Challenge #2 - Basic Algebra

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1 Introduction

Welcome to the Brookings Math Circle Summer 2018 Problem Sets! **Submissions for this problem set are due by Sunday 6/17 at 1:00 pm (the day of the second BMC class).** For a refresher on submission details or problem set details, see the introduction PDF linked to the Summer Challenge page of our website under Classes.

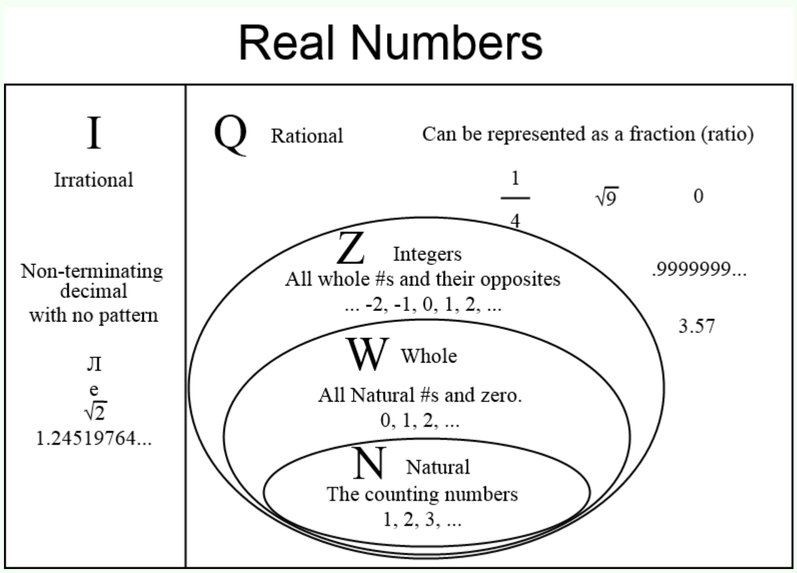
This handout and problem set will focus on basic algebra; in particular, we will cover constants, variables, single-variable equations and tricks, and polynomial basics. Next week we will delve into more advanced algebra, in particular linear equations and systems of equations, but with basics that you learn through this problem set, you will be able to tackle harder algebra problems soon! Keep calm and do the math!

Solutions to problems in the text are at the very back of this packet.

2 Constants

First, we must start with the types of numbers (or constants). Hopefully you are already familiar with most if not all of these terms. Notice how many of these groups fit into one another. Don't worry too much about the large letters; it's a shorthand but you will never see it in middle school math competitions, but you will see the words "real", "irrational", "rational", "integers", "whole", and "natural" quite frequently.

Theorem 2.1: Number Sets



3 Variables and Single-Variable Equations

Fundamentals are important in every branch of math, but perhaps even more so in algebra, which can be seen as a foundation to other branches. Algebra begins with the notion of variables and equations, which you may already be familiar with.

Prerequisites: The ability to do basic arithmetic following the order of operations.

Variables are unknown quantities. Typically, variables are represented by lowercase letters (such as x, y, a, or n), and less frequently you will see variables represented by uppercase letters (such as A or E) or symbols (such as $\diamond$ or $\clubsuit$). Typically, the question will require you to come up with a variable, which may represent the quantity you are solving for or another helpful quantity. Every time the same variable shows up in an expression, it represents the same quantity, but different variables may represent different quantities.

Equations include the equals sign "=" and show that the two sides are equal. The "golden rule" of algebra is that if you do something to one side of an equation, you must do the same thing to the other side. For example, if you multiply one side by 4, you must also multiply the other side by 4. The most basic equations include one variable and typically ask you to solve for that particular variable. You can check your work by plugging in the value you got for the variable and seeing if both sides are equal. Solve for each variable: (answers are at the back of this handout)

Problem 3.1: (Serena An)

(a) $4x - 17 = 7$
(b) $5a - 7 = 8 - a$
(c) $\frac{Z+5}{2} = 4$
(d) $2 - 4Q = 7$
(e) $3(\bigstar - 3) = 11 - 2\bigstar$
(f) $\frac{4}{5}\spadesuit - 5 = \frac{6-\spadesuit}{5}$
(g) $n^2 - 4 = 2n^2 - 13$
(h) $\sqrt{h+6} = 5$

Always simplify your final answer if it involves a fraction.

