Verification of Variational Source Conditions

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Outline

1. introduction, variational source conditions
2. linear inverse problems in Hilbert spaces
3. inverse problems in PDEs with distributed measurements
4. inverse scattering problems

framework

- $\mathbb{X}, \mathbb{Y}$ Hilbert (or Banach) spaces
- $F : D(F) \subset \mathbb{X} \to \mathbb{Y}$ forward operator
- $f^\dagger \in D(F)$ exact solution
- g^{obs} observed data, either
  - deterministic: $\|g^{\mathrm{obs}} - F(f^\dagger)\| \leq \delta$ or
  - stochastic: $g^{\mathrm{obs}} = F(f^\dagger) + \varepsilon W$, $W =$ white noise

Linear operators will be denoted by T instead of F.

source conditions

spectral source condition:

$$f^\dagger = \varphi\left(F'[f^\dagger]^* F'[f^\dagger]\right) w$$

variational source condition (VSC):

$$\forall f: \qquad \frac{1}{2}\left\|f^\dagger - f\right\|_{\mathbb{X}}^2 \leq \|f\|_{\mathbb{X}}^2 - \left\|f^\dagger\right\|_{\mathbb{X}}^2 + \psi\left(\left\|F(f) - F(f^\dagger)\right\|_{\mathbb{Y}}^2\right)$$

or equivalently

$$\forall f: \qquad 2\left\langle f^\dagger, f^\dagger - f\right\rangle \leq \frac{1}{2}\|f - f^\dagger\|^2 + \psi\left(\left\|F(f) - F(f^\dagger)\right\|_{\mathbb{Y}}^2\right).$$

First used (with $\psi(t) = c\sqrt{t}$) in

B. Hofmann, B. Kaltenbacher, C. Pöschl, and O. Scherzer. *A convergence rates result for Tikhonov regularization in Banach spaces with non-smooth operators*. **Inverse Problems** 23:987–1010, 2007.

Here $\varphi, \psi : [0, \infty) \to [0, \infty)$ are *index functions*, i.e. non-decreasing and vanishing at 0. ψ is assumed be concave.

advantages of variational vs. spectral source conditions

- simpler proofs
- VSCs do not involve F' $\rightsquigarrow$ no need of tangential cone condition or related conditions
- VSC work for Banach spaces and general data fidelities and penalties
- for linear operators in Hilbert spaces not only sufficient, but even necessary for certain rates of convergence

but

- so far few *verifiable* sufficient conditions for VSC for specific problems known

VSC vs. stability estimates

Let $K \subset \mathrm{dom}(F)$ be some smoothness class (e.g. a Sobolev ball).

$$
\begin{aligned}
&\text{(VSC)} \quad && \forall f^\dagger \in K \ \forall f \in \mathrm{dom}(F) \\
& && \frac{1}{2}\|f - f^\dagger\|^2 \leq \|f\|^2 - \|f^\dagger\|^2 + \psi\left(\|F(f) - F(f^\dagger)\|^2\right) \\
&\text{(St)} \quad && \forall f_1, f_2 \in K \\
& && \frac{1}{2}\|f_1 - f_2\|^2 \leq \psi\left(\|F(f_1) - F(f_2)\|^2\right)
\end{aligned}
$$

(VSC) $\Rightarrow$ (St):

- W.l.o.g. $\|f_1\| \geq \|f_2\|$. Choose $f_1 = f^\dagger$, $f_2 = f$.

(St) $\Rightarrow$ (VSC): not obvious!

- $\|f\| - \|f^\dagger\|$ may be negative.
- (VSC) must hold for all f in the larger set $\mathrm{dom}(F)$.

general strategy for verification of VSCs

rate of convergence determined by two factors:

- smoothness of the solution $f^\dagger$

  Suppose there exists a family of orthogonal projection operators $P_r \in \mathcal{L}(\mathbb{X})$ such that for all r

  $$\|f^\dagger - P_r f^\dagger\|_{\mathbb{X}} \le \kappa(r), \qquad \inf_r \kappa(r) = 0. \tag{H1}$$

- degree of ill-posedness of the operator

  Suppose that for all r and all f with $\|f^\dagger - f\| \le 4\|f^\dagger\|$

  $$\left\langle f^\dagger, P_r(f^\dagger - f)\right\rangle \le \lambda(r)\|F(f^\dagger) - F(f)\| + C\kappa(r)\|f^\dagger - f\|. \tag{H2}$$

