HW15: 11.11

Tuesday, December 12, 2023 8:31 PM

$\boxed{11.11: 1, 2, 3, 4, 5}$

① Use second degree Taylor polynomial for $\cos(x)$ near $x=0$. Use this to approximate $\cos(.1)$. Compute using calculator.

$$T_2(x) = \sum_{K=0}^{2} \frac{f^{(n)}(a)}{K!}(x-a)^K \qquad a=0$$

$$\left.\begin{aligned} f^{(0)}(0) &= \cos(0) = 1 \\ f^{(1)}(0) &= -\sin(0) = 0 \\ f^{(2)}(0) &= -\cos(0) = -1 \end{aligned}\right\} T_2(x) = 1 - \frac{x^2}{2}$$

$$T_2(.1) = 1 - \frac{(.1)^2}{2} = 1 - \frac{.001}{2} = 1 - .005$$

$$= .995$$

Calculator: $\cos(.1) = .995004$

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② Write third degree Taylor polynomial for $\sqrt{x}$ near $x=16$. Use this to approximate $\sqrt{15}$ and $\sqrt{17}$. Compute using Calculator.

$$T_3(x) = \sum_{K=0}^{3} \frac{f^{(n)}(a)}{K!}(x-a)^K \qquad a=16$$

$$\left.\begin{aligned} f^{(0)}(16) &= \sqrt{16} = 4 & 0! &= 1 \\ f^{(1)}(16) &= \frac{1}{2\sqrt{16}} = \frac{1}{8} & 1! &= 1 \\ f^{(2)}(16) &= -\frac{1}{4(16)^{\frac{3}{2}}} = -\frac{1}{256} & 2! &= 2 \\ f^{(3)}(16) &= \frac{3}{8(16)^{\frac{5}{2}}} = \frac{3}{8192} & 3! &= 6 \end{aligned}\right\} T_3(x) = 4 + \frac{(x-16)}{8} - \frac{(x-16)^2}{512} + \frac{(x-16)^3}{16384}$$

$$T_3(15) = 4 + \frac{(-1)}{8} - \frac{(-1)^2}{512} + \frac{(-1)^3}{16384} = 4 - \frac{1}{8} - \frac{1}{512} - \frac{1}{16384} = 3.872985$$

Calculator: $\sqrt{15} = 3.872983$

$$T_3(17) = 4 + \frac{1}{8} - \frac{1}{512} + \frac{1}{16384} = 4.1231079$$

Calculator: $\sqrt{17} = 4.1231056$

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(3) Write third degree Taylor Polynomial for $\sqrt[3]{x}$ near 27. Use this to approximate $\sqrt[3]{25}$ and $\sqrt[3]{29}$. Calculate using Calculator.

$$f^{(0)}(27) = \sqrt[3]{27} = 3 \qquad 0! = 1$$
$$f^{(1)}(27) = \frac{1}{3(27)^{\frac{2}{3}}} = \frac{1}{27} \qquad 1! = 1$$
$$f^{(2)}(27) = -\frac{2}{9(27)^{\frac{5}{3}}} = -\frac{2}{2187} \qquad 2! = 2$$
$$f^{(3)}(27) = \frac{10}{27(27)^{\frac{8}{3}}} = \frac{10}{177147} \qquad 3! = 6$$

$$T_3(x) = 3 + \frac{(x-27)}{27} - \frac{2(x-27)^2}{4374} + \frac{10(x-27)^3}{1062882}$$

$$\boxed{\begin{aligned} &T_3(25) = 3 + \frac{(-2)}{27} - \frac{2(-2)^2}{4374} + \frac{10(-2)^3}{1062882} = 3 - .074 - .00183 - .000075 = 2.924021 \\ &\text{Calculator: } \sqrt[3]{25} = 2.924017 \\ &T_3(29) = 3 + \frac{2}{27} - \frac{2(2)^2}{4374} + \frac{10(2)^3}{1062882} = 3 + .074 - .00183 + .000075 = 3.072320 \\ &\text{Calculator: } \sqrt[3]{29} = 3.072317 \end{aligned}}$$

(4) Write third degree Taylor Polynomial for e^x near 0. Use this to approximate $e^{-\frac{1}{2}}$. Use Alt. series estimation theorem to state upper bound for error of approximation.

$$f^{(0)}(0) = e^0 = 1 \qquad 0! = 1$$
$$f^{(1)}(0) = e^0 = 1 \qquad 1! = 1$$
$$f^{(2)}(0) = e^0 = 1 \qquad 2! = 2$$
$$f^{(3)}(0) = e^0 = 1 \qquad 3! = 6$$

$$T_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

$$T_3(-\tfrac{1}{2}) = 1 + (-\tfrac{1}{2}) + \frac{(-\frac{1}{2})^2}{2} + \frac{(-\frac{1}{2})^3}{6} = 1 - \frac{1}{2} + \frac{1}{8} - \frac{1}{48} = \frac{5}{8} - \frac{1}{48} = \frac{30}{48} - \frac{1}{48} = \frac{29}{48}$$

$$|e^{-\frac{1}{2}} - S_n| \le b_{n+1} \qquad b_{n+1} = \frac{x^4}{4!} = \frac{x^4}{24} \qquad b_{n+1}(-\tfrac{1}{2}) = \frac{(-\frac{1}{2})^4}{24} = \frac{\frac{1}{16}}{24} = \frac{1}{384}$$

$$\boxed{|e^{-\frac{1}{2}} - \tfrac{29}{48}| \le \frac{1}{384} \longleftarrow \text{upper bound on error for } T_3(-\tfrac{1}{2})}$$

(5) Suppose we approximate $\sin(x)$ with $x - \frac{x^3}{6}$. For what values of x does this approximation have error of less than .01. (10^{-2})

$$\left|\sin(x) - \left(x - \frac{x^3}{6}\right)\right| \le \frac{|x|^5}{5!} \le 10^{-2} \qquad T_3(x) = x - \frac{x^3}{6}$$

$$\frac{|x|^5}{120} \le \frac{1}{100} \Rightarrow |x|^5 \le \frac{6}{5} \Rightarrow |x| \le \sqrt[5]{\frac{6}{5}} = \text{bound for desired precision}$$

$$\boxed{\begin{aligned} &\therefore \text{ if } x \in \left[-\left(\tfrac{6}{5}\right)^{\frac{1}{5}}, \left(\tfrac{6}{5}\right)^{\frac{1}{5}}\right] \Rightarrow T_3(x) \text{ is within .01 of } \sin(x) \\ &\text{Calculator: } x \in [-1.037\ldots, 1.037\ldots] \Rightarrow T_3(x) \text{ is within .01 of } \sin(x) \end{aligned}}$$