EURO

Fundamental Quantum Computing Algorithms and Their Implementation in Qiskit

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Aims of the training

1. Familiarize you with the quantum computing
2. Show the possible advantage of quantum computers on specific tasks

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Timetable of day 2

9:00-10:30	Grover's algorithm
10:30-10:45	Break
10:45-12:00	Quantum Fourier Transform
12:00-13:00	Lunch Break
13:00-14:00	Quantum Phase Estimation
14:00-14:15	Break
14:15-15:45	Shor's algorithm
15:45	Q&A and Closing the event

Probably we will end sooner.

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Unstructured search

Let $\Sigma = \{0, 1\}$ denote the binary alphabet (throughout the lesson).
Suppose we're given a function

$$\boxed{f : \Sigma^n \rightarrow \Sigma}$$

that we can *compute efficiently.*
Our goal is to find a *solution*, which is a binary string $x \in \Sigma^n$ for which $f(x) = 1$.

Search

Input: $f : \Sigma^n \rightarrow \Sigma$
Output: a string $x \in \Sigma^n$ satisfying $f(x) = 1$, or "no solution" if no such strings exist.

This is *unstructured* search because f is arbitrary — there's *no promise* and we can't rely on it having a structure that makes finding solutions easy.

Search

Input: $f : \Sigma^n \to \Sigma$
Output: a string $x \in \Sigma^n$ satisfying $f(x) = 1$, or "no solution" if no such strings exist.

Hereafter let us write

$$\boxed{N = 2^n}$$

By iterating through all $x \in \Sigma^n$ and evaluating f on each one, we can solve Search with N queries.

This is the best we can do with a *deterministic* algorithm.

Probabilistic algorithms offer minor improvements, but still require a number of queries linear in N.

Grover's algorithm is a *quantum algorithm* for Search requiring $O(\sqrt{N})$ queries.

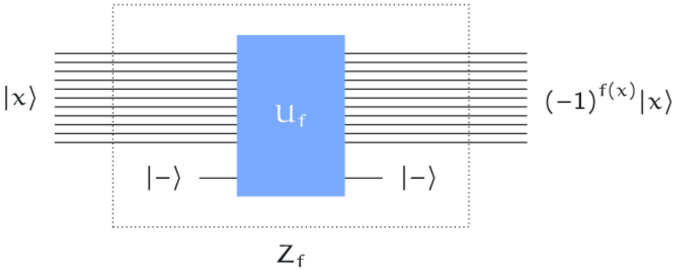
Phase query gates

We assume that we have access to the function $f : \Sigma^n \rightarrow \Sigma$ through a query gate:

$$\boxed{\mathbf{U_f} : |a\rangle|x\rangle \mapsto |a \oplus f(x)\rangle|x\rangle \quad (\text{for all } a \in \Sigma \text{ and } x \in \Sigma^n)}$$

(We can build a circuit for $\mathbf{U_f}$ given a Boolean circuit for f.)
The *phase query gate* for f operates like this:

$$\boxed{\mathbf{Z_f} : |x\rangle \mapsto (-1)^{f(x)}|x\rangle \quad (\text{for all } x \in \Sigma^n)}$$



Phase query gates

The *phase query gate* for f operates like this:

$$\boxed{\mathbf{Z_f} : |x\rangle \mapsto (-1)^{f(x)}|x\rangle \quad (\text{for all } x \in \Sigma^n)}$$

We're also going to need a phase query gate for the n-bit OR function:

$$\mathrm{OR}(x) = \begin{cases} 0 & x = 0^n \\ 1 & x \neq 0^n \end{cases} \quad (\text{for all } x \in \Sigma^n)$$

$$Z_{\mathrm{OR}}|x\rangle = \begin{cases} |x\rangle & x = 0^n \\ -|x\rangle & x \neq 0^n \end{cases} \quad (\text{for all } x \in \Sigma^n)$$

Grover's algorithm

1. *Initialize*: set n qubits to the state $H^{\otimes n}|0^n\rangle$.
2. *Iterate*: apply the *Grover operation* t times (for t to be specified later).
3. *Measure*: a standard basis measurement yields a candidate solution.

The Grover operation is defined like this:

$$G = H^{\otimes n} Z_{\mathrm{OR}} H^{\otimes n} Z_f$$

Z_f is the phase query gate for f and Z_{OR} is the phase query gate for the n-bit OR function.

Algorithm description

Grover's algorithm

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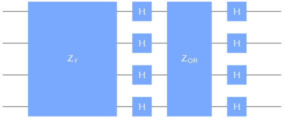

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A typical way that Grover's algorithm can be applied:

1. Choose the number of iterations t (next section).
2. Run Grover's algorithm with t iterations to get a candidate solution x.
3. Check the solution. If $f(x) = 1$ then output x, otherwise either run Grover's algorithm again (possibly with a different t) or report "no solutions."

Solutions and non-solutions

We'll refer to the n qubits being used for Grover's algorithm as a register Q.
We're interested in what happens when Q is initialized to the state $H^{\otimes n}|0^n\rangle$ and the Grover operation G is performed iteratively.

$$G = H^{\otimes n} Z_{\mathrm{OR}} H^{\otimes n} Z_f$$

These are the sets of non-solutions and solutions:

$$\mathcal{A}_0 = \{x \in \Sigma^n : f(x) = 0\}$$
$$\mathcal{A}_1 = \{x \in \Sigma^n : f(x) = 1\}$$

We will be interested in *uniform superpositions* over these sets:

$$|\mathcal{A}_0\rangle = \frac{1}{\sqrt{|\mathcal{A}_0|}} \sum_{x \in \mathcal{A}_0} |x\rangle$$
$$|\mathcal{A}_1\rangle = \frac{1}{\sqrt{|\mathcal{A}_1|}} \sum_{x \in \mathcal{A}_1} |x\rangle$$

Analysis: basic idea

$$\mathcal{A}_0 = \{x \in \Sigma^n : f(x) = 0\} \quad \mathcal{A}_1 = \{x \in \Sigma^n : f(x) = 1\}$$
$$|\mathcal{A}_0\rangle = \frac{1}{\sqrt{|\mathcal{A}_0|}} \sum_{x \in \mathcal{A}_0} |x\rangle \quad |\mathcal{A}_1\rangle = \frac{1}{\sqrt{|\mathcal{A}_1|}} \sum_{x \in \mathcal{A}_1} |x\rangle$$

The register **Q** is first initialized to this state:

$$|u\rangle = H^{\otimes n}|0^n\rangle = \frac{1}{\sqrt{N}} \sum_{x \in \Sigma^n} |x\rangle$$

This state is contained in the subspace spanned by $|\mathcal{A}_0\rangle$ and $|\mathcal{A}_1\rangle$:

$$|u\rangle = \sqrt{\frac{|\mathcal{A}_0|}{N}}|\mathcal{A}_0\rangle + \sqrt{\frac{|\mathcal{A}_1|}{N}}|\mathcal{A}_1\rangle$$

The state of **Q** *remains in this subspace* after every application of the Grover operation G.

