Orbital Mechanics for Engineers

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Chapter 1

Geometry

1.1 Area

1.1.1 Derivation

$$r = \frac{l}{e\cos\theta + 1}$$

$$A = \frac{1}{2}\int_{\theta_1}^{\theta_2} r^2 d\theta$$

$$A = \frac{1}{2}\int_{\theta_1}^{\theta_2} \left(\frac{l}{e\cos\theta + 1}\right)^2 d\theta$$

$$A = \frac{l^2}{2}\int_{\theta_1}^{\theta_2} \frac{1}{(e\cos\theta + 1)^2} d\theta$$

$$A = \frac{l^2}{2}\int_{\theta_1}^{\theta_2} \frac{(1-e^2)}{(1-e^2)(e\cos\theta + 1)^2} d\theta$$

$$A = \frac{l^2}{2}\int_{\theta_1}^{\theta_2} \frac{e\cos\theta + 1 - e^2 - e\cos\theta}{(1-e^2)(e\cos\theta + 1)^2} d\theta$$

$$A = \frac{l^2}{2}\int_{\theta_1}^{\theta_2} \left(\frac{e\cos\theta + 1}{(1-e^2)(e\cos\theta + 1)^2} - \frac{e^2 + e\cos\theta}{(1-e^2)(e\cos\theta + 1)^2}\right) d\theta$$

$$A = \frac{l^2}{2(1-e^2)}\int_{\theta_1}^{\theta_2} \left(\frac{1}{e\cos\theta + 1} - \frac{e^2 + e\cos\theta}{(e\cos\theta + 1)^2}\right) d\theta$$

$$A = \frac{l^2}{2(1-e^2)}\int_{\theta_1}^{\theta_2} \left(2 \cdot \frac{\frac{1}{2}}{e(\cos^2\left(\frac{\theta}{2}\right) - \sin^2\frac{\theta}{2}) + 1} - e \cdot \frac{e + \cos\theta}{(e\cos\theta + 1)^2}\right) d\theta$$

$$A = \frac{l^2}{2(1-e^2)} \int_{\theta_1}^{\theta_2} \left(\frac{2(1+e)^{\frac{-1}{2}}(1-e)^{\frac{1}{2}}}{(1+e)^{\frac{-1}{2}}(1-e)^{\frac{1}{2}}} \cdot \frac{\frac{1}{2}\sec^2\left(\frac{\theta}{2}\right)}{e(1-\tan^2\frac{\theta}{2}) + \sec^2\left(\frac{\theta}{2}\right)} - e \cdot \frac{e(\sin^2\theta + \cos^2\theta) + \cos\theta}{(e\cos\theta+1)^2} \right) d\theta$$

$$A = \frac{l^2}{2(1-e^2)} \int_{\theta_1}^{\theta_2} \left(\frac{2\sqrt{\frac{1-e}{1+e}}}{(1+e)^{\frac{-2}{2}}(1+e)^{\frac{1}{2}}(1-e)^{\frac{1}{2}}} \cdot \frac{\frac{1}{2}\sec^2\left(\frac{\theta}{2}\right)}{e - e\tan^2\frac{\theta}{2} + 1 + \tan^2\frac{\theta}{2}} - e \cdot \frac{e\cos^2\theta + \cos\theta + e\sin^2\theta}{(e\cos\theta+1)^2} \right) d\theta$$

$$A = \frac{l^2}{2(1-e^2)} \int_{\theta_1}^{\theta_2} \left(\frac{2}{(1+e)^{-1}(1-e^2)^{\frac{1}{2}}} \cdot \frac{\frac{1}{2}\sqrt{\frac{1-e}{1+e}}\sec^2\left(\frac{\theta}{2}\right)}{(1+e) + (1-e)\tan^2\frac{\theta}{2}} - e \cdot \frac{(e\cos\theta+1)(\cos\theta) + (e\sin\theta)(\sin\theta)}{(e\cos\theta+1)^2} \right) d\theta$$

$$A = \frac{l^2}{2(1-e^2)} \int_{\theta_1}^{\theta_2} \left(\frac{2}{\sqrt{1-e}} \cdot \frac{\frac{1}{2}\sqrt{\frac{1-e}{1+e}}\sec^2\left(\frac{\theta}{2}\right)}{1 + \frac{1-e}{1+e}\tan^2\frac{\theta}{2}} - e \cdot \frac{(e\cos\theta+1)(\cos\theta) + (e\sin\theta)(\sin\theta)}{(e\cos\theta+1)^2} \right) d\theta$$

