Week 1: Patterns

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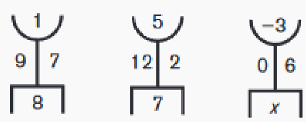
1 Exercises

If a problem doesn't have a source, it's from the Patterns Stretch in the 2017-18 MATHCOUNTS Handbook (Warm-up Problems 11-20). The apples are our own rating system of the "difficulty" – attempt all of them though! 3 apples is 'average'.

None of these problems require the use of a calculator! Email us if you attempt any of these throughout the week and get stuck.

Problem 1.1:

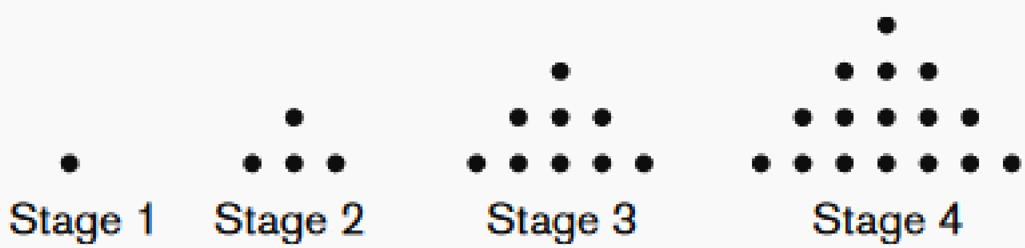
If the pattern continues, what is the value of x in the third figure?



Solution 1.1: The number on the top is the common difference between the other three numbers (going left to right), so the missing number is $\boxed{3}$.

Problem 1.2:

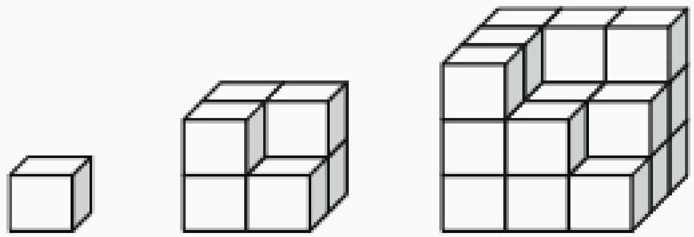
The first four stages of a dot pattern are shown. How many more dots are in the figure at Stage 47 than in the figure at Stage 27?

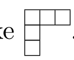
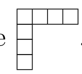


Solution 1.2: At Stage n, there are n^2 dots. This can be seen by cutting each triangle into two smaller right triangles that fit together in an $n \times n$ square. The answer is $47^2 - 27^2 = \boxed{1480}$.

Problem 1.3: MATHCOUNTS 15-16 Warm-up #152

If this pattern continues, how many cubes will be in the next figure?



Solution 1.3: To get from the first figure to the second, we add two levels of 3 cubes shaped like . To get from the second figure to the third, we add three levels of 5 cubes shaped like . Then, to get from the third figure to the fourth, we add four levels of 7 cubes shaped like . In all, there are $1 + 2 \cdot 3 + 3 \cdot 5 + 4 \cdot 7 = \boxed{50}$ cubes.

Problem 1.4:

The first three terms of a sequence are 1, 2 and 3. Each subsequent term is the sum of the three previous terms. What is the 11th term of this sequence?

Solution 1.4: We can just brute force this:

- The fourth term is $1 + 2 + 3 = 6$
- The fifth term is $2 + 3 + 6 = 11$
- The sixth term is $3 + 6 + 11 = 20$
- The seventh term is $6 + 11 + 20 = 37$
- The eighth term is $11 + 20 + 37 = 68$
- The ninth term is $20 + 37 + 68 = 125$
- The tenth term is $37 + 68 + 125 = 230$
- The eleventh term is $68 + 125 + 230 = \boxed{423}$

Problem 1.5:

What is the sum of the terms in the arithmetic sequence $2 + 5 + 8 + 11 + 14 + ... + 89 + 92$?

Solution 1.5: The average of all of the numbers in the arithmetic sequence is $\frac{2+92}{2} = 47$. Now we want to find the number of terms in the arithmetic sequence.

If we add one to each number, the series becomes 3, 6, 9, 12, 15, ... 93. If we divide every number by 3 now, we get the series 1, 2, 3, 4, 5 ... 31, so there are 31 numbers.

Therefore, the sum of all the numbers is $47 \times 31 = \boxed{1457}$.

Problem 1.6:

Three consecutive terms in an arithmetic sequence are x, $2x+11$ and $4x-3$. What is the constant difference between consecutive terms in this sequence?

Solution 1.6: Arithmetic sequences have a constant difference between two consecutive terms. Then

$$\begin{aligned}(4x-3)-(2x+11) &= (2x+11)-(x)\\ \implies 2x-14 &= x+11\\ \implies x &= 25.\end{aligned}$$

The constant difference is $2x-14 = x+11 = \boxed{36}$.

Problem 1.7:

What is the sum of the terms in the geometric sequence $1 + 4 + 16 + ... + 1024$?

Solution 1.7: We can just add $1+4+16+64+256+1024$ to get $\boxed{1365}$.

