Algebraic Manipulations

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1 Theory

1.1 Completing the Square

The idea behind completing the square is the formula

$$x^2 + 2ax + a^2 = (x + a)^2.$$

Whenever we have an expression of the form $x^2 + 2ax$ for a constant a and variable x, we can add a^2 in order to factor the expression.

Exercise 1.1: (2013 AMC 10)

Real numbers x and y satisfy the equation $x^2 + y^2 = 10x - 6y - 34$. What is $x + y$?

Solution 1.1: If we complete the square after bringing the x and y terms to the other side, we get

$$(x - 5)^2 + (y + 3)^2 = 0.$$

Squares of real numbers are nonnegative, so we need both $(x-5)^2$ and $(y+3)^2$ to be 0. This only happens when $x = 5$ and $y = -3$. Thus, $x + y = 5 + (-3) = \boxed{2}$.

Completing the square is related to the equation of a circle.

Theorem 1.1: Equation of a Circle

The equation of a circle centered at (a, b) and with radius r is

$$(x - a)^2 + (y - b)^2 = r^2.$$

Exercise 1.2: (Alcumus)

What is the area of the region defined by the equation $x^2 + y^2 - 7 = 2y - 8x + 1$?

Solution 1.2: We rewrite the equation as $x^2 + 8x + y^2 - 2y = 8$ and then complete the square, resulting in $(x + 4)^2 - 16 + (y - 1)^2 - 1 = 8$, or $(x + 4)^2 + (y - 1)^2 = 25$. This is the equation of a circle with center $(-4, 1)$ and radius 5, so the area of this region is $\pi r^2 = \pi(5)^2 = \boxed{25\pi}$.

1.2 Simon's Favorite Factoring Trick

The general statement of Simon's Favorite Factoring Trick is

$$xy + ax + by + ab = (x + a)(y + b).$$

Whenever we have an expression of the form $xy + ax + by$ for constants a, b and variables x, y, we can add ab in order to factor the expression. This can be thought of as "completing the rectangle", in analogy to "completing the square".

The following problem is from Richard Rusczyk's video at `https://www.youtube.com/watch?v=OnN3H7w2LnI`, which we recommend for you to watch after class!

Exercise 1.3: (AoPS)

Find all pairs of positive integers (m, n) that satisfy $mn + 3m - 8n = 59$.

Solution 1.3: We subtract 24 from both sides to get $mn + 3m - 8n - 24 = 35$. Now the left hand side factors as $(m-8)(n+3) = 35$. The possibilities for $(m-8, n+3)$ are $(35, 1), (7, 5), (5, 7)$, and $(1, 35)$. Since n is a positive integer, $n+3$ can't be 1. The other three cases lead to the solutions (m, n) of $(15, 2), (13, 4)$, and $(9, 32)$.

SFFT also works if the coefficient of xy is not 1.

Exercise 1.4: (Alcumus)

Find the ordered pair (m, n), where m, n are positive integers satisfying the following equation:

$$14mn = 55 - 7m - 2n$$

Solution 1.4: The given equation rearranges to $14mn + 7m + 2n + 1 = 56$, which can be factored to $(7m + 1)(2n + 1) = 56 = 2 \cdot 2 \cdot 2 \cdot 7$. Since n is a positive integer, we see that $2n + 1 > 1$ is odd. Examining the factors on the right side, we see we must have $2n + 1 = 7$, implying $7m + 1 = 2^3$. Solving, we find that $(m, n) = \boxed{(1, 3)}$.

1.3 Vieta's Relations

Vieta's relations relate the coefficients of a polynomial to the sums of its roots. The simplest case is with quadratic equations.

Theorem 1.2

The roots to the quadratic equation $ax^2 + bx + c = 0$ sum to $-\frac{b}{a}$ and multiply to $\frac{c}{a}$.

Exercise 1.5

The quadratic equation $k(x - m)(x - n) = 0$ has roots m and n. What happens when you expand the left hand side and solve for $m + n$ and mn?