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$$\|P_r(f^\dagger - f)\| \leq \lambda(r)\|F(P_r f^\dagger) - F(P_r f)\| \; \left\langle f^\dagger, P_r(f^\dagger - f)\right\rangle \leq \lambda(r)\|F(f^\dagger \tag{H2}$$

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Then a VSC holds true with the concave index function

$$\psi(t) := 2\inf_r \left[(C+1)^2\kappa(r)^2 + \lambda(r)\sqrt{t}\right].$$

proof

case $\|f - f^\dagger\| > 4\|f^\dagger\|$: VSC follows from

$$\left\langle f^\dagger, f^\dagger - f \right\rangle \leq \|f^\dagger\|\,\|f^\dagger - f\| < \frac{1}{4}\|f^\dagger - f\|^2,$$

case $\|f - f^\dagger\| \leq 4\|f^\dagger\|$:

$$\begin{aligned}
2\left\langle f^\dagger, f^\dagger - f \right\rangle &= 2\left\langle (I - P_r)f^\dagger, f^\dagger - f \right\rangle + 2\left\langle f^\dagger, P_r(f^\dagger - f) \right\rangle \\
&\leq 2(C+1)\kappa(r)\|f^\dagger - f\| + 2\lambda(r)\|F(f) - F(f^\dagger)\| \\
&\leq \frac{1}{2}\|f^\dagger - f\|^2 + 2(C+1)^2\kappa(r)^2 + 2\lambda(r)\|F(f) - F(f^\dagger)\|
\end{aligned}$$

Take the infimum of rhs over r with $t = \|F(f) - F(f^\dagger)\|^2$.

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linear operators in Hilbert spaces

setting: $\mathbb{X}, \mathbb{Y}$ Hilbert spaces, $F = T \in \mathcal{L}(\mathbb{X}, \mathbb{Y})$,
P_r spectral projections for T^*T, $r > 0$
$\Psi_\kappa(\delta^2) := \kappa(\Theta_\kappa^{-1}(\delta))^2$ with $\Theta_\kappa(t) := \sqrt{t}\kappa(t)$

methods: (iterated) Tikhonov, Landweber, Lardy, Showalter,
but not spectral cut-off

$$\mathrm{VSC}(\psi_\kappa) \overset{\kappa^2(t)/t^{1-\varepsilon}\text{ incr.}}{\Longleftrightarrow} \sup_{r>0} \frac{1}{\kappa(r)} \|f^\dagger - P_r f^\dagger\| < \infty \Leftrightarrow \sup_{\alpha>0} \frac{1}{\kappa(\alpha)} \|f_\alpha - f^\dagger\| < \infty$$

$$\Longleftrightarrow \quad \sup_{\delta>0} \frac{1}{\psi_\kappa(\delta)} \sup_{g^{\mathrm{obs}}} \inf_{\alpha>0} \|\widehat{f}_\alpha - f^\dagger\| < \infty$$

If $T^*T = \Lambda(-\Delta)$ with the Laplace-Beltrami operator Δ on a reasonable manifold $\mathcal{M}$:

$$\mathrm{VSC}(\psi_\kappa) \quad \Leftrightarrow \quad f^\dagger \in B^s_{2,\infty}(\mathcal{M}) \qquad \text{for a certain } \kappa = \kappa_s$$

$$f^\dagger \in R(\kappa(T^*T)) \quad \Leftrightarrow \quad f^\dagger \in W^s_2(\mathcal{M}) = B^s_{2,2}(\mathcal{M})$$

lit. on deterministic converse results

$$\mathrm{VSC}(\psi_\kappa) \overset{\kappa^2(t)/t^{1-\varepsilon}\text{ incr.}}{\Longleftrightarrow} \sup_{r>0} \frac{1}{\kappa(r)} \|f^\dagger - P_r f^\dagger\| < \infty \Leftrightarrow \sup_{\alpha>0} \frac{1}{\kappa(\alpha)} \|f_\alpha - f^\dagger\| < \infty$$

$$\Longleftrightarrow \quad \sup_{\delta>0} \frac{1}{\psi_\kappa(\delta)} \sup_{g^{\mathrm{obs}}} \inf_{\alpha>0} \|\widehat{f}_\alpha - f^\dagger\| < \infty$$

