Sample of the 1st short IQC test

(This is just a trial test.)

1. Show that $XYX = -Y$.

2. What quantum state will be at the output of the following quantum circuit? Write down the state vector in any usual form.

$$|0\rangle \;-\boxed{Z}-\boxed{H}-\boxed{X}-\boxed{H}-\; ?$$

3. How many $T^\dagger$ gates are needed to change the 1-qubit state $|+\rangle$ into the state $|-i\rangle$?

4. Create a matrix representation of a projector created from state $|+\rangle$. Then apply such a projector on a general one-qubit state $\alpha|0\rangle + \beta|0\rangle$

5. Draw a Bloch sphere and the resulting states of two following quantum circuits with the angles specified. Don't forget to describe the axis.

   - $$|0\rangle \;-\boxed{R_z(\pi)}-\boxed{R_x(-\pi/2)}-$$
   - $$|0\rangle \;-\boxed{H}-\boxed{R_y(\pi/4)}-$$

6. Starting from the 1-qubit state $|0\rangle$, can you create the following state using the gate set $\{H, Z, S, T\}$? If yes, show how. If no, explain why.

$$|\psi\rangle = \frac{1}{\sqrt{2}}\Big(|0\rangle + e^{i\frac{\pi}{2}}|1\rangle\Big)$$

7. In the case of a measurement of the following quantum state, with what probability will state 1 be detected. Assume, that the measurement is done in the basis $|0\rangle, |1\rangle$ (the usual one).

$$|\psi\rangle = \frac{1}{\sqrt{2}}\Big(|0\rangle + e^{-i\frac{\pi}{4}}|1\rangle\Big)$$

Solution with notes

1. Show that $XYX = -Y$:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

   Then

$$XYX = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = -\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = -Y.$$

2. You can perform matrix multiplication as was done in the previous task. Alternatively, use the knowledge of the one–qubit gates on the Bloch sphere:

$$|0\rangle \xrightarrow{Z} |0\rangle \xrightarrow{H} |+\rangle \xrightarrow{X} |+\rangle \xrightarrow{H} |0\rangle$$

3. The single-qubit T gate is defined as:

$$T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}.$$

   Its Hermitian conjugate is:

$$T^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\pi/4} \end{pmatrix}.$$

   $T^\dagger$ and T only differs by the sign of rotation angle, therefore $T^\dagger$ rotates the state in the opposite direction. The change of rotation direction by the change of sign of angle is general feature for all rotational gates. The correct answer is

4. Let solve the problem as it was given for the state $(\alpha + \beta)\,|0\rangle$:

$$|+\rangle = \tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad P = |+\rangle\langle+| = \tfrac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad |\psi\rangle = (\alpha+\beta)\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

$$P|\psi\rangle = \tfrac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}(\alpha+\beta)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \tfrac{\alpha+\beta}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{\alpha+\beta}{\sqrt{2}}|+\rangle.$$

   Now supose, that there was a typo in the question and the given state is $\alpha\,|0\rangle + \beta\,|1\rangle$: The projector is

$$P = |+\rangle\langle+| = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

   Apply to $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$:

$$P|\psi\rangle = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \alpha+\beta \\ \alpha+\beta \end{pmatrix} = \frac{\alpha+\beta}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{\alpha+\beta}{\sqrt{2}}|+\rangle.$$

5.
    - The answer is the state $|+i\rangle$. θ is equal to $\pi/2$ and ϕ is has the same value.
    - The state vector is equal to $[0.38268343 + 0.j, 0.92387953 + 0.j]$. The angles are: $\theta = 3/4\pi, \phi = 0$.

6. Yes, it can be created by the followitn circuit:

$$|0\rangle \; -\boxed{H}-\boxed{S}-$$

7. The probability is 50%. Just calculate the probability from α and β as defined in the first lecturers.

Advices for the preparation for the test:

- Familiarize with all forms of qubit state notation, especially ones with the Euler equation: $e^{i\phi} = cos(\phi) + i \cdot sin(\phi)$ and how to change between the notations.
- Understand the following:
    - bra-ket notation
    - Dot product (skalární součin)
    - Tensor product (tensorový součin)
    - Outer product (vnější součin)
    - Complex conjugate (komplexní sdružení) - using the Euler equation as well.
    - Hermitian conjugate (hermitovské sdružení) "†"
    - How is the probability of measuring, e.g. $|0\rangle$) calculated out of general $|\psi\rangle$)
- Get familiar with the basic gates: $I, X, Y, Z, H, S, T, R_x(\theta), R_y(\theta), R_z(\theta), CNOT, SWAP$. It's very handy to know what is the matrix forms, action on general one–qubit or two–qubit states, for one–qubit gates it's good to know what rotation the gates represents.
- Get familiar with the Bloch sphere: angles ϕ and θ, important points like $|+\rangle, |-i\rangle$ ..., operation of quantum gates on such a points.