HW: 6.3; 6.5; 7.1

Sunday, September 10, 2023 2:37 PM

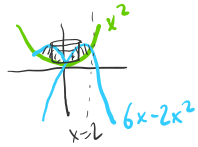
6.3: 1, 2, 3, 4, 5
6.5: 1, 2, 3
7.1: 1, 2, 3, 4, 5, 6, 7

$$dV_{shell} = 2\pi (x+a)(f(x) - g(x))\, dx$$
$$= 2\pi (r(x))(h(x))\, \Delta x$$

6.3: 1, 2, 3, 4, 5 } use the method of cylindrical shells to find the volume of the solid obtained by rotating the given region about the given line

(1) The region bounded by $y = x^2$ and $y = 6x - 2x^2$ about the y-axis

$y = x^2$
$y = 6x - 2x^2$



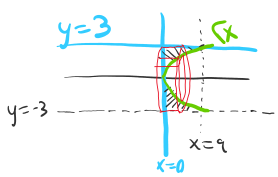
$6x - 2x^2 = 0$
$2x(3 - x) = 0$
$x = 0, 3$

$x^2 = 6x - 2x^2$
$3x^2 - 6x = 0$
$3x(x - 2) = 0$
$x = 0, 2$

$$V = 2\pi \int_0^2 x((6x - 2x^2) - x^2)\, dx = 2\pi \int_0^2 x(6x - 3x^2)\, dx = 2\pi \int_0^2 6x^2 - 3x^3\, dx$$

$$= 2\pi \left(\left[\frac{6x^3}{3} \Big|_0^2 \right] - \left[\frac{3x^4}{4} \Big|_0^2 \right] \right) = 2\pi (16 - 12) = \boxed{8\pi}$$

(2) $y = \sqrt{x}$ about the x-axis
$y = 3$
$x = 0$

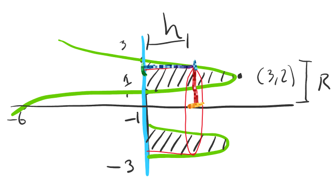


$h(y) = y^2 - 0$
$r = y$

$y = \sqrt{x}$
$y^2 = x$

$$V = 2\pi \int_0^3 y(y^2)\, dy = 2\pi \left[\frac{1}{4} y^4 \Big|_0^3 \right] = 2\pi \left(\frac{81}{4} \right) = \boxed{\frac{81\pi}{2}}$$

(3) $x = -3y^2 + 12y - 9$ about x-axis
$x = 0$



$-3(y^2 - 4y + 3) = 0$
$-3(y-3)(y-1) = 0$
$y = 3, 1$ @ $x = 0$
$x = -6$ @ $y = 0$

$$V = 2\pi \int_1^3 (y)(-3y^2 + 12y - 9)\, dy = 2\pi \int_1^3 (-3y^3 + 12y^2 - 9y)\, dy$$

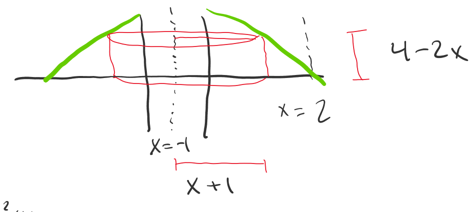
$-3(2^2 - 4(2) + 3) = 3$
$(3, 2)$

$$= 2\pi \left(-\frac{3}{4} y^4 + 4y^3 - \frac{9}{2} y^2 \Big|_1^3 \right) = 2\pi \left[\left(\frac{-3}{4}(81) + 4(27) - \frac{9}{2}(9) \right) - \left(-\frac{3}{4} + 4 - \frac{9}{2} \right) \right]$$

$$= 2\pi \left[\left(-\frac{243}{4} + \frac{432}{4} - \frac{162}{4} \right) - \left(-\frac{3}{4} + \frac{16}{4} - \frac{18}{4} \right) \right] = 2\pi \left(\frac{27}{4} + \frac{5}{4} \right) = 2\pi \left(\frac{32}{4} \right)$$