- A. Neubauer. *On converse and saturation results for Tikhonov regularization of linear ill-posed problems.* **SIAM J. Numer. Anal.**, 34:517–527, 1997.
- J. Flemming, B. Hofmann, P. Mathé. *Sharp converse results for the regularization error using distance functions.* **Inverse Problems** 27:025006, 2011.
- R. Andreev, P. Elbau, M. de Hoop, L. Qiu, O. Scherzer. *Generalized Convergence Rates Results for Linear Inverse Problems in Hilbert Spaces.* **Num. Funct. Anal. Optim.** 36:549–566, 2015.
- R. Andreev. *Tikhonov and Landweber convergence rates: characterization by interpolation spaces.* **Inverse Problems** 31:105007, 2015.
- V. Albani, P. Elbau, M.V. de Hoop, O. Scherzer. *Optimal convergence rates results for linear inverse problems in Hilbert spaces.* **Numer. Funct. Anal. Optim.**, 37:521–540, 2016.
- TH, F. Weidling. *Characterizations of variational source conditions, converse results, and maxisets of spectral regularization methods.* **SIAM J. Numer. Anal.** under revision.

statistical analogue: maxisets

- In statistics the largest set on which a given methods achieves a given rate is called a maxiset.
- Mainly studied for wavelet thresholding methods.

G. Kerkyacharian, D. Picard. *Density estimation by kernel and wavelets methods: optimality of Besov spaces.* **Statist. Probab. Lett.** 18:327-336, 1993.

G. Kerkyacharian, D. Picard. *Thresholding algorithms, maxisets and well-concentrated bases.* **Test**, 9:283–344, 2000.

V. Rivoirard. *Maxisets for linear procedures.* **Statist. Probab. Lett.** 67:267–275, 2004.

paradigm: Compare different estimators by comparing their maxisets for a given rate.

our result on maxisets

Assumption: $\exists C > 0, v \in C((0,\infty))\ \forall \alpha > 0$

$$\frac{1}{C} v(\alpha) \leq \mathbb{E}\left[\|\widehat{f}_\alpha - \mathbb{E}[\widehat{f}_\alpha]\|^2\right] \leq C v(\alpha)$$

and v does not grow faster than polynomially as $\alpha \to 0$.

This is a mild assumption, verified for many cases in

N. Bissantz, TH, A. Munk, F. Ruymgaart. *Convergence rates of general regularization methods for statistical inverse problems and applications.* **SIAM J. Numer. Anal.** 45:2610–2636, 2007.

Theorem

Let $\psi_{\kappa,v}(t) := \kappa\left(\Theta_{\kappa,v}^{-1}\left(\sqrt{t}\right)\right)^2$ *with* $\Theta_{\kappa,v}(\alpha) := \frac{\kappa(\alpha)}{v(\alpha)}$. *Then*

$$\sup_{\alpha>0} \frac{1}{\kappa(\alpha)}\|f_\alpha - f^\dagger\| < \infty \iff \sup_{\varepsilon>0} \frac{1}{\psi_{\kappa,v}(\varepsilon^2)} \inf_{\alpha>0} \mathbb{E}\left[\|\widehat{f}_\alpha - f^\dagger\|^2\right].$$

TH, F. Weidling. *Characterizations of variational source conditions, converse results, and maxisets of spectral regularization methods.* **SIAM J. Numer. Anal.** under revision.

A class of mildly ill-posed problems

Assumptions: $T : W_2^s(\mathcal{M}) \to W_2^{s+a}(\mathcal{M})$ is an isomorphism for some $a > 0$ and all $s \in \mathbb{R}$, $\mathbb{X} = \mathbb{Y} = L^2(\mathcal{M})$.