$$A = \frac{l^2}{2(1-e^2)} \int_{\theta_1}^{\theta_2} \left(\frac{2}{\sqrt{1-e}} \cdot \frac{\frac{1}{2}\sqrt{\frac{1-e}{1+e}}\sec^2\left(\frac{\theta}{2}\right)}{1 + (\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right))^2} - e \cdot \frac{(e\cos\theta+1)(\cos\theta) + (e\sin\theta)(\sin\theta)}{(e\cos\theta+1)^2} \right) d\theta$$

$u = \sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)$	$v = e\cos\theta + 1$	$w = \sin\theta$
$du = \frac{1}{2}\sqrt{\frac{1-e}{1+e}}\sec^2\left(\frac{\theta}{2}\right)d\theta$	$dv = -e\sin\theta d\theta$	$dw = \cos\theta d\theta$
$d\theta = 2\sqrt{\frac{1+e}{1-e}}\cos^2\left(\frac{\theta}{2}\right)du$	$d\theta = \frac{-\csc\theta}{e}dv$	$d\theta = \sec\theta dw$

$$A = \frac{l^2}{2(1-e^2)} \cdot \frac{2}{\sqrt{1-e^2}} \int_{\theta_1}^{\theta_2} \frac{\frac{1}{2}\sqrt{\frac{1-e}{1+e}}\sec^2\left(\frac{\theta}{2}\right)}{1 + (\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right))^2} d\theta - \frac{l^2}{2(1-e^2)} \cdot e \int_{\theta_1}^{\theta_2} \frac{(e\cos\theta+1)(\cos\theta) + (e\sin\theta)(\sin\theta)}{(e\cos\theta+1)^2} d\theta$$

$$A = \frac{l^2}{2(1-e^2)} \cdot \frac{2}{\sqrt{1-e^2}} \int_{\theta_1}^{\theta_2} \frac{1}{1+u^2} du - \frac{l^2}{2(1-e^2)} \cdot e \int_{\theta_1}^{\theta_2} \frac{(v)(dw) - (dv)(w)}{(v)^2} d\theta$$

$$\frac{d}{dx}[\arctan(x)] = \frac{1}{1+x^2}dx \qquad \frac{d}{dx}[\frac{f}{g}] = \frac{(g)\frac{d}{dx}[f] - \frac{d}{dx}[g](f)}{(g)^2}dx$$

$$A = \frac{l^2}{2(1-e^2)} \cdot \frac{2}{\sqrt{1-e^2}}[\arctan(u)]_{\theta_1}^{\theta_2} - \frac{l^2}{2(1-e^2)} \cdot e[\frac{w}{v}]_{\theta_1}^{\theta_2}$$

$$A = \frac{l^2}{2(1-e^2)}\left(\frac{2\arctan(u)}{\sqrt{1-e^2}} - \frac{ew}{v}\right)_{\theta_1}^{\theta_2}$$

$$A = \frac{l^2}{2(1-e^2)}\left(\frac{2\arctan(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right))}{\sqrt{1-e^2}} - \frac{e\sin\theta}{e\cos\theta+1}\right)_{\theta_1}^{\theta_2}$$

1.1.2 Elliptical Trajectories

$$A = \frac{l^2}{2(1-e^2)}\left(\frac{2\arctan\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)\right)}{\sqrt{1-e^2}} - \frac{e\sin\theta}{e\cos\theta+1}\right)_{\theta_1}^{\theta_2}$$

$$1 - e^2 > 0$$
$$e^2 < 1$$
$$e < 1$$

1.1.3 Hyperbolic Trajectories

$$i = \sqrt{-1}$$
$$\operatorname{arctanh} x = \frac{1}{i}\arctan(ix)$$

$$A = \frac{l^2}{2(1-e^2)}\left(\frac{2\arctan\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)\right)}{\sqrt{1-e^2}} - \frac{e\sin\theta}{e\cos\theta+1}\right)_{\theta_1}^{\theta_2}$$

