Sensitivity of LATE Estimates to Violations of the Monotonicity Assumption

Claudia Noack

CDSE, University of Mannheim

Heidelberg, July 19, 2018

Local Average Treatment Effect

- Let D denote a binary treatment variable, Y_1 the potential outcome variable in the presence and Y_0 in the absence of treatment
- Average Treatment Effect: $\Delta = \mathbb{E}[Y_1 - Y_0]$
- But Δ is unobserved due to endogeneity of the treatment status
- Let Z denote the instrumental variable and Y_{dz} the potential outcome

Table: Groups with a binary instrumental variable Z and treatment status D

	Always Takers (AT)	Never Takers (NT)	Compliers (CO)	Defiers (DF)
Z=1	D=1	D=0	D=1	D=0
Z=0	D=1	D=0	D=0	D=1

Local Average Treatment Effect II

Assumption 1 (LATE)

1. *Random assignment:* $(Y_{11}, Y_{01}, Y_{10}, Y_{00}, D_0, D_1) \perp Z$
2. *Exclusion restriction:* $Y_d = Y_{d0} = Y_{d1}$ *for* $d \in \{0, 1\}$*;*
3. *Relevance:* $\mathbb{P}(D = 1|Z = 1) > \mathbb{P}(D = 1|Z = 0)$
4. *Monotonicity:* $\mathbb{P}(D_1 \geq D_0) = 1$

- Monotonicity rules out the existence of defiers
- Suppose above assumptions hold. The Wald estimand β^{IV}

$$\beta^{IV} = \frac{cov(Z, Y)}{cov(Z, D)} = \Delta_{CO}$$

where Δ_{CO} denotes the average treatment effect of compliers

Motivating Example

Angrist and Evans (1998)

- Analysis of the impact of having a third child on mother's wages (endogeneity problem)
- Instrument: Having two children of the same sex
  - Exclusion restriction, random assignment and relevance assumption seem to be plausible
  - Monotonicity: all parents weakly prefer children of different sexes
- Sex preferences might be correlated with potential wages

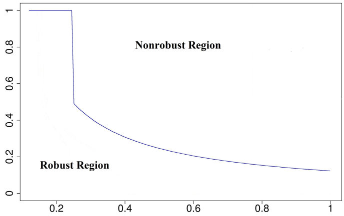
$\Rightarrow$ Validity of monotonicity and robustness of results might be questionable

Further Examples

Contribution

Sensitivity analysis of the monotonicity assumption

- Parameterization the degree of a violation of monotonicity into
  - Presence of defiers
  - Outcome heterogeneity of defiers and compliers
- Linking the degree of violation to implied treatment effects
- Identification of sensitivity parameters implying treatment effects being consistent with a certain empirical conclusion
- Approach: Identification of weakest sensitivity parameters using *breakdown frontiers*



Literature Review

Simple Case: Average Treatment Effect

Assumption 2

The IV satisfies random assignment, exclusion restriction and relevance.

From Angrist and Imbens (1995):

$$\beta^{IV} = \frac{\pi_{CO}}{\pi_{CO} - \pi_{DF}} \Delta_{CO} - \frac{\pi_{DF}}{\pi_{CO} - \pi_{DF}} \Delta_{DF}$$

Presence of Defiers

$$\lambda = \frac{\pi_{CO}}{\pi_{CO} - \pi_{DF}}$$

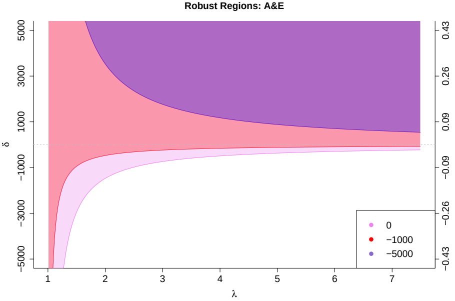
- $\lambda = 1$ absence of defiers; $\lambda \to \infty$ as many defiers as compliers

