CSCI 1515: Applied Cryptography

P. Miao

Spring 2023

These are lecture notes for CSCI 1515: Applied Cryptography taught at Brown University by Peihan Miao in the Spring of 2023.

These notes are taken by Jiahua Chen with gracious help and input from classmates and fellow TAs. Please direct any mistakes/errata to me via email, post a thread on Ed, or feel free to pull request or submit an issue to the notes repository (https://github.com/BrownAppliedCryptography/notes).

FYI, todo's are marked like this.

Notes last updated February 16, 2023.

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§1 January 26, 2023

§1.1 Introduction

If you find a lot of courses on cryptography online or at universities, you'll find a lot of theoretical content. However, it's helpful for students to get hands-on experience with cryptography:

- How cryptography has been used in practice,
- how cryptography will be used and implemented in the future.

The goal of this course is to

Introduction: Peihan Miao, please refer by Peihan (you will be corrected if you address as professor). Joined Brown last semester, taught a seminar on research topic (Secure Computation). Before that, was in Chicago, and Visa Research even before that. Really loved math and theoretical computer science, started PhD with theoretical computer science. Shifted to applied side of cryptography; it's exciting to see cryptography implemented in practice. However, realized that most crypto courses are very theoretical—there are not a lot of courses that build up advanced applications using the tools that have been set up.

For this course, it will be *much less* about math and proofs, and much more about how you can use these tools to do something more fun. It will be coding heavy, all projects will be implemented in C++ using crypto libraries. If, however, you are interested in the theoretical or mathematical side, you might consider other courses at Brown[1]

§1.2 Course Logistics

The course homepage is at https://brownappliedcryptography.github.io/.

Huge kudos to Nick and Jack for developing a new course from the ground up!

Please view the syllabus here. *If there are differences between this document and the syllabus, the syllabus trumps this document.*

The course is offered in-person in CIT 368, as well as synchronously over Zoom and recorded asynchronously (lectures posted online). You can do either[2] but try to attend online because there will be interaction! Lecture attendance is encouraged!

EdStem will be used for course questions, and **Gradescope** is used for assignments.

[1] CSCI 1510 and Math 1580 are good candidates. 1510 covers cryptographic proofs, and Math 1580 covers cryptography from a number theoretic perspective.

[2] It is a 9AM class...

For assignments: *Project 0* is a warmup for C++. Projects 1 & 2 is to develop secure communication or authentication systems using the underlying cryptographic libraries. The later project, 3, 4 & 5 will be on more advanced topics. The first 2 are more basic that are developed in practice, and the latter ones are more experimental in practice (so this is recent research in applied cryptography). The final project will be a combination of the existing projects or a project entirely new (separate from the earlier projects, but using the same cryptographic primitives).

Projects 1 through 5 will be accompanied with homework assignments (appropriately numbered 1 through 5) that develop a conceptual understanding of the materials.

Projects 1 & 2 are to be done individually, the later projects are done in pairs (you can choose to go solo if you so wish as well). You are encouraged to find partners earlier on to discuss and work with them from the beginning. You are *encouraged* to communicate with your partners on projects 1 & 2 so you gain a conceptual understanding. You should complete your own write-up and code for the first two projects, however.

There is an option to capstone this course, contact Peihan about this. It would also be best to find a partner who is also capstoning this course.

The following is the grading policy:

Type	*Percentage*
Project 0	5%
Projects 1 & 2	20% (10% each)
Projects 3, 4 & 5	45% (15% each)
Homeworks	20% (2% each)
Final Project	20%

You have 6 late days for *projects*, of which at most two can be used on a single project. Additionally, you have 3 late days for *homeworks*, of which at most one can be used on a single homework.

If you're sick, let Peihan know with a Dean's note.

§1.3 What is cryptography?

...or more importantly, what is cryptography used for?

At a high level, *cryptography is a set of techniques that protect information.*

Question. What are some cryptographic techniques used in practice and what does it achieve?

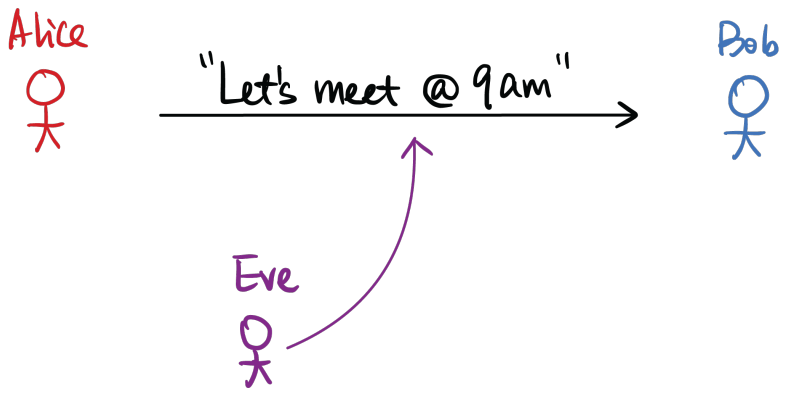
- Used in financial institutions.

  - When you make a purchase, you might not want people to see your bank balance, what else you have purchased, etc.

- Might be used in voting schemes. Making sure that ballots are kept private.
- Messaging systems, recently.
  - End-to-end texting.
- Storing medical records.
  - More generally, any sensitive information.
  - You don't want anyone else getting access to this sensitive information.
- Used in military or war.
  - Historically, it was used a lot in the military for secure communication.
- Authentication systems. Login systems.
  - There is authentication going on that ensures that *only you* can log in.

§1.4 Secure Communication

We'll start with the most primitive of cryptography: *secure communication*.



Assume Alice wants to communicate to Bob "Let's meet at 9am", what are some security guarantees we want?

- Eve cannot *see* the message from Alice to Bob.
- Eve cannot *alter* the message from Alice to Bob.

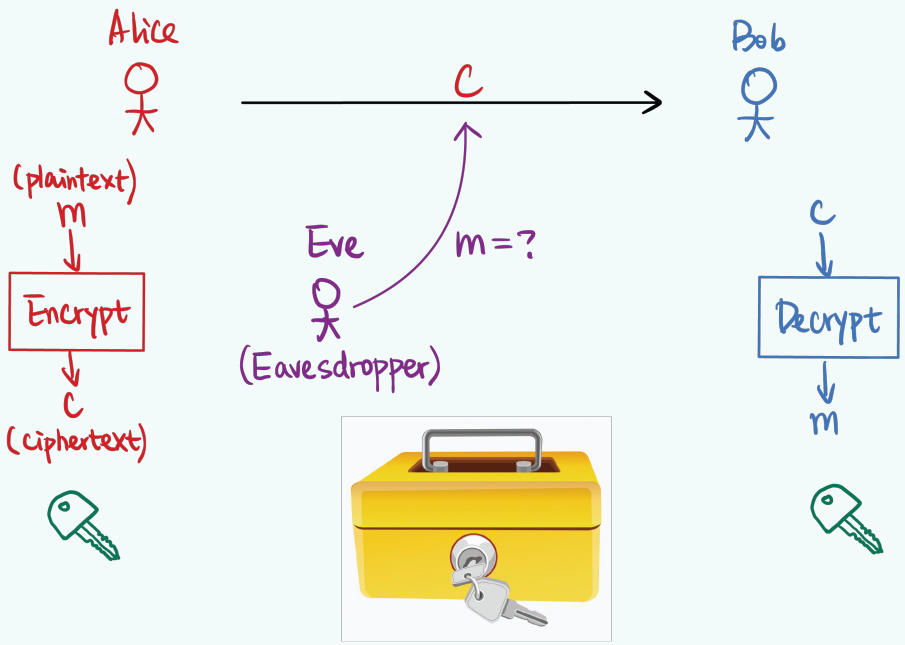
These two guarantees are the most important guarantees! The former is called message secrecy, the latter is called message integrity.

§1.4.1 Message Secrecy

Definition 1.1 (Message Secrecy)

We want cryptography to allow Alice to *encrypt* the message m (which we call *plaintext*) by running an algorithm that produces a *ciphertext* c.

Bob will be able to receive the ciphertext c and run a *decrypt* algorithm to produce the message m again. This is akin to a secure box that Alice locks up, and Bob unlocks, while Eve does not know the message.



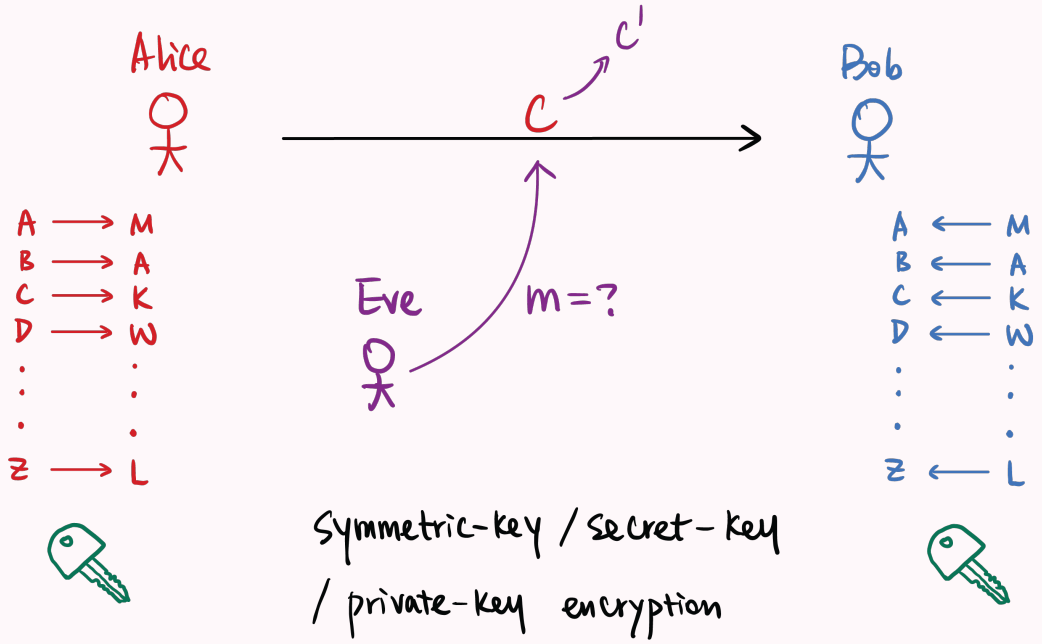
In this model, Eve is a weaker adversary, an *eavesdropper*. Eve can only see the message, not alter it.

Example 1.2 (Substitution Cipher)

The key that Alice and Bob jointly uses is a permutation mapping from $\{A \dots Z\} \to \{A \dots Z\}$.

This mapping is the *secret key*.

Bob also has the mapping, and takes the inverse of the permutation to retrieve the message.



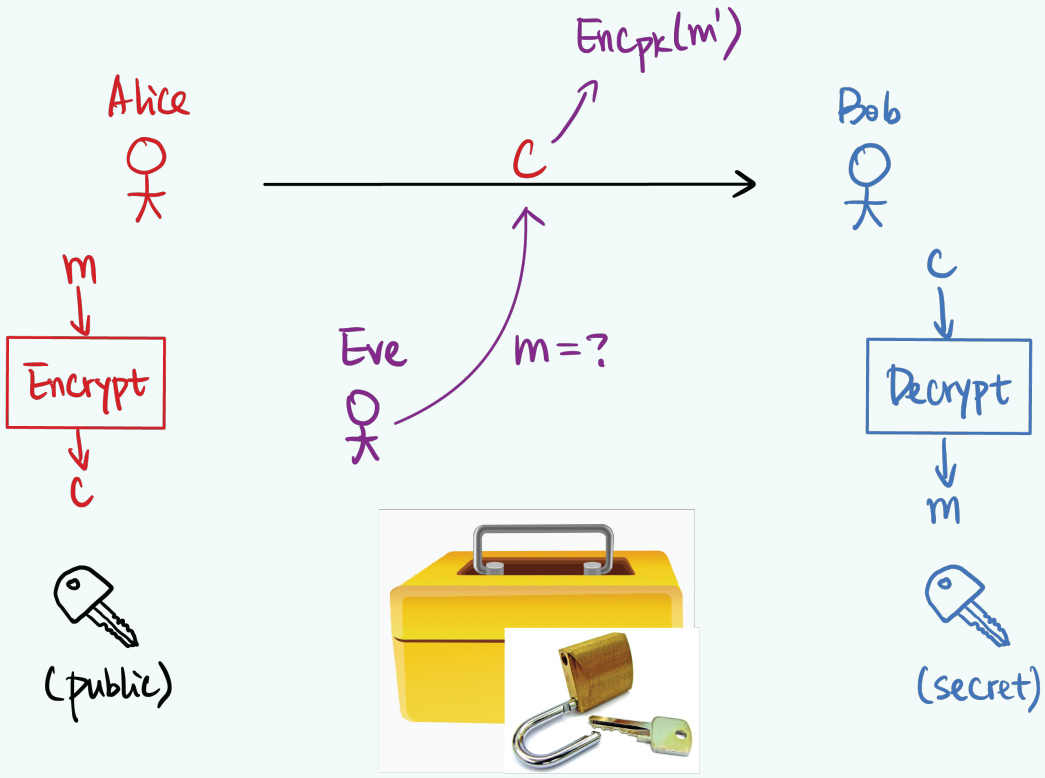
This scheme is not quite secure! *Why?*

You could guess a bunch of vowels and see what words could make up. If you have a long enough message, you can see which letters appear more often. We know that in English, the vowels appear more often; and you can make a lot of guesses.

Remark. This encryption scheme also requires that Alice and Bob meet up in person to exchange this shared private key. Schemes like this are called *symmetric-key, secret-key, or private-key encryption.* They need to meet up first to exchange secret keys.

Definition 1.3 (Public-key Encryption)

There is another primitive that is much more ideal/stronger, public-key encryption. Bob publishes both a *public* key and a *private* key. You can consider a lock where you don't need a key to lock it[3], and only Bob has the key to unlock it.

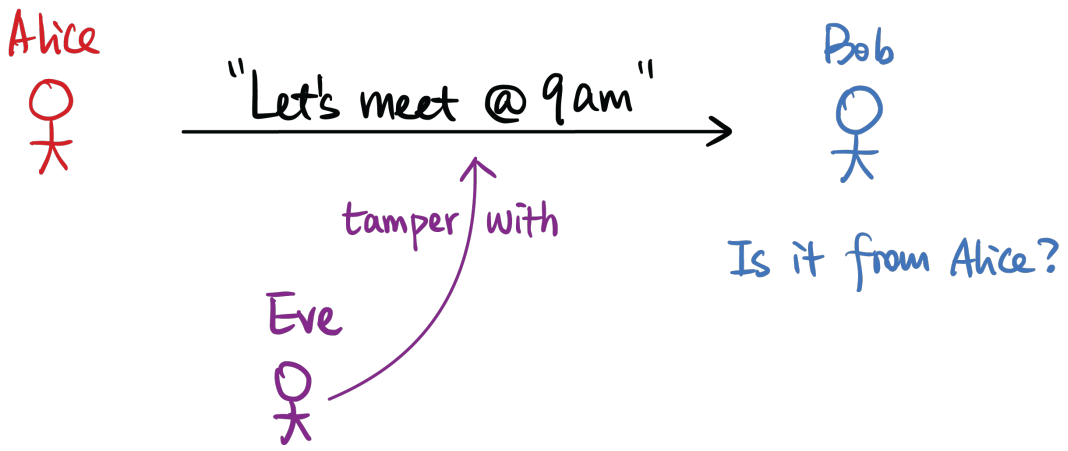


This is seemingly magic! Bob could publish a public key on his homepage, anyone can encrypt using a public key but only Bob can decrypt. *Stay tuned, we will see public-key encryption schemes next lecture!*

§1.4.2 Message Integrity

Alice wants to send a message to Bob again, but Eve is stronger! Eve can now tamper with the message.

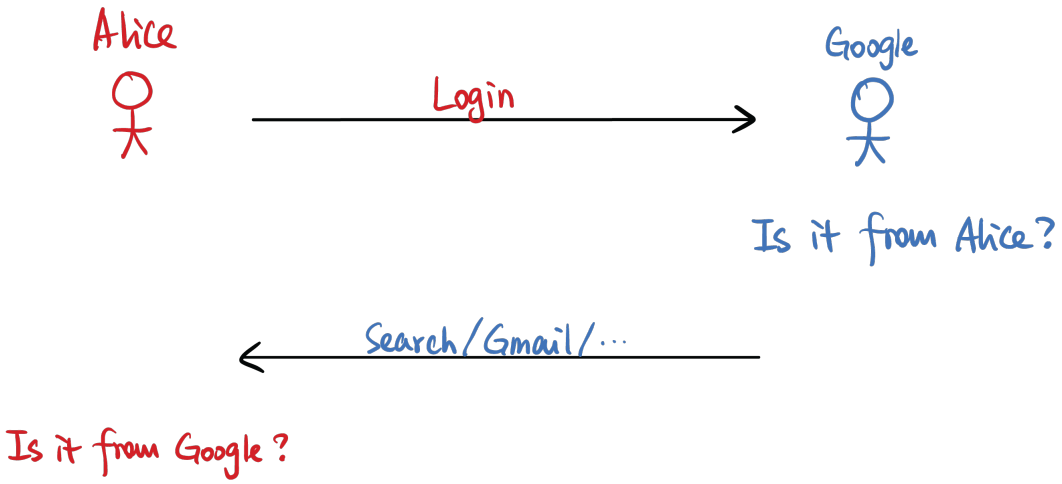
[3] You literally click it closed



Bob wants to ensure that the message *actually* comes from Alice. Does our previous scheme (of encrypting messages) solve this problem? Nope!

Eve can change the ciphertext to something else, they could pretend to be Alice. In secret-key schemes, if Eve figures out the secret-key, they can forge messages from Alice. Even if Eve doesn't know the underlying message, they could still change it to some other ciphertext which might be correlated to the original ciphertext, *without knowing the underlying message*. We'll see how Eve can meaningfully do this in some schemes. Alice could send a message "Let's meet at x AM" and Eve could tamper this to say "Let's meet at $x + 1$ AM."