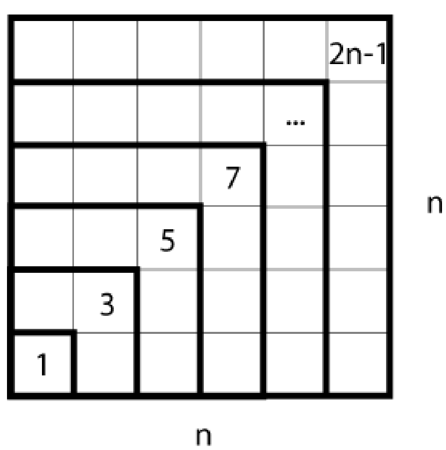
Alternatively, by the geometric sequence formula, $1+4^1+4^2+4^3+4^4+4^5 = \frac{4^6-1}{4-1} = \boxed{1365}$.

Problem 1.8:

What is the sum of the first 51 consecutive odd positive integers?

Solution 1.8: We compute $1 = 1, 1+3 = 4, 1+3+5 = 9$, and $1+3+5+7 = 16$, and guess that the sum of the first n consecutive odd positive integers is n^2. The answer is $51^2 = \boxed{2601}$.

To prove this, we can use a visualization of squares.



Problem 1.9:

What is the sum of the terms in the infinite series

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots$$

Solution 1.9: There's a sweet trick to infinite series: Let's set the entire infinite series to x:

$$x = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots$$

Now, since we are multiplying by $\frac{1}{2}$ each time, let's multiply the entire thing by the inverse of $\frac{1}{2}$, or 2.

$$2x = 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots$$

Notice a pattern? The second part is just x! We can substitute this back in:

$$2x = 2 + x$$

or, subtracting x from both sides, $x = \boxed{2}$.

Problem 1.10:

What is the sum in the infinite series

$$1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \dots$$

Solution 1.10: Let's set the entire infinite series to x:

$$x = 1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \dots$$

Now, since we are multiplying by $\frac{1}{4}$ each time, let's multiply the entire thing by the inverse of $\frac{1}{4}$, or 4.

$$4x = 4 + 1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \dots$$

Notice a pattern? The second part is just x! We can substitute this back in:

$$4x = 4 + x$$

or, subtracting x from both sides, $3x = 4$, so $x = \boxed{\frac{4}{3}}$.

Problem 1.11: MATHCOUNTS 2016-17 Warm-Up #11

What digit is in the units place in the product (don't calculate either term!)

$$3^{17} \cdot 7^{23}?$$

Solution 1.11: The units digit of powers of 3 repeats every four: 3, 9, 7, 1, 3, 9, 7, 1, etc. The units digit of powers of 7 also repeats every four: 7, 9, 3, 1, 7, 9, 3, 1, etc. Then the units digit of 3^{17} is 3 and the units digit of 7^{23} is 3, so the units digit of the product is $\boxed{9}$.

Problem 1.12:

Let $f(x) = 2x + 3$ and $f^2(x) = f(f(x)) = f(2x + 3) = 2(2x + 3) + 3 = 4x + 9$. If $f^5(x) = ax + b$, what is the value of $a + b$?

Solution 1.12: Note that $a + b = f^5(1)$. Then we can compute $f(1) = 5, f^2(1) = f(5) = 13, f^3(1) = f(13) = 29, f^4(1) = f(29) = 61$, and $f^5(1) = f(61) = \boxed{125}$.

Problem 1.13:

The degree measures of the interior angles of a quadrilateral form a geometric sequence whose terms have integer values and are all integer multiples of the first term. What is the largest possible degree measure of an angle in this quadrilateral?

Solution 1.13: Let the angles have measure a, ar, ar^2, ar^3, where $a$and r are a positive integers. Then $a(1+r+r^2+r^3) = 360$ and we want to maximize ar^3. We can find all values of r such that $1+r+r^2+r^3$ divides 360: $r = 1, 2, 3$. This leads to respective a values of 90, 24, and 9, and respective ar^3 values of 90, 192, and 243, so the answer is $\boxed{243}$.

Problem 1.14: MATHCOUNTS 2016-17 Warm-up #98

What is the maximum number of distinct intersections of 30 different coplanar[a] circles?

[a]Coplanar means in the same plane, if you don't know what it means ignore it – you'll get the same answer regardless

Solution 1.14: *(Official MATHCOUNTS Solution)* It's virtually impossible to draw 30 circles carefully enough to count the number of distinct intersections. You might try drawing one circle and then adding more circles, one at a time, and counting intersections to see if a pattern emerges. Doing so reveals that with 1 circle, there are 0 intersections; with 2 circles, there are, at most, 2 intersections; with 3 circles, there are, at most, 6 intersections; with 4 circles, there are, at most, 12 intersections. Organizing these findings in a table like the one shown, we see that for n circles, there appear to be $n(n-1)$ distinct intersections. Therefore, 30 circles will have, at most, $30 \cdot 29 = \boxed{870}$ intersections.

Alternatively, since every pair of circles intersects in, at most, 2 points, and there are ${}_{30}C_2$ ways to select a pair from a collection of 30 circles, it follows that there are $2 \cdot (30 \cdot 29)/2 = 30 \cdot 29 = \boxed{870}$ intersections.