Action of the Grover operation

We can better understand the Grover operation by splitting it into two parts:

$$G = \left(H^{\otimes n} Z_{\mathrm{OR}} H^{\otimes n}\right)\left(Z_f\right)$$

1. Recall that Z_f is defined like this:

$$Z_f|x\rangle = (-1)^{f(x)}|x\rangle \quad (\text{for all } x \in \Sigma^n)$$

Its action on $|\mathcal{A}_0\rangle$ and $|\mathcal{A}_1\rangle$ is simple:

$$Z_f|\mathcal{A}_0\rangle = |\mathcal{A}_0\rangle$$
$$Z_f|\mathcal{A}_1\rangle = -|\mathcal{A}_1\rangle$$

Action of the Grover operation

We can better understand the Grover operation by splitting it into two parts:

$$G = \left(H^{\otimes n} Z_{\mathrm{OR}} H^{\otimes n}\right)(Z_f)$$

2. The operation Z_{OR} is defined like this:

$$Z_{\mathrm{OR}}|x\rangle = \begin{cases} |x\rangle & x = 0^n \\ -|x\rangle & x \neq 0^n \end{cases} \quad \text{(for all } x \in \Sigma^n\text{)}$$

Here's an alternative way to express Z_{OR}:

$$Z_{\mathrm{OR}} = 2|0^n\rangle\langle 0^n| - I$$

Using this expression, we can write $H^{\otimes n} Z_{\mathrm{OR}} H^{\otimes n}$ like this:

$$H^{\otimes n} Z_{\mathrm{OR}} H^{\otimes n} = H^{\otimes n}(2|0^n\rangle\langle 0^n| - I)H^{\otimes n} = 2|u\rangle\langle u| - I$$

Action of the Grover operation

$$Z_f|\mathcal{A}_0\rangle = |\mathcal{A}_0\rangle$$
$$Z_f|\mathcal{A}_1\rangle = -|\mathcal{A}_1\rangle$$
$$|u\rangle = \sqrt{\frac{|\mathcal{A}_0|}{N}}|\mathcal{A}_0\rangle + \sqrt{\frac{|\mathcal{A}_1|}{N}}|\mathcal{A}_1\rangle$$

$$\begin{aligned}
G|\mathcal{A}_0\rangle &= (2|u\rangle\langle u| - I)Z_f|\mathcal{A}_0\rangle \\
&= (2|u\rangle\langle u| - I)|\mathcal{A}_0\rangle \\
&= 2\sqrt{\frac{|\mathcal{A}_0|}{N}}|u\rangle - |\mathcal{A}_0\rangle \\
&= 2\sqrt{\frac{|\mathcal{A}_0|}{N}}\left(\sqrt{\frac{|\mathcal{A}_0|}{N}}|\mathcal{A}_0\rangle + \sqrt{\frac{|\mathcal{A}_1|}{N}}|\mathcal{A}_1\rangle\right) - |\mathcal{A}_0\rangle \\
&= \frac{|\mathcal{A}_0| - |\mathcal{A}_1|}{N}|\mathcal{A}_0\rangle + \frac{2\sqrt{|\mathcal{A}_0||\mathcal{A}_1|}}{N}|\mathcal{A}_1\rangle
\end{aligned}$$

Action of the Grover operation

$$Z_f|\mathcal{A}_0\rangle = |\mathcal{A}_0\rangle$$
$$Z_f|\mathcal{A}_1\rangle = -|\mathcal{A}_1\rangle$$
$$|u\rangle = \sqrt{\frac{|\mathcal{A}_0|}{N}}|\mathcal{A}_0\rangle + \sqrt{\frac{|\mathcal{A}_1|}{N}}|\mathcal{A}_1\rangle$$

$$G|\mathcal{A}_0\rangle = \frac{|\mathcal{A}_0| - |\mathcal{A}_1|}{N}|\mathcal{A}_0\rangle + \frac{2\sqrt{|\mathcal{A}_0||\mathcal{A}_1|}}{N}|\mathcal{A}_1\rangle$$

$$\begin{aligned}
G|\mathcal{A}_1\rangle &= (2|u\rangle\langle u| - I)Z_f|\mathcal{A}_1\rangle \\
&= (1 - 2|u\rangle\langle u|)|\mathcal{A}_1\rangle \\
&= |\mathcal{A}_1\rangle - 2\sqrt{\frac{|\mathcal{A}_1|}{N}}|u\rangle \\
&= |\mathcal{A}_1\rangle - 2\sqrt{\frac{|\mathcal{A}_0|}{N}}\left(\sqrt{\frac{|\mathcal{A}_0|}{N}}|\mathcal{A}_0\rangle + \sqrt{\frac{|\mathcal{A}_1|}{N}}|\mathcal{A}_1\rangle\right)
\end{aligned}$$

Action of the Grover operation

$$Z_f|\mathcal{A}_0\rangle = |\mathcal{A}_0\rangle$$
$$Z_f|\mathcal{A}_1\rangle = -|\mathcal{A}_1\rangle$$
$$|u\rangle = \sqrt{\frac{|\mathcal{A}_0|}{N}}|\mathcal{A}_0\rangle + \sqrt{\frac{|\mathcal{A}_1|}{N}}|\mathcal{A}_1\rangle$$

$$G|\mathcal{A}_0\rangle = \frac{|\mathcal{A}_0| - |\mathcal{A}_1|}{N}|\mathcal{A}_0\rangle + \frac{2\sqrt{|\mathcal{A}_0||\mathcal{A}_1|}}{N}|\mathcal{A}_1\rangle$$
$$G|\mathcal{A}_1\rangle = -\frac{2\sqrt{|\mathcal{A}_0||\mathcal{A}_1|}}{N}|\mathcal{A}_0\rangle + \frac{|\mathcal{A}_0| - |\mathcal{A}_1|}{N}|\mathcal{A}_1\rangle$$

The action of G on $\text{span}\{|\mathcal{A}_0\rangle, |\mathcal{A}_1\rangle\}$ can be described by a 2×2 matrix:

$$M = \begin{pmatrix} \frac{|\mathcal{A}_0|-|\mathcal{A}_1|}{N} & -\frac{2\sqrt{|\mathcal{A}_0||\mathcal{A}_1|}}{N} \\ \frac{2\sqrt{|\mathcal{A}_0||\mathcal{A}_1|}}{N} & \frac{|\mathcal{A}_0|-|\mathcal{A}_1|}{N} \end{pmatrix}$$

Rotation by an angle

The action of G on $\text{span}\{|\mathcal{A}_0\rangle, |\mathcal{A}_1\rangle\}$ can be described by a 2×2 matrix:

$$M = \begin{pmatrix} \frac{|\mathcal{A}_0|-|\mathcal{A}_1|}{N} & -\frac{2\sqrt{|\mathcal{A}_0||\mathcal{A}_1|}}{N} \\ \frac{2\sqrt{|\mathcal{A}_0||\mathcal{A}_1|}}{N} & \frac{|\mathcal{A}_0|-|\mathcal{A}_1|}{N} \end{pmatrix} = \left(\begin{pmatrix} \sqrt{\frac{|\mathcal{A}_0|}{N}} & -\sqrt{\frac{|\mathcal{A}_1|}{N}} \\ \sqrt{\frac{|\mathcal{A}_1|}{N}} & \sqrt{\frac{|\mathcal{A}_0|}{N}} \end{pmatrix} \right)^2$$

This is a *rotation* matrix.

$$\begin{pmatrix} \sqrt{\frac{|\mathcal{A}_0|}{N}} & -\sqrt{\frac{|\mathcal{A}_1|}{N}} \\ \sqrt{\frac{|\mathcal{A}_1|}{N}} & \sqrt{\frac{|\mathcal{A}_0|}{N}} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \qquad \theta = \sin^{-1}\left(\sqrt{\frac{|\mathcal{A}_1|}{N}}\right)$$

$$M = \begin{pmatrix} \cos(2\theta) & -\sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) \end{pmatrix}$$