This is sort of an orthogonal problem to message secrecy. For example, when Alice logs in to Google, Google needs to verify that Alice actually is who she claims to be.



§1.5 Project Overview

0. Warm-up, you will implement some basic cryptographic schemes.
1. Secure Communication: what was just introduced.
2. Secure Authentication: how to authenticate yourself to a server, also mentioned just now.
3. Zero-Knowledge Proofs: we'll use ZKPs to implement a secure voting scheme.
4. Secure Multiparty Computation: we'll implement a way to run any function securely between two parties.
5. Fully Homomorphic Encryption: a form of post-quantum cryptography.

We'll quickly introduce the concepts used in the projects.

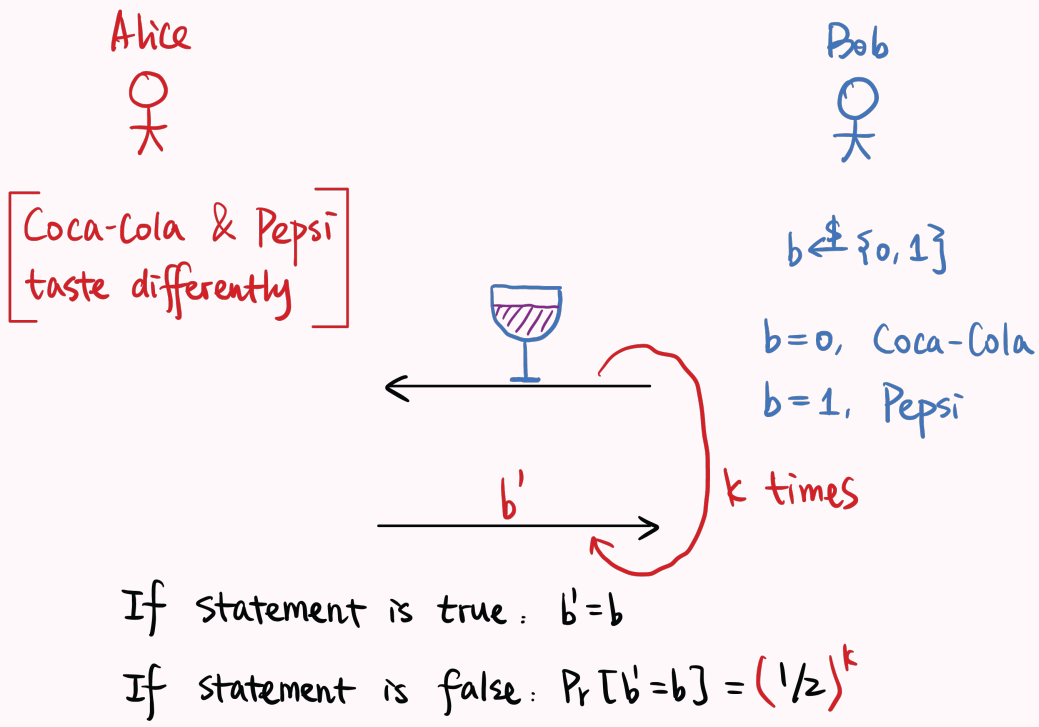
§1.5.1 Zero-Knowledge Proofs

This is to prove something without *revealing* any additional knowledge.

Example

Alice claims to be able to differentiate between Coca-Cola and Pepsi! She wants to prove this to Bob without revealing her secrets.

Bob will randomly sample a bit $b \xleftarrow{\$} \{0,1\}$, with $b = 0$ being Coca-Cola and $b = 1$ being Pepsi. Bob will let Alice taste this drink. Alice will give a guess b' of what drink it is.



If the statement is true, $b' = b$ (Alice gives the correct prediction).

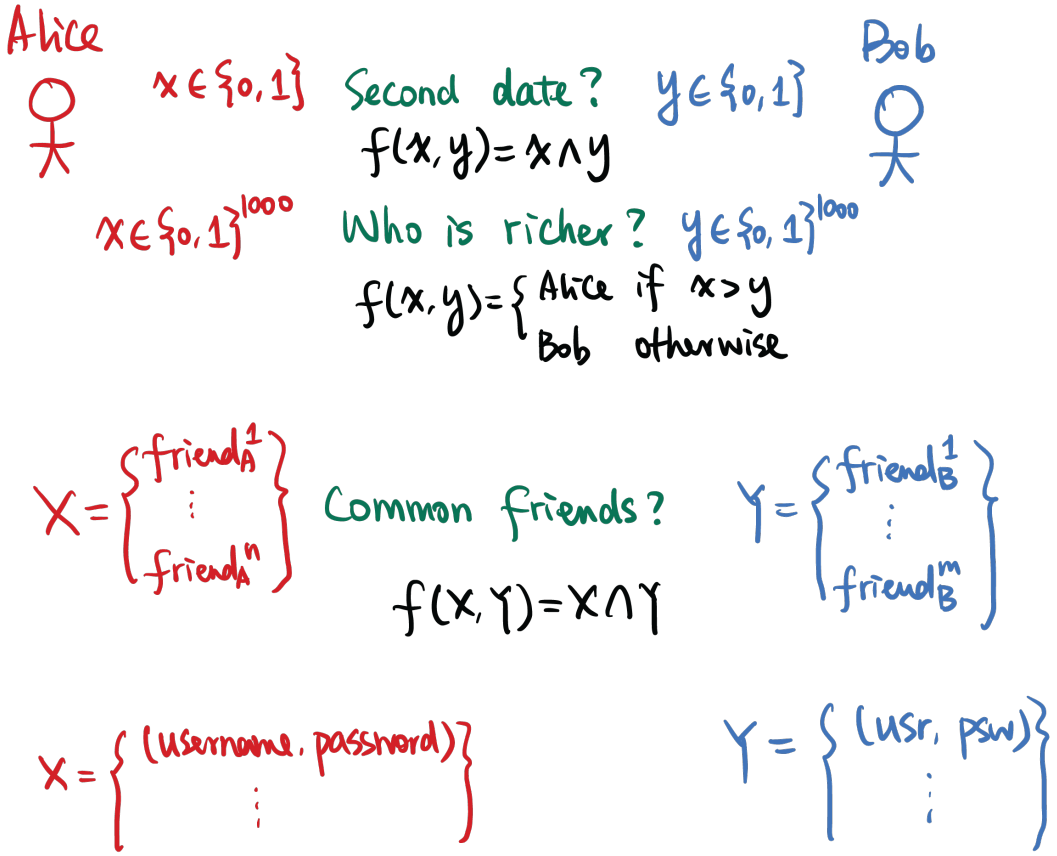
If the statement is false, $\Pr[b' = b] = \frac{1}{2}$ (Alice is guessing at 0.5 probability).

To enhance this, we can run this a total of k times. If we run it enough times, Bob will be more and more confident in believing this. Alice getting this correct by chance has a $\frac{1}{2^k}$ probability.

The key idea, however, is that Bob doesn't gain any knowledge of how Alice differentiates.

> **Remark.** This is a similar strategy in proving graph non-isomorphism.
>
> For people who have seen this before, generally speaking, any NP language can be proved in zero-knowledge. Alice has the *witness* to the membership in NP language.

§1.5.2 Secure Multi-Party Computation



Example (Secure AND)

Alice and Bob go on a first date, and they want to figure out whether they want to go on a second date. They will only go on a second date if and only if both agree to a second date.

How will they agree on this? They could tell each other, but this could be embarrassing. One way is for them to share with a third-party (this is what dating apps do!). However, there might not always be an appropriate third party (in healthcare examples, not everyone can be trusted with the data).

In this case, Alice has a choice bit $x \in \{0,1\}$ and Bob has a choice bit $y \in \{0,1\}$. They are trying to jointly compute $f(x,y) = x \wedge y$.

Example (Yao's Millionaires' Problem)

Perhaps, Alice and Bob wants to figure out who is richer. The inputs are $x \in \{0,1\}^{1000}$ and $y \in \{0,1\}^{1000}$ (for simplification, let's say they can express their wealth in 1000 bits). The

output is the person who has the max.

$$f(x,y) = \begin{cases} \text{Alice} & \text{if } x > y \\ \text{Bob} & \text{otherwise} \end{cases}$$

Example (Private Set Intersection)

Alice and Bob meet for the first time and want to determine which of their friends they share. However, they do not want to reveal who specifically are their friends.

X is a set of A's friends $X = \{\mathsf{friend}_A^1, \mathsf{friend}_A^2, \cdots, \mathsf{friend}_A^n\}$ and Bob also has a set $Y = \{\mathsf{friend}_B^1, \mathsf{friend}_B^2, \cdots, \mathsf{friend}_B^m\}$. They want to jointly compute

$$f(X,Y) = X \cap Y.$$

You might need to reveal the cardinality of these sets, but you could also pad them up to a maximum number of friends.

This has a lot of applications in practice! In Google Chrome, your browser will notify you that your password has been leaked on the internet, without having access to your passwords in the clear. X will be a set of *your* passwords, and Google will have a set Y of *leaked* passwords. The *intersection* of these sets are which passwords have been leaked over the internet, without revealing all passwords in the clear.

Question. Isn't the assumption that the size is revealed weaker than using a trusted third-party?

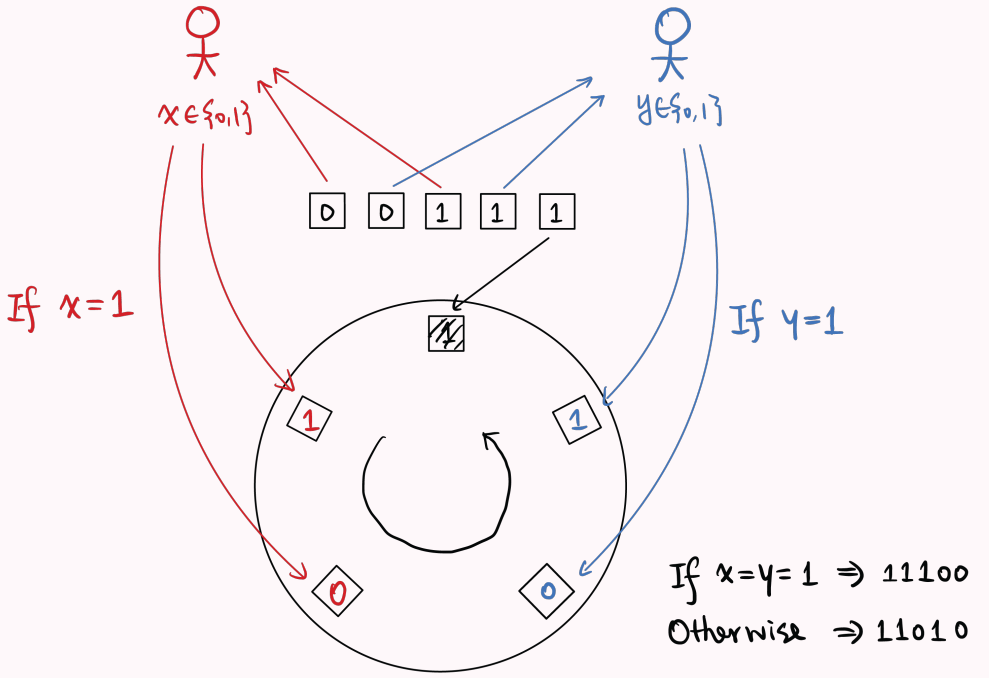
Yes, however in some cases (hospital health records), parties are legally obliged to keep data secure. We wish for security more than the secrecy of cardinality.

In the general case, Alice and Bob have some inputs x and y with bounded length, and they want to jointly compute some function f on these inputs. This is Secure Two-Party Computation. Furthermore, there could be multiple parties $x_1, \ldots, x_n$ that jointly compute $f(x_1, \ldots, x_n)$ that hides each input. This is Secure Multiparty Computation.

We'll explore a toy example with the bit-AND from the dating example.

Example (Private Dating)

Alice and Bob have choice bits $x \in \{0,1\}$ and $y \in \{0,1\}$ respectively. There is a *physical* round table with 5 identical slots, one already filled in with a 1 facing down.



Alice and Bob each have identical $0, 1$ cards (each of the 0 and 1 cards are indistinguishable from cards of the same value). Alice places her cards on the 2 slots in some order, and Bob does the same.

They then spin the table around and reveal all the cards, learning $x \wedge y$.

If $x = 1$, Alice places it as 1 on top of 0, and if $y = 1$, Bob places it as 1 on top of 0 as well. Otherwise, they flip them. If $x = y = 1$, then the 0's will be adjacent. If $x \neq y$, the order will be $1, 1, 0, 1, 0$ (the 0's are not adjacent), regardless of which of Alice or Bob produced $x = 0$ (or both!).

Question. If Alice puts 1, and the output is 0, she could infer the information from Bob. However, this is allowed. Whatever can be inferred from the desired output is inferred.

The more sensitive part is when if you put a 0, you don't learn whether the other party also put down a 0.

This is a toy example! It doesn't use cryptography at all! Two parties have to sit in front of a table. This is called card-based cryptography. We will be using more secure primitives.

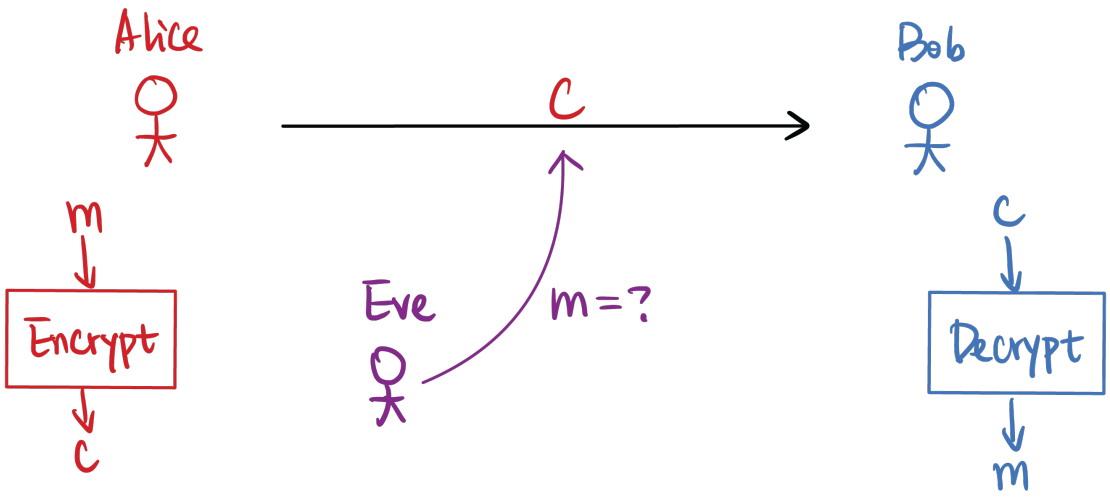
§1.5.3 Fully Homomorphic Encryption

We'll come back to the secure messaging example.

Alice wants to send Bob a message. She encrypts it somehow and sends a ciphertext $c_1 = \mathsf{Enc}(m_1)$. A nice feature for some encryption schemes is for Eve to do some computation homomorphically on the ciphertexts. Eve might possibly want to add ciphertexts (that leads to plaintext adding)

$$c_1 = \mathsf{Enc}(m_1), c_2 = \mathsf{Enc}(m_2) \Rightarrow c' = \mathsf{Enc}(m_1 + m_2)$$

or perhaps $c'' = \mathsf{Enc}(m_1 \cdot m_2)$, or compute arbitrary functions. *Sometimes*, this is simply adding $c_1 + c_2$, but usually not.

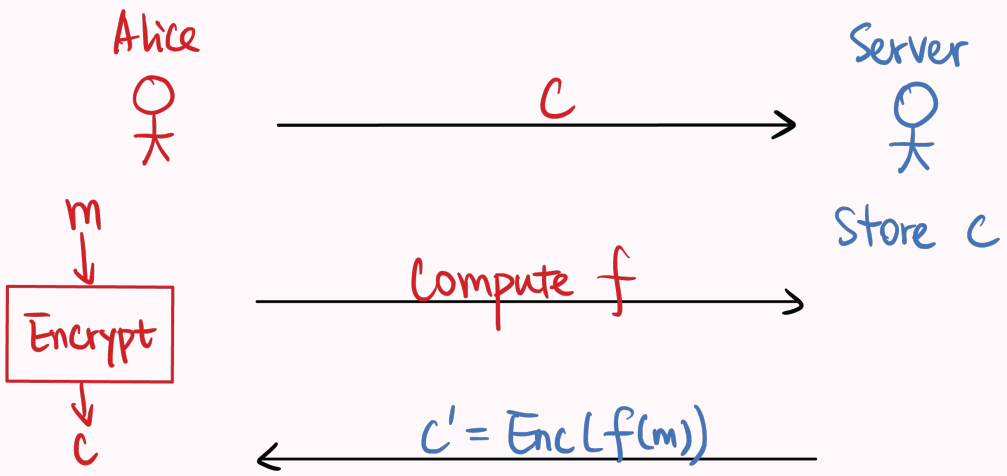


$$c_1 = \mathsf{Enc}(m_1)$$
$$c_2 = \mathsf{Enc}(m_2)$$
$$\Rightarrow \quad c' = \mathsf{Enc}(m_1 + m_2)$$
$$c'' = \mathsf{Enc}(m_1 \cdot m_2)$$

We want to hopefully compute any function in polynomial time!

Example (Outsourced Computation)

Alice has some messages but doesn't have enough compute. There is a server that has *a lot* of compute!



Alice encrypts her data and stores it in the server. At some point, Alice might want to compute a function on the encrypted data on the server, without the server revealing the original data.

This is an example of how fully homomorphic encryption can be useful.

Remark. This problem was not solved until 2009 (when Peihan started her undergrad). Theoretically, it doesn't even seem that possible! Being able to compute functions on ciphertexts that correspond to functions on plaintexts.

