HW4: 7.4; 7.8

Saturday, September 23, 2023 4:51 PM

7.4: 1, 2, 3, 4, 5
7.8: 1, 2, 3, 4, 5, 6, 7, 8, 9

① $\int \frac{y}{(y+4)(2y-1)}dy$ $\quad \frac{A}{(y+4)} + \frac{B}{(2y-1)} = \frac{y}{(y-4)(2y-1)}$ $\quad A(2y-1)+B(y+4) = y$

$A(2(-4)-1) + B(-4+4) = -4$
$A(-9) = -4$
$A = \frac{4}{9}$

$A(2(\frac{1}{2})-1) + B(\frac{1}{2}+4) = \frac{1}{2}$
$B(\frac{9}{2}) = \frac{1}{2}$
$B = \frac{2}{18} = \frac{1}{9}$

$$\int \frac{4}{9(y+4)} + \frac{1}{9(2y-1)}dy = \frac{4}{9}\ln|y+4| + \frac{1}{9}\left(\frac{1}{2}\right)\ln|2y-1| = \boxed{\frac{8\ln|y+4| + \ln|2y-1|}{18} + C}$$

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② $\int_1^2 \frac{3x^2+6x+2}{x^2+3x+2}dx$

$$\begin{array}{r} 3 \\ x^2+3x+2 \overline{) 3x^2+6x+2} \\ 3x^2+9x+6 \\ -3x-4 \end{array}$$

$3 + \frac{-3x-4}{x^2+3x+2}$

$$\int_1^2 3\,dx + \int_1^2 \frac{-(3x+4)}{x^2+3x+2}dx = (3(2)-3(1)) - \int_1^2 \frac{3x+4}{(x+1)(x+2)}dx = 3 - \int_1^2 \frac{3x+4}{(x+1)(x+2)}$$

$\frac{A}{(x+1)} + \frac{B}{(x+2)} = \frac{3x+4}{(x+1)(x+2)}$ $\quad A(x+2)+B(x+1) = 3x+4$

$A(-2+2) + B(-2+1) = 3(-2)+4$
$B(-1) = -2$
$B = 2$

$A(-1+2) + B(-1+1) = 3(-1)+4$
$A(1) = 1$
$A = 1$

$$3 - \int_1^2 \frac{3x+4}{(x+1)(x+2)}dx = 3 - \int_1^2\left(\frac{1}{x+1} + \frac{2}{x+2}\right)dx = 3 - \left(\ln|x+1| + 2\ln|x+2|\Big|_1^2\right)$$

$$3 - ((\ln 3 - \ln 2) + 2(\ln 4 - \ln 3)) = 3 - (\ln 3 - \ln 2 + 2\ln 4 - 2\ln 3) = 3 - (2\ln 4 - \ln 3 - \ln 2)$$

$= \boxed{3 - 2\ln 4 + \ln 3 + \ln 2}$

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(3) $\int_0^1 \frac{x^2+x+1}{(x+1)^2(x+2)}dx = \int_0^1 \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{(x+2)} dx$

$A(x+1)(x+2) + B(x+2) + C(x+1)^2 = x^2+x+1$

$x=-2 \quad A(-2+1)(-2+2) + B(-2+2) + C(-2+1)^2 = (-2)^2+(-2)+1$

$C(1) = 4-2+1 = 3$

$C=3$

$x=-1 \quad A(-1+1)(-1+2) + B(-1+2) + C(-1+1)^2 = (-1)^2-1+1$

$B(1) = 1-1+1 = 1$

$B=1$

$x=0 \quad A(1)(2) + B(2) + C = 1$

$2A + 2 + 3 = 1$

$A=-2$

$\int_0^1 \left(\frac{-2}{(x+1)} + \frac{1}{(x+1)^2} + \frac{3}{x+2}\right)dx = \left(-2\ln|x+1|\right)\Big|_0^1 + \left(-\frac{1}{x+1}\right)\Big|_0^1 + \left(3\ln|x+2|\right)\Big|_0^1$

