Problem Set #1

Serena An and Samyok Nepal–May 28, 2018

1 Introduction

Welcome to the first Brookings Math Circle Summer 2018 Problem Set! **Submissions are due by Sunday 6/10 at 1:00 pm (the day of the first BMC class).**

The Summer 2018 Problem Sets will run over 14 weeks, 6/3 to 9/9 (this first one is being released early). A new problem set and solutions to the previous problem set will be released on our website every Sunday afternoon at approximately 4:00 pm.

We hope you enjoy the weekly Summer 2018 Problem Sets! Keep calm and do the math!

2 Puzzle #1

Problem 2.1: (USAMTS)

Fill each white square in with a number so that each of the 27 three-digit numbers whose digits are all 1, 2, or 3 is used exactly once. For each pair of white squares sharing a side, the two numbers must have equal digits in exactly two of the three positions (ones, tens, hundreds). Some numbers have been given to you.

					212		
			213				123
						131	
		121			113		
				312			

111	112	~~113~~	211	~~212~~	~~213~~	311	~~312~~	313
~~121~~	122	~~123~~	221	222	223	321	322	323
~~131~~	132	133	231	232	233	331	332	333

Solution 2.1: There are many ways to go about this, which all involve some trial and error. If you are stuck, the best places to start are the square adjacent to 312 and 113, 113 and 131, or 123 and 131. These positions have the most restrictions. Another tip is to lightly pencil in possibilities for particular squares if there are 2-3 possible numbers for a square.

			233	232	212	222	122
231		223	213		112		123
331		221	211	311	111	131	133
332		121		313	113		132
333	323	321	322	312			

3 Bonus Problem Set #1

Problem 3.1: (AMC 8)

The sum of two prime numbers is 85. What is the product of these two prime numbers?

Solution 3.1: There is only one way two numbers add up to an odd number: one must be even and the other must be odd. Since there is only one even prime, the primes are 2 and 83. Thus, our answer is $2 \cdot 83 = \boxed{166}$.

Problem 3.2: (MATHCOUNTS)

A hexagon and pentagon share the property that the side lengths of each are consecutive integers, and the perimeter of each is 45 cm. What is the difference in length between the shortest side of the pentagon and the shortest side of the hexagon?

Solution 3.2: Let h be the shortest side of a hexagon, and p the shortest side of the pentagon. Since the sides are consecutive, we can write the two equations as follows:

$$h + (h+1) + (h+2) + (h+3) + (h+4) + (h+5) = 45$$
$$p + (p+1) + (p+2) + (p+3) + (p+4) = 45$$

These two can simplify down to

$$6h + 15 = 45$$
$$5p + 10 = 45$$

This leads to $h = 5$ and $p = 7$, so the difference between the shortest sides is $p - h = \boxed{2}$

Problem 3.3: (AMC 8)

How many integers between 1000 and 9999 have four distinct digits?

Solution 3.3: We count the number of possibilities for each digit and multiply. For the thousands digit, there are 9 possibilities, since any number from 1 to 9 (inclusive) works. For the hundreds digit, we have the ten digits 0 to 9 to consider, but since we can not repeat the digit used in the thousands place, there are 9 possibilities. In the tens place, we can consider the 10 digits 0 to 9, but we can't repeat the digits of the thousands or hundreds places so there are 8 possibilities. Using similar reasoning, there are 7 possibilities for the ones place.

Thus, the total number of possibilities is $9 \cdot 9 \cdot 8 \cdot 7 = \boxed{4536}$

Problem 3.4: (MATHCOUNTS)

What is the number of degrees of the acute angle formed by the minute and hour hands of a clock at 11:10 PM?

Solution 3.4: At 11:10, the minute hand is at the 2. The number of degrees in a full circle is 360, so we divide that by 12 to get the number of degrees from 12-1.

$$\frac{360}{12} = 30$$

However, we have to multiply by 2 because there are two hours between 12 and 2.

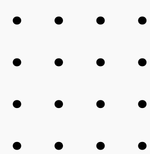
$$30 \cdot 2 = 60^\circ$$

Now, we have to find the number of degrees between the hour hand and the 12. This isn't just 30 degrees– the hour hand moves slightly every minute. However, we do know that over the course of 60 minutes, it moves 30 degrees. That means for every minute, the hour hand moves 0.5 degrees closer to the 12. Thus, 10 minutes after 11:00, the hour hand is 5 degrees closer to the 12. Our final answer then becomes:

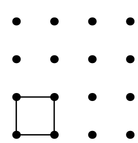
$$60 + 30 - 5 = \boxed{85^\circ}.$$

Problem 3.5: (CMMS)

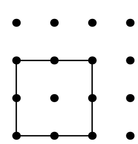
How many squares can be formed using four of the dots in the unit grid as vertices?



