Algebraic Manipulations

Serena An–8/10/19

1 Theory

1.1 Completing the Square

The idea behind completing the square is the formula

$$x^2 + 2ax + a^2 = (x + a)^2.$$

Whenever we have an expression of the form $x^2 + 2ax$ for a constant a and variable x, we can add a^2 in order to factor the expression.

Exercise 1.1: (2013 AMC 10)

Real numbers x and y satisfy the equation $x^2 + y^2 = 10x - 6y - 34$. What is $x + y$?

Solution 1.1: If we complete the square after bringing the x and y terms to the other side, we get

$$(x - 5)^2 + (y + 3)^2 = 0.$$

Squares of real numbers are nonnegative, so we need both $(x-5)^2$ and $(y+3)^2$ to be 0. This only happens when $x = 5$ and $y = -3$. Thus, $x + y = 5 + (-3) = \boxed{2}$.

Completing the square is related to the equation of a circle.

Theorem 1.1: Equation of a Circle

The equation of a circle centered at (a, b) and with radius r is

$$(x - a)^2 + (y - b)^2 = r^2.$$

Exercise 1.2: (Alcumus)

What is the area of the region defined by the equation $x^2 + y^2 - 7 = 2y - 8x + 1$?

Solution 1.2: We rewrite the equation as $x^2 + 8x + y^2 - 2y = 8$ and then complete the square, resulting in $(x + 4)^2 - 16 + (y - 1)^2 - 1 = 8$, or $(x + 4)^2 + (y - 1)^2 = 25$. This is the equation of a circle with center $(-4, 1)$ and radius 5, so the area of this region is $\pi r^2 = \pi(5)^2 = \boxed{25\pi}$.

1.2 Simon's Favorite Factoring Trick

The general statement of Simon's Favorite Factoring Trick is

$$xy + ax + by + ab = (x + a)(y + b).$$

Whenever we have an expression of the form $xy+ax+by$ for constants a, b and variables x, y, we can add ab in order to factor the expression. This can be thought of as "completing the rectangle", in analogy to "completing the square".

The following problem is from Richard Rusczyk's video at `https://www.youtube.com/watch?v=0nN3H7w2LnI`, which we recommend for you to watch after class!

Exercise 1.3: (AoPS)

Find all pairs of positive integers (m, n) that satisfy $mn + 3m - 8n = 59$.

Solution 1.3: We subtract 24 from both sides to get $mn+3m-8n-24 = 35$. Now the left hand side factors as $(m-8)(n+3) = 35$. The possibilities for $(m-8, n+3)$ are $(35, 1), (7, 5), (5, 7)$, and $(1, 35)$. Since n is a positive integer, $n+3$ can't be 1. The other three cases lead to the solutions (m, n) of $(15, 2), (13, 4)$, and $(9, 32)$.

SFFT also works if the coefficient of xy is not 1.

Exercise 1.4: (Alcumus)

Find the ordered pair (m, n), where m, n are positive integers satisfying the following equation:

$$14mn = 55 - 7m - 2n$$

Solution 1.4: The given equation rearranges to $14mn + 7m + 2n + 1 = 56$, which can be factored to $(7m+1)(2n+1) = 56 = 2 \cdot 2 \cdot 2 \cdot 7$. Since n is a positive integer, we see that $2n+1 > 1$ is odd. Examining the factors on the right side, we see we must have $2n + 1 = 7$, implying $7m + 1 = 2^3$. Solving, we find that $(m, n) = \boxed{(1, 3)}$.

1.3 Vieta's Relations

Vieta's relations relate the coefficients of a polynomial to the sums of its roots. The simplest case is with quadratic equations.

Theorem 1.2

The roots to the quadratic equation $ax^2 + bx + c = 0$ sum to $-\frac{b}{a}$ and multiply to $\frac{c}{a}$.

Exercise 1.5

The quadratic equation $k(x - m)(x - n) = 0$ has roots m and n. What happens when you expand the left hand side and solve for $m + n$ and mn?

Solution 1.5: Expanding, $k(x-m)(x-n) = k(x^2 - (m+n)x + mn) = kx^2 - k(m+n)x + kmn$. Matching coefficients to $ax^2 + bx + c$, we see that $m + n = -\frac{b}{a}$ and $mn = \frac{c}{a}$.

Theorem 1.3

In general, given a polynomial

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

the sum of the roots is $-\frac{a_{n-1}}{a_n}$ and the product of the roots is $(-1)^n \frac{a_0}{a_n}$.

Exercise 1.6

What is the sum of the squares of the roots of $x^2 - 7x + 9$?

Solution 1.6: Let the roots be r and s. Since $r^2 + s^2 = (r+s)^2 - 2rs$, $r + s = 7$, and $rs = 9$, we get that $r^2 + s^2 = 7^2 - 2 \cdot 9 = 31$.

1.4 Important Identities

There's no need to memorize the following; they are easily derived. The more problems you do involving algebraic manipulations, the more often you'll see these identities.

Theorem 1.4: Factorization

- $a^2 - b^2 = (a-b)(a+b)$ *(Difference of Squares)*
- $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$
- $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

Theorem 1.5: Expansion

- $(x+y)^2 = x^2 + 2xy + y^2$
- $(x-y)^2 = x^2 - 2xy + y^2$

Exercise 1.7: (108 Algebra Problems)

Factor $x^4 - 3x^2y^2 + y^4$.

Solution 1.7: We will try to complete the square first, by writing

$$x^4 - 3x^2y^2 + y^4 = x^4 - 2x^2y^2 + y^4 - x^2y^2 = (x^2 - y^2)^2 - (xy)^2.$$

Now by difference of squares, we obtain

$$x^4 - 3x^2y^2 + y^4 = (x^2 - y^2 - xy)(x^2 - y^2 + xy).$$

2 Problems

Problem 2.1: (Alcumus)

If $23 = x^4 + \frac{1}{x^4}$, then what is the value of $x^2 + \frac{1}{x^2}$?

Problem 2.2: (Alcumus)

Find the product of the roots of the equation

$$(2x^3 + x^2 - 8x + 20)(5x^3 - 25x^2 + 19) = 0.$$

Problem 2.3: (MATHCOUNTS 1999 National Team)

Given positive integers x and y such that $x \neq y$ and $\frac{1}{x} + \frac{1}{y} = \frac{1}{12}$, what is the smallest possible value for $x + y$?

Problem 2.4: (Sophie Germain Identity)

Factor the expression $a^4 + 4b^4$.

Problem 2.5: (Alcumus)

How many distinct rectangles are there with integer side lengths such that the numerical value of area of the rectangle in square units is equal to 5 times the numerical value of the perimeter in units? (Two rectangles are considered to be distinct if they are not congruent.)

Problem 2.6:

Two nonzero real numbers, x and y, satisfy $x^3 + 9x^2y + 11xy^2 - 21y^3 = 0$. What are all possible values of $\frac{x}{y}$?

Problem 2.7: (2000 AMC 10)

Two non-zero real numbers, a and b, satisfy $ab = a - b$. Find the smallest possible value of $\frac{a}{b} + \frac{b}{a} - ab$.

Problem 2.8: (Alcumus)

Let f be the function defined by $f(x) = x^3 - 49x^2 + 623x - 2015$, and let $g(x) = f(x+5)$. Compute the sum of the roots of g.

Problem 2.9: (Alcumus)

Let a, b, c be nonzero real numbers such that $a+b+c=2$ and $a^2+b^2+c^2=4$. Find

$$\frac{ab}{c}+\frac{ac}{b}+\frac{bc}{a}.$$