$= (-2\ln 2) + \left(-\frac{1}{2}+1\right) + (3\ln 3 - 3\ln 2) = -2\ln 2 + \frac{1}{2} + 3\ln 3 - 3\ln 2$

$\int_0^1 \frac{x^2+x+1}{(x+1)^2(x+2)}dx = \boxed{3\ln 3 - 5\ln 2 + \frac{1}{2}}$

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(4) $\int \frac{10}{(x-1)(x^2-9)}dx = \int \frac{10}{(x-1)(x+3)(x-3)}dx = \frac{A}{x-1} + \frac{B}{x+3} + \frac{C}{x-3}$

$A(x+3)(x-3) + B(x-1)(x-3) + C(x-1)(x+3) = 10$

$x=3 \quad A(3+3)(3-3) + B(3-1)(3-3) + C(3-1)(3+3) = 10$

$C(2)(6) = 10$

$C = \frac{10}{12} = \frac{5}{6}$

$x=-3 \quad A(-3+3)(-3-3) + B(-3-1)(-3-3) + C(-3-1)(-3+3) = 10$

$B(-4)(-6) = 10$

$B = \frac{10}{24} = \frac{5}{12}$

$x=1 \quad A(1+3)(1-3) + B(1-1)(1-3) + C(1-1)(1+3) = 10$

$A(4)(-2) = 10$

$A = -\frac{10}{8} = -\frac{5}{4}$

$\int \left(\frac{-5}{4(x-1)} + \frac{5}{12(x+3)} + \frac{5}{6(x-3)}\right)dx = -\frac{5}{4}\int \frac{1}{x-1}dx + \frac{5}{12}\int \frac{1}{x+3}dx + \frac{5}{6}\int \frac{1}{x-3}dx$

$= -\frac{5}{4}\ln|x-1| + \frac{5}{12}\ln|x+3| + \frac{5}{6}\ln|x-3| = \frac{-15\ln|x-1| + 5\ln|x+3| + 10\ln|x-3|}{12}$

$\int \frac{10}{(x-1)(x^2-9)}dx = \boxed{\frac{5\left(-3\ln|x-1| + \ln|x+3| + 2\ln|x-3|\right)}{12} + C}$

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5) $\int \frac{x^2+x+1}{(x^2+1)^2}dx = \int\left(\frac{A}{x^2+1} + \frac{B}{(x^2+1)^2}\right)dx$

$A(x^2+1) + B = x^2+x+1$

$x=i \quad A(i^2+1) + B = i^2+i+1$

$B = -1+i+1$

$B = i = x$

$B = x$

This seemed to work but there must be a better way that the Prof. wants

$A(x^2+1) + x = x^2+x+1$

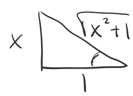
$\frac{A(x^2+1)}{x^2+1} = \frac{x^2+1}{x^2+1}$

$A = 1$

↑ EZ way is $\frac{Ax+B}{(x^2+1)} + \frac{Cx+D}{(x^2+1)^2}$

$\int \frac{x^2+x+1}{(x^2+1)^2}dx = \int\left(\frac{1}{x^2+1} + \frac{x}{(x^2+1)^2}\right)dx = \int \frac{1}{x^2+1}dx + \int \frac{x}{(x^2+1)^2}dx$

$\int \frac{1}{x^2+1}dx$



$\tan\theta = x$

$dx = \sec^2\theta\, d\theta$

$\theta = \tan^{-1}x$

$\cos\theta = \frac{1}{\sqrt{x^2+1}}$

$\cos^2\theta = \frac{1}{x^2+1}$

$\int \cancel{\cos^2\theta}\,\cancel{\sec^2\theta}\, d\theta = \int d\theta = \theta$

$\int \frac{1}{x^2+1} = \tan^{-1}x$ ← I won't forget this standard integral again :)

