Ruprecht-Karls-Universität Heidelberg

Pre-talk : Bayesian asymptotics and nonparametric inverse problems

In anticipation of Bartek Knapik talk

Xavier Loizeau

30^{th} of June 2016

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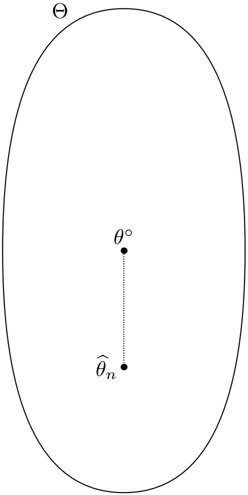
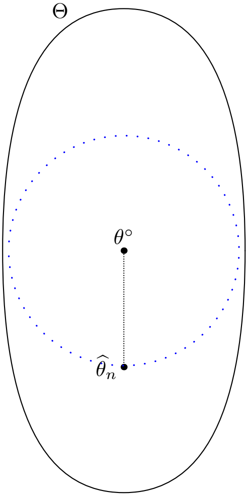
Point-wise comparison of frequentist estimators



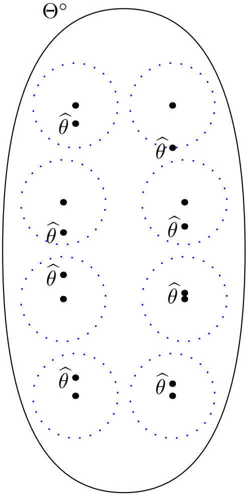
Point-wise comparison of frequentist estimators



- Measure the performance of a frequentist estimator $\widehat{\theta}_n$ using quadratic risk for a given parameter θ°

$$\mathbb{E}_{\theta^\circ}\left[d\left(\widehat{\theta}_n,\theta^\circ\right)^2\right]$$

Comparison of frequentist estimators over a subset



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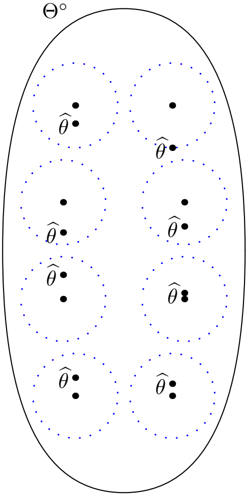

- Measure the performance of a frequentist estimator using maximal risk over a class Θ° of parameters

$$\sup_{\theta^\circ \in \Theta^\circ} \mathbb{E}_{\theta^\circ}\left[d\left(\widehat{\theta},\theta^\circ\right)^2\right]$$

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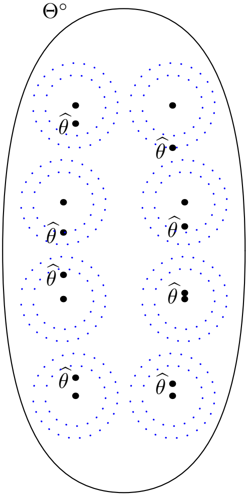

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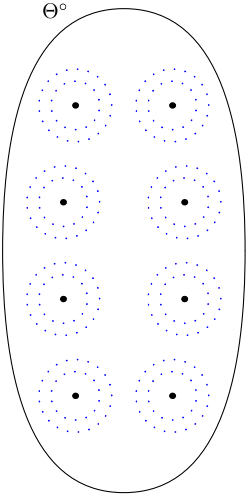

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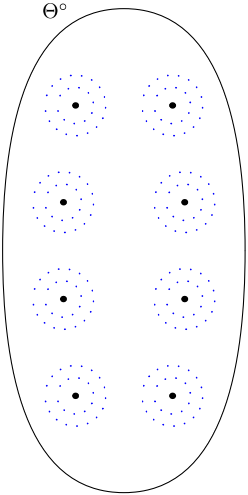

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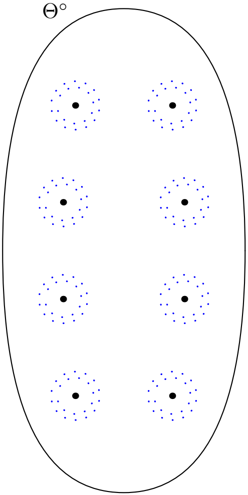

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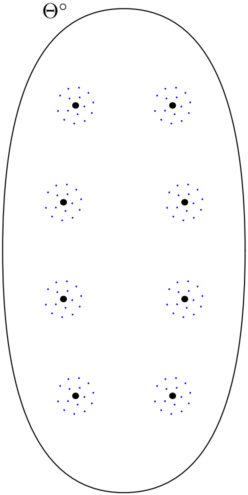