Rotation by an angle

$$M = \begin{pmatrix} \cos(2\theta) & -\sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) \end{pmatrix} \quad \theta = \sin^{-1}\left(\sqrt{\frac{|\mathcal{A}_1|}{N}}\right)$$

After the initialization step, this is the state of the register **Q**:

$$|u\rangle = \sqrt{\frac{|\mathcal{A}_0|}{N}}|\mathcal{A}_0\rangle + \sqrt{\frac{|\mathcal{A}_1|}{N}}|\mathcal{A}_1\rangle = \cos(\theta)|\mathcal{A}_0\rangle + \sin(\theta)|\mathcal{A}_1\rangle$$

Each time the Grover operation G is performed, the state of **Q** is rotated by an angle 2θ:

$$\begin{aligned} |u\rangle &= \cos(\theta)|\mathcal{A}_0\rangle + \sin(\theta)|\mathcal{A}_1\rangle \\ G|u\rangle &= \cos(3\theta)|\mathcal{A}_0\rangle + \sin(3\theta)|\mathcal{A}_1\rangle \\ G^2|u\rangle &= \cos(5\theta)|\mathcal{A}_0\rangle + \sin(5\theta)|\mathcal{A}_1\rangle \\ &\vdots \\ G^t|u\rangle &= \cos((2t+1)\theta)|\mathcal{A}_0\rangle + \sin((2t+1)\theta)|\mathcal{A}_1\rangle \end{aligned}$$

Geometric picture

Main idea

The operation $G = H^{\otimes n} Z_{OR} H^{\otimes n} Z_f$ is a composition of *two reflections:*

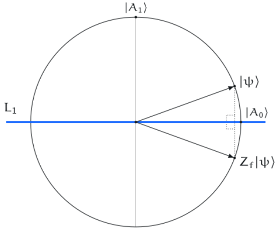
$$Z_f \quad \text{and} \quad H^{\otimes n} Z_{OR} H^{\otimes n}$$

Composing two reflections yields a *rotation.*

1. Recall that Z_f has this action on the 2-dimensional space spanned by $|A_0\rangle$ and $|A_1\rangle$:

$$Z_f|A_0\rangle = |A_0\rangle$$
$$Z_f|A_1\rangle = -|A_1\rangle$$

This is a *reflection* about the line L_1 parallel to $|A_0\rangle$.



The operation $G = H^{\otimes n} Z_{OR} H^{\otimes n} Z_f$ is a composition of *two reflections:*

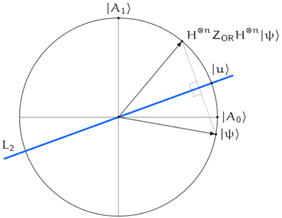
$$Z_f \quad \text{and} \quad H^{\otimes n} Z_{OR} H^{\otimes n}$$

Composing two reflections yields a *rotation.*

2. The operation $H^{\otimes n} Z_{OR} H^{\otimes n}$ can be expressed like this:

$$H^{\otimes n} Z_{OR} H^{\otimes n} = 2|u\rangle\langle u| - \mathbb{I}$$

Again this is a *reflection*, this time about the line L_2 parallel to $|u\rangle$.



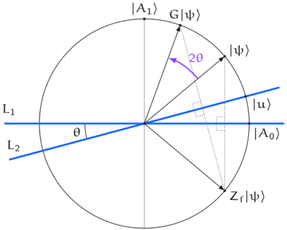
Main idea

The operation $G = H^{\otimes n} Z_{OR} H^{\otimes n} Z_f$ is a composition of *two reflections:*

$$Z_f \quad \text{and} \quad H^{\otimes n} Z_{OR} H^{\otimes n}$$

Composing two reflections yields a *rotation.*

When we compose two reflections, we obtain a *rotation* by twice the angle between the lines of reflection.



Setting the target

Consider any quantum state of this form:

$$\alpha|A_0\rangle + \beta|A_1\rangle$$

Measuring yields a solution $x \in A_1$ with probability $|\beta|^2$.

$$\alpha|A_0\rangle + \beta|A_1\rangle = \frac{\alpha}{\sqrt{|A_0|}} \sum_{x \in A_0} |x\rangle + \frac{\beta}{\sqrt{|A_1|}} \sum_{x \in A_1} |x\rangle$$

$$p(x) = \begin{cases} \frac{|\alpha|^2}{|A_0|}, & x \in A_0 \\ \frac{|\beta|^2}{|A_1|}, & x \in A_1 \end{cases}$$

$$\Pr(\text{outcome is in } A_1) = \sum_{x \in A_1} p(x) = |\beta|^2$$

Setting the target

Consider any quantum state of this form:

$$\alpha|A_0\rangle + \beta|A_1\rangle$$

Measuring yields a solution $x \in A_1$ with probability $|\beta|^2$.

The state of Q after t iterations in Grover's algorithm:

$$\cos((2t+1)\theta)|A_0\rangle + \sin((2t+1)\theta)|A_1\rangle \qquad \theta = \sin^{-1}\left(\sqrt{\frac{|A_1|}{N}}\right)$$

Measuring after t iterations gives an outcome $x \in A_1$ with probability:

$$\sin^2((2t+1)\theta)$$

We wish to maximize this probability—so we may view that $|A_1\rangle$ is our *target state.*

Setting the target

The state of Q after t iterations in Grover's algorithm:

$$\cos((2t+1)\theta)|A_0\rangle + \sin((2t+1)\theta)|A_1\rangle \qquad \theta = \sin^{-1}\left(\sqrt{\tfrac{|A_1|}{N}}\right)$$

Measuring after t iterations gives an outcome $x \in A_1$ with probability:

$$\sin^2((2t+1)\theta)$$

To make this probability close to 1 and minimize t, we will aim for:

$$(2t+1)\theta \approx \tfrac{\pi}{2} \quad \Leftrightarrow \quad t \approx \tfrac{\pi}{4\theta} - \tfrac{1}{2} \quad \text{closest integer} \quad \Rightarrow \quad t = \left\lfloor \tfrac{\pi}{4\theta} \right\rfloor$$

Important considerations:

- t must be an integer
- θ depends on the number of solutions $s = |A_1|$

Unique search

$$(2t+1)\theta \approx \frac{\pi}{2} \quad \Leftrightarrow \quad t = \left\lfloor \frac{\pi}{4\theta} \right\rfloor$$

Unique search

Input: $f : \Sigma^n \to \Sigma$
Promise: There is exactly one string $z \in \Sigma^n$ for which $f(z) = 1$, with $f(x) = 0$ for all strings $x \neq z$
Output: The string z

For Unique search we have $s = |A_1| = 1$ and therefore:

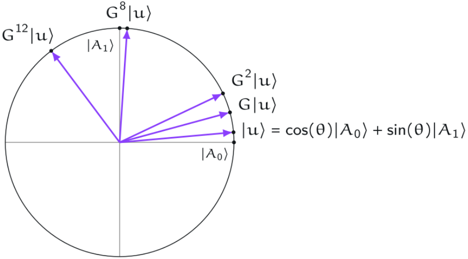
$$\theta = \sin^{-1}\left(\sqrt{\frac{1}{N}}\right) \approx \sqrt{\frac{1}{N}}$$

