Challenge #6 - Arithmetic

Serena An, Jan Park, and Divya Shyamal–August 31, 2018

1 Introduction

Welcome to the Brookings Math Circle Summer 2018 Problem Sets! **Submissions for this problem set are due by Friday 9/7 at 8:00 pm (the day *before* the fourth BMC class of the second session).** Solutions will be released online at our website at 8:00 pm Friday. For a refresher on submission details or problem set details, see the introduction PDF linked to the Summer Challenge page of our website under Classes.

This handout will teach arithmetic in different bases. Solutions to problems in the handout are at the end.

As always, there will also be a puzzle and summer challenge problems, with rules as usual. Keep calm and do the math!

2 Base Conversion Review

Exercise 2.1

Convert 724_8 to base 10.

Solution 3.1: $724_8 = 7 \cdot 8^2 + 2 \cdot 8 + 4 = \boxed{468_{10}}$.

Exercise 2.2

Convert 1580_{10} to base 12.

Solution 6.2: From the first handout, we know that we first must find the "expansion" of this number in powers of 12. Omitting the arithmetic, we have that $1580 = 10 \cdot 12^2 + 11 \cdot 12^1 + 8 \cdot 12^0$. But we run into a problem: 10 and 11 are not single digits! To remedy this, we will use letters. $10 = A$, $11 = B$, $12 = C$, and so on. Thus, the final answer is $\boxed{AB8}$.

Problem 2.1: (Divya Shyamal)

Convert 565_7 to a base 5 number.

Problem 2.2: (Divya Shyamal)

How many 5-digit base 3 numbers are there? (For example, one of them is 20001, 01001 while does not work).

Problem 2.3: (Divya Shyamal)

How many 3-digit numbers (in base 10) are 5-digit numbers when written in base 5?

3 Addition in Different Bases

Addition and subtraction in different bases uses the same processes as addition and subtraction in base 10. For the following theorems, assume that we are in base b.

Theorem 3.1: Addition in Different Bases

Begin by adding the ones digits. If the sum is less than or equal to $b-1$, write that digit in the ones place. If the sum is greater than or equal to b, then in base b, the sum would be of the form $\overline{1b_0}$ for some digit $0 \leq b_0 \leq b-1$. If this is the case, carry a 1 above the bs place and write b_0 in the ones place. Continue this process with the bs place, b^2s place, and so on.

Exercise 3.1

Calculate $531_8 + 246_8$.

Solution 3.1: We add the numbers just like how we would add the numbers if they were in base 10. There is no carrying in this problem because the numbers do not sum to more than 7. Our final answer is $\boxed{777_8}$.

$$\begin{array}{cccc} & 5 & 3 & 1 \\ + & 2 & 4 & 6 \\ \hline & 7 & 7 & 7 \end{array}$$

Exercise 3.2

Calculate $223_6 + 415_6$.

Solution 6.2: All numbers in this solution are in base 6. We start by adding the ones digits. $3+5=12$ Carrying the 1, the sixes digit is $1+2+1=4$. Adding the thirty-sixes digits, $2+4=10$. Our final answer is $\boxed{1042_6}$.

$$\begin{array}{cccc} & 2 & 2 & 3 \\ + & 4 & 1 & 5 \\ \hline 1 & 0 & 4 & 2 \end{array}$$

Problem 3.1

Add by first converting the numbers to base 10, adding in base 10, and then converting back to the original base. Then do the addition without converting the numbers to base 10.

(a) $240_5 + 113_5$

(b) $A17_{12} + 46_{12}$

Problem 3.2

Add without converting the numbers to base 10. If you wish, check your answers by converting to base 10.

(a) $12_4 + 32_4$

(b) $806_9 + 146_9$

(c) $928_{16} + 798_{16}$

(d) $6655_7 + 443_7 + 322_7 + 11_7$

4 Subtraction in Different Bases

Theorem 4.1: Subtraction in Different Bases

Suppose we are in base b. Let M and N be the numbers that we wish to use to calculate $M - N$ and let m and n be their respective ones digits. Begin by subtracting the n from m. If $m \geq n$, write the difference $m - n$ in the ones place. If $m < n$, then subtract 1 from M's bs place and write the value of $\overline{1m_b} - n_b$ in the ones place. Continue this process with the bs place, b^2s place, and so on.