Hopefully these problems were not bad. Maybe you even got many of them through guess and check or mental math! However, typically in math competitions, the answer will not be easily guessed and the problem will not be as easily done in your head. In addition, typically, you must set up the equation yourself. Thus, being able to write and solve equations is an essential skill that can get you far in math competitions. This will come with practice.

Warning:

Whenever we square an equation as a step in solving it, we must check that our solutions are valid. If a solution to the squared equation does not satisfy the original equation, it is an **extraneous** solution. Extraneous solutions are not valid solutions to the original equation. Basically, if a solution doesn't check (and typically doesn't make sense), it's extraneous.

For example, consider the equation $\sqrt{x} + 5 = 2$. Subtracting 5, we get $\sqrt{x} = -3$ which has no solutions. However, if we square $\sqrt{x} = -3$, we get $x = 9$, which doesn't satisfy the original equation of $\sqrt{x} + 5 = 2$. Thus 9 is an **extraneous** solution. The problem arises when we take the square root of $x = 9$ to evaluate $\sqrt{x}$ in the original equation. This gies us $\sqrt{x} = 3$ not $\sqrt{x} = -3$ because we define $\sqrt{x}$ for positive numbers x to be the *positive* number whose square is x.

A similar warning is to never forget the negative solution when you have an equation including x^2 such as $x^2 = 36$, which has solutions of $x = 6$ AND $x = -6$. *Typically*, negatives are not extraneous solutions when we have a quantity squared, but they are with square roots.

HINT: When labeling variables, try to keep them easy to remember, which typically means using the first letter of a word. For example, if you want to find the **h**eight of something, use h, or if you want to find **A**nn's age, use a. This keeps things easy to remember and will save you time, as you don't have to continuously wonder which variable is what.

4 Tricks for Solving Specific Problems

Note that consecutive integers can be written as $x, x+1, x+2, ...$, where x is the *smallest* integer in the list. However, some problems can be solved easier and faster by setting x to be the *middle* number. In addition, sometimes, the solving the problem may be faster if you *don't* use a variable!! You'll be able to recognize these special cases as you do more problems.

Solutions are in the back of the packet, once more.

Problem 4.1

The sum of 5 consecutive integers is 120. What is the largest of these integers?

Problem 4.2

The sum of 10 consecutive integers is 25. What is the smallest of these integers?

Problem 4.3

The product of 3 consecutive odd integers is 2145. What is the smallest of these integers?

5 Polynomial Basics

Before we get into linear equations, we need a few definitions.

A **polynomial** in one variable is an expression of the form

$$a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_1 x + a_0,$$

where x is a variable and each a_i is a constant. Each $a_i x^i$ is called a **term** and the a_i are called **coefficients**. The **degree** of the polynomial is the highest integer n such that its coefficient (a_n) is nonzero. Basically, the **degree** is the largest exponent.

There are some special names for polynomials of smaller degrees. A polynomial of degree 0 is **constant**, degree 1 is **linear**, degree 2 is **quadratic**, degree 3 is **cubic**, degree 4 is **quartic**, and for anything above degree 4, just call it a **polynomial**.

If you're wondering why degree 2 is *quad*ratic, it is because **quadra** refers to a square, and degree 2 polynomials include something squared. Similarly, *cubic* refers to a cube, and degree 3 polynomials include something cubed.

There a way to group polynomials based on the number of terms they have (this is not necessarily degree). A polynomial with 1 term is a **monomial**, 2 terms is a **binomial**, 3 terms is a **trinomial**. So $4x^7$ is a monomial of degree 7, $3x^2 + 1$ is a quadratic binomial, and $x^3 + 3x^2 + 6$ is a cubic trinomial.

Classify the following polynomials by degree and the number of terms. If there are more than three terms, just give the degree.

