Challenge #4 - Trail Mix

Serena An–August 12, 2018

1 Introduction

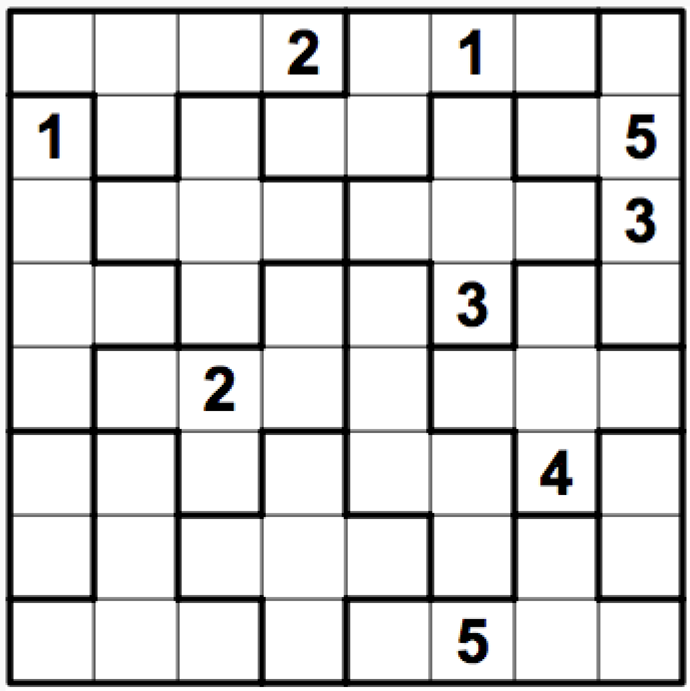
Welcome to the Brookings Math Circle Summer 2018 Problem Sets! **Submissions for this problem set are due by Saturday 8/18 at 1:30 pm (the day of the first BMC class of the second session).** For a refresher on submission details or problem set details, see the introduction PDF linked to the Summer Challenge page of our website under Classes.

This problem set will consist of a variety of problems from the four main subjects: algebra, counting and probability, number theory, and geometry. Each section consists of 5 problems, roughly ordered by difficulty. There are hints to selected problems in the back. After a problem, if there is *Hint:* followed by a number, the hint to that problem is by that number. Some problems are very hard, so don't worry if you can't solve many of them! In the upcoming class, we will discuss these problems.

As always, there will also be a puzzle and summer challenge problems, with rules as usual. Keep calm and do the math!

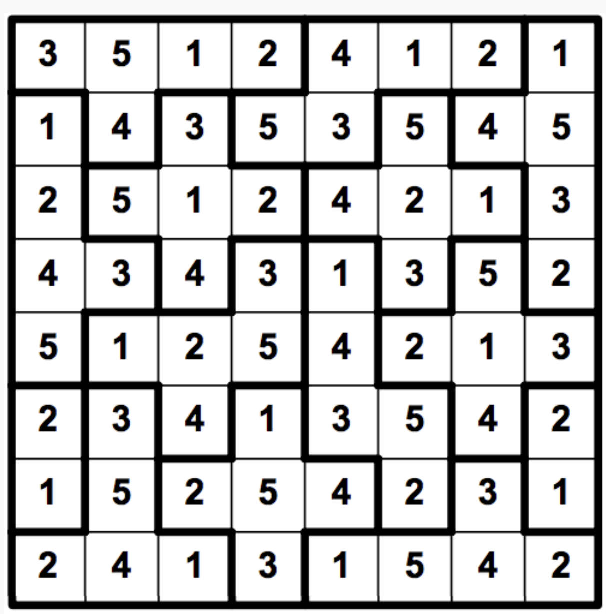
2 Puzzle #4

Problem 2.1: (krazydad.com)



The grid is subdivided into containers or cages, each of which is 1 to 5 cells in size. Fill each container with unique digits, counting up from 1. So for example a 2-square container contains the numbers 1 and 2. A 5-square container contains the numbers from 1 to 5. Adjacent (touching) cells may never contain the same number, and this includes diagonally adjacent cells. There is only one solution. Good luck!

Solution 2.1:



3 Algebra

Problem 3.1: (MATHCOUNTS National CD)

What is the slope of a line perpendicular to the hypotenuse of a triangle with vertices $X(-1,-1)$, $Y(2,-1)$ and $Z(-1,3)$? Express your answer as a common fraction.

