HW5: 9.1-9.3; 9.5

Monday, October 2, 2023 5:10 PM

9.1-9.3: 1, 2, 3, 4, 5, 6, 7, 8
9.5: 1, 2, 3, 4, 5, 6

(1) $y = -t\cos(t) - t$ is/isn't a solution of the diff. equation:

$y' = t\sin t - \cos t - 1$

$ty' = y + t^2\sin(t)$ with init. val. $y(\pi) = 0$

$t[t\sin t - \cos t - 1] = (-t\cos t - t) + t^2\sin(t)$

$t^2\sin t - t\cos t - t = t^2\sin t - t\cos t - t$ ∴ Valid general solution

$\boxed{0 = -\pi\overset{-1}{\cos(\pi)} - \pi = \pi - \pi \quad ∴\ y = -t\cos(t) - t \text{ is a valid solution with initial condition } y(\pi) = 0}$

(2) Which of the following are solutions of the diff. equation:

$y'' + y = \sin(x)$

(A) $y = \sin x$, $y' = \cos x$, $y'' = -\sin x$

(B) $y = \cos x$, $y' = -\sin x$, $y'' = -\cos x$

(C) $y = \frac{x}{2}\sin x$, $y' = \frac{x}{2}\cos x + \frac{1}{2}\sin x$, $y'' = \frac{x}{2}(-\sin x) + \frac{1}{2}\cos x + \frac{1}{2}\cos x$

(D) $y = -\frac{x}{2}\cos x$, $y' = \frac{x}{2}\sin x + (\frac{-1}{2})\cos x$, $y'' = \frac{x}{2}\cos x + \frac{1}{2}\sin x + \frac{1}{2}\sin x$

(A) $-\sin x + \sin x = \sin x$
$0 \neq \sin x$
X $\boxed{\text{not solution}}$

(B) $-\cos x + \cos x = \sin x$
$0 \neq \sin x$
X $\boxed{\text{not solution}}$

(C) $-\frac{x}{2}\sin x + \cos x + \frac{x}{2}\sin x = \sin x$
$\cos x \neq \sin x$
X $\boxed{\text{not solution}}$

(D) ✓ $\frac{x}{2}\cos x + \sin x - \frac{x}{2}\cos x = \sin x$
$\sin x = \sin x$
$\boxed{\text{Valid solution}}$

(3) Solve $\frac{dy}{dx} = x\sqrt{y}$

$\frac{dy}{\sqrt{y}} = x\,dx$

$\int \frac{1}{\sqrt{y}}\,dy = \int x\,dx$

$2\sqrt{y} = \frac{1}{2}x^2 + C$

$\sqrt{y} = \frac{1}{4}x^2 + \frac{C}{2}$

$y = (\frac{1}{4}x^2 + \frac{C}{2})^2$

$$\boxed{y = \frac{1}{16}x^4 + \frac{1}{4}x^2C + \frac{C^2}{4} \qquad y = \frac{1}{16}(x^4 + 4x^2C + 4C^2)}$$

(4) Solution that satisfies $y(0) = 1$

$\frac{dy}{dx} = \frac{x\sin x}{y}$

$u = x$, $dv = \sin x\,dx$
$du = dx$, $v = -\cos x$

$\int x\sin x\,dx = -x\cos x - \int -\cos x\,dx = \sin x - x\cos x + C$

$\int y\,dy = \int x\sin x\,dx$

$\frac{1}{2}y^2 = \sin x - x\cos x + C$ $\quad D = 2C$

$y^2 = 2\sin x - 2x\cos x + D$

$y = +\sqrt{2\sin x - 2x\cos x + D}$ or $y = -\sqrt{2\sin x - 2x\cos x + D}$

$1^2 = 2(\sin(0)) - 2(0)(\cos(0)) + D$

$1^2 = 0 - 0 + D$

$D = 1$ $\rightarrow$ $\boxed{y = \pm\sqrt{2\sin x - 2x\cos x + 1}}$

⑤ Solution that of the following diff. equation that satisfies $y(0) = 1$

$$x + 3y^2\sqrt{x^2+1}\,\frac{dy}{dx} = 0 \qquad -\int \frac{x}{\sqrt{x^2+1}}\,dx$$