Exercise 4.1

Calculate $744_8 - 534_8$.

Solution 4.1: We subtract the numbers just like how we would subtract the numbers in base 10. There is no "borrowing" of digits needed because in each place value, the digit of 744_8 is at least the digit of 534_8. Our final answer is $\boxed{210_8}$.

$$\begin{array}{cccc} & 7 & 4 & 4 \\ - & 5 & 3 & 4 \\ \hline & 2 & 1 & 0 \end{array}$$

Exercise 4.2

Calculate $503_6 - 214_6$.

Solution 4.2: All numbers in this solution are in base 6. Looking at the ones digits, because $3 < 4$, we must "borrow" a 1 from the sixes place of 503_6. However, the sixes place is 0, so "borrowing" a 1 leads to 4 and 5 in the thirty-sixes and sixes places respectively. Then $13 - 4 = 5$. Moving on to the sixes and thirty-sixes places, we can subtract with no issues: $5 - 1 = 4$ and $4 - 2 = 2$. The final answer is $\boxed{245_6}$.

$$\begin{array}{cccc} & 5 & 0 & 3 \\ - & 2 & 1 & 4 \\ \hline & 2 & 4 & 5 \end{array}$$

Problem 4.1

Subtract by first converting the numbers to base 10, subtracting in base 10, and then converting back to the original base. Then do the subtraction without converting the numbers to base 10.

(a) $726_9 - 513_9$

(b) $410_5 - 32_5$

Problem 4.2

Subtract without converting the numbers to base 10. Check your work with addition.

(a) $10011_2 - 1101_2$

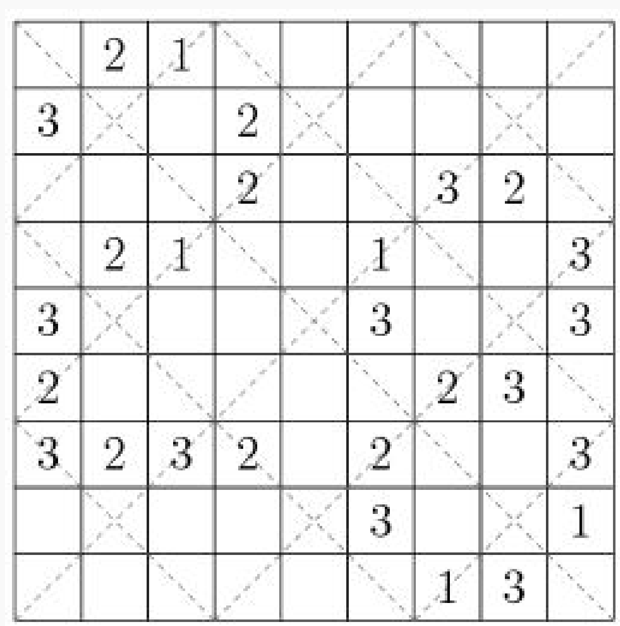
(b) $823_9 - 245_9$

(c) $18A_{12} - BA_{12}$

(d) $1CBA_{16} - DAC_{16}$

5 Puzzle #6

Problem 5.1: (USAMTS)



Fill in each cell of the grid with one of the numbers 1, 2, or 3. After all numbers are filled in, if a row, column, or any diagonal has a number of cells equal to a multiple of 3, then it must have the same amount of 1's, 2's, and 3's. (There are 10 such diagonals, and they are all marked in the grid by a gray dashed line.) Some numbers have been given to you. There is only one solution.

6 Summer Challenge #6

Problem 6.1: (Jan Park)

If a is the Least Common Multiple of $(48, 112)$ and b is the Greatest Common Divisor of $(48, 112)$, find $a - 2b$

Problem 6.2: (Jan Park)

Let $x\#y = x^2 + \frac{y}{x}$ for all positive integers x, y. Find $3\#(4\#8)$.

Problem 6.3: (Jan Park)

How many ways are there to arrange the 26 letters of the alphabet so that the exact substring $EMILY$ is included in the arrangement? (Write in simplest terms of factorials.)

