HW3: 7.2; 7.3

Sunday, September 17, 2023 7:37 PM

7.2: 1, 2, 3, 4, 5
7.3: 1, 2, 3, 4, 5, 6, 7, 8

Helpful Trig Identities

$\sin^2 x + \cos^2 x = 1$
$1 + \tan^2 x = \sec^2 x$

$\cos 2x = 2\sin x \cos x$
$\sin 2x = \cos^2 x - \sin^2 x$

$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$
$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$\sin(A+B) = \cos A \sin B + \sin A \cos B$
$\sin(A-B) = \cos A \sin B - \sin A \cos B$

$\cos(A+B) = \cos A \cos B + \sin A \sin B$
$\cos(A-B) = \cos A \cos B - \sin A \sin B$

$\cos x \sin x = \frac{1}{2}\sin 2x$

7.2: 1, 2, 3, 4, 5

① $\int \sin^3\theta \cos^4\theta \, d\theta = \int (1-\cos^2\theta)\sin\theta \cos^4\theta \, d\theta$

$u = \cos\theta$
$du = -\sin\theta \, d\theta \qquad d\theta = \frac{du}{-\sin\theta}$

$\int (1-u^2)\cancel{\sin\theta}\, u^4 \frac{du}{-\cancel{\sin\theta}} = \int (1-u^2)u^4(-du) = \int (-1+u^2)u^4 du = \int (u^6 - u^4)\,du$

$= \left[\frac{1}{7}u^7 - \frac{1}{5}u^5\right] = \boxed{\frac{1}{7}\cos^7\theta - \frac{1}{5}\cos^5\theta + C}$

② $\int_0^\pi \sin^2 t \cos^4 t \, dt = \int_0^\pi \frac{1}{2}(1-\cos 2t)\frac{1}{2}(1+\cos 2t)\frac{1}{2}(1+\cos 2t)\,dt$

$= \frac{1}{8}\int_0^\pi (1-\cos 2t)(1+\cos 2t)(1+\cos 2t)\,dt = \frac{1}{8}\int_0^\pi (1-\cos^2 2t)(1+\cos 2t)\,dt$

$= \frac{1}{8}\int_0^\pi \left(1-\frac{1}{2}(1+\cos 4t)\right)(1+\cos 2t)\,dt = \frac{1}{8}\int_0^\pi \left(\frac{1}{2}-\frac{1}{2}\cos 4t\right)(1+\cos 2t)\,dt = \frac{1}{16}\int_0^\pi (1-\cos 4t)(1+\cos 2t)\,dt$

$= \frac{1}{16}\int_0^\pi (1 + \cos 2t - \cos 4t - \cos 2t \cos 4t)\,dt$

$\cos 2t \cos 4t = \frac{1}{2}(\cos 6t + \cos 2t)$

$= \frac{1}{16}\int_0^\pi \left(1 + \cos 2t - \cos 4t - \frac{1}{2}\cos 6t - \frac{1}{2}\cos 2t\right)dt$

$= \frac{1}{16}\left(\int_0^\pi 1\,dt + \int_0^\pi \cos 2t\,dt - \int_0^\pi \cos 4t - \frac{1}{2}\int_0^\pi \cos 6t\,dt - \frac{1}{2}\int_0^\pi \cos 2t\,dt\right) = \frac{1}{16}\left(\pi + \frac{\sin 2t}{2}\Big|_0^\pi - \frac{\sin 4t}{4}\Big|_0^\pi - \frac{1}{2}\left(\frac{\sin 6t}{6}\Big|_0^\pi\right) - \frac{1}{2}\left(\frac{\sin 2t}{2}\Big|_0^\pi\right)\right)$

$= \frac{1}{16}\left(\pi + 0 - 0 - \frac{1}{2}(0) - \frac{1}{2}(0)\right) = \boxed{\frac{\pi}{16}}$

(3) $\int \tan^2\theta \sec^4\theta \, d\theta = \int \tan^2\theta \sec^2\theta (1+\tan^2\theta)\, d\theta$

