Pigeonhole Principle

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1 Introductory Problems

Problem 1.1. Suppose there are 101 pigeons and 100 holes. Prove that no matter how I place the pigeons into holes, at least one hole contains > 1 pigeon.

In general, if there are more pigeons than holes, at least one hole must contain > 1 pigeon.

Problem 1.2. I have a drawer with a large number of white, brown, and black socks. How many socks do I have to pull out of the drawer to ensure that I get a matching pair?

Problem 1.3. Given any 6 integers, prove that there are 2 of them whose difference is divisible by 5.

Problem 1.4. A subset B of the set of integers from 1 to 100, inclusive, has the property that no two elements of B sum to 125. What is the maximum possible number of elements in B? *(Source: AMC)*

Problem 1.5. Five points are chosen in a 2×2 square. Prove that two points are at most $\sqrt{2}$ apart.

Problem 1.6. Suppose that I place 25 balls into 6 boxes. Prove that one of the boxes must contain at least 5 balls.

Problem 1.7. Suppose that I place n balls into k boxes. What is the largest number m such that one of the boxes is guaranteed to contain at least m items?

The generalized **Pigeonhole Principle** states that if we place n pigeons into k holes, then at least one hole must contain at least $\left\lfloor \frac{n-1}{k} \right\rfloor + 1$ pigeons.

2 Advanced Problems

Problem 2.1. Given any set of 17 positive integers, prove that there is some subset of 5 of them whose sum is a multiple of 5.

Problem 2.2. Given any set of ten distinct 2-digit numbers, prove that there exist two disjoint subsets with the same sum. *(Source: IMO)*

Problem 2.3. Prove that for every prime p, there exists a Fibonacci number F_n divisible by p.

Problem 2.4. Prove that for every prime p except 2 and 5, there is a power of p that ends with the digits 0001.

Problem 2.5. Let r be any real number and let $n \geq 2$ be a positive integer. Show that at least one of $r, 2r, \ldots, (n-1)r$ differs from an integer by at most $\frac{1}{n}$.

Problem 2.6. Every point in the plane is colored either red, green, or blue. Prove that there exists a rectangle in the plane such that all four of its vertices are the same color. *(Source: USAMTS)*