Orbital Mechanics for Engineers

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Chapter 1

Introduction and Axioms

1.1 Intro to Orbits

1.2 Identities

Identities of an orbit:

The construction of an orbit (Orbital Trajectory) can be split into two parts: Geometry and Physics. Each of these takes some properties to determine the characteristics of the orbit. These properties are called "Identities", since combined they can identify a unique orbit.

The geometric identity of the orbit can be defined with almost any two properties of the geometry. The two we will be using are Eccentricity (used for defining the shape of the orbit) and Semi Latus Rectum (used for defining the scale of the orbit) These properties aren't traditionally used for orbits, but become extremely useful for both geometric construction and navigation. It is also possible to substitute one of these properties for Periapsis, since it is one of the few properties that share the universal applicability with the the other two properties, and is frequently used in navigation.

The physics identity is slightly easier to define, as it only requires a single property that is more standardized: the Standard Gravitational Parameter. This property allows the physics calculations to easily translate between gravitation and motion.

Geometric Identities:

- Eccentricity: The scalar metric of how far the conic section deviates from a circle; e
- Semi Latus Rectum: 1/2 The length of a chord intersecting the primary focus, perpendicular to the major axis; l
- Periapsis: The shortest distance between the orbit and the body at the prime focus; p

Physics Identities:

- Standard Gravitational Parameter: The product of the mass of the body being orbited and the Universal Gravitational Constant ; μ

Geometric Formula's used:

(Note: The following all have the center of the shape at the origin)

- Standard Circle Formula: $x^2 + y^2 = r^2$ $\{e = 0\}$
- Standard Ellipse Formula: $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ $\{0 < e < 1\}$
- Standard Parabola Formula: $y^2 = -4px$ $\{e = 1\}$
- Standard Hyperbola Formula: $\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1$ $\{e > 1\}$

Physics Formulas used:

- Conservation of Specific Mechanical Energy: $\epsilon = \epsilon_k + \epsilon_p$
- Conservation of Specific Angular Momentum: $\frac{L}{m_{satellite}} = rv_{\perp}$

Chapter 2

Geometry

2.1 Introduction to Conic Sections

Thanks to the work of Johannes Kepler and Sir Issac Newton, we know what the base geometries which orbiting bodies follow when only under the influence of gravity: **Conic Sections**. However, these base geometries aren't always 100% true to life. While quickly diminishing with distance, gravity's reach is infinite in the universe, and with all the mass surrounding us, those tiny pulls can add up to what is known as "Orbital Perturbations". Due to the difficulty in computing perturbations, it is better covered in a follow-up paper. For now it is sufficient to compute just the ideal trajectory. In this chapter, we will individually explore each conic section, their principal parts, and their limitations, before finding their commonalities to establish a universal geometric description of satellite motion.

There are four basic conic sections (in order of increasing eccentricity): **Circle**, **Ellipse**, **Parabola**, and **Hyperbola**. As Eccentricity increases, the commonalities between the shapes diminishes, most notably with the Parabola.

Common parts of conic sections:

- **Primary Focus**: The focus of the orbit where the center of mass of the orbit lies (usually just the planet or star being orbited).
- **Empty Focus***: The other focus of the orbit, which is typically unoccupied.
- **Center***: The center of the conic section; At the midpoint between the two foci; At the midpoint between Apoapsis and Periapsis.
- **Periapsis**: The point on the conic section that is closest to the primary focus; p
- **Apoapsis***: The point on the conic section that is farthest from the primary focus; The point closest to the empty focus.
- **True Anomaly**: The angle as measured from the primary focus, between Periapsis and a point on the conic section, going counter-clockwise (direction of motion); θ_t
- **Major Axis***: The longer axis of the orbit. It lies from Periapsis to Apoapsis.

- **Minor Axis***: The shorter axis of the orbit. It lies perpendicular to the Major axis, with its midpoint lying on the center.
- **Latus Rectum**: The chord intersecting the primary focus, running perpendicular to the Major Axis.
- **Eccentricity**: A scalar metric measuring how far a conic section/orbit deviates from a circle; e
- **Latus Scale**: A scalar metric used in conjunction with Eccentricity, as it is used to define several other properties in the orbit; The ratio between Semi Latus Rectum and Semi Major Axis; s
- **Semi Major Axis***: 1/2 the length of the Major Axis; a
- **Semi Minor Axis***: 1/2 the length of the Minor Axis; b
- **Semi Latus Rectum**: 1/2 the length of the Semi Latus Rectum; l
- **Linear Eccentricity***: The length between the center and either focus; The product of Semi Major Axis and Eccentricity; c
- **Radius of Prime Focus**: The distance from the Prime Focus to an arbitrary point; r_1
- **Radius of Empty Focus***: The distance from the Empty Focus to an arbitrary point; r_2
- **Constant of Radius**: The constant resulting from either adding or subtracting the Radii together.

Many of the above don't work in all circumstances. In the case of open trajectories (i.e. when $e \geq 1$) many of these items' meanings break down. Most notably for the Parabola ($e = 1$), many of these values have a limit of either infinity, or are undefined between positive and negative infinity, making their use difficult, or impossible in a general context. Values which break the orbital context at some point have been marked with an asterisk.

The reason why these values break down is because they are in some way defined by the Empty Focus. The issue here is that the Empty Focus does not always exist in the context of an orbiting body. When $e \geq 1$, the orbit is no longer a closed shape, and the points where $r_1 \geq r_2$ are no longer reachable. When this occurs, values like Apoapsis and Semi Major Axis are no longer suitable for navigation and make computation a headache, at best. For this reason we will only use values defined solely by their relationship to the Primary Focus, and those equations and formulas that do use these unstable values will be redefined to universally applicable formulae, using the values listed above without the asterisk.

For that reason, we will use Semi Latus Rectum and Eccentricity for our **geometric identities**, or the properties used to define an orbit. It is also possible to substitute one of these properties for Periapsis as it comes up frequently in navigation, and only two properties are required to calculate the geometry of an orbit.

There are two main classes of orbit geometries: **closed orbits**, which correspond to perpetually orbiting satellites, and **open orbits**, which correspond to escape trajectories; ones that will orbit once before being ejected away from the orbited body, never orbiting it again.

2.2 Ellipse

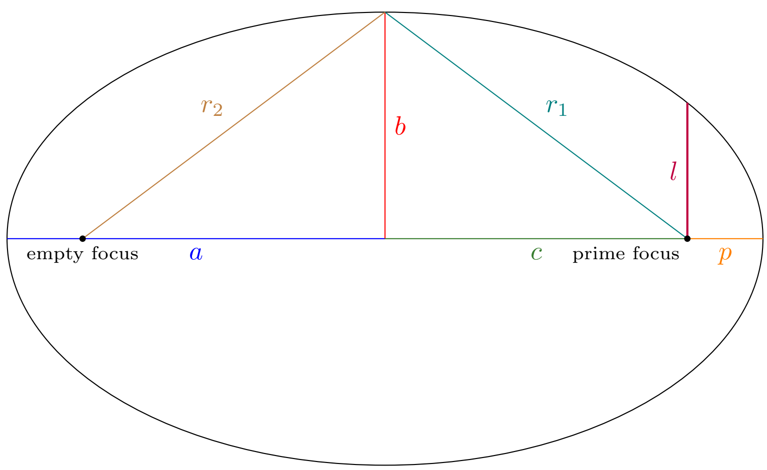
Principles

The first conic section we will examine is the ellipse, which is most characterized by its closed shape and oblong / "squished" appearance.

$$\text{Standard Formula: } \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

Local Properties

- Semi Latus Rectum: l
- Semi Major Axis: a
- Semi Minor Axis: b
- Linear Eccentricity: c
- Radius of Prime Focus: r_1
- Radius of Empty Focus: r_2



Defining principle: The distances between an arbitrary point on the ellipse, and the two foci, have a constant sum:

$$r_1 + r_2 = \textit{Constant of Radius}$$

The ***Constant of Radius*** for an Ellipse is very simple to define in terms In the case of the point being on the Periapsis of the Ellipse. The Primary Radius will be the difference of between the Semi Major Axis and the Linear Eccentricity, while the Empty Radius will be the sum of the Semi Major Axis and the Linear Eccentricity.

$$r_1 = a - c$$

$$r_2 = a + c$$

$$\textit{Constant of Radius} = (a - c) + (a + c)$$

$$\textit{Constant of Radius} = 2a$$

(The Constant of Radius applies not just to Periapsis, but to every point on the ellipse as well.)

Now that we have the defining principle defined algebraically, we can now also define the geometric identities: **Eccentricity** and **Semi Latus Rectum**. Because Semi Latus Rectum is perpendicular to the Major Axis, we can achieve this creating a right triangle, using Semi Latus Rectum as one leg, and the distance between the foci as the other leg, and using Pythagorean's Theorem.

$$r_1 = l$$
$$r_2 = 2a - l$$
$${r_1}^2 + (2c)^2 = {r_2}^2$$
$$l^2 + 4a^2e^2 = (2a - l)^2$$
$$l^2 + 4a^2e^2 = 4a^2 - 4la + l^2$$
$$4a^2e^2 = 4a^2 - 4la$$
$$ae^2 = a - l$$
$$l = a(1 - e^2)$$

Latus Scale is another scalar metric that appears often, and is convenient to have defined.

$$s = \frac{l}{a}$$
$$s = \frac{a(1 - e^2)}{a}$$
$$s = 1 - e^2$$
$$l = as$$

Periapsis can also serve as a geometric identity, so it is important to have it defined as well.

$$a = p + c \qquad\qquad p = a(1 - e)$$
$$p = a - c \qquad\qquad p = \frac{l}{1 + e}$$

Finally to define the last undefined value, **Semi Minor Axis**, we have the case of the covertex. At this point, both radial lines between the foci and the point form the hypotenuse of a right triangle with the legs being the Semi Minor Axis and Linear Eccentricity. Since they both form the hypotenuse of right triangles with the same legs, they must also be equal.

$$r_1 + r_2 = 2a \qquad\qquad {r_1}^2 = b^2 + c^2$$
$$r_2 = 2a - r_1 \qquad\qquad a^2 = b^2 + c^2$$
$$r_1 = r_2 \qquad\qquad b^2 = a^2 - c^2$$
$$r_1 + r_1 = 2a \qquad\qquad b = a\sqrt{1 - e^2}$$
$$r_1 = a \qquad\qquad b = a\sqrt{s}$$

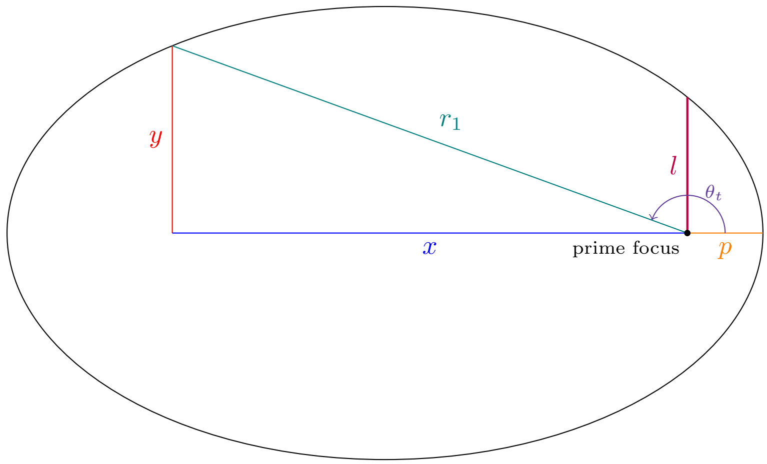
Pinpointing the domain of the ellipse can also help in ensuring we cover all cases. The main limitation of the standard ellipse formula is the value of Semi Minor Axis (b).

$$b = a\sqrt{s}$$
$$b = a\sqrt{1 - e^2}$$

$$\sqrt{1 - e^2} \text{ is defined for } 1 - e^2 \geq 0$$
$$1 - e^2 \geq 0$$
$$e^2 - 1 \leq 0$$
$$e^2 \leq 1$$
$$-1 \leq e \leq 1$$

Although technically $-1 \leq e \leq 1$, $e = 0$ and $e = 1$ are edge cases which are better described by other conic sections, and $e < 0$ is simply ignored since negative eccentricities operate identically to positive eccentricities, but flipped over the y axis. So the domain of this ellipse in this documentation is $0 < e < 1$.

