AM-GM

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1 Basics and Definitions

When two expressions are not necessarily equal, we can compare them using inequality signs. The two types of inequality signs are strict ($<$ and $>$) and nonstrict ($\leq$ and $\geq$). If we multiply or divide by a negative number, or take the reciprocal of both sides of an inequality, we "reverse" the inequality sign.

Definition 1.1. The **equality condition** of a nonstrict inequality is when equality holds (the two sides are equal). The equality condition of an inequality can be very useful when solving problems.

2 The Trivial Inequality

Theorem 2.1: Trivial Inequality

If x is a real number, then $x^2 \geq 0$. Equality holds if and only if $x = 0$.

Problem 2.1: (AoPS Intermediate Algebra)

Prove that if x and y are positive, then $\frac{1}{x} + \frac{1}{y} \geq \frac{4}{x+y}$.

3 AM-GM Inequality

The Arithmetic Mean Geometric Mean Inequality states that the arithmetic mean of any n *nonnegative* numbers is greater than or equal to the geometric mean of the numbers.

Theorem 3.1: AM-GM Inequality

If $a_1, a_2, ..., a_n$ are nonnegative, then

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}$$

Equality holds if and only if all a_i are equal.

Warning:

Remember, AM-GM will only work for **nonnegative** numbers.

Problem 3.1: AM-GM 2 Variables

Prove that $\frac{a+b}{2} \geq \sqrt{ab}$ for all $a, b \geq 0$.

Problem 3.2

Show that the sum of a positive number and its reciprocal is always greater than or equal to 2.

Problem 3.3: (AoPS)

Show that $\frac{a^2}{bc} + \frac{b^2}{ca} + \frac{c^2}{ab} \geq 3$.

Problem 3.4: (Mandelbrot)

Determine the minimum value of the sum $\frac{a}{2b} + \frac{b}{4c} + \frac{c}{8a}$, where a, b, and c are positive real numbers.

4 More Problems

Problem 4.1: (AoPS Intermediate Algebra)

Show that if x, y, and z are nonnegative, then $xy + yz + zx \geq x\sqrt{yz} + y\sqrt{zx} + z\sqrt{xy}$.

Problem 4.2: (AoPS Intermediate Algebra)

Show that $(x+y)(y+z)(z+x) \geq 8xyz$ for all nonnegative numbers x, y, and z.

Problem 4.3

Serena has 48 feet of fencing.
(a) What is the maximum area she can enclose?
(b) Can you generalize? If Serena has n feet of fencing, what is the maximum area she can enclose?

Problem 4.4

Samyok also has 48 feet of fencing. He plans to use one side of his house as a side of his enclosure, so he will need to form three sides with his fencing.
(a) What is the maximum area he can enclose?
(b) Can you generalize? If Samyok has n feet of fencing and only needs to form three sides, what is the maximum area he can enclose?