Solution 1.5: Expanding, $k(x-m)(x-n) = k(x^2-(m+n)x+mn) = kx^2 - k(m+n)x + kmn$. Matching coefficients to ax^2+bx+c, we see that $m+n = -\frac{b}{a}$ and $mn = \frac{c}{a}$.

Theorem 1.3

In general, given a polynomial

$$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0,$$

the sum of the roots is $-\frac{a_{n-1}}{a_n}$ and the product of the roots is $(-1)^n\frac{a_0}{a_n}$.

Exercise 1.6

What is the sum of the squares of the roots of x^2-7x+9?

Solution 1.6: Let the roots be r and s. Since $r^2+s^2 = (r+s)^2 - 2rs$, $r+s = 7$, and $rs = 9$, we get that $r^2+s^2 = 7^2 - 2\cdot 9 = 31$.

1.4 Important Identities

There's no need to memorize the following; they are easily derived. The more problems you do involving algebraic manipulations, the more often you'll see these identities.

Theorem 1.4: Factorization

- $a^2-b^2 = (a-b)(a+b)$ *(Difference of Squares)*
- $a^3-b^3 = (a-b)(a^2+ab+b^2)$
- $a^3+b^3 = (a+b)(a^2-ab+b^2)$

Theorem 1.5: Expansion

- $(x+y)^2 = x^2+2xy+y^2$
- $(x-y)^2 = x^2-2xy+y^2$

Exercise 1.7: (108 Algebra Problems)

Factor $x^4-3x^2y^2+y^4$.

Solution 1.7: We will try to complete the square first, by writing

$$x^4-3x^2y^2+y^4 = x^4-2x^2y^2+y^4-x^2y^2 = (x^2-y^2)^2-(xy)^2.$$

Now by difference of squares, we obtain

$$x^4-3x^2y^2+y^4 = (x^2-y^2-xy)(x^2-y^2+xy).$$

2 Problems

Problem 2.1: (Alcumus)

If $23 = x^4 + \frac{1}{x^4}$, then what is the value of $x^2 + \frac{1}{x^2}$?

Solution 2.1: Since $(x^2 + \frac{1}{x^2}) = x^4 + 2 + \frac{1}{x^4} = 25$, $x^2 + \frac{1}{x^2} = \boxed{5}$.

Problem 2.2: (Alcumus)

Find the product of the roots of the equation

$$(2x^3 + x^2 - 8x + 20)(5x^3 - 25x^2 + 19) = 0.$$

Solution 2.2: The left-hand side, when multiplied out, is a polynomial of degree 6. By Vieta's formulas, the product of the roots is determined by its x^6 coefficient and its constant term. The x^6 coefficient is $2 \cdot 5 = 10$ and the constant term is $20 \cdot 19 = 380$, so the product of the roots is $\frac{380}{10} = \boxed{38}$.

Problem 2.3: (MATHCOUNTS 1999 National Team)

Given positive integers x and y such that $x \neq y$ and $\frac{1}{x} + \frac{1}{y} = \frac{1}{12}$, what is the smallest possible value for $x + y$?

Solution 2.3: Simplifying, we have $12(x+y) = xy$, so $xy - 12x - 12y = 0$. Applying Simon's Favorite Factoring Trick by adding 144 to both sides, we get $xy - 12x - 12y + 144 = 144$, so

$$(x - 12)(y - 12) = 144.$$

Now we seek the minimal $x + y$, which occurs when $x - 12$ and $y - 12$ are as close to each other in value as possible. The two best candidates are $(x - 12, y - 12) = (18, 8)$ or $(16, 9)$, of which $(x, y) = (28, 21)$ attains the minimum sum of $\boxed{49}$.

Problem 2.4: (Sophie Germain Identity)

Factor the expression $a^4 + 4b^4$.

Solution 2.4: Since these two terms are in the expansion of $(a^2 + 2b^2)^2$, we rewrite the expression as

$$a^4 + 4b^4 = (a^2 + 2b^2)^2 - 4a^2b^2,$$

and since $4a^2b^2 = (2ab)^2$, we can apply the difference of squares identity, giving

$$(a^2 + 2b^2)^2 - (2ab)^2 = (a^2 + 2b^2 - 2ab)(a^2 + 2b^2 + 2ab).$$

Problem 2.5: (Alcumus)

How many distinct rectangles are there with integer side lengths such that the numerical value of area of the rectangle in square units is equal to 5 times the numerical value of the perimeter in units? (Two rectangles are considered to be distinct if they are not congruent.)

