Challenge #5 - Bases

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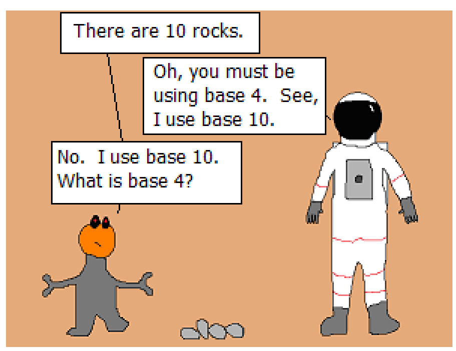
1 Introduction

Welcome to the Brookings Math Circle Summer 2018 Problem Sets! **Submissions for this problem set are due by Friday 8/31 at 8:00 pm (the day *before* the second BMC class of the second session).** Solutions will be released online at our website at 8:00 pm Friday. For a refresher on submission details or problem set details, see the introduction PDF linked to the Summer Challenge page of our website under Classes.

This handout will teach the basics of number bases. Solutions to problems in the handout are at the end.

As always, there will also be a puzzle and summer challenge problems, with rules as usual. Keep calm and do the math!

2 Introduction to Bases



Every base is base 10.

How many rocks are there? We would say that there are 4 (four) rocks, but why is that alien saying 10 rocks?

Our primary number system is the *base 10* system. This is probably because we have ten fingers and ten toes. However, other bases can be used to compute quantities. More popular examples of different bases are binary (base 2) and hexadecimal (base 16), which computers use, but any number greater than 1 can be a base! In fact, the subject of different bases is a rich source for many problems. All bases problems boil down to this: numbers can be expressed in different ways.

3 How Bases Work

Exercise 3.1

Write 5429 in expanded form.

Solution 3.1: 5429 expanded is $5000 + 400 + 20 + 9 = 5 \cdot 10^3 + 4 \cdot 10^2 + 2 \cdot 10^1 + 9 \cdot 10^0$.

The location of each digit determines its value. The first digit left of the decimal point is the *ones* place (10^0), the second digit left of the decimal point is the *tens* place (10^1), the third digit left of the decimal point is the *hundreds* place (10^2), and so on. In our expansion of 5429, 9 is multiplied by 10^0 because it is in the ones place, 2 is multiplied by 10^1 because it is in the tens place, 4 is multiplied by 10^2 because it is in the hundreds place, and 5 is multiplied by 10^3 because it is in the thousands place. In this problem, and in most of our computations, all the numbers we multiply by are powers of 10, meaning that we are using *base 10*. What if, instead of 10, we used a different number? We can extend this idea of "expanding numbers" to different bases.

Theorem 3.1: Converting from Other Bases to Base 10

Given a number $\overline{a_k a_{k-1} ... a_1 a_0}$ in base b, its base ten equivalent is

$$a_k \cdot b^k + a_{k-1} \cdot b^{k-1} + ... + a_1 \cdot b^1 + a_0 \cdot b^0.$$

The important idea to remember is that the rightmost digit is multiplied by b^0, the second rightmost digit is multiplied by b^1, and so on. Different bases are really not much of an extension of how base 10 works. Let's do an example.

Exercise 3.2

Convert the base 5 number 321 into base 10.

Solution 3.2: Instead of ones, tens, and hundreds, in base 5, the digits represent ones (5^0), fives (5^1), twenty fives (5^2). Thus, $321_5 = 3 \cdot 5^2 + 2 \cdot 5^1 + 1 \cdot 5^0 = 75 + 10 + 1 = \boxed{86}$ in base 10.

Now that we are working with different bases, we will introduce a convention. When a number is followed by a subscript, this subscript indicates what base that number is in. Numbers with no subscript should be assumed as base 10 numbers unless it is clear from the context that another base is being used. So, 321_5 represents the base 5 number 321.

Problem 3.1

Find the base 10 equivalents of the following numbers.

(a) 21_4

(b) 1001101_2

(c) 123_{16}

(d) 777_8

4 Base 10 to Different Bases

Now we know how to find the base 10 equivalent of a number given its representation in a different base. Converting from base 10 to different bases uses the exact same idea.

Theorem 4.1: Converting from Base 10 to Other Bases

Given a base 10 number N and a base c to convert into, this is the general way to convert N into base c.

Step 1: Identify the largest power of c, say c^k, that is less than or equal to N (i.e. $c^k \leq N < c^{k+1}$).