Problem 5.1

(a) $4x^2 + 19x - 26$

(b) $3 + 1 + 4 + 1 + 5 + 9 + 2 + 6 + 5$

(c) $20x^4 + 20x^7 + 3x$

(d) $5x^4 - 4x^3 + 3x^2 - 2x + 1$

(e) $10000000x - 987654321$

6 Puzzle #2

Problem 6.1: (USAMTS)

Fill in the spaces of the grid below with positive integers so that in each 2×2 square with top left number a, top right number b, bottom left number c, and bottom right number d, either $a + d = b + c$ or $ad = bc$. There is only one solution.

3	9			
	11		7	2
10				16
15				
20	36			32

Solution 6.1:

3	9	12	6	1
5	11	14	7	2
10	22	28	21	16
15	27	33	26	21
20	36	44	37	32

Start with the first two columns. 5 must be in the only blank space in the left most column. Then the possibilities for the squares in the second column 3rd from the bottom are 22 and 16, and 2nd from the bottom are 27 and 31. A bit of trial and error shows that the numbers must be 22 and 27.

Consider a 2x2 square, and suppose that $a < c$. Then if $a + d = b + c$, then $a + d = b + c > a + b$ so $d > b$. If $ad = bc$, then $ad = bc > ab$ so $d > b$. Either way, $d > b$. Thus, all the numbers in the second row must be greater than all the numbers in the first row, so the number in the upper right hand corner must be 1 and the number directly to the left must be 6. A bit of trial and error reveals that the numbers in the middle of the topmost and second topmost rows must be 12 and 14 respectively. Continue with more trial and error to fill in the entire table. Make conjectures for certain squares and if they're wrong, it's ok! A hint is that if you have three numbers filled in, say a, b, and c, if a does not divide bc evenly, you know that the numbers must satisfy $a + d = b + c$.

7 Bonus Problem Set #2

Problem 7.1: (AMC 8)

Four students take an exam. Three of their scores are $70, 80$, and 90. If the average of their four scores is 70, then what is the remaining score?

Solution 7.1: Let x be the fourth score. Since the average is 70, the sum of the four scores is $4 \cdot 70 = 280$. Since $x + 70 + 80 + 90 = 280$, $x = \boxed{40}$.

A way to do this in your head is to think about the difference between the first three scores and the average. 70 is 0 away, 80 is 10 above, and 90 is 20 above, so the remaining score must be 30 below the average, or $\boxed{40}$

Problem 7.2: (CMMS)

The height of a rectangle is three centimeters more than twice its length. If the perimeter of the rectangle is 60 cm, what is its area?

Solution 7.2: Let l be the length of the rectangle. Then the height is $2l + 3$. Solving for l in the perimeter equation $2l + 2(2l + 3) = 60$ gives $l = 9$. Thus, the height is $2 \cdot 9 + 3 = 21$. The area is therefore $9 \cdot 21 = \boxed{189 \text{ cm}^2}$.

Problem 7.3: (CMMS)

Molly's father James is three years less than three times her age. How many years from now will Molly's father be twice her age if he is 33 today?

Solution 7.3: Let m be Molly's current age. It is given that $3m - 3 = 33$, so $m = 12$. The difference between Molly and James's ages is 21. Thus, James will be twice Molly's age when Molly is 21 and James is 42. This will occur in $\boxed{9}$ years.

Problem 7.4: (AMC 8)

If $3^p + 3^4 = 90$, $2^r + 44 = 76$, and $5^3 + 6^s = 1421$, what is the product of p, r, and s?

Solution 7.4: We solve for each variable individually. Since $3^p + 3^4 = 90$, $3^p = 9$ and $p = 2$. Since $2^r + 44 = 76$, $2^r = 32$ and $r = 5$. Since $5^3 + 6^s = 1421$, $6^s = 1296$ and $s = 4$. Thus, the product of p, r, and s is $2 \cdot 5 \cdot 4 = \boxed{40}$.

Problem 7.5: (AIME)

Let x, y, and z all be greater than one. Let w be a positive integer such that

$$x^{24} = w, \ y^{40} = w, \text{ and } (xyz)^{12} = w.$$

Find a positive integer a such that $z^a = w$.