Solution 3.1: YZ is the hypotenuse and its slope is $\frac{3-(-1)}{-1-2} = -\frac{4}{3}$. The slope of a perpendicular line is $\boxed{\frac{3}{4}}$.

Problem 3.2: (MATHCOUNTS)

What is the value of the quotient $\frac{(1-\frac{1}{2})(1-\frac{2}{3})(1-\frac{3}{4})\cdot\ldots\cdot(1-\frac{99}{100})}{\frac{1}{101!}}$?

Note that $n! = n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot 1$ (pronounced "n factorial"), so for example, $4! = 4\cdot 3\cdot 2\cdot 1$.

Solution 3.2: The quotient is equivalent to $\frac{\frac{1}{2}\cdot\frac{1}{3}\cdot\ldots\cdot\frac{1}{100}}{\frac{1}{101!}}$, which is $\frac{\frac{1}{100!}}{\frac{1}{101!}}$. Clearing the fractions, we get $\frac{101!}{100!} = \boxed{101}$.

Problem 3.3: (AMC 10)

A rectangle with a diagonal of length x is twice as long as it is wide. What is the area of the rectangle? Express your answer in terms of x. *Hint: 1*

Solution 3.3: Let the width of the rectangle be w. Then the length is $2w$. Using the Pythagorean Theorem,

$$x^2 = w^2 + (2w)^2$$

$$x^2 = 5w^2$$

The area of the rectangle is $w \cdot 2w = 2w^2 = \boxed{\frac{2}{5}x^2}$.

Problem 3.4: (MPfG)

If a and b are nonzero real numbers such that $|a| \neq |b|$, compute the value of the expression

$$\left(\frac{b^2}{a^2} + \frac{a^2}{b^2} - 2\right) \times \left(\frac{a+b}{b-a} + \frac{b-a}{a+b}\right) \times \left(\frac{\frac{1}{a^2}+\frac{1}{b^2}}{\frac{1}{b^2}-\frac{1}{a^2}} - \frac{\frac{1}{b^2}-\frac{1}{a^2}}{\frac{1}{a^2}+\frac{1}{b^2}}\right).$$

Hint: 2

Solution 3.4: We use common denominators to combine terms in each pair of parentheses.

$$\frac{b^2}{a^2} + \frac{a^2}{b^2} - 2 = \frac{(a-b)^2(a+b)^2}{a^2b^2}$$

$$\frac{a+b}{b-a}+\frac{b-a}{a+b}=\frac{-2(a^2+b^2)}{(a-b)(a+b)}$$
$$\frac{\frac{1}{a^2}+\frac{1}{b^2}}{\frac{1}{b^2}-\frac{1}{a^2}}-\frac{\frac{1}{b^2}-\frac{1}{a^2}}{\frac{1}{a^2}+\frac{1}{b^2}}=\frac{4a^2b^2}{(a-b)(a+b)(a^2+b^2)}$$

Multiplying terms and simplifying, the answer is $\boxed{-8}$.

Note: With these types of problems, the final expression can not contain a or b since there is only one correct answer. Thus, another way to solve this is to plug in values for a and b and compute the final expression. This is often faster, provided that you choose a and b that would be easy to work with (say 1 and 2).

Problem 3.5: (AMC 10)

David drives from his home to the airport to catch a flight. He drives 35 miles in the first hour, but realizes that he will be 1 hour late if he continues at this speed. He increases his speed by 15 miles per hour for the rest of the way to the airport and arrives 30 minutes early. How many miles is the airport from his home? *Hint: 3*

Solution 3.5: Note that David drives at 50 miles per hour after the first hour and continues doing so until he arrives. Let d be the distance still needed to travel after the first 1 hour. If t is the amount of time until his flight, we get the equations $t=\dfrac{d}{50}+0.5$ and $t=\dfrac{d}{35}-1$. For the first equation, it will take $\frac{d}{50}$ hours to travel d miles at 50 miles per hour, and David will have an extra 0.5 hours. For the second equation, it will take $\frac{d}{35}$ hours to travel d miles at 35 miles per hour, and David will be 1 hour late.

Setting these equal and multiplying out denominators, we get $7d+525=10d$, or $d=175$. Now, we must add the extra 35 miles traveled in the first hour, giving a total of $\boxed{210}$ miles.

4 Counting & Probability

Problem 4.1: (AMC 10)

Jo and Blair take turns counting from 1 to one more than the last number said by the other person. Jo starts by saying "1", so Blair follows by saying "1, 2". Jo then says "1, 2, 3", and so on. What is the 53$^{\text{rd}}$ number said?

