Solving nonlinear inverse problems by sequential subspace optimization with an application to terahertz tomography

Anne Wald, Thomas Schuster

Department of Mathematics
Saarland University Saarbrücken

Workshop on *Inverse Problems: theory and statistical inference*

International Academic Forum Heidelberg, 29 October 2016

UNIVERSITÄT DES SAARLANDES

Professur für *Numerik*

Overview

Overview

Terahertz tomography - a short introduction

- novel imaging technique in nondestructive testing of plastics and ceramics
- non-ionizing electromagnetic radiation, frequency range 0.1 - 10 THz
- THz radiation: Gaussian beam
- **main goal:** reconstruction of complex refractive index $\tilde{n} = n + i\kappa$ to detect defects, inclusions or moisture contents from measurements of the resulting electric field

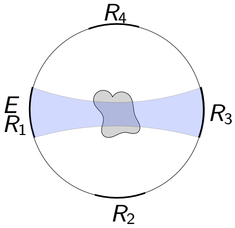


Figure: Sketch of a THz tomographic system with four receivers and one emitter

Gaussian beams - some properties

- solution of the paraxial Helmholtz equation
- transversal intensity profile: Gaussian function
- around beam waist: plane wave
- outside Rayleigh range: spherical wave

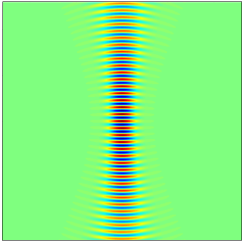
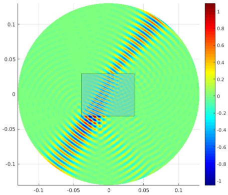


Figure: beam profile of a Gaussian beam in vacuum in the x-y-plane

- Combines elements of ray (X-ray CT) and wave (microwave CT) character

THz tomography

- refractive index n and extinction coefficient κ influence propagation of THz beam
- measured data include both transmitted and reflected field



Model assumptions

- fixed wave number k, non-conductive and non-magnetic materials,
- propagation of THz beam mainly in x-y-plane,
- dielectric permittivity ϵ fulfills

$$\nabla\big(\epsilon(\mathbf{x})\mathbf{E}(\mathbf{x})\big) \approx \epsilon(\mathbf{x})\nabla\mathbf{E}(\mathbf{x})$$

The forward problem in THz tomography

- $\Omega \subset \mathbb{R}^2$ an area with a C^1 boundary
- $m := 1 - \tilde{n}^2 \in (L^\infty(\Omega) \cap L^2_{\mathrm{comp}}(\Omega)) \subset L^2(\Omega)$ the complex refractive index

The Helmholtz equation

In Ω, the incident field u_i approximately fulfills

$$\Delta u_\mathrm{i} + k^2 u_\mathrm{i} \approx 0$$

and for the total field u we have

$$\Delta u + (1-m)k^2 u = 0.$$

The forward problem in THz tomography

- $\Omega \subset \mathbb{R}^2$ an area with a C^1 boundary
- $m := 1 - \tilde{n}^2 \in (L^\infty(\Omega) \cap L^2_{\mathrm{comp}}(\Omega)) \subset L^2(\Omega)$ the complex refractive index

The Helmholtz equation

In Ω, the incident field u_{i} approximately fulfills

$$\Delta u_{\mathrm{i}} + k^2 u_{\mathrm{i}} \approx 0$$

and for the total field u we have

$$\Delta u + (1-m)k^2 u = 0.$$

The scattered field u_{sc}

We obtain

$$\Delta u_{\mathrm{sc}} + k^2(1-m)u_{\mathrm{sc}} = k^2 m \cdot u_{\mathrm{i}} \qquad \text{in } \Omega.$$

The forward problem in THz tomography

- $\Omega \subset \mathbb{R}^2$ an area with a C^1 boundary
- $m := 1 - \tilde{n}^2 \in (L^\infty(\Omega) \cap L^2_{\mathrm{comp}}(\Omega)) \subset L^2(\Omega)$ the complex refractive index

The Helmholtz equation

In Ω, the incident field u_{i} approximately fulfills

$$\Delta u_{\mathrm{i}} + k^2 u_{\mathrm{i}} \approx 0$$

and for the total field u we have

$$\Delta u + (1 - m) k^2 u = 0.$$

The scattered field u_{sc}

We obtain

$$\Delta u_{\mathrm{sc}} + k^2 (1 - m) u_{\mathrm{sc}} = k^2 m \cdot u_{\mathrm{i}} \qquad \text{in } \Omega.$$