Outcome heterogeneity

$$\delta_a = \Delta_{CO} - \Delta_{DF}$$

Breakdown Frontier of Average Treatment Effect

$$BF_{ATE,CO}(\mu) = \left\{(\lambda, \delta_a) \in [1, \infty) \times \mathbb{R} : \delta_a = \frac{\mu - \beta^{IV}}{1 - \lambda}\right\}$$



Breakdown frontiers and robust regions for different claims of δ_a for $\lambda \in [1, 7.5]$

x-axis: λ, *Left y-axis:* δ_a *Right y-axis:* δ_a normalized by the sd of the outcome

More Interesting Case: Quantile Treatment Effect

Let $G_1(y) = \frac{cov(Z,\mathbf{1}[Y \leq y])}{cov(Z,D)}$. Based on Assumption 1 it holds that

$$G_d(y) = \frac{\pi_{CO}}{\pi_{CO} - \pi_{DF}} F_{Y_d^{CO}}(y) - \frac{\pi_{DF}}{\pi_{CO} - \pi_{DF}} F_{Y_d^{DF}}(y)$$

Presence of Defiers

$$\lambda = \frac{\pi_{CO}}{\pi_{CO} - \pi_{DF}}$$

Outcome Heterogeneity (Kolmogorov Smirnov Norm)

$$\sup_y \{ |F_{Y_d^{CO}}(y) - F_{Y_d^{DF}}(y)| \} \leq \delta_q \text{ with } \delta_q \in [0, 1]$$

Picture

Remark: One can further derive sharp upper and lower bounds of the prameter space. If parameter space is empty, Assumption 1 has to be violated.

Bounds

Bounding the Outcome Distributions

Theorem 1

Suppose Assumption 2 holds and the compliers outcome distributions is compatible with the sensitivity parameters. Denote the set of all CDFs on the support of the outcome variable Y by $\mathcal{F}_{\mathbb{Y}}$. Then, for $d \in \{0, 1\}$

$$\overline{F}_{Y_d^{co}}(y, \lambda, \delta_q) \leq F_{Y_d^{co}}(y) \leq \underline{F}_{Y_d^{co}}(y, \lambda, \delta_q).$$

Moreover, $\overline{F}_{Y_d^{co}}(y, \lambda, \delta_q)$ and $\underline{F}_{Y_d^{co}}(y, \lambda, \delta_q)$ are attainable as outcome CDFs. Thus, the bounds are sharp.

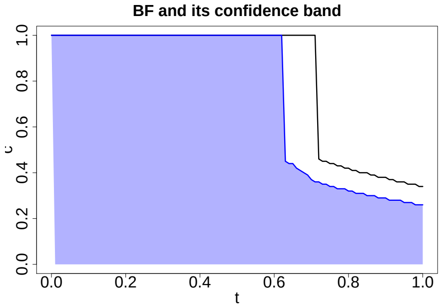
Bounds

Based on Theorem 1, one obtains quantile functions and quantile treatment effects $\underline{\Delta_{CO}}(\tau, \lambda, \delta_q)$. The breakdown frontier is given by:

$$BF_{QTE,CO}(\mu_\tau) = \{(\lambda, \delta_q) \in [0, 1]^2 : \delta_q = \arg \sup_{\delta_q^* \in [0,1]} \{\underline{\Delta_{CO}}(\tau, \lambda, \delta_q^*) \geq \mu_\tau\}\}$$

What is the Aim of Inference here?

$$\lim_{n \to \infty} \mathbb{P}(\widehat{RR}_{LB} \subseteq RR) \geq 1 - \alpha$$



Black line: Estimated breakdown frontier
Blue line: Confidence band at confidence level 0.9

Uniform Convergence

The breakdown frontier is a functional of some underlying parameters:

$$\phi : \ell^\infty(\mathbb{R}, \mathbb{R}^6) \to \ell^\infty([0,1]^2),\ \theta \mapsto \phi(\theta)$$