Solution 4.1: The number of numbers said is incremented by one each turn; that is, Jo says one number, then Blair says two numbers, then Jo says three numbers, etc. Thus, after nine turns, $1+2+3+4+5+6+7+8+9=45$ numbers have been said. In the tenth turn, the eighth number will be the 53rd number said, because $53-45=8$. Since we are starting from 1 every turn, the 53rd number said will be $\boxed{8}$.

Problem 4.2: (AMC 10)

A set of three points is randomly chosen from the grid shown. Each three point set has the same probability of being chosen. What is the probability that the points lie on the same straight line?

Solution 4.2: There are $\binom{9}{3} = \frac{9\cdot 87}{3\cdot 2\cdot 1} = 84$ ways to choose three points out of the 9 there. There are 8 combinations of dots such that they lie in a straight line: three vertical, three horizontal, and the diagonals. $\frac{8}{84} = \boxed{\frac{2}{21}}$.

Problem 4.3: (Serena An)

Serena selects 3 positive integers less than or equal to 10. What is the probability that their product is odd?

Solution 4.3: In order for the product to be odd, all 3 integers must be odd. There are 5 odd integers less than or equal to 10, and $\binom{5}{3} = \frac{5\cdot 4\cdot 3}{3!} = 10$ ways to choose 3 of them. There are $\binom{10}{3} = \frac{10\cdot 9\cdot 8}{3!} = 120$ ways to choose 3 integers out of 10. The probability of an odd product is $\frac{10}{120} = \boxed{\frac{1}{12}}$.

Problem 4.4: (AMC 10)

How many three-digit numbers satisfy the property that the middle digit is the average of the first and last digits? *Hint: 4*

Solution 4.4: The middle digit is uniquely defined by the first and third digits since it is half of their sum. In order for this to be possible, the sum of the first and third digits must be even. Since even numbers are formed either by adding two odd numbers or two even numbers, we can split our problem into 2 cases:

If both the first digit and the last digit are odd, then we have 1, 3, 5, 7, or 9 as choices for each of these digits, and there are $5 \cdot 5 = 25$ numbers in this case.

If both the first and last digits are even, then we have 2, 4, 6, 8 as our choices for the first digit and 0, 2, 4, 6, 8 for the third digit. There are $4 \cdot 5 = 20$ numbers here.

The total number of three-digit numbers is $20 + 25 = \boxed{45}$.

Problem 4.5: (MPfG)

Say that a 4-digit positive integer is mixed if it has 4 distinct digits, its leftmost digit is neither the biggest nor the smallest of the 4 digits, and its rightmost digit is not the smallest of the 4 digits. For example, 2013 is mixed. How many 4-digit positive integers are mixed?

Solution 4.5: The number of sets of 4 digits is $\binom{10}{4} = \frac{10\cdot9\cdot8\cdot7}{4!} = 210$. Consider a set of four distinct digits. The leftmost digit is neither the biggest nor the smallest, so there are 2 choices for it. (We don't have to worry about starting with a leading zero, because the leftmost digit is not the smallest.) Given that choice, the rightmost digit is neither the smallest nor the leftmost, so there are 2 choices for it. Given those choices, the second digit from the left is neither the leftmost nor the rightmost, so there are 2 choices for it. Finally, given those choices, the third digit from the left has only 1 choice, the remaining digit. Hence, given a set of 4 digits, there are $2 \cdot 2 \cdot 2 \cdot 1 = 8$ ways to order the digits to form a mixed integer. So the number of mixed integers is $\binom{10}{4} \cdot 8$, which is $210 \cdot 8$, or $\boxed{1680}$.

5 Number Theory

Problem 5.1: (AMC 10)

Three positive integers are each greater than 1, have a product of 27000, and are pairwise relatively prime. What is their sum? ("Pairwise relatively prime" means that any two of them share no common divisors besides 1.)

Solution 7.2: The prime factorization of 27000 is $2^3 \cdot 3^3 \cdot 5^3$. $2^3 = 8$, $3^3 = 27$, and $5^3 = 125$ are three factors that are pairwise relatively prime. Their sum is $8 + 27 + 125 = \boxed{160}$.

Problem 5.2: (MPfG)

Find the smallest two-digit positive integer that is a divisor of 201020112012.

Solution 5.2: 10 does not divide the number. 11 does not divide the number because $2+1+2+1+2+1-(0+0+0+1+0+2) = 6$ is not divisible by 11 (this is the alternating sum test for 11). 12 does divide the number because 4 divides it (last two digits are divisible by 4) and 3 divides it (sum of digits is divisible by 3). The answer is $\boxed{12}$.

