  $\frac{\partial}{\partial \Delta \varphi_m} D_M(\Delta \varphi_m, m')$ :

$$D_M(\Delta \varphi_m, m') = \begin{cases} f_M(2\pi(\frac{m' - m}{M} - 1 + \Delta \varphi_m)), & \frac{m' - 1}{M} < 0.5 \\ f_M(2\pi(\frac{m' - m}{M} + \Delta \varphi_m)), & \frac{m' - 1}{M} \geq 0.5 \end{cases}$$

其中:

$$f_M(x) = \frac{1}{\sqrt{M}} e^{jx(M-1)/2} \frac{\sin(Mx/2)}{\sin(x/2)}$$

对其求偏导:  $((u \pm v)' = u' \pm v', (\frac{u}{v})' = \frac{u'v - uv'}{v^2}), (uv)' = u'v + v'u$

$$\begin{aligned}f'_M(x) &\triangleq \frac{\partial}{\partial x} f_M(x) = \frac{1}{\sqrt{M}} \frac{j(M-1)}{2} e^{jx(M-1)/2} \frac{\sin(Mx/2)}{\sin(x/2)} \\ &\quad + \frac{1}{\sqrt{M}} e^{jx(M-1)/2} \frac{\frac{M}{2} \cos(Mx/2) \sin(x/2) - \frac{1}{2} \cos(x/2) \sin(Mx/2)}{\sin^2(x/2)} \\ \frac{\partial}{\partial \Delta \varphi_m} D_M(\Delta \varphi_m, m') &= \begin{cases} 2\pi f'_M(x)|_{x=2\pi(\frac{m'-m}{M}-1+\Delta\varphi_m)}, & \frac{m'-1}{M} < 0.5 \\ 2\pi f'_M(x)|_{x=2\pi(\frac{m'-m}{M}+\Delta\varphi_m)}, & \frac{m'-1}{M} \geq 0.5 \end{cases}\end{aligned}$$

定义辅助矩阵  $\mathbf{F}_4$ :

$$\mathbf{F}_4 = \underbrace{\begin{bmatrix} \Phi^H \end{bmatrix}_{t,:}}_{(1 \times M)} \mathbf{V}(\omega_l) \begin{bmatrix} D_M(\Delta\varphi_1, 1) & \cdots & \frac{\partial}{\partial \Delta\varphi_m} D_M(\Delta\varphi_m, 1) & \cdots & D_M(\Delta\varphi_M, 1) \\ D_M(\Delta\varphi_1, 2) & \cdots & \frac{\partial}{\partial \Delta\varphi_m} D_M(\Delta\varphi_m, 2) & \cdots & D_M(\Delta\varphi_M, 2) \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ D_M(\Delta\varphi_1, m') & \cdots & \frac{\partial}{\partial \Delta\varphi_m} D_M(\Delta\varphi_m, m') & \cdots & D_M(\Delta\varphi_M, m') \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ D_M(\Delta\varphi_1, M) & \cdots & \frac{\partial}{\partial \Delta\varphi_m} D_M(\Delta\varphi_m, M) & \cdots & D_M(\Delta\varphi_M, M) \end{bmatrix}$$

Partial derivative 2 最终写为：

$$\frac{\partial}{\partial \Delta\varphi_m} \mathbf{F}_{k,l,t} \mathbf{x}_k = \mathbf{F}_4 \mathbf{x}_k$$

定义  $\mathbf{F}_5 \triangleq \mathbf{F}_1^H \mathbf{F}_4$ ，则：，偏导可以表示为：

$$\begin{aligned} \frac{\partial}{\partial \Delta\varphi_m} \ln p(\mathbf{y} | \mathbf{x}, \boldsymbol{\kappa}; \Delta\varphi_m, \boldsymbol{\xi}_{1,-\Delta\varphi_m}^{(i)}) &= \sum_{k=1}^K \sum_{t=1}^{\tau} c_1 (c_2 - (\mathbf{F}_1 \mathbf{x}_k)^*) \cdot \mathbf{F}_4 \mathbf{x}_k \\ &= \sum_{k=1}^K \sum_{t=1}^{\tau} \left( c_1 c_2 \mathbf{F}_4 \mathbf{x}_k - c_1 \mathbf{x}_k^H \mathbf{F}_5 \mathbf{x}_k \right) \end{aligned}$$