$$= \boxed{16\pi}$$

(4) $y = 4-2x$ about $x = -1$
$y = 0$
$x = 0$



$4 - 2x = 0$
$-2x = 4$
$x = 2$

$$V = 2\pi \int_0^2 (x+1)(4-2x)\,dx \quad = 4x - 2x^2 + 4 - 2x$$

$$= 2\pi \int_0^2 (-2x^2 + 2x + 4)\,dx = -4\pi \int_0^2 (x^2 - x - 2)\,dx = -4\pi \left[\tfrac{1}{3}x^3 - \tfrac{1}{2}x^2 - 2x \Big|_0^2\right]$$

$$= -4\pi\left[\tfrac{1}{3}(8) - \tfrac{1}{2}(4) - 4\right] = -4\pi\left(\tfrac{16}{6} - \tfrac{12}{6} - \tfrac{24}{6}\right) = -4\pi\left(-\tfrac{20}{6}\right) = -4\pi\left(-\tfrac{10}{3}\right)$$

$$\boxed{V = \frac{40\pi}{3}}$$

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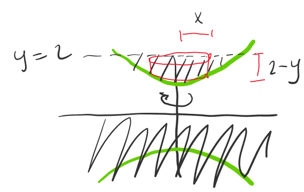
(5) $y^2 - x^2 = 1$ about the y-axis
$y = 2$



$0 - x^2 = 1$
$x^2 = -1$
$x = \sqrt{-1}$
only complex roots

$y^2 = 1 + x^2$
$y = \sqrt{1+x^2} = (1+x^2)^{\frac{1}{2}}$

$y^2 - 0 = 1 \quad y = \pm 1$

$2 = \sqrt{1+x^2}$
$4 = 1 + x^2$
$x = \sqrt{3}$

$$V = 2\pi \int_0^{\sqrt{3}} (x)\left(2 - \left((1+x^2)^{\frac{1}{2}}\right)\right)dx$$

$$= 2\pi \int_0^{\sqrt{3}} \left(2x - x\sqrt{1+x^2}\right)dx = 2\pi\left(\int_0^{\sqrt{3}} (2x)\,dx - \int_0^{\sqrt{3}} x(1+x^2)^{\frac{1}{2}}\,dx\right)$$

$$= 2\pi\left[x^2\Big|_0^{\sqrt{3}}\right] - 2\pi\int_0^{\sqrt{3}} x\sqrt{1+x^2}\,dx$$

$u = 1 + x^2 \qquad dx = \frac{du}{2x}$
$du = 2x\,dx$

$$= 2\pi(3) - \pi \int_0^{\sqrt{3}} u^{\frac{1}{2}}\,du = 6\pi - \pi\left[\tfrac{2}{3}u^{\frac{3}{2}}\Big|_0^{\sqrt{3}}\right] = 6\pi - \pi\left[\tfrac{2}{3}(1+x^2)^{\frac{3}{2}}\Big|_0^{\sqrt{3}}\right]$$

$$= 6\pi - \pi\left[\left(\tfrac{2}{3}\sqrt{64}\right) - \left(\tfrac{2}{3}\right)\right] = 6\pi - \pi\left(\tfrac{16}{3} - \tfrac{2}{3}\right) = 6\pi - \pi\left(\tfrac{14}{3}\right) = 6\pi - \frac{14\pi}{3}$$

$$\boxed{V = \frac{4}{3}\pi}$$

6.5: 1, 2, 3

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

(1) Find the average value of $f(x) = \sqrt{x}$ on the interval $[0,4]$

$$f_{avg} = \frac{1}{4-0}\int_0^4 \sqrt{x}\,dx = \frac{1}{4}\left(\frac{2x^{\frac{3}{2}}}{3}\Big|_0^4\right) = \frac{1}{4}\left(\frac{2\left(4^{\frac{1}{2}}\right)^{\frac{3}{1}}}{3}\right) = \frac{1}{4}\left(\frac{16}{3}\right) = \frac{16}{12}$$

$$= \boxed{\frac{4}{3}}$$

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(2) Find the average value of $f(x) = 1/x$ on the interval $[1,3]$

$$f_{avg} = \frac{1}{3-1}\int_1^3 \frac{1}{x}\,dx = \frac{1}{2}\left(\ln x\Big|_1^3\right) = \frac{1}{2}\left(\ln(3) - \ln(1)\right) = \frac{1}{2}\ln(3) - 0$$

$$= \boxed{\frac{\ln(3)}{2}}$$

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③ Find all numbers b such that the average value of $f(x) = 2 + 6x - 3x^2$ on the interval $[0, b]$ is 3

$$f_{avg} = \frac{1}{b-0}\int_0^b (2 + 6x - 3x^2)\,dx = 3$$

$$= \frac{1}{b}\left[2x\Big|_0^b + 3x^2\Big|_0^b - x^3\Big|_0^b\right] = \frac{1}{b}\left[2b + 3b^2 - b^3\right]$$

$$3 = 2 + 3b - b^2$$

$$b^2 - 3b + 1 = 0$$

$$b = \frac{3 \pm \sqrt{9-4}}{2} = \boxed{\frac{3+\sqrt{5}}{2} \text{ and } \frac{3-\sqrt{5}}{2}}$$