To construct fully-homomorphic encryption, we'll be using lattice-based cryptography. This is also the only cryptographic primitive that is quantum-secure[4].

§1.5.4 Further Topics

We might cover some other topics:

- Differential Privacy
- Crypto applications in machine learning
- Crypto techniques used in the blockchain[5]

What else would you like to learn? What else do you want to understand? Do go through the semester with these in mind! *How do I log into Google? How do I send messages to friends?*

Peihan will collect responses at the start and middle of the semester to shape course content.

[4] Everything before this can be broken if quantum computers become mainstream!
[5] One important techniques is Zero-Knowledge proofs, for example.

§1.6 A Quick Survey

Peihan conducted the following poll in-class to gauge content for future lectures. ***By all means, you don't need to know any/all of this going into this course! These will be self-contained in this course!***

Do you know what the following means?

- Polynomial-time algorithm.
- NP-hard problems.
- "a divides b" ($a \mid b$)
- GCDs
- (Extended) Euclidean Algorithms
- Groups
- One-Time pads
- RSA encryption/signature
- Diffie-Hellman Key Exchange
- SHA (hash functions)

§2 January 31, 2023

§2.1 Logistics

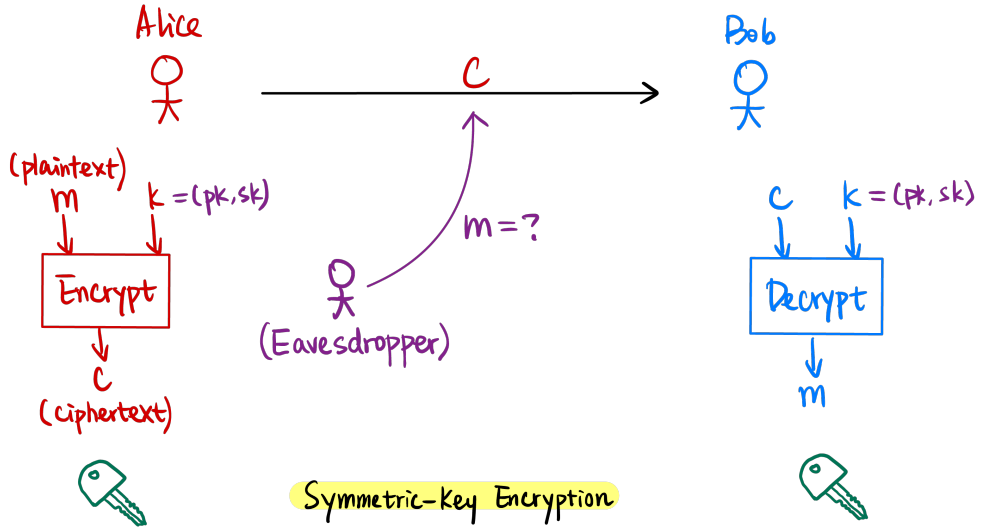
There's an EdStem post asking about topics you're interested, feel free to keep on posting!

We acknowledge the synchronization issues with Panopto. For now, if you want to watch the lecture recordings, you can use the Zoom link linked from the course home page. We can manually sync up EdStem but Panopto cannot be synced up, unfortunately.

§2.2 Encryption Schemes

This lecture we'll cover encryption schemes. We briefly mentioned what encryption schemes were last class, we'll dive into the technical content: how we construct them, assumptions, RSA, ElGamal, etc.

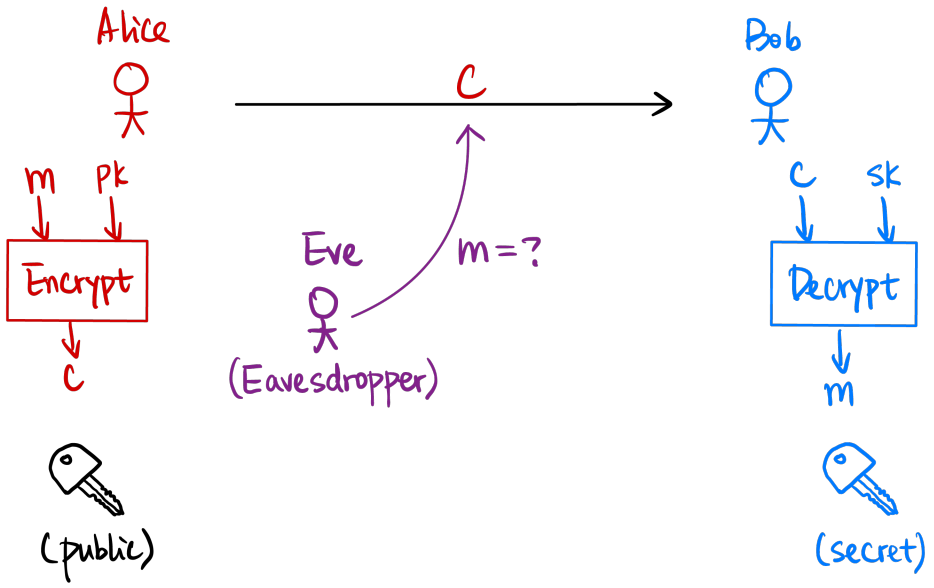
Fundamentally, an encryption scheme protects message secrecy. If Alice wants to communicate to Bob, Alice will encrypt a message (plaintext) using some key which gives her a ciphertext. Sending the ciphertext through Bob using a public channel, Bob can use the key to decrypt the ciphertext and recover the message. An eavesdropper in the middle will have no idea what message has been transmitted.



In this case, they are using a shared key, which we called secret-key encryption or symmetric-key encryption.

A stronger version of private-key encryption is called public-key encryption. Alice and Bob do not need to agree on a shared secret key beforehand. There is a keypair (pk, sk), a *public* and *private*

key.



§2.2.1 Syntax

Definition 2.1 (Symmetric-Key Encryption)

A symmetric-key encryption (SKE) scheme contains 3 algorithms, $\Pi = (\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec})$.

Generation. Generates key $k \leftarrow \mathsf{Gen}$.

Encryption. Encrypts message m with key k, $c \leftarrow \mathsf{Enc}(k, m)$, which we sometimes write as $\mathsf{Enc}_k(m)$.

Decryption. Decrypts using key k to retrieve message m, $m := \mathsf{Dec}(k, c)$, or written as $\mathsf{Dec}_k(c)$.

Note the notation $\leftarrow$ and $:=$ is different. In the case of generation and encryption, the produced key k or c follows a *distribution* (is randomly sampled), but we had better want decryption to be deterministic in producing the message.

Definition 2.2 (Public-Key Encryption)

A public-key encryption (PKE) scheme $\Pi = (\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec})$ contains the same 3 algorithms,

Generation. Generate keys $(pk, sk) \leftarrow \mathsf{Gen}$.

Encryption. Use the public key to encrypt, $c \leftarrow \mathsf{Enc}(pk, m)$ or $\mathsf{Enc}_{pk}(m)$.

Decryption. Use the secret key to decrypt, $m := \mathsf{Dec}(pk, c)$ or $\mathsf{Dec}_{sk}(c)$.

Question. If we can construct public-key encryption, why do we even bother with secret-key encryption? We could just use the (pk, sk) pair for our secret key, and this does the same thing.

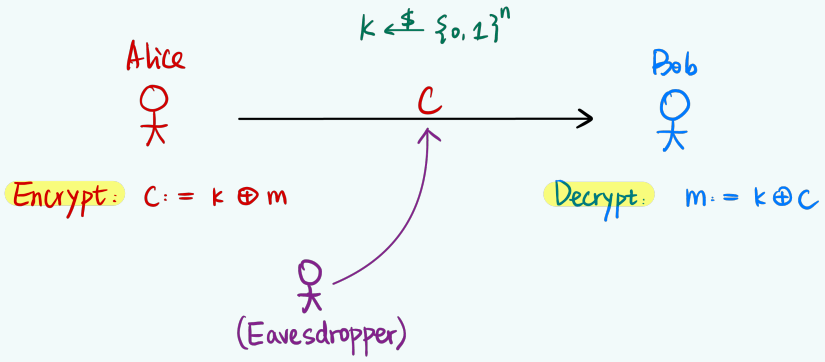
1. First of all, public-key encryption is almost always *more expensive*. Symmetric-key encryption schemes give us efficiency.

2. Public-key encryption relies on much stronger computational assumptions.

§2.2.2 Symmetric-Key Encryption Schemes

Definition 2.3 (One-Time Pad)

Secret key is a uniformly randomly sampled n bit string $k \xleftarrow{\$} \{0,1\}^n$.



Encryption. Alice uses the secret key and bitwise-XOR with the plaintext.

$$\begin{array}{rr} \text{secret key} & k = 0100101 \\ \oplus\ \text{plaintext} & m = 1001001 \\ \hline \text{ciphertext} & c = 1101100 \end{array}$$

Decryption. Bob uses the secret key and again bitwise-XOR with the ciphertext

$$\begin{array}{rr} \text{secret key} & k = 0100101 \\ \oplus\ \text{ciphertext} & c = 1101100 \\ \hline \text{plaintext} & c = 1001001 \end{array}$$

This is widely used in cryptography, called *masking* or *unmasking*.

Question. Why is this correct?

An XOR done twice with the same choice bit b is the identity. Or, an element is its own inverse with the XOR operator.

Question. Why is this secure?

We can think about this as the distribution of c. $\forall m \in \{0,1\}^n$, the encryption of m is uniform over $\{0,1\}^n$ (since k was uniform).

Another way to think about this is that for any two messages $m_0, m_1 \in \{0,1\}^n$, $\mathsf{Enc}_k(m_0) \equiv$

$\mathsf{Enc}_k(m_1)$. That is, the encryptions follow the *exact same* distribution. In this case, they are both uniform, but this is not always the case.

Question. Can we reuse k? Should we use the same key again to encrypt another message? Or, it is possible for the eavesdropper to extract information.

For example, $\mathsf{Enc}_k(m)$ is $c := k \oplus m$, and $\mathsf{Enc}_k(m')$ is $c' := k \oplus m'$. If the two messages are the same, the ciphertexts are the same.

By XOR c and c', we get

$$\begin{aligned} c \oplus c' &= (\cancel{k} \oplus m) \oplus (\cancel{k} \oplus m') \\ &= m \oplus m' \end{aligned}$$

This is why this is an *one-time* pad. This is a bit of an issue, to send an n-bit message, we need to agree on an n-bit message.

In fact, this is *the best* that we can do.

Theorem 2.4

Informally, for perfect (information-theoretic[6]) security, the key space must be at least as large as the message space.

$$|\mathcal{K}| \geq |\mathcal{M}|$$

where $\mathcal{K}$ is the key space and $\mathcal{M}$ is the message space.

Question. How can we circumvent this issue?

The high level idea is that we weaken our security guarantees *a little*. Instead of saying that they have to be *exactly the same* distribution, we say that they are *hard to distinguish* for an adversary with limited computational power. This is how modern cryptography gets around these lower bounds in classical cryptography. We can make *computational assumptions* about cryptography.

We can think about computational security,

Definition 2.5 (Computational Security)

We have computational security when two ciphertexts have distribution that cannot be distinguished using a polynomial-time algorithm.

[6]That the distributions of ciphertexts are identical, that $\mathsf{Enc}_k(m_0) \equiv \mathsf{Enc}_k(m_1)$.

Definition 2.6 (Polynomial-Time Algorithm)

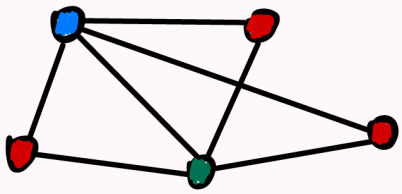
A polynomial time algorithm $A(x)$ is one that takes input x of length n, A's running time is $O(n^c)$ for a constant c.

Definition 2.7 (NP Problem)

A decision problem is in nondeterministic polynomial-time when its solution can be *verified* in polynomial time.

Example 2.8 (Graph 3-Coloring)

Given a graph, does it have a 3-coloring such that no two edges join the same color? For example,



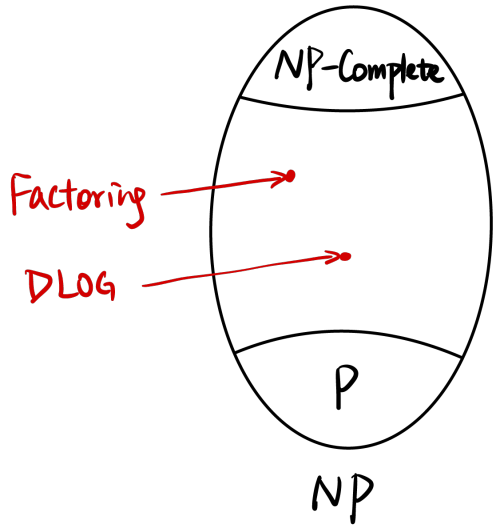
This can be *verified* in polynomial time (we can check if such a coloring is a valid 3-coloring), but it is computed in NP time.

Definition 2.9

An NP-complete problem is a "hardest" problem in NP. Every problem in NP is at least as hard as an NP-complete problem.

Right now, we assume $\mathsf{P} \neq \mathsf{NP}$. As of right now, there is no realistic algorithm that can solve any NP problem in polynomial-time.

Even further, we pick some problems not in NP-complete, not in P. We assume they are neither NP-complete nor in P (we don't yet have a reduction, but we don't know if one could exist) The reasoning behind using these problems is as we have no good cryptoscheme relying solely on NP-complete problems (we need something weaker).



Going back to our definition of computational security definition 2.5,

Definition (Computational Security)

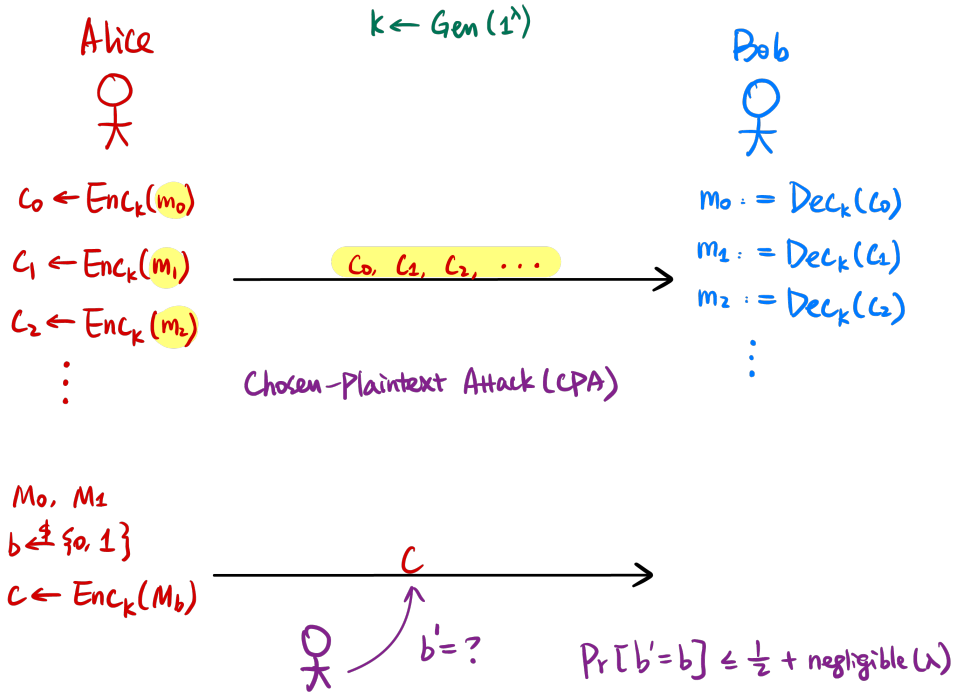
We say that the adversary is computationally bounded (is only a *polynomial-time algorithm*), that $\forall$probabilic poly-time algorithm $\mathcal{A}$,

$$\mathsf{Enc}_k(m_0) \overset{c}{\simeq} \mathsf{Enc}_k(m_1)$$

Where $\overset{c}{\simeq}$ is "computationally indistinguishable".

What does it mean for distributions to be "computationally indistinguishable"? Let's say Alice encrypts multiple messages $m_0, m_1, \ldots$ to Bob and produces $c_0, c_1, \ldots$. Even if Eve can see all plaintexts m_i in the open and ciphertexts c_i in the open, between known m_0, m_1 and randomly encrypting one of them $c \leftarrow \mathsf{Enc}_k(m_b)$ where $b \overset{\$}{\leftarrow} \{0,1\}$, the adversary cannot determine what the random choice bit b is. That is, $\Pr[b = b'] \leq \frac{1}{2} + \mathsf{negligible}(\lambda)$[7]. This is Chosen-Plaintext Attack (CPA) Security.

[7] λ is the security parameter, roughly a measure of how secure the protocol is. If it were exactly equal $\frac{1}{2}$, we have information-theoretic security.



For a key generated $k \leftarrow \mathsf{Gen}(1^\lambda)$.

Theoretically, for λ a security parameter and an adversary running in time $\mathsf{poly}(\lambda)$, the adversary should have distinguishing advantage $\mathsf{negligible}(\lambda)$ where

$$\mathsf{negligible}(\lambda) \ll \frac{1}{\lambda^c} \qquad \forall \text{ constant } c.$$

In practice, we set $\lambda = 128$. This means that the best algorithm to break the scheme (e.g. find the secret key) takes time $\sim 2^\lambda$. Currently, this is longer than the age of the universe.

Example 2.10

If the best algorithm is a brute-force search for k, what should our key length be?

It can just be a λ bit string.

Example

What if the best algorithm is no longer a brute-force search, but instead for a key length l takes $\sim \sqrt{2^l}$?

Our key length should be 2λ. Doing the math, we want $\sqrt{2^l} \equiv 2^\lambda$, solving for l gives 2λ.

Going back to the original problem of secret-key encryption, how can we use our newfound cryptographic constructions to improve this?

From a pseudorandom function/permutation (PRF/PRP), we can reuse our secret key by passing it through the pseudorandom function.