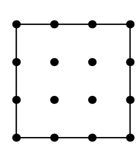
Solution 3.5: We can solve this via casework. There are 9 1x1 squares that look like this:



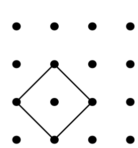
There are 4 2x2s:



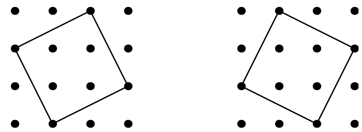
And 1 3x3:



However, there are also 4 $\sqrt{2}$x$\sqrt{2}$s:



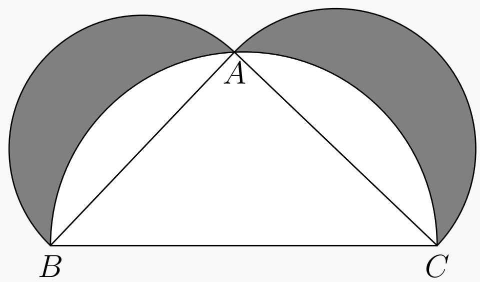
and 2 $\sqrt{5}$x$\sqrt{5}$s:



Thus, the total number of squares is $9 + 4 + 1 + 4 + 2 = \boxed{20}$.

Problem 3.6: (MPfG)

The figure below shows a triangle ABC with a semicircle on each of its three sides.



If $AB = 20$, $AC = 21$, and $BC = 29$, what is the area of the shaded region?

Solution 3.6: ABC is a right triangle because $21^2 + 20^2 = 29^2$. One way to find the area of the shaded region is to find the sum of the areas of the semicircles on the legs of the triangle and the triangle itself, and then subtract the area of the semicircle on the hypotenuse.

The area of the triangle is $\frac{20 \cdot 21}{2} = 210$ because the legs are the base and height of the triangle. The sum of the semicircles' areas is

$$\frac{\left(\frac{20}{2}\right)^2 \pi}{2} + \frac{\left(\frac{21}{2}\right)^2 \pi}{2} = \frac{400\pi}{8} + \frac{441\pi}{8} = \frac{841\pi}{8}.$$

The total area of the figure is thus $\frac{841\pi}{8} + 210$.

Now, the area of the largest semicircle of diameter 29 is $\frac{(\frac{29}{2})^2 \pi}{2} = \frac{841\pi}{8}$. Subtracting, we find that the area of the shaded area is $\boxed{210}$.

Note that this is exactly the area of triangle ABC. Try to prove that this is always the case with any right triangle ABC!

Problem 3.7: (AIME)

The AIME Triathlon consists of a half-mile swim, a 30-mile bicycle ride, and an eight-mile run. Tom swims, bicycles, and runs at constant rates. He runs fives times as fast as he swims, and he bicycles twice as fast as he runs. Tom completes the AIME Triathlon in four and a quarter hours. How many minutes does he spend bicycling?

Solution 3.7: Let r represent the rate Tom swims in miles per minute. Then $5r$ is the rate at which Tom runs in miles per minute, and $10r$ is the rate at which Tom bikes in miles per minute.

A half mile swim will take Tom $\frac{\frac{1}{2}}{r} = \frac{1}{2r}$ minutes. An eight-mile run will take $\frac{8}{5r}$ minutes. A 30-mile bicycle ride will take $\frac{30}{10r}$. Four and a quarter hours is $240 + 15 = 255$ minutes.

Then we have

$$\frac{1}{2r} + \frac{8}{5r} + \frac{30}{10r} = 255$$

We need a common denominator to add.

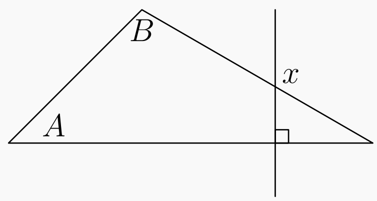
$$\frac{5}{10r} + \frac{16}{10r} + \frac{30}{10r} = \frac{51}{10r} = 255$$

Solving for r, we find $r = 1/50$, so the time Tom spends biking is $\frac{30}{(10)(1/50)} = \boxed{150}$ minutes.