$$A = \frac{l^2}{2(1-e^2)}\left(\frac{2\arctan\left(\sqrt{-1(\frac{e-1}{e+1})}\tan\left(\frac{\theta}{2}\right)\right)}{\sqrt{-1(e^2-1)}} - \frac{e\sin\theta}{e\cos\theta+1}\right)_{\theta_1}^{\theta_2}$$

$$A = \frac{l^2}{2(1-e^2)}\left(\frac{2\arctan\left(i\sqrt{\frac{e-1}{e+1}}\tan\left(\frac{\theta}{2}\right)\right)}{i\sqrt{e^2-1}} - \frac{e\sin\theta}{e\cos\theta+1}\right)_{\theta_1}^{\theta_2}$$

$$A = \frac{l^2}{2(1-e^2)}\left(\frac{2\operatorname{arctanh}\left(\sqrt{\frac{e-1}{e+1}}\tan\left(\frac{\theta}{2}\right)\right)}{\sqrt{e^2-1}} - \frac{e\sin\theta}{e\cos\theta+1}\right)_{\theta_1}^{\theta_2}$$

$$e^2 - 1 > 0$$
$$e^2 > 1$$
$$e > 1$$

1.1.4 Parabolic Trajectories

$$\text{L'Hosptial's Rule: if } \lim_{x\to a}\frac{f(a)}{g(a)} = \frac{0}{0}$$
$$\text{or } \lim_{x\to a}\frac{f(a)}{g(a)} = \frac{\infty}{\infty}$$
$$\text{then: } \lim_{x\to a}\frac{f(a)}{g(a)} = \frac{f'(a)}{g'(a)}$$

$$A = \frac{l^2}{2(1-e^2)}\left(\frac{2\arctan\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)\right)}{\sqrt{1-e^2}} - \frac{e\sin\theta}{e\cos\theta+1}\right)_{\theta_1}^{\theta_2}$$

$$A = \frac{l^2}{2}\left(\frac{2\arctan\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)\right)}{(1-e^2)^{\frac{3}{2}}} - \frac{e\sin\theta}{(1-e^2)(e\cos\theta+1)}\right)_{\theta_1}^{\theta_2}$$

$$A = \frac{l^2}{2}\left(\frac{2\arctan\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)\right)(e\cos\theta+1)}{(1-e^2)^{\frac{3}{2}}(e\cos\theta+1)} - \frac{e\sin\theta\cdot\sqrt{1-e^2}}{(1-e^2)^{\frac{3}{2}}(e\cos\theta+1)}\right)_{\theta_1}^{\theta_2}$$

$$A = \frac{l^2}{2}\left(\frac{2\arctan\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)\right)(e\cos\theta+1) - e\sin\theta\cdot\sqrt{1-e^2}}{(1-e^2)^{\frac{3}{2}}(e\cos\theta+1)}\right)_{\theta_1}^{\theta_2}$$

$$A = \frac{l^2}{2}\lim_{e\to 1}\left(\frac{2\arctan\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)\right)(e\cos\theta+1) - e\sin\theta\cdot\sqrt{1-e^2}}{(1-e^2)^{\frac{3}{2}}(e\cos\theta+1)}\right)_{\theta_1}^{\theta_2}$$

$$A = \frac{l^2}{2}\lim_{e\to 1}\left(\frac{2\arctan(\sqrt{\frac{1-1}{1+1}}\tan\left(\frac{\theta}{2}\right))(\cos\theta+1) - e\sin\theta\cdot\sqrt{1-1}}{(1-1)^{\frac{3}{2}}(\cos\theta+1)}\right)_{\theta_1}^{\theta_2}$$

$$A = \frac{l^2}{2}\lim_{e\to 1}\left(\frac{2\arctan(0\cdot\tan\left(\frac{\theta}{2}\right))(\cos\theta+1) - e\sin\theta\cdot 0}{0^{\frac{3}{2}}(\cos\theta+1)}\right)_{\theta_1}^{\theta_2}$$

$$A = \frac{l^2}{2}\lim_{e\to 1}\left(\frac{0-0}{0}\right)_{\theta_1}^{\theta_2}$$