$u = \tan\theta$
$du = \sec^2\theta \, d\theta$

$\int u^2 \cancel{\sec^2\theta} (1+u^2)\, du / \cancel{\sec^2\theta} = \int u^2(1+u^2)\, du = \int (u^2+u^4)\, du$

$= \frac{1}{3}u^3 + \frac{1}{5}u^5 = \boxed{\frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta + C}$

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(4) $\int \tan^5 x \sec^3 x \, dx = \int (\sec^2 x - 1)^2 \sec^2 x (\sec x \tan x)\, dx$

$u = \sec x$
$du = \sec x \tan x \, dx$

$\int (u^2-1)^2 u^2 (\cancel{\sec x \tan x})\, du / \cancel{\sec x \tan x} = \int (u^4 - 2u^2 - 1) u^2 \, du$

$= \int (u^6 - 2u^4 - u^2)\, du = \frac{1}{7}u^7 - \frac{2}{5}u^5 - \frac{1}{3}u^3 = \boxed{\frac{1}{7}\sec^7 x - \frac{2}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C}$

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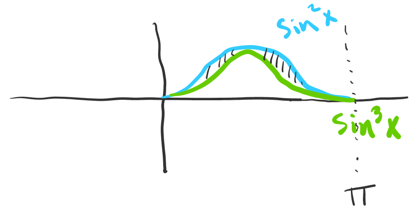
(5) Find area bounded by the given curves:

$y = \sin^2 x \qquad 0 \le x \le \pi$
$y = \sin^3 x$



$\int_0^\pi (\sin^2 x - \sin^3 x)\, dx = \int_0^\pi \sin^2 x \, dx - \int_0^\pi \sin^3 x \, dx$

$u = 2x$
$du = 2dx$

$\int_0^\pi \sin^2 x \, dx = \int_0^\pi \frac{1}{2}(1-\cos 2x)\, dx = \frac{1}{2}\int_0^\pi 1 \, dx - \frac{1}{2}\int_0^\pi \cos 2x \, dx = \frac{x}{2}\Big|_0^\pi - \left(\frac{1}{2}\cdot\frac{\sin 2x}{2}\right)\Big|_0^\pi \qquad \frac{\sin 2x}{4}\Big|_0^\pi$

$= \frac{\pi}{2} - 0 = \frac{\pi}{2}$

$\int_0^\pi \sin^3 x \, dx = \int_0^\pi \sin x (1-\cos^2 x)\, dx \qquad u = \cos x \quad du = -\sin x \, dx$

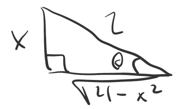
$\int \cancel{\sin x} (1-u^2)\, du / \cancel{-\sin x} = -\int (1-u^2)\, du = -\left(\int 1 \, du - \int u^2 du\right) = -u\Big|_{\square}^{\square} + \frac{1}{3}u^3\Big|_{\square}^{\square} = -\cos x\Big|_0^\pi + \frac{1}{3}\cos^3 x\Big|_0^\pi$

$= -(-1-1) + \left(-\frac{1}{3} - \frac{1}{3}\right) = 2 + -\frac{2}{3} = \frac{4}{3}$

Area $= \int_0^\pi \sin^2 x \, dx - \int_0^\pi \sin^3 x \, dx = \boxed{\frac{\pi}{2} - \frac{4}{3}}$

7.3: 1,2,3,4,5,6,7,8

① $\int \frac{1}{x^2\sqrt{4-x^2}}\,dx$ $x = 2\sin\theta$, $dx = 2\cos\theta\, d\theta$



