Challenge #4 - Trail Mix

Serena An–August 13, 2018

1 Introduction

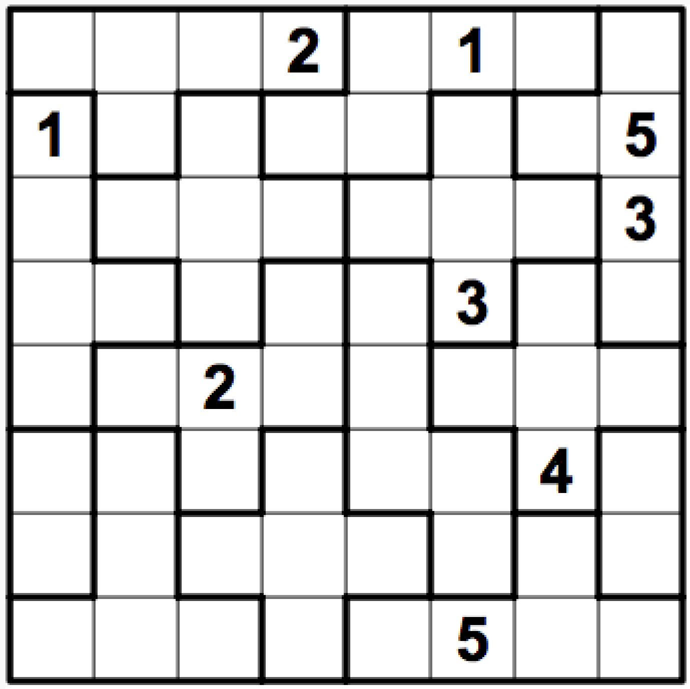
Welcome to the Brookings Math Circle Summer 2018 Problem Sets! **Submissions for this problem set are due by Saturday 8/18 at 1:30 pm (the day of the first BMC class of the second session).** For a refresher on submission details or problem set details, see the introduction PDF linked to the Summer Challenge page of our website under Classes.

This problem set will consist of a variety of problems from the four main subjects: algebra, counting and probability, number theory, and geometry. Each section consists of 5 problems, roughly ordered by difficulty. There are hints to selected problems in the back. After a problem, if there is *Hint:* followed by a number, the hint to that problem is by that number. Some problems are very hard, so don't worry if you can't solve many of them! In the upcoming class, we will discuss these problems.

As always, there will also be a puzzle and summer challenge problems, with rules as usual. Keep calm and do the math!

2 Puzzle #4

Problem 2.1: (krazydad.com)



The grid is subdivided into containers or cages, each of which is 1 to 5 cells in size. Fill each container with unique digits, counting up from 1. So for example a 2-square container contains the numbers 1 and 2. A 5-square container contains the numbers from 1 to 5. Adjacent (touching) cells may never contain the same number, and this includes diagonally adjacent cells. There is only one solution. Good luck!

3 Algebra

Problem 3.1: (MATHCOUNTS National CD)

What is the slope of a line perpendicular to the hypotenuse of a triangle with vertices $X(-1,-1)$, $Y(2,-1)$ and $Z(-1,3)$? Express your answer as a common fraction.

Problem 3.2: (MATHCOUNTS)

What is the value of the quotient $\frac{(1-\frac{1}{2})(1-\frac{2}{3})(1-\frac{3}{4})\cdot\ldots\cdot(1-\frac{99}{100})}{\frac{1}{101!}}$?

Note that $n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 1$ (pronounced "n factorial"), so for example, $4! = 4 \cdot 3 \cdot 2 \cdot 1$.

Problem 3.3: (AMC 10)

A rectangle with a diagonal of length x is twice as long as it is wide. What is the area of the rectangle? Express your answer in terms of x. *Hint: 1*

Problem 3.4: (MPfG)

If a and b are nonzero real numbers such that $|a| \neq |b|$, compute the value of the expression

$$\left(\frac{b^2}{a^2} + \frac{a^2}{b^2} - 2\right) \times \left(\frac{a+b}{b-a} + \frac{b-a}{a+b}\right) \times \left(\frac{\frac{1}{a^2} + \frac{1}{b^2}}{\frac{1}{b^2} - \frac{1}{a^2}} - \frac{\frac{1}{b^2} - \frac{1}{a^2}}{\frac{1}{a^2} + \frac{1}{b^2}}\right).$$

Hint: 2

Problem 3.5: (AMC 10)

David drives from his home to the airport to catch a flight. He drives 35 miles in the first hour, but realizes that he will be 1 hour late if he continues at this speed. He increases his speed by 15 miles per hour for the rest of the way to the airport and arrives 30 minutes early. How many miles is the airport from his home? *Hint: 3*

4 Counting & Probability

Problem 4.1: (AMC 10)

Jo and Blair take turns counting from 1 to one more than the last number said by the other person. Jo starts by saying "1", so Blair follows by saying "1, 2". Jo then says "1, 2, 3", and so on. What is the 53^{rd} number said?

Problem 4.2: (AMC 10)

A set of three points is randomly chosen from the grid shown. Each three point set has the same probability of being chosen. What is the probability that the points lie on the same straight line?

Problem 4.3: (Serena An)

Serena selects 3 positive integers less than or equal to 10. What is the probability that their product is odd?