Substituting $\theta \approx 1/\sqrt{N}$ into our expression for t gives:

$$t \approx \left\lfloor \frac{\pi}{4\sqrt{N}} \right\rfloor \quad \Leftarrow \quad \mathcal{O}(\sqrt{N}) \text{ queries}$$

Example: N = 128

$$\theta = \sin^{-1}\left(\frac{1}{\sqrt{N}}\right) = 0.0885...$$
$$t = \left\lfloor \frac{\pi}{4\theta} \right\rfloor = 8$$



Unique search

$$\theta = \sin^{-1}\left(\sqrt{\frac{1}{N}}\right) \qquad t = \left\lfloor \frac{\pi}{4\theta} \right\rfloor$$

Measuring after t iterations gives the (unique) outcome $x \in A_1$ with probability:

$$p(N, 1) = \sin^2((2t + 1)\theta)$$

Success probabilities for Unique search

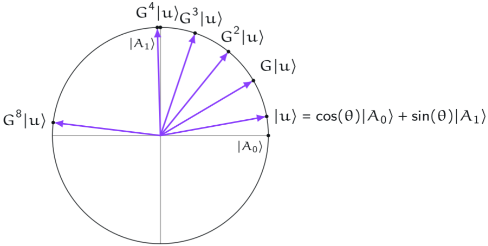
N	$p(N,1)$	N	$p(N,1)$
2	0.5	128	0.9956199
4	1.0	256	0.9999470
8	0.9453125	512	0.9994480
16	0.9613190	1024	0.9994612
32	0.9991823	2048	0.9999968
64	0.9965857	4096	0.9999453

It can be proved analytically that $p(N, 1) \geq 1 - \frac{1}{N}$.

Multiple solutions

Example: N = 128, s = 4

$$\theta = \sin^{-1}\left(\sqrt{\frac{s}{N}}\right) = 0.1777\cdots$$
$$t = \left\lfloor \frac{\pi}{4\theta} \right\rfloor = 4$$



Multiple solutions

$$\theta = \sin^{-1}\left(\sqrt{\tfrac{s}{N}}\right) \qquad t = \left\lfloor \tfrac{\pi}{4\theta} \right\rfloor$$

For every $s \in \{1, \ldots, N\}$, the probability $p(N, s)$ to find a solution satisfies:

$$p(N, s) \geq \max\left\{1 - \tfrac{s}{N}, \tfrac{s}{N}\right\}$$

Number of queries

$$\theta = \sin^{-1}\left(\sqrt{\tfrac{s}{N}}\right) \qquad t = \left\lfloor \tfrac{\pi}{4\theta} \right\rfloor$$

Each iteration of Grover's algorithm requires 1 query (or evaluation of f). How does the number of queries t depend on N and s?

$$\begin{gathered}
\sin^{-1}(x) \geq x \quad (\text{for every } x \in [0,1]) \\
\theta = \sin^{-1}\left(\sqrt{\tfrac{s}{N}}\right) \geq \sqrt{\tfrac{s}{N}} \\
t \leq \tfrac{\pi}{4\theta} \leq \tfrac{\pi}{4}\sqrt{\tfrac{N}{s}} \\
t = \mathcal{O}\left(\sqrt{\tfrac{N}{s}}\right)
\end{gathered}$$

Unknown number of solutions

What do we do if we don't know the number of solutions in advance?

A simple approach

Choose the number of iterations $t \in \{1, \ldots, \lfloor \pi\sqrt{N}/4 \rfloor\}$ *uniformly at random.*

- The probability to find a solution (if one exists) will be at least 40%. (Repeat several times to boost success probability.)
- The number of queries (or evaluations of f) is $O(\sqrt{N})$.

A more sophisticated approach

1. Set $T = 1$.
2. Run Grover's algorithm with $t \in \{1, \ldots, T\}$ chosen uniformly at random.
3. If a solution is found, output it and stop. Otherwise, increase T and return to step 2 (or report "no solut

- The rate of increase of T must be carefully balanced: slower rates require more queries, higher rates decrease success probability. $T \leftarrow \left\lceil \frac{5}{4} T \right\rceil$ works.
- If the number of solutions is $s \geq 1$, then the number of queries (or evaluations of f) required is $O(\sqrt{N/s})$. If there are no solutions, $O(\sqrt{N})$ queries are required.

Table of Contents

Spectral theorem for unitary matrices

The *spectral theorem* is an important fact in linear algebra. Here is a statement of a special case of this theorem, for *unitary matrices.*

Spectral theorem for unitary matrices

Suppose U is an $N \times N$ unitary matrix.
There exists an orthonormal basis $\{|\psi_1\rangle, \ldots, |\psi_N\rangle\}$ of vectors along with complex numbers

$$\lambda_1 = e^{2\pi i \theta_1}, \ldots, \lambda_N = e^{2\pi i \theta_N}$$

such that

$$U = \sum_{k=1}^{N} \lambda_k |\psi_k\rangle\langle\psi_k|$$

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such that

$$U = \sum_{k=1}^{N} \lambda_k |\psi_k\rangle\langle\psi_k|$$

Each vector $|\psi_k\rangle$ is an *eigenvector* of U having *eigenvalue* λ_k:

$$U|\psi_k\rangle = \lambda_k|\psi_k\rangle = e^{2\pi i\theta_k}|\psi_k\rangle$$

Phase estimation problem

In the phase estimation problem, we're given two things:

1. A description of a *unitary quantum circuit* on n qubits.
2. An n-qubit *quantum state* $|\psi\rangle$.

We're *promised* that $|\psi\rangle$ is an eigenvector of the unitary operation U described by the circuit, and our goal is to approximate the corresponding eigenvalue.

Phase estimation problem

Input: A unitary quantum circuit for an n-qubit operation U and an n-qubit quantum state $|\psi\rangle$
Promise: $|\psi\rangle$ is an eigenvector of U
Output: An approximation to the number $\theta \in [0, 1]$ satisfying

$$U|\psi\rangle = e^{2\pi i\theta}|\psi\rangle$$

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Phase estimation problem

Input: A unitary quantum circuit for an n-qubit operation U and an n-qubit quantum state $|\psi\rangle$
Promise: $|\psi\rangle$ is an eigenvector of U
Output: An approximation to the number $\theta \in [0, 1]$ satisfying

$$U|\psi\rangle = e^{2\pi i\theta}|\psi\rangle$$

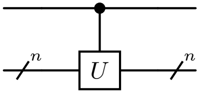
We can approximate θ by a fraction:

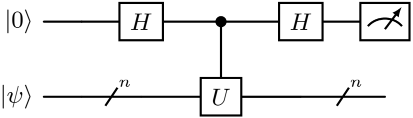
$$\theta \approx \frac{y}{2^m}$$

for $y \in \{0, 1, \ldots, 2^m - 1\}$. This approximation is taken *modulo 1*.