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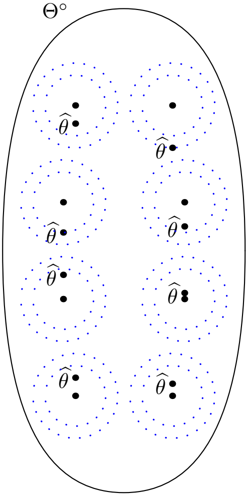

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- ... which is reached by an estimator $\widehat{\theta}$

$$\sup_{\theta^0 \in \Theta^\circ} \mathbb{E}_{\theta^\circ}\left[d\left(\widehat{\theta},\theta^\circ\right)^2\right] \leq C_2 \cdot \mathscr{R}_n^\star\left(\Theta^\circ\right)$$

Contents

Let $(\Theta, \mathfrak{A})$ be a measurable space
$\boldsymbol{\theta}$ is a random variable $(\Omega, \mathfrak{F}) \to (\Theta, \mathfrak{A})$

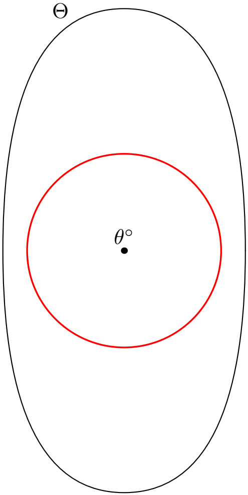
$$\boldsymbol{\theta} \sim \Pi$$

Denote by p_θ the density of Π with respect to a measure μ
Posterior distribution

$$\forall \mathfrak{B} \in \mathfrak{A} \quad \Pi_{\boldsymbol{\theta}|Y}(\mathfrak{B}) = \frac{\int_{\mathfrak{B}} p_{\boldsymbol{\theta}}(Y) d\Pi(\boldsymbol{\theta})}{\int_{\Theta} p_{\boldsymbol{\theta}}(X) d\Pi(\boldsymbol{\theta})}$$

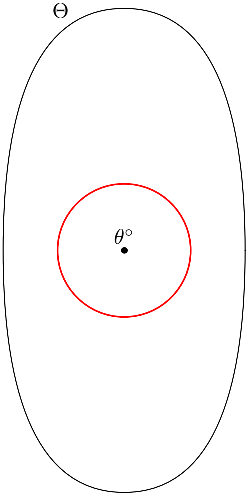
Taking a frequentist point of view

- θ° the true parameter
- Is $\Pi_{\boldsymbol{\theta}|Y}$ shrinking around θ° as ϵ tends to 0 ?
- How fast ?



- Concentration rate $(\phi_\epsilon)_{\epsilon\in\mathbb{R}_+}$

$$\exists c \in \mathbb{R}_+, \quad \lim_{\epsilon\to 0} \mathbb{E}_{\theta^\circ}\left[\Pi_{\boldsymbol{\theta}|Y}\left(d\left(\boldsymbol{\theta},\theta^\circ\right)^2 \geq c\phi_\epsilon\right)\right] = 0$$



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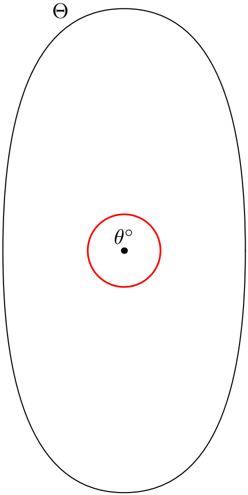

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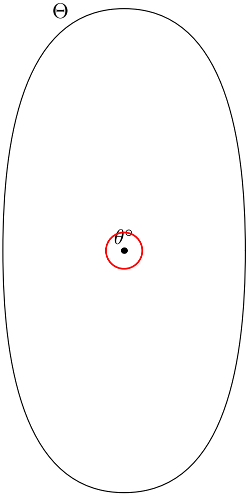

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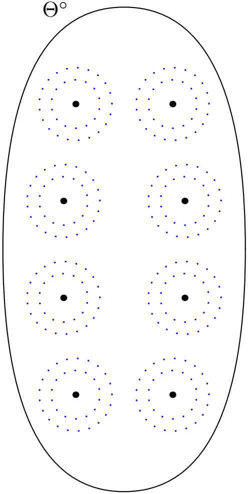

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- Uniform concentration rate $(\phi_\epsilon)_{\epsilon\in\mathbb{R}_+}$ over a subset on $\Theta^\circ \subset \Theta$

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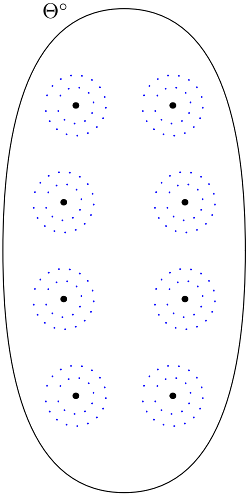
Bayesian paradigm
Comparing prior choices