The forward problem in THz tomography

Rotation of the tomograph

- $\vartheta_j = (j-1)\frac{2\pi}{J}$, $j = 1, \ldots, J \in \mathbb{N}$, the positions of the tomograph
- u_{i}^j the incident field (Gaussian beam with wave number k) which depends on the position of the tomograph

In view of a later reconstruction: Use **first order scattering boundary conditions** instead of Sommerfeld radiation conditions.
The superposition principle yields the total field

$$u_{\mathrm{sc}}^j + u_{\mathrm{i}}^j = u^j \quad \text{in } \Omega,$$

where

$$\begin{aligned} \Delta u_{\mathrm{sc}}^j + k^2(1-m)u_{\mathrm{sc}}^j &= k^2 m \cdot u_{\mathrm{i}}^j \quad \text{in } \Omega, \\ \frac{\partial u_{\mathrm{sc}}^j}{\partial \mathbf{n}} - iku_{\mathrm{sc}}^j &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1}$$

The forward problem in THz tomography

Rotation of the tomograph

- $\vartheta_j = (j-1)\frac{2\pi}{J}$, $j = 1, ..., J \in \mathbb{N}$, the positions of the tomograph
- u_{i}^j the incident field (Gaussian beam with wave number k) which depends on the position of the tomograph

In view of a later reconstruction: Use **first order scattering boundary conditions** instead of Sommerfeld radiation conditions.
The superposition principle yields the total field

$$u_{\mathrm{sc}}^j + u_{\mathrm{i}}^j = u^j \quad \text{in } \Omega,$$

where

$$\begin{aligned} \Delta u_{\mathrm{sc}}^j + k^2(1-m)u_{\mathrm{sc}}^j &= k^2 m \cdot u_{\mathrm{i}}^j \quad \text{in } \Omega, \\ \frac{\partial u_{\mathrm{sc}}^j}{\partial \mathbf{n}} - iku_{\mathrm{sc}}^j &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1}$$

The forward problem in THz tomography

We define the (nonlinear) forward operator

$$F = \left(F^1, ..., F^J\right) \,:\, L^\infty(\Omega) \cap L^2_{\mathrm{komp}}(\Omega) \to \mathbb{C}^{N\times J},$$

where $F^j(m) := Q^j \gamma S^j(m)$ with

(i) the scattering map

$$S^j \,:\, L^\infty(\Omega) \cap L^2_{\mathrm{comp}}(\Omega) \to H^1(\Omega), \qquad S^j(m) := u^j$$

in (1),

(ii) the trace operator $\gamma \,:\, H^1(\Omega) \to H^{1/2}(\partial\Omega)$, $\qquad \gamma u = u|_{\partial\Omega}$,

The forward problem in THz tomography

We define the (nonlinear) forward operator

$$F = \left(F^1, \ldots, F^J\right) : L^\infty(\Omega) \cap L^2_{\mathrm{komp}}(\Omega) \to \mathbb{C}^{N \times J},$$

where $F^j(m) := Q^j \gamma S^j(m)$ with

(i) the scattering map

$$S^j : L^\infty(\Omega) \cap L^2_{\mathrm{comp}}(\Omega) \to H^1(\Omega), \qquad S^j(m) := u^j$$

in (1),

(ii) the trace operator $\gamma : H^1(\Omega) \to H^{1/2}(\partial\Omega)$, $\qquad \gamma u = u|_{\partial\Omega}$,

(iii) and a linear observation operator $Q^j : H^{1/2}(\partial\Omega) \to \mathbb{C}^N$,

$$y^j_\nu = (Q^j v)_\nu = \int_{\partial\Omega} \chi_{E^j_\nu}(\mathbf{x}) v(\mathbf{x}) \,\mathrm{d}s_{\mathbf{x}}, \quad \nu = 1, \ldots, N.$$

The forward problem in THz tomography

We define the (nonlinear) forward operator

$$F = \left(F^1, \dots, F^J\right) \,:\, L^\infty(\Omega) \cap L^2_{\mathrm{komp}}(\Omega) \to \mathbb{C}^{N\times J},$$

where $F^j(m) := Q^j \gamma S^j(m)$ with

(i) the scattering map

$$S^j \,:\, L^\infty(\Omega) \cap L^2_{\mathrm{comp}}(\Omega) \to H^1(\Omega), \qquad S^j(m) := u^j$$

in (1),

(ii) the trace operator $\gamma \,:\, H^1(\Omega) \to H^{1/2}(\partial\Omega)$, $\qquad \gamma u = u|_{\partial\Omega}$,