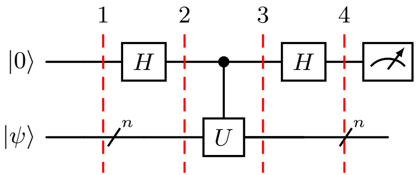
Warm-up: using the phase kickback

Given a circuit for U, we can create a circuit for a controlled-U operation:





Warm-up: using the phase kickback II

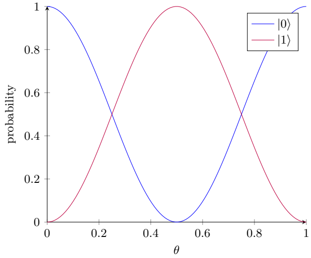


$$
\begin{aligned}
|\pi_1\rangle &= |\psi\rangle|0\rangle \\
|\pi_2\rangle &= \tfrac{1}{\sqrt{2}}|\psi\rangle|0\rangle + \tfrac{1}{\sqrt{2}}|\psi\rangle|1\rangle \\
|\pi_3\rangle &= \tfrac{1}{\sqrt{2}}\,|\psi\rangle\,|0\rangle + \tfrac{1}{\sqrt{2}}(U\,|\psi\rangle)\,|1\rangle = |\psi\rangle \otimes \left(\tfrac{1}{\sqrt{2}}\,|0\rangle + \tfrac{e^{2\pi i\theta}}{\sqrt{2}}\,|1\rangle\right) \\
|\pi_4\rangle &= |\psi\rangle \otimes \left(\tfrac{1+e^{2\pi i\theta}}{2}|0\rangle + \tfrac{1-e^{2\pi i\theta}}{2}|1\rangle\right)
\end{aligned}
$$

$$|\pi_4\rangle = |\psi\rangle \otimes \left(\frac{1+e^{2\pi i\theta}}{2}|0\rangle + \frac{1-e^{2\pi i\theta}}{2}|1\rangle \right)$$

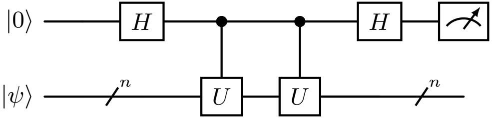
$$p_0 = \left| \frac{1+e^{2\pi i\theta}}{2} \right|^2 = \cos^2(\pi\theta) \quad p_1 = \left| \frac{1-e^{2\pi i\theta}}{2} \right|^2 = \sin^2(\pi\theta)$$

Measuring the top qubit yields the outcomes 0 and 1 with these probabilities:

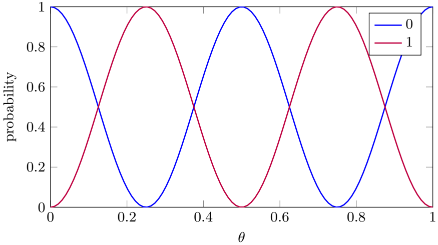


Iterating the unitary operation

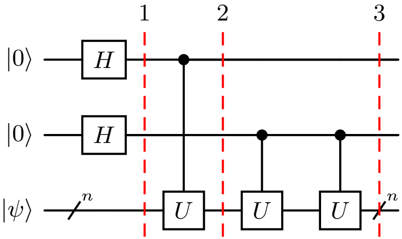
How can we learn more about θ? One possibility is to apply the controlled-U operation twice (or multiple times):



Performing the controlled-U operation twice has the effect of squaring the eigenvalue:



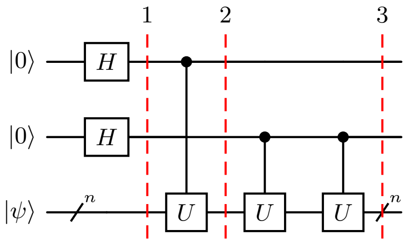
Two control qubits



$$|\pi_1\rangle = |\psi\rangle \otimes \frac{1}{2} \sum_{a_0=0}^{1} \sum_{a_1=0}^{1} |a_1 a_0\rangle$$

$$|\pi_2\rangle = |\psi\rangle \otimes \frac{1}{2} \sum_{a_0=0}^{1} \sum_{a_1=0}^{1} e^{2\pi i a_0 \theta} |a_1 a_0\rangle$$

Two control qubits



$$|\pi_3\rangle = |\psi\rangle \otimes \frac{1}{2} \sum_{a_0=0}^{1} \sum_{a_1=0}^{1} e^{2\pi i (2a_1 + a_0)\theta} |a_1 a_0\rangle \tag{1}$$

$$= |\psi\rangle \otimes \frac{1}{2} \sum_{x=0}^{3} e^{2\pi i x\theta} |x\rangle \tag{2}$$

Two control qubits

What can we learn about θ from this state? Suppose we're promised that $\theta = \frac{y}{4}$ for $y \in \{0,1,2,3\}$. Can we figure out which one it is?
Define a two-qubit state for each possibility:

$$|\Phi_y\rangle = \frac{1}{2}\sum_{x=0}^{3} e^{2\pi i \frac{xy}{4}} |x\rangle$$

$$|\Phi_0\rangle = \frac{1}{2}|0\rangle + \frac{1}{2}|1\rangle + \frac{1}{2}|2\rangle + \frac{1}{2}|3\rangle \qquad |\Phi_1\rangle = \frac{1}{2}|0\rangle + \frac{i}{2}|1\rangle - \frac{1}{2}|2\rangle - \frac{i}{2}|3\rangle$$

$$|\Phi_2\rangle = \frac{1}{2}|0\rangle - \frac{1}{2}|1\rangle + \frac{1}{2}|2\rangle - \frac{1}{2}|3\rangle \qquad |\Phi_3\rangle = \frac{1}{2}|0\rangle - \frac{i}{2}|1\rangle - \frac{1}{2}|2\rangle + \frac{i}{2}|3\rangle$$

These vectors are *orthonormal*—so they can be discriminated perfectly by a projective measurement.

Two control qubits

Unitary Matrix Representation

$$V = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix} \tag{3}$$

Action of the Unitary Matrix

$$V\,|y\rangle = |\Phi_y\rangle \quad \text{(for every } y \in \{0,1,2,3\}\text{)} \tag{4}$$

Inverse Operation

We can identify y by performing the inverse of V and then a standard basis measurement:

$$V^\dagger\,|\Phi_y\rangle = |y\rangle \quad \text{(for every } y \in \{0,1,2,3\}\text{)} \tag{5}$$

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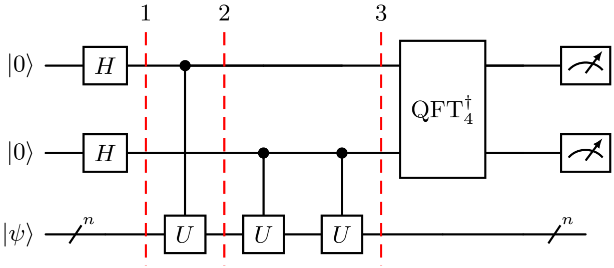
Two-qubit phase estimation

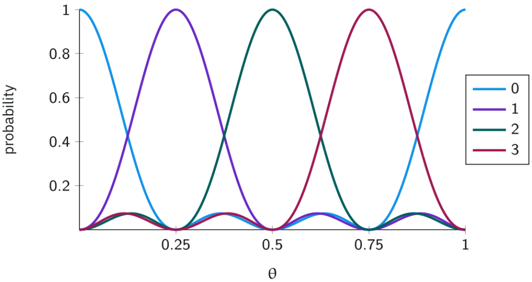
This matrix is associated with the *discrete Fourier transform* (for 4 dimensions). When we think about this matrix as a unitary operation, we call it the *quantum Fourier transform*.