$\tan\theta = x$

$\cos\theta = \frac{1}{\sqrt{x^2+1}}$

$dx = \sec^2\theta\, d\theta$

$$3y^2\frac{dy}{dx} = \frac{-x}{\sqrt{x^2+1}} \qquad -\int \tan\theta\, \cancel{\cos\theta}\, \sec^2\theta\, d\theta$$

$$3\int y^2\,dy = \int \frac{-x}{\sqrt{x^2+1}}\,dx \qquad -\int \sec\theta\tan\theta\, d\theta = -\sec\theta = -\sqrt{x^2+1} + C$$

$$y^3 = -\sqrt{x^2+1} + C \xrightarrow{y(0)=1} 1^3 = -\sqrt{x^2+1} + C$$

$$y = \sqrt[3]{-\sqrt{x^2+1} + C} \qquad 1 = -\sqrt{1} + C$$

$$1 = -1 + C$$

$$\boxed{y = \sqrt[3]{-\sqrt{x^2+1} + 2}} \qquad C = 2$$

⑥ $\frac{dC}{dt} = r - kC$ $\qquad r$ = rate administered $\qquad k$ is a constant

a) Solve diff. equation with $C(0) = C_0$ // Find $C(t)$

$$\frac{dy}{dx} + P(x)y = Q(y) \qquad \frac{dC}{dt} + kC = r$$

$$I(x) = e^{\int k\,dt} = e^{kt} \qquad I\frac{dC}{dt} + IkC = Ir$$

$$e^{kt}\frac{dC}{dt} + e^{kt}kC = re^{kt}$$

$u = kt \qquad dt = \frac{du}{k}$

$du = k\,dt$

$$\int (e^{kt}\cdot C)' = \int re^{kt}\,dt$$

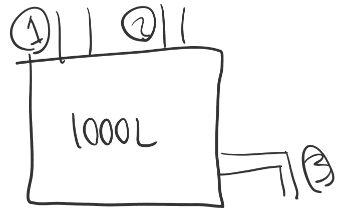
$$\frac{e^{kt}}{e^{kt}}C = \frac{r}{k}e^{kt} + D$$

a) $\boxed{C(t) = \frac{r}{k} + \frac{D}{e^{kt}}}$ $\qquad E = e^{-kt}$

$$C(t) = \frac{r}{k} + E$$

b) $\boxed{\lim_{t\to\infty}\left(\frac{D}{e^{kt}}\right) = 0 \quad \therefore \quad C(t) = \frac{r}{k} \text{ as } t\to\infty}$

7. A tank contains 1000L of pure water. Brine containing .05kg of salt per liter of water enters the tank at a rate of 5L/min. Brine containing .04kg of salt per liter of water enters the tank at a rate of 10L/min. The solution is kept well mixed and drains from the tank at a rate of 15L/m



Water in/out

$V_1 = 5 \frac{L}{m}$

$V_2 = 10 \frac{L}{m}$

$V_3 = 15 \, L/m$

Salt conc.

$\rho_1 = .05 \frac{kg}{L}$

$\rho_2 = .04 \frac{kg}{L}$

$\rho_1 V_1 = .25 \frac{kg}{m}$

$\rho_2 V_2 = .40 \frac{kg}{m}$

(a) How much salt is in the tank after t minutes?
(b) How much salt is in the tank after one hour?

$$\frac{\partial S}{\partial t} = (\text{rate in}) - (\text{rate out})$$

$$\frac{dS}{dt} = .65 - \frac{S}{1000}(15)$$

$$\frac{\partial S}{\partial t} + \underbrace{\frac{15}{1000}}_{P} S = \underbrace{.65}_{Q} \qquad I(x) = e^{\int \frac{15}{1000} dt} = e^{\frac{15}{1000}t}$$

$$e^{\frac{15}{1000}t}\frac{\partial S}{\partial t} + e^{\frac{15}{1000}t}\left(\frac{15}{1000}\right)S = e^{\frac{15}{1000}t}(.65)$$

$$\int \left(e^{\frac{15}{1000}t}(S)\right)' = \int e^{\frac{15}{1000}t}(.65)\,dt$$