(iii) and a linear observation operator $Q^j \,:\, H^{1/2}(\partial\Omega) \to \mathbb{C}^N$,

$$y^j_\nu = (Q^j v)_\nu = \int_{\partial\Omega} \chi_{E^j_\nu}(\mathbf{x}) v(\mathbf{x})\,\mathrm{d}s_{\mathbf{x}}, \quad \nu = 1, \dots, N.$$

Some properties of the scattering map S

- S is well-defined and nonlinear, i. e. for $m \in L^\infty(\Omega) \cap L^2_{\mathrm{comp}}(\Omega)$ the scattering problem has a weak solution $u \in H^1(\Omega)$.

- S is continuous with respect to m, such that

$$\|S(m_1) - S(m_2)\|_{H^1(\Omega)} \leq C\|m_1 - m_2\|_{L^\infty(\Omega)}\|u_{\mathrm{i}}\|_{L^2(\Omega)}$$

for $m_1, m_2 \in (L^\infty(\Omega) \cap L^2_{\mathrm{comp}}(\Omega))$.

Some properties of the scattering map S

- S is well-defined and nonlinear, i. e. for $m \in L^\infty(\Omega) \cap L^2_{\mathrm{comp}}(\Omega)$ the scattering problem has a weak solution $u \in H^1(\Omega)$.

- S is continuous with respect to m, such that

$$\|S(m_1) - S(m_2)\|_{H^1(\Omega)} \leq C\|m_1 - m_2\|_{L^\infty(\Omega)}\|u_{\mathrm{i}}\|_{L^2(\Omega)}$$

for $m_1, m_2 \in (L^\infty(\Omega) \cap L^2_{\mathrm{comp}}(\Omega))$.

- S ist Fréchet differentiable w. r. t. m with Fréchet derivative $S'(m) : (L^\infty(\Omega) \cap L^2_{\mathrm{komp}}(\Omega)) \to H^1(\Omega),\ S'(m)\delta m = w$, where

$$\Delta w + k^2(1-m)w = k^2 \delta m \cdot S(m) \quad \text{in } \Omega,$$
$$\frac{\partial w}{\partial \mathbf{n}} - ikw = 0 \quad \text{auf } \partial\Omega.$$

Some properties of the scattering map S

- S is well-defined and nonlinear, i. e. for $m \in L^\infty(\Omega) \cap L^2_{\mathrm{comp}}(\Omega)$ the scattering problem has a weak solution $u \in H^1(\Omega)$.

- S is continuous with respect to m, such that

$$\|S(m_1) - S(m_2)\|_{H^1(\Omega)} \leq C\|m_1 - m_2\|_{L^\infty(\Omega)}\|u_{\mathrm{i}}\|_{L^2(\Omega)}$$

  for $m_1, m_2 \in (L^\infty(\Omega) \cap L^2_{\mathrm{comp}}(\Omega))$.

- S ist Fréchet differentiable w. r. t. m with Fréchet derivative $S'(m) : (L^\infty(\Omega) \cap L^2_{\mathrm{komp}}(\Omega)) \to H^1(\Omega),\ S'(m)\delta m = w$, where

$$\Delta w + k^2(1-m)w = k^2 \delta m \cdot S(m) \quad \text{in } \Omega,$$
$$\frac{\partial w}{\partial \mathbf{n}} - ikw = 0 \quad \text{auf } \partial\Omega.$$

Reconstruction of the complex refractive index from noisy data y^δ

Iterative solution of the inverse problem

Given noisy data y^δ, find a regularized solution $m^+ \in L^2(\Omega)$ of $F(m) = y$.

Nonlinear Landweber method

Take the mean of the directions corresponding to the different positions $j = 1, ..., J$:

$$m_{n+1}^\delta = m_n^\delta - \omega \cdot \frac{1}{J} \sum_{j=1}^{J} \left(\left(F^j \right)' (m_n^\delta) \right)^* \left(F^j(m_n^\delta) - y^\delta \right).$$

A priori information

We assume the outer boundaries of the object are known.