Step 2. Find the largest multiple of c^k, say $m \cdot c^k$, that is less than or equal to N (i.e. $m \cdot c^k \leq N < (m+1) \cdot c^k$). m is always less than or equal to $c-1$, for if $m \geq c$, then c^{k+1} would fit inside N!

Step 3. Subtract $m \cdot c^k$ from N.

Step 4. Repeat steps $1-3$, with the number you found in Step 3. Each time, the value of k, the largest power of c will decrease. Repeat until you get 0 in Step 3.

Step 5. Add all numbers in the form $m_i \cdot c^i$ computed in part 2 to get the expansion. The expansion $m_k \cdot c^k + m_{k-1} \cdot c^{k-1} + ... + m_1 \cdot c^1 + m_0 \cdot c_0$ can be written as the number $\overline{m_k m_{k-1} ... m_1 m_0}$ in base c, where m_i, the largest power of c^i calculated in Step 2 (this may be 0), is the $i+1$th rightmost digit.

This is very important, so we'll say it again:

Warning:

The largest digit that can be used in base c is $c-1$. Do you understand why you would never need a digit larger than $c-1$?

Don't worry if these steps don't quite make sense yet. The process should become clear with a few examples and problems. Also, you will soon see the word "fit" as shorthand for "is less than or equal to".

Exercise 4.1

Find the base 4 expansion of 520_{10}.

Solution 4.1:

Step 1: We will first identify the largest power of 4 that fits into 520. How about $4^4 = 256$? Indeed, 256 fits, but maybe we can go higher. How about $4^5 = 1024$? 1024 is too high, so we have to settle for 4^4.

Step 2: We check how many multiples of 4^4 fit. One multiple certainly fits, as $256 \leq 520$. How about

two multiples? This also fits, as $\mathbf{2} \cdot \mathbf{4^4} = 512 \leq 520$. However, 3 multiples is too much; $3 \cdot 4^4 = 768 > 520$.
Step 3: We subtract $2 \cdot 4^4$ from 520 to get $520 - 2 \cdot 4^4 = 520 - 512 = 8$.
Step 4: We now repeat the process with 8. What is the largest power of 4 that fits into 8? It is simply $4^1 = 4$. Two multiples of 4 fit into 8, as $8 = \mathbf{2} \cdot \mathbf{4^1}$. Subtracting, we have $8 - 2 \cdot 4^1 = 0$, so we are done with Step 4.
Step 5: Adding the bold numbers, we get the expansion $520 = 2 \cdot 4^4 + 2 \cdot 4^1$ (do a quick check to convince yourself!). This is written as 20020_4.

Problem 4.1

Convert the base 10 number 729 into the following bases.
(a) 9
(b) 7
(c) 5
(d) 2

5 Problems

Problem 5.1: (Serena An)

What digits are allowed in base n?

Problem 5.2: (Serena An)

Find the base 6 equivalent of 2018_9.

Problem 5.3: (Serena An)

Find the base 10 equivalents of 1_2, 11_2, 111_2, 1111_2, and 11111_2. Do you see a pattern?

Problem 5.4: (CMMS)

What is the base 10 value of the greatest five-digit base 4 integer?

Problem 5.5: (Serena An)

How many zeros are in the base 3 representation of the base 10 number 27^{10}?

Problem 5.6: (Divya Shyamal)

Find the number b such that 22_{10} written in base b satisifies the following:
(a) The number in base b has two digits.
(b) The first digit (leftmost) is twice the second digit (rightmost).

6 Puzzle #5

Problem 6.1: (krazydad.com)

				3		3	2		
2	1	1	1	3		3			2
		2	3	2					2
2	1				3	3			
		3	2					3	
	1		2	3	2			2	
2	1	3	1	2	1	0	2		
	2		2	3		2	0	2	3
		2				3	2		2
3		3			2				

The Slitherlink puzzle consists of a grid of dots, with some clue cells containing numbers. You connect horizontally or vertically adjacent dots to form a meandering path that forms a single loop or "Slitherlink." The loop must not have any branches and must not cross itself. The clue numbers indicate how many lines surround the cell. Empty cells may be surrounded by any number of lines (from 0 to 3). There is one unique solution, and you should be able to find it without guessing. You may find it helpful to make Xs between dots that cannot be connected.

7 Summer Challenge #5

Problem 7.1: (Serena An)

What is the product of the distinct positive integer factors of 36? Express your answer as a power of 6.