$u = \frac{15}{1000}t \quad du = \frac{15}{1000}dt$

$dt = \frac{1000}{15}du$

$$\frac{65}{100}\int e^{\frac{15}{1000}t}dt$$

$$\frac{65}{100}\int e^{u}du\left(\frac{1000}{15}\right)$$

$$\frac{65000}{1500}e^{\frac{15}{1000}t} = \frac{130}{3}e^{\frac{15}{1000}t}$$

$$e^{\frac{3}{200}t}(S) = \frac{130}{3}e^{\frac{3}{200}t} + C$$

$$S(t) = \frac{130}{3} + \frac{C}{e^{\frac{3}{200}t}}$$

$$S(t) = \frac{130}{3} + Ce^{-\frac{3}{200}t} \qquad S(0) = 0 \text{ b/c pure water}$$

$$0 = \frac{130}{3} + \frac{C}{e^0} = \frac{130}{3} + C \Rightarrow C = -\frac{130}{3}$$

$$S(t) = \frac{130}{3} - \frac{130}{3}e^{\frac{-3}{200}t}$$

a) $\boxed{S(t) = \frac{130}{3}\left(1 - e^{\frac{-3}{200}t}\right)}$

b) $S(60) = \frac{130}{3}\left(1 - e^{-\frac{3}{200}(60)}\right) = \boxed{\frac{130}{3}\left(1 - e^{\frac{-9}{10}}\right) = \frac{130}{3} - \frac{130}{3}e^{\frac{-9}{10}}}$ cannot simplify further without calc.

Calculator $\rightarrow$ $\boxed{\frac{130}{3}(.59343...) \approx 25.72}$

(8) Solve the diff. equation $\frac{dY}{dt} = kY(1-Y)$ with init. cond. $Y(0) = Y_0$

$$\frac{dY}{Y(1-Y)} = k\,dt$$

$$\int \frac{1}{Y(1-Y)}\,dY = \int k\,dt$$

$$\int \frac{1}{Y(1-Y)}\,dY = -\int \frac{1}{Y(Y-1)} \qquad \frac{A}{Y} + \frac{B}{Y-1}$$

$$A(Y-1) + BY = 1$$
$$A(0-1) + B(0) = 1$$
$$A = -1$$
$$A(1-1) + B1 = 1$$
$$B = 1$$

$$-\left(\int \frac{-1}{Y}\,dY + \int \frac{1}{Y-1}\,dY\right)$$

$$-\left(-\ln|Y| + \ln|Y-1|\right) = \ln|Y| - \ln|Y-1| + C$$

$$\ln|Y| - \ln|Y-1| = kt + C$$

$$\ln\left|\frac{Y}{Y-1}\right| = kt + C$$

$$\frac{Y}{Y-1} = e^{kt+C}$$

$$(Y-1)\frac{Y}{Y-1} = e^{kt+C}(Y-1)$$

$$Y = e^{kt+C}(Y) - e^{kt+C}$$

$$Y - e^{kt+C}(Y) = -e^{kt+C}$$

$$Y(1-e^{kt+C}) = -e^{kt+C}$$

$$Y = \frac{-e^{kt+C}}{(1-e^{kt+C})}$$

$$Y(0)=Y_0 \rightarrow Y_0 = \frac{-e^{0+C}}{(1-e^{0+C})}$$

$$Y_0 = \frac{-e^C}{1-e^C}$$

$$Y_0 - Y_0 e^C = -e^C$$

$$Y_0 = Y_0 e^C - e^C$$

$$Y_0 = e^C(Y_0 - 1)$$

$$\ln(Y_0) = \ln(e^C(Y_0-1))$$

$$\ln(Y_0) = \ln(e^C) + \ln(Y_0 - 1)$$

$$C = \ln(Y_0) - \ln(Y_0 - 1)$$

$$C = \ln\left(\frac{Y_0}{Y_0-1}\right)$$

$$Y = -\frac{e^{kt + \left(\ln\left(\frac{Y_0}{Y_0-1}\right)\right)}}{\left(1 - e^{kt + \left(\ln\left(\frac{Y_0}{Y_0-1}\right)\right)}\right)}$$

$$\boxed{Y = -\frac{e^{kt}\left(\frac{Y_0}{Y_0-1}\right)}{\left(1 - e^{kt}\left(\frac{Y_0}{Y_0-1}\right)\right)}}$$

9.5: 1, 2, 3, 4, 5, 6

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(1) Decide if the diff. equation is linear: $Y' - X = Y\tan(x)$

$$\underset{\substack{A(x)}}{1}Y' - Y\underset{\substack{B(x)\\P(x)}}{\tan x} = \underset{\substack{C(x)\\Q(x)}}{x}$$