Problem 4.4: (AMC 10)

How many three-digit numbers satisfy the property that the middle digit is the average of the first and last digits? *Hint: 4*

Problem 4.5: (MPfG)

Say that a 4-digit positive integer is mixed if it has 4 distinct digits, its leftmost digit is neither the biggest nor the smallest of the 4 digits, and its rightmost digit is not the smallest of the 4 digits. For example, 2013 is mixed. How many 4-digit positive integers are mixed?

5 Number Theory

Problem 5.1: (AMC 10)

Three positive integers are each greater than 1, have a product of 27000, and are pairwise relatively prime. What is their sum? ("Pairwise relatively prime" means that any two of them share no common divisors besides 1.)

Problem 5.2: (MPfG)

Find the smallest two-digit positive integer that is a divisor of 201020112012.

Problem 5.3: (MPfG)

If m and n are integers such that $3m + 4n = 100$, what is the smallest possible value of $|m - n|$?

Problem 5.4: (MATHCOUNTS)

How many integers n satisfy the condition $100 < n < 200$ and the condition n has the same remainder whether it is divided by 6 or 8?

Problem 5.5: (Serena An)

Let $S(n)$ equal the sum of the digits of n. Let $S^x(n)$ equal $S(S(...S(n)...))$, where the number of Ss is x. Evaluate $S^{100}(2^{2018})$. *Hints: 5, 6*

6 Geometry

Problem 6.1: (AMC 8)

In rectangle $ABCD$, $AB = 6$ and $AD = 8$. Point M is the midpoint of $\overline{AD}$. What is the area of $\triangle AMC$?

Problem 6.2: (AMC 10)

The two legs of a right triangle, which are altitudes, have lengths $2\sqrt{3}$ and 6. How long is the third altitude of the triangle?

Problem 6.3: (MATHCOUNTS)

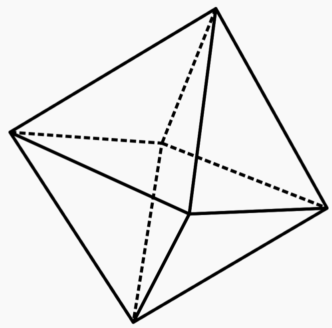
Circles O and P are tangent to the x-axis at $(4, 0)$ and $(13, 0)$, respectively. respectively. Point $A(4, 8)$ is on circle O and point $B(13, 10)$ in on circle P. Line segment OP connects the center of circle O to the center of circle P what is the slops of segment OP? Express your answer as a common fraction.

Problem 6.4: (AMC 10)

The y-intercepts, P and Q, of two perpendicular lines intersecting at the point $A(6, 8)$ have a sum of zero. What is the area of $\triangle APQ$? *Hint: 7*

Problem 6.5: (Serena An)

The sides of an octahedron are all 1. What is the volume of the octahedron?



(An octahedron is an 12-sided 3D figure. It is two square pyramids glued together at the bases.)

7 Summer Challenge #4

Problem 7.1: (Serena An)

For what two digit number N is N equal to 8 times the sum of its digits?

Problem 7.2: (Serena An)

There are 20 acts in a talent show and each ordering of acts is equally likely. What is the probability that the didgeridoo act is before the piano violin duet and the comedy skit?

Problem 7.3: (Serena An)

A plane has a constant speed p and flies from Seattle to Boston, and then from Boston back to Seattle. Assume there is wind of a positive, constant speed w from Seattle to Boston throughout the trip, so the plane's speed from Seattle to Boston is $p+w$ and from Boston to Seattle is $p-w$. Is the total flight time with wind more or less than if there was no wind?

Problem 7.4: (Serena An)

Triangle ABC has area 80. The midpoints of AB, BC, and CA are C_1, A_1, and B_1, respectively. The midpoints of A_1B_1, B_1C_1, and C_1A_1 are C_2, A_2, and B_2, respectively. What is the area of triangle $A_2B_2C_2$?

Problem 7.5: (Serena An)

(a) How many ways are there to color the vertices of a hexagon using two colors if rotations and reflections are considered distinct?

(b) How many ways are there to color the vertices of an n-gon using two colors if rotations and reflections are considered distinct? Express your answer in terms of n.

(c) How many ways are there to color the vertices of a hexagon using two colors if rotations are *not* considered distinct and reflections are considered distinct?

Reminder: Write a full solution to this problem, not just the answer. Even if you do not have a full solution, or a solution for every part, type/write up what you have and you may receive partial credit!

Problem 7.6:

Write your own problem from any subject area and of any difficulty! Problems should be fun to solve! Don't expect to be able to instantly come up with good problems; it will typically take awhile and some thought. Submitting one problem will earn **3** points and submitting two problems will earn **6** points. Additional submissions will not earn additional points, although they are welcomed! You may submit as many problems as you'd like to serena.an@gmail.com. These problems may appear on future problem sets, and possibly even a BMC created mock competition (such as an AMC 8)! There will be more discussion of this in class.

8 Hints to Selected Problems

1. Let the width of the rectangle be w. Use the Pythagorean Theorem.
2. Remember that $a^2 - b^2 = (a+b)(a-b)$. Clear denominators and simplify carefully.
3. Let d be the distance still needed to travel after the first 1 hour. If t is the amount of time until his flight after the first hour, write two equations for t in terms of d.
4. The middle digit is determined by the first and last digits, if the sum of the first and last digits is even.
5. $S(n)$ is significantly smaller than n. When does $S(n) = n$?
6. If you apply $S(n)$ enough times, you will end up with a one digit number. Which one? Experiment with some n and see if you can find the pattern.
7. The distances from the midpoint of the hypotenuse to all three vertices of a right triangle are all equal.