- $\theta = \left(F_{Y_{11}}, F_{Y_{10}}, F_{Y_{01}}, F_{Y_{00}}, G_1^+, G_0^+\right)$, where G_d^+ is a functional of outcome densities
- ϕ consists of among others mimimum, maximum, supremum, infimum operators

Theorem 2

Suppose Assumption 1 and additional regularity conditions are satisfied.

$$\sqrt{n}(\phi(\hat{\theta}) - \phi(\theta_0)) \xrightarrow{d} \phi'_{\theta_0}(\sqrt{n}(\hat{\theta} - \theta_0)) \equiv \mathcal{Z}_1,$$

where $\mathcal{Z}_1$ is a tight random element of $\ell^\infty([0,1]^2)$.

Outline of Proof

Step 1: Gaussianity of estimators of underlying parameters

- $G_d^+(y) = \int_{-\infty}^{y} \max\{0, f_1(z) - f_2(z)\}dz$, where $f_1(z)$ and $f_2(z)$ are densities
- $\sqrt{n}\left(\hat{G}_d^+(y) - G_d^+(y)\right) \to \mathbb{G}$, where $\mathbb{G}$ is a Gaussian process (similar to Anderson, Linton, and Whang, 2010)

Assumptions

Step 2: Hadamard Directional Differentiability

- ϕ is a composition of HDD functionals
- Chain rule for HDD functionals (Masten and Poirier, 2017)
- Delta-method for HDD functionals (Fang and Santos, 2016; Shapiro, 1991): If ϕ is HDD and $\sqrt{n}(\hat{\theta} - \theta_0) \to \mathbb{G}_0$ where $\mathbb{G}_0$ is a Gaussian process,

$$\sqrt{n}\left(\phi(\hat{\theta}) - \phi(\theta_0)\right) \xrightarrow{d} \phi'_{\theta_0}(\mathbb{G}_0) \equiv \mathcal{Z}$$

where $\mathcal{Z}$ is a tight random process with continuous path but unknown analytical distribution.

Illustration Definition

Bootstrap

- $\mathcal{Z}_1$ has unknown analytical distribution
- Analytical consistent estimates of ϕ'_{θ_0} difficult to obtain
- If there exists a consistent estimator of ϕ'_{θ_0}, bootstrap consistency follows from Fang and Santos (2016)
- A consistent estimator is proposed by Hong and Li (2016):

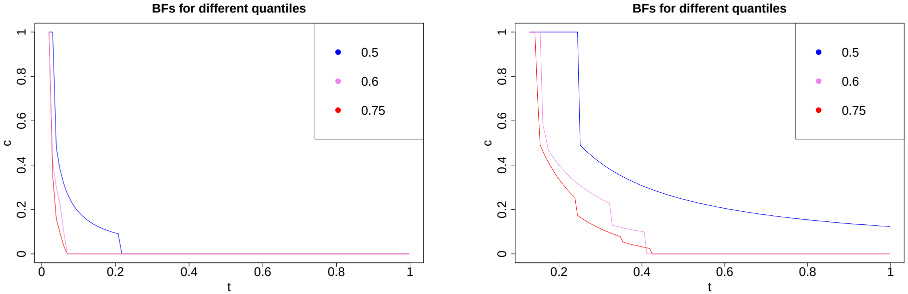
$$\tilde{\phi}'_{\theta_0}(\sqrt{n}(\hat{\theta}^* - \hat{\theta})) = \frac{\phi(\hat{\theta} + \varepsilon_n \sqrt{n}(\hat{\theta}^* - \hat{\theta})) - \phi(\hat{\theta})}{\varepsilon_n},$$

The tuning parameter ε_n is crucial.