Radius As True Anomaly



Defining a function for radius using True Anomaly is a pretty straight forward process. Since these models of orbits always have:

- Periapsis directly to the right of the primary focus, on the x axis;
- True Anomaly is defined as zero at periapsis;
- True Anomaly is measured counter clockwise (the direction of motion for all of these models);

So it's just a straight foward conversion from Cartesisan to Polar coordinates.

$$\text{Standard Ellipse Formula: } \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

$$\text{Focus-Offset Ellipse Formula: } \left(\frac{x+c}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

$$\frac{(x+c)^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$b^2\frac{(x+c)^2}{a^2} + y^2 = b^2$$

$$s(x+c)^2 + y^2 = b^2$$

$$s(x^2 + 2xc + c^2) + y^2 - b^2 = 0$$

$$sx^2 + y^2 + 2xaes + a^2e^2s - a^2s = 0$$

$$s(r_1\cos\theta_t)^2 + (r_1\sin\theta_t)^2 + 2(r_1\cos\theta_t)aes + a^2s(e^2-1) = 0$$

$${r_1}^2(s\cos^2\theta_t + \sin^2\theta_t) + r_1(2aes\cos\theta_t) + (-a^2s^2) = 0$$

$${r_1}^2(\cos^2\theta_t - e^2\cos^2\theta_t + \sin^2\theta_t) + r_1(2aes\cos\theta_t) + (-a^2s^2) = 0$$

$$r_1{}^2(1 - e^2\cos^2\theta_t) + r_1(2aes\cos\theta_t) + (-a^2s^2) = 0$$
$$r_1{}^2(e^2\cos^2\theta_t - 1) + r_1(-2aes\cos\theta_t) + (a^2s^2) = 0$$

$$r_1 = \frac{2aes\cos\theta_t \pm \sqrt{(-2aes\cos\theta_t)^2 - 4(e^2\cos^2\theta_t - 1)(a^2s^2)}}{2(e^2\cos^2\theta_t - 1)}$$

$$r_1 = \frac{2aes\cos\theta_t \pm \sqrt{4a^2s^2(e^2\cos^2\theta_t) + 4a^2s^2(1 - e^2\cos^2\theta_t)}}{2(e^2\cos^2\theta_t - 1)}$$

$$r_1 = \frac{2aes\cos\theta_t \pm 2as\sqrt{(e^2\cos^2\theta_t) + (1 - e^2\cos^2\theta_t)}}{2(e^2\cos^2\theta_t - 1)}$$

$$r_1 = \frac{as(e\cos\theta_t \pm \sqrt{1})}{e^2\cos^2\theta_t - 1^2}$$

$$r_1 = \frac{l(e\cos\theta_t \pm 1)}{(e\cos\theta_t - 1)(e\cos\theta_t + 1)}$$

$$r_1 = \frac{l(e\cos\theta_t + 1)}{(e\cos\theta_t - 1)(e\cos\theta_t + 1)}$$
$$r_1 = \frac{l}{e\cos\theta_t - 1}$$
$$p = \frac{a(1+e)(1-e)}{e\cos 0 - 1}$$
$$p = \frac{-a(1+e)(e-1)}{e-1}$$
$$p = -a(1+e)$$

[REJECT]

$$r_1 = \frac{l(e\cos\theta_t - 1)}{(e\cos\theta_t - 1)(e\cos\theta_t + 1)}$$
$$r_1 = \frac{l}{e\cos\theta_t + 1}$$
$$p = \frac{a(1+e)(1-e)}{e\cos 0 + 1}$$
$$p = \frac{a(1+e)(1-e)}{e+1}$$
$$p = a(1-e)$$

[ACCEPT]

Ellipse Primary Radius Formula: $r_1 = \dfrac{l}{e\cos\theta_t + 1}$

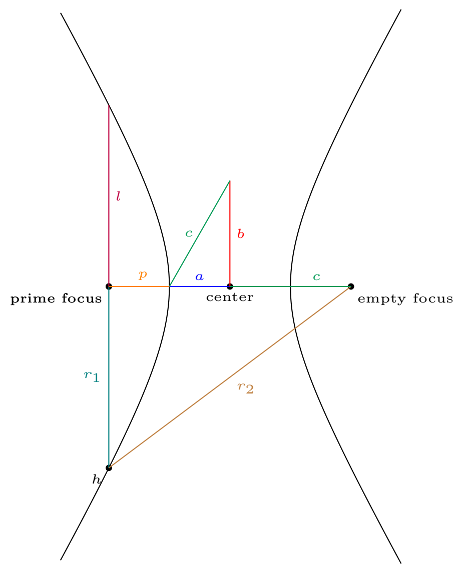
2.3 Hyperbola

Principles

The next conic section we'll cover is the **Hyperbola**. Its main characteristics are its two mirrored sections and **open shape**. Due to this shape, it is considered an **escape trajectory**, with the section opposite periapsis being mostly imaginary since it is not accessible.

$$\text{Standard Formula: } \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1$$

- Semi Latus Rectum: l
- Semi Major Axis: a
- Semi Minor Axis: b
- Linear Eccentricity: c
- Radius of Prime Focus: r_1
- Radius of Empty Focus: r_2



Defining principle: The distances between an arbitrary point on the hyperbola, and the two foci, have a constant difference:

$$|r_1 - r_2| = \textit{Constant of Radius}$$

The ***Constant of Radius*** for a Hyperbola is very simple to define in terms In the case of the point being on the Periapsis of the Hyperbola. The Primary Radius will be the difference of between the Semi Major Axis and the Linear Eccentricity, while the Empty Radius will be the sum of the Semi Major Axis and the Linear Eccentricity.

$$p = c - a$$

$$r_1 = c - a$$

$$r_2 = c + a$$

$$|r_1 - r_2| = \textit{Constant of Radius}$$

$$\textit{Constant of Radius} = |(c - a) - (c + a)|$$

$$\textit{Constant of Radius} = |-2a|$$

$$\textit{Constant of Radius} = 2a$$

(The Constant of Radius applies not just to Periapsis, but to every point on the ellipse as well.)

Now that we have the defining principle defined algebraically, we can now also define the geometric identities: **Eccentricity** and **Semi Latus Rectum**. Because Semi Latus Rectum is perpendicular to the Major Axis, we can achieve this creating a right triangle, using Semi Latus Rectum as one leg, and the distance between the foci as the other leg, and using Pythagorean's Theorem.

$$r_1 = l$$

$$r_2 = 2a + l$$

$${r_1}^2 + (2c)^2 = {r_2}^2$$

$$l^2 + 4a^2e^2 = (2a + l)^2$$

$$l^2 + 4a^2e^2 = 4a^2 + 4la + l^2$$

$$4a^2e^2 = 4a^2 + 4la$$

$$ae^2 = a + l$$

$$l = a(e^2 - 1)$$

Latus Scale is another scalar metric that appears often, and is convenient to have defined.

$$s = \frac{l}{a}$$

$$s = \frac{a(e^2 - 1)}{a}$$

$$s = e^2 - 1$$

$$l = as$$

As we can see above for Semi Minor axis and Semi Latus Rectum, anywhere where Latus Scale was part of a definition in the ellipse, the hyperbolic equivalent is negative Latus Scale. This explains why everything about the hyperbola seems a little backwards, or inside-out compared to the ellipse; because almost all values have been sign flipped. This is the main reason why coming up with a unified set of formulae for conic sections is so important, and so tedious.

There is no analogous case of the point being on the covertex for the hyperbola, like on the ellipse. Instead we must use a known coordinate pair and plug it back into the standard formula to find the relationship for Semi Minor Axis. The easiest example to use is Periapsis.

$$\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1$$
$$\left(\frac{y}{b}\right)^2 = \left(\frac{x}{a}\right)^2 - 1$$
$$\frac{y^2}{b^2} = \frac{x^2 - a^2}{a^2}$$
$$\frac{b^2}{y^2} = \frac{a^2}{x^2 - a^2}$$
$$b^2 = \frac{y^2 a^2}{x^2 - a^2}$$

$$b^2 = \frac{(l)^2 a^2}{c^2 - a^2}$$
$$b^2 = \frac{l^2}{e^2 - 1}$$
$$b^2 = \frac{a^2(e^2-1)^2}{e^2 - 1}$$
$$b^2 = a^2(e^2 - 1)$$
$$b = a\sqrt{e^2 - 1}$$
$$b = a\sqrt{s}$$

Pinpointing the domain of the ellipse can also help in ensuring we cover all cases. The main limitation of the standard ellipse formula is the value of Semi Minor Axis (b).

$$b = a\sqrt{s}$$
$$b = a\sqrt{e^2 - 1}$$

$$\sqrt{e^2 - 1} \text{ is defined for } e^2 - 1 \geq 0$$
$$e^2 - 1 \geq 0$$
$$e^2 \geq 1$$
$$e \leq -1 \text{ or } e \geq 1$$

Although technically $e \leq -1$ or $e \geq 1$, $e = 1$ is an edge case which is better described by other conic section, and $e < 0$ is simply ignored since negative eccentricities operate identically to positive eccentricities, but flipped over the y axis. So the domain of this hyperbola in this documentation is $e > 1$.

However, Hyperbolas have a unique problem: Not all values of True Anomaly make geometric, physical, or navigational sense. Some True Anomaly values outside a certain range on the left arc are infinitely far away, and the entirety of the right half of the Hyperbola makes no sense as there is no gravitational force to bring it there, not even mentioning the fact if you could, it'd take an infinite amount of time.

To remedy this, we must define our domain of True Anomaly as well as Eccentricity. The point where we can cut off our trajectory is where the distance from the orbited body to the craft is infinite.

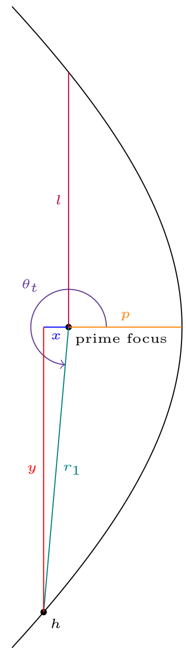
We already have a radius function from the ellipse and as we will see in next section, it is congruent in the hyperbola as well.

$$\theta_{\text{lim}} : \text{limit of } \theta_t \text{ domain}$$
$$m_{lim} : \text{Slope form of } \theta_t$$

$$r_1 = \frac{l}{e\cos\theta_t + 1}$$
$$e\cos\theta_t + 1 = \frac{l}{r_1}$$
$$\cos\theta_t = \frac{l}{er_1} - \frac{1}{e}$$
$$\theta_t = \pm\arccos\left(\frac{l}{er_1} - \frac{1}{e}\right)$$
$$\theta_{\text{lim}} = \pm\lim_{r_1\to\infty}\arccos\left(\frac{l}{e\cdot\infty} - \frac{1}{e}\right)$$

$$\theta_{\text{lim}} = \pm\lim_{r_1\to\infty}\arccos\left(0 - \frac{1}{e}\right)$$
$$\theta_{\text{lim}} = \pm\arccos\left(-\frac{1}{e}\right)$$
$$m_{lim} = \pm\tan\left(\arccos\left(-\frac{1}{e}\right)\right)$$
$$m_{lim} = \pm\sqrt{e^2 - 1}$$
$$m_{lim} = \pm\sqrt{s}$$

Radius As True Anomaly



Defining a function for radius using True Anomaly is a pretty straight forward process. Since these models of orbits always have:

- Periapsis directly to the right of the primary focus, on the x axis;
- True Anomaly is defined as zero at periapsis;
- True Anomaly is measured counter clockwise (the direction of motion for all of these models);

So it's just a straight forward conversion from Cartesian to Polar coordinates.

$$\text{Standard Hyperbola Formula: } \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1$$

$$\text{Focus-Offset Hyperbola Formula: } \left(\frac{x-c}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1$$

$$\frac{(x-c)^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$b^2\frac{(x-c)^2}{a^2} - y^2 = b^2$$

$$s(x-c)^2 - y^2 = b^2$$

$$s(x^2 - 2xc + c^2) - y^2 - b^2 = 0$$

$$sx^2 - y^2 - 2xaes + a^2e^2s - a^2s = 0$$

$$s(r_1\cos\theta_t)^2 - (r_1\sin\theta_t)^2 - 2(r_1\cos\theta_t)aes + a^2s(e^2-1) = 0$$
$$r_1{}^2(s\cos^2\theta_t - \sin^2\theta_t) + r_1(-2aes\cos\theta_t) + (a^2s^2) = 0$$
$$r_1{}^2(e^2\cos^2\theta_t - \cos^2\theta_t - \sin^2\theta_t) + r_1(-2aes\cos\theta_t) + (a^2s^2) = 0$$
$$r_1{}^2(e^2\cos^2\theta_t - 1) + r_1(-2aes\cos\theta_t) + (a^2s^2) = 0$$

$$r_1 = \frac{2aes\cos\theta_t \pm \sqrt{(2aes\cos\theta_t)^2 - 4(e^2\cos^2\theta_t-1)(a^2s^2)}}{2(e^2\cos^2\theta_t - 1)}$$

$$r_1 = \frac{2aes\cos\theta_t \pm \sqrt{4a^2s^2(e^2\cos^2\theta_t) + 4a^2s^2(1-e^2\cos^2\theta_t)}}{2(e^2\cos^2\theta_t - 1)}$$

$$r_1 = \frac{2aes\cos\theta_t \pm 2as\sqrt{(e^2\cos^2\theta_t) + (1-e^2\cos^2\theta_t)}}{2(e^2\cos^2\theta_t - 1)}$$

$$r_1 = \frac{as(e\cos\theta_t \pm \sqrt{1})}{e^2\cos^2\theta_t - 1^2}$$

$$r_1 = \frac{l(e\cos\theta_t \pm 1)}{(e\cos\theta_t - 1)(e\cos\theta_t + 1)}$$

$$r_1 = \frac{l(e\cos\theta_t + 1)}{(e\cos\theta_t - 1)(e\cos\theta_t + 1)}$$
$$r_1 = \frac{l}{e\cos\theta_t - 1}$$
$$p = \frac{a(e^2-1)}{e\cos 0 - 1}$$
$$p = \frac{a(e+1)(e-1)}{e-1}$$
$$p = a(e+1)$$

[REJECT]

$$r_1 = \frac{l(e\cos\theta_t - 1)}{(e\cos\theta_t - 1)(e\cos\theta_t + 1)}$$
$$r_1 = \frac{l}{e\cos\theta_t + 1}$$
$$p = \frac{a(e^2-1)}{e\cos 0 + 1}$$
$$p = \frac{a(e+1)(e-1)}{e+1}$$
$$p = a(e-1)$$

[ACCEPT]

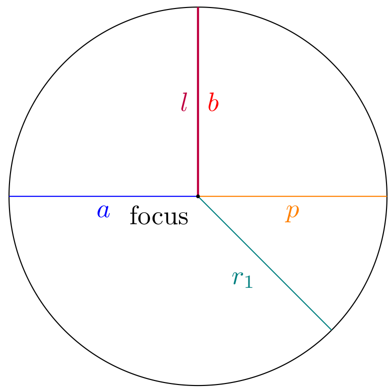
Hyperbola Primary Radius Formula: $r_1 = \dfrac{l}{e\cos\theta_t + 1}$

(Here we can also see that the radius function is in fact congruent with the Ellipse radius function, as stated in the last part of the previous section.)

2.3.1 Circle

Principles

The **Circle** is perhaps the most well known Conic Section and by far the simplest. Because of this simplicity, satellites often are put in as circular orbits as possible.



Defining Property: All points are equidistant from the center.

$$r = constant$$

$$r = r_1 = r_2$$

$$r = a = b$$

$$x^2 + y^2 = r^2$$

$$x^2 + y^2 = {r_1}^2$$

$$\left(\frac{x}{r_1}\right)^2 + \left(\frac{y}{r_1}\right)^2 = 1$$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

Here we can see that a circle is actually just a special case of an ellipse, where Semi Major Axis equals Semi Minor Axis. We can take advantage of this fact to find/verify the domain of the shape.

$$a = b$$

$$b = a\sqrt{1 - e^2}$$

$$a = a\sqrt{1 - e^2}$$

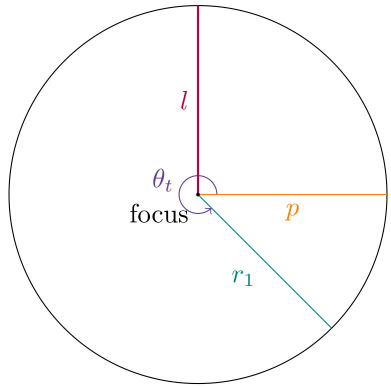
$$1^2 = 1 - e^2$$

$$e = 0$$

Not only is a circle just an ellipse with equivalent Major and Minor Axis, we can prove that it is an ellipse with an eccentricity of zero.

