Challenge #2 - Basic Algebra

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1 Introduction

Welcome to the Brookings Math Circle Summer 2018 Problem Sets! **Submissions for this problem set are due by Sunday 6/17 at 1:00 pm (the day of the second BMC class).** For a refresher on submission details or problem set details, see the introduction PDF linked to the Summer Challenge page of our website under Classes.

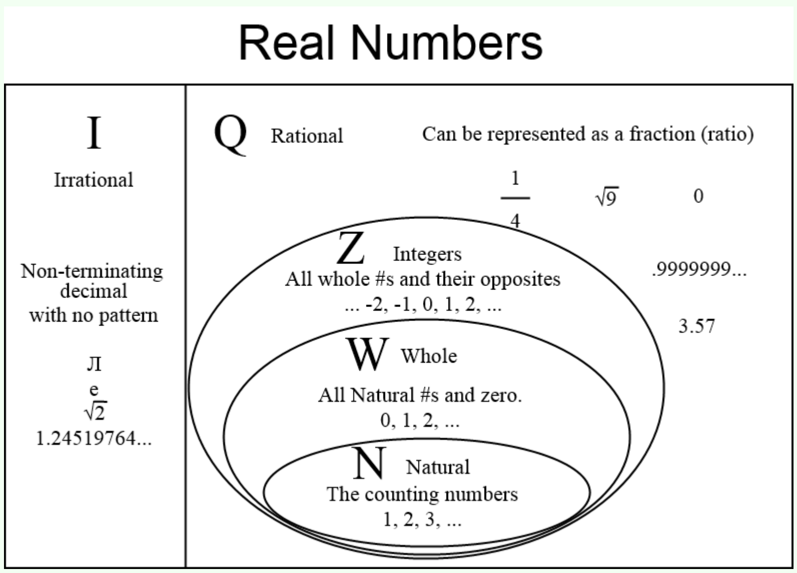
This handout and problem set will focus on basic algebra; in particular, we will cover constants, variables, single-variable equations and tricks, and polynomial basics. Next week we will delve into more advanced algebra, in particular linear equations and systems of equations, but with basics that you learn through this problem set, you will be able to tackle harder algebra problems soon! Keep calm and do the math!

Solutions to problems in the text are at the very back of this packet.

2 Constants

First, we must start with the types of numbers (or constants). Hopefully you are already familiar with most if not all of these terms. Notice how many of these groups fit into one another. Don't worry too much about the large letters; it's a shorthand but you will never see it in middle school math competitions, but you will see the words "real", "irrational", "rational", "integers", "whole", and "natural" quite frequently.

Theorem 2.1: Number Sets



3 Variables and Single-Variable Equations

Fundamentals are important in every branch of math, but perhaps even more so in algebra, which can be seen as a foundation to other branches. Algebra begins with the notion of variables and equations, which you may already be familiar with.

Prerequisites: The ability to do basic arithmetic following the order of operations.

Variables are unknown quantities. Typically, variables are represented by lowercase letters (such as x, y, a, or n), and less frequently you will see variables represented by uppercase letters (such as A or E) or symbols (such as $\diamond$ or $\clubsuit$). Typically, the question will require you to come up with a variable, which may represent the quantity you are solving for or another helpful quantity. Every time the same variable shows up in an expression, it represents the same quantity, but different variables may represent different quantities.

Equations include the equals sign "=" and show that the two sides are equal. The "golden rule" of algebra is that if you do something to one side of an equation, you must do the same thing to the other side. For example, if you multiply one side by 4, you must also multiply the other side by 4. The most basic equations include one variable and typically ask you to solve for that particular variable. You can check your work by plugging in the value you got for the variable and seeing if both sides are equal. Solve for each variable: (answers are at the back of this handout)

Problem 3.1: (Serena An)

(a) $4x - 17 = 7$
(b) $5a - 7 = 8 - a$
(c) $\frac{Z+5}{2} = 4$
(d) $2 - 4Q = 7$
(e) $3(\bigstar - 3) = 11 - 2\bigstar$
(f) $\frac{4}{5}\spadesuit - 5 = \frac{6-\spadesuit}{5}$
(g) $n^2 - 4 = 2n^2 - 13$
(h) $\sqrt{h+6} = 5$

Always simplify your final answer if it involves a fraction.

Hopefully these problems were not bad. Maybe you even got many of them through guess and check or mental math! However, typically in math competitions, the answer will not be easily guessed and the problem will not be as easily done in your head. In addition, typically, you must set up the equation yourself. Thus, being able to write and solve equations is an essential skill that can get you far in math competitions. This will come with practice.

Warning:

Whenever we square an equation as a step in solving it, we must check that our solutions are valid. If a solution to the squared equation does not satisfy the original equation, it is an **extraneous** solution. Extraneous solutions are not valid solutions to the original equation. Basically, if a solution doesn't check (and typically doesn't make sense), it's extraneous.