$\Rightarrow$ Critical values can be bootstrapped to construct uniform confidence bands

Uniform Confidence Bands

Application to Angrist and Evans (1998)



Breakdown Frontier for $\overline{\Delta}_{CO}(\tau) \leq 0$
Left panel $\lambda \in [1, 6.06]$, *Right* panel $\lambda \in [1, 1.78]$

Thanks for your attention !

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- BOUNDS ON THE TREATMENT EFFECTS: Angrist, Imbens, and Rubin (1996), Balke and Pearl (1997), Klein (2007), Manski and Pepper (2000), Huber and Mellace (2012), Small and Tan (2007), Huber and Mellace (2017)
- REINTERPRETATION OF WALD-ESTIMAND: De Chaisemartin (2017)
- SENSITIVITY ANALYSIS: Huber (2014), Fiorini and Stevens (2016)

Breakdown Frontier

- Horowitz and Manski (1995); Kline and Santos (2013); Masten and Poirier (2017); Stoye (2005, 2010)

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Empirical Papers Suffering from a Potential Violation of Monotonicity

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Bounding Parameter Space

Sensitivity Parameter λ

- $\pi_{CO} - \pi_{DF} = \mathbb{P}(D = 1|Z = 1) - \mathbb{P}(D = 1|Z = 0)$
- $\overline{\pi}_{CO} = \min\{\mathbb{P}(D = 1|Z = 1), \mathbb{P}(D = 0|Z = 0)\}$
- $\underline{\pi}_{CO} = \max\{\sup_{B\in\mathscr{B}}\{\mathbb{P}(Y \in B, D = 1|Z = 1) - \mathbb{P}(Y \in B, D = 1|Z = 0)\}, \sup_{B\in\mathscr{B}}\{\mathbb{P}(Y \in B, D = 0|Z = 0) - \mathbb{P}(Y \in B, D = 0|Z = 1)\}\}.$

Lemma 1

Let $\underline{\lambda} = \frac{\underline{\pi}_{CO}}{\pi_{CO}-\pi_{DF}}$ and $\overline{\lambda} = \frac{\overline{\pi}_{CO}}{\pi_{CO}-\pi_{DF}}$. If Assumption 1 holds, $\underline{\lambda} \leq \lambda \leq \overline{\lambda}$. These bounds are sharp.

Sensitivity Parameter δ_q

The domain of δ_q is bounded for a given value of λ by

$$\delta_q \geq \psi(\lambda)$$

$\psi(\lambda)$ is a known functional of the outcome distributions and densities

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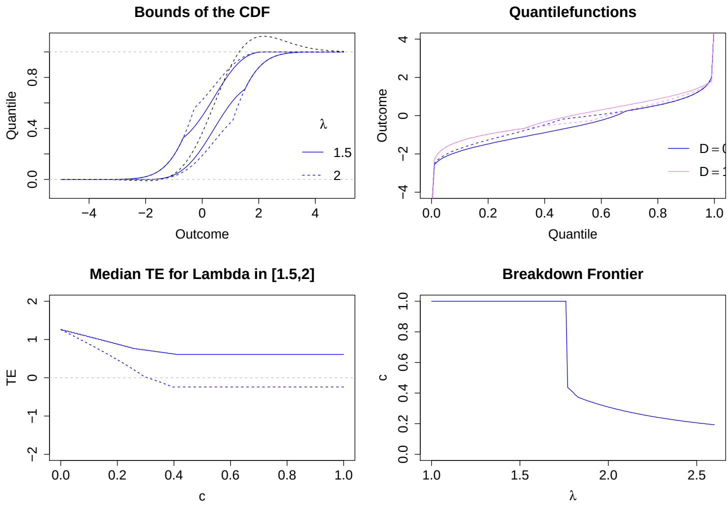
Derivation of the Breakdown Frontier



