Week 4: PIE

Samyok Nepal–8/3/19

1 What is PIE?

PIE, or the ***P***rinciple of ***I***nclusion-***E***xclusion, is a very powerful counting tool that has it's roots from very simple venn diagrams.

All problems from today are from AoPS's Intermediate C&P if a source is not mentioned.

Problem 1.1

If 20 girls are on my school's soccer team, 25 girls are on my school's hockey team, and 11 are on both, how many girls play only one sport?

Problem 1.2

How many 6 digit numbers start or end with an even digit?

Problem 1.3

There are three languages offered at a school: Spanish, French, and German.

- There are 57 kids in at least one foreign language class
- There are 29 kids in the Spanish Class
- There are 34 kids in the French Class
- There are 33 kids in the German Class
- 15 are taking both Spanish and French
- 16 are taking both French and German
- 12 are taking both Spanish and German

How many students are taking all three languages?

Problem 1.4

How many positive integers less than 1000 are divisible by neither 2, 3, nor 5?

Probability Stretch

1. ______ % Petra randomly selects a card from a standard deck of 52 playing cards. What is the percent probability that the card shows a red number greater than 6? Express your answer to the nearest hundredth.

2. ______ Max has eight identical cups. Each cup contains a different combination of nickels, dimes and quarters, each totaling 45 cents. Max randomly selects a cup. What is the probability that the cup he selects contains at least three dimes? Express your answer as a common fraction.

3. ______ A bag contains five chips numbered 2 through 6. Danya draws chips from the bag one at a time and sets them aside. After each draw, she totals the numbers on all the chips she has already drawn. What is the probability that at any point in this process her total will equal 10? Express your answer as a decimal to the nearest tenth.

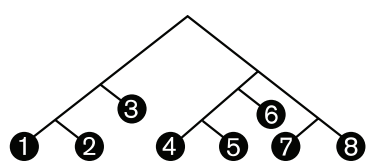
4. ______ A drawer contains five socks: two green and three blue. What is the probability that two socks pulled out of the drawer at random will match? Express your answer as a common fraction.

5. ______ A penny, a nickel and a dime are flipped. What is the probability that at least two coins land heads up and one of them is the nickel? Express your answer as a common fraction.

6. ______ % When the circuit containing blinking lights A and B is turned on, lights A and B blink together. Then A blinks once every 5 seconds and B blinks once every 11 seconds. Lindsey looks at the two lights just in time to see A blink alone. What is the percent probability that the next light to blink will be A blinking alone?

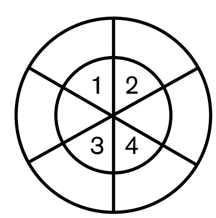
7. ______ % What is the percent probability that a randomly selected multiple of 3 less than or equal to 3000 is also a multiple of 5?

8. ______ Starting at the top and selecting paths randomly as you move downward, what is the probability of ending at an odd number? Express your answer as a common fraction.



9. ______ A five-digit number is made by randomly ordering the digits 1, 2, 3, 4 and 5. What is the probability that this number is divisible by 4? Express your answer as a common fraction.

10. ______ Pierre throws darts that land randomly in the dartboard shown here. The dartboard is a circle of radius 2 units, with an inner circle of radius 1 unit. Both circles are divided into six congruent sectors. What is the probability that a dart Pierre throws will land in one of the four inner numbered sectors? Express your answer as a decimal to the nearest hundredth.



Here are a couple of problem-solving concepts regarding PIE:

> **Concept:** Messy casework often means that it's simpler to use complementary counting and PIE.

> **Concept:** If a problem asks you to count how many items have "at least" one property, that's a good sign that you may want to use PIE. Similarly, if a problem asks you to count how many items have "none" of several properties, that may be a sign to use complementary counting with PIE.

> **Concept:** Don't memorize a "formula" for PIE. Instead, think about how many times each item is counted, and make sure that each item is counted once and only once.