$\cos\theta = \frac{\sqrt{4-x^2}}{2}$
$2\cos\theta = \sqrt{4-x^2}$

$$\int \frac{1}{x^2\sqrt{4-x^2}}\,dx = \int \frac{1}{4\sin^2\theta\,\cancel{2\cos\theta}} \cdot \cancel{2\cos\theta}\, d\theta = \int \frac{1}{4\sin^2\theta}\, d\theta = \frac{1}{4}\int \csc^2\theta\, d\theta$$

$$= \frac{1}{4}(-\cot\theta) = -\frac{1}{4}\cot\theta = -\frac{1}{4}\,\frac{\cos\theta}{\sin\theta} = -\frac{1}{4}\left(\frac{\sqrt{4-x^2}}{2}\right)\left(\frac{1}{\frac{x}{2}}\right) = \frac{-\sqrt{4-x^2}}{8}\left(\frac{2}{x}\right)$$

$$= \boxed{\frac{-\sqrt{4-x^2}}{4x} + C}$$

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② $\int \frac{x^3}{\sqrt{4+x^2}}\,dx$ $x = 2\tan\theta$, $dx = 2\sec^2\theta\, d\theta$

$\cos\theta = \frac{2}{\sqrt{4+x^2}}$ $\frac{1}{2}\cos\theta = \frac{1}{\sqrt{4+x^2}}$
$2\sec\theta = \sqrt{4+x^2}$
$\sec\theta = \frac{\sqrt{4+x^2}}{2}$

$$\int \frac{x^3}{\sqrt{4+x^2}}\,dx = \int 8\tan^3\theta \frac{1}{\cancel{2\sec\theta}}\,\cancel{2}\sec^{\cancel{2}}\theta\, d\theta = 8\int \tan^3\theta\sec\theta\, d\theta$$

$$= 8\int (\sec^2\theta - 1)\sec\theta\tan\theta\, d\theta \qquad u = \sec\theta \qquad du = \sec\theta\tan\theta\, d\theta$$

$$= 8\int (u^2-1)\,du = 8\left(\int u^2 du - \int 1\, du\right) = 8\left(\frac{1}{3}u^3 - u\right) = 8\left(\frac{1}{3}\sec^3\theta - \sec\theta\right)$$

$$= \frac{8}{3}\sec^3\theta - 8\sec\theta = \frac{8}{3}\left(\frac{\sqrt{4+x^2}}{2}\right)^3 - 8\left(\frac{\sqrt{4+x^2}}{2}\right) = \frac{8}{3}\left(\frac{(4+x^2)\sqrt{4+x^2}}{8}\right) - 4\sqrt{4+x^2} = \frac{8(4+x^2)\sqrt{4+x^2}}{24} - 4\sqrt{4+x^2}$$

$$= \frac{(32+8x^2)\sqrt{4+x^2} - 96\sqrt{4+x^2}}{24} = \frac{32\sqrt{4+x^2} + 8x^2\sqrt{4+x^2} - 96\sqrt{4+x^2}}{24} = \frac{8x^2\sqrt{4+x^2} - 64\sqrt{4+x^2}}{24}$$

$$= \frac{8\sqrt{4+x^2}\,(x^2-8)}{24} = \boxed{\frac{\sqrt{4+x^2}\,(x^2-8)}{3} + C}$$

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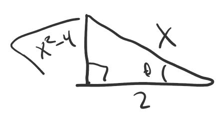
③ $\int \frac{\sqrt{x^2-4}}{x}\,dx$ $x = 2\sec\theta$, $dx = 2\sec\theta\tan\theta\, d\theta$



$\tan\theta = \frac{\sqrt{x^2-4}}{2}$
$2\tan\theta = \sqrt{x^2-4}$

$$\int \frac{\sqrt{x^2-4}}{x}\,dx = \int 2\tan\theta \cdot \frac{1}{\cancel{2\sec\theta}} \cdot \cancel{2\sec\theta}\tan\theta\, d\theta = 2\int \tan^2\theta\, d\theta = 2\int (\sec^2\theta - 1)\, d\theta$$

$$= 2\left(\int \sec^2\theta\, d\theta - \int 1\, d\theta\right) = 2(\tan\theta - \theta) = 2\tan\theta - 2\theta$$