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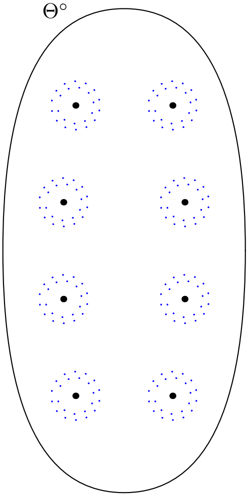

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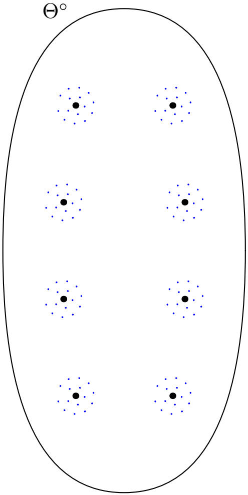

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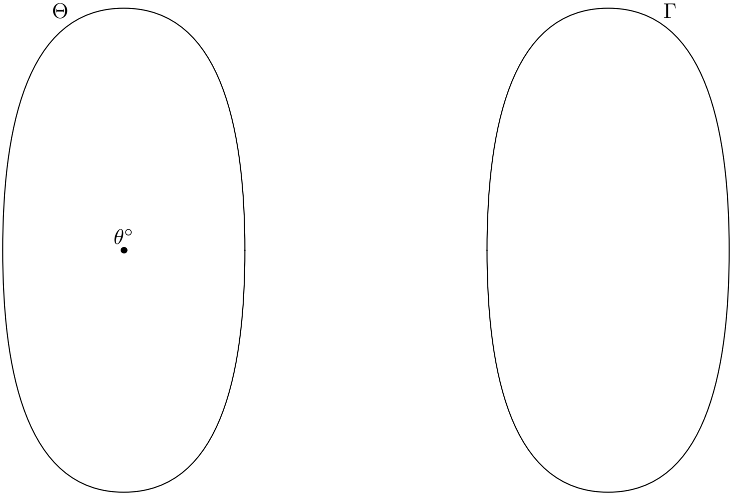

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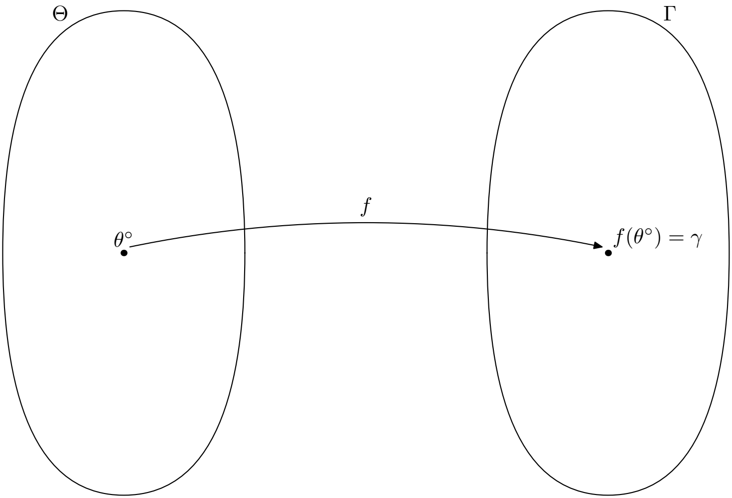

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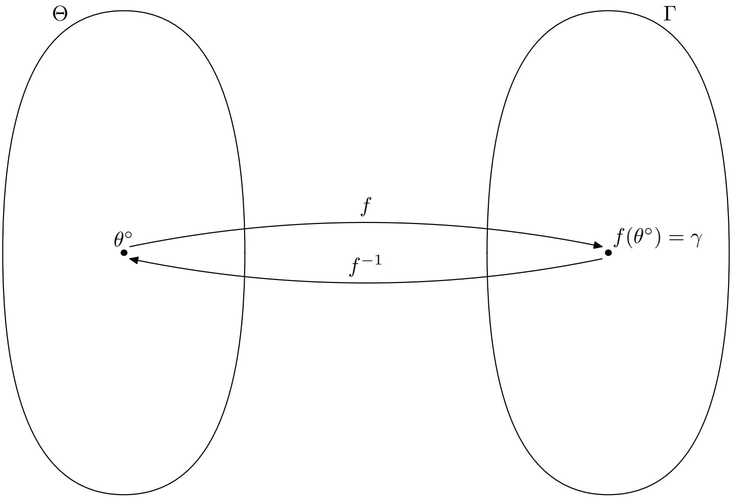