对其进行积分有：

$$\begin{aligned} &\int q(\mathbf{x}) \frac{\partial}{\partial \Delta\varphi_m} \ln p(\mathbf{y} | \mathbf{x}, \boldsymbol{\kappa}; \Delta\varphi_m, \boldsymbol{\xi}_{1,-\Delta\varphi_m}^{(i)}) d\mathbf{x} \\ &= \sum_{k=1}^K \sum_{l=1}^L \sum_{t=1}^{\tau} \int q(\mathbf{x}_k) \left( c_1 c_2 \mathbf{F}_4 \mathbf{x}_k - c_1 \mathbf{x}_k^H \mathbf{F}_5 \mathbf{x}_k \right) d\mathbf{x}_k \\ &= \sum_{k=1}^K \sum_{l=1}^L \sum_{t=1}^{\tau} \left\{ \underbrace{c_1 c_2 \mathbf{F}_4 \int q(\mathbf{x}_k) \mathbf{x}_k \cdot d\mathbf{x}_k}_{\text{integration 1}} - c_1 \underbrace{\int q(\mathbf{x}_k) \mathbf{x}_k^H \mathbf{F}_5 \mathbf{x}_k \cdot d\mathbf{x}_k}_{\text{integration 2}} \right\} \end{aligned}$$

对于integration 1来说：

$$\begin{aligned} \mathbf{F}_4 \int q(\mathbf{x}_k) \mathbf{x}_k \cdot d\mathbf{x}_k &= \mathbf{F}_4 \cdot \mathbb{E}_q[\mathbf{x}_k] \\ &= \mathbf{F}_4 \cdot \underbrace{\boldsymbol{\mu}_k}_{\text{the parameter in E-step}} \end{aligned}$$

对于integration 2来说：

$$\begin{aligned} \int q(\mathbf{x}_k) \mathbf{x}_k^H \mathbf{F}_5 \mathbf{x}_k \cdot d\mathbf{x}_k &= \mathbb{E}_q[\mathbf{x}_k^H \mathbf{F}_5 \mathbf{x}_k] \\ &= \text{Tr}(\mathbf{F}_5 \boldsymbol{\Sigma}_k) + \boldsymbol{\mu}_k^H \mathbf{F}_5 \boldsymbol{\mu}_k \end{aligned}$$

最终，将所有变量代换：

$$\begin{aligned} \frac{\partial}{\partial \Delta\varphi_m} \hat{u}^{EM} \left( \Delta\varphi_m, \boldsymbol{\xi}_{-\Delta\varphi_m}^{(i)}; \Delta\varphi_m^{(i)}, \boldsymbol{\xi}_{-\Delta\varphi_m}^{(i)} \right) &= \\ \sum_{k=1}^K \sum_{l=1}^L \sum_{t=1}^{\tau} \left\{ c_1 c_2 \mathbf{F}_4 \boldsymbol{\mu}_k - c_1 \left( \text{Tr}(\mathbf{F}_5 \boldsymbol{\Sigma}_k) + \boldsymbol{\mu}_k^H \mathbf{F}_5 \boldsymbol{\mu}_k \right) \right\} \end{aligned}$$

## 结论：

对于  $\omega_l$

$$\begin{aligned} \frac{\partial}{\partial \omega_l} \hat{u}^{EM} \left( \omega_l, \boldsymbol{\xi}_{-\omega_l}^{(i)}; \omega_l^{(i)}, \boldsymbol{\xi}_{-\omega_l}^{(i)} \right) &= \\ \sum_{k=1}^K \sum_{t=1}^{\tau} \left( c_1 c_2 \mathbf{F}_2 \boldsymbol{\mu}_k - c_1 \left( \text{Tr}(\mathbf{F}_3 \boldsymbol{\Sigma}_k) + \boldsymbol{\mu}_k^H \mathbf{F}_3 \boldsymbol{\mu}_k \right) \right) \\ &\propto \sum_{k=1}^K \sum_{t=1}^{\tau} \left( c_2 \mathbf{F}_2 \boldsymbol{\mu}_k - \text{Tr}(\mathbf{F}_3 \boldsymbol{\Sigma}_k) - \boldsymbol{\mu}_k^H \mathbf{F}_3 \boldsymbol{\mu}_k \right) \end{aligned}$$

