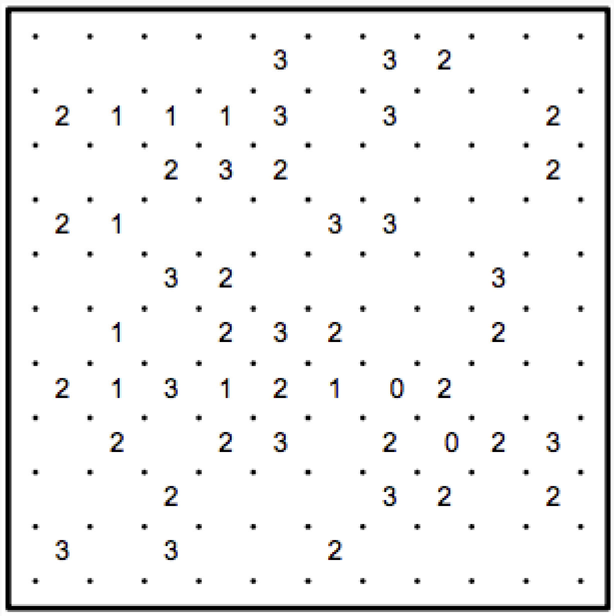
Challenge #5 - Bases

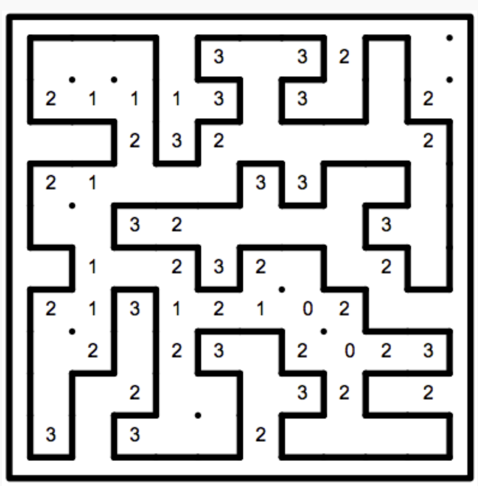
Serena An and Divya Shyamal–August 24, 2018

1 Puzzle #5

Problem 1.1: (krazydad.com)



Solution 1.1:



2 Summer Challenge #5

Problem 2.1: (Serena An)

What is the product of the distinct positive integer factors of 36? Express your answer as a power of 6.

Solution 2.1: You could find all factors of 36 and do the multiplication, but here's a slicker solution. $36 = 2^2 \cdot 3^2$ has $3 \cdot 3 = 9$ factors. (In general, to quickly find the number of factors of a number, write out the prime factorization and add 1 to each exponent and multiply. Email/talk to me about this if it is new; I can explain why it works.) 8 of the factors pair up with another factor to multiply to 36. The last factor is 6. Thus, the product is $36^4 \cdot 6 = \boxed{6^9}$.

Problem 2.2: (Serena An)

Five people of different heights get in a line. What is the probability that the shortest person is immediately behind the tallest person?

Solution 2.2: There are $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ ways to order 5 people in a line. The tallest and shortest people can be next to each other in 4 ways: in spots 1 and 2, spots 2 and 3, spots 3 and 4, and spots 4 and 5. In each of these cases, there are $3! = 3 \cdot 2 \cdot 1 = 6$ ways to rearrange the other 3 people. Thus, there are 24 arrangements satisfying the problem and the answer is $\frac{24}{120} = \boxed{\frac{1}{5}}$.

Another way to think about this problem is that there is a $\frac{4}{5}$ chance that the tallest person is not last in line. Then given that the tallest person is not last in line, there is a $\frac{1}{4}$ probability that the shortest person is immediately behind the tallest person (equal probabilities for the remaining 4 people). The final answer is $\frac{4}{5} \cdot \frac{1}{4} = \boxed{\frac{1}{5}}$.

