Assignment 3

1. $y''' + 6y'' + y' - 34y = 0$, $y_1 = e^{-4} \cos x$

$m^3 + 6m^2 + m - 34 = 0$, $m = 2$

$$\begin{array}{c|cccc} 2 & 1 & 6 & 1 & -34 \\ & & 2 & 16 & 34 \\ \hline & 1 & 8 & 17 & 0 \end{array}$$

$m^2 + 8m + 17 = 0$

$$\frac{-8 \pm \sqrt{64 - 4(1)(17)}}{2}$$

$$\frac{-8 \pm 2\sqrt{-1}}{2} = -4 \pm \sqrt{-1}$$

$$\boxed{y = c_1 e^{2x} + e^{-4x}\left[c_2 \sin(x) + c_3 \cos(x)\right]}$$

2. $e^{(3-5i)x}$

$= e^{3x} \cdot e^{-5ix}$

$\boxed{= e^{3x}\left[\cos(5x) - i\sin(5x)\right]}$

3. $y'' + 4y' + 20y = 3\cos(4x) - 2x\sin(4x)$

particular:

$$3\cos(4x) - 2x\sin(4x) = A\cos(4x) + B\sin(4x) + Cx\cos(4x) + Dx\sin(4x)$$

$$y_p' = -4A\sin(4x) + 4B\cos(4x) + C\cos(4x) - 4Cx\sin(4x) + D\sin(4x) + 4Dx\cos(4x)$$
$$= (-4A + D)\sin(4x) + (4B + C)\cos(4x) - 4Cx\sin(4x) + 4Dx\cos(4x)$$

$$y_p'' = (-16A + 4D)\cos(4x) + (-16B - 4C)\sin(4x) - 4C\sin(4x) + 4D\cos(4x)$$
$$- 16Cx\cos(4x) - 16Dx\sin(4x)$$
$$= (-16A + 8D)\cos(4x) + (-16B - 8C)\sin(4x) - 16Cx\cos(4x) - 16Dx\sin(4x)$$

1. $-3 = (-16A + 8D) + 4(4B + C) + 20(A)$

2. $0 = (-16B - 8C) + 4(-4A + D) + 20(B)$

3. $0 = -16C + 4(4D) + 20(C)$

4. $-2 = -16D + 4(-4C) + 20(D)$

$\Rightarrow$ 4 unknowns and 4 eq'n $\therefore$ you can solve

Homogeneous:

$y'' + 4y' + 20y \Rightarrow m^2 + 4m + 20 = 0$

$$\frac{-4 \pm \sqrt{16 - 80}}{2} = -2 \pm 4i$$

$\Rightarrow e^{-2x}[C_1\sin(4x) + C_2\cos(4x)] \Rightarrow$ solution to homogeneous

$\therefore$ True, you can use the method of undetermined coefficients to solve.

4. $\boxed{\text{False}}$

- When dealing with duplicates, you multiply one term by x. By doing this, substituting back into the equation will produce the correct equation. If y_p contains a duplicate, the same rules stand. We only would multiply y_p by x^2 if a term has already been multiplied by x because duplicates were already seen.

5.

$\sin \pi = 0$

$\Rightarrow 0 \neq \pi$

- If θ was close to 0, $\sin \theta \simeq \theta$, but since θ is close to π, $\sin \theta \not\simeq \theta$

$\boxed{\therefore \text{False}}$

6.

a) $3y'' + x^2y' - 3y = 0$

- homogeneous: $g(x) = 0$

b) $3y'' + x^2y' - 3y = 2y$

$3y'' + x^2y' - 5y = 0$

- Homogeneous: $g(x) = 0$

c) $3y'' + x^2y' - 3y = x^3$

- Nonhomogeneous: $g(x) = x^3$

d) $3y'' + x^2y' = 3y$

$3y'' + x^2y' - 3y = 0$

- Homogeneous: $g(x) = 0$

e) $3y'' + x^2y' - 3y = 1$

- Nonhomogeneous: $g(x) = 1$

f) $3y'' + x^2y' - 3 = 0$

$3y'' + x^2y' = 3$

- Nonhomogeneous: $g(x) = 3$

7.

$$\frac{dN}{dt} = rN\left(1-\frac{N}{K}\right)$$

$$\frac{dN}{dt} = rN - r\frac{N^2}{K}$$

$$\frac{dN}{dt} - rN = -\frac{r}{K}N^2 \qquad \text{let } u = N^{1-2} = N^{-1}$$
$$du = -N^{-2}\,dN$$

$$\frac{1}{N^2}\frac{dN}{dt} - \frac{1}{N}r = -\frac{r}{K}$$

$$-\frac{du}{dt} - ur = -\frac{r}{K} \qquad \mu(t) = e^{\int r\,dt} = e^{rt}$$

$$\int \frac{d}{dt}\left[ue^{rt}\right] = \int \frac{r}{K}e^{rt}$$

$$ue^{rt} = \frac{1}{K}e^{rt} + C$$

$$u = \frac{1}{K} + Ce^{-rt}$$

$$N(t) = \frac{1}{\frac{1}{K} + Ce^{-rt}}$$

$$N(t) = \frac{K}{1 + KCe^{-rt}}$$

- K is the limiting number and $= 50{,}000$

$$N(t) = \frac{50{,}000}{1 + 50{,}000\,Ce^{-rt}}$$

$$N(0) = 500 = \frac{50{,}000}{1 + 50{,}000\,C}$$

$$1 + 50{,}000C = 100$$
$$C = 1.98\times10^{-3} \Rightarrow KC = 99$$

$$N(1) = 1000 = \frac{50{,}000}{1 + 99e^{-r}}$$

$$1 + 99e^{-r} = 50$$
$$e^{-r} = \frac{49}{99}$$
$$\Rightarrow r = -\ln\left(\frac{49}{99}\right)$$