The current practical construction for PRD/PRP is called the block cipher, and the standardized implementation is AES[8]

It is a computational assumption[9] that the AES construction is secure, and the best attack is currently a brute-force search (in both classical and quantum computing realms).

§2.2.3 Public-Key Encryption Schemes

Using computational assumptions, we explore some public-key encryption schemes.

RSA Encryption. This is based on factoring/RSA assumption, that factoring large numbers is hard.

ElGamal Encryption. This is based on the discrete logarithm/Diffie-Hellman Assumption, that finding discrete logs in $\mathbb{Z}_p$ is hard.

Lattice-Based Encryption. The previous two schemes are not quantum secure. Quantum computation will break these schemes. Lattice-based encryption schemes are post-quantum secure. They are associated with the difficulty of finding 'short' vectors in lattices[10].

Another thing worth mentioning is that

Theorem 2.11

(Very informally,) It is impossible to construct PKE from SKE in a black-box way. This is called "black-box separation".

We first need to define a bit of number-theory background.

Definition 2.12

We denote $a \mid b$ as a divides b, that is, there is integer c such that $b = a \cdot c$.

Definition 2.13

The $\gcd(a, b)$ is the greatest common divisor of a, b. If $\gcd(a, b) = 1$, then a, b are coprime.

Question. How do we compute gcd? What is its time complexity?

[8] Determined via a competition for such an algorithm in the early 2000s.
[9] Based on heuristic, not involving any number theory!
[10] Covered later in class, we focus on the first two now.

Example

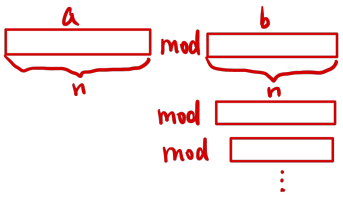
We use the Euclidean Algorithm. Take $\gcd(12, 17)$,

$$\begin{aligned} 17 &\mod 12 = 5 \\ 12 &\mod 5 = 2 \\ 5 &\mod 2 = 1 \\ 2 &\mod 1 = 0 \end{aligned}$$

or take $\gcd(12, 18)$

$$\begin{aligned} 18 &\mod 12 = 6 \\ 12 &\mod 6 = 0 \end{aligned}$$

If we have two bitstrings of length n bits, what is the running time of the Euclidan Algorithm?



Very informally, we see that every step, the length of a, b decrease by approximately 1 bit. Then, finding gcd is roughly order $O(n)$.

Definition 2.14 (Mod)

$a \mod N$ is the remainder of a when divided by N.

$a \equiv b \pmod N$ means when a and b are congruent modulo N. That is, $a \mod N = b \mod N$.

Question. How might we compute $a^b \mod N$? What is the time complexity? Let a, b, N be n-bit strings.

Naïvely, we can repeatedly multiply. But this takes b steps (2^n).

We can 'repeatedly square'. For example, we can get to a^8 faster by getting a^2, squaring to get a^4, and again to get a^8. We can take the bitstring of b and determine how to compute this.

Example

If $b = 100101_2$, we take $a \cdot a^4 \cdot a^{32} \mod N$ which can be calculated recursively (an example is given in the first assignment).

The time complexity of this is order $O(n)$ for n-bit a, b, N[11].

Theorem 2.15 (Bezout's Theorem, *roughly*)

If $\gcd(a, N) = 1$, then $\exists b$ such that

$$a \cdot b \equiv 1 \pmod{N}.$$

This is to say, a is invertible modulo N. b is its inverse, denoted as a^{-1}.

Question. How do we compute b?

We can use the Extended Euclidean Algorithm!

Example

We write linear equations of a and N that sum to 1, using our previous Euclidean Algorithm. Take the previous example $\gcd(12, 17)$,

$$\begin{aligned} 17 &\mod 12 = 5 \\ 12 &\mod 5 = 2 \\ 5 &\mod 2 = 1 \\ 2 &\mod 1 = 0 \end{aligned}$$

We write this as

$$\begin{aligned} 5 &= 17 - 12 \cdot 1 \\ 2 &= 12 - 5 \cdot 2 = 12 \cdot x + 17 \cdot y \\ 1 &= 5 - 2 \cdot 2 = 12 \cdot x' + 17 \cdot y' \end{aligned}$$

where we substitute the linear combination of 5 into 5 on line 2, substitute linear combination of 2 into 2 on line 1, each producing another linear combination of $12, 17$.

If $\gcd(a, N) = 1$, we use the Extended Euclidean Algorithm to write $1 = a \cdot x + N \cdot y$, then $1 \equiv a \cdot x \pmod{N}$.

[11] Not exactly order n, we should add the complexity of multiplication. However, this should be bounded by N since we can log at every step.

Definition 2.16 (Group of Units mod N)

We have set

$$\mathbb{Z}_N^\times := \{a \mid a \in [1, N-1], \gcd(a, N) = 1\}$$

which is the group of units modulo N (they are units since they all have an inverse by above).

Definition 2.17 (Euler's Phi Function)

Euler's phi (or totient) function, $\phi(N)$, counts the number of elements in this set. That is, $\phi(N) = |\mathbb{Z}_N^\times|$.

Theorem 2.18 (Euler's Theorem)

For all a, N where $\gcd(a, N) = 1$, we have that

$$a^{\phi(N)} \equiv 1 \pmod{N}.$$

With this, we can start talking about RSA.

§2.2.4 RSA

We first define the RSA assumption.

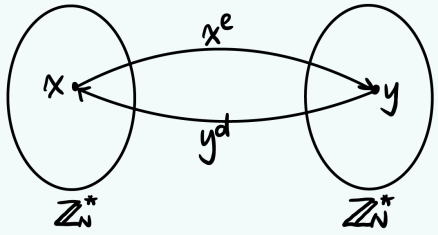
Definition 2.19 (Factoring Assumption)

Given two n-bit primes p, q, we compute $N = p \cdot q$. Given N, it's computationally hard to find p and q (classically).

Definition 2.20 (RSA Assumption)

Given two n-bit primes, we again compute $N = p \cdot q$, where $\phi(N) = (p-1)(q-1)$. We choose an e such that $\gcd(e, \phi(N)) = 1$ and compute $d \equiv e^{-1} \pmod{\phi(N)}$.

Given N and a random $y \xleftarrow{\$} \mathbb{Z}_N^\times$, it's computationally hard to find x such that $x^e \equiv y \pmod{N}$.



However, given p, q, it's easy to find d. We know $\phi(N) = (p-1)(q-1)$, so we can compute d from e by running the Extended Euclidean Algorithm. Then, taking $(x^e)^d \equiv x^{ed} \equiv x$ which allows us to extract x again.

Encrypting is exactly raising by power d, and decrypting is raising again by power e.

Remaining questions:

- How can we generate primes p, q?
- How can we pick e such that $\gcd(e, \phi(N)) = 1$?
- What security issues can you see?

We'll continue next class.

§3 February 2, 2023

§3.1 RSA Encryption, *continued*

Recall: that the RSA encryption algorithm contains 3 components:

$\mathsf{Gen}(1^\lambda)$: Generate two n-bit primes p, q. We compute $N = p \cdot q$ and $\phi(N) = (p-1)(q-1)$. Choose e such that $gcd(e, \phi(N)) = 1$. We compute $d = e^{-1} \mod \phi(N)$. Our public key $pk = (N, e)$, our secret key is $sk = d$.

$\mathsf{Enc}_{pk}(m)$: $c = m^e \mod N$.

$\mathsf{Dec}_{sk}(c)$: $m = c^d \mod N$.

We have a few remaining questions:

1. How do we generate 2 primes p, q?
2. How do we choose such an e?
3. How do we compute $d = e^{-1} \mod \phi(N)$?
4. How do we efficiently compute $m^e \mod N$ and $c^d \mod N$.

How do we resolve these issues to ensure the Gen step is efficient (polynomial time).

1. We pick an arbitrary number p and check for primality efficiently (using Miller Rabin, a probabilistic primality test). We pick random numbers until they are prime. Since primes are 'pretty dense' in the integers, this can be done efficiently.
2. We can also guess! Since we're unsure whether coprime numbers are dense, we can pick small prime e.
3. We can compute d using the Extended Euclidean Algorithm.
4. We can repeatedly square (using fast power algorithm).

A note that we have correctness with this scheme: $(m^e)^d = m^{ed} = m \mod N$.

Question. Still, are there any security issues?

- It relies on factoring being difficult (this is the computational assumption). Post-quantum, Shor's Algorithm will break RSA.

- Recall last lecture that CPA (Chosen-Plaintext Attack) security was defined as an adversary not being able to discern between an encryption of m_0 and m_1, *knowing* m_0 and m_1 in the clear.

  Eve could just encrypt m_0 and m_1 themselves using public e, and discern which of the plaintexts the ciphertext corresponds to. Using RSA, you *really* have to be careful. For RSA, this is a very concrete attack.

  The concrete reason is that the encryption algorithm Enc is *deterministic*. If you encrypt the same message twice, it will be the same ciphertext. We really want to be sure that $m \xleftarrow{\$} \mathbb{Z}_N^\times$ (that it has enough entropy).

  Returning on the RSA assumption. It's crucial that the $y \xleftarrow{\$} \mathbb{Z}_N^\times$ is randomly sampled.

Question. In practice, how can RSA be useful with these limitations?

As long as we pick the plaintext which is randomly sampled, security for RSA holds.

> **Remark.** In practice, we usually set length of p and q to be 1024 bits, and the key length is 2048 bits. This is because of better algorithms for finding big primes.

Question. Asked on Ed: What happens if $p \mid m$ (or $q \mid m$)?

Correctness still holds[12].

However, security will be broken. If $p \mid (m^2 \mod N)$, then $gcd(c, N) = p$ will factor N.

However, this is *very* unlikely! Sampling $m \xleftarrow{\$} [1, N-1]$, then

$$\Pr[p \mid m] \equiv \frac{q}{N} = \frac{1}{p} = \frac{1}{2^{\theta(\lambda)}} = \mathsf{negligible}(\lambda)$$

When sending k messages,

$$\Pr[p \mid m \text{ for any } m] \leq \frac{k}{p} \equiv \frac{\mathsf{poly}(\lambda)}{2^{\theta(\lambda)}}$$

which is still negligible. Put simply, Alice has just as much chance to break RSA by randomly factoring as she is to send a message that is a multiple of p or q.

[12] Can do out, or by using Chinese Remainder Theorem to see that it preserves the qualities we need mod p and mod q.

§3.2 Intro to Group Theory

Definition 3.1 (Group)

A group is a set $\mathbb{G}$ along with a binary operation $\circ$ with properties:

Closure. $\forall g, h \in \mathbb{G}$, $g \circ h \in \mathbb{G}$.

Existence of an identity. $\exists e \in \mathbb{G}$ such that $\forall g \in \mathbb{G}$, $e \circ g = g \circ e = g$.

Existence of inverse. $\forall g \in \mathbb{G}$, $\exists h \in \mathbb{G}$ such that $g \circ h = h \circ g = e$. We denote the inverse of g as g^{-1}.

Associativity. $\forall g_1, g_2, g_3 \in \mathbb{G}$, $(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$.

We say a group is additionally *Abelian* if it satisfies

Commutativity. $\forall g, h \in \mathbb{G}$, $g \circ h = h \circ g$.

For a finite group, we use $|\mathbb{G}|$ to denote its *order*.

Example 3.2

$(\mathbb{Z}, +)$ is an Abelian group.

We can check so: two integers sum to an integer, identity is 0, the inverse of a is $-a$, addition is associative and commutative.

$(\mathbb{Z}, \cdot)$ is not a group.

$(\mathbb{Z}_N^\times, \cdot)$ is an Abelian group ($\cdot$ is multiplication mod N).

Definition 3.3 (Cyclic Group)

Let $\mathbb{G}$ be a group of order m. We denote

$$\langle g \rangle := \{e = g^0, g^1, g^2, \ldots, g^{m-1}\}.$$

$\mathbb{G}$ is a cyclic group if $\exists g \in \mathbb{G}$ such that $\langle g \rangle = \mathbb{G}$. g is called a generator of $\mathbb{G}$.

Example

$\mathbb{Z}_p^\times$ (for prime p) is a cyclic group of order $p - 1$[13].

$$\mathbb{Z}_7^\times = \{3^0 = 1, 3^1, 3^2 = 2, 3^3 = 6, 3^4 = 5, 3^5 = 5\}.$$

Question. How do we find a generator?

For every element, we can continue taking powers until $g^\alpha = 1$ for some α. We hope that $\alpha = p - 1$ (the order of g is the order of the group), but we know at least $\alpha \mid p - 1$.

§3.3 Computational Assumptions

We have a few assumptions we make called the Diffie-Hellman Assumptions, in order of **weakest to strongest**[14] assumptions.

Let $(\mathbb{G}, q, g) \leftarrow \mathcal{G}(1^\lambda)$ be a cyclic group $\mathbb{G}$ or order q (a $\theta(\lambda)$-bit integer) with generator g. For integer groups, keys are usually 2048-bits. For elliptic curve groups, keys are usually 256-bits.

Definition 3.4 (Discrete Lgoarithm (DLOG) Assumption)

Let $x \xleftarrow{\$} \mathbb{Z}_q$. We compute $h = g^x$.

Given $(\mathbb{G}, q, g, h)$, it's computationally hard to find the exponent x (classically).

Definition 3.5 (Computational Diffie-Hellman (CDH) Assumption)

$x, y \xleftarrow{\$} \mathbb{Z}_q$, compute $h_1 = g^x$, $h_2 = g^y$.

Given $(\mathbb{G}, q, g, h_1, h_2)$, it's computationally hard to find g^{xy}.

Definition 3.6 (Decisional Diffie-Hellman (DDH) Assumption)

$x, y, z \xleftarrow{\$} \mathbb{Z}_q$. Compute $h_1 = g^x$, $h_2 = g^y$.

Given $(\mathbb{G}, q, g, h_1, h_2)$, it's computationally hard to distinguish between g^{xy} and g^z.

$$(g^x, g^y, g^{xy}) \overset{c}{\simeq} (g^x, g^y, g^z).$$

§3.4 ElGamal Encryption

The ElGamal encryption scheme involves the following:

[13] A proof of this extends beyond the scope of this course, but you are recommended to check out Math 1560 (Number Theory) or Math 1580 (Cryptography). You can take this on good faith.

[14] If one can solve DLOG, we can solve CDH. Given CDH, we can solve DDH. This is why CDH is *stronger* than DDH, and DDH is *stronger* than DLOG. It's not necessarily true the other way around (similar to factoring and DSA assumptions).

$\mathsf{Gen}(1^\lambda)$: We generate a group $(\mathbb{G}, q, g) \leftarrow \mathcal{G}(1^\lambda)$. We sample $x \xleftarrow{\$} \mathbb{Z}_q$, compute $h = g^x$. Our public key is $pk = (\mathbb{G}, q, g, h)$, secret key $sk = x$.

$\mathsf{Enc}_{pk}(m)$: We have $m \in \mathbb{G}$. We sample $y \xleftarrow{\$} \mathbb{Z}_q$. Our ciphertext is $c = \langle g^y, h^y \cdot m \rangle$. Note that $h = g^x$, so $g^{xy} \overset{c}{\simeq} g^z$ is a one-time pad for our message m.

$\mathsf{Dec}_{sk}(c)$: To decrypt $c = \langle c_1, c_2 \rangle$, we raise

$$c_1^x = (g^y)^x = g^{xy}$$
$$m = \frac{g^{xy} \cdot m}{g^{xy}} = c_2 \cdot (c_1^x)^{-1}.$$

Notes about ElGamal:

- Our group can be reused! We can use a public group that is fixed. In fact, there are *popular* groups out there used in practice. Some of these are Elliptic Curve groups which are much more efficient than integer groups. You don't need to use the details, yet you can use it! You can use any group, so long as the group satisfies the DDH assumption.
- Similar to RSA, this is breakable post-quantum. Given Shor's Algorithm, we can break discrete log.

§3.5 Secure Key Exchange

Using DDH, we can construct something very important, *secure key exchange*.

Definition 3.7 (Secure Key Exchange)

Alice and Bob sends messages back and forth, and at the end of the protocol, can agree on a shared key.

An eavesdropper looking at said communications cannot figure out what shared key they came up with.

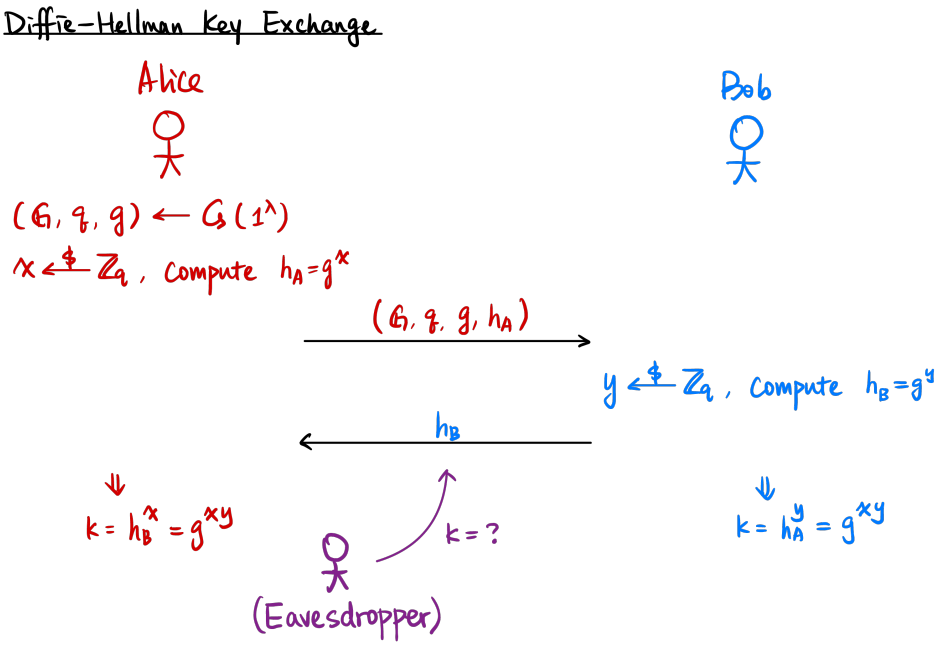
Theorem 3.8

Informally, It's impossible to construct secure key exchange from secret-key encryption in a black-box way.