Let me know if you are unfamiliar with divisibility rules. They are quick to learn and very useful.

Problem 5.3: (MPfG)

If m and n are integers such that $3m + 4n = 100$, what is the smallest possible value of $|m - n|$?

Solution 5.3: One pair (m, n) that satisfies the constraints is $(0, 25)$. We get other pairs by repeatedly adding 4 to m and subtracting 3 from n. So the list of pairs is ...,$(0, 25)$, $(4, 22)$, $(8, 19)$, $(12, 16)$, $(16, 13)$, $(20, 10)$, ... The corresponding difference $m - n$ is ..., -25, -18, -11, -4, 3, 10, ... As we see, the differences go up by 7. The smallest absolute difference is $\boxed{3}$.

Problem 5.4: (MATHCOUNTS)

How many integers n satisfy the condition $100 < n < 200$ and the condition n has the same remainder whether it is divided by 6 or 8?

Solution 5.4: n must leave a remainder of 0, 1, 2, 3, 4, or 5 when divided by both 6 and 8 (note that its remainder can't be 6 or 7, because although those are achievable when dividing by 8, those remainders are not achievable when dividing by 6). The least common multiple of 6 and 8 is 24. Thus, n must leave a remainder of 0, 1, 2, 3, 4, or 5 when divided by 24. The integers in the range $[101, 199]$ (brackets denote a range including the endpoints, which are 101 and 199 in this case) satisfying that are 101, $[120, 125]$, $[144, 149]$, $[168, 173]$, and $[192, 197]$, for a total of $1 + 4 \cdot 6 = \boxed{25}$ integers.

Problem 5.5: (Serena An)

Let $S(n)$ equal the sum of the digits of n. Let $S^x(n)$ equal $S(S(...S(n)...))$, where the number of Ss is x. Evaluate $S^{100}(2^{2018})$. *Hints: 5, 6*

Solution 5.5: Note that $S(n)$ is significantly smaller than n for larger n. Also, $S(n) = n$ for one digit numbers. Thus, the end result of 100 applications of S will be a one digit number.

$S(n)$ and n leave the same remainder when divided by 9. If $n = a_i 10^i + a_{i-1}10^{i+1} + ... + a_1 10^1 + a_0$, the remainder it leaves when divided by 9 is the same as the remainder $n - (a_i(10^i - 1) + a_{i-1}(10^{i-1} - 1) + ... + a_i(10^1 - 1))$ because $10^j - 1 = 99...9$ is divisible by 9. This expression is exactly $a_i + a_{i-1} + ... + a_1 + a_0 = S(n)$.

Thus, we evaluate the remainder of 2^{2018} when divided by 9. After repeatedly multiplying by 2 a few times, we see that the remainders cycle every six terms: 2, 4, 8, 7, 5, 1, 2, 4, etc. 2018 leaves a remainder of 2 when divided by 6, so the 2018th term in this sequence is equal to the second term, which is $\boxed{4}$.

6 Geometry

Problem 6.1: (AMC 8)

In rectangle $ABCD$, $AB = 6$ and $AD = 8$. Point M is the midpoint of $\overline{AD}$. What is the area of $\triangle AMC$?

Solution 6.1: Use the triangle area formula for triangles: $A = \frac{bh}{2}$, where A is the area, b is the base, and h is the height. This equation gives us $A = \frac{4 \cdot 6}{2} = \frac{24}{2} = \boxed{12}$.

Problem 6.2: (AMC 10)

The two legs of a right triangle, which are altitudes, have lengths $2\sqrt{3}$ and 6. How long is the third altitude of the triangle?

Solution 6.2: The area of the triangle is $\frac{1}{2} \cdot 2\sqrt{3} \cdot 6 = 6\sqrt{3}$. By the Pythagorean Theorem, the length of the hypotenuse is $\sqrt{(2\sqrt{3})^2 + 6^2} = \sqrt{48} = 4\sqrt{3}$. Dropping the altitude from the right angle to the hypotenuse, we can calculate the area in another way.

Let h be the length of the third altitude of the triangle. We have $4\sqrt{3}h = 2\sqrt{3} \times 6 = 12\sqrt{3} \implies h = \boxed{\textbf{(C) } 3}$

Problem 6.3: (MATHCOUNTS)

Circles O and P are tangent to the x-axis at $(4, 0)$ and $(13, 0)$, respectively. respectively. Point $A(4, 8)$ is on circle O and point $B(13, 10)$ in on circle P. Line segment OP connects the center of circle O to the center of circle P what is the slops of segment OP? Express your answer as a common fraction.