Solution 2.5: Let the side lengths of the rectangle be a and b with $a \leq b$. Then $ab = 10(a+b)$. Expanding and moving all the terms to the left hand side gives $ab - 10a - 10b = 0$. We apply Simon's Favorite Factoring Trick and add 100 to both sides to allow us to factor the left hand side:

$$ab - 10a - 10b + 100 = (a-10)(b-10) = 100$$

From this, we know that $(a-10, b-10)$ must be a pair of factors of 100. Consequently, the pairs (a, b) that provide different areas are $(11, 110)$, $(12, 60)$, $(14, 35)$, $(15, 30)$, and $(20, 20)$. There are therefore $\boxed{5}$ distinct rectangles with the desired property.

Problem 2.6:

Two nonzero real numbers, x and y, satisfy $x^3 + 9x^2y + 11xy^2 - 21y^3 = 0$. What are all possible values of $\frac{x}{y}$?

Since y is nonzero, we divide the equation by y^3 to get $\frac{x^3}{y^3} + 9\frac{x^2}{y^2} + 11\frac{x}{y} - 21 = 0$. Letting $r = \frac{x}{y}$, we get the equation $r^3 + 9r^2 + 11r - 21 = 0$, and the left hand side factors as $(r-1)(r+3)(r+7)$, so $\frac{x}{y}$ can be $\boxed{1, -3, \text{or } -7}$.

Problem 2.7: (2000 AMC 10)

Two non-zero real numbers, a and b, satisfy $ab = a - b$. Find the smallest possible value of $\frac{a}{b} + \frac{b}{a} - ab$.

Solution 2.7: Find the common denominator and replace the ab in the numerator with $a - b$ to get

$$\begin{aligned}\frac{a}{b} + \frac{b}{a} - ab &= \frac{a^2 + b^2 - (ab)^2}{ab} \\ &= \frac{a^2 + b^2 - (a-b)^2}{ab} \\ &= \frac{a^2 + b^2 - (a^2 - 2ab + b^2)}{ab} \\ &= \frac{2ab}{ab} = 2.\end{aligned}$$

Hence the minimum possible value is the only possible value, $\boxed{2}$.

Problem 2.8: (Alcumus)

Let f be the function defined by $f(x) = x^3 - 49x^2 + 623x - 2015$, and let $g(x) = f(x+5)$. Compute the sum of the roots of g.

Solution 2.8: Let a, b, c be the roots of $x^3 - 49x^2 + 623x - 2015$. Then by Vieta's formulas, $a+b+c = 49$. The roots of $g(x) = f(x+5)$ are $a-5$, $b-5$, and $c-5$, and their sum is $a+b+c-15 = 49-15 = \boxed{34}$.

Problem 2.9: (Alcumus)

Let a, b, c be nonzero real numbers such that $a+b+c = 2$ and $a^2+b^2+c^2 = 4$. Find

$$\frac{ab}{c} + \frac{ac}{b} + \frac{bc}{a}.$$

Solution 2.9: We want

$$\frac{ab}{c} + \frac{ac}{b} + \frac{bc}{a} = \frac{a^2b^2 + a^2c^2 + b^2c^2}{abc}.$$

Squaring the equation $a+b+c = 2$, we get

$$a^2 + b^2 + c^2 + 2(ab + ac + bc) = 4.$$

Since $a^2 + b^2 + c^2 = 4$, $ab + ac + bc = 0$.

Squaring the equation $ab + ac + bc = 0$, we get

$$a^2b^2 + a^2c^2 + b^2c^2 + 2abc(a+b+c) = 0.$$

Since $a+b+c = 2$, $a^2b^2 + a^2c^2 + b^2c^2 + 4abc = 0$. Therefore,

$$\frac{a^2b^2 + a^2c^2 + b^2c^2}{abc} = \boxed{-4}.$$