Question. How do we build a key exchange from public-key encryption?

Bob generates a keypair (pk, sk). Alice generates a shared key $k \xleftarrow{\$} \{0,1\}^\lambda$, and sends $\mathsf{Enc}_{pk}(k)$ to Bob.

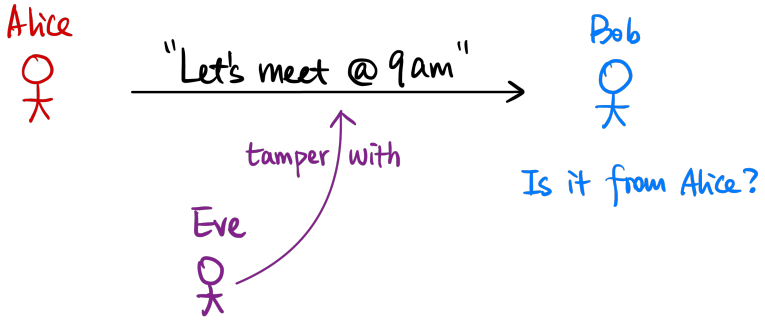
Using Diffie-Hellman, it's very easy. We have group $(\mathbb{G}, q, g) \leftarrow \mathcal{G}(1^\lambda)$. Alice samples $x \xleftarrow{\$} \mathbb{Z}_q$ and sends g^x. Bob also samples $y \xleftarrow{\$} \mathbb{Z}_q$ and sends g^y. Both Alice and Bob compute $g^{xy} = (g^x)^y = (g^y)^x$.



What happens in practice is that parties run Diffie-Hellman key exchange to agree on a shared key. Using that shared key, they run symmetric-key encryption. This gives us efficiency. Additionally, private-key encryptions don't rely on heavy assumptions on the security of protocols (such as the DDH, RSA assumptions).

§3.6 Message Integrity

Alice sends a message to Bob, how does Bob ensure that the message came from Alice?



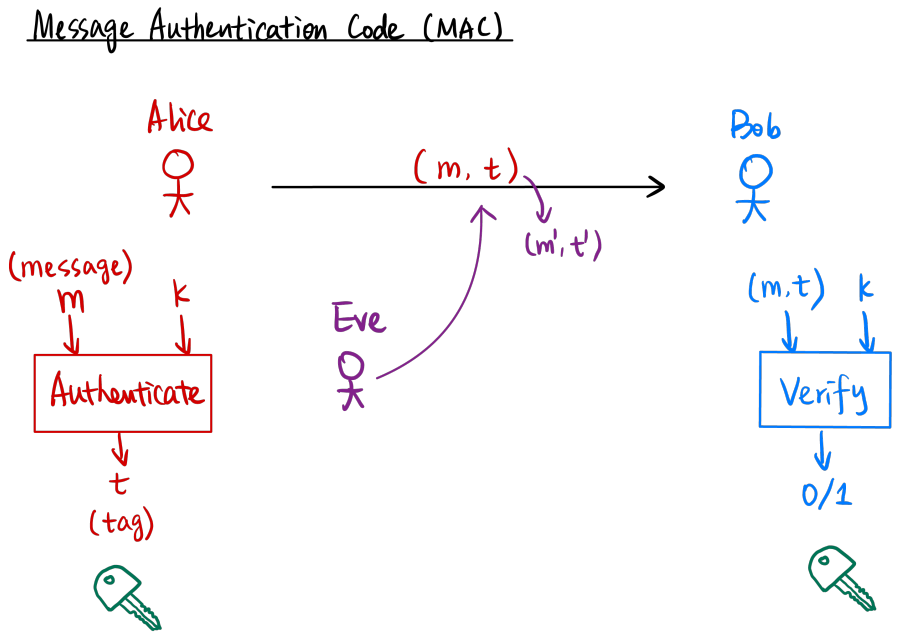
We can build up another line of protocols to ensure message integrity. It's similar to encryption, but the parties run 2 algorithms: *Authenticate* and *Verify*.

Using a message m, Alice can generate a *tag* or *signature*, and Bob can verify (m, t) is either valid or invalid.

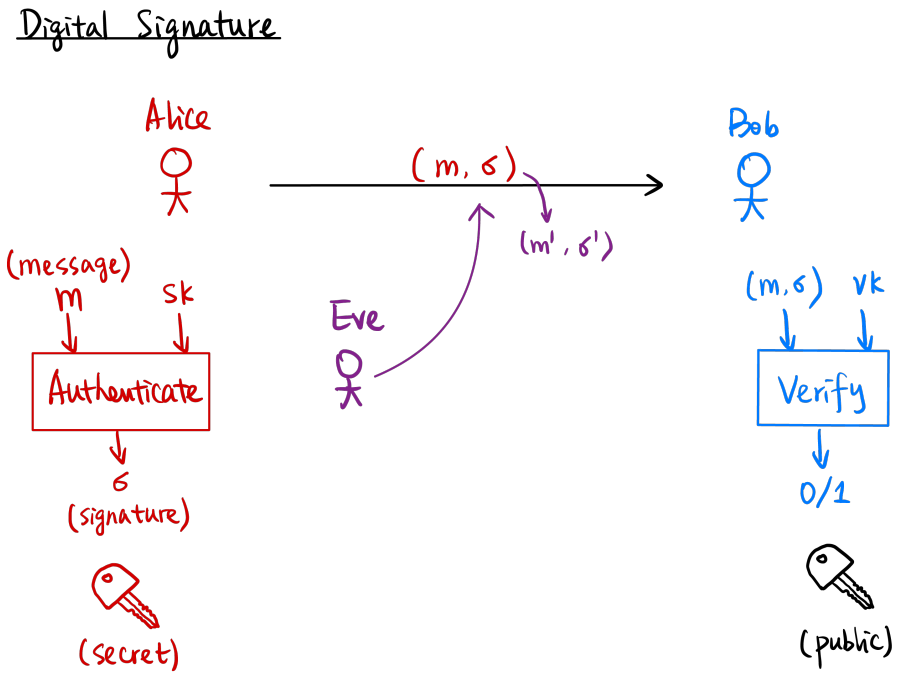
Our adversary has been upgraded to an Eve who can now tamper with messages.

Similarly to encryption, we have symmetric-key and public-key encryption.

Using a shared key k, Alice can authenticate m using k to get a tag k. Similarly, Bob can verify whether (m, t) is valid using k. This is called a Message Authentication Code.



Using a public key vk (verification key) and private key sk (secret/signing key), Alice can sign a message m using signing key sk to get a *signature* σ. Bob verifies (m, σ) is valid using vk. This is called a Digital Signature.



§3.6.1 Syntax

The following is the syntax we use for MACs and digital signatures.

A message authentication code (MAC) scheme consists of $\Pi = (\mathsf{Gen}, \mathsf{Mac}, \mathsf{Verify})$.

Generation. $k \leftarrow \mathsf{Gen}(1^\lambda)$.

Authentication. $t \leftarrow \mathsf{Mac}_k(m)$.

Verification $0/1 := \mathsf{Verify}_k(m, t)$.

A digital signature scheme consists of $\Pi = (\mathsf{Gen}, \mathsf{Sign}, \mathsf{Verify})$.

Generation. $(sk, vk) \leftarrow \mathsf{Gen}(1^\lambda)$.

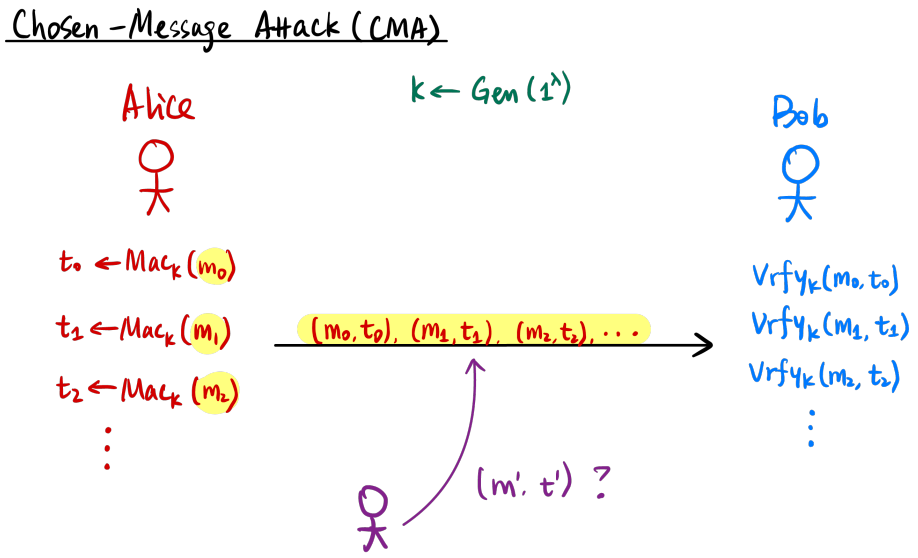
Authentication. $\sigma \leftarrow \mathsf{Sign}_{sk}(m)$.

Verification $0/1 := \mathsf{Verify}_{vk}(m, \sigma)$.

§3.6.2 Chosen-Message Attack

Similar to chosen-plaintext attack from encryption, we have chosen-message attack security. An adversary chooses a number of messages to generate signatures or tags for. After that, the adversary

will try to generate another valid pair of message and tag. We want to make sure that generating a new pair of message and tag is hard.



§3.6.3 Constructions

Very briefly, we discuss constructions for MAC and digital signatures.

Using block ciphers, we have CBC-MAC. Using a hash function, we have HMAC.

For digital signatures, we have RSA which relies on the RSA assumption, or DSA which relies on discrete-log algorithms. There are also lattice signature schemes for post-quantum digital signatures.

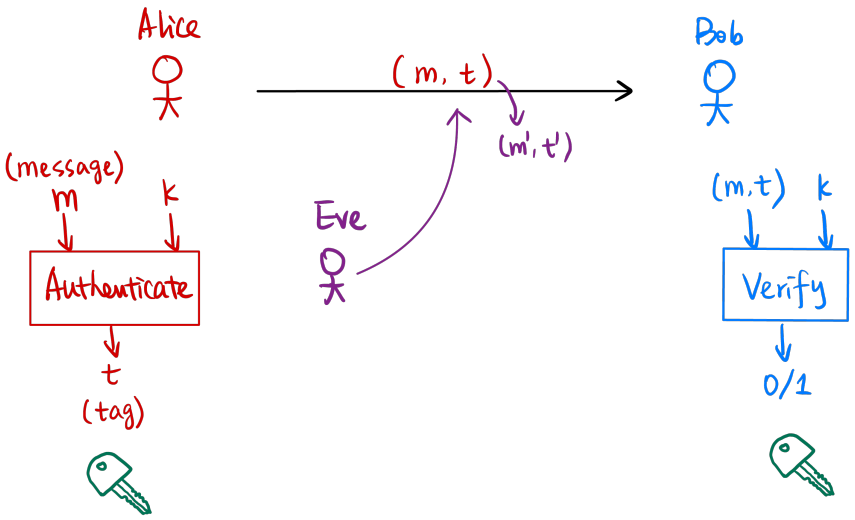
§4 February 7, 2023

§4.1 Message Integrity, *reviewed*

Last lecture, we discussed methods of authenticating a message. The symmetric-key version is called a MAC (message authentication code), the public-key version is called a digital signature. Let's review what we covered last time.

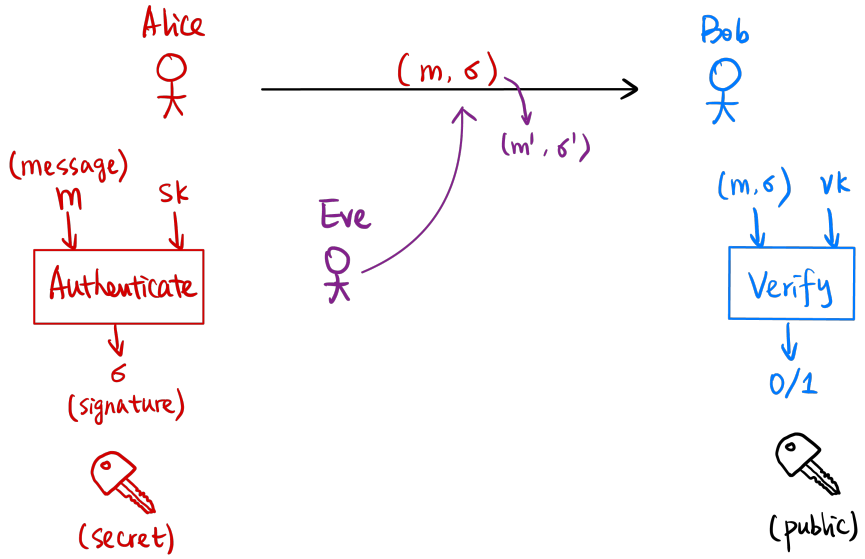
§4.1.1 Message Authentication Code

To authenticate a message, Alice will use the private key k to tag a message m with a tag t. Bob will verify that (m, t) is valid with key k.



§4.1.2 Digital Signature

In the public-key version, Alice has a secret key sk to sign message m with signature σ. Bob (or anyone) can verify with the public key pk that (m, σ) is a valid signature.



§4.1.3 Syntax

Recall the syntax of MAC and digital signatures (see section 3.6.1).

A message authentication code (MAC) scheme consists of $\Pi = (\mathsf{Gen}, \mathsf{Mac}, \mathsf{Verify})$.

Generation. $k \leftarrow \mathsf{Gen}(1^\lambda)$.

Authentication. $t \leftarrow \mathsf{Mac}_k(m)$.

Verification $0/1 := \mathsf{Verify}_k(m, t)$.

A digital signature scheme consists of $\Pi = (\mathsf{Gen}, \mathsf{Sign}, \mathsf{Verify})$.

Generation. $(sk, vk) \leftarrow \mathsf{Gen}(1^\lambda)$.

Authentication. $\sigma \leftarrow \mathsf{Sign}_{sk}(m)$.

Verification $0/1 := \mathsf{Verify}_{vk}(m, \sigma)$.

§4.1.4 Constructions

We can construct MAC pratically using

- Block Cipher: CBC-MAC
- Hash Functions: HMAC

We can construct digital signatures using

- RSA signature: RSA Assumption.
- DSA signature: Discrete Log Assumption
- Lattice-Based Encryption Schemes (post-quantum secure).

§4.2 RSA Signatures

Our RSA signatures algorithm works very similarly to RSA encryption.

We generate two n-bit primes p, q. Compute $N := p \cdot q$ and $\phi(N) = (p-1)(q-1)$. Again choose e with $\gcd(e, \phi(N)) = 1$ and invert $d = e^{-1} \mod \phi(N)$. Given N and a random $y \xleftarrow{\$} \mathbb{Z}_N^\times$, it's computationally hard to find x such that $x^e \equiv y \mod N$.

Similarly, $sk := d$ and $vk := (N, e)$. To sign, we compute

$$\mathsf{Sign}_{sk}(m) := m^d \mod N.$$

To verify, we compute

$$\mathsf{Verify}_{vk}(m, \sigma) := \sigma^e \stackrel{?}{\equiv} m \mod N.$$

Question. Are there any security issues with RSA as we have constructed it so far?

Thinking back to our definition of chosen-message attack, if Eve knows many messages and signatures, she can compute another pair of valid message and keys. If we have messages

$$m_0, \sigma_0 = m_0^d \mod N$$
$$m_1, \sigma_1 = m_0^d \mod N$$

We can compute $m^* := m_0 \cdot m_1$ and $\sigma^* := \sigma_0 \cdot \sigma_1 = (m_0 \cdot m_1)^d \mod N$.

We can do linear combinations of messages, as well as raising messages to arbitrary exponents, and we can get other messages with valid signatures.

There is an easy solution, however. We can hash our message m before we sign, like so

$$\mathsf{Sign}_{sk}(m) := H(m)^d \mod N$$
$$\mathsf{Verify}_{vk}(m, \sigma) := \sigma^e \stackrel{?}{\equiv} H(m) \mod N$$

where H is a hash function[15]. This is a commonly known technique called 'hash-and-sign'.

[15] A hash function is, briefly, a function that produces some random output that is hard to compute the inverse of.

§4.3 DSA Signatures

DSA signatures are a bit more involved; they rely on the discrete log assumption and that it is hard to compute discrete logs in $\mathbb{Z}_p^\times$ and $\mathbb{Z}_q^\times$ a subgroup.

We give this definition using integer groups. There is also ECDSA which uses elliptic curve groups, which is used in cryptocurrencies[16]. Elliptic curves are much more efficient (especially when chosen correctly) and provide similar security guarantees.

$\mathsf{Gen}(1^k)$: We generate an n-bit[17] prime p and an m-bit[18] prime q such that $q \mid (p-1)$. We pick a random $\alpha \in \mathbb{Z}_p^\times$ such that $g = \alpha^{\frac{p-1}{q}} \mod p \neq 1$[19]

We pick $x \xleftarrow{\$} \mathbb{Z}_q^\times$ and compute $h = g^x \mod p$.

Our verification key is $vk := (p, q, g, h)$ and our signing key is $sk = x$.

$\mathsf{Sign}_{sk}(m)$: Sample $y \xleftarrow{\$} \mathbb{Z}_q^\times$, compute $r = (g^y \mod p) \mod q$. Compute $s = y^{-1} \cdot (H(m) + x \cdot r) \mod q$ and $\sigma := (r, s)$.