Additionally, since all points are equidistant to the focus, the Semi Latus Rectum is also equal to radius.

2.3.2 Radius as True Anomaly



Since we know that a circle is an ellipse with a specific value of eccentricity ($e = 0$), we can verify that the Ellipse Primary Radius Formula is valid for a circle.

$$l = r_1$$

$$r_1 = \frac{l}{e\cos\theta_t + 1}$$

$$r_1 = \frac{r_1}{(0)\cos\theta_t + 1}$$

$$r_1 = \frac{r_1}{1}$$

$$r_1 = r_1$$

Circle Primary Radius Formula: $r_1 = \dfrac{l}{e\cos\theta_t + 1}$

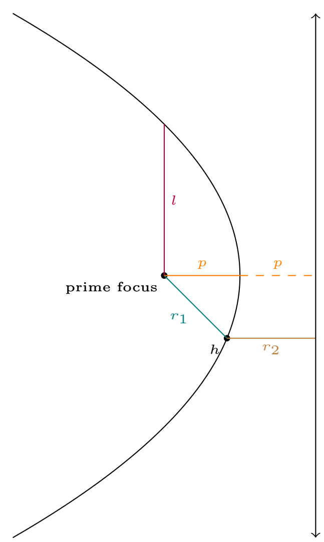
2.4 Parabola

Principles

The **Parabola** is the case between the Ellipse and Hyperbola. Right at the moment when a closed elliptical trajectory opens up to become an escape trajectory. Because it is the moment when a satellite escapes the gravitational confines of the body its orbiting, its farthest point should be at infinity.

$$\text{Standard Formula: } y^2 = -4px$$

- Semi Latus Rectum: l
- Periapsis: p
- Radius of Prime Focus: r_1
- Radius of Directrix: r_2



Defining principle: The distance between an arbitrary point on the Parabola and the prime focus, and the distance between the same arbitrary point and the directrix, are equal:

$$r_1 = r_2$$

$$\textit{Constant of Radius} = r_1 - r_2$$

$$\textit{Constant of Radius} = 0$$

Unlike the Hyperbola, the Parabola cannot use the values defined by the Empty Focus *at all*, since the Empty Focus is infinitely far away from the periapsis. Instead of using the Empty Focus to determine the relationship of Prime Radius, we will use a directrix opposite the Prime Focus from Periapsis. For **Semi Latus Rectum** we can simply plug and play with the given formula.

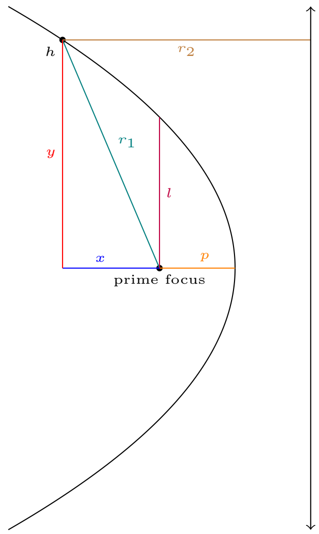
$$y^2 = -4px$$
$$l^2 = -4p(-p)$$
$$l^2 = 4p^2$$
$$l = 2p$$

Pinpointing the domain of the Parabola can also help in ensuring we cover all cases. However, unlike Hyperbolas and Ellipses, Parabolas do no have a Semi Minor Axis with a convenient limit to derive. Instead we can use the consistent relationship found in the other 3 conic sections to derive its domain with respect to eccentricity.

$$l = p(1 + e)$$
$$2p = p(1 + e)$$
$$2 = 1 + e$$
$$e = 1$$

Although technically $e \leq -1$ or $e \geq 1$, $e = 1$ is an edge case which is better described by other conic section, and $e < 0$ is simply ignored since negative eccentricities operate identically to positive eccentricities, but flipped over the y axis. So the domain of this hyperbola in this documentation is $e > 1$.

Radius As True Anomaly



Defining a function for radius using True Anomaly is a pretty straight forward process. Since these model of orbits always have:

- Periapsis directly to the right of the primary focus, on the x axis;
- True Anomaly is defined as zero at periapsis;
- True Anomaly is measured counter clockwise (the direction of motion for all of these models);

So it's just a straight forward conversion from Cartesian to Polar coordinates.

$$\text{Standard Parabola Formula: } y^2 = -4px$$

$$\text{Focus-Offset Parabola Formula: } y^2 = -4p(x - p)$$

$$y^2 + 4p(x - p) = 0$$

$$y^2 + 4px - 4p^2 = 0$$

$$(r_1 \sin\theta_t)^2 + 4p(r_1 \cos\theta_t) - 4p^2 = 0$$

$$r_1{}^2(\sin^2\theta_t) + r_1(4p\cos\theta_t) + (-4p^2) = 0$$

$$r_1 = \frac{-(4p\cos\theta_t) \pm \sqrt{(4p\cos\theta_t)^2 + 4(\sin^2\theta_t)(-4p^2)}}{2(\sin^2\theta_t)}$$

$$r_1 = \frac{-4p\cos\theta_t \pm \sqrt{16p^2\cos^2\theta_t + 16p^2\sin^2\theta_t}}{2(1 - \cos^2\theta_t)}$$

$$r_1 = \frac{-4p\cos\theta_t \pm 4p\sqrt{\cos^2\theta_t + \sin^2\theta_t}}{-2(\cos^2\theta_t - 1)}$$

$$r_1 = \frac{2p(\cos\theta_t \pm \sqrt{1})}{\cos^2\theta_t - 1^2}$$

$$r_1 = \frac{l(\cos\theta_t \pm 1)}{(\cos\theta_t - 1)(\cos\theta_t + 1)}$$

$$r_1 = \frac{l(\cos\theta_t + 1)}{(\cos\theta_t - 1)(\cos\theta_t + 1)}$$

$$r_1 = \frac{l}{\cos\theta_t - 1}$$

$$p = \frac{2p}{\cos 0 - 1}$$

$$p = \frac{2p}{1 - 1}$$

$$p = \frac{2p}{0}$$

[REJECT]

$$r_1 = \frac{l(\cos\theta_t - 1)}{(\cos\theta_t - 1)(\cos\theta_t + 1)}$$

$$r_1 = \frac{l}{\cos\theta_t + 1}$$

$$p = \frac{2p}{\cos 0 + 1}$$

$$p = \frac{2p}{1 + 1}$$

$$p = \frac{2p}{2}$$

[ACCEPT]

$$r_1 = \frac{l}{\cos\theta_t + 1}$$

$$e = 1$$

Parabola Primary Radius Formula: $r_1 = \dfrac{l}{e\cos\theta_t + 1}$

2.5 Area

2.5.1 Derivation

$$r = \frac{l}{e\cos\theta + 1}$$

$$A = \frac{1}{2}\int_{\theta_1}^{\theta_2} r^2 d\theta$$

$$A = \frac{1}{2}\int_{\theta_1}^{\theta_2} \left(\frac{l}{e\cos\theta + 1}\right)^2 d\theta$$

$$A = \frac{l^2}{2}\int_{\theta_1}^{\theta_2} \frac{1}{(e\cos\theta + 1)^2} d\theta$$

$$A = \frac{l^2}{2}\int_{\theta_1}^{\theta_2} \frac{(1-e^2)}{(1-e^2)(e\cos\theta + 1)^2} d\theta$$

$$A = \frac{l^2}{2}\int_{\theta_1}^{\theta_2} \frac{e\cos\theta + 1 - e^2 - e\cos\theta}{(1-e^2)(e\cos\theta + 1)^2} d\theta$$

$$A = \frac{l^2}{2}\int_{\theta_1}^{\theta_2} \left(\frac{e\cos\theta + 1}{(1-e^2)(e\cos\theta + 1)^2} - \frac{e^2 + e\cos\theta}{(1-e^2)(e\cos\theta + 1)^2}\right) d\theta$$

$$A = \frac{l^2}{2(1-e^2)}\int_{\theta_1}^{\theta_2} \left(\frac{1}{e\cos\theta + 1} - \frac{e^2 + e\cos\theta}{(e\cos\theta + 1)^2}\right) d\theta$$

$$A = \frac{l^2}{2(1-e^2)}\int_{\theta_1}^{\theta_2} \left(2\cdot\frac{\frac{1}{2}}{e(\cos^2\left(\frac{\theta}{2}\right) - \sin^2\frac{\theta}{2}) + 1} - e\cdot\frac{e + \cos\theta}{(e\cos\theta + 1)^2}\right) d\theta$$

$$A = \frac{l^2}{2(1-e^2)}\int_{\theta_1}^{\theta_2} \left(\frac{2(1+e)^{\frac{-1}{2}}(1-e)^{\frac{1}{2}}}{(1+e)^{\frac{-1}{2}}(1-e)^{\frac{1}{2}}}\cdot\frac{\frac{1}{2}\sec^2\left(\frac{\theta}{2}\right)}{e(1-\tan^2\frac{\theta}{2}) + \sec^2\left(\frac{\theta}{2}\right)} - e\cdot\frac{e(\sin^2\theta + \cos^2\theta) + \cos\theta}{(e\cos\theta + 1)^2}\right) d\theta$$

$$A = \frac{l^2}{2(1-e^2)}\int_{\theta_1}^{\theta_2} \left(\frac{2\sqrt{\frac{1-e}{1+e}}}{(1+e)^{\frac{-2}{2}}(1+e)^{\frac{1}{2}}(1-e)^{\frac{1}{2}}}\cdot\frac{\frac{1}{2}\sec^2\left(\frac{\theta}{2}\right)}{e - e\tan^2\frac{\theta}{2} + 1 + \tan^2\frac{\theta}{2}} - e\cdot\frac{e\cos^2\theta + \cos\theta + e\sin^2\theta}{(e\cos\theta + 1)^2}\right) d\theta$$

$$A = \frac{l^2}{2(1-e^2)}\int_{\theta_1}^{\theta_2} \left(\frac{2}{(1+e)^{-1}(1-e^2)^{\frac{1}{2}}}\cdot\frac{\frac{1}{2}\sqrt{\frac{1-e}{1+e}}\sec^2\left(\frac{\theta}{2}\right)}{(1+e) + (1-e)\tan^2\frac{\theta}{2}} - e\cdot\frac{(e\cos\theta + 1)(\cos\theta) + (e\sin\theta)(\sin\theta)}{(e\cos\theta + 1)^2}\right) d\theta$$

$$A = \frac{l^2}{2(1-e^2)}\int_{\theta_1}^{\theta_2} \left(\frac{2}{\sqrt{1-e}}\cdot\frac{\frac{1}{2}\sqrt{\frac{1-e}{1+e}}\sec^2\left(\frac{\theta}{2}\right)}{1 + \frac{1-e}{1+e}\tan^2\frac{\theta}{2}} - e\cdot\frac{(e\cos\theta + 1)(\cos\theta) + (e\sin\theta)(\sin\theta)}{(e\cos\theta + 1)^2}\right) d\theta$$

$$A = \frac{l^2}{2(1-e^2)} \int_{\theta_1}^{\theta_2} \left(\frac{2}{\sqrt{1-e}} \cdot \frac{\frac{1}{2}\sqrt{\frac{1-e}{1+e}}\sec^2\left(\frac{\theta}{2}\right)}{1+(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right))^2} - e \cdot \frac{(e\cos\theta+1)(\cos\theta)+(e\sin\theta)(\sin\theta)}{(e\cos\theta+1)^2} \right) d\theta$$

$$u = \sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right) \qquad v = e\cos\theta + 1 \qquad w = \sin\theta$$

$$du = \frac{1}{2}\sqrt{\frac{1-e}{1+e}}\sec^2\left(\frac{\theta}{2}\right)d\theta \qquad dv = -e\sin\theta d\theta \qquad dw = \cos\theta d\theta$$

$$d\theta = 2\sqrt{\frac{1+e}{1-e}}\cos^2\left(\frac{\theta}{2}\right)du \qquad d\theta = \frac{-\csc\theta}{e}dv \qquad d\theta = \sec\theta dw$$

$$A = \frac{l^2}{2(1-e^2)} \cdot \frac{2}{\sqrt{1-e^2}} \int_{\theta_1}^{\theta_2} \frac{\frac{1}{2}\sqrt{\frac{1-e}{1+e}}\sec^2\left(\frac{\theta}{2}\right)}{1+(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right))^2} d\theta - \frac{l^2}{2(1-e^2)} \cdot e \int_{\theta_1}^{\theta_2} \frac{(e\cos\theta+1)(\cos\theta)+(e\sin\theta)(\sin\theta)}{(e\cos\theta+1)^2} d\theta$$

$$A = \frac{l^2}{2(1-e^2)} \cdot \frac{2}{\sqrt{1-e^2}} \int_{\theta_1}^{\theta_2} \frac{1}{1+u^2}du - \frac{l^2}{2(1-e^2)} \cdot e \int_{\theta_1}^{\theta_2} \frac{(v)(dw)-(dv)(w)}{(v)^2}d\theta$$

$$\frac{d}{dx}[\arctan(x)] = \frac{1}{1+x^2}dx \qquad \frac{d}{dx}[\frac{f}{g}] = \frac{(g)\frac{d}{dx}[f] - \frac{d}{dx}[g](f)}{(g)^2}dx$$

$$A = \frac{l^2}{2(1-e^2)} \cdot \frac{2}{\sqrt{1-e^2}}[\arctan(u)]_{\theta_1}^{\theta_2} - \frac{l^2}{2(1-e^2)} \cdot e[\frac{w}{v}]_{\theta_1}^{\theta_2}$$

$$A = \frac{l^2}{2(1-e^2)} \left(\frac{2\arctan(u)}{\sqrt{1-e^2}} - \frac{ew}{v} \right)_{\theta_1}^{\theta_2}$$

$$A = \frac{l^2}{2(1-e^2)} \left(\frac{2\arctan(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right))}{\sqrt{1-e^2}} - \frac{e\sin\theta}{e\cos\theta+1} \right)_{\theta_1}^{\theta_2}$$