Problem 6.4: (Jan Park)

How many ways are there to distribute 6 candies between Alice, Elena, Rachel, and Susan, where the candies are indistinguishable?

Problem 6.5: (Serena An)

Alison has an analog clock whose hands have the following lengths: 6 inches (the hour hand), 9 inches (the minute hand), and 12 inches (the second hand). How far does the tip of each hand travel in 1 day?
(Based off of MPfG 2012)

Reminder: Write a full solution to this problem, not just the answer. Even if you do not have a full solution, or a solution for every part, type/write up what you have and you may receive partial credit!

7 Solutions

Solution 2.1: We will first convert 565_7 to base 10 and then convert that number to base 5. $565_7 = 5 \cdot 7^2 + 6 \cdot 7 + 5 = 292_{10} = 2 \cdot 5^3 + 1 \cdot 5^2 + 3 \cdot 5 + 2 = \boxed{2132_5}$.

Solution 2.2: The only digits allowed in base 3 are 0, 1, and 2. The left most digit can be 1 or 2. The other 4 digits can be 0, 1, or 2. Thus, the number of 5-digit base 3 numbers is $2 \cdot 3^4 = \boxed{162}$.

Solution 2.3: The smallest 5-digit base 5 integer is $10000 = 625_{10}$ and the largest 5-digit base 5 integer is $44444_5 = 3124$. The 5-digit base 5 integers that are 3-digit base 10 numbers are the numbers between 625 and 999 (inclusive) for $\boxed{375}$ numbers.

Solution 3.1:

(a) $240_5 = 70_{10}$ and $113_5 = 35_{10}$ so $240_5 + 113_5 = 70_{10} + 33_{10} = 103_{10} = 4 \cdot 5^2 + 0 \cdot 5^1 + 3 \cdot 5^0 = \boxed{403_5}$. Now we will add without converting to base 10. Adding the ones digits, $0_5 + 3_5 = 3_5$. Adding the fives digits, $4_5 + 1_5 = 10_5$ so we write a 0 in the fives place and carry a 1 to the twenty-fives place. Adding the twenty-fives digits, we get $1_5 + 2_5 + 1_5 = 4_5$. The final answer is $\boxed{403_5}$.

(b) $A17_{12} = 10 \cdot 12^2 + 1 \cdot 12 + 7 = 1459_{10}$ and $46_{12} = 4 \cdot 12 + 6 = 54_{10}$ so $A17_{12} + 46_{12} = 1459_{10} + 54_{10} = 1513_10 = 10 \cdot 12^2 + 6 \cdot 12 + 1 = \boxed{A61_{12}}$. Now we will add without converting to base 10. Adding the ones digits, $6_{12} + 7_{12} = 11_{12}$ so we write a 1 in the ones place and carry a 1 to the twelves place. Adding the twelves digits, $1_{12} + 1_{12} + 4_{12} = 6_{12}$. The one hundred forty-fours digit is just A, so our final answer is $\boxed{A61_{12}}$.

Solution 3.2:

(a) 100_4

(b) 1053_9

(c) $10C0_{16}$

(d) 11064_7

If you would like to see the detailed steps for any of these, please email Serena!

Solution 4.1:

(a) $726_9 = 7 \cdot 9^2 + 2 \cdot 9 + 6 = 591_{10}$ and $513_9 = 5 \cdot 9^2 + 1 \cdot 9 + 3 = 417_{10}$ so $726_9 - 513_9 = 591_{10} - 417_{10} = 174_{10} = 2 \cdot 9^2 + 1 \cdot 9 + 3 = \boxed{213_9}$.

(b) $410_5 = 4 \cdot 5^2 + 1 \cdot 5 = 105_{10}$ and $32_5 = 3 \cdot 5 + 2 = 17_{10}$, so $410_5 - 32_5 = 105_{10} - 17_{10} = 88_{10} = 3 \cdot 5^2 + 2 \cdot 5 + 3 = \boxed{323_5}$.

Solution 4.2:

(a) 110_2

(b) 567_9

(c) 90_{12}

(d) $F0E_{16}$

Once again, if you would like to see the detailed steps for any of these, please email Serena!