Modular Arithmetic

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1 Divisibility Rules

We present the common divisibility rules for all numbers from 2 to 11. The more difficult proofs will be tackled next with with the *Introduction to Modular Arithmetic* handout.

Theorem 1.1: Divisibility Rules for 2, 5, 10

A number is divisible by

- 2 if it ends in 0, 2, 4, 6, or 8 (even numbers)
- 5 if it ends in 0 or 5
- 10 if it ends in 0

Problem 1.1

Prime factorize 800000.

Theorem 1.2: Divisibility Rules for 4, 8

A number is divisible by

- 4 if its last 2 digits are divisible by 4
- 8 if its last 3 digits are divisible by 8

For divisibility by 4, usually you can figure out whether or not the last two digits are divisible by 4 by inspection (just looking at it) or by some quick division. A number must be divisible by 4 in order for it to be divisible by 8.

Exercise 1.1

Prove the divisibility rules for 4 and 8.

Solution 1.1: 4 divides 100, and all multiples of 100 (i.e. numbers ending in 00), so we only need to look at the last 2 digits. Similarly, 8 divides 1000 (but not 100) and all multiples of 1000 (i.e. numbers ending in 000), so we only need to look at the last 3 digits.

Problem 1.2

Determine which of the following are divisible by 4. Then determine which are divisible by 8.

- 198
- 476
- 936
- 111104

Theorem 1.3: Divisibility Rules for 3, 6, 9

A number is divisible by

- 3 if the sum of its digits is a multiple of 3
- 6 if it is divisible by 2 and 3
- 9 if the sum of its digits is a multiple of 9

Problem 1.3

Determine which of the following are divisible by 3. Then determine which are divisible by 9.

- 483
- 2957
- 314159265
- 111111111

Problem 1.4: (MATHCOUNTS 2009 Chapter Countdown)

In the six-digit integer $3A6,792$, what is the largest digit A so that the six-digit integer will be divisible by 3?

Theorem 1.4: Divisibility Rule for 7

Usually, it is easier to just do the division by 7 than to use this divisibility rule. This rule is most useful with 3 digit numbers (sometimes 4 digits), but with anything above that, the rule tends to be a bit cumbersome.

1. Remove the units digit from the number and double the units digit.
2. Subtract twice the units digit from the truncated original number.
3. Repeat this process as many times as needed (usually until you end up with a 1 or 2 digit number). If this smaller number is divisible by 7 (note that 0 is divisible by 7), then the original number was also divisible by 7.

Exercise 1.2

Show that 483903 is divisible by 7.

Solution 1.2:

1. $48390 - 2 \cdot 3 = 48384$
2. $4838 - 2 * 4 = 4830$
3. $483 - 2 * 0 = 483$ (Alternatively, we could have realized that 7 does not divide 10, so we can divide the number by 10.)
4. $48 - 2 * 3 = 42$

We know that 42 is divisible by 7, so our original number, 483903, is also divisible by 7. We could have continued the algorithm one more step to get $4 - 2 * 2 = 0$, which is also clearly divisible by 7.

Problem 1.5

Determine which of the following are divisible by 7.

- 259
- 536
- 1001
- 111111

Theorem 1.5: Divisibility by 11

Sum the alternating digits and subtract these two sums (order doesn't matter). If the final answer is divisible by 11 (remember that 0 is divisible by 11), the original number is also divisible by 11.

Exercise 1.3

Show that 918,291 is divisible by 11.

Solution 1.3: We sum the alternating digits to get the two sums $9 + 8 + 9 = 26$ and $1 + 2 + 1 = 4$. Their positive difference is 22, which is divisible by 11, so 918,291 is also divisible by 11.

Problem 1.6

Determine which of the following are divisible by 11.

- 396
- 1001
- 8592
- 212121

2 Bonus Problems

Problem 2.1: (Alcumus)

If an integer ends in the digit 0 and the sum of its digits is divisible by 3, then how many of the numbers 2, 3, 4, 5, 6, 8, 9 necessarily divide it?

Problem 2.2: (Alcumus)

What is the least four-digit positive integer, with all different digits, that is divisible by each of its digits?

Problem 2.3: (Alcumus)

Let A be a digit. If the 7-digit number 353808A is divisible by 2, 3, 4, 5, 6, 8, and 9, then what is A?

Problem 2.4

Prime factorize 52446240.

Problem 2.5: (Alcumus)

The number $5\,41G\,507\,2H6$ is divisible by 72. If G and H each represent a single digit, what is the sum of all **distinct** possible values of the product GH? (Count each possible value of GH only once, even if it results from multiple G, H pairs.)

Problem 2.6: (MATHCOUNTS 2002 National Team)

The 9-digit number $abb, aba, ba3$ is a multiple of 99 for some pair of digits a and b such that $b > a$. What is $b - a$?