2.5.2 Elliptical Trajectories

$$A = \frac{l^2}{2(1-e^2)} \left(\frac{2\arctan\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)\right)}{\sqrt{1-e^2}} - \frac{e\sin\theta}{e\cos\theta+1} \right)_{\theta_1}^{\theta_2}$$

$$1 - e^2 > 0$$
$$e^2 < 1$$
$$e < 1$$

2.5.3 Hyperbolic Trajectories

$$i = \sqrt{-1}$$
$$\operatorname{arctanh} x = \frac{1}{i}\arctan(ix)$$

$$A = \frac{l^2}{2(1-e^2)} \left(\frac{2\arctan\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)\right)}{\sqrt{1-e^2}} - \frac{e\sin\theta}{e\cos\theta+1} \right)_{\theta_1}^{\theta_2}$$

$$A = \frac{l^2}{2(1-e^2)} \left(\frac{2\arctan\left(\sqrt{-1(\frac{e-1}{e+1})}\tan\left(\frac{\theta}{2}\right)\right)}{\sqrt{-1(e^2-1)}} - \frac{e\sin\theta}{e\cos\theta+1} \right)_{\theta_1}^{\theta_2}$$

$$A = \frac{l^2}{2(1-e^2)} \left(\frac{2\arctan\left(i\sqrt{\frac{e-1}{e+1}}\tan\left(\frac{\theta}{2}\right)\right)}{i\sqrt{e^2-1}} - \frac{e\sin\theta}{e\cos\theta+1} \right)_{\theta_1}^{\theta_2}$$

$$A = \frac{l^2}{2(1-e^2)} \left(\frac{2\operatorname{arctanh}\left(\sqrt{\frac{e-1}{e+1}}\tan\left(\frac{\theta}{2}\right)\right)}{\sqrt{e^2-1}} - \frac{e\sin\theta}{e\cos\theta+1} \right)_{\theta_1}^{\theta_2}$$

$$e^2 - 1 > 0$$
$$e^2 > 1$$
$$e > 1$$

2.5.4 Parabolic Trajectories

$$\text{L'Hosptial's Rule: if } \lim_{x\to a}\frac{f(a)}{g(a)}=\frac{0}{0}$$
$$\text{or } \lim_{x\to a}\frac{f(a)}{g(a)}=\frac{\infty}{\infty}$$
$$\text{then: } \lim_{x\to a}\frac{f(a)}{g(a)}=\frac{f'(a)}{g'(a)}$$

$$A=\frac{l^2}{2(1-e^2)}\left(\frac{2\arctan\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)\right)}{\sqrt{1-e^2}}-\frac{e\sin\theta}{e\cos\theta+1}\right)_{\theta_1}^{\theta_2}$$
$$A=\frac{l^2}{2}\left(\frac{2\arctan\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)\right)}{(1-e^2)^{\frac{3}{2}}}-\frac{e\sin\theta}{(1-e^2)(e\cos\theta+1)}\right)_{\theta_1}^{\theta_2}$$
$$A=\frac{l^2}{2}\left(\frac{2\arctan\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)\right)(e\cos\theta+1)}{(1-e^2)^{\frac{3}{2}}(e\cos\theta+1)}-\frac{e\sin\theta\cdot\sqrt{1-e^2}}{(1-e^2)^{\frac{3}{2}}(e\cos\theta+1)}\right)_{\theta_1}^{\theta_2}$$
$$A=\frac{l^2}{2}\left(\frac{2\arctan\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)\right)(e\cos\theta+1)-e\sin\theta\cdot\sqrt{1-e^2}}{(1-e^2)^{\frac{3}{2}}(e\cos\theta+1)}\right)_{\theta_1}^{\theta_2}$$
$$A=\frac{l^2}{2}\lim_{e\to 1}\left(\frac{2\arctan\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)\right)(e\cos\theta+1)-e\sin\theta\cdot\sqrt{1-e^2}}{(1-e^2)^{\frac{3}{2}}(e\cos\theta+1)}\right)_{\theta_1}^{\theta_2}$$
$$A=\frac{l^2}{2}\lim_{e\to 1}\left(\frac{2\arctan(\sqrt{\frac{1-1}{1+1}}\tan\left(\frac{\theta}{2}\right))(\cos\theta+1)-e\sin\theta\cdot\sqrt{1-1}}{(1-1)^{\frac{3}{2}}(\cos\theta+1)}\right)_{\theta_1}^{\theta_2}$$
$$A=\frac{l^2}{2}\lim_{e\to 1}\left(\frac{2\arctan(0\cdot\tan\left(\frac{\theta}{2}\right))(\cos\theta+1)-e\sin\theta\cdot 0}{0^{\frac{3}{2}}(\cos\theta+1)}\right)_{\theta_1}^{\theta_2}$$
$$A=\frac{l^2}{2}\lim_{e\to 1}\left(\frac{0-0}{0}\right)_{\theta_1}^{\theta_2}$$

L'Hospital 1:

$$A=\frac{l^2}{2}\lim_{e\to 1}\left(\frac{\frac{d}{de}\left[2\cdot\arctan\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)\right)(e\cos\theta+1)-\sin\theta\cdot\left(e\sqrt{1-e^2}\right)\right]}{\frac{d}{de}[(1-e^2)^{\frac{3}{2}}(e\cos\theta+1)]}\right)_{\theta_1}^{\theta_2}$$

$$A = \frac{l^2}{2}\lim_{e\to 1}\left(\frac{2\cdot\left[\frac{\frac{\frac{(-1)(1+e)-(1-e)(1)}{(1+e)^2}}{2\sqrt{\frac{1-e}{1+e}}}\tan\left(\frac{\theta}{2}\right)(e\cos\theta+1)}{1+\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)\right)^2}+\arctan\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)\right)(\cos\theta)\right]-\sin\theta\cdot\left[\sqrt{1-e^2}+\frac{-2e^2}{2\sqrt{1-e^2}}\right]}{\left[\frac{3}{2}(1-e^2)^{\frac{1}{2}}(-2e)(e\cos\theta+1)+(1-e^2)^{\frac{3}{2}}(\cos\theta)\right]}\right)_{\theta_1}^{\theta_2}$$

$$A = \frac{l^2}{2}\lim_{e\to 1}\left(\frac{2\cdot\left[\frac{(-(1+e)-(1-e))\tan\left(\frac{\theta}{2}\right)(e\cos\theta+1)}{2(1+e)^2\sqrt{\frac{1-e}{1+e}}\cdot\left(1+\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)\right)^2\right)}+\arctan\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)\right)(\cos\theta)\right]-\sin\theta\cdot\left[\frac{1-e^2}{\sqrt{1-e^2}}+\frac{-e^2}{\sqrt{1-e^2}}\right]}{\sqrt{1-e^2}\left[-3e(e\cos\theta+1)+(1-e^2)(\cos\theta)\right]}\right)_{\theta_1}^{\theta_2}$$

$$A = \frac{l^2}{2}\lim_{e\to 1}\left(\frac{\frac{(-1-e-1+e)\tan\left(\frac{\theta}{2}\right)(e\cos\theta+1)}{(1+e)^2\sqrt{\frac{1-e}{1+e}}\cdot\left(1+(\frac{1-e}{1+e})\tan^2\left(\frac{\theta}{2}\right)\right)}+2\cos\theta\cdot\arctan\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)\right)-\sin\theta\cdot\left(\frac{1-2e^2}{\sqrt{1-e^2}}\right)}{\sqrt{1-e^2}(-3e(e\cos\theta+1)+(1-e^2)(\cos\theta))}\right)_{\theta_1}^{\theta_2}$$

$$A = \frac{l^2}{2}\lim_{e\to 1}\left(\frac{\frac{-2\tan\left(\frac{\theta}{2}\right)(e\cos\theta+1)}{\sqrt{1-e^2}\cdot\left((1+e)+(1-e)\tan^2\left(\frac{\theta}{2}\right)\right)}+\frac{2\cos\theta\cdot\sqrt{1-e^2}\cdot\arctan\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)\right)}{\sqrt{1-e^2}}-\sin\theta\cdot\left(\frac{1-2e^2}{\sqrt{1-e^2}}\right)}{\sqrt{1-e^2}(-3e^2\cos\theta-3e+(1-e^2)\cos\theta)}\right)_{\theta_1}^{\theta_2}$$

$$A = \frac{l^2}{2}\lim_{e\to 1}\left(\frac{-2\tan\left(\frac{\theta}{2}\right)\cdot\frac{e\cos\theta+1}{(1+e)+(1-e)\tan^2\left(\frac{\theta}{2}\right)}+2\cos\theta\sqrt{1-e^2}\cdot\arctan\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)\right)-\sin\theta\cdot(1-2e^2)}{(1-e^2)(-3e^2\cos\theta-3e)+(1-e^2)^2\cos\theta}\right)_{\theta_1}^{\theta_2}$$

L'Hospital 2:

$$A = \frac{l^2}{2}\lim_{e\to 1}\left(\frac{\frac{d}{de}\left[-2\tan\left(\frac{\theta}{2}\right)\frac{e\cos\theta+1}{(1+e)+(1-e)\tan^2\left(\frac{\theta}{2}\right)}+2\cos\theta\cdot\sqrt{1-e^2}\arctan\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)\right)-\sin\theta\cdot(1-2e^2)\right]}{\frac{d}{de}[(1-e^2)(-3e^2\cos\theta-3e)+(1-e^2)^2\cos\theta]}\right)_{\theta_1}^{\theta_2}$$

$$A = \frac{l^2}{2}\left(\frac{\left[-2\tan\left(\frac{\theta}{2}\right)\frac{\left((1+e)+(1-e)\tan^2\left(\frac{\theta}{2}\right)\right)(\cos\theta)-\left(1-\tan^2\left(\frac{\theta}{2}\right)\right)(e\cos\theta+1)}{\left((1+e)+(1-e)\tan^2\left(\frac{\theta}{2}\right)\right)^2}\right.}{(-2e)(-3e^2\cos\theta-3e)+(1-e^2)(-6e\cos\theta-3)+2(1-e^2)(-2e)\cos\theta} + \frac{\left. 2\cos\theta\cdot\left(\sqrt{1-e^2}\,\frac{\frac{\frac{(-1)(1+e)-(1-e)(1)}{(1+e)^2}}{2\sqrt{\frac{1-e}{1+e}}}\tan\left(\frac{\theta}{2}\right)}{1+\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)\right)^2}+\frac{-2e}{2\sqrt{1-e^2}}\arctan\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)\right)\right)-\sin\theta(-4e^2)\right]}{(-2e)(-3e^2\cos\theta-3e)+(1-e^2)(-6e\cos\theta-3)+2(1-e^2)(-2e)\cos\theta}\right)_{\theta_1}^{\theta_2}$$

$$A = \frac{l^2}{2}\left(\frac{-2\tan\left(\frac{\theta}{2}\right)\frac{\left((1+e)+(1-e)\tan^2\left(\frac{\theta}{2}\right)\right)(\cos\theta)-\left(1-\tan^2\left(\frac{\theta}{2}\right)\right)(e\cos\theta+1)}{\left((1+e)+(1-e)\tan^2\left(\frac{\theta}{2}\right)\right)^2}}{6e^3\cos\theta+6e^2-6e\cos\theta-3+6e^3\cos\theta+3e^2-4e\cos\theta+4e^3\cos\theta} + \frac{2\cos\theta\cdot\left(\sqrt{1-e}\cdot\sqrt{1+e}\,\frac{(-1)(1+e)-(1-e)(1)\tan\left(\frac{\theta}{2}\right)}{2(1+e)^2\sqrt{\frac{1-e}{1+e}}\cdot\left(1+\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)\right)^2\right)}+\frac{-2e}{2\sqrt{1-e^2}}\arctan\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)\right)\right)+4e^2\sin\theta}{6e^3\cos\theta+6e^2-6e\cos\theta-3+6e^3\cos\theta+3e^2-4e\cos\theta+4e^3\cos\theta}\right)_{\theta_1}^{\theta_2}$$

$$A = \frac{l^2}{2}\left(\frac{-2\tan\left(\frac{\theta}{2}\right)\frac{\left(\tan^2\left(\frac{\theta}{2}\right)-1\right)(e\cos\theta+1)}{\left((1+e)+(1-e)\tan^2\left(\frac{\theta}{2}\right)\right)^2}-2\tan\left(\frac{\theta}{2}\right)\frac{((1+e)+(1-e)\tan^2\left(\frac{\theta}{2}\right))(\cos\theta)}{\left((1+e)+(1-e)\tan^2\left(\frac{\theta}{2}\right)\right)^2}}{16e^3\cos\theta+9e^2-10e\cos\theta-3} + \frac{2\cos\theta\cdot\left(\frac{-2\tan\left(\frac{\theta}{2}\right)}{2(1+e)\left(1+\left(\frac{1-e}{1+e}\right)\tan^2\left(\frac{\theta}{2}\right)\right)}+\frac{-2e}{2\sqrt{1-e^2}}\arctan\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)\right)\right)+4e^2\sin\theta}{16e^3\cos\theta+9e^2-10e\cos\theta-3}\right)_{\theta_1}^{\theta_2}$$

$$A = \frac{l^2}{2}\left(\frac{-2\tan\left(\frac{\theta}{2}\right)\frac{\left(\tan^2\left(\frac{\theta}{2}\right)-1\right)(e\cos\theta+1)}{\left((1+e)+(1-e)\tan^2\left(\frac{\theta}{2}\right)\right)^2}-\frac{2\cos\theta\cdot\tan\left(\frac{\theta}{2}\right)}{(1+e)+(1-e)\tan^2\left(\frac{\theta}{2}\right)}-\frac{2\cos\theta\cdot\tan\left(\frac{\theta}{2}\right)}{(1+e)+(1-e)\tan^2\left(\frac{\theta}{2}\right)}-\frac{2e(\cos\theta)}{\sqrt{1-e^2}}\arctan\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)\right)+4e^2\sin\theta}{16e^3\cos\theta+9e^2-10e\cos\theta-3}\right)_{\theta_1}^{\theta_2}$$

$$A = \frac{l^2}{2}\left(\frac{-2\tan\left(\frac{\theta}{2}\right)\frac{\left(\tan^2\left(\frac{\theta}{2}\right)-1\right)(e\cos\theta+1)}{\left((1+e)+(1-e)\tan^2\left(\frac{\theta}{2}\right)\right)^2}-\frac{4\cos\theta\cdot\tan\left(\frac{\theta}{2}\right)}{(1+e)+(1-e)\tan^2\left(\frac{\theta}{2}\right)}-\frac{2e\cdot\cos\theta}{\sqrt{1-e^2}}\arctan\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)\right)+4e^2\sin\theta}{16e^3\cos\theta+9e^2-10e\cos\theta-3}\right)_{\theta_1}^{\theta_2}$$