其中:

$$c_1 = \frac{\kappa_{k,l,t}}{2}, c_2 = y_{k,l,t}^*$$

$$\mathbf{F}_1 \triangleq \mathbf{F}_{k,l,t},$$

$$\mathbf{F}_2 \triangleq \sum_{m=1}^M e^{j\vartheta_{l,m}} \left( -j2\pi(M-1) \right) e^{-j2\pi(m-1)\omega_l} [\mathbf{U}_M]_{m,:} \mathbf{D}_M(\Delta\varphi)$$

$$\mathbf{F}_3 \triangleq \mathbf{F}_1^H \mathbf{F}_2$$

对于 $\Delta\varphi_m$

$$\begin{aligned} \frac{\partial}{\partial \Delta\varphi_m} \hat{u}^{EM} \left( \Delta\varphi_m, \boldsymbol{\xi}_{-\Delta\varphi_m}^{(i)}; \Delta\varphi_m, \boldsymbol{\xi}_{-\Delta\varphi_m}^{(i)} \right) = \\ \sum_{k=1}^K \sum_{l=1}^L \sum_{t=1}^{\tau} \left\{ c_1 c_2 \mathbf{F}_4 \boldsymbol{\mu}_k - c_1 \left( \text{Tr}(\mathbf{F}_5 \boldsymbol{\Sigma}_k) + \boldsymbol{\mu}_k^H \mathbf{F}_5 \boldsymbol{\mu}_k \right) \right\} \\ \propto \sum_{k=1}^K \sum_{l=1}^L \sum_{t=1}^{\tau} \left\{ c_2 \mathbf{F}_4 \boldsymbol{\mu}_k - \text{Tr}(\mathbf{F}_5 \boldsymbol{\Sigma}_k) - \boldsymbol{\mu}_k^H \mathbf{F}_5 \boldsymbol{\mu}_k \right\} \end{aligned}$$

其中:

$$c_1 = \frac{\kappa_{k,l,t}}{2}, c_2 = y_{k,l,t}^*$$

$$\mathbf{F}_4 = \underbrace{\left[ \boldsymbol{\Phi}^H \right]_{t,:}}_{(1 \times M)} \begin{bmatrix} D_M(\Delta\varphi_1, 1) & \cdots & \frac{\partial}{\partial \Delta\varphi_m} D_M(\Delta\varphi_m, 1) & \cdots & D_M(\Delta\varphi_M, 1) \\ D_M(\Delta\varphi_1, 2) & \cdots & \frac{\partial}{\partial \Delta\varphi_m} D_M(\Delta\varphi_m, 2) & \cdots & D_M(\Delta\varphi_M, 2) \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ D_M(\Delta\varphi_1, m') & \cdots & \frac{\partial}{\partial \Delta\varphi_m} D_M(\Delta\varphi_m, m') & \cdots & D_M(\Delta\varphi_M, m') \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ D_M(\Delta\varphi_1, M) & \cdots & \frac{\partial}{\partial \Delta\varphi_m} D_M(\Delta\varphi_m, M) & \cdots & D_M(\Delta\varphi_M, M) \end{bmatrix}$$

$$\mathbf{F}_5 \triangleq \mathbf{F}_1^H \mathbf{F}_4$$

where:

$$\frac{\partial}{\partial \Delta\varphi_m} D_M(\Delta\varphi_m, m') = \begin{cases} 2\pi f'_M(x)|_{x=2\pi(\frac{m'-m}{M}-1+\Delta\varphi_m)} & , \frac{m'-1}{M} < 0.5 \\ 2\pi f'_M(x)|_{x=2\pi(\frac{m'-m}{M}+\Delta\varphi_m)} & , \frac{m'-1}{M} \geq 0.5 \end{cases}$$

and

$$\begin{aligned} f'_M(x) \triangleq \frac{\partial}{\partial x} f_M(x) &= \frac{1}{\sqrt{M}} \frac{j(M-1)}{2} e^{jx(M-1)/2} \frac{\sin(Mx/2)}{\sin(x/2)} \\ &+ \frac{1}{\sqrt{M}} e^{jx(M-1)/2} \frac{\frac{M}{2} \cos(Mx/2) \sin(x/2) - \frac{1}{2} \cos(x/2) \sin(Mx/2)}{\sin^2(x/2)} \end{aligned}$$