$\mathsf{Verify}_{vk}(m, \sigma)$: We can compute

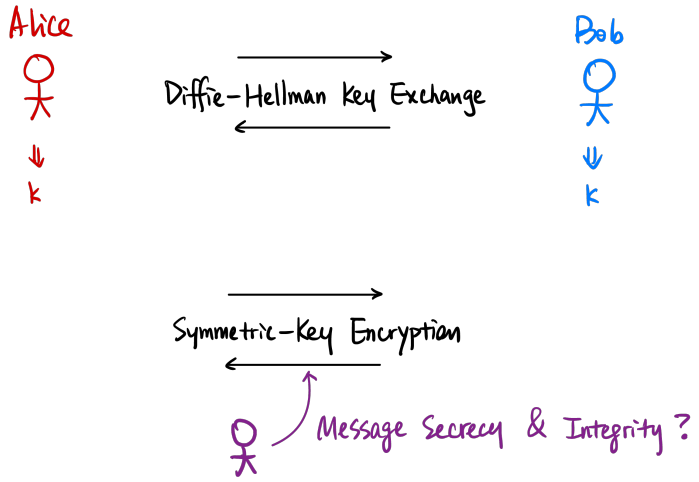
$$\begin{aligned} w &= s^{-1} \mod q \\ u_1 &= H(m) \cdot w \mod q \\ u_2 &= r \cdot w \mod q \end{aligned}$$

and verify $r \stackrel{?}{=} (g^{u_1} \cdot h^{u_2} \mod p) \mod q$.

§4.4 Authenticated Encryption

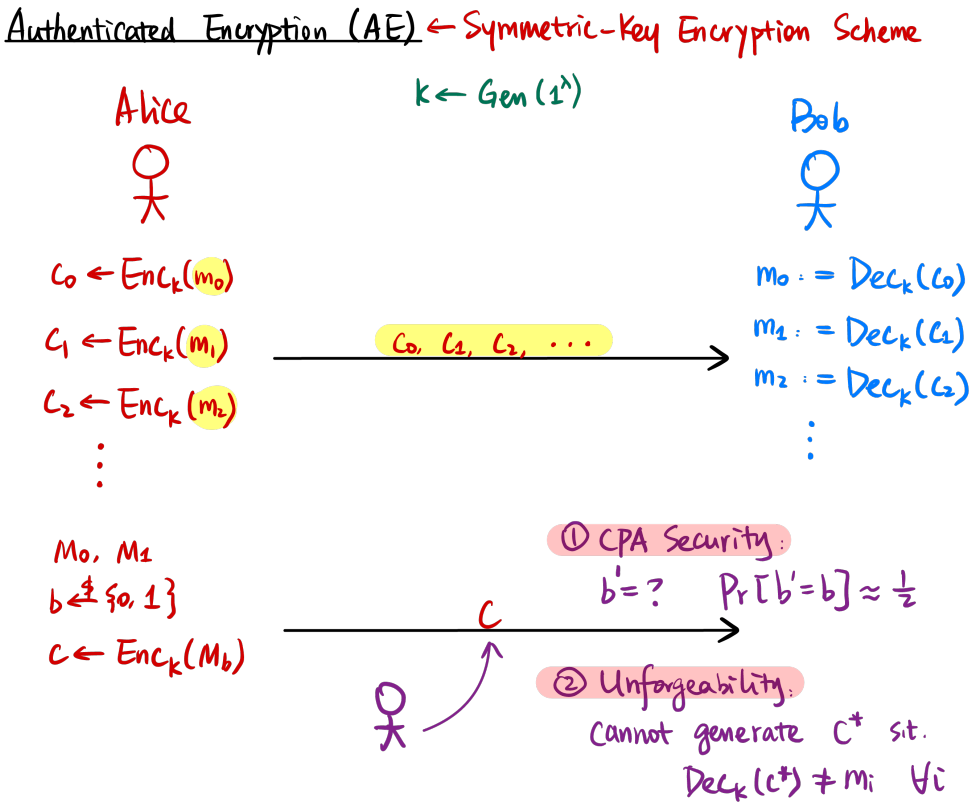
Generally, Alice and Bob will first perform a Diffie-Hellman key exchange, then use that shared key to conduct Symmetric-Key Encryption.

[16] The specific curve used by Bitcoin, for example, is called **secp256k1**.
[17] Usually, 2048.
[18] Usually, 256. We assume discrete log in *both* of these groups.
[19] This means that $\langle g \rangle$ has order q, a subgroup in $\mathbb{Z}_p^{-1}$. $g^q \equiv \alpha^{p-1} \equiv 1 \mod p$.



In reality, we want to achieve both message secrecy and integrity *at the same time*. For this, we can introduce Authenticated Encryption.

Our security definition is that our adversary can see the encryptions of many messages m_0, m_1, m_2. We want **CPA security**: given an encryption of either m_0, m_1, our adversary cannot distinguish between encryptions of the two. *Additionally*, we want the property of **unforgeability**, that our adversary cannot generate a c^* that is a valid encryption, such that $\mathsf{Dec}_k(c^*) \neq m_i$ for any i.

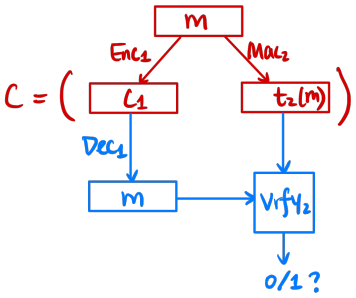


Now that we have two new primitives, we can construct Authenticated Encryption schemes.

§4.4.1 Encrypt-and-MAC?

Given a CPA-secure SKE scheme $\Pi_1(\mathsf{Gen}_1, \mathsf{Enc}_1, \mathsf{Dec}_1)$ and a CMA-secure MAC scheme $\Pi_2 = (\mathsf{Gen}_2, \mathsf{Mac}_2, \mathsf{Verify}_2)$.

We construct an AE scheme $\Pi = (\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec})$ by composing encryption and MAC. We encrypt the plaintext and also compute the MAC the *plaintext*.



$\mathsf{Gen}(1^\lambda)$: $k_1 := \mathsf{Gen}_1(1^\lambda)$, $k_2 := \mathsf{Gen}_2(1^\lambda)$, output (k_1, k_2).

$\mathsf{Enc}(m)$: To encrypt, we first encrypt ciphertext $c_1 := \mathsf{Enc}_1(k_1, m)$ and sign message (in plaintext) $t_2 := \mathsf{Mac}_2(k_2, m)$ and output (c_1, t_2).

$\mathsf{Dec}(m)$: We have ciphertext $c = (c_1, t_2)$. Our message is $m := \mathsf{Dec}_1(k_1, c_1)$, and our verification bit $b := \mathsf{Verify}_1(k_2, (m, t_2))$. If $b = 1$, output m, otherwise we output $\perp$.

Question. Is this scheme secure? Assuming the CPA-secure SKE scheme and CMA-secure MAC scheme, does this give us both CPA-security and unforgeability?

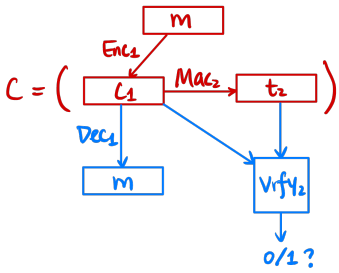
MAC gives you *unforgeability*—it doesn't even try to hide the message at all. It's possible that the MAC scheme reveals the message in the clear. For example, we might have a MAC scheme that includes the message in the signature *in the clear* (which is still secure)!

Since MAC doesn't try to hide the message. If our MAC reveals something about our message, our composed scheme Π doesn't give us CPA-security. You might still be able to infer something about the message.

We try something else...

§4.4.2 Encrypt-then-MAC

We encrypt first, then we MAC on the *ciphertext*.



$\mathsf{Gen}(1^\lambda)$: $k_1 := \mathsf{Gen}_1(1^\lambda)$, $k_2 := \mathsf{Gen}_2(1^\lambda)$, output (k_1, k_2).

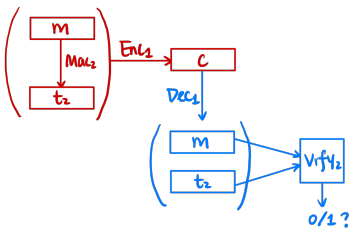
$\mathsf{Enc}(m)$: To encrypt, we first encrypt ciphertext $c_1 := \mathsf{Enc}_1(k_1, m)$ and sign message (in ciphertext) $t_2 := \mathsf{Mac}_2(k_2, c_1)$ and output (c_1, t_2).

$\mathsf{Dec}(m)$: We have ciphertext $c = (c_1, t_2)$. Our message is $m := \mathsf{Dec}_1(k_1, c_1)$, and our verification bit $b := \mathsf{Verify}_1(k_2, (c_1, t_2))$. If $b = 1$, output m, otherwise we output $\perp$.

You can prove that Encrypt-then-MAC schemes are CPA-secure and unforgeable.

§4.4.3 MAC-then-Encrypt

Similarly, we can also MAC first, encrypt the entire ciphertext and tag concatenated.



$\mathsf{Gen}(1^\lambda)$: $k_1 := \mathsf{Gen}_1(1^\lambda)$, $k_2 := \mathsf{Gen}_2(1^\lambda)$, output (k_1, k_2).

$\mathsf{Enc}(m)$: To encrypt, we first encrypt ciphertext $c_1 := \mathsf{Enc}_1(k_1, m)$ and sign message (in plaintext) $t_2 := \mathsf{Mac}_2(k_2, m||t_2)$ and output (c_1, t_2).

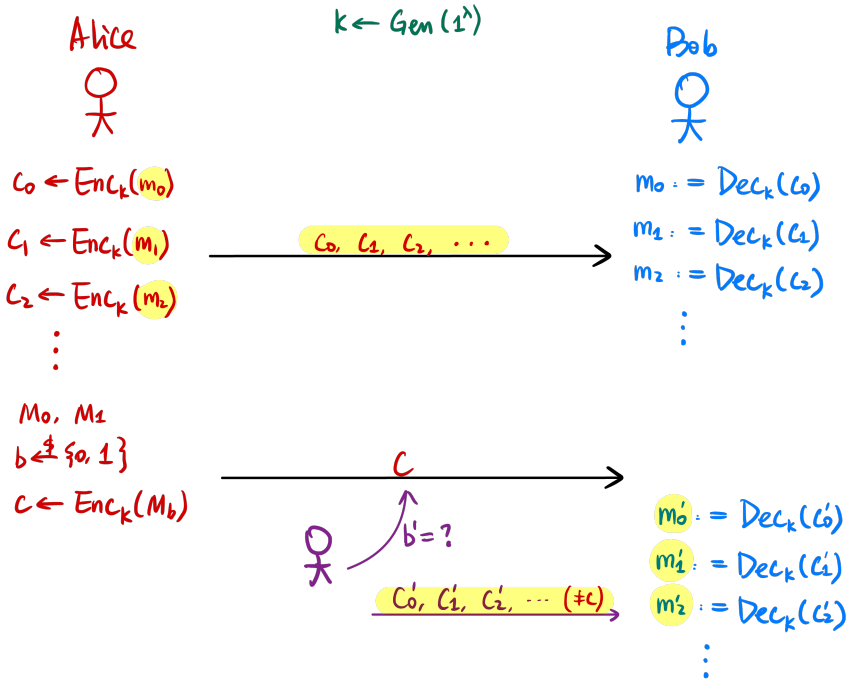
$\mathsf{Dec}(m)$: We have ciphertext $c = (c_1, t_2)$. Our message is $m||t_2 := \mathsf{Dec}_1(k_1, c_1)$, and our verification bit $b := \mathsf{Verify}_1(k_2, (m, t_2))$. If $b = 1$, output m, otherwise we output $\perp$.

Question. Is this secure?

This doesn't satisfy a stronger security definition called Chosen Ciphertext Attack (CCA) security. We might be able to forge ciphertexts that decrypt to valid message and tags.

§4.4.4 Chosen Ciphertext Attack Security

On top of CMA security, the adversary can now request Alice to decrypt ciphertexts $c_0, c_1, \ldots$.



We can prove that MAC-then-Encrypt is not CCA secure.

The moral of this is that **you should always use Encrypt-then-MAC.**

§4.5 A Summary So Far

To summarize, here's all we've covered so far:

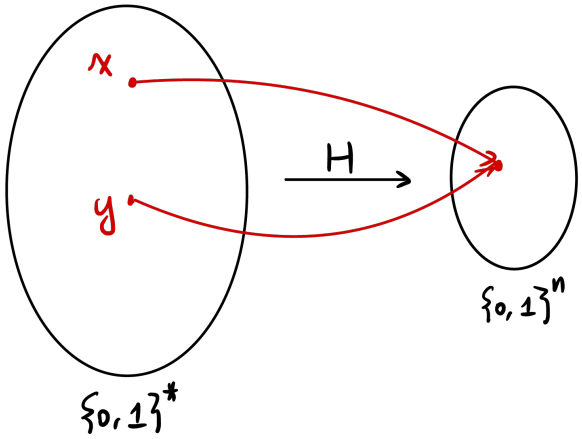
	Symmetric-KePy	**Public-Key**
Message Secrecy	Primitive: SKE Construction: Block Cipher	Primitive: PKE Constructions: RSA/ElGamal
Message Integrity	Primitive: MAC Constructions: CBC-MAC/HMAC	Primitive: Signature Constructions: RSA/DSA
Secrecy & Integrity	Primitive: AE Construction: Encrypt-then-MAC	
Key Exchange		Construction: Diffie-Hellman
Important Tool	Primitive: Hash function Construction: SHA	

§4.6 Hash Function

A hash function is a public function

$$H : 0,1^* \to 0,1^n$$

where n is order $\Theta(\lambda)$.



We want our hash function to be collision-resistant. That is, it's computationally hard to find $x, y \in \{0,1\}^*$ such that $x \neq y$ yet $H(x) = H(y)$ (which is called a collision).

How might one find a collision for function $H : \{0,1\}^* \to \{0,1\}^n$. We can try $H(x_1), H(x_2), \ldots, H(x_q)$.

If $H(x_1)$ outputs a random value, $0,1^n$, what is the probability of finding a collision?

If $q = 2^n + 1$, our probability is exactly 1 (by pigeon-hole). If $q = 2$, our probability is $\frac{1}{2^\lambda}$ (we have to get it right on the first try). What q do we need for a 'reasonable' probability?

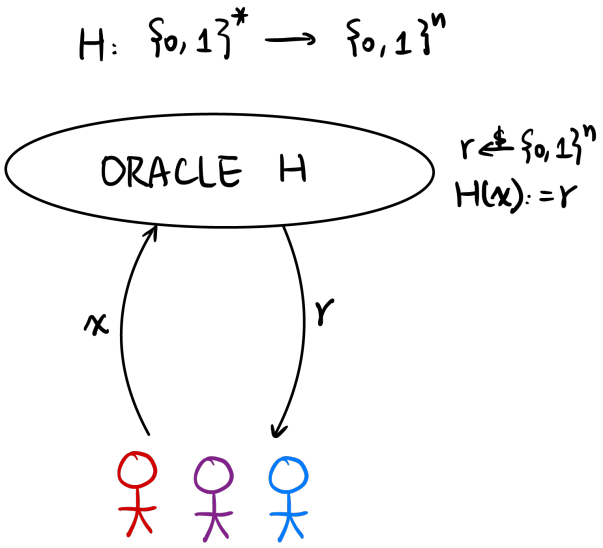
Remark. This is related to the birthday problem. If there are q students in a class, assume each student's birthday is a random day $y_i \xleftarrow{\$} [365]$. What is the probability of a collision? $q = 366$ gives 1, $q = 23$ gives around 50%, and $q = 70$ gives roughly 99.9%.

We can apply this trick to our hash function. If $y_i \xleftarrow{\$} [N]$, then $q = N + 1$ gives us 100%, but $q = \sqrt{N}$ gives 50% probability.

Knowing this, we want $n = 2\lambda$ (output length of hash function). If $\lambda = 128$, we want n to be around 256.

§4.6.1 Random Oracle Model

Another way to model a hash function is the *Random Oracle Model*. We think of our hash function to be an oracle (in the sky) that can *only* take input and a random output (and if you give it the same input twice, the same output).



There are proofs that state that no hash functions can be a random oracle. There are schemes that can be secure in the random oracle model, but are not using hash functions[20].

In reality, hash functions are *about as good as*[21] random oracles. Thinking of our hash functions as random oracles gives us a good intuitive understanding of how hash functions can be used in our schemes.

In this model, the best thing that an attacker can do is to try inputs and query for outputs.

[20] Some constructions don't rely on this model.
[21] But can never be...

§4.6.2 Constructions for Hash Function

MDS. Output length 128-bit. Best known attack is in 2^{16}. A collision was found in 2004.

And we also have Secure Hash Functions (SHA), founded by NIST.

SHA-0. Standardized in 1993. Output length is 160-bit. Best known attack is in 2^{39}.

SHA-1. Standardized in 1995. Output length is 160-bit. Best known attack is in 2^{63}, and a collision *was* found in 2017.

SHA-2. Standardized in 2001. Output length of 224, 256, 284, 512-bit. The most commonly used is SHA-256.

SHA-3. There was a competition from 2007-2012 for new hash functions. SHA-3 was released in 2015, and has output length 224, 256, 2384, 512-bit. This is *completely different* from SHA-2.

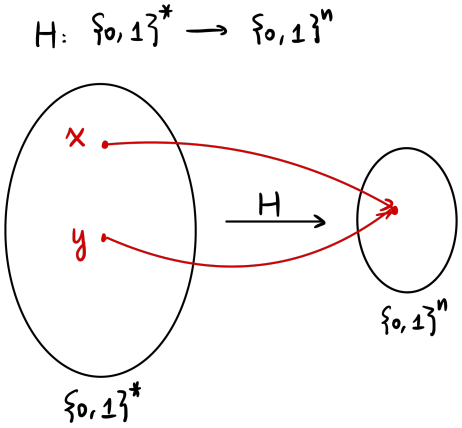
> **Remark.** The folklore is that during a session at a cryptography conference, a mathematician, Xiaoyun Wang, presented slide-after-slide of attacks on MDS and SHA-0, astounding the audience.

§5 February 9, 2023

Logistics: Please let us know if you have outstanding Ed or Gradescope issues!