Examples: injective elliptic pseudodifferential operators, certain convolution operators (if $\mathcal{M} = (\mathbb{S}^1)^d$ or $\mathcal{M} = \mathbb{R}^d$, compositions of such operators

Theorem

The following statements are equivalent for $u \in (0, a)$:

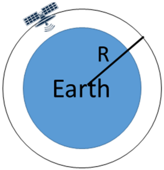
1. $f^\dagger \in B^u_{2,\infty}(\mathcal{M})$.
2. $f^\dagger$ *satisfies VSC with* $\psi(t) = Ct^{\frac{u}{u+a}}$, $C > 0$.
3. $\sup_{g^{\mathrm{obs}}} \inf_\alpha \|\widehat{f}_\alpha - f^\dagger\|_{L^2} = \mathcal{O}\left(\delta^{\frac{u}{u+a}}\right)$ *as* $\delta \to 0$.
4. $\inf_{\alpha>0} \mathbb{E}\left[\left\|\widehat{f}_\alpha - f^\dagger\right\|_{L^2}^2\right]^{1/2} = \mathcal{O}\left(\varepsilon^{\frac{u}{u+a+d/2}}\right)$ *as* $\varepsilon \to 0$.

Moreover,

$$f^\dagger \in R\left((T^*T)^{\frac{u}{2a}}\right) \Leftrightarrow f^\dagger \in W_2^u(\mathcal{M}).$$

satellite gradiometry

$$\begin{aligned} \Delta u &= 0 && \text{in } \{x \in \mathbb{R}^3 : |x| > 1\} \\ |u| &\to 0, && |x| \to \infty \\ u &= f && \text{on } \mathbb{S}^2 \end{aligned}$$



unknown: gravitational potential $u = f$ on surface of the Earth

observations: $\frac{\partial^2 u}{\partial r^2}$, $r = |x|$ at $R\mathbb{S}^2$

Theorem: The following statements are equivalent for $\beta > 0$:

1. $f^\dagger \in B^\beta_{2,\infty}(\mathcal{M})$.
2. $f^\dagger$ satisfies a VSC with $\psi_\beta(t) = C \log(3 + t^{-1})^{-2\beta}$, $C > 0$.
3. $\sup \inf_\alpha \|\widehat{f}_\alpha - f^\dagger\| = \mathcal{O}\left(\log(\delta^{-1})^{-\beta}\right)$ as $\delta \to 0$.
4. $\inf_{\alpha>0} \mathbb{E}\left[\left\|\widehat{f}_\alpha - f^\dagger\right\|^2\right]^{1/2} = \mathcal{O}\left(\log(\varepsilon^{-1})^{-\beta}\right)$ as $\varepsilon \to 0$.

Moreover, $f^\dagger \in R(\psi_\beta(T^*T)) \Leftrightarrow f^\dagger \in W^\beta_2(\mathcal{M})$.

similar results for backward and sideways heat equation

Is $B^s_{2,\infty}(\mathcal{M}) \setminus B^s_{2,2}(\mathcal{M})$ relevant?

Yes, it is!

- For an interval $I \subset \mathbb{R}$ the difference set $B^{1/2}_{2,\infty}(I) \setminus B^{1/2}_{2,2}(I)$ contains smooth functions with finitely many jumps.
- $B^{3/2}_{2,\infty}(I) \setminus B^{3/2}_{2,2}(I)$ contains smooth functions with finitely many kinks.

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Historically, the spaces $B^s_{p,\infty}$ were studied well before the spaces $W^s_p = B^s_{p,p}$, $s \in (0, \infty) \setminus \mathbb{N}$, and are sometimes called Nikol′skiĭ-spaces.

S.M. Nikol′skiĭ. *Inequalities for entire functions of finite degree and their application in the theory of differentiable functions of several variables.* **Trudy Mat. Inst. Steklov.** 38:244–278, 1951.

Slobodeckij, L. *Generalized Sobolev spaces and their applications to boundary value problems of partial differential equations.* **Gos. Ped. Inst. Ucep. Zap.** 197:54–112, 1958.

O.V. Besov. *On some families of functional spaces. Imbedding and extension theorems.* **Dokl. Akad. Nauk SSSR** 126:1163–1165, 1959.

equivalent norms for Nikol′skiĭ spaces

Let $V_1 \subset V_2 \subset \cdots \subset L^p(\Omega)$ be either

- dyadic spline subspaces
- hierarchical finite element subspaces
- wavelet subspaces

and $P_j : L^p(\Omega) \to V_j$, $j \in \mathbb{N}$ bounded (quasi-)projections. Then

$$\|f\|_{L^p} + \sup_{j \in \mathbb{N}} 2^{js} \|(I - P_j) f\|_{L^p(\Omega)}$$

defines an equivalent norm on $B^s_{p,\infty}(\Omega)$ for $s \in (0, s_0]$ and $p \in [1, \infty]$.