L'Hospital 1:

$$A = \frac{l^2}{2}\lim_{e\to 1}\left(\frac{\frac{d}{de}\left[2\cdot\arctan\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)\right)(e\cos\theta+1) - \sin\theta\cdot\left(e\sqrt{1-e^2}\right)\right]}{\frac{d}{de}[(1-e^2)^{\frac{3}{2}}(e\cos\theta+1)]}\right)_{\theta_1}^{\theta_2}$$

$$A = \frac{l^2}{2} \lim_{e\to 1} \left(\frac{2 \cdot \left[\frac{\frac{\frac{(-1)(1+e)-(1-e)(1)}{(1+e)^2}}{2\sqrt{\frac{1-e}{1+e}}} \tan\left(\frac{\theta}{2}\right)(e\cos\theta+1)}{1+\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)\right)^2} + \arctan\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)\right)(\cos\theta) \right] - \sin\theta \cdot \left[\sqrt{1-e^2} + \frac{-2e^2}{2\sqrt{1-e^2}}\right]}{\left[\frac{3}{2}(1-e^2)^{\frac{1}{2}}(-2e)(e\cos\theta+1) + (1-e^2)^{\frac{3}{2}}(\cos\theta)\right]} \right)_{\theta_1}^{\theta_2}$$

$$A = \frac{l^2}{2} \lim_{e\to 1} \left(\frac{2 \cdot \left[\frac{(-(1+e)-(1-e))\tan\left(\frac{\theta}{2}\right)(e\cos\theta+1)}{2(1+e)^2\sqrt{\frac{1-e}{1+e}}\cdot\left(1+\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)\right)^2\right)} + \arctan\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)\right)(\cos\theta) \right] - \sin\theta \cdot \left[\frac{1-e^2}{\sqrt{1-e^2}} + \frac{-e^2}{\sqrt{1-e^2}}\right]}{\sqrt{1-e^2}\left[-3e(e\cos\theta+1) + (1-e^2)(\cos\theta)\right]} \right)_{\theta_1}^{\theta_2}$$

$$A = \frac{l^2}{2} \lim_{e\to 1} \left(\frac{\frac{(-1-e-1+e)\tan\left(\frac{\theta}{2}\right)(e\cos\theta+1)}{(1+e)^2\sqrt{\frac{1-e}{1+e}}\cdot\left(1+(\frac{1-e}{1+e})\tan^2\left(\frac{\theta}{2}\right)\right)} + 2\cos\theta\cdot\arctan\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)\right) - \sin\theta\cdot\left(\frac{1-2e^2}{\sqrt{1-e^2}}\right)}{\sqrt{1-e^2}(-3e(e\cos\theta+1)+(1-e^2)(\cos\theta))} \right)_{\theta_1}^{\theta_2}$$

$$A = \frac{l^2}{2} \lim_{e\to 1} \left(\frac{\frac{-2\tan\left(\frac{\theta}{2}\right)(e\cos\theta+1)}{\sqrt{1-e^2}\cdot\left((1+e)+(1-e)\tan^2\left(\frac{\theta}{2}\right)\right)} + \frac{2\cos\theta\cdot\sqrt{1-e^2}\cdot\arctan\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)\right)}{\sqrt{1-e^2}} - \sin\theta\cdot\left(\frac{1-2e^2}{\sqrt{1-e^2}}\right)}{\sqrt{1-e^2}(-3e^2\cos\theta-3e+(1-e^2)\cos\theta)} \right)_{\theta_1}^{\theta_2}$$

$$A = \frac{l^2}{2} \lim_{e\to 1} \left(\frac{-2\tan\left(\frac{\theta}{2}\right)\cdot\frac{e\cos\theta+1}{(1+e)+(1-e)\tan^2\left(\frac{\theta}{2}\right)} + 2\cos\theta\sqrt{1-e^2}\cdot\arctan\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)\right) - \sin\theta\cdot(1-2e^2)}{(1-e^2)(-3e^2\cos\theta-3e)+(1-e^2)^2\cos\theta} \right)_{\theta_1}^{\theta_2}$$

