HW9: 10.3; 10.4

Tuesday, October 31, 2023 6:07 PM

10.3: 2, 3, 4, 5, 6, 7, 8, 9, 10

$x = r\cos(\theta)$
$y = r\sin(\theta)$
$\theta = \tan^{-1}(\frac{x}{y})$

Polar | Cartesian
$(r,\theta) = (r\cos\theta, r\sin\theta)$

(2) Plot each point and find Cartesian coordinates of the point.

a) $(r,\theta) = (4, \frac{4\pi}{3}) = (4\cos(\frac{4\pi}{3}), 4\sin(\frac{4\pi}{3})) = (4(-.5), 4(-\frac{\sqrt{3}}{2})) = \boxed{(-2, -2\sqrt{3})}$

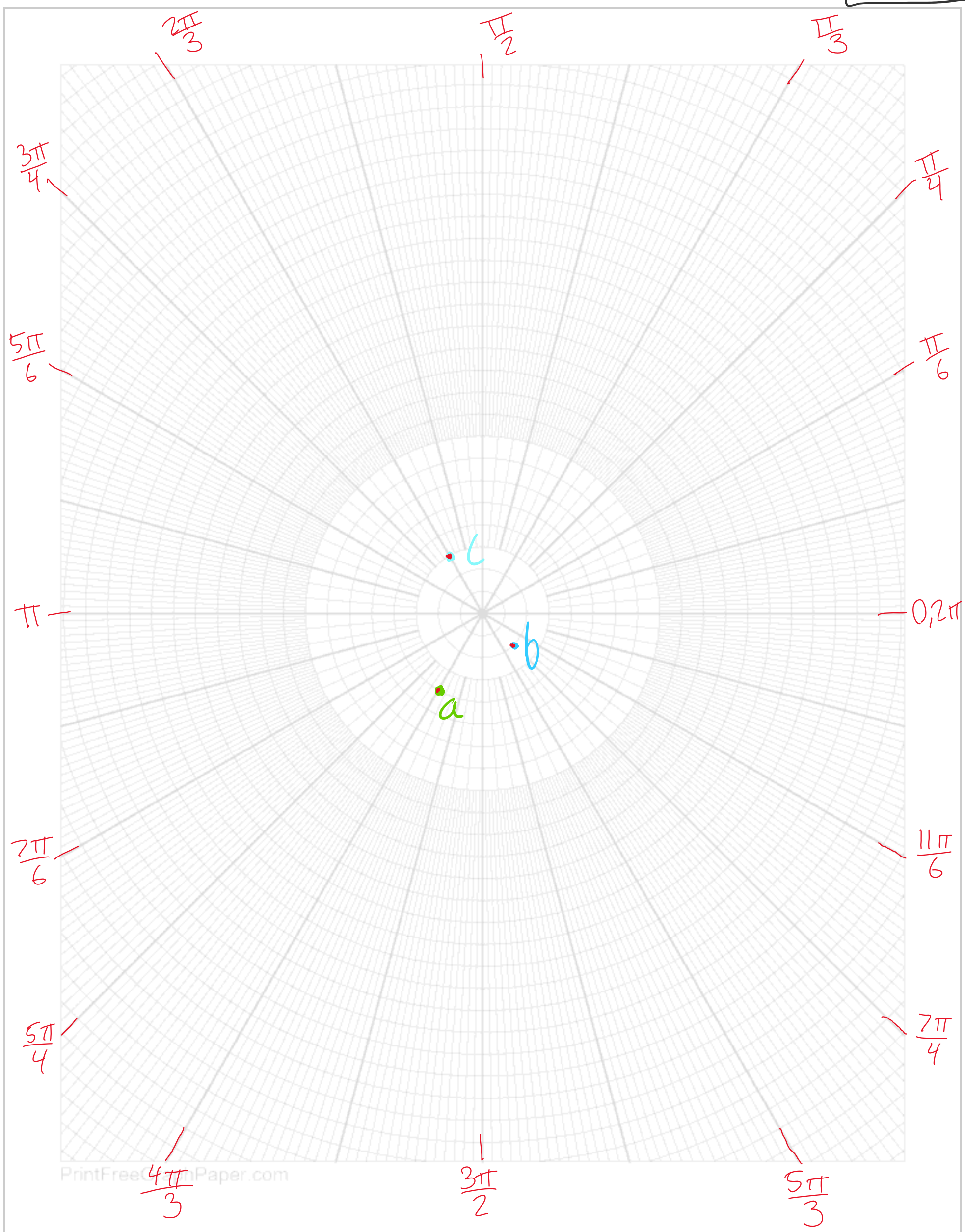
b) $(r,\theta) = (-2, \frac{3\pi}{4}) = (-2\cos(\frac{3\pi}{4}), -2\sin(\frac{3\pi}{4})) = (-2(-\frac{\sqrt{2}}{2}), -2(\frac{\sqrt{2}}{2})) = \boxed{(\sqrt{2}, -\sqrt{2})}$