R.A. DeVore, V.A. Popov. *Interpolation of Besov spaces.* **Trans. Amer. Math. Soc.**, 305:397–414, 1988.

P. Oswald. *Multilevel finite element approximation.* Teubner, Stuttgart, 1994.

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a semilinear equation

forward problem:

$$
\begin{aligned}
-\Delta u + \xi(u) &= f && \text{in } \Omega \\
u &= 0 && \text{on } \partial\Omega
\end{aligned}
$$

where ξ is Lipschitz continuous and increasing.
forward operator: $F : L^2(\Omega) \to L^2(\Omega)$, $F(f) := u$.
F turns out to be monotone, i.e. $\langle F(f_1) - F(f_2), f_1 - f_2 \rangle \geq 0$.

VSC for monotone operators (monVSC)

$$
\left\langle f^\dagger - \bar{f}, f^\dagger - f \right\rangle \leq \frac{1}{2}\|f^\dagger - f\|^2 + \psi\left(\left\langle F(f) - F(f^\dagger), f - f^\dagger \right\rangle\right)
$$

implies convergence rates for Lavrentiev regularization

$$
F(f_\alpha) + \alpha(f_\alpha - \bar{f}) = u^\delta.
$$

B. Hofmann, B. Kaltenbacher, E. Resmerita. *Lavrentiev's regularization method in Hilbert spaces revisited.* **Inv. Probl. Imag.** 10:741–764, 2016.

verification of monVSC

Theorem (George, TH, Jidesh, 2016)

Assume that

$$f^\dagger - \overline{f} \in \mathring{B}^s_{2,\infty}(\Omega)$$

and $s \in (0,1)$. *Set* $\rho := \|f^\dagger - \overline{f}\|_{B^s_{2,\infty}(\Omega)}$. *Then (monVSC) holds true with*

$$\psi(t) = C\rho^{\frac{2}{s+1}} t^{\frac{s}{s+1}}$$

and a constant C *independent of* ρ. *This implies*

$$\|f^\delta_{\alpha_{\mathrm{opt}}} - f^\dagger\|_{L^2} \le C' \rho^{\frac{2}{s+2}} \delta^{\frac{s}{s+2}}.$$

- Improves rates by Hofmann, Kaltenbacher & Resmerita (not only by replacing Sobolev by Nikol′skiĭ spaces).
- Rates can be shown to be optimal using entropy arguments.

related results

- identification of reaction coefficient c in

$$-\Delta u + cu = f.$$

- identification of drift term f in stationary Focker Planck equation

$$\begin{aligned} \operatorname{div}\left(-fu + \sigma^2 \operatorname{grad} u\right) &= 0 \qquad \text{in } \mathbb{R}^d, \\ \int_{\mathbb{R}^3} u(x)\,dx &= 1. \end{aligned}$$

joint work with Fabian Dunker

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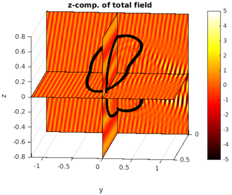
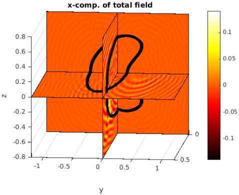
forward medium scattering problem





Given a refractive index $n = 1 - f$ and one or several incident wave(s) u^{i} solving the Helmholtz equation $\Delta u^{\mathrm{i}} + \kappa^2 u^{\mathrm{i}} = 0$, determine the total field(s) $u = u^{\mathrm{i}} + u^{\mathrm{s}}$ such that

$$\begin{aligned} &\Delta u + \kappa^2 n u = 0, && \text{in } \mathbb{R}^3, \\ &\frac{\partial u^{\mathrm{s}}}{\partial r} - i\kappa u^{\mathrm{s}} = \mathcal{O}\left(r^{-2}\right) && \text{as } r = |x| \to \infty. \end{aligned}$$

Assumptions on contrast $f = 1 - n$:

$$f \in \mathbb{D} := \left\{ f \in L^\infty(\mathbb{R}^3) : \begin{array}{l} \Im(f) \leq 0, \Re(f) \leq 1, \\ \operatorname{supp}(f) \subset \mathfrak{B}(\pi) \end{array} \right\}.$$