L'Hospital 2:

$$A = \frac{l^2}{2} \lim_{e\to 1} \left(\frac{\frac{d}{de}\left[-2\tan\left(\frac{\theta}{2}\right)\frac{e\cos\theta+1}{(1+e)+(1-e)\tan^2\left(\frac{\theta}{2}\right)} + 2\cos\theta\cdot\sqrt{1-e^2}\arctan\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)\right) - \sin\theta\cdot(1-2e^2)\right]}{\frac{d}{de}[(1-e^2)(-3e^2\cos\theta-3e)+(1-e^2)^2\cos\theta]} \right)_{\theta_1}^{\theta_2}$$

$$A = \frac{l^2}{2}\left(\frac{\Bigg[-2\tan\left(\frac{\theta}{2}\right)\frac{\left((1+e)+(1-e)\tan^2\left(\frac{\theta}{2}\right)\right)(\cos\theta)-\left(1-\tan^2\left(\frac{\theta}{2}\right)\right)(e\cos\theta+1)}{\left((1+e)+(1-e)\tan^2\left(\frac{\theta}{2}\right)\right)^2}}{(-2e)(-3e^2\cos\theta-3e)+(1-e^2)(-6e\cos\theta-3)+2(1-e^2)(-2e)\cos\theta}\right.$$

$$\left.+\frac{2\cos\theta\cdot\left(\sqrt{1-e^2}\frac{\frac{\frac{(-1)(1+e)-(1-e)(1)}{(1+e)^2}}{2\sqrt{\frac{1-e}{1+e}}}\tan\left(\frac{\theta}{2}\right)}{1+\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)\right)^2}+\frac{-2e}{2\sqrt{1-e^2}}\arctan\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)\right)\right)-\sin\theta(-4e^2)\Bigg]}{(-2e)(-3e^2\cos\theta-3e)+(1-e^2)(-6e\cos\theta-3)+2(1-e^2)(-2e)\cos\theta}\right)_{\theta_1}^{\theta_2}$$

$$A = \frac{l^2}{2}\left(\frac{-2\tan\left(\frac{\theta}{2}\right)\frac{\left((1+e)+(1-e)\tan^2\left(\frac{\theta}{2}\right)\right)(\cos\theta)-\left(1-\tan^2\left(\frac{\theta}{2}\right)\right)(e\cos\theta+1)}{\left((1+e)+(1-e)\tan^2\left(\frac{\theta}{2}\right)\right)^2}}{6e^3\cos\theta+6e^2-6e\cos\theta-3+6e^3\cos\theta+3e^2-4e\cos\theta+4e^3\cos\theta}\right.$$

$$\left.+\frac{2\cos\theta\cdot\left(\sqrt{1-e}\cdot\sqrt{1+e}\frac{(-1)(1+e)-(1-e)(1)\tan\left(\frac{\theta}{2}\right)}{2(1+e)^2\sqrt{\frac{1-e}{1+e}}\cdot\left(1+\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)\right)^2\right)}+\frac{-2e}{2\sqrt{1-e^2}}\arctan\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)\right)\right)+4e^2\sin\theta}{6e^3\cos\theta+6e^2-6e\cos\theta-3+6e^3\cos\theta+3e^2-4e\cos\theta+4e^3\cos\theta}\right)_{\theta_1}^{\theta_2}$$

$$A = \frac{l^2}{2}\left(\frac{-2\tan\left(\frac{\theta}{2}\right)\frac{\left(\tan^2\left(\frac{\theta}{2}\right)-1\right)(e\cos\theta+1)}{\left((1+e)+(1-e)\tan^2\left(\frac{\theta}{2}\right)\right)^2}-2\tan\left(\frac{\theta}{2}\right)\frac{((1+e)+(1-e)\tan^2\left(\frac{\theta}{2}\right))(\cos\theta)}{\left((1+e)+(1-e)\tan^2\left(\frac{\theta}{2}\right)\right)^2}}{16e^3\cos\theta+9e^2-10e\cos\theta-3}\right.$$

$$\left.+\frac{2\cos\theta\cdot\left(\frac{-2\tan\left(\frac{\theta}{2}\right)}{2(1+e)\left(1+\left(\frac{1-e}{1+e}\right)\tan^2\left(\frac{\theta}{2}\right)\right)}+\frac{-2e}{2\sqrt{1-e^2}}\arctan\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)\right)\right)+4e^2\sin\theta}{16e^3\cos\theta+9e^2-10e\cos\theta-3}\right)_{\theta_1}^{\theta_2}$$