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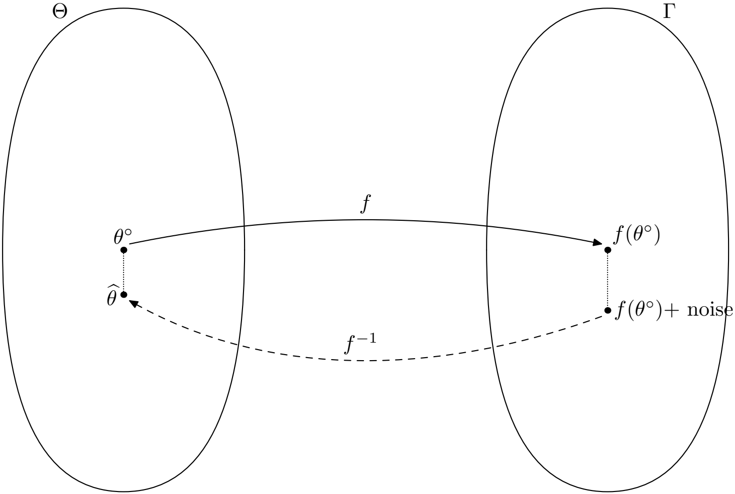

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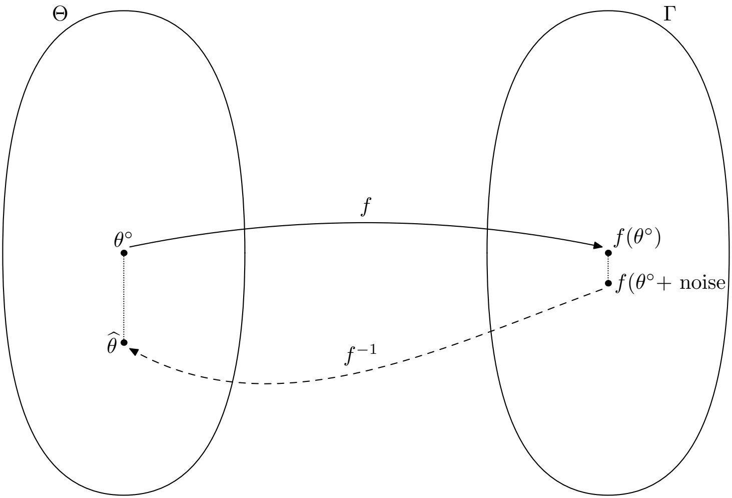

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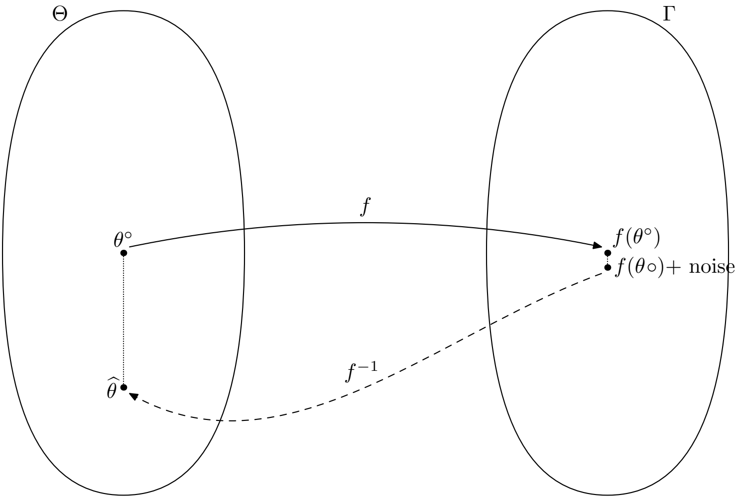

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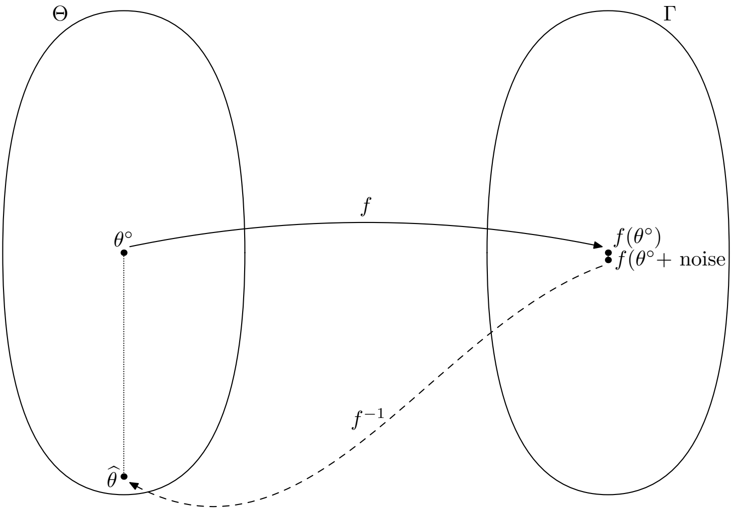

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