Substitution 1:

$$A=\frac{l^2}{2}\left(\frac{-2\tan\left(\frac{\theta}{2}\right)\frac{\left(\tan^2\left(\frac{\theta}{2}\right)-1\right)(\cos\theta+1)}{((1+1)+(1-1)\tan^2\left(\frac{\theta}{2}\right))^2}-\frac{4\cos\theta\cdot\tan\left(\frac{\theta}{2}\right)}{(1+1)+(1-1)\tan^2\left(\frac{\theta}{2}\right)}-\frac{2\cos\theta}{\sqrt{1-1}}\arctan\left(\sqrt{\frac{1-1}{1+1}}\tan\left(\frac{\theta}{2}\right)\right)+4\sin\theta}{16\cos\theta+9-10\cos\theta-3}\right)_{\theta_1}^{\theta_2}$$

$$A=\frac{l^2}{2}\left(\frac{-2\tan\left(\frac{\theta}{2}\right)\frac{\left(\tan^2\left(\frac{\theta}{2}\right)-1\right)(\cos\theta+1)}{((2)+(0)\tan^2\left(\frac{\theta}{2}\right))^2}-\frac{4\cos\theta\cdot\tan\left(\frac{\theta}{2}\right)}{(2)+(0)\tan^2\left(\frac{\theta}{2}\right)}-\frac{2\cos\theta}{\sqrt{0}}\arctan\left(\sqrt{\frac{0}{2}}\tan\left(\frac{\theta}{2}\right)\right)+4\sin\theta}{6\cos\theta+6}\right)_{\theta_1}^{\theta_2}$$

$$A=\frac{l^2}{2}\left(\frac{2\tan\left(\frac{\theta}{2}\right)\frac{\left(\tan^2\left(\frac{\theta}{2}\right)-1\right)(\cos\theta+1)}{2^2}+\frac{4\cos\theta\cdot\tan\left(\frac{\theta}{2}\right)}{2}+\frac{2\cos\theta\cdot\arctan(0)}{0}-4\sin\theta}{-6(\cos\theta+1)}\right)_{\theta_1}^{\theta_2}$$

$$A=\frac{l^2}{2}\left(\frac{\tan\left(\frac{\theta}{2}\right)\left(\tan^2\left(\frac{\theta}{2}\right)-1\right)(\cos\theta+1)+4\cos\theta\cdot\tan\left(\frac{\theta}{2}\right)+\frac{0}{0}-8\sin\theta}{-12(\cos\theta+1)}\right)_{\theta_1}^{\theta_2}$$

only the arctan term doesn't exist at $e=1$. L'Hospital 3:

$$A=\frac{l^2}{2}\left(\frac{-2\tan\left(\frac{\theta}{2}\right)\frac{\left(\tan^2\left(\frac{\theta}{2}\right)-1\right)(e\cos\theta+1)}{\left((1+e)+(1-e)\tan^2\left(\frac{\theta}{2}\right)\right)^2}-\frac{4\cos\theta\cdot\tan\left(\frac{\theta}{2}\right)}{(1+e)+(1-e)\tan^2\left(\frac{\theta}{2}\right)}+4\sin\theta e^2-\frac{2e(\cos\theta)}{\sqrt{1-e^2}}\arctan\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)\right)}{16e^3\cos\theta+9e^2-10e\cos\theta-3}\right)_{\theta_1}^{\theta_2}$$

$$A=\frac{l^2}{2}\left(\frac{\tan\left(\frac{\theta}{2}\right)\left(\tan^2\left(\frac{\theta}{2}\right)-1\right)(\cos\theta+1)+4\cos\theta\cdot\tan\left(\frac{\theta}{2}\right)-8\sin\theta}{-12(\cos\theta+1)}+\frac{4\cos\theta\cdot\frac{d}{de}\left[e\arctan\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)\right)\right]}{\frac{d}{de}[\sqrt{1-e^2}(-32e^3\cos\theta-18e^2+20e\cos\theta+6)]}\right)_{\theta_1}^{\theta_2}$$

$$A=\frac{l^2}{2}\left(\frac{\tan\left(\frac{\theta}{2}\right)\left(\tan^2\left(\frac{\theta}{2}\right)-1\right)(\cos\theta+1)+4\cos\theta\cdot\tan\left(\frac{\theta}{2}\right)-8\sin\theta}{-12(\cos\theta+1)}\right.$$
$$\left.+\frac{4\cos\theta\cdot\left[\frac{\frac{\frac{(-1)(1+e)-(1-e)(1)}{(1+e)^2}}{2\sqrt{\frac{1-e}{1+e}}}\tan\left(\frac{\theta}{2}\right)}{1+\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)\right)^2}+\arctan\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)\right)\right]}{\left[\frac{-e}{\sqrt{1-e^2}}(-32e^3\cos\theta-18e^2+20e\cos\theta+6)+\sqrt{1-e^2}(-96e^2\cos\theta-36e+20\cos\theta)\right]}\right)_{\theta_1}^{\theta_2}$$

$$A=\frac{l^2}{2}\left(\frac{\tan\left(\frac{\theta}{2}\right)\left(\tan^2\left(\frac{\theta}{2}\right)-1\right)(\cos\theta+1)+4\cos\theta\cdot\tan\left(\frac{\theta}{2}\right)-8\sin\theta}{-12(\cos\theta+1)}\right.$$
$$\left.+\frac{4\cos\theta\cdot\left[\frac{-2\tan\left(\frac{\theta}{2}\right)}{2(1+e)^2\sqrt{\frac{1-e}{1+e}}\cdot(1+(\frac{1-e}{1+e})\tan^2\left(\frac{\theta}{2}\right)}+\arctan\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)\right)\right]}{\frac{-e}{\sqrt{1-e^2}}(-32e^3\cos\theta-18e^2+20e\cos\theta+6)+\sqrt{1-e^2}(-96e^2\cos\theta-36e+20\cos\theta)}\right)_{\theta_1}^{\theta_2}$$

$$A = \frac{l^2}{2}\left(\frac{\tan\left(\frac{\theta}{2}\right)\left(\tan^2\left(\frac{\theta}{2}\right)-1\right)(\cos\theta+1)+4\cos\theta\cdot\tan\left(\frac{\theta}{2}\right)-8\sin\theta}{-12(\cos\theta+1)}\right.$$
$$\left.+\frac{4\cos\theta\cdot\sqrt{1-e^2}\left[\frac{-\tan\left(\frac{\theta}{2}\right)}{\sqrt{1+e}\sqrt{1-e}((1+e)+(1-e)\tan^2\left(\frac{\theta}{2}\right))}+\arctan\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)\right)\right]}{\sqrt{1-e^2}(\frac{1}{\sqrt{1-e^2}}(32e^4\cos\theta+18e^3-20e^2\cos\theta-6e)+\sqrt{1-e^2}(-96e^2\cos\theta-36e+20\cos\theta))}\right)_{\theta_1}^{\theta_2}$$

$$A = \frac{l^2}{2}\left(\frac{\tan\left(\frac{\theta}{2}\right)\left(\tan^2\left(\frac{\theta}{2}\right)-1\right)(\cos\theta+1)+4\cos\theta\cdot\tan\left(\frac{\theta}{2}\right)-8\sin\theta}{-12(\cos\theta+1)}\right.$$
$$\left.+\frac{4\cos\theta\cdot\left[\frac{-\tan\left(\frac{\theta}{2}\right)}{(1+e)+(1-e)\tan^2\left(\frac{\theta}{2}\right)}+\sqrt{1-e^2}\cdot\arctan\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)\right)\right]}{(32e^4\cos\theta+18e^3-20e^2\cos\theta-6e)+(1-e^2)(-96e^2\cos\theta-36e+20\cos\theta)}\right)_{\theta_1}^{\theta_2}$$

Substitution 2:

$$A = \frac{l^2}{2}\left(\frac{\tan\left(\frac{\theta}{2}\right)\left(\tan^2\left(\frac{\theta}{2}\right)-1\right)(\cos\theta+1)+4\cos\theta\cdot\tan\left(\frac{\theta}{2}\right)-8\sin\theta}{-12(\cos\theta+1)}+\frac{4\cos\theta\cdot\left[\frac{-\tan\left(\frac{\theta}{2}\right)}{(1+1)+(1-1)\tan^2\left(\frac{\theta}{2}\right)}+\sqrt{1-1}\cdot\arctan\left(\sqrt{\frac{1-1}{1+1}}\tan\left(\frac{\theta}{2}\right)\right)\right]}{(32\cos\theta+18-20\cos\theta-6)+(1-1)(-96\cos\theta-36+20\cos\theta)}\right)_{\theta_1}^{\theta_2}$$

$$A = \frac{l^2}{2}\left(\frac{\tan\left(\frac{\theta}{2}\right)\left(\tan^2\left(\frac{\theta}{2}\right)-1\right)(\cos\theta+1)+4\cos\theta\cdot\tan\left(\frac{\theta}{2}\right)-8\sin\theta}{-12(\cos\theta+1)}+\frac{4\cos\theta\left[\frac{-\tan\left(\frac{\theta}{2}\right)}{2+0\tan^2\left(\frac{\theta}{2}\right)}+\sqrt{0}\arctan\left(\sqrt{\frac{0}{2}}\tan\left(\frac{\theta}{2}\right)\right)\right]}{(12\cos\theta+12)+(0)(-96\cos\theta-36+20\cos\theta)}\right)_{\theta_1}^{\theta_2}$$

$$A = \frac{l^2}{2}\left(\frac{\tan\left(\frac{\theta}{2}\right)\left(\tan^2\left(\frac{\theta}{2}\right)-1\right)(\cos\theta+1)+4\cos\theta\cdot\tan\left(\frac{\theta}{2}\right)-8\sin\theta}{-12(\cos\theta+1)}+\frac{4\cos\theta\cdot\frac{-\tan\left(\frac{\theta}{2}\right)}{2}}{12\cos\theta+12}\right)_{\theta_1}^{\theta_2}$$

$$A = \frac{l^2}{2}\left(\frac{\tan\left(\frac{\theta}{2}\right)\left(\tan^2\left(\frac{\theta}{2}\right)-1\right)(\cos\theta+1)+4\cos\theta\cdot\tan\left(\frac{\theta}{2}\right)-8\sin\theta}{-12(\cos\theta+1)}+\frac{2\cos\theta\cdot\tan\left(\frac{\theta}{2}\right)}{-12(\cos\theta+1)}\right)_{\theta_1}^{\theta_2}$$

$$A = \frac{l^2}{2}\left(\frac{\tan\left(\frac{\theta}{2}\right)\left(\tan^2\left(\frac{\theta}{2}\right)-1\right)(\cos\theta+1)+6(\cos\theta)\tan\left(\frac{\theta}{2}\right)-8\sin\theta}{-12(\cos\theta+1)}\right)_{\theta_1}^{\theta_2}$$

$e = 1$

2.6 Conclusion

When translating from Cartesian to Polar, all conic sections have extremely similar properties, with the main exception of Semi Latus scale, and the properties defined by Semi Latus Scale (Marked by a '*' in section 2.1).

Common properties:

- l : Identity
- e : Identity
- $p = \frac{l}{1+e}$
- $r_1 = \frac{l}{e\cos\theta_t + 1}$

Area Formulae:

- Elliptical Area: $A = \frac{l^2}{2(1-e^2)} \left(\frac{2\arctan\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)\right)}{\sqrt{1-e^2}} - \frac{e\sin\theta}{e\cos\theta+1} \right)_{\theta_1}^{\theta_2}$
- Hyperbolic Area: $A = \frac{l^2}{2(1-e^2)} \left(\frac{2\,\mathrm{arctanh}\left(\sqrt{\frac{e-1}{e+1}}\tan\left(\frac{\theta}{2}\right)\right)}{\sqrt{e^2-1}} - \frac{e\sin\theta}{e\cos\theta+1} \right)_{\theta_1}^{\theta_2}$
- Parabolic Area: $A = \frac{l^2}{2} \left(\frac{\tan\left(\frac{\theta}{2}\right)\left(\tan^2\left(\frac{\theta}{2}\right)-1\right)(\cos\theta+1)+6\cos\theta\cdot\tan\left(\frac{\theta}{2}\right)-8\sin\theta}{-12(\cos\theta+1)} \right)_{\theta_1}^{\theta_2}$

Chapter 3

Physics

3.1 Direction from Change in Position

At any given point, for a non-zero velocity, the Change in Position has only one direction. Therefore, the Change in Position with respect to time (aka, Velocity) and the Change in Position with respect to θ_t must share a common direction if they're at the same point.

$$r = \frac{l}{e\cos\theta_t + 1}$$

$$\vec{r} = (\vec{r}_x, \vec{r}_y)$$

$$\vec{v} = \frac{dr}{dt}$$

$$\vec{v} = v \cdot \hat{v}$$

$$\hat{v} = \frac{(d\vec{r}_x, d\vec{r}_y)}{\sqrt{d\vec{r}_x^2 + d\vec{r}_y^2}}$$

$$\vec{r}_x = r \cdot \cos\theta_t$$

$$d\vec{r}_x = \frac{d}{d\theta_t}\left[\frac{l\cos\theta_t}{e\cos\theta_t + 1}\right]$$

$$d\vec{r}_x = \frac{(-l\sin\theta_t)(e\cos\theta_t + 1) - (l\cos\theta_t)(-e\sin\theta_t)}{(e\cos\theta_t + 1)^2}$$

$$d\vec{r}_x = \frac{-l\sin\theta_t}{(e\cos\theta_t + 1)^2}$$

$$d\vec{r}_x = \frac{l}{e\cos\theta_t + 1} \cdot \frac{-\sin\theta_t}{e\cos\theta_t + 1}$$

$$d\vec{r}_x = r \cdot \frac{-\sin\theta_t}{e\cos\theta_t + 1}$$

$$\vec{r}_y = r \cdot \sin\theta_t$$

$$d\vec{r}_y = \frac{d}{d\theta_t}\left[\frac{l\sin\theta_t}{e\cos\theta_t + 1}\right]$$

$$d\vec{r}_y = \frac{(l\cos\theta_t)(e\cos\theta_t + 1) - (l\sin\theta_t)(-e\sin\theta_t)}{(e\cos\theta_t + 1)^2}$$

$$d\vec{r}_y = \frac{le(\cos^2\theta_t + \sin^2\theta_t) + l\cos\theta_t}{(e\cos\theta_t + 1)^2}$$

$$d\vec{r}_y = \frac{l}{e\cos\theta_t + 1} \cdot \frac{(\cos\theta_t + e)}{e\cos\theta_t + 1}$$

$$d\vec{r}_y = r \cdot \frac{\cos\theta_t + e}{e\cos\theta_t + 1}$$

$$\hat{v} = \frac{\left(r \cdot \frac{-\sin\theta_t}{e\cos\theta_t+1}, r \cdot \frac{\cos\theta_t+e}{e\cos\theta_t+1}\right)}{\sqrt{\left(r \cdot \frac{-\sin\theta_t}{e\cos\theta_t+1}\right)^2 + \left(r \cdot \frac{\cos\theta_t+e}{e\cos\theta_t+1}\right)^2}}$$

$$\hat{v} = \frac{r\left(\frac{-\sin\theta_t}{e\cos\theta_t+1}, \frac{\cos\theta_t+e}{e\cos\theta_t+1}\right)}{r\sqrt{\left(\frac{-\sin\theta_t}{e\cos\theta_t+1}\right)^2 + \left(\frac{\cos\theta_t+e}{e\cos\theta_t+1}\right)^2}}$$

$$\hat{v} = \frac{(e\cos\theta_t+1)^{-1}(-\sin\theta_t, \cos\theta_t+e)}{(e\cos\theta_t+1)^{-1}\sqrt{(-\sin\theta_t)^2+(\cos\theta_t+e)^2}}$$

$$\hat{v} = \frac{(-\sin\theta_t, \cos\theta_t+e)}{\sqrt{\sin^2\theta_t+\cos^2\theta_t+2e\cos\theta_t+e^2}}$$

$$\hat{v} = \frac{(-\sin\theta_t, \cos\theta_t+e)}{\sqrt{e^2+2e\cos\theta_t+1}}$$

3.2 Angular Momentum

Orbiting bodies in this physical model (without perturbations) experience no torque, as the only force they experience is gravity, which is parallel with their radius vector. Since it does not experience a torque, Angular Momentum must be conserved.

