Matrix Operator

Operator-Sum Representation Method

All gates can be written in form of summation of density matrix.

Single Qubit Gates

$X = |1\rangle\langle 0|+|0\rangle\langle 1|$

Lets try to understand what X expends as,

$$X = |1\rangle\langle 0|+|0\rangle\langle 1|$$

First we start with simple understanding that a qubit can take only 2 state as $|0\rangle$ and
$|1\rangle$ with property,

$$\begin{aligned}\langle j|i\rangle &= 1 \text{ if } i = j\\ &= 0 \text{ if } i \neq j\end{aligned}$$

So, applying X on $|0\rangle$ will give us,

$$\begin{aligned}(|1\rangle\langle 0|+|0\rangle\langle 1|).|0\rangle &= |1\rangle\langle 0|0\rangle+|0\rangle\langle 1|0\rangle\\ &= |1\rangle*1+|0\rangle*0\\ &= |1\rangle\end{aligned}$$

applying X on $|1\rangle$,

$$\begin{aligned}(|1\rangle\langle 0|+|0\rangle\langle 1|).|1\rangle &= |1\rangle\langle 0|1\rangle+|0\rangle\langle 1|1\rangle\\ &= |1\rangle*0+|0\rangle*1\\ &= |0\rangle\end{aligned}$$

So, $X = |1\rangle\langle 0|+|0\rangle\langle 1|$. Lets see what it means in matrix representation.

$$|1\rangle\langle 0| = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}$$

$$|0\rangle\langle 1| = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}$$

$$X = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$

Similarly H has the decomposition as $|+\rangle\langle 0|+|-\rangle\langle 1|$.

2 Qubit Gates

$CX(0,1) = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X$

cx has a control bit and when it has the up state x gate acts on the target qubit.

So when $q_0 = |0\rangle$ the second term vanishes and the first terms return the same state as $|0\rangle \otimes |q1\rangle$, but when $q_0 = |1\rangle$ the first term vanishes and X acts on $|q1\rangle$ giving us the Final state as $|1\rangle \otimes X.|q1\rangle$.

Lets see if the qubits are not consecutive.

CX(0,2), here we have control as q0 and target as q2.

Lets first take looks on different sets on input,
Case 1: control qubit is $|0\rangle$

> For this we dont need to change anything in q2. And on top of that we don't need to have any effect on q1 so we make Identity act on it keeping it unchanged.
> Formulating it we get,

$$|0\rangle\langle 0| \otimes I \otimes I$$

Case 2: control qubit is $|1\rangle$

> For this we have to apply a X gate on q2 keeping q0 and q1 unaffected.
> Formulating it we get,

$$|1\rangle\langle 1| \otimes I \otimes X$$

As we don't have any more cases to conclude we just take the summation of all the cases.

$$CX(0,2) = |0\rangle\langle 0| \otimes I \otimes I + |1\rangle\langle 1| \otimes I \otimes X$$

What if we have m-qubits circuit and CX(a,b). what would be its evolution matrix.

For CX(a,b)
First point to consider is all qubits except q_a and q_b will remain unchanged, so we can simply swap its $|q_i\rangle\langle q_i|$ with I.

Case 1: control qubit is $|0\rangle$
As control qubit is not triggered target qubit remains unchanged.

$$I\otimes\ldots.\otimes\underbrace{|0\rangle\langle 0|}_{\text{control qubit}}\otimes I\otimes\ldots\otimes I\otimes \underbrace{I}_{\text{target qubit}}\otimes I\otimes\ldots\otimes I$$

Case 1: control qubit is $|1\rangle$
For this case we need to apply X gate on the target qubit,

$$I\otimes\ldots.\otimes\underbrace{|1\rangle\langle 1|}_{\text{control qubit}}\otimes I\otimes\ldots\otimes I\otimes \underbrace{X}_{\text{target qubit}}\otimes I\otimes\ldots\otimes I$$

As the control has only 2 state we have completed all the cases, giving us the Final Matrix Operator as ,

$$\begin{aligned} O = {} & I\otimes\ldots\otimes|0\rangle\langle 0|\otimes I\otimes\ldots\otimes I\otimes I\otimes\ldots\otimes I \\ & +I\otimes\ldots\otimes|1\rangle\langle 1|\otimes I\otimes\ldots\otimes I\otimes X\otimes\ldots\otimes I \end{aligned}$$

For CCX in m-qubit circuit.