Solution 6.3: Since the circle O is tangent to the x-axis at $(4, 0)$, and $(4, 8)$ is on the circle, the center of circle O is the midpoint $(4, 4)$. Similarly, the center of circle P is the midpoint of $(13, 0)$ and $(13, 10)$, which is $(13, 5)$. The slope of the line connecting these two points is $\frac{5-4}{13-4} = \boxed{\frac{1}{9}}$.

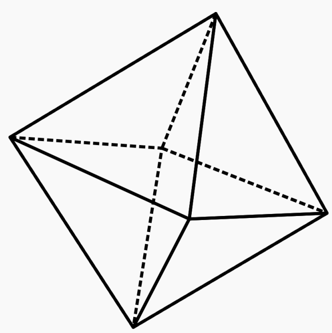
Problem 6.4: (AMC 10)

The y-intercepts, P and Q, of two perpendicular lines intersecting at the point $A(6, 8)$ have a sum of zero. What is the area of $\triangle APQ$? *Hint: 7*

Solution 6.4: Let O be the origin. Since the y-intercepts sum to 0, O is the midpoint of PQ. Because APQ is a right triangle, $OP = OQ = OA$. By the Pythagorean Theorem, OA is $\sqrt{6^2 + 8^2} = 10$, and so $PQ = 20$. PQ is on the y-axis. The distance from A to the y-axis is 6, so the area of $\triangle APQ$ is $\frac{20 \cdot 6}{2} = \boxed{60}$.

Problem 6.5: (Serena An)

The sides of an octahedron are all 1. What is the volume of the octahedron?



(An octahedron is an 12-sided 3D figure. It is two square pyramids glued together at the bases.)

Solution 6.5: We will find the volume of one square pyramid and then multiply by 2 for the volume of the octahedron. The base of the square pyramid has area 1. Let the apex of the square pyramid be A, the center of the square base be B, and a vertex of the square base be C. Then $\triangle ABC$ is right with $\angle B = 90°$ and AB is the height of the pyramid (which is what we want to solve for). The diagonal of a square with side length 1 is $\sqrt{1^2 + 1^2} = \sqrt{2}$, and since BC is half of the diagonal, $BC = \frac{\sqrt{2}}{2}$. $AC = 1$, as it is a side of the square pyramid. By the Pythagorean Theorem, $AB = \sqrt{AC^2 - BC^2} = \sqrt{1 - \frac{1}{2}} = \frac{\sqrt{2}}{2}$. The volume of the square pyramid is thus $\frac{1 \cdot \frac{\sqrt{2}}{2}}{3} = \frac{\sqrt{2}}{6}$. The volume of the octahedron is $\boxed{\frac{\sqrt{2}}{3}}$.

7 Summer Challenge #4

Problem 7.1: (Serena An)

For what two digit number N is N equal to 8 times the sum of its digits?

Solution 7.1: Let $N = ab$, where a and b are the tens and ones digits, respectively. Then from the problem, we get the equation $10a + b = 8(a + b)$. Simplifying, $2a = 7b$. Since 7 must divide the left side and a is a one digit number, $a = 7$, and so $b = 2$. Thus, the number is $\boxed{72}$.

Problem 7.2: (Serena An)

There are 20 acts in a talent show and each ordering of acts is equally likely. What is the probability that the didgeridoo act is before the piano violin duet and the comedy skit?

Solution 7.2: Focus on just the three acts. In an arbitrary ordering, one must be first, and the probability of each being first is the same. Thus, the probability of the didgeridoo act being before the other two is $\boxed{\dfrac{1}{3}}$.

Problem 7.3: (Serena An)

A plane has a constant speed p and flies from Seattle to Boston, and then from Boston back to Seattle. Assume there is wind of a positive, constant speed w from Seattle to Boston throughout the trip, so the plane's speed from Seattle to Boston is $p + w$ and from Boston to Seattle is $p - w$. Is the total flight time with wind more or less than if there was no wind?

Solution 7.3: The answer is $\boxed{\text{more}}$. We can show this using logical reasoning or algebra.

Logical Reasoning: Whether the answer is more or less should hold true for all constants w. Suppose the wind speed is very very close to the plane speed. Then the way there may be fast, but the way back will be extremely slow. Thus, the total flight time with wind is more than with no wind.