The complete circuit for learning $y \in \{0, 1, 2, 3\}$ when $\theta = y/4$:



Two-qubit phase estimation

The outcome probabilities when we run the circuit, as a function of θ:



Quantum Fourier Transform

The quantum Fourier transform is defined for each positive integer N:

$$\mathrm{QFT}_N = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} e^{2\pi i \frac{xy}{N}} |x\rangle\langle y|$$

$$\mathrm{QFT}_N |y\rangle = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} e^{2\pi i \frac{xy}{N}} |x\rangle$$

$$\mathrm{QFT}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = H$$

$$\mathrm{QFT}_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \frac{-1+i\sqrt{3}}{2} & \frac{-1-i\sqrt{3}}{2} \\ 1 & \frac{-1-i\sqrt{3}}{2} & \frac{-1+i\sqrt{3}}{2} \end{pmatrix}$$

Quantum Fourier Transform

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$$\mathrm{QFT}_N = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} e^{2\pi i \frac{xy}{N}} |x\rangle\langle y|$$

$$\mathrm{QFT}_N |y\rangle = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} e^{2\pi i \frac{xy}{N}} |x\rangle$$

$$\mathrm{QFT}_4 = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}$$

Quantum Fourier Transform

$$\mathrm{QFT}_8 = \frac{1}{2\sqrt{2}} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \frac{1+i}{\sqrt{2}} & i & \frac{-1+i}{\sqrt{2}} & -1 & \frac{-1-i}{\sqrt{2}} & -i & \frac{1-i}{\sqrt{2}} \\ 1 & i & -1 & -i & 1 & i & -1 & -i \\ 1 & \frac{-1+i}{\sqrt{2}} & -i & \frac{1+i}{\sqrt{2}} & -1 & \frac{1-i}{\sqrt{2}} & i & \frac{-1-i}{\sqrt{2}} \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & \frac{-1-i}{\sqrt{2}} & -i & \frac{1-i}{\sqrt{2}} & -1 & \frac{1+i}{\sqrt{2}} & i & \frac{-1+i}{\sqrt{2}} \\ 1 & -i & -1 & i & 1 & -i & -1 & i \\ 1 & \frac{1-i}{\sqrt{2}} & -i & \frac{-1-i}{\sqrt{2}} & -1 & \frac{-1+i}{\sqrt{2}} & i & \frac{1+i}{\sqrt{2}} \end{pmatrix}$$

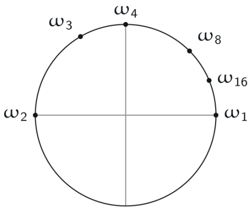
Quantum Fourier Transform

The quantum Fourier transform is defined for each positive integer N as follows.

$$\mathrm{QFT}_N = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} e^{2\pi i \frac{xy}{N}} |x\rangle\langle y| = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} \omega_N^{xy} |x\rangle\langle y|$$

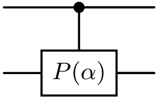
Useful shorthand notation:

$$\omega_N = e^{\frac{2\pi i}{N}} = \cos\left(\frac{2\pi}{N}\right) + i \sin\left(\frac{2\pi}{N}\right)$$

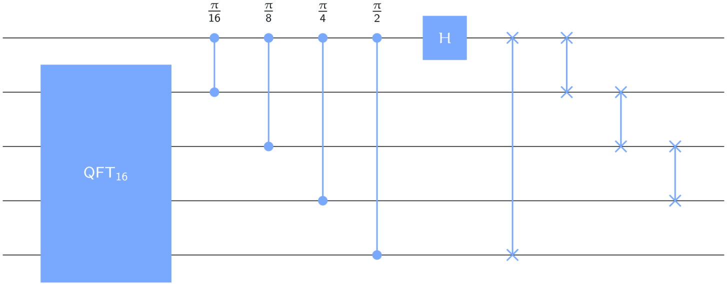


Circuits for the QFT

We can implement $\mathbf{QFT}_N$ efficiently with a quantum circuit when N is a power of 2.
The implementation makes use of **controlled-phase** gates:



$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i\alpha} \end{pmatrix}$$



Cost analysis

Let s_m denote the number of gates we need for m qubits.

- For $m = 1$, a single Hadamard gate is required.
- For $m \geq 2$, these are the gates required:
  - s_{m-1} gates for the QFT on $m-1$ qubits
  - $m-1$ controlled phase gates
  - $m-1$ swap gates
  - 1 Hadamard gate

$$s_m = \begin{cases} 1 & m = 1 \\ s_{m-1} + 2m - 1 & m \geq 2 \end{cases}$$

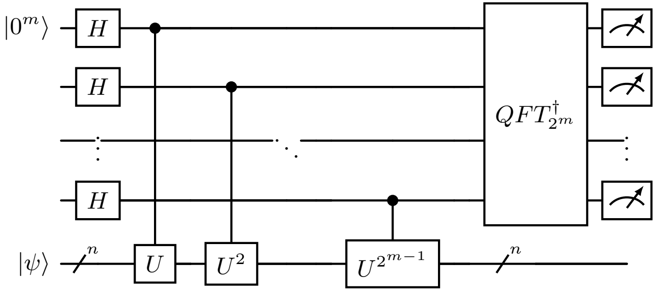
This is a ***recurrence relation*** with a closed-form solution:

$$s_m = \sum_{k=1}^{m} (2k-1) = m^2$$

Additional remarks:

- The number of swap gates can be reduced.
- Approximations to QFT_{2^m} can be done at lower cost (and lower depth).

The general phase-estimation procedure, for any choice of m:

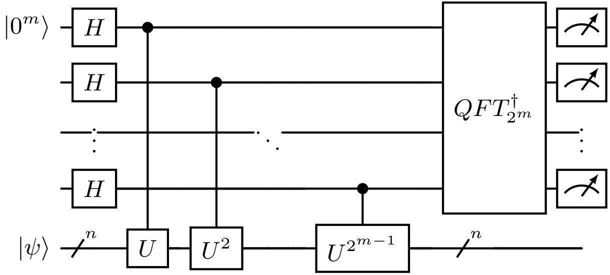


Warning

If we perform each U^k-operation by repeating a controlled-U operation k times, increasing the number of control qubits m comes at a ***high cost***.

Phase estimation procedure

The general phase-estimation procedure, for any choice of m:



$$|\pi\rangle = |\psi\rangle \otimes \frac{1}{2^m} \sum_{y=0}^{2^m-1} \sum_{x=0}^{2^m-1} e^{2\pi i x(\theta - y/2^m)} |y\rangle \tag{6}$$

$$p_y = \left| \frac{1}{2^m} \sum_{x=0}^{2^m-1} e^{2\pi i x(\theta - y/2^m)} \right|^2 \tag{7}$$

Phase estimation procedure

$$p_y = \left| \frac{1}{2^m} \sum_{x=0}^{2^m-1} e^{2\pi i x(\theta - y/2^m)} \right|^2 \tag{8}$$

Best approximations

Suppose $\frac{y}{2^m}$ is the ***best approximation*** to θ:

$$\left|\theta - \frac{y}{2^m}\right|_1 \leq 2^{-(m+1)}$$

Then the probability to measure y will be relatively high:

$$p_y \geq \frac{4}{\pi^2} \approx 0.405$$

Worse approximations

Suppose there's a ***better approximation*** to θ between $\frac{y}{2^m}$ and θ:

$$\left|\theta - \frac{y}{2^m}\right|_1 \geq 2^{-m}$$

Then the probability to measure y will be relatively low:

$$p_y \leq \frac{1}{4}$$

Phase Estimation Accuracy

To obtain an approximation $\frac{y}{2^m}$ that is ***very likely*** to satisfy

$$\left|\theta - \frac{y}{2^m}\right|_1 < 2^{-m}$$

we can run the phase estimation procedure using m control qubits ***several times*** and take y to be the ***mode*** of the outcomes.

