Equations and Graphing

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1 Introduction

This handout is an improved version of parts of the BMC 2018 handouts **Basic Algebra** and **Graphing**. Both handouts (with and without solutions) can be found at `https://www.brookingsmathcircle.org/more/archive/2018/summer-challenge`.

2 Single Variable Equations

Variables are unknown quantities that are typically represented by lowercase letters. A fundamental skill for contest math is the ability to write and solve equations.

Exercise 2.1: (CMMS)

Angela is three years older than twice her brother Thomas' age. If Angela is 17, how old is Thomas?

Exercise 2.2

Solve for each variable.

1. $4x - 17 = 7$
2. $5a - 7 = 8 - a$
3. $\frac{z+5}{2} = 4$
4. $n^2 - 4 = 2n^2 - 13$

Exercise 2.3

The sum of 5 consecutive integers is 120. What is the largest of these integers?

Solution 2.3: The mean of the 5 integers is $\frac{120}{5} = 24$, so that must be the middle integer. Finally, we add 2 to get $\boxed{26}$ as our answer.

Exercise 2.4: (AMC 8)

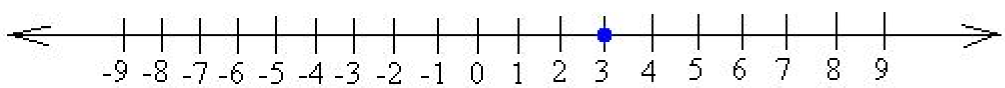
Four students take an exam. Three of their scores are $70, 80,$ and 90. If the average of their four scores is 70, then what is the remaining score?

Solution 2.4: Let x be the fourth score. Since the average is 70, the sum of the four scores is $4 \cdot 70 = 280$. Since $x + 70 + 80 + 90 = 280$, $x = \boxed{40}$.

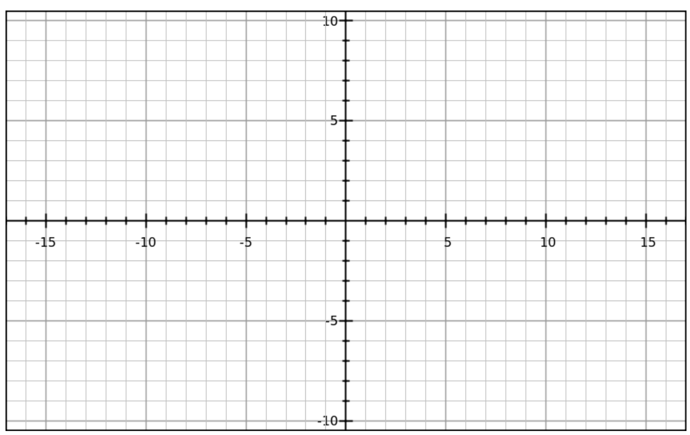
A way to do this in your head is to think about the difference between the first three scores and the average. 70 is 0 away, 80 is 10 above, and 90 is 20 above, so the remaining score must be 30 below the average, or $\boxed{40}$

3 Graphing

Imagine a number line. We can do some things on this number line; for example, if we have the equation $x = 3$, we can represent x by a dot on the number line.



What if we want to have an equation with two variables? To visually represent what's going on, let's stack two number lines. We'll call the entire region a **graph**. The horizontal axis is called the x **axis**, and the vertical axis is called the y **axis**.



Head over to `desmos.com/calculator`.

Problem 3.1

Graph points (x, y) that satisfy the following equations. Any observations?

1. $y = 2x + 5$
2. $y = -\frac{1}{2}x + 5$
3. $y = 3x - 1$
4. $x = 2$

4 Problems

Problem 4.1:

The sum of 10 consecutive integers is 25. What is the smallest of these integers?

Solution 4.1: The mean of these integers is $\frac{25}{10} = \frac{5}{2} = 2.5$. Because there are 10 consecutive integers, 5 integers are less than 2.5 and 5 integers are greater than 2.5. The middle-most two integers must be 2 and 3. Then the smallest integer is 4 less than 2, or $\boxed{-2}$.

Problem 4.2: (CMMS)

The height of a rectangle is three centimeters more than twice its length. If the perimeter of the rectangle is 60 cm, what is its area?

Solution 4.2: Let l be the length of the rectangle. Then the height is $2l + 3$. Solving for l in the perimeter equation $2l + 2(2l + 3) = 60$ gives $l = 9$. Thus, the height is $2 \cdot 9 + 3 = 21$. The area is therefore $9 \cdot 21 = \boxed{189 \text{ cm}^2}$.

Problem 4.3: (CMMS)

Molly's father James is three years less than three times her age. How many years from now will Molly's father be twice her age if he is 33 today?

Solution 4.3: Let m be Molly's current age. It is given that $3m - 3 = 33$, so $m = 12$. The difference between Molly and James's ages is 21. Thus, James will be twice Molly's age when Molly is 21 and James is 42. This will occur in $\boxed{9}$ years.

Problem 4.4: (AMC 8)

If $3^p + 3^4 = 90$, $2^r + 44 = 76$, and $5^3 + 6^s = 1421$, what is the product of p, r, and s?

Solution 4.4: We solve for each variable individually. Since $3^p + 3^4 = 90$, $3^p = 9$ and $p = 2$. Since $2^r + 44 = 76$, $2^r = 32$ and $r = 5$. Since $5^3 + 6^s = 1421$, $6^s = 1296$ and $s = 4$. Thus, the product of p, r, and s is $2 \cdot 5 \cdot 4 = \boxed{40}$.

Problem 4.5:

The sum of Ann, Ben, and Cathy's ages is 35. If Ben is 18 years older than half of Ann's age and Cathy is 3 years older than twice Ann's age, find the sum of Ben and Cathy's ages.

Solution 4.5: Let a be Ann's age. Then Ben's age is $\frac{1}{2}a + 18$ and Cathy's age is $2a + 3$. The sum of the three ages is 35, so $a + \frac{1}{2}a + 18 + 2a + 3 = \frac{7}{2}a + 21 = 35$. Then $\frac{7}{2}a = 14$ so $a = 4$. Don't stop here; remember what the problem is asking! The sum of Ben in Cathy's ages is $35 - a = \boxed{31}$. Alternatively, you could find Ben's age to be 20 and Cathy's age to be 11, and then sum to get $\boxed{31}$, although it is a bit more work.