The adjoint linearized problem

The linearity of the trace and observation operator yields

$$F'(m)\delta m = Q\gamma S'(m)\delta m.$$

We define the continuous linear operator

$$\begin{aligned} T(m) \,:\, L^{\infty}(\Omega) \cap L^2_{\mathrm{comp}}(\Omega) &\to H^{1/2}(\partial\Omega), \\ T(m)\delta m &= \gamma S'(m)\delta m. \end{aligned}$$

The adjoint linearized forward operator

We obtain

$$F'(m)^*\beta = \left(Q\gamma S'(m)\right)^*\beta = T(m)^* Q^*\beta,$$

where

$$Q^* \,:\, \mathbb{C}^N \to H^{1/2}(\partial\Omega)^*,\ Q^*\beta = \sum_{\nu=1}^{N} \beta_\nu \chi_{E_\nu}.$$

The adjoint linearized problem

For some $R \in H^{1/2}(\partial\Omega)^*$ there exists a $\phi \in H^1(\Omega)$, such that

$$T(m)^* R = -k^2 \overline{S(m)} \phi$$

holds, fulfilling

$$\begin{aligned} \Delta\phi + k^2(1 - \overline{m})\phi &= 0 \quad \text{in } \Omega, \\ \frac{\partial\phi}{\partial\mathbf{n}} + ik\phi &= -R \quad \text{on } \partial\Omega. \end{aligned}$$

We thus have

$$F'(m)^*\beta = T(m)^* Q^* \beta = -k^2 \overline{S(m)} \phi,$$

where

$$\begin{aligned} \Delta\phi + k^2(1 - \overline{m})\phi &= 0 \quad \text{in } \Omega, \\ \frac{\partial\phi}{\partial\mathbf{n}} + ik\phi &= -\sum_{\nu=1}^{N} \beta_\nu \chi_{E_\nu} \quad \text{on } \partial\Omega. \end{aligned}$$

Reconstruction from synthetic data with a nonlinear Landweber method

Example: Testing of a plastic block with a hole

- 10 receivers, 60 different angles of tomograph
- relaxation parameter $\omega = 0.1$

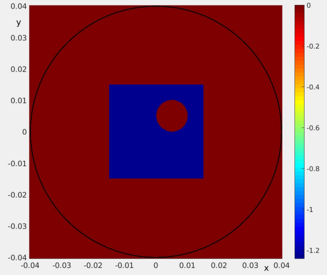


Figure: Refractive index of a tested plastic block with $m = 1 - \tilde{n}^2 = -1.24 - i \cdot 0.3$

Reconstruction from synthetic data with a nonlinear Landweber method

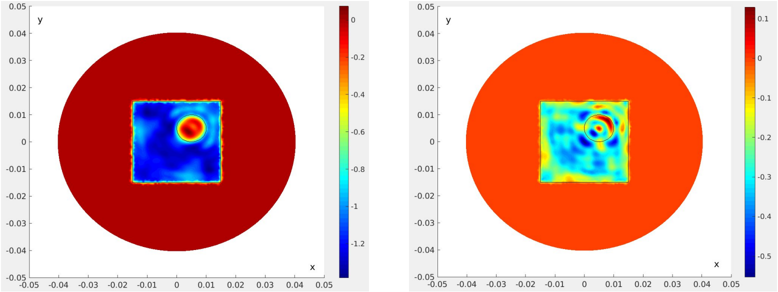


Figure: Reconstruction of m from synthetic noisy data with 5% noise (left: real part $Re(m)$, right: imaginary part $Im(m)$) after 350 iterations

Exact value:
$m = -1.24 - i \cdot 0.3$

Overview

Sequential subspace optimization (SESOP) - motivation

THz tomography

- Landweber method: two evaluations of a boundary value problem for each position in each step
- very long run-time of reconstruction
- insufficient reconstruction of imaginary part (extinction coefficient κ)

Sequential subspace optimization

- **Main idea:** reduce number of iterations by increasing the search space
- reuse former search directions (gradients) to avoid additional evaluations of the BVP
- **Goal:** extension of SESOP for nonlinear inverse problems $F(x) = y$

SESOP for linear inverse problems

Let X, Y be real Hilbert spaces, $A : X \to Y$ a linear operator and

$$M_{Ax=y} := \{x \in X : Ax = y\}.$$

Calculate

$$x_{n+1} := x_n - \sum_{i \in I_n} t_{n,i} A^* w_{n,i},$$

where the parameters $t_{n,i}$ with $t_n := (t_{n,i})_{i \in I_n} \in \mathbb{R}^{|I_n|}$ minimize the function

$$h_n(t) := \frac{1}{2} \left\| x_n - \sum_{i \in I_n} t_i A^* w_{n,i} \right\|^2 + \sum_{i \in I_n} t_i \langle w_{n,i}, y \rangle.$$