(The eigenvector $|\psi\rangle$ is unchanged by the procedure and can be reused as many times as needed.)

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The order-finding problem

For each positive integer N, we define

$$\mathbb{Z}_N = \{0, 1, \ldots, N-1\}$$

For instance, $\mathbb{Z}_1 = \{0\}$, $\mathbb{Z}_2 = \{0, 1\}$, $\mathbb{Z}_3 = \{0, 1, 2\}$, and so on.
We can view arithmetic operations on $\mathbb{Z}_N$ as being defined modulo N.

Example

Let $N = 7$. We have $3 \cdot 5 = 15$, which leaves a remainder of 1 when divided by 7.
This is often expressed like this:

$$3 \cdot 5 \equiv 1 \pmod{7}$$

We can also simply write $3 \cdot 5 = 1$ when it's clear we're working in $\mathbb{Z}_7$.
The elements $a \in \mathbb{Z}_N$ that satisfy $\gcd(a, N) = 1$ are special.

$$\mathbb{Z}_N^* = \{a \in \mathbb{Z}_N : \gcd(a, N) = 1\}$$
$$\mathbb{Z}_{21}^* = \{1, 2, 4, 5, 8, 10, 11, 13, 16, 17, 19, 20\}$$

The order-finding problem

Fact

For every $a \in \mathbb{Z}_N^*$ there must exist a positive integer k such that $a^k = 1$. The smallest such k is called the ***order*** of a in $\mathbb{Z}_N^*$.

Example

For $N = 21$, these are the smallest powers for which this works:

$$\begin{array}{cccc} 1^1 = 1 & 5^6 = 1 & 11^6 = 1 & 17^6 = 1 \\ 2^6 = 1 & 8^2 = 1 & 13^2 = 1 & 19^6 = 1 \\ 4^3 = 1 & 10^6 = 1 & 16^3 = 1 & 20^2 = 1 \end{array}$$

Order-finding problem

Input: Positive integers a and N with $\gcd(a, N) = 1$.
Output: The smallest positive integer r such that $a^r \equiv 1 \pmod{N}$.

No efficient classical algorithm for this problem is known — an efficient algorithm for order-finding implies an efficient algorithm for integer factorization.

Order-finding by phase-estimation

To connect the order-finding problem to phase estimation, consider a system whose classical state set is $\mathbb{Z}_N$.
For a given element $a \in \mathbb{Z}_N^*$, define an operation as follows:

$$\mathcal{M}_a|x\rangle = |ax\rangle \quad (\text{for each } x \in \mathbb{Z}_N)$$

This is a ***unitary operation***—but only because $\gcd(a, N) = 1$!

Example

Let $N = 15$ and $a = 2$. The operation $\mathcal{M}_a$ has this action:

$$\begin{array}{lll}
\mathcal{M}_2|0\rangle = |0\rangle & \mathcal{M}_2|5\rangle = |10\rangle & \mathcal{M}_2|10\rangle = |5\rangle \\
\mathcal{M}_2|1\rangle = |2\rangle & \mathcal{M}_2|6\rangle = |12\rangle & \mathcal{M}_2|11\rangle = |7\rangle \\
\mathcal{M}_2|2\rangle = |4\rangle & \mathcal{M}_2|7\rangle = |14\rangle & \mathcal{M}_2|12\rangle = |9\rangle \\
\mathcal{M}_2|3\rangle = |6\rangle & \mathcal{M}_2|8\rangle = |1\rangle & \mathcal{M}_2|13\rangle = |11\rangle \\
\mathcal{M}_2|4\rangle = |8\rangle & \mathcal{M}_2|9\rangle = |3\rangle & \mathcal{M}_2|14\rangle = |13\rangle
\end{array}$$

Order-finding by phase-estimation

To connect the order-finding problem to phase estimation, consider a system whose classical state set is $\mathbb{Z}_N$.
For a given element $a \in \mathbb{Z}_N^*$, define an operation as follows:

$$\mathcal{M}_a|x\rangle = |ax\rangle \quad (\text{for each } x \in \mathbb{Z}_N)$$

This is a ***unitary operation***—but only because $\gcd(a, N) = 1$!

Main idea

The ***eigenvalues*** of $\mathcal{M}_a$ are closely connected with the ***order*** of a.

By approximating certain eigenvalues with enough precision using phase estimation, we'll be able to compute the order.

Eigenvectors and eigenvalues

This is an eigenvector of $\mathcal{M}_a$:

$$|\psi_0\rangle = \frac{|1\rangle + |a\rangle + \cdots + |a^{r-1}\rangle}{\sqrt{r}}$$

The associated eigenvalue is 1:

$$\mathcal{M}_a|\psi_0\rangle = \frac{|a\rangle + |a^2\rangle + \cdots + |a^r\rangle}{\sqrt{r}} = \frac{|a\rangle + \cdots + |a^{r-1}\rangle + |1\rangle}{\sqrt{r}} = |\psi_0\rangle$$

To identify more eigenvectors, first recall that

$$\omega_r = e^{2\pi i/r}$$

This is another eigenvector of $\mathcal{M}_a$:

$$|\psi_1\rangle = \frac{|1\rangle + \omega_r^{-1}|a\rangle + \cdots + \omega_r^{-(r-1)}|a^{r-1}\rangle}{\sqrt{r}}$$

Eigenvectors and eigenvalues

$$
\begin{aligned}
\mathcal{M}_a|\psi_1\rangle &= \frac{|a\rangle + \omega_r^{-1}|a^2\rangle + \cdots + \omega_r^{-(r-1)}|a^r\rangle}{\sqrt{r}} \\
&= \frac{\omega_r|1\rangle + |a\rangle + \omega_r^{-1}|a^2\rangle + \cdots + \omega_r^{-(r-2)}|a^{r-1}\rangle}{\sqrt{r}} \\
&= \omega_r \left(\frac{|1\rangle + \omega_r^{-1}|a\rangle + \omega_r^{-2}|a^2\rangle + \cdots + \omega_r^{-(r-1)}|a^{r-1}\rangle}{\sqrt{r}} \right) \\
&= \omega_r|\psi_1\rangle
\end{aligned}
$$

Additional eigenvectors can be identified by similar reasoning...
For each $j \in \{0, \ldots, r-1\}$, this is an eigenvector of $\mathcal{M}_a$:

$$
|\psi_j\rangle = \frac{|1\rangle + \omega_r^{-j}|a\rangle + \cdots + \omega_r^{-j(r-1)}|a^{r-1}\rangle}{\sqrt{r}}
$$

$$
\mathcal{M}_a|\psi_j\rangle = \omega_r^j|\psi_j\rangle
$$

A convenient eigenvector

$$|\psi_1\rangle = \frac{|1\rangle + \omega_r^{-1}|a\rangle + \cdots + \omega_r^{-(r-1)}|a^{r-1}\rangle}{\sqrt{r}}$$

$$\mathcal{M}_a|\psi_1\rangle = \omega_r|\psi_1\rangle = e^{2\pi i \frac{1}{r}}|\psi_1\rangle$$

Suppose we're given $|\psi_1\rangle$ as a quantum state. We can attempt to learn r as follows:

1. Perform phase estimation on the state $|\psi_1\rangle$ and a quantum circuit implementing $\mathcal{M}_a$. The outcome is an approximation $y/2^m \approx 1/r$.
2. Output $2^m/y$ rounded to the nearest integer:

$$\text{round}\left(\frac{2^m}{y}\right) = \left\lfloor \frac{2^m}{y} + \frac{1}{2} \right\rfloor$$

How much precision do we need to correctly determine r?

$$\left|\frac{y}{2^m} - \frac{1}{r}\right| \leq \frac{1}{2N^2} \quad \Rightarrow \quad \text{round}\left(\frac{2^m}{y}\right) = r$$

Choosing $m = 2\log(N) + 1$ in phase estimation makes such an approximation likely.