Since Angular Momentum is conserved, that means it must remain constant for the entirety of the orbit. It must also mean **Specific Angular Momentum** (h) is conserved, or Angular Momentum per Unit Mass. This allows for ignoring mass in calculations.

Specific Angular Momentum is the cross product of radius and velocity. Because radius and velocity vectors are both on the x-y/θ-r plane, we can simplify cross product to the determinant.

$$h_p = \vec{r} \times \vec{v} = \vec{r}_x \cdot \vec{v}_y - \vec{r}_y \cdot \vec{v}_x$$

$$h_p = (p\cos\theta_t)\left(v_p \cdot \frac{\cos\theta_t+e}{\sqrt{e^2+2e\cos\theta_t+1}}\right) - (p\sin\theta_t)(v_p \cdot \hat{v}_y)$$

$$h_p = (p\cos 0)\left(v_p \cdot \frac{\cos 0+e}{\sqrt{e^2+2e\cos 0+1}}\right) - (p\sin 0)(v_p \cdot \hat{v}_y)$$

$$h_p = \left(pv_p \cdot \frac{1+e}{\sqrt{e^2+2e+1}}\right) - (0)(v_p \cdot \hat{v}_y)$$

$$h_p = \left(pv_p \cdot \frac{1+e}{\sqrt{(e+1)^2}}\right)$$

$$h_p = pv_p$$

$$h_l = (l\cos\theta_t)(v_l \cdot \hat{v}_x) - (l\sin\theta_t)\left(v_l \cdot \frac{-\sin\theta_t}{\sqrt{e^2+2e\cos\theta_t+1}}\right)$$

$$h_l = \left(l\cos\left(\frac{\pi}{2}\right)\right)(v_l \cdot \hat{v}_x) - \left(l\sin\left(\frac{\pi}{2}\right)\right)\left(v_l \cdot \frac{-\sin\left(\frac{\pi}{2}\right)}{\sqrt{e^2+2e\cos\left(\frac{\pi}{2}\right)+1}}\right)$$

$$h_l = (0)(v_l \cdot \hat{v}_x) - (l)\left(v_l \cdot \frac{-1}{\sqrt{e^2+1}}\right)$$

$$h_l = \frac{lv_l}{\sqrt{1+e^2}}$$

$$h = pv_p = \frac{lv_l}{\sqrt{1+e^2}}$$

$$l = p(1+e)$$

$$pv_p = \frac{p(1+e)v_l}{\sqrt{(1+e)(1-e)}}$$

$$v_p = \frac{(1+e)v_l}{\sqrt{1+e^2}}$$

3.3 Total Energy and Velocity

Similarly to Conservation of Special Angular Momentum, **Conservation of Specific Energy** can be used to calculate the speed of the object orbiting. Together with direction, a velocity vector can be constructed. Like Specific Angular Momentum, Specific Energy (ϵ_{t}) is constant throughout the orbit, even though its components change. It's equal to the sum of Specific Kinetic Energy (ϵ_{k}) and Specific Gravitational Potential Energy (ϵ_{gp}).

$$\epsilon_{\mathrm{t}} = \epsilon_k + \epsilon_{gp}$$

$$\epsilon_k = \frac{1}{2}v^2$$

$$\epsilon_{gp} = \int_0^r \left(\frac{\mu}{r^2}\right) dr$$

$$\epsilon_{gp} = \frac{-\mu}{r}$$

$$\epsilon_{\mathrm{t}} = \frac{v^2}{2} - \frac{\mu}{r}$$

$$\epsilon_{\mathrm{t}} = \frac{v_l^2}{2} - \frac{\mu}{l} = \frac{v_p^2}{2} - \frac{\mu}{p}$$

$$\frac{v_l^2}{2} - \frac{v_p^2}{2} = \frac{\mu}{l} - \frac{\mu}{p}$$

$$v_l^2 - \left(\frac{v_l(1+e)}{\sqrt{1+e^2}}\right)^2 = 2\mu\left(\frac{1}{l} - \frac{1+e}{l}\right)$$

$$v_l^2\left(1 - \frac{(1+e)^2}{1+e^2}\right) = 2\mu\left(\frac{-e}{l}\right)$$

$$v_l^2 \cdot \frac{(1+e^2)-(1+e)^2}{1+e^2} = 2\mu(\frac{-e}{l})$$

$$v_l^2 \cdot \frac{1+e^2-1-2e-e^2}{1+e^2} = 2\mu\left(\frac{-e}{l}\right)$$

$$v_l^2 \cdot \frac{-2e}{1+e^2} = 2\mu\left(\frac{-e}{l}\right)$$

$$v_l^2 = 2\mu\left(\frac{-e}{l}\right)\left(\frac{1+e^2}{-2e}\right)$$

$$v_l = \sqrt{\mu\left(\frac{1+e^2}{l}\right)}$$

$$\epsilon_{\mathrm{t}} = \frac{v_l^2}{2} - \frac{\mu}{l}$$

$$\epsilon_{\mathrm{t}} = \frac{\left(\sqrt{\mu\left(\frac{1+e^2}{l}\right)}\right)^2}{2} - \frac{\mu}{l}$$

$$\epsilon_{\mathrm{t}} = \frac{\mu\left(\frac{1+e^2}{l}\right)}{2} - \frac{\mu}{l}$$

$$\epsilon_{\mathrm{t}} = \frac{\mu(1+e^2)}{2l} - \frac{2\mu}{2l}$$

$$\epsilon_{\mathrm{t}} = \frac{\mu(e^2-1)}{2l}$$

$$\epsilon_{\mathrm{t}} = \epsilon_k + \epsilon_{gp}$$

$$\frac{\mu(e^2-1)}{2l} = \frac{v^2}{2} + \frac{-\mu}{r}$$

$$\frac{\mu(e^2-1)}{l} = v^2 + \frac{-2\mu}{r}$$

$$v^2 = \frac{\mu(e^2-1)}{l} + \frac{2\mu}{r}$$

$$v = \sqrt{\mu\left(\frac{2}{r} + \frac{e^2-1}{l}\right)}$$

$$h = pv_p = p\sqrt{\mu\left(\frac{2}{p} + \frac{e^2-1}{l}\right)}$$

$$h = pv_p = p\sqrt{\mu\left(\frac{2(1+e)}{l} + \frac{e^2-1}{l}\right)}$$

$$h = \frac{l}{1+e}\sqrt{\mu\left(\frac{1+2e+e^2}{l}\right)}$$

$$h = \sqrt{l\mu}$$

3.4 Areal Velocity

The final physics piece in the puzzle is Areal Velocity. Once this is established, it becomes possible to determine angular displacement based on a given passage of time. The area covered by an orbiting body in an infinitesimal amount of time can be closely modeled by a triangle. A convenient property of the cross product of two vectors is that the resulting magnitude equals the area made by a parallelogram, with side lengths equaling to the two vectors. The area of a triangle with the same side lengths is simply half that of the parallelogram.

If the first vector is the initial position r, and the second vector is the displacement vector d, then their area is

$$\Delta A = \frac{r \times d}{2}$$

Further, the displacement can be rewritten as the product of a velocity and a time over which the displacement occurred.

$$\Delta A = \frac{r \times (v \cdot \Delta t)}{2}$$

And Finally, the Areal Velocity can be determined by dividing by the same time step.

$$\Delta A = \frac{r \times (v \cdot \Delta t)}{2\Delta t}$$

$$\Delta A = \frac{r \times v}{2} \cdot \left(\frac{\Delta t}{\Delta t}\right)$$

$$\Delta A = \frac{r \times v}{2} = \frac{h}{2}$$

$$\Delta A = \frac{\sqrt{l\mu}}{2}$$

As we can see, Areal velocity is simply just half of the Specific Angular Momentum of an orbiting body.

Unfortunately, there is no way to analytically solve for angular displacement, as the Area formulae derived in the Area Section of the Geometry chapter (Sections 2.5 and 2.6) are not analytically solvable for θ_t. It can be done numerically, however.

3.5 Change in Position Over Time

While a convenient, analytic solution does not exist for determining position as a function of time, it is possible to calculate it numerically, using an iterative approach. The Newton-Raphson Method is ideal for this scenario, as the derivative of area (r^2) never equals zero and should no overshoot as both the Area formula and the derivative are continuous. The following iterative formula should work for the Newton-Raphson Method.
Let:

- $A(\theta_t)$: The area function for this orbit
- X : Terminating condition
- T : Target area
- θ_0 : The true anomaly of the initial position.
- Δt : The elapsed time passed since the orbiting body was at the initial position, in seconds

Firstly, set a target area to iterate towards. This is simply set by summing the area corresponding to the initial position with the expected change in area. Ensure that T is within the domain of the orbit.

$$T = A(\theta_t) + \Delta A \cdot \Delta t$$

Next, determine the terminating condition. The method used here will have it set so that the position is accurate down to the nearest millisecond, however this can be variable..

$$X = \Delta A \cdot 10^{-3} s$$

Whenever the absolute value of the difference between target area and the calculated area is less than T, we can terminate the iterative process. Lastly in the setup, all that's left is an initial guess. The initial position is a very good place to start.

$$\textbf{While } X < |A(\theta_n) - T|$$

$$\theta_{n+1} = \theta_n - \frac{A(\theta_n) - T}{r(\theta_n)^2}$$

For those more familiar with pseudo-code:

$\theta_t = \theta_0$

While X is less than the absolute difference between $A(\theta_t)$ and T

Calculate the next iteration of θ_t:
$\theta_t = \theta_t - \frac{A(\theta_t) - T}{r(\theta_t)^2}$

Chapter 4

Orientation

With everything outlined in the first 3 chapters, it's possible to create a complete model of a 2 body orbital system***. However there is a glaring limitation to the model as it is now: It can only describe orbits in a single plane. Without additional work, this model can't describe the difference between a polar orbit and an equatorial orbit if they have the same shape, size, and orbit the same body. The solution to this is encoding into this model a way to describe any configuration of rotation. Note: Everything in this chapter will be describing right-handed orientations. Section x.x will cover left-handed conversions.

4.1 Quaternions

Quaternions are by far the simplest way to encode rotation in a 3D space without fiddling with Matrices (which we will have to eventually) or Euler Angles (can cause gimbal lock).

4.1.1 Construction

Unit Quaternions are specifically what is simplest for rotation. They only require 2 parameters to construct, and have 4 outputs:

Inputs:

- θ : amount to rotate by, counter clockwise, in radians
- $\vec{a}$: vector to rotate by θ

Outputs:

- w : real component
- x : $\hat{i}$ component
- y : $\hat{j}$ component
- z : $\hat{k}$ component

Mathematically, A Quaternion is represented as the sum of these output components:

$$Q_n = w + x\hat{i} + y\hat{j} + z\hat{k}$$

The unfortunate drawback to Quaternions is that they're not easily readable at a glance; it's hard to tell what they're doing. But this is made up for by they're ease of use and construction, if you can keep track of them properly.

Constructing a Unit Quaternion for rotation is pretty straightforward:

$$
\begin{aligned}
w &= \cos\left(\frac{\theta}{2}\right) \\
x &= \sin\left(\frac{\theta}{2}\right) \cdot \frac{\vec{a}_x}{\sqrt{\vec{a}_x^2 + \vec{a}_y^2 + \vec{a}_z^2}} \\
y &= \sin\left(\frac{\theta}{2}\right) \cdot \frac{\vec{a}_y}{\sqrt{\vec{a}_x^2 + \vec{a}_y^2 + \vec{a}_z^2}} \\
z &= \sin\left(\frac{\theta}{2}\right) \cdot \frac{\vec{a}_z}{\sqrt{\vec{a}_x^2 + \vec{a}_y^2 + \vec{a}_z^2}}
\end{aligned}
$$

Another useful construct for rotating points/vectors by Quaternions is the **Quaternion Conjugate**. They are simply a Quaternion, where the θ input has been negated before construction. The are denoted by putting an asterisk after the symbol for the original Quaternion (EX: $Q \rightarrow Q^*$).

$$
\begin{aligned}
w &= \cos\left(\frac{-\theta}{2}\right) = \cos\theta \\
x &= \sin\left(\frac{-\theta}{2}\right) \cdot \frac{\vec{a}_x}{\sqrt{\vec{a}_x^2 + \vec{a}_y^2 + \vec{a}_z^2}} = -\sin\theta \cdot \frac{\vec{a}_x}{\sqrt{\vec{a}_x^2 + \vec{a}_y^2 + \vec{a}_z^2}} \\
y &= \sin\left(\frac{-\theta}{2}\right) \cdot \frac{\vec{a}_y}{\sqrt{\vec{a}_x^2 + \vec{a}_y^2 + \vec{a}_z^2}} = -\sin\theta \cdot \frac{\vec{a}_y}{\sqrt{\vec{a}_x^2 + \vec{a}_y^2 + \vec{a}_z^2}} \\
z &= \sin\left(\frac{-\theta}{2}\right) \cdot \frac{\vec{a}_z}{\sqrt{\vec{a}_x^2 + \vec{a}_y^2 + \vec{a}_z^2}} = -\sin\theta \cdot \frac{\vec{a}_z}{\sqrt{\vec{a}_x^2 + \vec{a}_y^2 + \vec{a}_z^2}}
\end{aligned}
$$

Lastly, if you have a vector that you wish to treat like a Quaternion, for purposes such as multiplication, it is even simpler: treat it identically to Quaternion, with a real component of zero. **Note:** this will **not** be a Unit Quaternion unless the original vector is a unit vector.