If not otherwise stated: $c = 1$, $\tau = 0.5$, and $\delta_\tau \geq 0$

Illustration of the KS-Norm

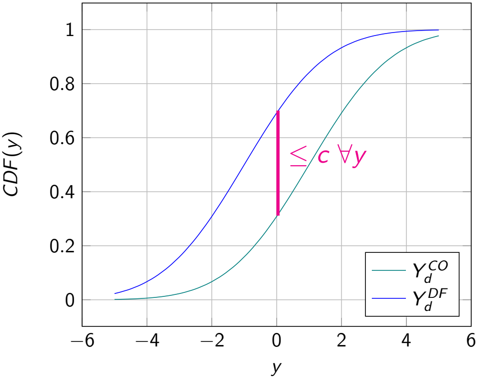


Figure: Visualization of the KS-Assumption

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Bounds of the Distributions - Definition

Let $G_d^{sup}(y) = \sup_{\tilde{y} \leq y} G_d(\tilde{y}),\ G_d^{inf}(y) = \inf_{\tilde{y} \geq y} G_d(\tilde{y}),\ G_d^{+}(y) = \int_{\underline{y}}^{y} \max\{0, \partial G_d(y) \backslash \partial y\}$

$$\overline{H}(y,t,c) = \frac{F_{Y_{dd}}(y)}{\pi_{CO}} - \max\{0, G_d^{sup}(y) + (1-\lambda)c, \frac{G_d^{sup}(y)}{\lambda}, \frac{F_{Y_{dd}}(y) - \pi_d}{\pi_{CO}}\}$$
$$+ \sup_{\tilde{y} \geq y} \left(\frac{G_d^{+}(\tilde{y})}{\lambda} - \inf_{\hat{y} \leq \tilde{y}} \left(\frac{G_d^{+}(\hat{y})}{\lambda} - \max\{0, G_d^{sup}(\hat{y}) + (1-\lambda)c, \frac{G_d^{sup}(\hat{y})}{\lambda}, \frac{F_{Y_{dd}}(\hat{y}) - \pi_d}{\pi_{CO}}\} \right) - \frac{F_{Y_{dd}}(\tilde{y})}{\pi_{CO}} \right)$$

$$\underline{H}(y,t,c) = \min\{1, G_d^{inf}(y) - (1-\lambda)c, \frac{G_d^{inf}(y) - (1-\lambda)}{\lambda}, \frac{F_{Y_{dd}}(y)}{\pi_{CO}}\} - \frac{G_d^{+}(y)}{\lambda}$$
$$- \inf_{\tilde{y} \geq y} \left(-\frac{G_d^{+}(\tilde{y})}{\lambda} + \frac{F_{Y_{dd}}(\tilde{y})}{\pi_{CO}} + \inf_{\hat{y} \leq \tilde{y}} \left(-\frac{F_{Y_{dd}}(\hat{y})}{\pi_{CO}} + \min\{1, G_d^{inf}(\hat{y}) - (1-\lambda)c, \frac{G_d^{inf}(\hat{y}) - (1-\lambda)}{\lambda}, \frac{F_{Y_{dd}}(\hat{y})}{\pi_{CO}}\} \right) \right)$$

Then

$$\overline{F}_{Y_d^{CO}}(y,t,c) = \max\{0, G_d^{sup}(y) + (1-\lambda)c, \frac{G_d^{sup}(y)}{\lambda}, \frac{F_{Y_{dd}}(y) - \pi_d}{\pi_{CO}}\} + \overline{H}(y,t,c)$$
$$\underline{F}_{Y_d^{CO}}(y,t,c) = \min\{1, G_d^{inf}(y) - (1-\lambda)c, \frac{G_d^{inf}(y) - (1-\lambda)}{\lambda}, \frac{F_{Y_{dd}}(y)}{\pi_{CO}}\} - \underline{H}(y,t,c).$$

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Outcome Distributions of the Other Groups

$$\overline{F}_{Y_d^{DF}}(y,t,c) = \frac{1}{1-\lambda}\left(G_d(y) - \lambda \overline{F}_{Y_d^{CO}}(y,t,c)\right)$$

$$\overline{F}_{Y_1^{AT}}(y,t,c) = \frac{F_{Y_{11}}(y) - \pi_{CO}\underline{F}_{Y_1^{CO}}(y,t,c)}{\pi_{AT}}$$

$$\overline{F}_{Y_0^{NT}}(y,t,c) = \frac{F_{Y_{00}}(y) - \pi_{CO}\underline{F}_{Y_0^{CO}}(y,t,c)}{\pi_{NT}}$$

Lower bounds are obtained by replacing upper with lower bounds and vice versa.