SESOP for linear inverse problems

Let X, Y be real Hilbert spaces, $A : X \to Y$ a linear operator and

$$M_{Ax=y} := \{x \in X \,:\, Ax = y\}.$$

Calculate

$$x_{n+1} := x_n - \sum_{i \in I_n} t_{n,i} A^* w_{n,i},$$

where the parameters $t_{n,i}$ with $t_n := (t_{n,i})_{i \in I_n} \in \mathbb{R}^{|I_n|}$ minimize the function

$$h_n(t) := \frac{1}{2} \left\| x_n - \sum_{i \in I_n} t_i A^* w_{n,i} \right\|^2 + \sum_{i \in I_n} t_i \langle w_{n,i}, y \rangle.$$

This is equivalent to the calculation of the projection

$$x_{n+1} = P_{\bigcap_{i \in I_n} H_{n,i}}(x_n)$$

of x_n onto the intersection of the hyperplanes

$$H_{n,i} := \{x \in X \,:\, \langle A^* w_{n,i}, x \rangle - \langle w_{n,i}, y \rangle = 0\} \supseteq M_{Ax=y}.$$

SESOP for linear inverse problems

Let X, Y be real Hilbert spaces, $A : X \to Y$ a linear operator and

$$M_{Ax=y} := \{x \in X \; : \; Ax = y\} .$$

Calculate

$$x_{n+1} := x_n - \sum_{i \in I_n} t_{n,i} A^* w_{n,i},$$

where the parameters $t_{n,i}$ with $t_n := (t_{n,i})_{i \in I_n} \in \mathbb{R}^{|I_n|}$ minimize the function

$$h_n(t) := \frac{1}{2} \left\| x_n - \sum_{i \in I_n} t_i A^* w_{n,i} \right\|^2 + \sum_{i \in I_n} t_i \langle w_{n,i}, y \rangle .$$

This is equivalent to the calculation of the projection

$$x_{n+1} = P_{\bigcap_{i \in I_n} H_{n,i}}(x_n)$$

of x_n onto the intersection of the hyperplanes

$$H_{n,i} := \{x \in X \; : \; \langle A^* w_{n,i}, x \rangle - \langle w_{n,i}, y \rangle = 0\} \; \supseteq M_{Ax=y} .$$

Regularizing SESOP for linear inverse problems

Let us assume that only noisy data y^δ with noise level δ is given, i. e.

$$\left\| y^\delta - y \right\| \leq \delta.$$

Basic idea

Replace hyperplanes $H_{n,i}$ by stripes

$$H\left(u_{n,i}^\delta, \alpha_{n,i}^\delta, \xi_{n,i}^\delta\right) := \left\{ x \in X \;:\; \left| \left\langle u_{n,i}^\delta, x \right\rangle - \alpha_{n,i}^\delta \right| \leq \xi_{n,i}^\delta \right\},$$

that contain $M_{Ax=y}$ and have a finite width $\xi_{n,i}^\delta$ in the range of the noise level:

$$\begin{aligned} u_{n,i}^\delta &:= A^* w_{n,i}^\delta, \\ \alpha_{n,i}^\delta &:= \left\langle w_{n,i}^\delta, y^\delta \right\rangle, \\ \xi_{n,i}^\delta &:= \delta \left\| w_{n,i}^\delta \right\|. \end{aligned}$$

SESOP for nonlinear inverse problems

We consider the nonlinear operator equation

$$F(x) = y,$$

where $F : \mathcal{D}(F) \subset X \to Y$ is a nonlinear operator between Hilbert spaces X and Y.

Again, we denote the solution set by

$$M_{F(x)=y} := \{x \in X : F(x) = y\}.$$

Main idea for nonlinear SESOP / RESESOP

Assume F fulfills the tangential cone condition

$$\|F(x) - F(\tilde{x}) - F'(x)(x - \tilde{x})\| \leq c_{\mathrm{tc}} \|F(x) - F(\tilde{x})\|.$$

- Project sequentially onto stripes that are defined with respect to the current iterate and take into account the cone condition!

SESOP for nonlinear inverse problems

We consider the nonlinear operator equation

$$F(x) = y,$$

where $F : \mathcal{D}(F) \subset X \to Y$ is a nonlinear operator between Hilbert spaces X and Y.
Again, we denote the solution set by

$$M_{F(x)=y} := \{x \in X \,:\, F(x) = y\}\,.$$

Main idea for nonlinear SESOP / RESESOP

Assume F fulfills the tangential cone condition

$$\left\| F(x) - F(\tilde{x}) - F'(x)(x - \tilde{x}) \right\| \leq c_{\mathrm{tc}} \left\| F(x) - F(\tilde{x}) \right\|.$$

- Project sequentially onto stripes that are defined with respect to the current iterate and take into account the cone condition!