We also saw one general good piece of advice when trying to do proofs:

> **Concept:** In proof problems, one way to start is by listing what you know and what you're trying to prove.

REVIEW PROBLEMS

3.25 How many 4-letter words (consisting of any sequence of 4 letters, possibly repeated) start or end with a vowel? (For the purposes of this problem, consider A, E, I, O, and U to be vowels, and consider Y to be a consonant.)

3.26 How many 3-digit numbers have two consecutive digits the same?

3.27 Is it possible that among a group of 20 ninth-graders, 15 of them play lacrosse, 12 of them play soccer, and 6 of them play both? Why or why not?

3.28 When I go to work, there's a 20% probability that I'll forget my office keys, and a 30% probability that I'll forget my wallet. If there's a 5% probability that I forget both, then what's the probability that I arrive at work with both my keys and my wallet?

3.29 How many 4-letter "words" (any combination of 4 letters) have no two consecutive letters identical?

(a) Solve the problem using PIE.

(b) Solve the problem using constructive counting.

(c) Can you algebraically explain why your two answers from (a) and (b) are the same? (Of course we know that they must be the same, since they're just two different ways of counting the same thing, but can you explain it in terms of algebra?)

3.30 Sam can only remember 10-digit numbers if the first four digits are either exactly the same as the next four digits of the number or the last four digits of the number. For example, Sam can remember 1234123456 and 3444533444, but not 3344443334. How many 10-digit numbers can Sam remember?

3.31 My state uses a sequence of three letters followed by a sequence of three numbers as its standard license plate pattern (for example, GMQ829). Given that each three-letter three-digit arrangement is equally likely, find the probability that such a license plate will contain at least one palindrome (a three-letter arrangement or a three-digit arrangement that reads the same left-to-right as it does right-to-left). *(Source: AIME)*

3.32 Let $\pi = (x_1, x_2, x_3, x_4, x_5, x_6, x_7)$ be a permutation of the numbers $(1, 2, 3, 4, 5, 6, 7)$ (recall that a permutation is a rearrangement of the numbers, in which each number appears exactly once). Find the number of such permutations π in which $x_n = n$ for some odd integer n.

3.33 Twenty five of King Arthur's knights are seated at their customary round table. Three of them are chosen—all choices being equally likely—and are sent off to slay a troublesome dragon. Find the probability that at least two of the three had been sitting next to each other. *(Source: AIME)*

3.34 What is the probability that a 13-card bridge hand (dealt at random from a standard 52-card deck) has a *void* (meaning it has no cards of some suit)?

3.35 How many 5-digit sequences have a digit that appears at least 3 times? (For instance, 03005 and 22922 are examples of such sequences.)

Challenge Problems

3.36 Consider two events A and B. Find $P(A \text{ or } B)$ in terms of $P(A)$, $P(B)$, and $P(A \text{ and } B)$. **Hints:** 275

3.37 There are N students at Grant High School. Let $S(F)$ be the number of students at Grant who speak French, $S(J)$ be the number of students who speak Japanese, and $S(A)$ be the number who speak Arabic. Let $S(AF)$ be the number who speak both French and Arabic; define $S(AJ)$ and $S(FJ)$ similarly. Prove that $3N + S(AF) + S(AJ) + S(FJ) \geq 2S(A) + 2S(F) + 2S(J)$. **Hints:** 167

3.38 Given any set S, let $s(S)$ be the number of subsets of S (including S and the empty set). If X, Y, and Z are sets such that $s(X) + s(Y) + s(Z) = s(X \cup Y \cup Z)$ and $\#(X) = \#(Y) = 100$, what is the minimum possible value of $\#(X \cap Y \cap Z)$? *(Source: AMC)* **Hints:** 193, 98

3.39

(a) Prove that for any positive integer k less than 9,

$$9^k - \binom{9}{1}8^k + \binom{9}{2}7^k - \cdots - \binom{9}{7}2^k + \binom{9}{8} = 0.$$

(b) What happens if $k = 9$?

Hints: 19, 209, 233