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Bounds of the Distributions - Graphical Illustration

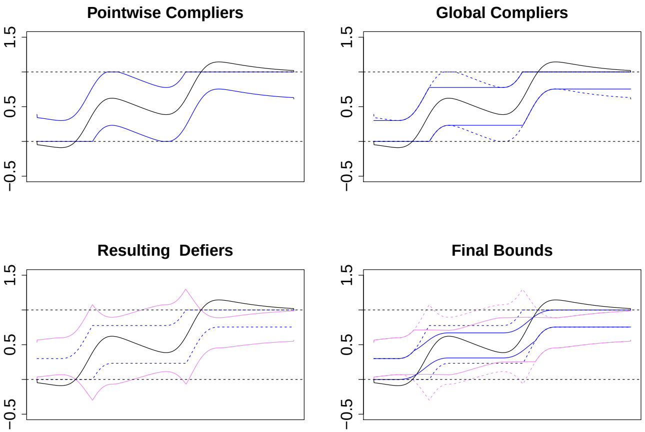

Figure: Derivation of the Bounds

Black solid line: Observed Function $G_d(y)$, *Blue lines:* Compliers, *Violet lines:* Defiers

Assumption 2 (Anderson et al., 2010)

1. $\{(Y_i, D_i, Z_i)\}_{i=1}^n$ are identically and independently distributed according to the distribution of (Y, D, Z).
2. The kernel is a 2nd order kernel function, being symmetric around zero, integrates to 1, two-times continuously differentiable, of bounded variation and zero-valued off $[-0.5, 0.5]$.
3. The bandwidth satisfies: (a) $nh^4 \to 0$, (b) $nh^2 \to \infty$, (c) $nh/\log(n) \to \infty$, (d) $nh^{(1+\gamma)/\gamma}/(\log(n)^{(1+2\gamma)/\gamma} \to \infty$, where $\gamma > 1$ is a constant satisfying (iv).
4. The densities $f_{YD|Z=z}(y_1|Z = z)$ and $f_{Y(1-D)|Z=z}(y_0|Z = z)$ are bounded, absolutely continuous and two times continuously differentiable with uniformly bounded derivatives for $z \in \{0, 1\}$.
5. $\mu(\{x \in [\underline{y}, y] : 0 < |f_1(x) - f_2(x)| < \varepsilon\}) = O_p(\varepsilon^\gamma)$ as $\varepsilon \to 0$ for some constant $\gamma > 1$ and uniformly over all y.
6. The outcome variable Y has compact support $\mathbb{Y}$.

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Hadamard Directional Differentiability

Definitions 3

Let $\mathbb{D}$ and $\mathbb{E}$ be Banach spaces with norms $\|\cdot\|_{\mathbb{D}}$ and $\|\cdot\|_{\mathbb{E}}$. Let $\mathbb{D}_\phi \subseteq \mathbb{D}$ and $\mathbb{D}_0 \subseteq \mathbb{D}$. The map $\phi : \mathbb{D}_\phi \to \mathbb{E}$ is Hadamard directional differentiable at $\theta \in \mathbb{D}_\phi$ tangentially to $\mathbb{D}_0$ if there is a continuous map $\phi'_\theta : \mathbb{D}_0 \to \mathbb{E}$ such that

$$\lim_{n\to\infty} \| \frac{\phi(\theta + t_n h_n) - \phi(\theta)}{t_n} - \phi'_\theta(h)\|_{\mathbb{E}} = 0$$

for all sequences $\{h_n\} \subset \mathbb{D}$ and $\{t_n\} \in \mathbb{R}_+$ such that $t_n \to 0$, $\|h_n - h\|_{\mathbb{D}} \to 0$, $h \in \mathbb{D}_0$ as $n \to \infty$ and $\theta + t_n h_n \in \mathbb{D}_\phi$ for all n.