$$A = \frac{l^2}{2}\left(\frac{-2\tan\left(\frac{\theta}{2}\right)\frac{\left(\tan^2\left(\frac{\theta}{2}\right)-1\right)(e\cos\theta+1)}{\left((1+e)+(1-e)\tan^2\left(\frac{\theta}{2}\right)\right)^2}-\frac{2\cos\theta\cdot\tan\left(\frac{\theta}{2}\right)}{(1+e)+(1-e)\tan^2\left(\frac{\theta}{2}\right)}-\frac{2\cos\theta\cdot\tan\left(\frac{\theta}{2}\right)}{(1+e)+(1-e)\tan^2\left(\frac{\theta}{2}\right)}-\frac{2e(\cos\theta)}{\sqrt{1-e^2}}\arctan\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)\right)+4e^2\sin\theta}{16e^3\cos\theta+9e^2-10e\cos\theta-3}\right)_{\theta_1}^{\theta_2}$$

$$A = \frac{l^2}{2}\left(\frac{-2\tan\left(\frac{\theta}{2}\right)\frac{\left(\tan^2\left(\frac{\theta}{2}\right)-1\right)(e\cos\theta+1)}{\left((1+e)+(1-e)\tan^2\left(\frac{\theta}{2}\right)\right)^2}-\frac{4\cos\theta\cdot\tan\left(\frac{\theta}{2}\right)}{(1+e)+(1-e)\tan^2\left(\frac{\theta}{2}\right)}-\frac{2e\cdot\cos\theta}{\sqrt{1-e^2}}\arctan\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)\right)+4e^2\sin\theta}{16e^3\cos\theta+9e^2-10e\cos\theta-3}\right)_{\theta_1}^{\theta_2}$$

Substitution 1:

$$A=\frac{l^2}{2}\left(\frac{-2\tan\left(\frac{\theta}{2}\right)\frac{\left(\tan^2\left(\frac{\theta}{2}\right)-1\right)(\cos\theta+1)}{((1+1)+(1-1)\tan^2\left(\frac{\theta}{2}\right))^2}-\frac{4\cos\theta\cdot\tan\left(\frac{\theta}{2}\right)}{(1+1)+(1-1)\tan^2\left(\frac{\theta}{2}\right)}-\frac{2\cos\theta}{\sqrt{1-1}}\arctan\left(\sqrt{\frac{1-1}{1+1}}\tan\left(\frac{\theta}{2}\right)\right)+4\sin\theta}{16\cos\theta+9-10\cos\theta-3}\right)_{\theta_1}^{\theta_2}$$

$$A=\frac{l^2}{2}\left(\frac{-2\tan\left(\frac{\theta}{2}\right)\frac{\left(\tan^2\left(\frac{\theta}{2}\right)-1\right)(\cos\theta+1)}{((2)+(0)\tan^2\left(\frac{\theta}{2}\right))^2}-\frac{4\cos\theta\cdot\tan\left(\frac{\theta}{2}\right)}{(2)+(0)\tan^2\left(\frac{\theta}{2}\right)}-\frac{2\cos\theta}{\sqrt{0}}\arctan\left(\sqrt{\frac{0}{2}}\tan\left(\frac{\theta}{2}\right)\right)+4\sin\theta}{6\cos\theta+6}\right)_{\theta_1}^{\theta_2}$$

$$A=\frac{l^2}{2}\left(\frac{2\tan\left(\frac{\theta}{2}\right)\frac{\left(\tan^2\left(\frac{\theta}{2}\right)-1\right)(\cos\theta+1)}{2^2}+\frac{4\cos\theta\cdot\tan\left(\frac{\theta}{2}\right)}{2}+\frac{2\cos\theta\cdot\arctan(0)}{0}-4\sin\theta}{-6(\cos\theta+1)}\right)_{\theta_1}^{\theta_2}$$

$$A=\frac{l^2}{2}\left(\frac{\tan\left(\frac{\theta}{2}\right)\left(\tan^2\left(\frac{\theta}{2}\right)-1\right)(\cos\theta+1)+4\cos\theta\cdot\tan\left(\frac{\theta}{2}\right)+\frac{0}{0}-8\sin\theta}{-12(\cos\theta+1)}\right)_{\theta_1}^{\theta_2}$$

only the arctan term doesn't exist at $e=1$. L'Hospital 3:

$$A=\frac{l^2}{2}\left(\frac{-2\tan\left(\frac{\theta}{2}\right)\frac{\left(\tan^2\left(\frac{\theta}{2}\right)-1\right)(e\cos\theta+1)}{\left((1+e)+(1-e)\tan^2\left(\frac{\theta}{2}\right)\right)^2}-\frac{4\cos\theta\cdot\tan\left(\frac{\theta}{2}\right)}{(1+e)+(1-e)\tan^2\left(\frac{\theta}{2}\right)}+4\sin\theta e^2-\frac{2e(\cos\theta)}{\sqrt{1-e^2}}\arctan\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)\right)}{16e^3\cos\theta+9e^2-10e\cos\theta-3}\right)_{\theta_1}^{\theta_2}$$

$$A=\frac{l^2}{2}\left(\frac{\tan\left(\frac{\theta}{2}\right)\left(\tan^2\left(\frac{\theta}{2}\right)-1\right)(\cos\theta+1)+4\cos\theta\cdot\tan\left(\frac{\theta}{2}\right)-8\sin\theta}{-12(\cos\theta+1)}+\frac{4\cos\theta\cdot\frac{d}{de}\left[e\arctan\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)\right)\right]}{\frac{d}{de}[\sqrt{1-e^2}(-32e^3\cos\theta-18e^2+20e\cos\theta+6)]}\right)_{\theta_1}^{\theta_2}$$

$$A=\frac{l^2}{2}\left(\frac{\tan\left(\frac{\theta}{2}\right)\left(\tan^2\left(\frac{\theta}{2}\right)-1\right)(\cos\theta+1)+4\cos\theta\cdot\tan\left(\frac{\theta}{2}\right)-8\sin\theta}{-12(\cos\theta+1)}\right.$$
$$\left.+\frac{4\cos\theta\cdot\left[\frac{\frac{\frac{(-1)(1+e)-(1-e)(1)}{(1+e)^2}}{2\sqrt{\frac{1-e}{1+e}}}\tan\left(\frac{\theta}{2}\right)}{1+\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)\right)^2}+\arctan\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)\right)\right]}{\left[\frac{-e}{\sqrt{1-e^2}}(-32e^3\cos\theta-18e^2+20e\cos\theta+6)+\sqrt{1-e^2}(-96e^2\cos\theta-36e+20\cos\theta)\right]}\right)_{\theta_1}^{\theta_2}$$

$$A=\frac{l^2}{2}\left(\frac{\tan\left(\frac{\theta}{2}\right)\left(\tan^2\left(\frac{\theta}{2}\right)-1\right)(\cos\theta+1)+4\cos\theta\cdot\tan\left(\frac{\theta}{2}\right)-8\sin\theta}{-12(\cos\theta+1)}\right.$$
$$\left.+\frac{4\cos\theta\cdot\left[\frac{-2\tan\left(\frac{\theta}{2}\right)}{2(1+e)^2\sqrt{\frac{1-e}{1+e}}\cdot(1+(\frac{1-e}{1+e})\tan^2\left(\frac{\theta}{2}\right)}+\arctan\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)\right)\right]}{\frac{-e}{\sqrt{1-e^2}}(-32e^3\cos\theta-18e^2+20e\cos\theta+6)+\sqrt{1-e^2}(-96e^2\cos\theta-36e+20\cos\theta)}\right)_{\theta_1}^{\theta_2}$$

$$A = \frac{l^2}{2}\left(\frac{\tan\left(\frac{\theta}{2}\right)\left(\tan^2\left(\frac{\theta}{2}\right)-1\right)(\cos\theta+1)+4\cos\theta\cdot\tan\left(\frac{\theta}{2}\right)-8\sin\theta}{-12(\cos\theta+1)}\right.$$
$$\left.+\frac{4\cos\theta\cdot\sqrt{1-e^2}\left[\frac{-\tan\left(\frac{\theta}{2}\right)}{\sqrt{1+e}\sqrt{1-e}((1+e)+(1-e)\tan^2\left(\frac{\theta}{2}\right))}+\arctan\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)\right)\right]}{\sqrt{1-e^2}(\frac{1}{\sqrt{1-e^2}}(32e^4\cos\theta+18e^3-20e^2\cos\theta-6e)+\sqrt{1-e^2}(-96e^2\cos\theta-36e+20\cos\theta))}\right)_{\theta_1}^{\theta_2}$$