Interpretation

SESOP for nonlinear problems

At iteration $n \in \mathbb{N}$, we project sequentially onto stripes

$$H_{n,i} := \Big\{ x \in X \ : \big| \langle F'(x_i)^* w_{n,i}, x_i - x \rangle - \langle w_{n,i}, F(x_i) - y \rangle \big| \\ \leq c_{\mathrm{tc}} \|w_{n,i}\| \|R_i\| \Big\},$$

where $i \in I_n$ and I_n is some finite index set.

RESESOP for nonlinear problems

In the case of noisy data, define

$$H^\delta_{n,i} := H(u^\delta_{n,i}, \alpha^\delta_{n,i}, \xi^\delta_{n,i}) := \Big\{ x \in X \ : \ \Big| \Big\langle u^\delta_{n,i}, x \Big\rangle - \alpha^\delta_{n,i} \Big| \leq \xi^\delta_{n,i} \Big\}$$

with the search direction $u^\delta_{n,i} := F'(x^\delta_i)^* w^\delta_{n,i}$ and

$$\alpha^\delta_{n,i} := \Big\langle F'(x^\delta_i)^* w^\delta_{n,i}, x^\delta_i \Big\rangle - \Big\langle w^\delta_{n,i}, F(x^\delta_i) - y^\delta \Big\rangle,$$

$$\xi^\delta_{n,i} := \Big(\delta + c_{\mathrm{tc}}(\|R^\delta_i\| + \delta) \Big) \|w^\delta_{k,i}\|.$$

Interpretation

SESOP for nonlinear problems

At iteration $n \in \mathbb{N}$, we project sequentially onto stripes

$$H_{n,i} := \Big\{ x \in X \, : \big| \langle F'(x_i)^* w_{n,i}, x_i - x \rangle - \langle w_{n,i}, F(x_i) - y \rangle \big| \\ \leq c_{\mathrm{tc}} \|w_{n,i}\| \|R_i\| \Big\},$$

where $i \in I_n$ and I_n is some finite index set.

RESESOP for nonlinear problems

In the case of noisy data, define

$$H^\delta_{n,i} := H(u^\delta_{n,i}, \alpha^\delta_{n,i}, \xi^\delta_{n,i}) := \Big\{ x \in X \, : \, \Big| \Big\langle u^\delta_{n,i}, x \Big\rangle - \alpha^\delta_{n,i} \Big| \leq \xi^\delta_{n,i} \Big\}$$

with the search direction $u^\delta_{n,i} := F'(x^\delta_i)^* w^\delta_{n,i}$ and

$$\alpha^\delta_{n,i} := \Big\langle F'(x^\delta_i)^* w^\delta_{n,i}, x^\delta_i \Big\rangle - \Big\langle w^\delta_{n,i}, F(x^\delta_i) - y^\delta \Big\rangle,$$

$$\xi^\delta_{n,i} := \Big(\delta + c_{\mathrm{tc}}(\|R^\delta_i\| + \delta) \Big) \|w^\delta_{k,i}\|.$$

An algorithm with two search directions

Choose a starting value $x_0^\delta := x_0 \in X$.
In the first step ($n = 0$), choose the search direction u_0^δ.
In the following steps ($n \geq 1$): Choose the search directions $\{u_n^\delta, u_{n-1}^\delta\}$, where

$$u_n^\delta := F'(x_n^\delta)^* w_n^\delta,$$
$$w_n^\delta := R_n^\delta := F(x_n^\delta) - y^\delta.$$

Define $H_{-1}^\delta := X$ and for $n \in \mathbb{N}$ the stripes $H_n^\delta := H(u_n^\delta, \alpha_n^\delta, \xi_n^\delta)$ with

$$\alpha_n^\delta := \left\langle u_n^\delta, x_n^\delta \right\rangle - \|R_n^\delta\|^2,$$
$$\xi_n^\delta := \|R_n^\delta\| \big(\delta + c_{\mathrm{tc}}(\|R_n^\delta\| + \delta)\big).$$

Stopping rule: Choose the discrepancy principle with tolerance parameter

$$\tau > \frac{1 + c_{\mathrm{tc}}}{1 - c_{\mathrm{tc}}}.$$

An algorithm with two search directions

As long as $\|R_n^\delta\| > \tau\delta$, we have

$$x_n^\delta \in H_>(u_n^\delta, \alpha_n^\delta + \xi_n^\delta) \cap H_{n-1}^\delta,$$

and thus we calculate the new iterate x_{n+1}^δ according to the following two steps:

(i) Calculate

$$\tilde{x}_{n+1}^\delta := P_{H(u_n^\delta, \alpha_n^\delta + \xi_n^\delta)}(x_n^\delta) = x_n^\delta - \frac{\langle u_n^\delta, x_n^\delta \rangle - (\alpha_n^\delta + \xi_n^\delta)}{\|u_n^\delta\|^2} u_n^\delta.$$

Is $\tilde{x}_{n+1}^\delta \in H_{n-1}^\delta$, we have $\tilde{x}_{n+1}^\delta = P_{H_n^\delta \cap H_{n-1}^\delta}(x_n^\delta)$ and we are done.
Otherwise, go to step (ii).

An algorithm with two search directions

As long as $\|R_n^\delta\| > \tau\delta$, we have

$$x_n^\delta \in H_>(u_n^\delta, \alpha_n^\delta + \xi_n^\delta) \cap H_{n-1}^\delta,$$

and thus we calculate the new iterate x_{n+1}^δ according to the following two steps:

(i) Calculate

$$\tilde{x}_{n+1}^\delta := P_{H(u_n^\delta, \alpha_n^\delta + \xi_n^\delta)}(x_n^\delta) = x_n^\delta - \frac{\langle u_n^\delta, x_n^\delta \rangle - (\alpha_n^\delta + \xi_n^\delta)}{\|u_n^\delta\|^2} u_n^\delta.$$

Is $\tilde{x}_{n+1}^\delta \in H_{n-1}^\delta$, we have $\tilde{x}_{n+1}^\delta = P_{H_n^\delta \cap H_{n-1}^\delta}(x_n^\delta)$ and we are done. Otherwise, go to step (ii).

An algorithm with two search directions

(ii) First: Decide whether we have

$$\tilde{x}^\delta_{n+1} \in H_>(u^\delta_{n-1}, \alpha^\delta_{n-1} + \xi^\delta_{n-1})$$

or

$$\tilde{x}^\delta_{n+1} \in H_<(u^\delta_{n-1}, \alpha^\delta_{n-1} - \xi^\delta_{n-1}).$$

Calculate accordingly

$$x^\delta_{n+1} := P_{H(u^\delta_n, \alpha^\delta_n + \xi^\delta_n) \cap H(u^\delta_{n-1}, \alpha^\delta_{n-1} \pm \xi^\delta_{n-1})}(\tilde{x}^\delta_{n+1}).$$

We then have $x^\delta_{n+1} = P_{H^\delta_n \cap H^\delta_{n-1}}(x^\delta_n)$ and all $z \in M_{F(x)=y}$ fulfill

$$\left\| z - x^\delta_{n+1} \right\|^2 = \left\| z - x^\delta_n \right\|^2 - S^\delta_n.$$

An algorithm with two search directions

(ii) First: Decide whether we have

$$\tilde{x}_{n+1}^{\delta} \in H_{>}(u_{n-1}^{\delta}, \alpha_{n-1}^{\delta} + \xi_{n-1}^{\delta})$$

or

$$\tilde{x}_{n+1}^{\delta} \in H_{<}(u_{n-1}^{\delta}, \alpha_{n-1}^{\delta} - \xi_{n-1}^{\delta}).$$

Calculate accordingly

$$x_{n+1}^{\delta} := P_{H(u_n^{\delta}, \alpha_n^{\delta} + \xi_n^{\delta}) \cap H(u_{n-1}^{\delta}, \alpha_{n-1}^{\delta} \pm \xi_{n-1}^{\delta})}(\tilde{x}_{n+1}^{\delta}).$$

We then have $x_{n+1}^{\delta} = P_{H_n^{\delta} \cap H_{n-1}^{\delta}}(x_n^{\delta})$ and all $z \in M_{F(x)=y}$ fulfill

$$\left\| z - x_{n+1}^{\delta} \right\|^2 = \left\| z - x_n^{\delta} \right\|^2 - S_n^{\delta}.$$

Theorem (Wald, S. 2016)

(i) Together with the discrepancy principle, the algorithm with two search directions yields a finite stopping index $n_*(\delta)$.

(ii) The iterates $\{x_n^\delta\}_{n\in\mathbb{N}}$ depend, for a fixed $n \in \mathbb{N}$, continuously on the data y^δ and we have

$$x_n^\delta \to x_n \quad \text{for } \delta \to 0.$$

Theorem (Wald, S. 2016)

(i) Together with the discrepancy principle, the algorithm with two search directions yields a finite stopping index $n_*(\delta)$.