c) $(r,\theta) = (-3, -\frac{\pi}{3}) = (-3\cos(\frac{-\pi}{3}), -3\sin(\frac{-\pi}{3})) = (-3(.5), -3(-\frac{\sqrt{3}}{2})) = \boxed{(-\frac{3}{2}, \frac{3\sqrt{3}}{2})}$

a) $(4, \frac{4\pi}{3})$

b) $(-2, \frac{3\pi}{4})$

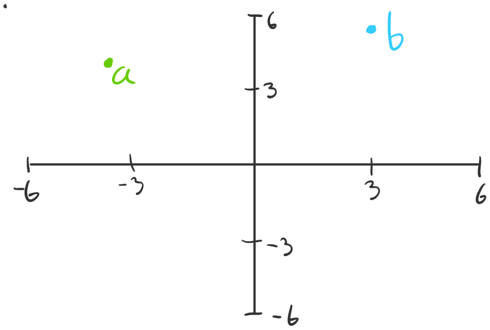
c) $(-3, -\frac{\pi}{3})$



③ For each Cartesian Coordinate (x,y), plot it, find polar coordinates (r_1,θ_1) for $r_1>0$ and $0\le\theta_1<2\pi$. And, find polar coordinates (r_2,θ_2) for $r_2<0$ and $0\le\theta_2<2\pi$.

a) $(x,y) = (-4,4)$

b) $(x,y) = (3,3\sqrt{3})$



$r^2 = x^2+y^2 \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$

a) $(x,y)=(-4,4) \rightarrow r^2=(-4)^2+(4)^2 \Rightarrow r^2 = 32 \Rightarrow r = \pm\sqrt{32} = \pm 4\sqrt{2}$

for $r = +4\sqrt{2}$ $\qquad \theta = \tan^{-1}(-1) = \frac{3\pi}{4}$

for $r = -4\sqrt{2}$ $\qquad \theta = \frac{3\pi}{4}+\pi = \frac{7\pi}{4}$

$$\boxed{\begin{array}{l}(r,\theta) = \left(4\sqrt{2}, \frac{3\pi}{4}\right)\\ (r,\theta) = \left(-4\sqrt{2}, \frac{7\pi}{4}\right)\end{array}}$$

b) $(x,y) = (3,3\sqrt{3}) \rightarrow r^2 = 3^2 + (3\sqrt{3})^2 \Rightarrow r^2 = 36 \Rightarrow r = \pm 6$

for $r=+6$ $\qquad \theta = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$

for $r=-6$ $\qquad \theta = \frac{\pi}{3}+\pi = \frac{4\pi}{3}$

$$\boxed{\begin{array}{l}(r,\theta) = \left(6, \frac{\pi}{3}\right)\\ (r,\theta) = \left(-6, \frac{4\pi}{3}\right)\end{array}}$$

Q4 and Q5 on separate graph paper

⑥ Find the slope of the line tangent to the curve:

$r = \cos(\theta/3)$ @ $\theta = \pi$

@ π $\rightarrow$ $x = r\cos\theta = \cos\left(\frac{\pi}{3}\right)\cos(\pi) = \left(\frac{1}{2}\right)(-1) = -\frac{1}{2}$