$$A = \frac{l^2}{2}\left(\frac{\tan\left(\frac{\theta}{2}\right)\left(\tan^2\left(\frac{\theta}{2}\right)-1\right)(\cos\theta+1)+4\cos\theta\cdot\tan\left(\frac{\theta}{2}\right)-8\sin\theta}{-12(\cos\theta+1)}\right.$$
$$\left.+\frac{4\cos\theta\cdot\left[\frac{-\tan\left(\frac{\theta}{2}\right)}{(1+e)+(1-e)\tan^2\left(\frac{\theta}{2}\right)}+\sqrt{1-e^2}\cdot\arctan\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)\right)\right]}{(32e^4\cos\theta+18e^3-20e^2\cos\theta-6e)+(1-e^2)(-96e^2\cos\theta-36e+20\cos\theta)}\right)_{\theta_1}^{\theta_2}$$

Substitution 2:

$$A = \frac{l^2}{2}\left(\frac{\tan\left(\frac{\theta}{2}\right)\left(\tan^2\left(\frac{\theta}{2}\right)-1\right)(\cos\theta+1)+4\cos\theta\cdot\tan\left(\frac{\theta}{2}\right)-8\sin\theta}{-12(\cos\theta+1)}+\frac{4\cos\theta\cdot\left[\frac{-\tan\left(\frac{\theta}{2}\right)}{(1+1)+(1-1)\tan^2\left(\frac{\theta}{2}\right)}+\sqrt{1-1}\cdot\arctan\left(\sqrt{\frac{1-1}{1+1}}\tan\left(\frac{\theta}{2}\right)\right)\right]}{(32\cos\theta+18-20\cos\theta-6)+(1-1)(-96\cos\theta-36+20\cos\theta)}\right)_{\theta_1}^{\theta_2}$$

$$A = \frac{l^2}{2}\left(\frac{\tan\left(\frac{\theta}{2}\right)\left(\tan^2\left(\frac{\theta}{2}\right)-1\right)(\cos\theta+1)+4\cos\theta\cdot\tan\left(\frac{\theta}{2}\right)-8\sin\theta}{-12(\cos\theta+1)}+\frac{4\cos\theta\left[\frac{-\tan\left(\frac{\theta}{2}\right)}{2+0\tan^2\left(\frac{\theta}{2}\right)}+\sqrt{0}\arctan\left(\sqrt{\frac{0}{2}}\tan\left(\frac{\theta}{2}\right)\right)\right]}{(12\cos\theta+12)+(0)(-96\cos\theta-36+20\cos\theta)}\right)_{\theta_1}^{\theta_2}$$

$$A = \frac{l^2}{2}\left(\frac{\tan\left(\frac{\theta}{2}\right)\left(\tan^2\left(\frac{\theta}{2}\right)-1\right)(\cos\theta+1)+4\cos\theta\cdot\tan\left(\frac{\theta}{2}\right)-8\sin\theta}{-12(\cos\theta+1)}+\frac{4\cos\theta\cdot\frac{-\tan\left(\frac{\theta}{2}\right)}{2}}{12\cos\theta+12}\right)_{\theta_1}^{\theta_2}$$

$$A = \frac{l^2}{2}\left(\frac{\tan\left(\frac{\theta}{2}\right)\left(\tan^2\left(\frac{\theta}{2}\right)-1\right)(\cos\theta+1)+4\cos\theta\cdot\tan\left(\frac{\theta}{2}\right)-8\sin\theta}{-12(\cos\theta+1)}+\frac{2\cos\theta\cdot\tan\left(\frac{\theta}{2}\right)}{-12(\cos\theta+1)}\right)_{\theta_1}^{\theta_2}$$

$$A = \frac{l^2}{2}\left(\frac{\tan\left(\frac{\theta}{2}\right)\left(\tan^2\left(\frac{\theta}{2}\right)-1\right)(\cos\theta+1)+6(\cos\theta)\tan\left(\frac{\theta}{2}\right)-8\sin\theta}{-12(\cos\theta+1)}\right)_{\theta_1}^{\theta_2}$$

$e = 1$