Counting and Probability

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1 Counting

Exercise 1.1

A chapter starts on page 47 and ends on page 65. How many pages long is the chapter?

Solution 1.1: $(65 - 47) + 1 = \boxed{19}$

Exercise 1.2

How many multiples of 4 are greater than 100 and less than 200?

Solution 1.2: We want to count multiples of 4 from 104 to 196 (inclusive). Dividing by 4, we get the numbers 26 to 49, and there are $(49 - 26) + 1 = \boxed{24}$ numbers.

Exercise 1.3: (CMMS)

Jonathan starts counting at 130 and counts by fives. What is the 13th number that Jonathan says?

Solution 1.3: Starting at 130, we add 5 once to get the 2nd number, twice to get the 3rd number, and so on. We add 5 twelve times to get the 13th number, which is $130 + 5 \cdot 12 = \boxed{190}$.

When choosing k items, in order, from a group of n (without replacement), we have

$$n(n-1)(n-2)\cdots(n-k+1)$$

choices. This is called a **permutation**.

Exercise 1.4

If you have 6 shirts and 5 ties, how many shirt-and-tie outfits can you make?

Solution 1.4: $6 \cdot 5 = \boxed{30}$.

Exercise 1.5

There are 8 sprinters in the Olympic 100-meter finals. The gold medal goes to first place, silver to second, and bronze to third. How many ways can the medals be awarded?

Solution 1.5: There are 8 choices for first place, 7 choices for second place (after first has been chosen), and 6 choices for third place (after first and second have been chosen). In all, there are $8 \cdot 7 \cdot 6 = \boxed{336}$

ways to award the medals.

Exercise 1.6: (Alcumus)

How many distinct arrangements of the letters in the word "monkey"' are there?

Solution 1.6: Since all letters are distinct, we have 6 choices for the first letter, 5 choices for the second letter, and so on until we have 1 choice for the sixth letter. There are $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = \boxed{720}$ distinct arrangements.

2 Probability

Consider a set of (finite) possible outcomes. If all outcomes are equally likely, then the probability that A occurs is

$$P(A) = \frac{\text{number of ways A can occur}}{\text{total number of outcomes}}.$$

Exercise 2.1

If a fair coin is flipped once, what is the probability that it lands on heads?

Solution 2.1: There are two equally likely possibilities for the result of the coin flip: heads or tails. The probability is $\boxed{\frac{1}{2}}$.

Exercise 2.2

If a fair coin is flipped twice, what is the probability that exactly one head is flipped?

Solution 2.2: There are four equally likely possibilities for two coin flips: HH, HT, TH, TT. Two of these, namely HT and TH, include exactly one head, so the probability is $\frac{2}{4} = \boxed{\frac{1}{2}}$.

Exercise 2.3

Three coins are flipped and the number of heads is recorded. What is the probability that exactly one head is recorded?

Solution 2.3: There are four possibilities for the number of heads recorded: 0, 1, 2, 3. However, these outcomes are not equally likely, so the probability is not simply $\frac{1}{4}$. There are $2^3 = 8$ total possibilities for the outcomes of three coin flips. Out of these, three have exactly one head: HTT, THT, TTH. Thus, the probability is $\boxed{\frac{3}{8}}$.

If two events A and B can happen at the same time, we compute the probability that both happen as

$$P(A \text{ and } B) = P(A) \cdot P(B).$$

Exercise 2.4: (CMMS)

You roll a pair of dice; one red and the other green. What is the probability of rolling a five on the red die and an even number on the green die?

Solution 2.4: There is a $\frac{1}{6}$ probability of rolling a five on the red die and a $\frac{1}{2}$ probability of rolling an even number on the green die. Multiplying, there is a $\boxed{\frac{1}{12}}$ chance that both events occur.

Exercise 2.5: 13

You draw three cards from a standard deck with replacement. What is the probability that the first card is a spade, the second card is an ace, and the third card is the ace of spades?

Solution ??: There is a $\frac{1}{4}$ probability that the first card is a spade, a $\frac{1}{13}$ probability the second card is an ace, and a $\frac{1}{52}$ probability the third card is the ace of spades. Multiplying, the probability is $\boxed{\frac{1}{2704}}$.

3 Simpson's Paradox

Problem 3.1: Simpson's Paradox

(a) Consider a black box with 5 red and 6 green balls and a white box with 3 red and 4 green balls. From which of these two boxes is the probability of drawing a red ball greater?

(b) Consider a second black box with 6 red and 3 green balls and a second white box with 9 red and 5 green balls. From which of these two boxes is the probability of drawing a red ball greater?

(c) Consider adding the contents of the second black box to the first black box and adding the contents of the second white box to the first white box. Now, from which of these two boxes is the probability of drawing a red ball greater?

Solution 3.1:

(a) The probabilities of drawing a red ball in the black and white boxes are $\frac{5}{11} = 0.455$ and $\frac{3}{7} = 0.429$, respectively, so the probability of drawing a red ball is greater in the black box.

(b) The probabilities of drawing a red ball in the second black and second white boxes are $\frac{2}{3} = 0.667$ and $\frac{9}{14} = 0.643$, respectively, so the probability of drawing a red ball is greater in the second black box.

(c) Combining, there are now 11 red and 9 green balls in the black box and 12 red and 9 green balls in the white box. Intuitively, the probability of drawing a red ball from the combined black box should be greater than the probability of drawing a red ball from the combined white box. However, the black and white boxes probabilities are now $\frac{11}{20} = 0.55$ and $\frac{12}{21} = 0.571$, respectively, and thus the probability of drawing a red ball is greater in the combined white box.

This is an example of **Simpson's paradox**. Simpson's paradox occurs when groups of data show one particular trend, but the trend is reversed when the groups of data are combined.

Why does Simpson's paradox occur? One would expect that larger probabilities in each case leads to larger probabilities overall. However, this is only true when the case sizes are equal. In the above example, there are 11 and 7 balls in the first boxes and 9 and 14 balls in the second boxes, which are somewhat large differences in the number of balls.

One real life example is batting averages. David Justice had a higher batting average than Derek Jeter in both 1995 and 1996, but across the two years, Derek Jeter's batting average was higher. Another example is University of California admission rates. A study showed that men were accepted at higher rates than women (44% to 35%), but in each department, women were usually accepted at a rate at least of the men. This was due to more women than men applying to lower admittance departments.