$\tan^{-1}x + \int \frac{x}{(x^2+1)^2}dx \qquad u = x^2+1 \quad du = 2x\,dx \qquad \tan^{-1}x + \int \frac{\cancel{x}}{u^2}\frac{du}{2\cancel{x}} = \tan^{-1}x + \frac{1}{2}\int \frac{1}{u^2}du$

$= \tan^{-1} + \frac{1}{2}\left(-\frac{1}{u}\right) = \tan^{-1}x + \frac{1}{2}\left(-\frac{1}{x^2+1}\right) = \boxed{\tan^{-1} - \frac{1}{2x^2+2} + C}$

7.8: 1,2,3,4,5,6,7,8,9

Evaluate the following integrals or show that they do not converge

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1) $\int_3^\infty \frac{1}{(x-2)^{3/2}}dx = \lim_{t\to\infty}\left(\int_3^t \frac{1}{(x-2)^{3/2}}dx\right)$

$\int \frac{1}{(x-2)^{3/2}}dx \qquad u = x-2 \quad du = dx \qquad \int \frac{1}{u^{3/2}}du = -\frac{2}{u^{1/2}} = -\frac{2}{\sqrt{x-2}}$

$\lim_{t\to\infty}\left(\frac{2}{\sqrt{x-2}}\Big|_3^t\right) = \boxed{\lim_{t\to\infty}\left(\frac{1}{2\sqrt{t-3}} - \frac{2}{\sqrt{1}}\right) \quad \therefore \text{converges to } 2}$

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2) $\int_0^\infty \frac{1}{(1+x)^{1/4}}dx = \lim_{t\to\infty}\left(\int_0^t \frac{1}{(1+x)^{1/4}}dx\right)$

$\int \frac{1}{(1+x)^{1/4}}dx \qquad u = 1+x \quad du = dx \qquad \int \frac{1}{u^{1/4}}du = \frac{4u^{3/4}}{3} = \frac{4(1+x)^{3/4}}{3}$

$\lim_{t\to\infty}\left(\frac{4(1+x)^{3/4}}{3}\Big|_0^t\right) = \boxed{\lim_{t\to\infty}\left(\frac{4(t+1)^{3/4}}{3} - \frac{4(1)}{3}\right) \quad \therefore \text{Diverges}}$

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③ $\int_{-\infty}^{\infty} xe^{-x^2}dx = \int_{-\infty}^{0} xe^{-x^2}dx + \int_{0}^{\infty} xe^{-x^2}dx$

$\int xe^{-x^2}dx$ $\quad u = x^2$, $du = 2x\,dx$ $\quad \int x e^{-u}\frac{du}{2x} = \frac{1}{2}\int e^{-u}du = -\frac{1}{2}e^{-u} = -\frac{1}{2}e^{-x^2}+C$

$\lim_{t\to\infty}\left(-\frac{1}{2}e^{-x^2}\Big|_0^t\right) = \lim_{t\to\infty}\left(-\frac{1}{2}e^{-t^2}+\frac{1}{2}\right) = \lim_{t\to\infty}\left(-\frac{1}{2}\left(\frac{1}{e^{t^2}}\right)^{\to 0}+\frac{1}{2}\right)$ $\therefore$ converges to $\frac{1}{2}$

$\lim_{t\to\infty}\left(-\int_0^{\infty} xe^{-x^2}dx\right) = \lim_{t\to\infty}\left(-\left(-\frac{1}{2}e^{-x^2}\right)\Big|_0^t\right) = \lim_{t\to\infty}\left(\frac{1}{2}\frac{1}{e^{t^2}}^{\to 0} - \frac{1}{2}\right)$ $\therefore$ converges to $-\frac{1}{2}$

$-\frac{1}{2}+\frac{1}{2} = 0$ $\quad \boxed{\therefore \int_{-\infty}^{\infty} xe^{-x^2}dx \text{ converges to } 0}$