Yes, it is a first order linear diff. equation in the form $A(x)Y' + B(x)Y = C(x)$

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(2) Solve the diff. equation $Y' + Y = 1$

② Solve the diff. equation $y' + y = 1$

$I(x) = e^{\int dx} = e^x$

$e^x y' + e^x y = e^x$

$\int (e^x y)' dx = \int e^x dx$

$e^x y = e^x + C$ $\longrightarrow$ $\boxed{y = 1 + Ce^{-x}}$

$\frac{dy}{dx}(e^x y) = e^x y' + e^x y$

③ Solve the diff. equation $2xy' + y = 2\sqrt{x}$

$y' + \frac{1}{2x} y = \frac{\sqrt{x}}{x}$

$I(x) = e^{\int \frac{1}{2x} dx} = e^{\frac{\ln x}{2}} = (e^{\ln x})^{\frac{1}{2}} = \sqrt{x}$

$\sqrt{x}\, y' + \frac{1}{2\sqrt{x}} y = 1$

$\frac{d}{dx}(\sqrt{x}\, y) = \sqrt{x}\, y' + \frac{1}{2\sqrt{x}} y$

$\int (\sqrt{x}\, y)' = \int dx$

$\sqrt{x}\, y = x + C$

$\boxed{y = \sqrt{x} + \frac{C}{\sqrt{x}}}$

④ Find the sol. of the diff. equation: $t^3 \frac{dy}{dt} + 3t^2 y = \cos t$ with $y(\pi) = 0$

$t^3 \frac{dy}{dt} + 3t^2 y = \cos(t)$

$\int (t^3 y)' = \int \cos(t)$

$t^3 y = \sin(t) + C$

$y = \frac{\sin(t)}{t^3} + Ct^{-3}$ $\longrightarrow$ $0 = \frac{\sin(\pi)}{\pi^3} + \frac{C}{\pi^3}$

$0 = \frac{0}{\pi^3} + \frac{C}{\pi^3}$

$C = 0$

$\boxed{y = \frac{\sin(t)}{t^3}}$

⑤ Find the sol. of the diff. equation: $xy' + y = x\ln x$ with $y(1) = 0$

$\int (xy)' dx = \int x \ln x \, dx$

$\int x \ln x \, dx$ $\quad u = \ln x \quad dv = x\,dx$

$du = \frac{1}{x} dx \quad v = \frac{1}{2}x^2$

$\frac{1}{2}x^2 \ln x - \int \frac{1}{2} x^{\cancel{2}} \frac{1}{\cancel{x}} dx = \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x\, dx = \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2$

$xy = \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C$

$y = \frac{1}{2} x \ln x - \frac{1}{4}x + \frac{C}{x}$

$y = \frac{1}{4}x(2\ln x - 1) + \frac{C}{x} \xrightarrow{y(1)=0} 0 = \frac{1}{4}(2\ln(1) - 1) + C = \frac{1}{4}(0-1) + C = -\frac{1}{4} + C$

$C = \frac{1}{4}$

$\boxed{y = \frac{1}{4}x(2\ln x - 1) + \frac{1}{4x}}$

$$\boxed{y = \frac{1}{4} x (2\ln x - 1) + \frac{}{4x}}$$

⑥ Solve the second order diff. equation: $xy'' + 2y' = 12x^2$ substituting $u = y'$

$xu' + 2u = 12x^2$

$u' + \frac{2}{x} u = 12x$

$I(x) = e^{\int \frac{2}{x} dx} = e^{2\ln x} = (e^{\ln x})^2 = x^2$

$x^2 u' + 2xu = 12x^3$

$\int (x^2 u)' dx = \int 12x^3 dx$

$x^2 u = 3x^4 + C$

$y' = 3x^2 + \frac{C}{x^2}$ $\qquad y' = \frac{dy}{dx}$

$\int dy = \int 3x^2 dx + \int \frac{C}{x^2} dx$

$y = x^3 + \left(-\frac{C}{x}\right) + D$

$\boxed{y = x^3 - \frac{1}{x} C + D}$ or w/ $E = -C$ $\boxed{y = x^3 + \frac{1}{x} E + D}$