7.1: 1, 2, 3, 4, 5, 6, 7

$$\int u\,dv = u \cdot v - \int v\,du$$

$$\int_a^b u\,dv = u(x)v(x)\Big|_a^b - \int_a^b v\,du$$

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① $\int \sqrt{x}\ln(x)\,dx$

$u = \ln x \qquad dv = \sqrt{x}\,dx$

$du = \frac{1}{x}dx \qquad v = \frac{2}{3}x^{\frac{3}{2}}$

$x^{\frac{3}{2}} \cdot x^{-\frac{2}{2}} = x^{\frac{1}{2}}$

$$\int u\,dv = \ln x \cdot \frac{2}{3}x^{\frac{3}{2}} - \int \frac{2}{3}x^{\frac{3}{2}}\left(\frac{1}{x}\right)dx = \ln x \cdot \frac{2}{3}x^{\frac{3}{2}} - \frac{2}{3}\int \sqrt{x}\,dx$$

$$= \ln x \cdot \frac{2}{3}x^{\frac{3}{2}} - \frac{2}{3}\left(\frac{2}{3}x^{\frac{3}{2}}\right) + C = \frac{2\ln x\,(x^{\frac{3}{2}})}{3} - \frac{4x^{\frac{3}{2}}}{9} + C$$

$$= \frac{6\ln x\,(x^{\frac{3}{2}}) - 4x^{\frac{3}{2}}}{9} + C = \boxed{\frac{2x^{\frac{3}{2}}(3\ln x - 2)}{9} + C}$$

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② $\int t^2 \sin(\beta t)\,dt$

$u = t^2 \qquad dv = \sin(\beta t)$

$du = 2t\,dt \qquad v = \frac{-\cos(\beta t)}{\beta}$

$$\int u\,dv = t^2 \cdot \frac{-\cos(\beta t)}{\beta} - \int \frac{-\cos(\beta t)}{\beta}(2t)\,dt = \frac{-t^2\cos(\beta t)}{\beta} - \left(\frac{-2}{\beta}\right)\int \cos(\beta t)\,t$$

$$= \frac{-t^2\cos(\beta t)}{\beta} - \left(\frac{-2}{\beta}\right)\int \cos(\beta t)\,t\,dt$$

$u = t \qquad dv = \cos(\beta t)$

$du = dt \qquad v = \frac{\sin(\beta t)}{\beta}$

$$= \frac{-t^2\cos(\beta t)}{\beta} - \left(\frac{-2}{\beta}\right)\left(\frac{t\sin(\beta t)}{\beta} - \int \frac{\sin(\beta t)}{\beta}\,dt\right)$$

$$= \frac{-t^2\cos(\beta t)}{\beta} - \left(\frac{-2}{\beta}\right)\left(\frac{t\sin(\beta t)}{\beta} - \left(\frac{1}{\beta}\right)\int \sin\beta t\,dt\right) = \frac{-t^2\cos(\beta t)}{\beta} - \left(\frac{-2}{\beta}\right)\left(\frac{t\sin(\beta t)}{\beta} - \frac{1}{\beta}\left(\frac{-\cos\beta t}{\beta}\right)\right)$$

$$= \frac{-t^2\cos(\beta t)}{\beta} - \left(\frac{-2}{\beta}\right)\left(\frac{\beta t\sin(\beta t) + \cos(\beta t)}{\beta^2}\right) \Rightarrow \frac{-2\beta t\sin(\beta t) + 2\cos(\beta t)}{\beta^3}$$

$$= \frac{-t^2\cos(\beta t)}{\beta} + \frac{2\beta t\sin(\beta t) + 2\cos(\beta t)}{\beta^3} = \frac{-\beta^2 t^2\cos(\beta t) + 2\beta t\sin(\beta t) + 2\cos(\beta t)}{\beta^3}$$