$\searrow$ $y = r\sin\theta = \cos\left(\frac{\pi}{3}\right)\sin(\pi) = \left(\frac{1}{2}\right)(0) = 0$

$$\frac{dy}{d\theta} = \frac{\partial}{\partial\theta}\left(\cos\left(\frac{\theta}{3}\right)\sin(\theta)\right) = \cos\left(\frac{\theta}{3}\right)\cos(\theta) - \sin(\theta)\frac{1}{3}\sin\left(\frac{\theta}{3}\right)$$

$$\frac{dx}{d\theta} = \frac{\partial}{\partial\theta}\left(\cos\left(\frac{\theta}{3}\right)\cos(\theta)\right) = \cos\left(\frac{\theta}{3}\right)(-\sin(\theta)) - \cos(\theta)\frac{1}{3}\sin\left(\frac{\theta}{3}\right)$$

$$\left.\frac{dy}{dx}\right|_{\theta=\pi} = \left.\frac{\cos\left(\frac{\theta}{3}\right)\cos\theta - \frac{1}{3}\sin\theta\sin\left(\frac{\theta}{3}\right)}{-\cos\left(\frac{\theta}{3}\right)\sin\theta - \frac{1}{3}\cos\theta\sin\left(\frac{\theta}{3}\right)}\right|_{\theta=\pi} = \frac{(.5)(-1) - \cancel{\frac{1}{3}(0)\left(\frac{\sqrt{3}}{2}\right)}}{-\cancel{(.5)(0)} - \frac{1}{3}(-1)\left(\frac{\sqrt{3}}{2}\right)} = \frac{-\frac{1}{2}}{\frac{1}{3}\left(\frac{\sqrt{3}}{2}\right)} = \frac{-\frac{1}{2}}{\frac{\sqrt{3}}{6}}$$

$$= -\frac{1}{2}\cdot\left(\frac{6}{\sqrt{3}}\right) = \frac{-3}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}} = \frac{-3\sqrt{3}}{3} = -\sqrt{3}$$

$\boxed{\text{Slope of tangent line @ } \theta=\pi = -\sqrt{3}}$

⑦ For what values of θ does the curve $r = 15\sin\theta + 15\cos\theta$ have Horizontal or Verticle tangent lines?

When $\frac{dy}{dx} = 0$ or $\frac{dy}{dx}$ is undefined

$x = r\cos\theta = (15\sin\theta + 15\cos\theta)\cos\theta = 15\sin\theta\cos\theta + 15\cos^2\theta$

$y = r\sin\theta = (15\sin\theta + 15\cos\theta)\sin\theta = 15\sin^2\theta + 15\sin\theta\cos\theta$

$\frac{dx}{d\theta} = -15\sin^2\theta + 15\cos^2\theta - 30\cos\theta\sin\theta = -15(\sin^2\theta - \cos^2\theta + 2\cos\theta\sin\theta)$

$= -15(2\cos\theta\sin\theta - (\cos^2\theta - \sin^2\theta)) = -15(\sin 2\theta - \cos 2\theta)$

$\frac{dy}{d\theta} = 30\sin\theta\cos\theta + (-15\sin^2\theta) + 15\cos^2\theta = 15(2\sin\theta\cos\theta + \cos^2\theta - \sin^2\theta) = 15(\sin 2\theta + \cos 2\theta)$

$\frac{dy}{dx} = \frac{15(\sin 2\theta + \cos 2\theta)}{-15(\sin 2\theta - \cos 2\theta)} = \frac{\sin 2\theta + \cos 2\theta}{-\sin 2\theta + \cos 2\theta}$ when $\sin 2\theta + \cos 2\theta = 0 \Rightarrow$ Horiz. lines
when $-\sin 2\theta + \cos 2\theta = 0 \Rightarrow$ vert. lines

$-\sin 2\theta + \cos 2\theta = 0$ when $\theta = \frac{\pi}{8} + n\pi$ $\therefore$ $\boxed{\text{curve has Vert. tangent lines @ } \theta = \frac{\pi}{8} + n\pi}$

$\sin 2\theta + \cos 2\theta = 0$ when $\theta = \frac{3\pi}{8} + n\pi$ $\therefore$ $\boxed{\text{Curve has Horiz. tangent lines @ } \theta = \frac{3\pi}{8} + n\pi}$