CCX decomposes as,

$$\begin{aligned} CXX(0,1,2) = {} & |0\rangle\langle 0|\otimes|0\rangle\langle 0|\otimes I+|1\rangle\langle 1|\otimes|0\rangle\langle 0|\otimes I \\ & + |0\rangle\langle 0|\otimes|1\rangle\langle 1|\otimes I+|1\rangle\langle 1|\otimes|1\rangle\langle 1|\otimes X \end{aligned}$$

Applying on m-qubit gate with arbitrary target qubits as, CXX(a,b,c)

$$\begin{aligned} CXX(a,b,c) = {} & I\otimes\ldots\otimes|0\rangle\langle 0|\otimes I\otimes\ldots\otimes I\otimes|0\rangle\langle 0|\otimes I\otimes\ldots\otimes I\otimes\ldots\otimes I \\ & + I\otimes\ldots\otimes|1\rangle\langle 1|\otimes I\otimes\ldots\otimes I\otimes|0\rangle\langle 0|\otimes I\otimes\ldots\otimes I\otimes\ldots\otimes I \\ & + I\otimes\ldots\otimes|0\rangle\langle 0|\otimes I\otimes\ldots\otimes I\otimes|1\rangle\langle 1|\otimes I\otimes\ldots\otimes I\otimes\ldots\otimes I \\ & + I\otimes\ldots\otimes\underbrace{|1\rangle\langle 1|}_{\text{control qubit}}\otimes I\otimes\ldots\otimes I\otimes\underbrace{|1\rangle\langle 1|}_{\text{control qubit}}\otimes I\otimes\ldots\otimes \underbrace{X}_{\text{target qubit}}\otimes\ldots\otimes I \end{aligned}$$

For m-qubits circuit and n-qubit gate

Lets say that the n-qubit gate taking n_1 control bit and n-n_1 target bit has a decomposition,

$$M(0,1,\ldots n-1) = \sum_{i=1}^{c}\underbrace{|q_{0,i}\rangle\langle q_{0,i}|\otimes\ldots\otimes|q_{n1-1,i}\rangle\langle q_{n1-1,i}|}_{\text{control qubits}}$$

$$\otimes\underbrace{U3(\theta_{n1,i},\phi_{n1,i},\lambda_{n1,i})\otimes\ldots\otimes U3(\theta_{n-1,i},\phi_{n-1,i},\lambda_{n-1,i})}_{\text{target qubits}}$$

here c is total no of cases.

Lets take cxx as an example to see what the generalised equation gives us
c = 4

$$M(0,1,2) = |q_{0,1}\rangle\langle q_{0,1}|\otimes|q_{1,1}\rangle\langle q_{1,1}|\otimes U3(\theta_{2,1},\phi_{2,1},\lambda_{2,1})$$
$$+ |q_{0,2}\rangle\langle q_{0,2}|\otimes|q_{1,2}\rangle\langle q_{1,2}|\otimes U3(\theta_{2,2},\phi_{2,2},\lambda_{2,2})$$
$$+ |q_{0,3}\rangle\langle q_{0,3}|\otimes|q_{1,3}\rangle\langle q_{1,3}|\otimes U3(\theta_{2,3},\phi_{2,3},\lambda_{2,3})$$
$$+ |q_{0,4}\rangle\langle q_{0,4}|\otimes|q_{1,4}\rangle\langle q_{1,4}|\otimes U3(\theta_{2,4},\phi_{2,4},\lambda_{2,4})$$

$$M(0,1,2) = |0\rangle\langle 0|\otimes|0\rangle\langle 0|\otimes U3(0,0,0)$$
$$+ |0\rangle\langle 0|\otimes|1\rangle\langle 1|\otimes U3(0,0,0)$$
$$+ |1\rangle\langle 1|\otimes|0\rangle\langle 0|\otimes U3(0,0,0)$$
$$+ |1\rangle\langle 1|\otimes|1\rangle\langle 1|\otimes U3(\pi,0,\pi)$$

$$M(0,1,2) = |0\rangle\langle 0|\otimes|0\rangle\langle 0|\otimes I + |1\rangle\langle 1|\otimes|0\rangle\langle 0|\otimes I$$
$$+ |0\rangle\langle 0|\otimes|1\rangle\langle 1|\otimes I + |1\rangle\langle 1|\otimes|1\rangle\langle 1|\otimes X$$

Moving Forward if we have

$$M(0,1,...n-1) = \sum_{i=1} \underbrace{|q_{0,i}\rangle\langle q_{0,i}|\otimes...\otimes|q_{n1-1,i}\rangle\langle q_{n1-1,i}|}_{\text{control qubits}}$$
$$\underbrace{\otimes U3(\theta_{n1,i},\phi_{n1,i},\lambda_{n1,i})\otimes...\otimes U3(\theta_{n-1,i},\phi_{n-1,i},\lambda_{n-1,i})}_{\text{target qubits}}$$

and we want to apply it on m- qubit circuit we follow the rules same CXX in m-qubit circuit.

$$\mathbf{O(t_0,t_1,..,t_{n1-1},...t_{n-1}) = \sum I\otimes...\otimes|q_{t0,i}\rangle\langle q_{t0,i}|\otimes I\otimes..}$$
$$\mathbf{..\otimes I\otimes|q_{n1-1,i}\rangle\langle q_{n1-1,i}|\otimes I\otimes..}$$
$$\mathbf{..\otimes I\otimes U3(\theta_{n1,i},\phi_{n1,i},\lambda_{n1,i})\otimes I\otimes..}$$
$$\mathbf{..\otimes I\otimes U3(\theta_{n-1,i},\phi_{n-1,i},\lambda_{n-1,i})\otimes...\otimes I}$$

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