Algebra: If x is the distance between Seattle and Boston, the times it takes the plane with and without wind are $\frac{x}{p+w} + \frac{x}{p-w} = \frac{2px}{(p+w)(p-w)}$ and $\frac{x}{p} + \frac{x}{p} = \frac{2x}{p} = \frac{2px}{p^2}$. Since $p^2 > (p + w)(p - w) = p^2 - w^2$, $\frac{2px}{p^2} < \frac{2px}{(p+w)(p-w)}$ and the total flight time is more with wind.

Problem 7.4: (Serena An)

Triangle ABC has area 80. The midpoints of AB, BC, and CA are C_1, A_1, and B_1, respectively. The midpoints of A_1B_1, B_1C_1, and C_1A_1 are C_2, A_2, and B_2, respectively. What is the area of triangle $A_2B_2C_2$?

Solution 7.4: Due to midpoints, $A_1B_1 = \frac{1}{2}AB$, $B_1C_1 = \frac{1}{2}BC$, and $C_1A_1 = \frac{1}{2}CA$. Because the side lengths of $A_1B_1C_1$ and ABC are in a $\frac{1}{2}$ ratio, the areas of $A_1B_1C_1$ and ABC are in a $(\frac{1}{2})^2 = \frac{1}{4}$ ratio.

Similarly, the areas of $A_2B_2C_2$ and $A_1B_1C_1$ are in a $\frac{1}{4}$ ratio, so the areas of $A_2B_2C_2$ and ABC are in a $\frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$ ratio. Thus, the area of $A_2B_2C_2$ is $\frac{80}{16} = \boxed{5}$.

Problem 7.5: (Serena An)

(a) How many ways are there to color the vertices of a hexagon using two colors if rotations and reflections are considered distinct?

(b) How many ways are there to color the vertices of an n-gon using two colors if rotations and reflections are considered distinct? Express your answer in terms of n.

(c) How many ways are there to color the vertices of a hexagon using two colors if rotations are *not* considered distinct and reflections are considered distinct?

Reminder: Write a full solution to this problem, not just the answer. Even if you do not have a full solution, or a solution for every part, type/write up what you have and you may receive partial credit!

Solution 7.5:

(a) There are 2 ways to color each of the six vertices, so there are $2^6 = \boxed{64}$ ways.

(b) Generalizing, there are 2 ways to color each of the n vertices, so there are $\boxed{2^n}$ ways.

(c) Suppose our colors are red and blue. We do casework based on the number of red vertices. If there are 6 red vertices, there is 1 way to color. If there are 5 red vertices, there is 1 way to color (rotations are considered the same). If there are 4 red vertices, the two blue vertices can be adjacent, 1 red vertex apart, or 2 red vertices apart for a total of 3 ways. If there are 3 red vertices, casework shows that there are 4 ways (rrrbbb, rrbrbb, rrbbrb, rbrbrb). For the cases of 2, 1, and 0 red vertices, we mirror the cases of 2, 1, and 0 blue vertices (which correspond to 4, 5, and 6 red vertices) for 3, 1, and 1 ways to color. The total number of colorings is $1 + 1 + 3 + 4 + 3 + 1 + 1 = \boxed{14}$.

Problem 7.6:

Write your own problem from any subject area and of any difficulty! Problems should be fun to solve! Don't expect to be able to instantly come up with good problems; it will typically take awhile and some thought. Submitting one problem will earn **3** points and submitting two problems will earn **6** points. Additional submissions will not earn additional points, although they are welcomed! You may submit as many problems as you'd like to serena.an@gmail.com. These problems may appear on future problem sets, and possibly even a BMC created mock competition (such as an AMC 8)! There will be more discussion of this in class.

8 Hints to Selected Problems

1. Let the width of the rectangle be w. Use the Pythagorean Theorem.

2. Remember that $a^2 - b^2 = (a+b)(a-b)$. Clear denominators and simplify carefully.

3. Let d be the distance still needed to travel after the first 1 hour. If t is the amount of time until his flight after the first hour, write two equations for t in terms of d.

4. The middle digit is determined by the first and last digits, if the sum of the first and last digits is even.

5. $S(n)$ is significantly smaller than n. When does $S(n) = n$?

6. If you apply $S(n)$ enough times, you will end up with a one digit number. Which one? Experiment with some n and see if you can find the pattern.

7. The distances from the midpoint of the hypotenuse to all three vertices of a right triangle are all equal.