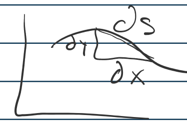
HW8: 10.2

Tuesday, October 24, 2023 8:05 PM

$\boxed{10.2: 7, 8, 9}$

Find the length of each arc.



$\partial s^2 = \partial x^2 + \partial y^2$

$\partial s^2 = (x'(t)\,\partial t)^2 + (y'(t)\,\partial t)^2$

$\partial s = \sqrt{\left(\frac{\partial x}{\partial t}\right)^2 + \left(\frac{\partial y}{\partial t}\right)^2}\,\partial t$

7) $x(t) = 1 + 3t^2$ $\quad 0 \le t \le 1$
$y(t) = 4 + 2t^2$

$L = \int_0^1 \partial s \qquad \partial s = \sqrt{\left(\frac{\partial x}{\partial t}\right)^2 + \left(\frac{\partial y}{\partial t}\right)^2}\,\partial t$

$\frac{dx}{\partial t} = 6t$

$\frac{\partial y}{\partial t} = 4t$

$L = \int_0^1 \sqrt{(6t)^2 + (4t)^2}\,\partial t = \int_0^1 \sqrt{36t^2 + 16t^2}\,\partial t = \int_0^1 \sqrt{52t^2}\,\partial t$

$= \int_0^1 (\sqrt{52}\,t)\,\partial t = \frac{\sqrt{52}}{2} t^2 \Big|_0^1 = \frac{\sqrt{4(13)}}{2} = \boxed{\sqrt{13}}$

8) $x(t) = e^t - t$ $\quad 0 \le t \le 2$
$y(t) = 4e^{t/2}$

$L = \int_0^2 \sqrt{\left(\frac{\partial x}{\partial t}\right)^2 + \left(\frac{\partial y}{\partial t}\right)^2}\,\partial t$

$\partial x/\partial t = e^t - 1$

$\partial y/\partial t = 2e^{t/2}$

$L = \int_0^2 \sqrt{(e^t - 1)^2 + 4e^t}\,\partial t = \int_0^2 \sqrt{e^{2t} - 2e^t + 1 + 4e^t}\,\partial t$

$= \int_0^2 \sqrt{e^{2t} + 2e^t + 1}\,\partial t = \int_0^2 \sqrt{(e^t + 1)^2}\,\partial t = \int_0^2 (e^t + 1)\,\partial t = e^t + t \Big|_0^2$

$= (e^2 + 2) - (e^0 + 0) = e^2 + 2 - 1 = \boxed{e^2 + 1}$

9) Find the distance traveled by a particle with position $(x,y) = (\sin^2(t), \cos^2(t))$ from time $t=0$ to time $t=3\pi$. How does this compare to the length of the arc on which the particle travels?

$x(t) = \sin^2(t)$

$y(t) = \cos^2(t)$

$dx/dt = 2\sin(t)\cos(t) = \sin(2t)$

$dy/dt = -2\cos(t)\sin(t) = -\sin(2t)$

$$L_{Dist} = \int ds$$

$$L_{Dist} = \int \sqrt{(2\sin t\cos t)^2 + (-2\cos t\sin t)^2}\, dt = \int \sqrt{4\sin^2 t\cos^2 t + 4\cos^2 t\sin^2 t}\, dt$$

$$= \int \sqrt{8\sin^2 t\cos^2 t}\, dt = \sqrt{8}\int \sin t\cos t\, dt \qquad u = \sin t \quad du = \cos t\, dt \quad dt = \frac{du}{\cos t}$$

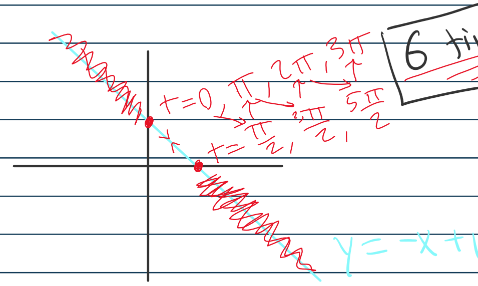
$$= \sqrt{8}\int u \cancel{\cos t}\frac{du}{\cancel{\cos t}} = \sqrt{8}\int u\, du = \sqrt{8}\left[\tfrac{1}{2}u^2\right] = \boxed{\sqrt{8}\left[\tfrac{1}{2}\sin^2 t\right] = \int ds}$$

$x(t) = \sin^2(t)$

$y(t) = \cos^2(t)$

$$x + y = \sin^2(t) + \cos^2(t) = 1$$

$$y = -x + 1$$



$\boxed{\text{6 times}}$

@ $t=0$ $\quad x = \sin^2(0) = 0$, $\quad y = \cos^2(0) = 1$

@ $t=3\pi$ $\quad x = \sin^2(3\pi) = 0$, $\quad y = \cos^2(3\pi) = 1$

@ $t=\pi$ $\quad x = \sin^2(\pi) = 0$, $\quad y = \cos^2(\pi) = 1$

@ $t=\frac{\pi}{2}$ $\quad x = \sin^2(\frac{\pi}{2}) = 1$, $\quad y = \cos^2(\frac{\pi}{2}) = 0$

Solved above: $\int_a^b ds = \sqrt{8}\left[\tfrac{1}{2}\sin^2 t\right]_a^b$

$$\text{Arc Length (no traceback)} = L = \int_0^{\frac{\pi}{2}} ds = \sqrt{8}\left[\tfrac{1}{2}\sin^2(\tfrac{\pi}{2}) - \tfrac{1}{2}\sin^2(0)\right]$$

$$\sqrt{8}\left[\tfrac{1}{2} - 0\right] = \sqrt{8}\cdot\tfrac{1}{2} = \frac{2\sqrt{2}}{2} = \boxed{\sqrt{2}}$$

$$\text{Total Distance traveled by particle} = 6 \times L_{arc} = 6\times\sqrt{2} = \boxed{6\sqrt{2}}$$

Comparison: Particle travels 6 times the length of the arc