Solution 7.5: We try to use a common exponent. Lets make all the exponents on the right 120.

$$x^{24} = w \quad \to \quad x^{120} = w^5 \tag{1}$$
$$y^{40} = w \quad \to \quad y^{120} = w^3 \tag{2}$$
$$x^{12}y^{12}z^{12} = w \quad \to \quad x^{120}y^{120}z^{120} = w^{10} \tag{3}$$

Multiplying the first two equations together gives:

$$x^{120}y^{120} = w^8 \tag{4}$$

We can take the third equation and divide it by the fourth to get:

$$z^{120} = w^2 \tag{5}$$

Square rooting both sides, we get

$$z^{60} = w \implies \boxed{a = 60}.$$

Problem 7.6: (Intermediate Algebra)

Find all z such that $\frac{1}{\sqrt{z-3}-1} + 2 = \frac{3}{\sqrt{z-3}-1}$.

Solution 7.6: First, note that the denominators of the fractions are the same. So we can combine the fractions by subtracting $\frac{1}{\sqrt{z-3}-1}$ from both sides to get $2 = \frac{2}{\sqrt{z-3}-1}$. We get rid of the fraction by multiplying both sides by $\sqrt{z-3}-1$ which gives us $2\sqrt{z-3}-2 = 2$. Adding 2 to both sides, then dividing by 2 to isolate $\sqrt{z-3}$ gives us $\frac{z-3}{=}2$. We get rid of the square root sign by squaring both sides, which yields $z - 3 = 4$, so $z = \boxed{7}$.

Remember the warning; whenever we square an equation as a step in solving it, we have to make sure that our solutions are valid (and so we don't have an *extraneous* solution). Plugging in $z = 7$ to the original equation, we get $\sqrt{7-3}-1 = 1$, so our equation is $1 + 2 = 3$, which is true. So, $z = \boxed{7}$ is the only solution to the equation.

Problem 7.7: (Intermediate Algebra)

Find all values of c such that the equation $\frac{3}{2-\frac{1}{x}} = c$ has no solutions for x.

Solution 7.7: (From Intermediate Algebra) We want to know when there are no solutions for x, so we solve the equation for x in terms of c. Then we can use the resulting expression to determine what values of c fail to give us a value for x.

Multiplying both sides of the equation by $2 - \frac{1}{x}$ gives us $3 = 2c - \frac{c}{x}$. We isolate the term with x by adding $\frac{c}{x} - 3$ to both sides, which gives $\frac{c}{x} = 2c - 3$. Multiplying both sides by x and dividing both sides by $2c - 3$ gives $\frac{c}{2c-3} = x$. We can't have $c = \boxed{\frac{3}{2}}$, because then the denominator would be 0. It appears that for any other value of c, we can use $x = \frac{c}{2c-3}$ to find the solution for x that results for that value of c. But we have to be careful - we can't have a value of x that results in dividing by 0 in our original equation, so we can't have $x = 0$ and we can't have $2 - \frac{1}{x} = 0$, or $x = \frac{1}{2}$. Therefore, we must check for values of c in $x = \frac{c}{2c-3}$ that give $x = 0$ or $x = \frac{1}{2}$.

If $x = 0$ in $x = \frac{c}{2c-3}$, then we must have $c = \boxed{0}$, so it appears that we can't have $c = 0$ in our original equation. Indeed, letting $c = 0$ gives us $\frac{3}{2-\frac{1}{x}} = 0$, which clearly has no solution because the numerator is not 0 for any x. If $x = \frac{1}{2}$ in $x = \frac{c}{2c-3}$, then we have $\frac{1}{2} = \frac{c}{2c-3}$. Cross multiplying gives $2c - 3 = 2c$, which has no solutions. Therefore, there are no values of c for which $x = \frac{1}{2}$.

In conclusion, the only values of c for which there is no solution for x in the original equation are $c = \boxed{0}$ and $c = \boxed{\frac{3}{2}}$. For all other values of c, we can use $x = \frac{c}{2c-3}$ to find the x that satisfies the original equation for that value of c.

Whew, that was a pretty long solution! It's okay if you don't get every step the first time through; try reading it again!