For example, consider the equation $\sqrt{x} + 5 = 2$. Subtracting 5, we get $\sqrt{x} = -3$ which has no solutions. However, if we square $\sqrt{x} = -3$, we get $x = 9$, which doesn't satisfy the original equation of $\sqrt{x} + 5 = 2$. Thus 9 is an **extraneous** solution. The problem arises when we take the square root of $x = 9$ to evaluate $\sqrt{x}$ in the original equation. This gies us $\sqrt{x} = 3$ not $\sqrt{x} = -3$ because we define $\sqrt{x}$ for positive numbers x to be the *positive* number whose square is x.

A similar warning is to never forget the negative solution when you have an equation including x^2 such as $x^2 = 36$, which has solutions of $x = 6$ AND $x = -6$. *Typically*, negatives are not extraneous solutions when we have a quantity squared, but they are with square roots.

HINT: When labeling variables, try to keep them easy to remember, which typically means using the first letter of a word. For example, if you want to find the **h**eight of something, use h, or if you want to find **A**nn's age, use a. This keeps things easy to remember and will save you time, as you don't have to continuously wonder which variable is what.

4 Tricks for Solving Specific Problems

Note that consecutive integers can be written as $x, x+1, x+2, ...$, where x is the *smallest* integer in the list. However, some problems can be solved easier and faster by setting x to be the *middle* number. In addition, sometimes, the solving the problem may be faster if you *don't* use a variable!! You'll be able to recognize these special cases as you do more problems.

Solutions are in the back of the packet, once more.

Problem 4.1

The sum of 5 consecutive integers is 120. What is the largest of these integers?

Problem 4.2

The sum of 10 consecutive integers is 25. What is the smallest of these integers?

Problem 4.3

The product of 3 consecutive odd integers is 2145. What is the smallest of these integers?

5 Polynomial Basics

Before we get into linear equations, we need a few definitions.

A **polynomial** in one variable is an expression of the form

$$a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_1 x + a_0,$$

where x is a variable and each a_i is a constant. Each $a_i x^i$ is called a **term** and the a_i are called **coefficients**. The **degree** of the polynomial is the highest integer n such that its coefficient (a_n) is nonzero. Basically, the **degree** is the largest exponent.

There are some special names for polynomials of smaller degrees. A polynomial of degree 0 is **constant**, degree 1 is **linear**, degree 2 is **quadratic**, degree 3 is **cubic**, degree 4 is **quartic**, and for anything above degree 4, just call it a **polynomial**.

If you're wondering why degree 2 is *quad*ratic, it is because **quadra** refers to a square, and degree 2 polynomials include something squared. Similarly, *cubic* refers to a cube, and degree 3 polynomials include something cubed.

There a way to group polynomials based on the number of terms they have (this is not necessarily degree). A polynomial with 1 term is a **monomial**, 2 terms is a **binomial**, 3 terms is a **trinomial**. So $4x^7$ is a monomial of degree 7, $3x^2 + 1$ is a quadratic binomial, and $x^3 + 3x^2 + 6$ is a cubic trinomial.

Classify the following polynomials by degree and the number of terms. If there are more than three terms, just give the degree.

Problem 5.1

(a) $4x^2 + 19x - 26$

(b) $3 + 1 + 4 + 1 + 5 + 9 + 2 + 6 + 5$

(c) $20x^4 + 20x^7 + 3x$

(d) $5x^4 - 4x^3 + 3x^2 - 2x + 1$

(e) $10000000x - 987654321$

6 Puzzle #2

Problem 6.1: (USAMTS)

Fill in the spaces of the grid below with positive integers so that in each 2×2 square with top left number a, top right number b, bottom left number c, and bottom right number d, either $a + d = b + c$ or $ad = bc$. There is only one solution.

3	9			
	11		7	2
10				16
15				
20	36			32

7 Bonus Problem Set #2

Problem 7.1: (AMC 8)

Four students take an exam. Three of their scores are $70, 80$, and 90. If the average of their four scores is 70, then what is the remaining score?

Problem 7.2: (CMMS)

The height of a rectangle is three centimeters more than twice its length. If the perimeter of the rectangle is 60 cm, what is its area?

Problem 7.3: (CMMS)

Molly's father James is three years less than three times her age. How many years from now will Molly's father be twice her age if he is 33 today?

Problem 7.4: (AMC 8)

If $3^p + 3^4 = 90$, $2^r + 44 = 76$, and $5^3 + 6^s = 1421$, what is the product of p, r, and s?

Problem 7.5: (AIME)

Let x, y, and z all be greater than one. Let w be a positive integer such that

$$x^{24} = w, \; y^{40} = w, \; \text{and } (xyz)^{12} = w.$$

Find a positive integer a such that $z^a = w$.

Problem 7.6: (Intermediate Algebra)

Find all z such that $\frac{1}{\sqrt{z-3}-1} + 2 = \frac{3}{\sqrt{z-3}-1}$.

Problem 7.7: (Intermediate Algebra)

Find all values of c such that the equation $\frac{3}{2-\frac{1}{x}} = c$ has no solutions for x.

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Problem 8.1: (Serena An)

If $x = 3$, $y = 4$, and $z = 5$, what is the value of $\sqrt{x^2 + y^2} + 5z^2 - (x^2 + y^2)$?