$\tan\theta = \frac{\sqrt{x^2-4}}{2}$
thus... $\tan^{-1}\left(\frac{\sqrt{x^2-4}}{2}\right) = \theta$

$$\int \frac{\sqrt{x^2-4}}{x}\,dx = \boxed{\sqrt{x^2-4} - 2\left(\tan^{-1}\left(\frac{\sqrt{x^2-4}}{2}\right)\right) + C}$$

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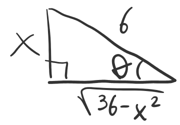
④ $\int_0^3 \frac{x}{\sqrt{36-x^2}}\,dx$



$\sin\theta = \frac{x}{6}$ $\quad 6\sin\theta = x$ $\quad dx = 6\cos\theta\,d\theta$

$\cos\theta = \frac{\sqrt{36-x^2}}{6}$ $\quad \frac{1}{6}\sec\theta = \frac{1}{\sqrt{36-x^2}}$

$$\int_{\square}^{\square} 6\sin\theta\left(\frac{1}{6}\cancel{\sec\theta}\right)6\cancel{\cos\theta}\,d\theta = 6\int_{\square}^{\square}\sin\theta\,d\theta = 6\left(-\cos\theta\Big|_{\square}^{\square}\right)$$

$$= 6\left(-\frac{\sqrt{36-x^2}}{6}\Big|_0^3\right) = 6\left(\left(-\frac{\sqrt{27}}{6}\right)-\left(-\frac{6}{6}\right)\right) = 6\left(\frac{-\sqrt{27}+6}{6}\right)$$

$$\int_0^3 \frac{x}{\sqrt{36-x^2}}\,dx = \boxed{6-\sqrt{27}}$$

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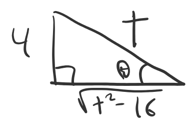
⑤ $\int \frac{1}{t^2\sqrt{t^2-16}}\,dt$



$\sin\theta = \frac{4}{t}$ $\quad \frac{1}{16}\sin^2\theta = \frac{1}{t^2}$

$\tan\theta = \frac{4}{\sqrt{t^2-16}}$ $\quad \frac{1}{4}\tan\theta = \frac{1}{\sqrt{t^2-16}}$ $\quad \cos\theta = \frac{\sqrt{t^2-16}}{t}$

$4\csc\theta = t$

$dt = -4\csc\theta\cot\theta\,d\theta$

$$\int\frac{1}{t^2\sqrt{t^2-16}}\,dt = \int \frac{1}{16}\sin^2\theta\,\frac{1}{4}\tan\theta\,-4\csc\theta\cot\theta\,d\theta = -\frac{1}{16}\int\sin^2\theta\,\cancel{\tan\theta}\left(\frac{1}{\cancel{\sin\theta}}\right)\left(\frac{1}{\cancel{\tan\theta}}\right)d\theta$$

$$-\frac{1}{16}\int\sin\theta\,d\theta = -\frac{1}{16}\left(-\cos\theta\right)+C = -\frac{1}{16}\left(-\frac{\sqrt{t^2-16}}{t}\right)+C = \boxed{\frac{\sqrt{t^2-16}}{16t}+C}$$

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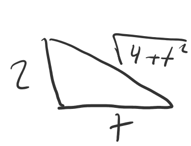
⑥ $\int_0^2 \frac{1}{\sqrt{4+t^2}}\,dt$



$\sin\theta = \frac{2}{\sqrt{4+t^2}}$ $\quad \frac{1}{2}\sin\theta = \frac{1}{\sqrt{4+t^2}}$

$\tan\theta = \frac{2}{t}$ $\quad 2\cot\theta = t$ $\quad dt = -2\csc^2\theta\,d\theta$

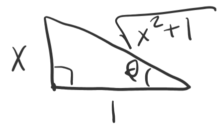
$$\int_0^2\frac{1}{\sqrt{4+t^2}}\,dt = \int_{\square}^{\square}\frac{1}{2}\sin\theta\,-2\csc^2\theta\,d\theta = -\int_{\square}^{\square}\cancel{\sin\theta}\left(\frac{1}{\cancel{\sin\theta}}\right)^2 d\theta = -\int_{\square}^{\square}\csc\theta\,d\theta$$