$$= -\frac{t^2\cos(\beta t)}{\beta} + \frac{2\beta t\sin(\beta t) + 2\cos(\beta t)}{\beta^3} = \frac{-\beta^2 t^2\cos(\beta t) + 2\beta t\sin(\beta t) + 2\cos(\beta t)}{\beta^3}$$

$$= \frac{2\cos(\beta t) - \beta^2 t^2\cos(\beta t) + 2\beta t\sin(\beta t)}{\beta^3} = \boxed{\frac{\cos(\beta t)\left(2-\beta^2 t^2\right) + 2\beta t\sin(\beta t)}{\beta^3} + C}$$

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③ $\int \cos^{-1}(x)\,dx$

$u = \cos^{-1}(x) \qquad dv = dx$

$du = -\frac{1}{\sqrt{1-x^2}}\,dx \qquad v = x$

$$\int u\,dv = \cos^{-1}(x)\,x - \int \frac{-x}{\sqrt{1-x^2}}\,dx$$

$u = 1-x^2 \qquad dx = \frac{du}{-2x}$

$du = -2x\,dx$

$$= x\cos^{-1}(x) - (-1)\int \frac{\cancel{x}}{\sqrt{u}}\,du \cdot \frac{1}{-2\cancel{x}} = x\cos^{-1}(x) - \left(\frac{1}{2}\right)\int \frac{1}{\sqrt{u}}\,du = x\cos^{-1}(x) - \left(\frac{1}{2}\right)\left(2\sqrt{u}\right)$$

$$= x\cos^{-1}(x) - \left(\frac{1}{2}\right)\left(2\sqrt{1-x^2}\right) = \boxed{x\cos^{-1}(x) - \sqrt{1-x^2} + C}$$

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④ $\int (\ln(x))^2\,dx$

$u = \ln^2 x \qquad dv = dx$

$du = \frac{2\ln x}{x}\,dx \qquad v = x$

$$\int u\,dv = x\ln^2 x - \int \cancel{x}\left(\frac{2\ln x}{\cancel{x}}\right)dx = x\ln^2 x - (2)\int \ln(x)$$

$u = \ln x \qquad dv = dx$

$du = \frac{1}{x}\,dx \qquad v = x$

$$= x\ln^2 x - (2)\left(x\ln x - \int \cancel{x}\cdot\frac{1}{\cancel{x}}\,dx\right)$$

$$= x\ln^2 x - \left((2)\left(x\ln x - x\right)\right) = x\ln^2 x - 2x\ln x + 2x$$

$$= \boxed{x\left(\ln^2 x - 2\ln x + 2\right) + C}$$

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⑤ $\int e^{2x}\sin(3x)\,dx = I$

Let $\theta = x$

$u = \sin(3x)$ $\quad dv = e^{2x}\,dx$

$du = 3\cos(3x)\,dx$ $\quad v = \frac{1}{2}e^{2x}$

$$\int u\,dv = \sin 3x\left(\frac{e^{2x}}{2}\right) - \int \frac{e^{2x}}{2}\,(3\cos(3x))\,dx$$

$u = 3\cos(3x)$ $\quad dv = \frac{e^{2x}}{2}$

$du = -9\sin(3x)\,dx$ $\quad v = \frac{e^{2x}}{4}$

$$= \sin 3x\left(\frac{e^{2x}}{2}\right) - \left(3\cos(3x)\left(\frac{e^{2x}}{4}\right) - \int \frac{e^{2x}}{4}\left(-9\sin(3x)\right)dx\right)$$

$$\frac{4}{4}I = \sin 3x\left(\frac{e^{2x}}{2}\right) - \left(3\cos(3x)\left(\frac{e^{2x}}{4}\right) + \frac{9}{4}\int e^{2x}\sin 3x\,dx\right) = \sin 3x\left(\frac{e^{2x}}{2}\right) - 3\cos(3x)\left(\frac{e^{2x}}{4}\right) - \frac{9}{4}I$$

$$\frac{13}{4}I = \sin 3x\left(\frac{e^{2x}}{2}\right) - 3\cos(3x)\left(\frac{e^{2x}}{4}\right)$$

$$I = \left(\frac{2e^{2x}\sin 3x - 3e^{2x}\cos 3x}{\cancel{4}}\right)\frac{\cancel{4}}{13} = \frac{2e^{2x}\sin 3x - 3e^{2x}\cos 3x}{13}$$

$$= \boxed{\frac{e^{2x}\left(2\sin(3x) - 3\cos(3x)\right)}{13} + C}$$

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⑥ $\int_0^{2\pi} t^2\sin(2t)\,dt$