(ii) The iterates $\{x_n^\delta\}_{n\in\mathbb{N}}$ depend, for a fixed $n \in \mathbb{N}$, continuously on the data y^δ and we have

$$x_n^\delta \to x_n \quad \text{for } \delta \to 0.$$

(iii) The algorithm yields a regularized solution $x_{n_*(\delta)}^\delta$ of the nonlinear problem $F(x) = y$, if noisy data y^δ are given, i. e.

$$x_{n_*(\delta)}^\delta \to x^+ \quad \text{for } \delta \to 0,$$

if there is only one solution $x^+ \in B_\rho(x_0)$ and if $\lim_{\delta\to 0} \left|t_{n,i}^\delta\right| < t$ holds for all $n \in \mathbb{N}$ and $i \in I_n^\delta = \{n-1, n\}$ for some $t > 0$.

Theorem (Wald, S. 2016)

(i) Together with the discrepancy principle, the algorithm with two search directions yields a finite stopping index $n_*(\delta)$.

(ii) The iterates $\left\{x_n^\delta\right\}_{n\in\mathbb{N}}$ depend, for a fixed $n \in \mathbb{N}$, continuously on the data y^δ and we have

$$x_n^\delta \to x_n \quad \text{for } \delta \to 0.$$

(iii) The algorithm yields a regularized solution $x_{n_*(\delta)}^\delta$ of the nonlinear problem $F(x) = y$, if noisy data y^δ are given, i. e.

$$x_{n_*(\delta)}^\delta \to x^+ \quad \text{for } \delta \to 0,$$

if there is only one solution $x^+ \in B_\rho(x_0)$ and if $\lim_{\delta\to 0}\left|t_{n,i}^\delta\right| < t$ holds for all $n \in \mathbb{N}$ and $i \in I_n^\delta = \{n-1, n\}$ for some $t > 0$.

Reconstruction from synthetic data with the RESESOP method

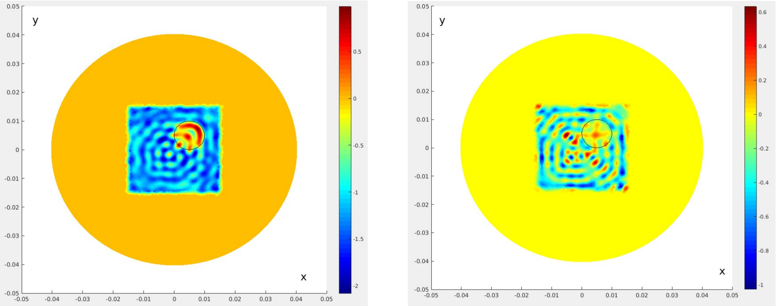


Figure: Reconstruction of m from synthetic noisy data with 5% noise (left: real part $Re(m)$, right: imaginary part $Im(m)$) after 50 iterations with the RESESOP method using two search directions

Exact value:

$m = -1.24 - i \cdot 0.3$

Overview

Conclusion

- Modelling of the inverse problem in THz tomography with the Helmholtz equation
- Reconstruction of the complex refractive index
- Inclusion of the geometry of the THz beam
- Modification of SESOP and RESESOP techniques for use in nonlinear inverse problems
- Analysis of the new method and proof of convergence and regularizing properties

Outlook

- Adaption and application of SESOP in THz tomography and comparison to Landweber method
- Development of hybrid reconstruction algorithm to further reducing computation time

References

- G. Bao and P. Li, Inverse medium scattering for the Helmholtz equation at fixed frequency, *Inverse Problems* **21** (2005), no. 5, 1621-1641
- M. Hanke, A. Neubauer and O. Scherzer, A convergence analysis of the Landweber iteration for nonlinear ill-posed problems, *Numer. Math.* **72** (1995), no. 1, 21-37
- F. Schöpfer, A. K. Louis and T. Schuster, Metric and Bregman projections onto affine subspaces and their computation via sequential subspace optimization methods, *J. Inverse Ill-Posed Probl.* **16** (2008), no. 5, 479-506
- J. Tepe, T. Schuster and B. Littau, A modified algebraic reconstruction technique taking refraction into account with an application in terahertz tomography, preprint, arXiv:1601.04496, (2016)
- A. Wald, T. Schuster, Sequential subspace optimization for nonlinear inverse problems, *J. Inverse Ill-Posed Probl.* (2016), aop

Thank you for your attention!