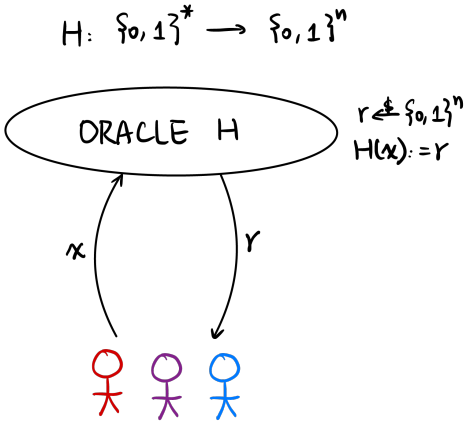
§5.1 Hash Functions, *continued*

Recall that we defined a hash function to be a function for which it is computationally hard to find a collision. That is, finding two distinct strings x, y such that $H(x) = H(y)$ is computationally difficult.



One model that we sometimes use to facilitate our analysis is the random oracle model. We assume our hash function is a random oracle 'in the sky'that produces random hashes.

By guessing, analyzing it via the birthday problem, we require time approximately $\sqrt{n}$.



§5.1.1 Constructions

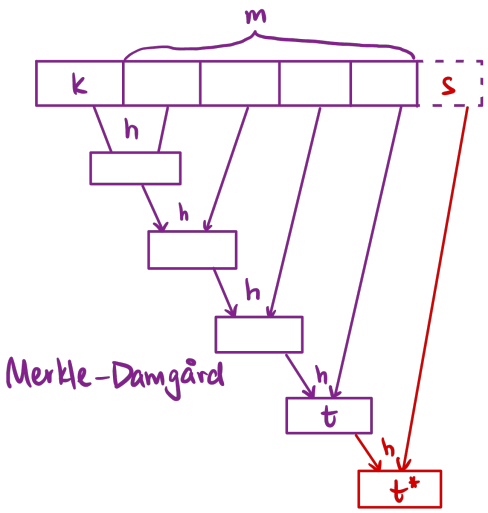
The hash function constructions that are still in practice (and unbroken) are SHA-2 and SHA-3.

§5.1.2 Applications

HMAC. We can use a hash function to conduct a MAC. Computing a tag involves computing the hash function on the key appended to the message $(k||m)$. It is computationally difficult to find another $k||m'$ that produces the same hash. This is a scheme that looks like

$$\mathsf{Mac}_k(m) = H(k||m).$$

However, an adversary could potentially attach some additional s to m to produce $m' = m||s$ such that they can easily compute $\mathsf{tag}' = H(\mathsf{tag}||s)$. This is due to the Merkle-Damgård construction of SHA-2, which associatively tags blocks of the message one-by-one.



Therefore, in practice, we use a nested MAC like

$$\mathsf{Mac}_k(m) = H(k||H(k||m))$$

and just to be sure (that we're not reusing the key), we produce k_1, k_2 as such

$$\mathsf{HMAC}_k(m) = H(k_1||H(k_2||m))$$

such that $k_1 = k \oplus \mathsf{opad}$ and $k_2 = k \oplus \mathsf{ipad}$, some one-time pads.

Hash-and-Sign. There are some other applications of a hash function. We've seen before with RSA that we want to Hash-and-Sign, removing any homomorphism that an adversary could exploit. Additionally, this allows us to sign larger messages since they are constant size after hashing.

Password Authentication. Another application is password authentication. Instead of storing plaintext passwords on servers, websites can store a hash of the password instead. This means that the passwords are not compromised even if the server is compromised.

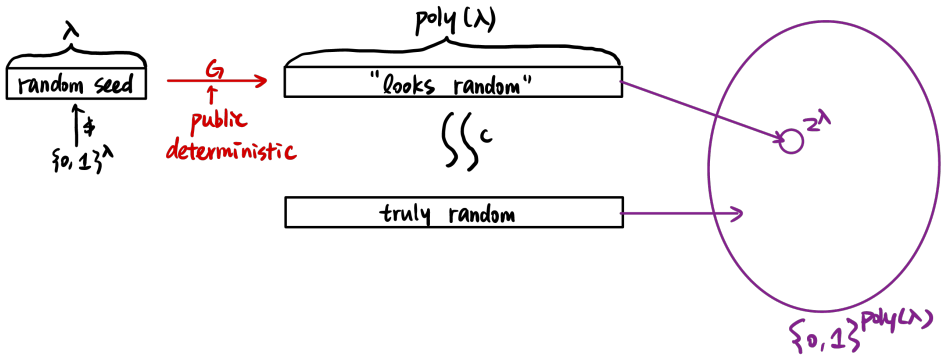
Deduplicate Files. We can also use hash functions to deduplicate files. We can hash two files to produce identifiers h_1 and h_2. If $h_1 \neq h_2$, this implies $D_1 \neq D_2$. If $h_1 = h_2$, it almost always[22] implies that $D_1 = D_2$.

HKDF (Key Derivation Function). We can derive more keys from a shared key, essentially using a hash function as a pseudorandom generator (PRG).

For example, if there is g^{ab} shared key, we can do

$$\mathsf{HMAC}(g^{ab}, \mathsf{salt})$$

Using a random seed, and adding a public deterministic salt G, we can generate a random[23] string.



Given a hash function H, we can generate a PRG easily for any length string by generating

$$\begin{aligned} \mathsf{seed} \xleftarrow{\$} \{0,1\}^\lambda \longrightarrow & H(\mathsf{seed}||00\cdots00) \\ & H(\mathsf{seed}||00\cdots01) \\ & H(\mathsf{seed}||00\cdots10) \\ & \vdots \end{aligned}$$

We can take a bit of randomness (like the way we move our mouse, type keyboard, system properties) and generate our seed.

Fast Membership Proof (Merkle Tree). Using hash functions, we can generate Merkle Trees to prove membership. In blockchains, this is equivalent to checking if a transaction occurred.

[22] If they are not equal, we've found a collision for our hash function, which is extremely unlikely.
[23] Computationally random, because if our computational power were to be unbounded, we can try all strings.

SKE Scheme? Could we use this to encrypt? If we have a secret key $k \xleftarrow{\$} \{0,1\}^\lambda$, can we just encrypt by

$$\mathsf{Enc}_k(m) = H(k||m)$$

Well, we can't decrypt for one without having unbounded computational power. If our plaintext m comes from a small set, like $\{0, \ldots, 10\}$, we could decrypt properly. However, this is not CPA-secure, since the adversary could just query for all the messages.

> **Remark 5.1.** In general, all deterministic encryption schemes are not CPA-secure.

§5.2 Putting it Together: Secure Communication

This is essentially what we want to do in the second project.

We use Diffie-Hellman Key Exchange between Alice and Bob to get shared g^{ab}. Hashing the shared key using an HKDF, we can get shraed key $k = (k_1, k_2)$ (one for AES one for HMAC). Then, they perform authenticated encryption, namely Encrypt-then-MAC.

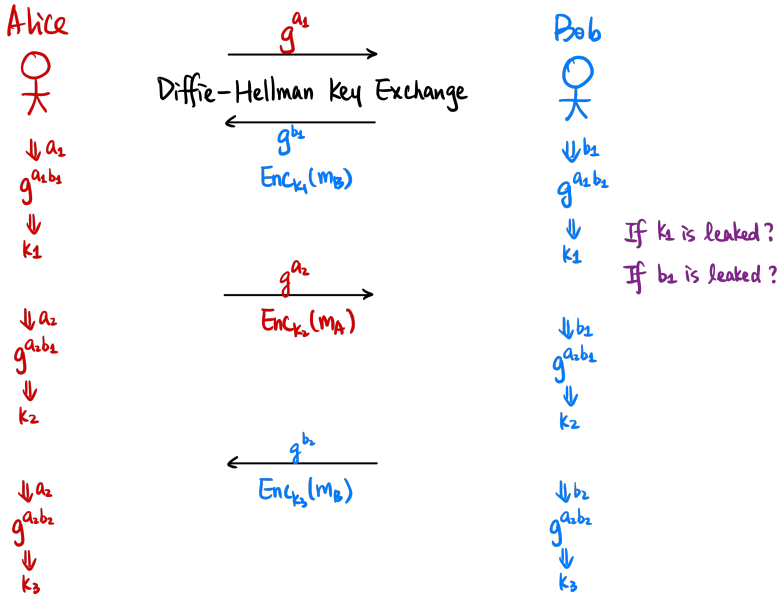
Question. Are there any issues with this scheme?

An Eve could pretend to be Alice to Bob and Bob to Alice, fudging up their shared keys. This is called a *Man-in-the-Middle* attack.

§5.2.1 Diffie-Hellman Ratchet

What if a secret key gets leaked, or cracked? One simple way to fix this is to perform a Diffie-Hellman key exchange on every message. However, this incurs additionall communications costs.

Here's another idea: with every new message (when the direction of communications shifts), the party sending the message sends a new Diffie-Hellman public key for themselves. For example, if Bob is sending a message to Alice and he knows Alice's public key g^{a_1} and his previous secret was b_1 (hence shared $g^{a_1 b_1}$), Bob will generate new key b_2, g^{b_2} and encrypt using $g^{a_1 b_2}$, sending g^{b_2} as public to Alice. Alice can recompute the shared key before decrypting.



This is the protocol used in the Signal messaging app, and is what you will implement for Project 1.

Question. What if k_1 is leaked?

We might have leaked one key, but the other keys are still computationally hard to compute. $k_1 = g^{a_1 b_1}$ is known, but it's equivalent to DDH to compute $g^{a_1 b_2}$ or other keys.

Question. What if b_1 is leaked?

We can compute key $k_1 = g^{a_1 b_1}$ and $k_2 = g^{a_2 b_1}$, but no further keys are leaked, and the next round of communications (after Bob refreshes his private key b_2) is still secure.

§5.3 Block Cipher

To summarize, here's what we've seen so far (this table should be familiar):

	Symmetric-Key	**Public-Key**
Message Secrecy	Primitive: SKE Construction: **Block Cipher**	Primitive: PKE Constructions: RSA/ElGamal
Message Integrity	Primitive: MAC Constructions: CBC-MAC/HMAC	Primitive: Signature Constructions: RSA/DSA
Secrecy & Integrity	Primitive: AE Construction: Encrypt-then-MAC	
Key Exchange		Construction: Diffie-Hellman
Important Tool	Primitive: Hash function Construction: SHA	

The only thing we haven't seen thus far is a block cipher. We first start with the definitions.

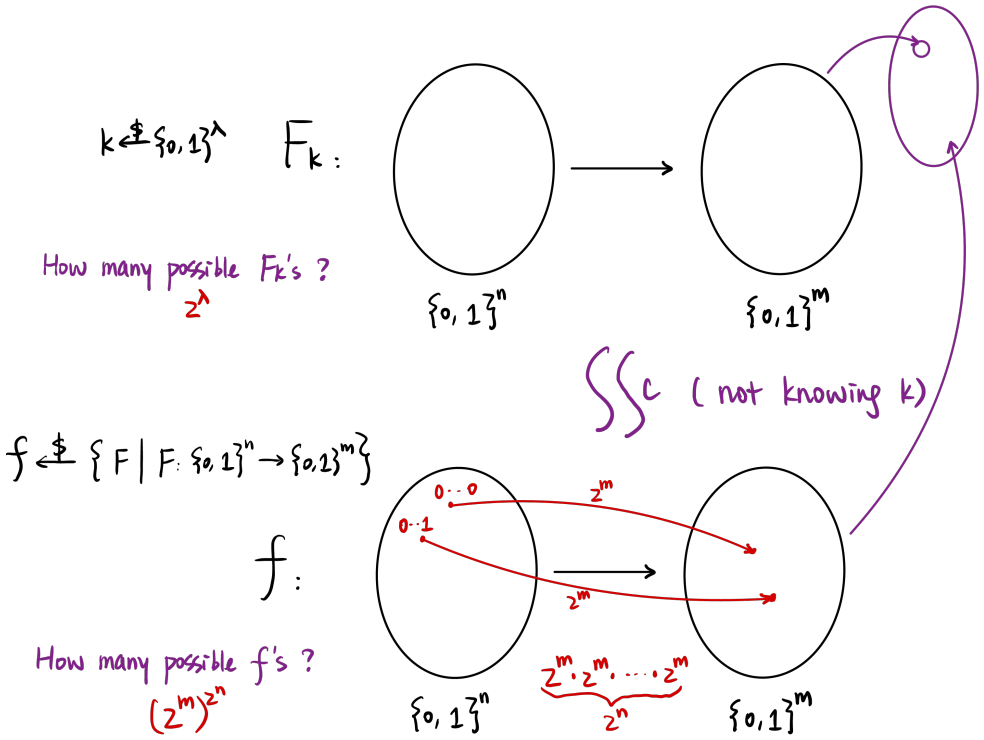
We saw earlier that a Pseudorandom Generator (PRG) produces a string that looks random. We also have Pseudorandom Functions (PRF), which are 'random-looking' functions.

§5.3.1 Pseudorandom Function (PRF)

Our Pseudorandom Function F is a keyed function[24] $F : \{0,1\}^\lambda \times \{0,1\}^n \to \{0,1\}^m$, F will take key k and input x to produce output y, $F(k,x) = y$.

Without knowing our key k, F_k is computationally indistinguishable from some random $f \xleftarrow{\$} \{F \mid F\{0,1\}^n \to \{0,1\}^m\}$.

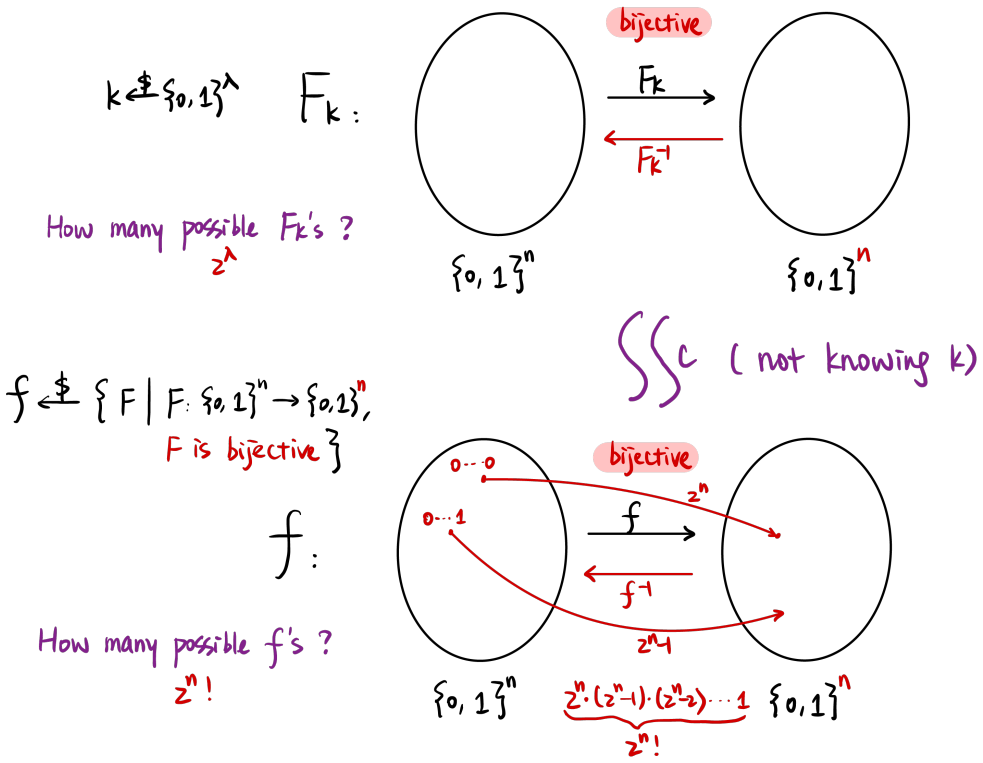
[24] In deterministic polynomial-time.



We have 2^λ possible F_k's, and we have $(2^m)^{2^n}$ possible functions f. A computationally unbounded adversary could try all possible functions and distinguish our function, since F_k lives in a subset of the space of f. However, in reality, we can assume that F_k is computationally indistinguishable from any generic function.

§5.3.2 Pseudorandom Permutation (PRP)

A further assumption is that our function is a bijection. F_k is a keyed function from $F_k : \{0,1\}^n \to \{0,1\}^n$. We still have 2^λ possible F_k's since there are 2^λ



Question. Again, how many possible f's are there?

Our first string has 2^n choices to map to, our second choice has $2^n - 1$, so there are

$$(2^n)(2^n - 1)(2^n - 2) \cdots 1 = 2^n!$$

Still, this is a much larger number than 2^λ, so we still make a computational assumption that our keyed function F_k is still computationally indistinguishable from a random function f.

§5.3.3 Block Cipher Definition

A block cipher is a function

$$F : \{0,1\}^\lambda \times \{0,1\}^n \to \{0,1\}^n$$

where λ is the key length and n is the block length. A block cipher is assumed to be a pseudorandom permutation (PRP).

Our practical construction is the Advanced Encryption Standard (AES).

- Our key and block sizes are $\lambda = n = 128$.
- This was standardized by NIST in 2001.

- There was a competition from 1997-2000 to come up with a block cipher scheme. It is said that the end of the competition, competitors were simply trying to attack each other's schemes. AES was eventually selected for its efficiency.

- Before AES, we used the Data Encryption Standard (DES) with $\lambda = 56$ and $n = 64$. The best attack is *still* a brute-force search, but key and block lengths are relatively fixed (cannot be extended).

Currently, the best attack for AES is still a brute-force search, which takes time 2^{128}.