Q8 and Q9 and Q10 are on separate graph paper

10.4: 1, 2, 3 | Find the areas of the following regions

① One loop of the curve $r = 4\cos(3\theta)$ ← 3-petal flower

$\cos 3\theta = 0$ when $3\theta = \frac{\pi}{2}, \frac{3\pi}{2} \Rightarrow \theta = \frac{\pi}{6}, \frac{3\pi}{6}, \ldots$

$u = 6\theta$ $d\theta = \frac{du}{6}$
$du = 6d\theta$

$A = \frac{1}{2}\int_{\theta_a}^{\theta_b} r^2 d\theta = \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} 16\cos^2(3\theta)\, d\theta = 8\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1}{2}(1 + \cos(6\theta))\, d\theta$

$= 4\left[\theta\Big|_{-\frac{\pi}{6}}^{\frac{\pi}{6}} + \frac{1}{6}\sin(6\theta)\Big|_{-\frac{\pi}{6}}^{\frac{\pi}{6}}\right] = 4\left[\left(\frac{\pi}{6} - \left(-\frac{\pi}{6}\right)\right) + (0 - 0)\right] = 4\left(\frac{\pi}{3}\right) = \boxed{\frac{4\pi}{3}}$

② Inside the circle $r = 4\sin\theta$ but outside $r = 2$

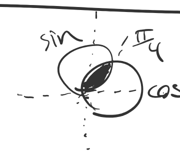
$4\sin\theta = 2$ when $\sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6} \ldots$

$A = \frac{1}{2}\int_{\theta_a}^{\theta_b} r^2 d\theta = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} 16\sin^2\theta\, d\theta = 4\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (1 - \cos(2\theta))\, d\theta = 4\left[\theta\Big|_{\frac{\pi}{6}}^{\frac{5\pi}{6}} - \frac{1}{2}\sin(2\theta)\Big|_{\frac{\pi}{6}}^{\frac{5\pi}{6}}\right]$

$= 4\left[\left(\frac{5\pi}{6} - \frac{\pi}{6}\right) - \left(\frac{1}{2}\left(\frac{-\sqrt{3}}{2}\right) - \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)\right)\right] = 4\left[\frac{2\pi}{3} - \left(-\frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4}\right)\right] = 4\left[\frac{2\pi}{3} - \left(-\frac{\sqrt{3}}{2}\right)\right] = 4\left[\frac{4\pi}{6} + \frac{3\sqrt{3}}{6}\right]$

$= 4\left(\frac{4\pi + 3\sqrt{3}}{6}\right) = \boxed{\frac{8\pi + 6\sqrt{3}}{3}}$

③ Intersection of the circles $r = 3\sin\theta$, $r = 3\cos\theta$



$3\sin\theta = 3\cos\theta$ when $\theta = \frac{\pi}{4}, \frac{5\pi}{4}, \ldots$

$A = \frac{1}{2}\int_{\theta_a}^{\theta_b} r^2 d\theta$ $\quad A_{\text{intersection}} = \frac{1}{2}\int_0^{\frac{\pi}{4}} 9\sin^2\theta d\theta + \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 9\cos^2\theta d\theta$

$= \frac{9}{4}\int_0^{\frac{\pi}{4}}(1-\cos 2\theta)d\theta + \frac{9}{4}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(1+\cos 2\theta)d\theta = \frac{9}{4}\left[\theta\Big|_0^{\frac{\pi}{4}} - \frac{1}{2}\sin 2\theta\Big|_0^{\frac{\pi}{4}}\right] + \frac{9}{4}\left[\theta\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} + \frac{1}{2}\sin 2\theta\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}\right]$

$= \frac{9}{4}\left[\left(\frac{\pi}{4}-0\right) - \left(\frac{1}{2}(1) - \cancel{\frac{1}{2}(0)}\right)\right] + \frac{9}{4}\left[\left(\frac{2\pi}{4} - \frac{\pi}{4}\right) + \left(\cancel{\frac{1}{2}(0)} - \frac{1}{2}(1)\right)\right] = \frac{9}{4}\left[\frac{\pi}{4} - \frac{1}{2}\right] + \frac{9}{4}\left[\frac{\pi}{4} - \frac{1}{2}\right]$

$= \frac{9}{2}\left[\frac{\pi}{4} - \frac{1}{2}\right] = \frac{9\pi}{8} - \frac{18}{8} = \boxed{\frac{9\pi - 18}{8}}$