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④ $\int_1^{\infty}\frac{1}{x^2+x}dx = \lim_{t\to\infty}\left(\int_1^t \frac{1}{x^2+x}dx\right)$

$\int \frac{1}{x^2+x}dx = \int\frac{1}{x(x+1)}dx = \int \frac{A}{x}+\frac{B}{x+1}dx$ $\quad A(x+1)+Bx = 1$, $A(-1+1)+B(-1)=1$, $B=-1$ $\quad A(1)+B(0)=1$, $A=1$

$\int\frac{1}{x}dx - \int\frac{1}{x+1}dx = \ln x - \ln|x+1|$

$\lim_{t\to\infty}\left(\ln|x|-\ln|x+1|\Big|_1^t\right) = \lim_{t\to\infty}\left((\ln t - \ln|t+1|)-(0-\ln 2)\right) = \lim_{t\to\infty}\left(\ln t - \ln|t+1| + \ln 2\right)$

$= \boxed{\lim_{t\to\infty}\left(\ln\left|\frac{t}{t+1}\right|^{LH\,\frac{\infty}{\infty}} + \ln 2\right) = \lim_{t\to\infty}\left(\ln\left|\frac{1}{1}\right| + \ln 2\right) \therefore \text{ converges to } \ln 2}$

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⑤ $\int_0^5 \frac{1}{(5-x)^{\frac{1}{3}}}dx = \lim_{t\to 5^-}\left(\int_0^t \frac{1}{(5-x)^{\frac{1}{3}}}dx\right)$

$\int\frac{1}{(5-x)^{\frac{1}{3}}}dx = -\frac{3(5-x)^{\frac{2}{3}}}{2}$ $\quad \lim_{t\to 5^-}\left(-\frac{3(5-x)^{\frac{2}{3}}}{2}\Big|_0^t\right) = \lim_{t\to 5^-}\left(-\frac{3(5-t)^{\frac{2}{3}}}{2}+\frac{3(5)^{\frac{2}{3}}}{2}\right)$

$\boxed{\lim_{t\to 5^-}\left(\frac{3(5)^{\frac{2}{3}}}{2} - \frac{3(5-t)^{\frac{2}{3}}}{2}^{\to 0}\right) \therefore \text{ converges to } \frac{3(5)^{\frac{2}{3}}}{2}}$

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(6) $\int_0^9 \frac{1}{(x-1)^{\frac{1}{3}}} dx = \lim_{t\to 1^-}\left(\int_0^t \frac{1}{(x-1)^{\frac{1}{3}}} dx\right) + \lim_{t\to 1^+}\left(\int_t^9 \frac{1}{(x-1)^{\frac{1}{3}}} dx\right)$

$\int \frac{1}{(x-1)^{\frac{1}{3}}} dx = \frac{3(x-1)^{\frac{2}{3}}}{2} + C$

$\lim_{t\to 1^-}\left(\frac{3(x-1)^{\frac{2}{3}}}{2}\Big|_0^t\right) = \lim_{t\to 1^-}\left(\frac{3(t-1)^{\frac{2}{3}}}{2} - \frac{3(-1)^{\frac{2}{3}}}{2}\right) = \lim_{t\to 1^-}\left(\frac{3\overbrace{(t-1)^{\frac{2}{3}}}^{\to 0}}{2} - \frac{3}{2}\right) \therefore$ converges to $-\frac{3}{2}$

$\lim_{t\to 1^+}\left(\frac{3(x-1)^{\frac{2}{3}}}{2}\Big|_t^9\right) = \lim_{t\to 1^+}\left(\frac{3(8)^{\frac{2}{3}}}{2} - \frac{3(t-1)^{\frac{2}{3}}}{2}\right) = \lim_{t\to 1^+}\left(\frac{3(4)}{2} - \frac{3\overbrace{(t-1)^{\frac{2}{3}}}^{\to 0}}{2}\right) \therefore$ converges to $\frac{12}{2}$

$-\frac{3}{2} + \frac{12}{2} = \frac{9}{2}$ $\boxed{\therefore \int_0^9 \frac{1}{(x-1)^{\frac{1}{3}}} dx \text{ converges to } \frac{9}{2}}$

Decide whether the following integrals converge or not by comparison to simpler integrals. Do not evaluate the integrals.