$u = t^2$ $\quad dv = \sin 2t\,dt$

$du = 2t\,dt$ $\quad v = \frac{-\cos 2t}{2}$

$$\boxed{\int_a^b u\,dv = u(x)v(x)\Big|_a^b - \int_a^b v\,du}$$

$$\int_0^{2\pi} t^2\sin 2t\,dt = \left[\frac{-t^2\cos 2t}{2}\Big|_0^{2\pi}\right] - \int_0^{2\pi}\frac{-\cos 2t}{\cancel{2}}\,\cancel{2}t\,dt$$

$$= \left[\frac{-t^2\cos 2t}{2}\Big|_0^{2\pi}\right] - (-1)\int_0^{2\pi} t\cos 2t\,dt$$

$u = t$ $\quad dv = \cos 2t\,dt$

$du = dt$ $\quad v = \frac{\sin 2x}{2}$

$$= \left[\frac{-4\pi^2(1)}{2}\right] - (-1)\left[t\left(\frac{\sin 2x}{2}\right)\Big|_0^{2\pi}\right] - \int_0^{2\pi}\frac{\sin 2x}{2}\,dt$$

$$= \left[-2\pi^2\right] - 0 - \left[\frac{-\cos 2x}{4}\Big|_0^{2\pi}\right] = -2\pi^2 - \left(-\frac{1}{4} - \left(\frac{-1}{4}\right)\right)$$

$$= \boxed{-2\pi^2}$$

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7 Find the area of the region bounded by the following 2 curves:

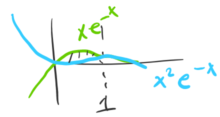
$y = x^2 e^{-x}$ $\quad \frac{x^2}{e^x}$

$y = xe^{-x}$

$x^2 e^{-x} = x e^{-x}$

$x e^{-x} = e^{-x}$

$x = 1$



$$\int_0^1 (xe^{-x})dx - \int_0^1 (x^2 e^{-x})dx$$

$$\int_0^1 (xe^{-x})dx$$

$u = x \qquad dv = e^{-x}dx$

$du = dx \qquad v = -e^{-x}$

$$\int_0^1 u\,dv = \left(-xe^{-x}\Big|_0^1\right) - \int_0^1 -e^{-x}dx = \left(-xe^{-x}\Big|_0^1\right) - \left(e^{-x}\Big|_0^1\right) = \left(-\frac{1}{e}\right) - \left(\frac{1}{e} - 1\right)$$

$$= \frac{-2}{e} + 1$$

$$\int_0^1 (x^2 e^{-x})dx$$

$u = x^2 \qquad dv = e^{-x}dx$

$du = 2x\,dx \qquad v = -e^{-x}$

$$\int_0^1 u\,dv = \left(-(x^2)e^{-x}\Big|_0^1\right) - \int_0^1 -e^{-x}2x\,dx$$

$u = x \qquad dv = -e^{-x}dx$

$du = dx \qquad v = e^{-x}$

$$= \left(-\frac{1}{e}\right) - (2)\left(\left(xe^{-x}\Big|_0^1\right) - \int_0^1 e^{-x}dx\right) = \frac{-1}{e} - (2)\left(\frac{1}{e} - \left(-e^{-x}\Big|_0^1\right)\right)$$

$$= \frac{-1}{e} - (2)\left(\frac{1}{e} - \left(-\frac{1}{e} + 1\right)\right) = \frac{-1}{e} - (2)\left(\frac{1}{e} + \frac{1}{e} - 1\right) = \frac{-1}{e} - \left(\frac{2}{e} + \frac{2}{e} - 2\right)$$

$$= -\frac{1}{e} - \frac{2}{e} - \frac{2}{e} + 2 = \frac{-5}{e} + 2$$

$$\text{Area} = \left(\frac{-2}{e} + 1\right) - \left(\frac{-5}{e} + 2\right) = \boxed{\frac{3}{e} - 1}$$