8 Summer Challenge #2

Problem 8.1: (Serena An)

If $x = 3$, $y = 4$, and $z = 5$, what is the value of $\sqrt{x^2+y^2} + 5z^2 - (x^2+y^2)$?

Solution 8.1: Plugging in the values for the variables, we get $\sqrt{x^2+y^2}+5z^2-(x^2+y^2) = \sqrt{3^2+4^2} + 5\cdot 5^2 - (3^2+4^2) = 5 + 125 - 25 = \boxed{105}$.

Problem 8.2: (Serena An)

The sum of Ann, Ben, and Cathy's ages is 35. If Ben is 18 years older than half of Ann's age and Cathy is 3 years older than twice Ann's age, find the sum of Ben and Cathy's ages.

Solution 8.2: Let a be Ann's age. Then Ben's age is $\frac{1}{2}a + 18$ and Cathy's age is $2a + 3$. The sum of the three ages is 35, so $a + \frac{1}{2}a + 18 + 2a + 3 = \frac{7}{2}a + 21 = 35$. Then $\frac{7}{2}a = 14$ so $a = 4$. Don't stop here though, remember what the problem is asking! The sum of Ben in Cathy's ages is $35 - a = \boxed{31}$. Alternatively, you could find Ben's age to be 20 and Cathy's age to be 11, and then sum to get $\boxed{31}$, although it is a bit more work.

Problem 8.3: (Serena An)

What's the smallest positive integer that is the product of 3 consecutive integers, one of which is a prime, another one of which is a multiple of 5, and the last one is a multiple of 8 (in some order)?

Solution 8.3: We consider the most restrictive condition on one of the three integers which is the multiple of 8. We first try letting one of the factors be 8. Then the closest multiple of 5 is 10, but we can't use the sequence 8, 9, 10 because 9 is not prime. Next, we try letting one of the factors be 16. We see that 15, 16, and 17 work, and considering all sequences with 16, this is the only sequence that will work (with 14, 15, 16, 14 is not prime, and with 16, 17, 18, there is no multiple of 5). Thus, the smallest positive integer is $15 \cdot 16 \cdot 17 = \boxed{4080}$.

Problem 8.4: (Serena An)

If $\sqrt{x + \sqrt{x + \sqrt{x + \ldots}}} = 12$, what is x?

Solution 8.4: Squaring both sides, we get $x + \sqrt{x + \sqrt{x + \sqrt{x + \ldots}}} = 144$. Now we subtract $\sqrt{x + \sqrt{x + \sqrt{x + \ldots}}} = 12$ from both sides to get $x = 144 - 12 = \boxed{132}$.

Problem 8.5: (Serena An)

A list of 5 positive integers has a mean of 5 and a unique mode of 2. What is the greatest possible number in the list? (The unique mode is the only number that is repeated the most times; so basically, there are more 2s in the list than any other number.)

Reminder: Write a full solution to this problem, not just the answer. Even if you do not have a full solution, type/write up what you have and you may receive partial credit!

Solution 8.5: Because 2 is the unique mode, it must be repeated at least twice. So our list so far contains 2, 2. Since we want our largest number to be as large as possible, we want to minimize the other numbers. So let's put a 1 in our list, which is now 1, 2, 2. Adding a 1 a first time seemed like a good idea; what if we try it again? Then our list would be 1, 1, 2, 2, but since 2 is the unique mode, we need more 2s than 1s, so we would need another 2 and our list would become 1, 1, 2, 2, 2. However, this doesn't have a mean of 5! Thus, we can't include another 1. The next best option would be to include another 2 for the list to become 1, 2, 2, 2. Since the mean is 5, the sum of all 5 numbers is $5 \cdot 5 = 25$, so the last number, and the largest possible number, is $25 - 1 - 2 - 2 - 2 = \boxed{18}$.

These types of mean/median/mode/range problems occur *a lot* in MATHCOUNTS competitions, especially problems asking to to maximize or minimize a number. Be able to recognize them quickly and be sure to solve them confidently. What Serena often does when encountering a problem like this is to make N blanks, where N is the number of numbers in the list and then try to fill in the blanks with as small or as large of numbers as possible (if the problem is asking for the largest possible or smallest possible element, respectively). Also, always remember that if the mean of N numbers is m, the sum of the N integers is $N \cdot m$.