$$\boxed{N(t) = \frac{50{,}000}{1 + 99e^{\ln\left(\frac{49}{99}\right)t}}}$$

8. $y'' - e^{-y'} = 0$, let $u = y'$,
$$\frac{du}{dx} = y''$$

$$\frac{du}{dx} - e^{-u} = 0$$

$$\int du\,e^{u} = \int dx$$

$$e^{u} = x + c_1$$

$$u = \ln(x + c_1) = y'$$

$$y = (x+c_1)\ln(x+c_1) - (x+c_1) + c$$

$$\Rightarrow c_1 + c = c_2$$

$$\Rightarrow \boxed{y = (x+c_1)\ln(x+c_1) - x + c_2}$$

9: $2x^2y'' + 5xy' + \frac{25}{8}y = 0$

let $y = x^r$

$y' = rx^{r-1}$

$y'' = r(r-1)x^{r-2}$

$2x^2(r(r-1)x^{r-2}) + 5x(rx^{r-1}) + \frac{25}{8}x^r = 0$

$2x^2[(r^2-r)x^r \cdot x^{-2}] + 5x(rx^r \cdot x^{-1} + \frac{25}{8}x^r = 0$

$x^r[2(r^2-r) + 5r + \frac{25}{8}] = 0$

$x^r[2r^2 + 3r + \frac{25}{8}] = 0$

$2r^2 + 3r + \frac{25}{8} = 0$

$r = \frac{-3 \pm \sqrt{9-4(2)(\frac{25}{8})}}{4}$

$= \frac{-3 \pm 4i}{4}$

$r = -\frac{3}{4} \pm i$

$\Rightarrow \boxed{y = x^{-\frac{3}{4}}[C_1 \cos(\ln(x)) + C_2 \sin(\ln(x))]}$

10.

$m = 3\,kg$, Spring constant $= 27\,N/m$, damping proportional to $18\times$ instantaneous velocity

$\Rightarrow 3\frac{d^2x}{dt^2} + 18\frac{dx}{dt} + 27x = 0$

$\frac{d^2x}{dt} + 6\frac{dx}{dt} + 9 = 0$

$n^2 + 6n + 9 = 0$

$(n+3)^2 = 0$

$\Rightarrow x(t) = c_1 e^{-3t} + c_2 t e^{-3t}$

$x(0) = \boxed{175 = c_1}$

$x'(t) = -3c_1 e^{-3t} - 3c_2 t e^{-3t} + c_2 e^{-3t}$

$x'(0) = 0 = -3c_1 + c_2$

$c_2 = 3(175) = 525$

$\therefore x(t) = 175e^{-3t} + 525t\,e^{-3t}$

$x(4) = 175e^{-12} + 525(4)e^{-12}$

$\boxed{x(4) = 0.014\ cm}$

11.

$4y^{(4)} + 12y'' + 9y = 0$

$4m^4 + 12m^2 + 9 = 0$, let $u = m^2$

$4u^2 + 12u + 9 = 0$

$$\frac{-12 \pm \sqrt{144 - 4(9)(4)}}{8} = \frac{-12 \pm \sqrt{0}}{8} = -\frac{3}{2} = u$$

$\Rightarrow u = m^2 = -\frac{3}{2}$

$m = \pm \frac{\sqrt{6}}{2} i$

$$\boxed{\Rightarrow y(t) = c_1 \cos\left(\frac{\sqrt{6}}{2} x\right) + c_2 \sin\left(\frac{\sqrt{6}}{2} x\right) + c_3 x \cos\left(\frac{\sqrt{6}}{2} x\right) + c_4 x \sin\left(\frac{\sqrt{6}}{2} x\right)}$$

12. $y'' + 2y' - 8y = -9\cos x - 2\sin x, \; y(0) = 4, \; y'(0) = -6$

Homogeneous:

$y'' + 2y' - 8y = 0$

$m^2 + 2m - 8 = 0$

$(m+4)(m-2) = 0 \Rightarrow m = 2, -4$

$\Rightarrow C_1 e^{2x} + C_2 e^{-4x} = 0$

particular:

$y_p = -9\cos x - 2\sin x = A\cos x + B\sin x$

$y'_p = -A\sin x + B\cos x$

$y''_p = -A\cos x - B\sin x$

$-A\cos x - B\sin x - 2A\sin x + 2B\cos x - 8A\cos x - 8B\sin x$

$(-9A + 2B)\cos x + (-9B - 2A)\sin x$

$-9A + 2B = -9 \qquad -9B - \frac{4}{9}B - 2 = -2$

$A = \frac{2}{9}B + 1 \qquad B = 0$

$\Rightarrow A = 1$

Compete:

$y = C_1 e^{2x} + C_2 e^{-4x} + \cos(x)$

$4 = C_1 + C_2 + 1$

$3 = C_1 + C_2$

$y' = 2C_1 e^{2x} + -4C_2 e^{-4x} - \sin(x)$

$-6 = 2C_1 - 4C_2$

$-6 = -4C_2 + 2(3) - 2C_2$

$-12 = -16C_2$

$C_2 = 2, \; C_1 = 1$

$\boxed{\Rightarrow y = e^{2x} + 2e^{-4x} + \cos(x)}$