(7) $\int_1^\infty \frac{1+\sin^2 x}{\sqrt{x}} dx$ $\quad 0 \le \sin^2 x \le 1$ so... $\frac{1}{\sqrt{x}} \le \frac{1+\sin^2 x}{\sqrt{x}} \le \frac{2}{\sqrt{x}}$

$\int_1^\infty \frac{1}{\sqrt{x}} dx = \lim_{t\to\infty}\left(2\sqrt{x}\Big|_1^t\right) = \boxed{\lim_{t\to\infty}\left(2\overbrace{\sqrt{t}}^{\to\infty} - 2\right) = \infty \therefore \int_1^\infty \frac{1+\sin^2 x}{\sqrt{x}} dx \text{ diverges}}$

(8) $\int_0^1 \frac{\sec^2 x}{x\sqrt{x}} dx$ $\quad \sec^2 x \ge 1$ for $[0,1]$

$\frac{1}{x(x)^{\frac{1}{2}}} = \frac{1}{x^{\frac{3}{2}}} \le \frac{\sec^2 x}{x\sqrt{x}}$ so if $\int_0^1 \frac{1}{x^{\frac{3}{2}}} dx$ diverges then $\int_0^1 \frac{\sec^2 x}{x\sqrt{x}} dx$ also

$\lim_{t\to 0^+}\left(\int_t^1 \frac{1}{x^{\frac{3}{2}}} dx\right) = \lim_{t\to 0^+}\left(-\frac{2}{x^{\frac{1}{2}}}\Big|_t^1\right) = \lim_{t\to 0^+}\left(-2 + \frac{2}{t^{\frac{1}{2}}}\right) \therefore$ diverges

$\frac{2}{.0000...01} \to \infty \Rightarrow \frac{1}{0}$ undefined

Given $\frac{1}{x^{\frac{3}{2}}} \le \frac{\sec^2 x}{x\sqrt{x}}$ for all x in $[0,1]$ and $\int_0^1 \frac{1}{x^{\frac{3}{2}}} dx$ diverges... then $\int_0^1 \frac{\sec^2 x}{x\sqrt{x}}$ diverges

⑨ $\int_0^\infty \frac{1}{x^2+x^{\frac{1}{2}}}\,dx = \int_0^1 \frac{1}{x^2+x^{\frac{1}{2}}}\,dx + \int_1^\infty \frac{1}{x^2+x^{\frac{1}{2}}}\,dx$

$\int_1^\infty \frac{1}{x^2+x^{\frac{1}{2}}}\,dx \le \int_1^\infty \frac{1}{x^2}\,dx$ ← passed P-integral test ∴ converges

$\int_0^1 \frac{1}{x^2+x^{\frac{1}{2}}}\,dx \le \int_0^1 \frac{1}{x^{\frac{1}{2}}}\,dx \qquad \lim_{t\to 0^+}\left(\int_t^1 \frac{1}{x^{\frac{1}{2}}}\,dx\right) = \lim_{t\to 0^+}\left(2\sqrt{x}\,\Big|_t^1\right) = \lim_{t\to 0^+}\left(2\sqrt{1} - \overset{\to 0}{2\sqrt{t}}\right) = 2$

∴ converges

So... given $\int_0^1 \frac{1}{x^2+x^{\frac{1}{2}}}\,dx$ converges and $\int_1^\infty \frac{1}{x^2+x^{\frac{1}{2}}}\,dx$ converges then...
$\int_0^\infty \frac{1}{x^2+x^{\frac{1}{2}}}\,dx$ must also converge