Problem 7.2: (Serena An)

Five people of different heights get in a line. What is the probability that the shortest person is immediately behind the tallest person?

Problem 7.3: (Serena An)

For how many positive values of n are both $\frac{n}{2}$ and $3n$ three digit base 5 integers?

Problem 7.4: (Serena An)

Triangles BMC and FUN are drawn in regular hexagon $BFMUCN$ with side length 6. What is the area of the overlap of the triangles?

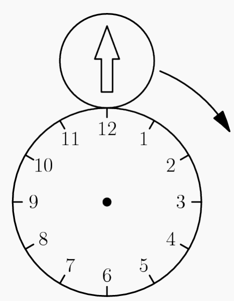
Problem 7.5: (Serena An)

The diagram below shows the circular face of a clock with radius 20 cm and a circular disk with radius 10 cm externally tangent to the clock face at 12 o'clock. The disk has an arrow painted on it, initially pointing in the upward vertical direction. The disk rolls clockwise around the clock face.

(a) At what point on the clock will the disk be tangent to when the arrow is pointing left for the first time?

(b) How many full rotations of the disk are needed to travel rotate around the clock face once?

(c) When the disk rotates completely around the clock face once, how many times is the arrow pointing left?



(Problem idea from AMC 10)

Reminder: Write a full solution to this problem, not just the answer. Even if you do not have a full solution, or a solution for every part, type/write up what you have and you may receive partial credit!

8 Solutions

Solution 3.1:

(a) $21_4 = 2 \cdot 4^1 + 1 \cdot 4^0 = \boxed{9}$

(b) $1001101_2 = 1 \cdot 2^6 + 1 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2^0 = \boxed{77}$

(c) $123_{16} = 1 \cdot 16^2 + 2 \cdot 16^1 + 3 \cdot 16^0 = \boxed{291}$

(d) $777_8 = 7 \cdot 8^2 + 7 \cdot 8^1 + 7 \cdot 8^0 = 1000_8 - 1_8 = 1 \cdot 8^3 - 1 \cdot 8^0 = \boxed{511}$

Solution 4.1:

(a) $729 = 1 \cdot 9^3 = \boxed{1000_9}$

(b) $729 = 2 \cdot 7^3 + 6 \cdot 7^1 + 1 \cdot 7^0 = \boxed{2061_7}$

(c) $729 = 1 \cdot 5^4 + 4 \cdot 5^2 + 4 \cdot 5^0 = \boxed{10404_5}$

(d) $729 = 1 \cdot 2^9 + 1 \cdot 2^7 + 1 \cdot 2^6 + 1 \cdot 2^4 + 1 \cdot 2^3 + 1 \cdot 2^0 = \boxed{1011011001}$

Solution 5.1:

The allowed digits in base n are the digits 0 through $n-1$.

Solution 5.2:

We will first convert 2018_9 into base 10, and then convert that number into base 6. $2018_9 = 2 \cdot 729 + 1 \cdot 9^1 + 8 \cdot 9^0 = 1475 = 1 \cdot 6^4 + 4 \cdot 6^2 + 5 \cdot 6^1 + 5 \cdot 6^0 = \boxed{10455_6}$.

Solution 5.3:

The respective base 10 equivalents are 1, 3, 7, 15, and 31. The pattern is that all of these numbers are one less than a power of 2. Do you know why? We will discuss this next week with a handout on arithmetic operations in different bases.

Solution 5.4:

$33333_4 = 100000_4 - 1_4 = 4^5 - 1 = \boxed{1023}$

Solution 5.5:

$27^{10} = 3^{30}$. In base 3, 3^{30} is represented by a 1 in the 31st rightmost place and 30 following 0s, so the answer is $\boxed{30}$.

Solution 5.6:

Let a be the rightmost digit, so the leftmost digit is $2a$. Then we have the equation $2a \cdot b + a = 22$. Factoring, $a(2b+1) = 22$. The factors of 22 are 1, 2, 11, and 22. If $a = 22$, $2b + 1 = 1$ and $b = 0$, which is impossible. If $a = 11$, $2b + 1 = 2$ and $b = \frac{1}{2}$, which is also impossible. If $a = 2$, $2b + 1 = 11$ and $b = 5$. This is indeed a solution because $42_5 = 22_{10}$. If $a = 1$, $2b+1 = 22$ and $b = \frac{21}{2}$, which is impossible. Thus, the only solution is $\boxed{b = 5}$