9 Solutions to Selected Problems

Problem 9.1: (Serena An)

(a) $4x - 17 = 7$
(b) $5a - 7 = 8 - a$
(c) $\frac{Z+5}{2} = 4$
(d) $2 - 4Q = 7$
(e) $3(\bigstar - 3) = 11 - 2\bigstar$
(f) $\frac{4}{5}\spadesuit - 5 = \frac{6-\spadesuit}{5}$
(g) $n^2 - 4 = 2n^2 - 13$
(h) $\sqrt{h+6} = 5$

Solution 9.1:

(a) $x = 6$
(b) $a = \frac{5}{2}$
(c) $Z = 3$
(d) $Q = -\frac{5}{4}$
(e) $\bigstar = 4$
(f) $\spadesuit = \frac{31}{5}$
(g) $n = 3$ or -3 (don't forget both solutions!)
(h) $h = 19$

Problem 9.2

The sum of 5 consecutive integers is 120. What is the largest of these integers?

Solution 9.2::

Quick way: Since we have five numbers, we can set the middle number to x. We now have the list $x-2, x-1, x, x+1, x+2$ as our five consecutive numbers. Adding these up, we have $5x = 120$. Thus, $x = 24$. However, remember we set our *middle* number to x, so adding two, we get that $\boxed{26}$ is the largest of these integers.

Even quicker way: The mean of the 5 integers is $\frac{120}{5} = 24$, so that must be the middle integer. Finally, we add 2 to get $\boxed{26}$ as our answer.

Slower way: Let l be the largest integer. Then the five integers are $l-4$, $l-3$, $l-2$, $l-1$, and l. Summing, we get $5l - 10 = 120$ and $l = \boxed{26}$.

With this problem, there wasn't too much of a time difference between the "quick" and "slower" ways, although the "quick" way would have likely been easier to do in your head. The "quick" and "quicker" ways have the exact same idea, but, the "quicker" way didn't explicitly use a variable or list out the entire sequence in terms of that variable.

Problem 9.3

The sum of 10 consecutive integers is 25. What is the smallest of these integers?

Solution 9.3:: The mean of these integers is $\frac{25}{10} = \frac{5}{2} = 2.5$. Because there are 10 consecutive integers, 5 integers are less than 2.5 and 5 integers are greater than 2.5. The middle-most two integers must be 2 and 3. Then the smallest integer is 4 less than 2, or $\boxed{-2}$.

Notice that the problem did not say that the integers had to be positive.

Problem 9.4

The product of 3 consecutive odd integers is 2145. What is the smallest of these integers?

Solution 9.4:: This problem is different from the previous ones, but nevertheless, let's try the usual approach. x is again the middle number, and the three consecutive odd integers are $x-2, x, x+2$. However, multiplying $(x-2)x(x+2)$ and then trying to find x is not only going to be really tedious, slow, and boring, it's not really going to help us.

Instead, we first prime factorize 2145, by first trying small primes. We see that that $\frac{2145}{3} = 715$. Also, $\frac{715}{5} = 143$. Finally, we know that $143 = 11 \cdot 13$. Thus, the complete prime factorization of 2145 is $3 \cdot 5 \cdot 11 \cdot 13$. Writing this as the product of 3 consecutive odd integers, $2145 = 11 \cdot 13 \cdot 15$, so our answer is $\boxed{11}$.

In this problem, we didn't use a variable, and that's OK. Variables are typically very helpful, but not in every single instance. If working with variables on a problem gets too messy and tedious, step back and try a different approach!

Problem 9.5

(a) $4x^2 + 19x - 26$

(b) $3 + 1 + 4 + 1 + 5 + 9 + 2 + 6 + 5$

(c) $20x^4 + 20x^7 + 3x$

(d) $5x^4 - 4x^3 + 3x^2 - 2x + 1$

(e) $10000000x - 987654321$

Solution 9.5:

(a) Quadratic trinomial

(b) Constant monomial

(c) Degree 7 trinomial

(d) Quartic

(e) Linear binomial