If ϕ'_θ is a linear map, ϕ_θ is Hadamard differentiable.

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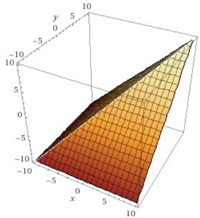
Hadamard Directional Differentiability (HDD)



Let $\theta = (\theta_1, \theta_2) \in \mathbb{R}^2$ and $\phi(\theta) = \min\{\theta_1, \theta_2\}$. ϕ is not differentiable if $\theta_1 = \theta_2$.

ϕ is HDD, if its directional derivative at θ in direction $h = (h_1, h_2) \in \mathbb{R}^2$ exists and is continuous $\forall h_n \to h$ and $t_n \searrow 0$

$$\begin{aligned}\phi'_\theta(h) &= \lim_{n\to\infty} \frac{1}{t_n} \left(\phi(\theta + t_n h_n) - \phi(\theta)\right) \\ &= \mathbb{1}[\theta_1 < \theta_2]h_1 + \mathbb{1}[\theta_1 > \theta_2]h_2 + \mathbb{1}[\theta_1 = \theta_2]\min\{h_1, h_2\}\end{aligned}$$

Further Examples

- $\phi(\theta_1, \theta_2) = \theta_1 + \theta_2$
- $\phi(\theta_1, \theta_2) = \frac{\theta_1}{\theta_2}$ if $\theta_2 \neq 0$
- $\phi(\theta_1, \theta_2) = \max\{\theta_1, \theta_2\}$
- $\phi(\theta) = \inf_{\tau \in \mathcal{T}} \theta(\tau)$

Applications

- Chain rule (Masten and Poirier, 2017; Shapiro, 1991)
- Delta-method (Fang and Santos, 2016)

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Uniform Confidence Bands

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{t\in[0,1]} \sqrt{n} \left(\widehat{LB} - BP_{LQTE,CO}(t,\mu_\tau) \right) \leq 0 \right) \geq 1-\alpha.$$

Let us simplify the problem by only considering

$$\widehat{LB} = \widehat{BP}_{LQTE,CO}(\mu_t, t) - \hat{k}(t)$$

Given that $\hat{k}(t)$ satisfies the probability constraint, one can either assume

- *Constant band size*: $\hat{k}(t) = k$
- *Maximal size of robust region*: $\min \int_0^1 \hat{k}(t)dt$
- *Relative constant bandwidth*: $\min k$ such that $\hat{k}(t) = \frac{k}{\widehat{BF}(t,\mu_\tau)}$
- *Relative maximal robust region*: $\hat{k}(t) = \arg\min_{\tilde{k}(t)} \int_0^1 \frac{\tilde{k}(t)}{\widehat{BP}(t)} dt$
- *Maxmin*: $\hat{k}(t)$ such that $\hat{k}(t) = \arg\min_{\tilde{k}(t)} \min_t \widehat{BP}(t) - \tilde{k}(t)$

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Summary Statistics: Angrist and Evans (1998)

Data and Sample Restrictions

- Data set from 1980 restricted to women at the age of $20-36$ with at least two children
- Further restriction to women being white and having their first child at the age of $19-25$
- Outcome of interest: Wage

OLS and TSLS

Model	Estimate of D	Standard Deviation	Sample Size
OLS	-2892	47	211,983
TSLS	-1575	699	211,983
TSLS (A&E)	-1390	555	390,000