For a vector $\vec{V}$:

$$
\begin{aligned}
w &= 0 \\
x &= \vec{V}_x \hat{\imath} \\
y &= \vec{V}_y \hat{\jmath} \\
z &= \vec{V}_z \hat{k}
\end{aligned}
$$

4.1.2 Multiplication

Another drawback to using Quaternions is trying to multiply them by hand. It's a simple operation of multiply two 4th degree polynomials, but since they're 4th degree polynomials, it can be extraordinarily tedious. Luckily this is a calculation that is very, very easy to do by a computer.

There is one catch to multiplying Quaternions: order matters. The multiplication of the $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ factors are ***non-commutative***. This means that multiplying entire Quaternions is also ***non-commutative***: $Q_1Q_2 \neq Q_2Q_1$. It is very similar to cross products in linear algebra:

- $\hat{\imath}\hat{\jmath} = \hat{k}$
- $\hat{\jmath}\hat{k} = \hat{\imath}$
- $\hat{k}\hat{\imath} = \hat{\jmath}$
- $\hat{\imath}\hat{k} = -\hat{\jmath}$
- $\hat{k}\hat{\jmath} = -\hat{\imath}$
- $\hat{\jmath}\hat{\imath} = -\hat{k}$
- $\hat{\imath}\hat{\imath} = -1$
- $\hat{\jmath}\hat{\jmath} = -1$
- $\hat{k}\hat{k} = -1$

An easy shorthand for memorizing whether a product is positive or negative is to imagine them in a row as $\hat{\imath}\hat{\jmath}\hat{k}\hat{\imath}$. Take the factors you are trying to multiply and see if they match that order. If they do, then the product is positive. If you need to reverse the order of your factors to get it to match, then their product is negative. The product of such Quaternion multiplication is called the **Hamilton Product** of the two Quaternions. The Hamilton Product of Q_AQ_B is a Quaternion representing a rotation of Q_B, followed by Q_A.

EXAMPLE:

$$Q_A = A_w + A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$$
$$Q_B = B_w + B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}$$
$$Q_AQ_B = \left(A_w + A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}\right) \cdot \left(B_w + B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}\right)$$

$$\begin{aligned} Q_AQ_B = & A_wB_w + A_x\hat{\imath}B_x\hat{\imath} + A_y\hat{\jmath}B_y\hat{\jmath} + A_z\hat{k}B_z\hat{k} \\ & + A_wB_x\hat{\imath} + A_x\hat{\imath}B_w + A_y\hat{\jmath}B_z\hat{k} + A_z\hat{k}B_y\hat{\jmath} \\ & + A_wB_y\hat{\jmath} + A_y\hat{\jmath}B_w + A_z\hat{k}B_x\hat{\imath} + A_x\hat{\imath}B_z\hat{k} \\ & + A_wB_z\hat{k} + A_z\hat{k}B_w + A_x\hat{\imath}B_y\hat{k} + A_y\hat{\jmath}B_x\hat{\imath} \end{aligned}$$

$$\begin{aligned} Q_AQ_B = & \left(A_wB_w - A_xB_x - A_yB_y - A_zB_z\right) \\ & + \left(A_wB_x + A_xB_w + A_yB_z - A_zB_y\right)\hat{\imath} \\ & + \left(A_wB_y + A_yB_w + A_zB_x - A_xB_z\right)\hat{\jmath} \\ & + \left(A_wB_z + A_zB_w + A_xB_y - A_yB_x\right)\hat{k} \end{aligned}$$

4.1.3 Rotation

Quaternions make rotating a point/vector in 3D space very simple, since it only requires Quaternion Multiplication. If want to rotate a vector V by a Quaternion Q_R, to get its image V', it is simply a composition of 2 Quaternion Multiplications.

$$Q_{V'} = Q_R Q_V Q_R^*$$

The Quaternion $Q_{V'}$ will be a Quaternion with a real component of zero, making it very simple to convert back to a 3D vector. Simply correspond the x,y,z components of the Quaternion to the respective x,y,z components of a Vector.

4.1.4 Matrix Conversion

Lastly, there are some cases where Matrices are easier to deal with than Quaternions.

$$Q_R Q_V Q_R^* = \left(R_w + R_x\hat{\imath} + R_y\hat{\jmath} + R_z\hat{k}\right) \cdot \left(V_w + V_x\hat{\imath} + V_y\hat{\jmath} + V_z\hat{k}\right) \cdot \left(R_w - R_x\hat{\imath} - R_y\hat{\jmath} - R_z\hat{k}\right)$$

$$\begin{aligned} Q_R Q_V Q_R^* = \Big(& (R_wV_w - R_xV_x - R_yV_y - R_zV_z) + (R_wV_x + R_xV_w + R_yV_z - R_zV_y)\,\hat{\imath} \\ &+ (R_wV_y + R_yV_w + R_zV_x - R_xV_z)\,\hat{\jmath} + (R_wV_z + R_zV_w + R_xV_y - R_yV_x)\,\hat{k}\Big) \\ &\cdot \left(R_w - R_x\hat{\imath} - R_y\hat{\jmath} - R_z\hat{k}\right) \end{aligned}$$

$$\begin{aligned} Q_R Q_V Q_R^* = &\Big((R_wV_w - R_xV_x - R_yV_y - R_zV_z) + (R_wV_x + R_xV_w + R_yV_z - R_zV_y)\,\hat{\imath} \\ &+ (R_wV_y + R_yV_w + R_zV_x - R_xV_z)\,\hat{\jmath} + (R_wV_z + R_zV_w + R_xV_y - R_yV_x)\,\hat{k}\Big)R_w \\ -&\Big((R_wV_w - R_xV_x - R_yV_y - R_zV_z) + (R_wV_x + R_xV_w + R_yV_z - R_zV_y)\,\hat{\imath} \\ &+ (R_wV_y + R_yV_w + R_zV_x - R_xV_z)\,\hat{\jmath} + (R_wV_z + R_zV_w + R_xV_y - R_yV_x)\,\hat{k}\Big)R_x\hat{\imath} \\ -&\Big((R_wV_w - R_xV_x - R_yV_y - R_zV_z) + (R_wV_x + R_xV_w + R_yV_z - R_zV_y)\,\hat{\imath} \\ &+ (R_wV_y + R_yV_w + R_zV_x - R_xV_z)\,\hat{\jmath} + (R_wV_z + R_zV_w + R_xV_y - R_yV_x)\,\hat{k}\Big)R_y\hat{\jmath} \\ -&\Big((R_wV_w - R_xV_x - R_yV_y - R_zV_z) + (R_wV_x + R_xV_w + R_yV_z - R_zV_y)\,\hat{\imath} \\ &+ (R_wV_y + R_yV_w + R_zV_x - R_xV_z)\,\hat{\jmath} + (R_wV_z + R_zV_w + R_xV_y - R_yV_x)\,\hat{k}\Big)R_z\hat{k} \end{aligned}$$

$$
\begin{aligned}
Q_R Q_V Q_R^* = & R_w \left(R_w V_w - R_x V_x - R_y V_y - R_z V_z\right) + R_w \left(R_w V_x + R_x V_w + R_y V_z - R_z V_y\right) \hat{\imath} \\
& + R_w \left(R_w V_y + R_y V_w + R_z V_x - R_x V_z\right) \hat{\jmath} + R_w \left(R_w V_z + R_z V_w + R_x V_y - R_y V_x\right) \hat{k} \\
& - R_x \left(R_w V_w - R_x V_x - R_y V_y - R_z V_z\right) \hat{\imath} - R_x \left(R_w V_x + R_x V_w + R_y V_z - R_z V_y\right) \hat{\imath}\hat{\imath} \\
& - R_x \left(R_w V_y + R_y V_w + R_z V_x - R_x V_z\right) \hat{\jmath}\hat{\imath} - R_x \left(R_w V_z + R_z V_w + R_x V_y - R_y V_x\right) \hat{k}\hat{\imath} \\
& - R_y \left(R_w V_w - R_x V_x - R_y V_y - R_z V_z\right) \hat{\jmath} - R_y \left(R_w V_x + R_x V_w + R_y V_z - R_z V_y\right) \hat{\imath}\hat{\jmath} \\
& - R_y \left(R_w V_y + R_y V_w + R_z V_x - R_x V_z\right) \hat{\jmath}\hat{\jmath} - R_y \left(R_w V_z + R_z V_w + R_x V_y - R_y V_x\right) \hat{k}\hat{\jmath} \\
& - R_z \left(R_w V_w - R_x V_x - R_y V_y - R_z V_z\right) \hat{k} - R_z \left(R_w V_x + R_x V_w + R_y V_z - R_z V_y\right) \hat{\imath}\hat{k} \\
& - R_z \left(R_w V_y + R_y V_w + R_z V_x - R_x V_z\right) \hat{\jmath}\hat{k} - R_z \left(R_w V_z + R_z V_w + R_x V_y - R_y V_x\right) \hat{k}\hat{k}
\end{aligned}
$$

$$
\begin{aligned}
Q_R Q_V Q_R^* = & \left(R_w^2 V_w - R_w R_x V_x - R_w R_y V_y - R_w R_z V_z\right) + \left(R_w^2 V_x + R_w R_x V_w + R_w R_y V_z - R_w R_z V_y\right) \hat{\imath} \\
& + \left(R_w^2 V_y + R_w R_y V_w + R_w R_z V_x - R_w R_x V_z\right) \hat{\jmath} + \left(R_w^2 V_z + R_w R_z V_w + R_w R_x V_y - R_w R_y V_x\right) \hat{k} \\
& - \left(R_w R_x V_w - R_x^2 V_x - R_x R_y V_y - R_x R_z V_z\right) \hat{\imath} + \left(R_w R_x V_x + R_x^2 V_w + R_x R_y V_z - R_x R_z V_y\right) \\
& + \left(R_w R_x V_y + R_x R_y V_w + R_x R_z V_x - R_x^2 V_z\right) \hat{k} - \left(R_w R_x V_z + R_x R_z V_w + R_x^2 V_y - R_x R_y V_x\right) \hat{\jmath} \\
& - \left(R_w R_y V_w - R_x R_y V_x - R_y^2 V_y - R_y R_z V_z\right) \hat{\jmath} - \left(R_w R_y V_x + R_x R_y V_w + R_y^2 V_z - R_y R_z V_y\right) \hat{k} \\
& + \left(R_w R_y V_y + R_y^2 V_w + R_y R_z V_x - R_x R_y V_z\right) + \left(R_w R_y V_z + R_y R_z V_w + R_x R_y V_y - R_y^2 V_x\right) \hat{\imath} \\
& - \left(R_w R_z V_w - R_x R_z V_x - R_y R_z V_y - R_z^2 V_z\right) \hat{k} + \left(R_w R_z V_x + R_x R_z V_w + R_y R_z V_z - R_z^2 V_y\right) \hat{\jmath} \\
& - \left(R_w R_z V_y + R_y R_z V_w + R_z^2 V_x - R_x R_z V_z\right) \hat{\imath} + \left(R_w R_z V_z + R_z^2 V_w + R_x R_z V_y - R_y R_z V_x\right)
\end{aligned}
$$

$$
\begin{aligned}
Q_R Q_V Q_R^* = & \left(R_w^2 V_w - R_w R_x V_x - R_w R_y V_y - R_w R_z V_z\right) + \left(R_w^2 V_x + R_w R_x V_w + R_w R_y V_z - R_w R_z V_y\right) \hat{\imath} \\
& + \left(R_w^2 V_y + R_w R_y V_w + R_w R_z V_x - R_w R_x V_z\right) \hat{\jmath} + \left(R_w^2 V_z + R_w R_z V_w + R_w R_x V_y - R_w R_y V_x\right) \hat{k} \\
& + \left(-R_w R_x V_w + R_x^2 V_x + R_x R_y V_y + R_x R_z V_z\right) \hat{\imath} + \left(R_w R_x V_x + R_x^2 V_w + R_x R_y V_z - R_x R_z V_y\right) \\
& + \left(R_w R_x V_y + R_x R_y V_w + R_x R_z V_x - R_x^2 V_z\right) \hat{k} + \left(-R_w R_x V_z - R_x R_z V_w - R_x^2 V_y + R_x R_y V_x\right) \hat{\jmath} \\
& + \left(-R_w R_y V_w + R_x R_y V_x + R_y^2 V_y + R_y R_z V_z\right) \hat{\jmath} + \left(-R_w R_y V_x - R_x R_y V_w - R_y^2 V_z + R_y R_z V_y\right) \hat{k} \\
& + \left(R_w R_y V_y + R_y^2 V_w + R_y R_z V_x - R_x R_y V_z\right) + \left(R_w R_y V_z + R_y R_z V_w + R_x R_y V_y - R_y^2 V_x\right) \hat{\imath} \\
& + \left(-R_w R_z V_w + R_x R_z V_x + R_y R_z V_y + R_z^2 V_z\right) \hat{k} + \left(R_w R_z V_x + R_x R_z V_w + R_y R_z V_z - R_z^2 V_y\right) \hat{\jmath} \\
& + \left(-R_w R_z V_y - R_y R_z V_w - R_z^2 V_x + R_x R_z V_z\right) \hat{\imath} + \left(R_w R_z V_z + R_z^2 V_w + R_x R_z V_y - R_y R_z V_x\right)
\end{aligned}
$$

