Divisibility Rules

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1 Theory

We present the common divisibility rules for all numbers from 2 to 11.

Theorem 1.1: Divisibility Rules for 2, 5, 10

A number is divisible by

- 2 if it ends in 0, 2, 4, 6, or 8 (even numbers)
- 5 if it ends in 0 or 5
- 10 if it ends in 0

Problem 1.1

Prime factorize 800000.

Theorem 1.2: Divisibility Rules for 4, 8

A number is divisible by

- 4 if its last 2 digits are divisible by 4
- 8 if its last 3 digits are divisible by 8

Exercise 1.1

Prove the divisibility rules for 4 and 8.

Solution 1.1: 4 divides 100, and all multiples of 100 (i.e. numbers ending in 00), so we only need to look at the last 2 digits. Similarly, 8 divides 1000 (but not 100) and all multiples of 1000 (i.e. numbers ending in 000), so we only need to look at the last 3 digits.

Problem 1.2

Determine which of the following are divisible by 4. Then determine which are divisible by 8.

- 198
- 476
- 936
- 111104

Theorem 1.3: Divisibility Rules for 3, 6, 9

A number is divisible by

- 3 if the sum of its digits is a multiple of 3
- 6 if it is divisible by 2 and 3
- 9 if the sum of its digits is a multiple of 9

Problem 1.3

Determine which of the following are divisible by 3. Then determine which are divisible by 9.

- 483
- 2957
- 314159265
- 111111111

Problem 1.4: (MATHCOUNTS 2009 Chapter Countdown)

In the six-digit integer $3A6,792$, what is the largest digit A so that the six-digit integer will be divisible by 3?

Theorem 1.4: Divisibility Rule for 7

1. Remove the units digit from the number and double the units digit.
2. Subtract twice the units digit from the truncated original number.
3. Repeat this process as many times as needed (usually until you end up with a 1 or 2 digit number). If this smaller number is divisible by 7 (note that 0 is divisible by 7), then the original number was also divisible by 7.

Usually, it is easier to just do the division by 7 than to use this divisibility rule. This rule is most useful with 3 to 4 digit numbers, but with anything above that, the rule tends to be a bit cumbersome.

Exercise 1.2

Show that 483903 is divisible by 7.

Solution 1.2:

1. $48390 - 2 \cdot 3 = 48384$
2. $4838 - 2 * 4 = 4830$
3. $483 - 2 * 0 = 483$ (Alternatively, we could have realized that 7 does not divide 10, so we can divide the number by 10.)
4. $48 - 2 * 3 = 42$

We know that 42 is divisible by 7, so our original number, 483903, is also divisible by 7. We could have continued the algorithm one more step to get $4 - 2 * 2 = 0$, which is also divisible by 7.

Problem 1.5

Show that 111111 is divisible by 7.

Theorem 1.5: Divisibility by 11

Sum the alternating digits and subtract these two sums (order doesn't matter). If 11 divides the final answer, then 11 also divides the original number

Exercise 1.3

Show that 918,291 is divisible by 11.

Solution 1.3: We sum the alternating digits to get the two sums $9 + 8 + 9 = 26$ and $1 + 2 + 1 = 4$. Their positive difference is 22, which is divisible by 11, so 918,291 is also divisible by 11.

Problem 1.6

Determine which of the following are divisible by 11.

- 396
- 1001
- 8592
- 212121

2 Problems

Problem 2.1: (Alcumus)

If an integer ends in the digit 0 and the sum of its digits is divisible by 3, then how many of the numbers 2, 3, 4, 5, 6, 8, 9 necessarily divide it?

Solution 2.1: Since it ends with 0, 2 and 5 must divide it. Since the sum of its digits is divisible by 3, 3 must divide it. Since 2 and 3 divide it, 6 must also divide it. In all, $\boxed{4}$ numbers divide this integer.

Problem 2.2: (Alcumus)

What is the least four-digit positive integer, with all different digits, that is divisible by each of its digits?

Solution 2.4: The smallest four-digit positive integer with all different digits is 1234. This doesn't work, and neither does 1235. However, $\boxed{1236}$ does work!

Problem 2.3: (Alcumus)

Let A be a digit. If the 7-digit number 353808A is divisible by 2, 3, 4, 5, 6, 8, and 9, then what is A?

Solution 2.3: Since 2 and 5 divide it, it must end in $\boxed{0}$.

Problem 2.4

Prime factorize 52446240.

Solution 2.4: $\boxed{2^5 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11^2 \cdot 43}$.

Problem 2.5: (Alcumus)

The number $5\,41G\,507\,2H6$ is divisible by 72. If G and H each represent a single digit, what is the sum of all **distinct** possible values of the product GH? (Count each possible value of GH only once, even if it results from multiple G, H pairs.)

Solution 2.5: Since the number is divisible by 8, $2H6$ is divisible by 8, leading to the cases of $H = 1, 5, 9$. Since the number is divisible by 9, 9 divides $5+4+1+G+5+0+7+2+H+6 = 30+G+H$. When $H = 1$, $G = 5$. When $H = 5$, $G = 1$. When $H = 9$, $G = 6$. The two distinct possible values of GH are 5 and 54, and their sum is $\boxed{59}$.

Problem 2.6: (MATHCOUNTS 2002 National Team)

The 9-digit number $abb, aba, ba3$ is a multiple of 99 for some pair of digits a and b such that $b > a$. What is $b - a$?

Solution 2.6: To be divisible by 9, we need that 9 divides $a+b+b+a+b+a+b+a+3 = 4a+4b+3$. To be divisible by 11, we need that 11 divides $(a+b+b+b+3)-(b+a+a+a) = 2b-2a+3$. Then

$2b-2a+3$ can be 11, leading to $b-a=\boxed{4}$. Note: the only solution (a,b) is $(1,5)$.