CO-Version of $\int \sec x\,dx$

$$= -\left(-\ln|\csc\theta+\cot\theta|\Big|_{\square}^{\square}\right) = \left(\ln\left|\frac{\sqrt{4+t^2}}{2}+\frac{t}{2}\right|\right)\Big|_0^2 = \left(\ln\left|\frac{\sqrt{8}}{2}+1\right| - \cancel{\ln|1|}\right)$$

$$\int_0^2\frac{1}{\sqrt{4+t^2}}\,dt = \boxed{\ln\left|\frac{\sqrt{8}+2}{2}\right|}$$

7) $\int_0^1 \frac{1}{(x^2+1)^2}\,dx$



$\tan\theta = x$
$dx = \sec^2\theta\, d\theta$

$\cos\theta = \frac{1}{\sqrt{x^2+1}}$
$\cos^4\theta = \frac{1}{(x^2+1)^2}$

$$\int_0^1 \frac{1}{(x^2+1)^2}\,dx = \int_{\square}^{\square} \cos^4\theta \cancel{\sec^2}\theta\, d\theta = \int_{\square}^{\square} \cos^2\theta\, d\theta = \int_{\square}^{\square} \frac{1}{2}\left(1+\cos 2\theta\right) d\theta$$

$$= \frac{1}{2}\int_{\square}^{\square} 1\, d\theta + \int_{\square}^{\square} \cos 2\theta\, d\theta = \frac{1}{2}\left(\theta\Big|_{\square}^{\square} + \left(\frac{1}{2}\sin 2\theta\Big|_{\square}^{\square}\right)\right)$$

$\sin 2\theta = 2\sin\theta\cos\theta$
$\sin\theta = \frac{x}{\sqrt{x^2+1}}$
$\sin 2\theta = 2\left(\frac{x}{\sqrt{x^2+1}}\right)\left(\frac{1}{\sqrt{x^2+1}}\right)$
$\sin 2\theta = 2x / x^2+1$

$$= \frac{1}{2}\left(\cos^{-1}\left(\frac{1}{\sqrt{x^2+1}}\right)\Big|_0^1 + \left(\frac{2x}{2x^2+2}\Big|_0^1\right)\right) = \frac{1}{2}\left(\left(\frac{\pi}{4}-0\right) + \left(\frac{1}{2}-0\right)\right)$$

$\theta = \cos^{-1}\frac{1}{\sqrt{x^2+1}}$

$\frac{\pi}{4}$

$$= \frac{1}{2}\left(\frac{\pi}{4}+\frac{1}{2}\right) = \frac{\pi}{8}+\frac{1}{4} \qquad \int_0^1 \frac{1}{(x^2+1)^2}\,dx = \boxed{\frac{\pi+2}{8}}$$

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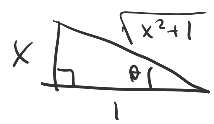
8) $\int \frac{x}{\sqrt{x^2+1}}\,dx$



$\sin\theta = \frac{x}{\sqrt{x^2+1}}$
$\cos\theta = \frac{1}{\sqrt{x^2+1}}$

$\tan\theta = x$
$dx = \sec^2\theta\, d\theta$

$$\int \frac{x}{\sqrt{x^2+1}}\,dx = \int \sin\theta \sec^2\theta\, d\theta = \int \sin\theta \frac{1}{\cos^2\theta}\, d\theta$$

$u = \cos\theta$
$du = -\sin\theta\, d\theta$
$d\theta = \frac{du}{-\sin\theta}$

$$\int \cancel{\sin\theta}\frac{1}{u^2}\frac{du}{-\cancel{\sin\theta}} = -\int \frac{1}{u^2}\,du = -\left(-\frac{1}{u}\right) = \frac{1}{\cos\theta}$$

$$\int \frac{x}{\sqrt{x^2+1}} = \boxed{\sqrt{x^2+1} + C}$$