$$
\begin{aligned}
Q_R Q_V Q_R^* = \Big(& R_w^2 V_w - R_w R_x V_x - R_w R_y V_y - R_w R_z V_z + R_w R_x V_x + R_x^2 V_w + R_x R_y V_z - R_x R_z V_y \\
& + R_w R_y V_y + R_y^2 V_w + R_y R_z V_x - R_x R_y V_z + R_w R_z V_z + R_z^2 V_w + R_x R_z V_y - R_y R_z V_x\Big) \\
+\Big(& R_w^2 V_x + R_w R_x V_w + R_w R_y V_z - R_w R_z V_y - R_w R_x V_w + R_x^2 V_x + R_x R_y V_y + R_x R_z V_z \\
& + R_w R_y V_z + R_y R_z V_w + R_x R_y V_y - R_y^2 V_x - R_w R_z V_y - R_y R_z V_w - R_z^2 V_x + R_x R_z V_z\Big)\hat{i} \\
+\Big(& R_w^2 V_y + R_w R_y V_w + R_w R_z V_x - R_w R_x V_z - R_w R_x V_z - R_x R_z V_w - R_x^2 V_y + R_x R_y V_x \\
& - R_w R_y V_w + R_x R_y V_x + R_y^2 V_y + R_y R_z V_z + R_w R_z V_x + R_x R_z V_w + R_y R_z V_z - R_z^2 V_y\Big)\hat{j} \\
+\Big(& R_w^2 V_z + R_w R_z V_w + R_w R_x V_y - R_w R_y V_x + R_w R_x V_y + R_x R_y V_w + R_x R_z V_x - R_x^2 V_z \\
& - R_w R_y V_x - R_x R_y V_w - R_y^2 V_z + R_y R_z V_y - R_w R_z V_w + R_x R_z V_x + R_y R_z V_y + R_z^2 V_z\Big)\hat{k}
\end{aligned}
$$

$$
\begin{aligned}
Q_R Q_V Q_R^* =& \Big(V_w(R_w^2 + R_x^2 + R_y^2 + R_z^2)\Big) \\
&+\Big(V_x(R_w^2 + R_x^2 - R_y^2 - R_z^2) + 2V_y(R_x R_y - 2R_w R_z) + 2V_z(R_w R_y + 2R_x R_z)\Big)\hat{i} \\
&+\Big(2V_x(R_w R_z + 2R_x R_y) + V_y(R_w^2 - R_x^2 + R_y^2 - R_z^2) + 2V_z(R_y R_z - 2R_w R_x)\Big)\hat{j} \\
&+\Big(2V_x(R_x R_z - R_w R_y) + 2V_y(R_w R_x + R_y R_z) + V_z(R_w^2 - R_x^2 - R_y^2 + R_z^2\Big)\hat{k}
\end{aligned}
$$

Now if we plug in the Unit Vectors for x,y and z, we can construct the a general matrix conversion.

$$
Q_x = 0 + 1\hat{i} + 0\hat{j} + 0\hat{k} = \hat{i} \quad Q_y = 0 + 0\hat{i} + 1\hat{j} + 0\hat{k} = \hat{j} \quad Q_z = 0 + 0\hat{i} + 0\hat{j} + 1\hat{k} = \hat{k}
$$

$$
\begin{aligned}
Q_R Q_x Q_R^* =& \Big(0 \cdot (R_w^2 + R_x^2 + R_y^2 + R_z^2)\Big) \\
&+\Big(1 \cdot (R_w^2 + R_x^2 - R_y^2 - R_z^2) + 2 \cdot 0 \cdot (R_x R_y - R_w R_z) + 2 \cdot 0 \cdot (R_w R_y + R_x R_z)\Big)\hat{i} \\
&+\Big(2 \cdot 1 \cdot (R_w R_z + R_x R_y) + 0 \cdot (R_w^2 - R_x^2 + R_y^2 - R_z^2) + 2 \cdot 0 \cdot (R_y R_z - R_w R_x)\Big)\hat{j} \\
&+\Big(2 \cdot 1 \cdot (R_x R_z - R_w R_y) + 2 \cdot 0 \cdot (R_w R_x + R_y R_z) + 0 \cdot (R_w^2 - R_x^2 - R_y^2 + R_z^2)\Big)\hat{k}
\end{aligned}
$$

$$
Q_R Q_x Q_R^* = (R_w^2 + R_x^2 - R_y^2 - R_z^2)\hat{i} + 2(R_w R_z + 2R_x R_y)\hat{j} + 2(R_x R_z - R_w R_y)\hat{k}
$$

$$
\begin{aligned}
Q_R Q_y Q_R^* =& \Big(0 \cdot (R_w^2 + R_x^2 + R_y^2 + R_z^2)\Big) \\
&+ \Big(0 \cdot (R_w^2 + R_x^2 - R_y^2 - R_z^2) + 2 \cdot 1 \cdot (R_x R_y - R_w R_z) + 2 \cdot 0 \cdot (R_w R_y + R_x R_z)\Big)\hat{\imath} \\
&+ \Big(2 \cdot 0 \cdot (R_w R_z + R_x R_y) + 1 \cdot (R_w^2 - R_x^2 + R_y^2 - R_z^2) + 2 \cdot 0 \cdot (R_y R_z - R_w R_x)\Big)\hat{\jmath} \\
&+ \Big(2 \cdot 0 \cdot (R_x R_z - R_w R_y) + 2 \cdot 1 \cdot (R_w R_x + R_y R_z) + 0 \cdot (R_w^2 - R_x^2 - R_y^2 + R_z^2)\Big)\hat{k}
\end{aligned}
$$

$$
Q_R Q_y Q_R^* = 2(R_x R_y - R_w R_z)\hat{\imath} + (R_w^2 - R_x^2 + R_y^2 - R_z^2)\hat{\jmath} + 2(R_w R_x + R_y R_z)\hat{k}
$$

$$
\begin{aligned}
Q_R Q_z Q_R^* =& \Big(0 \cdot (R_w^2 + R_x^2 + R_y^2 + R_z^2)\Big) \\
&+ \Big(0 \cdot (R_w^2 + R_x^2 - R_y^2 - R_z^2) + 2 \cdot 0 \cdot (R_x R_y - R_w R_z) + 2 \cdot 1 \cdot (R_w R_y + R_x R_z)\Big)\hat{\imath} \\
&+ \Big(2 \cdot 0 \cdot (R_w R_z + R_x R_y) + 0 \cdot (R_w^2 - R_x^2 + R_y^2 - R_z^2) + 2 \cdot 1 \cdot (R_y R_z - R_w R_x)\Big)\hat{\jmath} \\
&+ \Big(2 \cdot 0 \cdot (R_x R_z - R_w R_y) + 2 \cdot 0 \cdot (R_w R_x + R_y R_z) + 1 \cdot (R_w^2 - R_x^2 - R_y^2 + R_z^2)\Big)\hat{k}
\end{aligned}
$$

$$
Q_R Q_z Q_R^* = 2(R_w R_y + 2R_x R_z)\hat{\imath} + 2(R_y R_z - 2R_w R_x)\hat{\jmath} + (R_w^2 - R_x^2 - R_y^2 + R_z^2)\hat{k}
$$

$$
\begin{bmatrix}
R_w^2 + R_x^2 - R_y^2 - R_z^2 & 2(R_x R_y - R_w R_z) & 2(R_w R_y + R_x R_z) \\
2(R_w R_z + R_x R_y) & R_w^2 - R_x^2 + R_y^2 - R_z^2 & 2(R_y R_z - R_w R_x) \\
2(R_x R_z - R_w R_y) & 2(R_w R_x + R_y R_z) & R_w^2 - R_x^2 - R_y^2 + R_z^2
\end{bmatrix}
$$

4.2 Orientation Formats

4.2.1 Orientation in Space

Attempting to orient yourself in space is notoriously difficult. Without gravity consistently acting in one direction, there is no natural reference point for which way "down" is, let alone left or right. There is a relatively straightforward way to side-step the issue: Convention. As with all things defined by convention, there are always multiple, often competing standard that frequently (annoyingly, and inconsistently) change based on scenario.

With the many competing conventions for defining spatial coordinates, and which direction is even defined as "up", this model of orientation will allow for defining both "up" and "across" arbitrarily. Defining and using these two directions, a **Plane of Reference** can be constructed, and the orbital plane can be described relative to that Plane of Reference.

Let:

- $\hat{u}$: up vector
- $\hat{d}$: reference vector
- $\hat{f}$: forward vector
- W : world matrix

Constructing A Plane of Reference is very simple. $\hat{u}$ is orthogonal to the Plane of Reference, and defines which way is "up". $\hat{d}$ is on the Plane of Reference and acts as a meridian; it can act as a fixed point to help triangulate position. Lastly, $\hat{f}$ is orthogonal to both $\hat{u}$ and $\hat{d}$. It can be defined as such to complete a right-handed set:

$$\hat{f} = \hat{u} \times \hat{d}$$

Using these three vectors, the world matrix can defined such that it maps the $\hat{x}$,$\hat{y}$,$\hat{z}$ axes to the $\hat{d}$,$\hat{f}$,$\hat{u}$ axes, respectively. It is now possible to work with the x axis pointing toward the reference direction, and the z axis pointing up, which can simplify things a great deal.

$$W = \begin{bmatrix} \hat{d}_x & \hat{f}_x & \hat{u}_x \\ \hat{d}_y & \hat{f}_y & \hat{u}_y \\ \hat{d}_z & \hat{f}_z & \hat{u}_z \end{bmatrix}$$

4.2.2 Orbital Elements

Perhaps the more popular way to parameterize orbits is through **Orbital Elements**. Orbital Elements provide an easy to read way to uniquely describe an orbit. Three of the elements are used to describe Geometry and Physics: commonly, **Semi Major Axis**, **Eccentricity**, and **Standard Gravitational Parameter**; the other three elements are used to describe Rotation: **Inclination**, **Longitude of Ascending Node**, and **Argument of Periapsis**.

Most of these are based off of a point called the **Ascending Node**. There are two nodes for any orbit where the inclination is non-zero; they are the **Ascending Node** and the **Descending Node**. They mark where the Orbital Plane intersect the Plane of Reference. As the name suggests, the Descending Node marks when the orbiting body crosses the reference plane to the side opposite the up vector, $\hat{u}$. The Ascending Node conversely marks when the orbiting body crosses to the same side of Plane of Reference as the up vector, $\hat{u}$.

Let:

- I : Inclination; how tilted the orbital plane is from the Plane of Reference as measured from the $\vec{N}_a$; angular distance between $\hat{u}$ and h
- L : Longitude of Ascending Node; The angular displacement from the reference direction, to the Ascending Node, measured counter-clock wise.
- A : Argument of Periapsis
- $\vec{N}_a$: Ascending Node

4.2.3 Orbital State Vectors

4.2.4 Orbital Matrix

4.3 Constructions

4.3.1 Orbital Elements

$$Q_I = \cos\left(\frac{I}{2}\right) + \sin\left(\frac{I}{2}\right)\left(\cos(-A)\hat{\imath} + \sin(-A)\hat{\jmath} + 0\hat{k}\right)$$

$$Q_L = \cos\left(\frac{L+A}{2}\right) + \sin\left(\frac{L+A}{2}\right)\left(0\hat{\imath} + 0\hat{\jmath} + 1\hat{k}\right)$$

$$Q_E = Q_L Q_I$$

$$W(\theta) = \begin{bmatrix} \hat{d}_x & \hat{f}_x & \hat{u}_x \\ \hat{d}_y & \hat{f}_y & \hat{u}_y \\ \hat{d}_z & \hat{f}_z & \hat{u}_z \end{bmatrix} \begin{bmatrix} Q_{Ew}^2 + Q_{Ex}^2 - Q_{Ey}^2 - Q_{Ez}^2 & 2(Q_{Ex}Q_{Ey} - Q_{Ew}Q_{Ez}) & 2(Q_{Ew}Q_{Ey} + Q_{Ex}Q_{Ez}) \\ 2(Q_{Ew}Q_{Ez} + Q_{Ex}Q_{Ey}) & Q_{Ew}^2 - Q_{Ex}^2 + Q_{Ey}^2 - Q_{Ez}^2 & 2(Q_{Ey}Q_{Ez} - Q_{Ew}Q_{Ex}) \\ 2(Q_{Ex}Q_{Ez} - Q_{Ew}Q_{Ey}) & 2(Q_{Ew}Q_{Ex} + Q_{Ey}Q_{Ez}) & Q_{Ew}^2 - Q_{Ex}^2 - Q_{Ey}^2 + Q_{Ez}^2 \end{bmatrix} \begin{bmatrix} \frac{l\cos\theta}{e\cos\theta+1} \\ \frac{l\sin\theta}{e\cos\theta+1} \\ 0 \end{bmatrix}$$

4.3.2 Orbital State Vectors

$$h = r \times v$$

$$l = \frac{h^2}{\mu}$$

$$e = \sqrt{l\left(\frac{v^2}{\mu} - \frac{2}{r}\right) + 1}$$

$$\theta_t = \begin{cases} r \cdot v \geq 0 & +\arccos\left(\frac{l-r}{e \cdot r}\right) \\ r \cdot v < 0 & -\arccos\left(\frac{l-r}{e \cdot r}\right) \end{cases}$$

$$Q_p = \cos\left(\frac{-\theta_t}{2}\right) + \sin\left(\frac{-\theta_t}{2}\right)\left(\hat{h}_x\hat{\imath} + \hat{h}_y\hat{\jmath} + \hat{h}_z\hat{k}\right)$$

$$Q_l = \cos\left(\frac{90-\theta_t}{2}\right) + \sin\left(\frac{90-\theta_t}{2}\right)\left(\hat{h}_x\hat{i} + \hat{h}_y\hat{j} + \hat{h}_z\hat{k}\right)$$

$$\hat{p} = Q_p r Q_p^*$$

$$\hat{l} = Q_l r Q_l^*$$

$$W(\theta) = \begin{bmatrix} \hat{p}_x & \hat{l}_x & \hat{h}_x \\ \hat{p}_y & \hat{l}_y & \hat{h}_y \\ \hat{p}_z & \hat{l}_z & \hat{h}_z \end{bmatrix} \cdot \begin{bmatrix} \frac{l\cos\theta}{e\cos\theta+1} \\ \frac{l\sin\theta}{e\cos\theta+1} \\ 0 \end{bmatrix}$$

Elements to State Vectors

4.4 Conversions

4.4.1 Left Handed Space