Here $\mathfrak{B}(R) := \{x : |x| \leq R\}$ for $R > 0$.

inverse problem for near field data

Incident fields are point sources

$$u_y^{\mathrm{i}}(x) = \frac{1}{4\pi} \frac{e^{i\kappa|x-y|}}{|x-y|} \qquad \text{at } y \in \partial\mathfrak{B}(R)$$

with $R > \pi$. Data are corresponding total fields

$$w_f(x,y) = u_y^{\mathrm{i}}(x) + u_y^{\mathrm{s}}(x), \quad (x,y) \in R\mathbb{S}^2 \times R\mathbb{S}^2.$$

forward operator:

$$F : \mathbb{D} \to L^2\left((R\mathbb{S}^2)^2\right), \quad f \mapsto w_f.$$

VSC for near field data

Theorem

Let $\mathbb{X} = H_0^m(\mathcal{B}(\pi))$ and assume that $\frac{3}{2} < m < s$, $s \neq 2m + 3/2$ and $f^\dagger \in \mathbb{D}$ satisfies $\|f^\dagger\|_{B^s_{2,\infty}} \leq C_s$ for some $C_s \geq 0$. Then a VSC holds true for the operator F with ψ given by

$$\psi(t) := A\left(\ln(3 + t^{-1})\right)^{-2\mu}, \qquad \mu := \min\left\{1, \frac{s-m}{m+3/2}\right\},$$

where the constant $A > 0$ depends only on m, s, C_s, κ, and R.

Corollary (convergence rate)

For nonlinear Tikhonov regularization and α_* chosen, e.g., by the discrepancy principle we have

$$\|\widehat{f}_{\alpha_*} - f^\dagger\|_{H^m} = \mathcal{O}\left(\ln \delta^{-1}\right)^{-\mu}, \qquad \delta \to 0.$$

TH, F. Weidling. *Verification of a variational source condition for acoustic inverse medium scattering problems.* **Inverse Problems** 31:075006 , 2015.

geometrical optics solutions

Our main tool are solutions to $\Delta u + \kappa^2 nu = 0$ of the form

$u(x) = e^{i\zeta\cdot x}(1 + v(x))$ with $\zeta \in \mathbb{C}^3,\ \zeta\cdot\zeta = \sum_{j=1}^{3} \zeta_j^2 = \kappa^2$

with "small" v.

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bounds on Fourier coefficients

Lemma

Let $C_m > 0$, $m > 3/2$, $\pi < R < R'$ and f_1 and f_2 be contrasts with $f_j \in \mathbb{D}$ and $\|f_j\|_{H^m} \leq C_m$ for $j = 1, 2$. Define

$$t_0 := 2\kappa^2 \frac{R'}{\pi} M_{\mathrm{em}} C_m$$

where M_{em} is the norm of $H^m \hookrightarrow L^\infty$. Let $t \geq t_0$ and $1 \leq r \leq 2\sqrt{\kappa^2 + t^2}$. Then there exists a constant $C > 0$ depending only on m, R, R', κ and r such that for all $\gamma \in \mathbb{Z}^3$ satisfying $|\gamma| \leq r$ we have

$$\left|\widehat{f_1}(\gamma) - \widehat{f_2}(\gamma)\right| \leq Ce^{4R't} \|F(f_1) - F(f_2)\|_{L^2} + \frac{C}{t} \|f_1 - f_2\|_{H^m}$$

bounds on Fourier coefficients

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for trigonometric projections P_r.

extensions

- far field data
- electromagnetic scattering problems
- explicit dependence on $\kappa \rightsquigarrow$ Hölder-logarithmic sc
- Banach norms as penalties

F. Weidling, TH. *Variational source conditions and stability estimates for inverse electromagnetic medium scattering problems.* arXiv 1512.06586, 2015.

Conclusions

- VSCs are always necessary and sufficent for certain rates of convergenc for linear inverse problems and most spectral regularization methods.
- For a number of relevant inverse problems VSC can be characterized by Besov-Nikol′skiĭ spaces.
- Certain conditional stability estimates for inverse problems in PDEs can be sharpened to VSCs.

Thank you for your attention!