A random eigenvector

$$|\psi_j\rangle = \frac{|1\rangle + \omega_r^{-j}|a\rangle + \cdots + \omega_r^{-j(r-1)}|a^{r-1}\rangle}{\sqrt{r}}$$

$$\mathcal{M}_a|\psi_j\rangle = \omega_r^j|\psi_j\rangle = e^{2\pi i \frac{j}{r}}|\psi_j\rangle$$

Suppose we're given $|\psi_j\rangle$ as a quantum state for a ***random choice*** of $j \in \{0, \dots, r-1\}$. We can attempt to learn j/r as follows:

1. Perform phase estimation on the state $|\psi_j\rangle$ and a quantum circuit implementing $\mathcal{M}_a$. The outcome is an approximation $y/2^m \approx j/r$.
2. Among the fractions u/v in lowest terms satisfying $u, v \in \{0, \dots, N-1\}$ and $v \neq 0$, output the one closest to $y/2^m$. This can be done efficiently using the ***continued fraction algorithm.***

How much precision do we need to correctly determine $u/v = j/r$?

$$\left| \frac{y}{2^m} - \frac{j}{r} \right| \leq \frac{1}{2N^2} \quad \Rightarrow \quad \frac{u}{v} = \frac{j}{r}$$

Choosing $m = 2\log(N) + 1$ for phase estimation makes such an approximation likely.
We might get unlucky: j could have common factors with r.

A random eigenvector

$$|\psi_j\rangle = \frac{|1\rangle + \omega_r^{-j}|a\rangle + \cdots + \omega_r^{-j(r-1)}|a^{r-1}\rangle}{\sqrt{r}}$$

$$\mathcal{M}_a|\psi_j\rangle = \omega_r^j|\psi_j\rangle = e^{2\pi i \frac{j}{r}}|\psi_j\rangle$$

Suppose we're given $|\psi_j\rangle$ as a quantum state for a ***random choice*** of $j \in \{0, \ldots, r-1\}$. We can attempt to learn j/r as follows:

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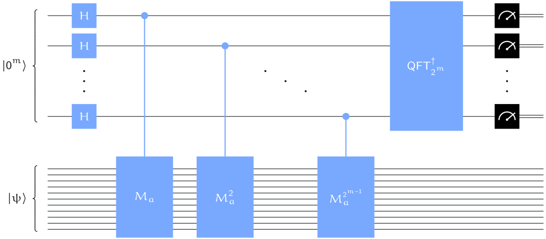
$$\left|\frac{y}{2^m} - \frac{j}{r}\right| \leq \frac{1}{2N^2} \quad \Rightarrow \quad \frac{u}{v} = \frac{j}{r}$$

If we can draw ***independent samples***, for $j \in \{0, \ldots, r-1\}$ is chosen uniformly, we can recover r with high probability by computing the ***least common multiple*** of the values of v we observed.

Implementation

To find the order of $a \in \mathbb{Z}_N^*$, we apply phase estimation to the operation $\mathcal{M}_a$. Let's measure the cost as a function of $n = \lg(N)$.



Cost for each controlled unitary

We can implement $\mathcal{M}_a$ at cost $O(n^2)$.

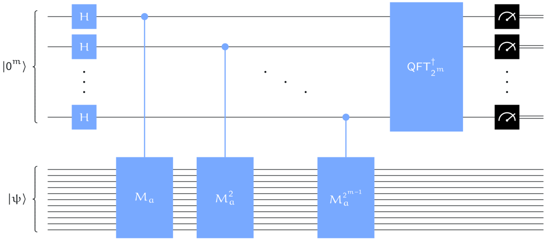
We need to implement $\mathcal{M}_a^k$ for each $k = 1, 2, 4, 8, \ldots, 2^{m-1}$. Each $\mathcal{M}_a^k$ can be implemented as follows:

- Compute $b = a^k \mod N$.
- Use a circuit for $\mathcal{M}_b$.

The cost to implement $\mathcal{M}_b = \mathcal{M}_a^k$ is $O(n^2)$.

Implementation

To find the order of $a \in \mathbb{Z}_N^*$, we apply phase estimation to the operation $\mathcal{M}_a$. Let's measure the cost as a function of $n = \lg(N)$.

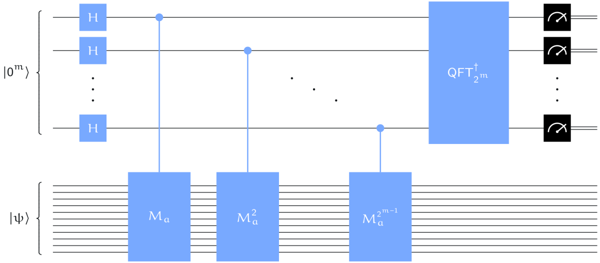


Cost for phase estimation

- m Hadamard gates: cost $O(n)$
- m controlled unitary operations: cost $O(n^3)$
- Quantum Fourier transform: cost $O(n^2)$

Total cost: $O(n^3)$

Implementation



Remaining issue: getting one of the eigenvectors $|\psi_0\rangle, \dots, |\psi_{r-1}\rangle$.
Solution: replace the eigenvector $|\psi\rangle$ with the state $|1\rangle$.
This works because of the following equation:

$$|1\rangle = \frac{|\psi_0\rangle + \cdots + |\psi_{r-1}\rangle}{\sqrt{r}}$$

The outcome is the same as if we chose $j \in \{0, 1, \dots, r-1\}$ uniformly and used $|\psi\rangle = |\psi_j\rangle$.

Factoring through order-finding

The following method succeeds in finding a factor of N with probability at least $1/2$, provided N is odd and not a prime power.

Factor-finding method

1. Choose $a \in \{2, \ldots, N-1\}$ at random.
2. Compute $d = \gcd(a, N)$. If $d \geq 2$, then output d and stop.
3. ***Compute the order*** r of a modulo N.
4. If r is even, then compute $d = \gcd(a^{r/2} - 1, N)$. If $d \geq 2$, output d and stop.
5. If this step is reached, the method has failed.

Main idea

1. By the definition of the order, we know that

$$a^r \equiv 1 \pmod{N} \iff N \text{ divides } a^r - 1$$

2. If r is even, then

$$a^r - 1 = (a^{r/2} + 1)(a^{r/2} - 1)$$

Each prime dividing N must therefore divide either $(a^{r/2} + 1)$ or $(a^{r/2} - 1)$.
For a random a, at least one of the prime factors of N is likely to divide $(a^{r/2} - 1)$.

Table of Contents

Thank you for your attention

Thanks to the IBM Quantum for great learning materials.