Problem 2.3: (Serena An)

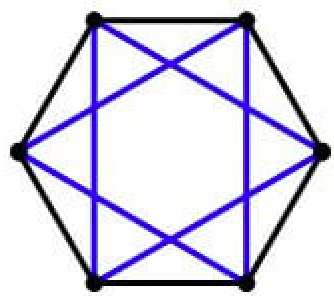
For how many positive values of n are both $\frac{n}{2}$ and $3n$ three digit base 5 integers?

Solution 2.3: The smallest 3-digit base 5 integer is $100_5 = 25$. The largest 3-digit base 5 integer is $444_5 = 624$. The fact that $\frac{n}{2}$ and $3n$ are both 3-digit base 5 integers means that $25 \le \frac{n}{2} \le 624$ and $25 \le 3n \le 624$. The first inequality is equivalent to $50 \le n \le 1248$. The second inequality is equivalent to $8.\overline{33} \le n \le 208$. The n that satisfy all of these conditions is $50 \le n \le 208$. Thus, there are $\boxed{159}$ n such that $\frac{n}{2}$ and $3n$ are 2-digit base 5 integers.

Problem 2.4: (Serena An)

Triangles BMC and FUN are drawn in regular hexagon $BFMUCN$ with side length 6. What is the area of the overlap of the triangles?

Solution 2.4: The triangles are equilateral triangles, and the overlap is a regular hexagon. Using the Pythagorean Theorem (or if you are familiar with 30-60-90 triangles), $BC = \sqrt{BU^2 - CU^2} = \sqrt{12^2 - 6^2} = \sqrt{108} = 6\sqrt{3}$ because the length of the long diagonal is 12 and the length of one side is 6. The length of one side of the hexagon, is $\frac{1}{3}$ of that, or $2\sqrt{3}$. The formula for the area of an equilateral triangle is $\frac{s^2\sqrt{3}}{4}$, so the area of the hexagon is $6 \cdot \frac{(2\sqrt{3})^2\sqrt{3}}{4} = \boxed{18\sqrt{3}}$.



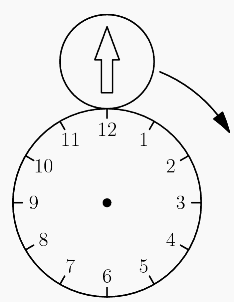
Problem 2.5: (Serena An)

The diagram below shows the circular face of a clock with radius 20 cm and a circular disk with radius 10 cm externally tangent to the clock face at 12 o'clock. The disk has an arrow painted on it, initially pointing in the upward vertical direction. The disk rolls clockwise around the clock face.

(a) At what point on the clock will the disk be tangent to when the arrow is pointing left for the first time?

(b) How many full rotations of the disk are needed to travel rotate around the clock face once?

(c) When the disk rotates completely around the clock face once, how many times is the arrow pointing left?



(Problem idea from AMC 10)

Reminder: Write a full solution to this problem, not just the answer. Even if you do not have a full solution, or a solution for every part, type/write up what you have and you may receive partial credit!

Solution 2.5: (a) When the disk makes half a rotation, it is tangent to the $\boxed{3}$, and the arrow is pointing left.

(b) The circumference of the clock is twice that of the disk, so $\boxed{2}$ full rotations are needed.

(c) Consider rotating the disk completely around the clock. At first, it may seem like the arrow will point each direction twice as well. However, this is not the case. At the 3, the arrow is pointing left and has made $\frac{3}{4}$ of a full rotation. Thus, multiplying that by 4, when the disk rotates completely around the clock, the arrow will have made 3 rotations and will have pointed left 3 times. Try modeling this with paper or coins!

In general, rotating a circle around another circle will lead to an "extra" rotation. This is the distinction between measuring rotations from the number of times a particular point on the disk touches the clock and rotations from the number of times a particular point on the disk is pointing some way, not necessarily touching the clock. This may be hard to wrap your head around (pun intended) at first, but that's okay.