Problem 8.2: (Serena An)

The sum of Ann, Ben, and Cathy's ages is 35. If Ben is 18 years older than half of Ann's age and Cathy is 3 years older than twice Ann's age, find the sum of Ben and Cathy's ages.

Problem 8.3: (Serena An)

What's the smallest positive integer that is the product of 3 consecutive integers, one of which is a prime, another one of which is a multiple of 5, and the last one is a multiple of 8 (in some order)?

Problem 8.4: (Serena An)

If $\sqrt{x + \sqrt{x + \sqrt{x + ...}}} = 12$, what is x?

Problem 8.5: (Serena An)

A list of 5 positive integers has a mean of 5 and a unique mode of 2. What is the greatest possible number in the list? (The unique mode is the only number that is repeated the most times; so basically, there are more 2s in the list than any other number.)

Reminder: Write a full solution to this problem, not just the answer. Even if you do not have a full solution, type/write up what you have and you may receive partial credit!

9 Solutions to Selected Problems

Problem 9.1: (Serena An)

(a) $4x - 17 = 7$

(b) $5a - 7 = 8 - a$

(c) $\frac{Z+5}{2} = 4$

(d) $2 - 4Q = 7$

(e) $3(\bigstar - 3) = 11 - 2\bigstar$

(f) $\frac{4}{5}\spadesuit - 5 = \frac{6-\spadesuit}{5}$

(g) $n^2 - 4 = 2n^2 - 13$

(h) $\sqrt{h+6} = 5$

Solution 9.1:

(a) $x = 6$

(b) $a = \frac{5}{2}$

(c) $Z = 3$

(d) $Q = -\frac{5}{4}$

(e) $\bigstar = 4$

(f) $\spadesuit = \frac{31}{5}$

(g) $n = 3$ or -3 (don't forget both solutions!)

(h) $h = 19$

Problem 9.2

The sum of 5 consecutive integers is 120. What is the largest of these integers?

Solution 9.2::

Quick way: Since we have five numbers, we can set the middle number to x. We now have the list $x-2, x-1, x, x+1, x+2$ as our five consecutive numbers. Adding these up, we have $5x = 120$. Thus, $x = 24$. However, remember we set our *middle* number to x, so adding two, we get that $\boxed{26}$ is the largest of these integers.

Even quicker way: The mean of the 5 integers is $\frac{120}{5} = 24$, so that must be the middle integer. Finally, we add 2 to get $\boxed{26}$ as our answer.

Slower way: Let l be the largest integer. Then the five integers are $l-4$, $l-3$, $l-2$, $l-1$, and l. Summing, we get $5l - 10 = 120$ and $l = \boxed{26}$.

With this problem, there wasn't too much of a time difference between the "quick" and "slower" ways, although the "quick" way would have likely been easier to do in your head. The "quick" and "quicker" ways have the exact same idea, but, the "quicker" way didn't explicitly use a variable or list out the entire sequence in terms of that variable.

Problem 9.3

The sum of 10 consecutive integers is 25. What is the smallest of these integers?

Solution 9.3:: The mean of these integers is $\frac{25}{10} = \frac{5}{2} = 2.5$. Because there are 10 consecutive integers, 5 integers are less than 2.5 and 5 integers are greater than 2.5. The middle-most two integers must be 2 and 3. Then the smallest integer is 4 less than 2, or $\boxed{-2}$.

Notice that the problem did not say that the integers had to be positive.

Problem 9.4

The product of 3 consecutive odd integers is 2145. What is the smallest of these integers?

Solution 9.4:: This problem is different from the previous ones, but nevertheless, let's try the usual approach. x is again the middle number, and the three consecutive odd integers are $x-2, x, x+2$. However, multiplying $(x-2)x(x+2)$ and then trying to find x is not only going to be really tedious, slow, and boring, it's not really going to help us.

Instead, we first prime factorize 2145, by first trying small primes. We see that that $\frac{2145}{3} = 715$. Also, $\frac{715}{5} = 143$. Finally, we know that $143 = 11 \cdot 13$. Thus, the complete prime factorization of 2145 is $3 \cdot 5 \cdot 11 \cdot 13$. Writing this as the product of 3 consecutive odd integers, $2145 = 11 \cdot 13 \cdot 15$, so our answer is $\boxed{11}$.

In this problem, we didn't use a variable, and that's OK. Variables are typically very helpful, but not in every single instance. If working with variables on a problem gets too messy and tedious, step back and try a different approach!

Problem 9.5

(a) $4x^2 + 19x - 26$

(b) $3 + 1 + 4 + 1 + 5 + 9 + 2 + 6 + 5$

(c) $20x^4 + 20x^7 + 3x$

(d) $5x^4 - 4x^3 + 3x^2 - 2x + 1$

(e) $10000000x - 987654321$

Solution 9.5:

(a) Quadratic trinomial

(b) Constant monomial

(c) Degree 7 trinomial

(d) Quartic

(e) Linear binomial