4 Summer Challenge #1

Problem 4.1: (Samyok Nepal)

What is angle x if $\angle A$ is $45°$ and $\angle B$ is $105°$?



Solution 4.1: Since the angles of a triangle add up to $180°$, the third angle of the large triangle is $180 - 45 - 105 = 30°$. Now focus on the right triangle. The third angle of the right triangle, the supplement of x, is $180 - 90 - 30 = 60°$. Thus, $x = 180 - 60 = \boxed{120°}$.

Problem 4.2: (Serena An)

What is the units digit of

$$(12^2 + 23^2 + 34^2 + 45^2 + 56^2 + 67^2 + 78^2 + 89^2)^2?$$

Solution 4.2: Since we only need to find the units digit, we only need to take into consideration the unit digits of the 7 numbers. The units digit of 12^2 is $2^2 = 4$. The units digit of 23^2 is $3^2 = 9$. Continuing, the units digit of the given expression is equivalent to the units digit of $(4+9+6+5+6+9+4+1)^2 = 44^2$. The units digit of 44^2 is 6, so our final answer is $\boxed{6}$.

Problem 4.3: (Serena An)

Serena flips 1000 fair coins while Samyok flips 999 fair coins. What is the probability that Serena flips more heads than Samyok? Express your answer as a common fraction.

Solution 4.3: Imagine Serena and Samyok flipping 999 coins simultaneously. The expected number of heads each receives is equal since they have flipped the same number of coins. However, Serena has one more coin to flip. The probability that it lands on heads, and thus Serena flips more heads, is $\boxed{\dfrac{1}{2}}$.

Advanced Algebraic Solution 4.3: Let P be the probability of Serena has flipped more heads in the first 999 flips. By symmetry, P is also the probability that Samyok has flipped more coins in 999 flips. The probability that they have flipped an equal number of heads is then $1 - 2P$. Thus, the probability of Serena winning (not tying) is the sum of the probabilities of already flipping more coins in the first 999 flips (P) and being in a tie situation and flipping a heads on the last flip ($(1 - 2P) \cdot \frac{1}{2}$). $P + \frac{1}{2} \cdot (1 - 2P) = P + \frac{1}{2} - P = \boxed{\dfrac{1}{2}}$.

Problem 4.4: (Serena An)

If $x^{x^{x^{\cdots}}} = 2$, what is x^4?

Solution 4.4: We can rewrite the given equation as

$$x^{\left(x^{x^{x^{\cdots}}}\right)} = 2$$

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Note that this is equivalent to $x^2 = 2$. Squaring both sides, we get $x^4 = \boxed{4}$.

Problem 4.5: (Samyok Nepal)

Samyok has a special spinner. On this spinner, there are 4 numbers: 1, 2, 3, and 4. However, the probability of getting a number is proportional to that number. For example, the probability of getting a 2 is half of the probability of getting a 4. Samyok spins the spinner twice. What is the probability that the sum of the two spins is 4?

Reminder: Write a full solution to this problem, not just the answer. Even if you do not have a full solution, type/write up what you have and you may receive partial credit!

Solution 4.5: Let the probability that the spinner lands on the number 1 be x. Then, because the probabilities are proportional, the probability that the spinner lands on 2 is $2x$, on 3 is $3x$, and 4 is $4x$. We also know that the sum of all these probabilities is 1, so $x + 2x + 3x + 4x = 10x = 1$ and $x = \frac{1}{10}$. So, the probability of landing on 1 is $\frac{1}{10}$, on 2 is $\frac{2}{10}$, on 3 is $\frac{3}{10}$, and on 4 is $\frac{4}{10}$. There are three ways to spin a sum of 4 (taking into consideration order): a 1 and then a 3, a 2 and then a 2, and a 3 and then a 1. The probabilities are, respectively, $\frac{1}{10} \cdot \frac{3}{10} = \frac{3}{100}$, $\frac{2}{10} \cdot \frac{2}{10} = \frac{4}{100}$, and $\frac{3}{10} \cdot \frac{1}{10} = \frac{3}{100}$. The sum of these probabilities is the final answer: $\frac{3}{100} + \frac{4}{100} + \frac{3}{100} = \frac{10}{100} = \boxed{\frac{1}{10}}$.