§5.3.4 Block Cipher Modes of Operation

We have block cipher

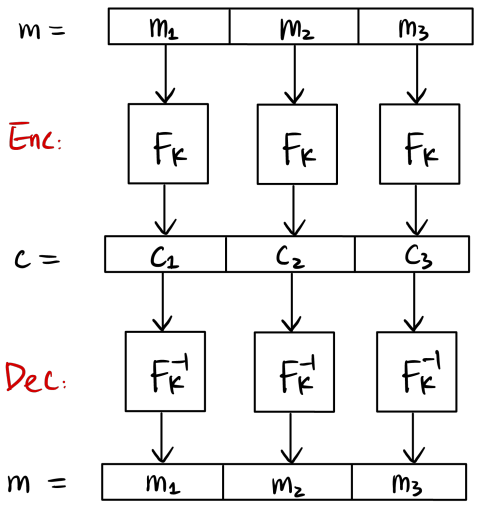
$$F : \{0,1\}^\lambda \times \{0,1\}^\lambda \to \{0,1\}^\lambda$$

We want to construct an SKE scheme from F for arbitrary-length messages. We have some $k \xleftarrow{\$} \{0,1\}^\lambda$. We encrypt with $\mathsf{Enc}_k(m)$ and decrypt $\mathsf{Dec}_k(c)$.

Our goal is to construct an SKE scheme that is CPA (Chosen Plaintext Attack) secure.

Electronic Code Book (ECB) Mode:

The easiest solution is to split up our message into blocks, and run our function F on each of those blocks.

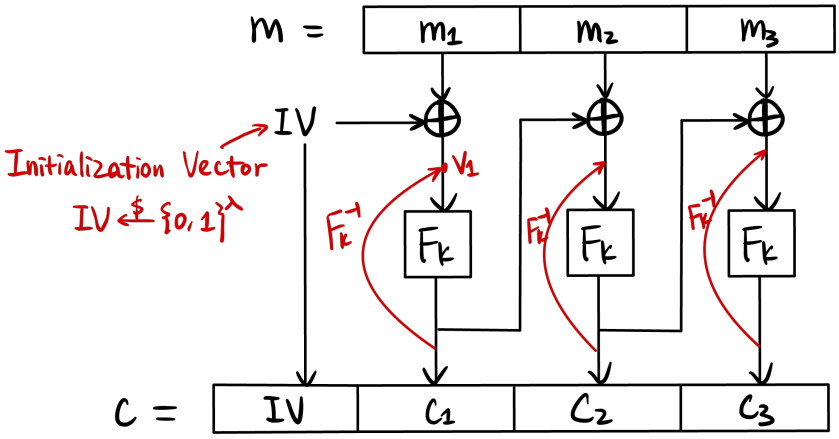


However, this is not CPA secure, since each block is deterministically computed.

Cipher Block Chaining (CBC) Mode:

We can do something else to ensure each block's plaintext is different using an Initialization Vector (IV), sampled from $\{0,1\}^\lambda$.

After every block, we XOR that block's c_i with the next block's message m_{i+1}.



Question. How do we decrypt this?

We can decrypt the first block, then XOR c_i with $F_k^{-1}(c_{i+1})$.

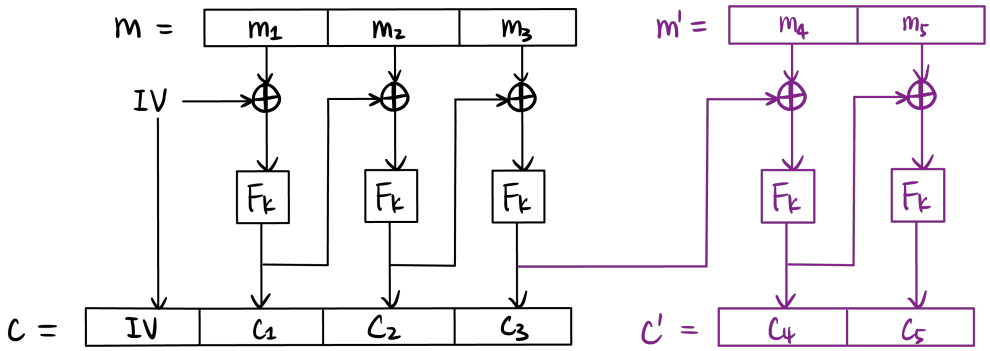
Question. Is this secure?

Assuming F_k is a valid pseudorandom permutation, this is. We'll elide the proof here.

Question. Can we parallelize this? Especially the F_k of F_k^{-1} steps?

We can't in the case of encryption. For decryption, we can perform F_k^{-1} all at once, and do all the XOR operations in series.

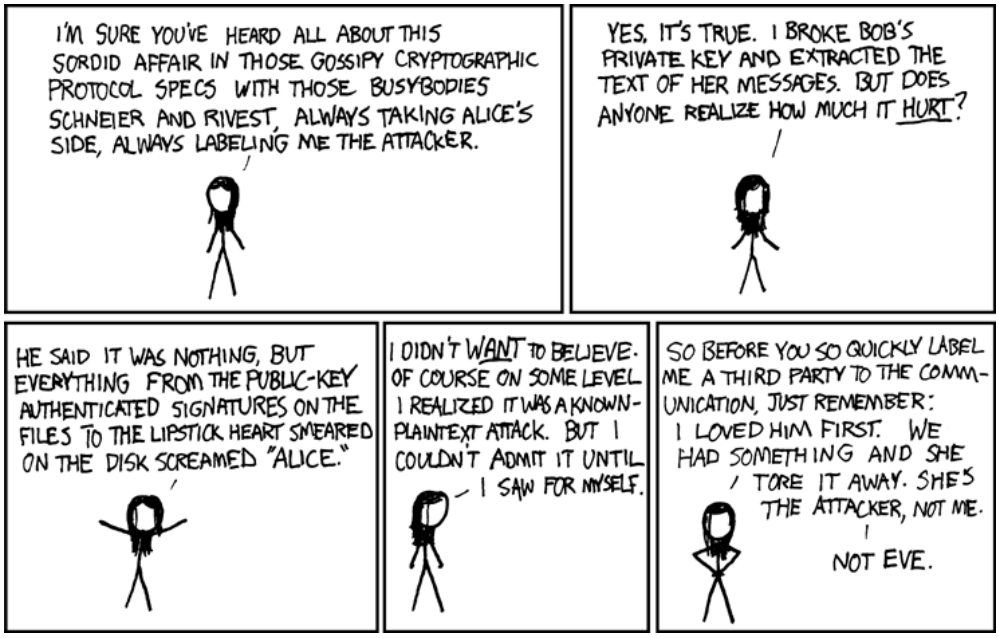
Chained Cipher Block Chaining (Chained-CBC) Mode: We *could* also use the previous messages' c values as the IV for future messages.



Question. Is this now CPA-secure?

We'll continue this next time...

§6 February 14, 2023



xkcd 177. Happy Valentine's Day! Stay safe by selecting authenticated encryption schemes (see section 4.4) that are CPA secure and unforgeable to communicate with your date.

§6.1 Block Ciphers, *continued*

Looking back on section 4.5, the last outstanding primitive was the block cipher. We saw this last lecture, we'll continue discussing the block cipher.

Recall that we had seen pseudorandom functions which are keyed functions that are computationally indistinguishable from *all* random functions from $\{0,1\}^n \to \{0,1\}^m$. A stronger form of pseudorandom functions are pseudorandom permutations: a keyed bijective map between $\{0,1\}^n \to \{0,1\}^n$ that is computational indistinguishable from pseudorandom permutations.

Block ciphers are a special form of pseudorandom permutation. It is a keyed function

$$F : \{0,1\}^\lambda \times \{0,1\}^n \to \{0,1\}^n$$

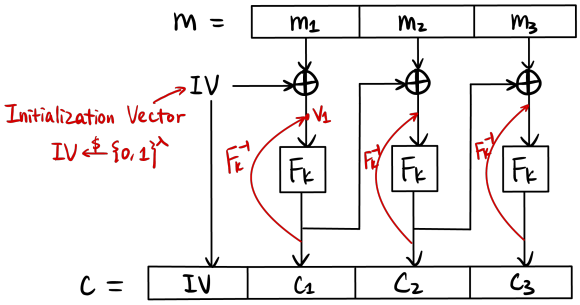
where λ is the key length and n is the block length. The practical construction of which is AES, which takes blocks of $n = 128$ and key length $\lambda = 128, 192, 256$ as choices.

§6.1.1 Modes of Operation

You can see section 5.3.4 for a recap as well.

Electronic Code Book (ECB) Mode: We will run our block cipher on each block of our message individually. However, this is not CPA secure, since encryptions are deterministic. We need to 'seed' our encryption with some random value.

Cipher Block Chaining (CBC) Mode: Instead of running on our block cipher on each block individually, every block will get an additional *initialization vector* IV, which is XORed onto each message before running the block cipher.



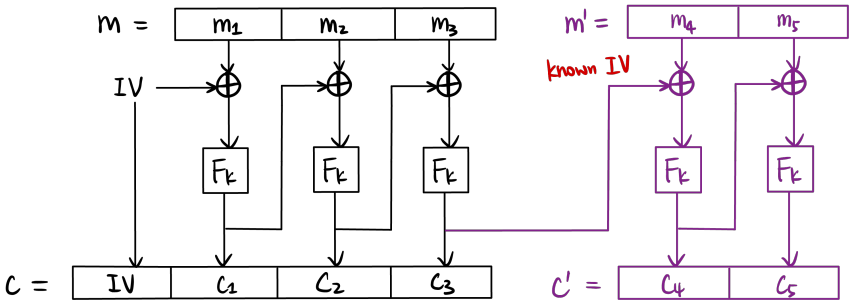
We waved our hand over the fact that this is CPA secure—but it relies on the initialization vector being random.

What if our IV is not randomly sampled? Consider an IV that is *different* but not randomly sampled. For example, the IV is $0\cdots00$ for the first message, $0\cdots01$ for the second message, and so on. Do we still have security?

Unfortunately not. Say m_1 is XORed onto $0\cdots01$, an adversary under CPA can choose plaintext that is m_1 with its last bit flipped, such that v_1 is manipulated and the block cipher is again deterministic.

It is crucial that IV is randomly selected, and that the next IVs for future blocks (of the same message) are also pseudorandom (that are the previous ciphertext, which is okay).

Chained Cipher Block Chaining (Chained-CBC) Mode: We touched on this earlier, but there is a mode of operation of CBC that feeds the last cipher block as the new IV for the next message.



Similar to the case earlier, an adversary here can select a next message *based on* their knowledge of the previous ciphertext and hence the upcoming IV.

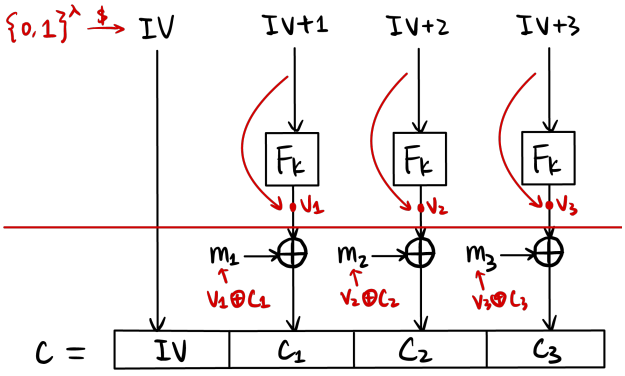
This makes chained-CBC *very subtly* different than CBC. If we squint our eyes enough, it just looks like sending a single message using CBC mode. The key difference is that between rounds of communication m and m', an adversary could influence m' given the knowledge of the previous round.

> **Remark.** Another note that this is *very subtle*! To the extent that when *Signal* was being developed, the course staff initially wrote the solution using Chained-CBC mode. This highlights the difficulty in creating real-world cryptographic systems!

The following will be new modes not covered last lecture:

Counter (CTR) Mode: Instead of chaining each successive IV from the previous block ciphertexts, we'll encrypt *only* the IV $\xleftarrow{\$} \{0,1\}^\lambda$, and XOR the encrypted $F_k(\text{IV}+i)$ to mask m_i, like a one-time pad.

Another way to think about the CTR mode is that we're using F_k and a random IV to generate a long enough one-time pad to pad the entire message.



How do we decrypt? Since we know the first IV, we can compute the one-time pads $F_k(\text{IV}+i)$ and XOR with m_is. This scheme is valid.

Is this CPA secure? The XOR *after* F_k might throw you off and cast doubt in your mind. However, this mode of operation is (!) CPA-secure. Even if we know IV, $\text{IV}+1, \text{IV}+2, \cdots$, we can't figure out the output of F_k that becomes our one-time pad (to do so constradicts the CPA security of our block cipher). The CPA security of each F_k being pseudorandom guarantees the CPA security of this scheme.

What about a "stateful CTR mode" which just increments IV every successive time? Instead of sending a new IV for the next message, we'll just increment the IV from before. Similar to Chained-CBC mode, the adversary will know the IV that is going into the next message. However, this

doesn't *really* help the adversary. They've never seen those encrypted IV values before, and hence cannot modify the message given this information.

This is a distinction from last time, where the IV was XORed onto the message directly, which could be tampered with by an adversary who knows the IV.

What if IV is not randomly sampled? Nothing really breaks down, unlike the previous case. We just want to make sure that two IVs are not reused and don't collide. If IVs collide, two blocks will have the same one-time pad, which is potentially a problem. This doesn't prevent us from using $0\cdots00, 0\cdots01, 0\cdots10, \ldots$ as our IV values at all. In practice, however, they are still randomly sampled to prevent collisions.

Can we parallelize this? Yes, we can compute $F_k(\text{IV}+i)$ in parallel and XOR onto each block. Similar for encryption and decryption.

Can we construct a PRG from a PRF? Using a seed (IV, k), we can generate an $n\lambda$ bit string

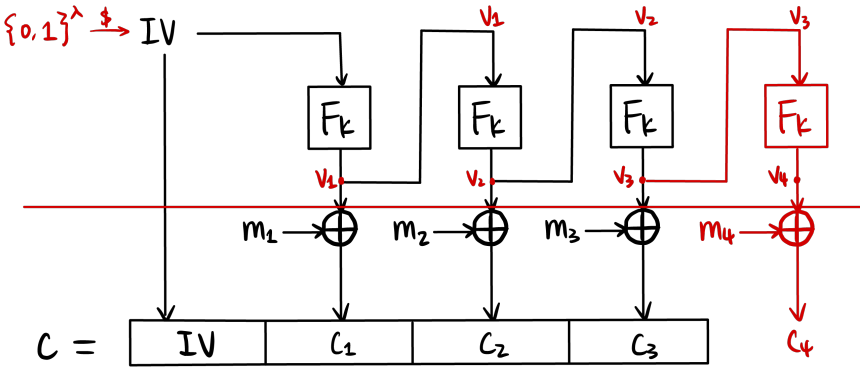
$$G(k||\text{IV}) = F_k(\text{IV})||F_k(\text{IV}+1)||F_k(\text{IV}+2)||\cdots$$

In fact, we can get rid of IV entirely and start at 0,

$$G(k) = F_k(0)||F_k(1)||F_k(2)||\cdots$$

Counter mode essentially uses this PRG with private k to generate a long one-time pad which is used to pad the message. Another note is that in this mode, we don't even require a pseudorandom permutation, since we don't need to invert the function at any point.

Output Feedback (OFB) Mode: This is a mix of CBC and CTR modes. Successive one-time pad blocks are fed into the next F_k as IV, and they are XORed with the message after encryption.



We have the same questions. *How do we decrypt? Is this CPA secure? Is a "stateful" version of OFB secure? Can we use this to construct a PRG?*

We can decrypt similarly: we decrypt the first block, get the IV for the next block and continue on. All security is guaranteed by the same reasoning as in counter mode: we know IV but still cannot

compute $F_k(\text{IV})$. Similar to counter mode, this is another form of PRG (which chains successive blocks instead of using IVs in series) that generates a long one-time pad. Again, our IV doesn't need to be randomly sampled, but it should not collide with previous IV values.

A difference to counter mode is that we cannot parallelize this scheme. However, in both CTR and OFB modes, we can precompute the entire one-time pad in both encryption and decryption to happen in the offline phase. The online phase (when parties are communicating) is limited to cheap XOR operations.

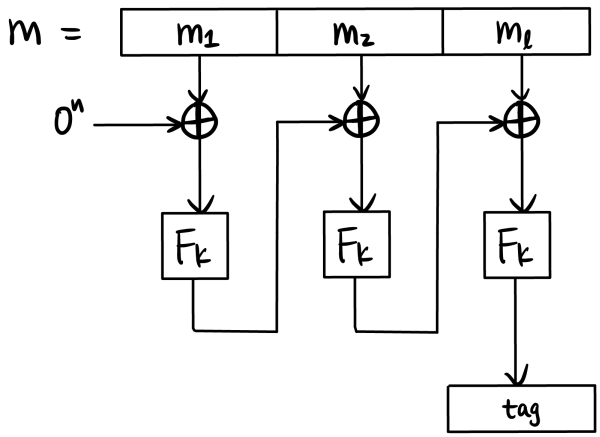
Question. We've listed *a lot* of benefits to counter mode or output feedback mode. Why do people use CBC mode at all?

We've seen how things can go wrong catastrophically[25]. This is more true for counter mode than CBC mode. If our IV is reused in counter mode, our entire one-time pad has been exposed previously[26]. However, if our IV is reused in CBC mode, the worst that could happen is something akin to ECB mode, and no messages are compromised.

At the end of the day, *engineers are quite oblivious to cryptographic schemes*! Libraries only specify for *some key* and *some IV*, so it is exceedingly easy to screw up your cryptograhic scheme by reusing IVs, etc. CBC mode is simply more foolproof and incurs better outcomes in case it is used incorrectly[27].

§6.1.2 CBC-MAC

We can use block ciphers to construct a MAC scheme. Splitting up our message into blocks, we feed blocks into F_k and chain to next blocks. In the end, the final cipher output is our tag.



[25] We nearly made mistakes in this course!
[26] XORing our ciphertexts will give $m \text{XOR} m'$.
[27] However, if Peihan were to implement a block cipher scheme herself, would opt for counter mode.

How do we verify? We can just Mac the message again and check that the tag matches. If F_k is invertible, we can also go the other way.

Is this CMA secure?

- Fixed-length messages of length $l \cdot n$? Yes, since we can only query for fixed-length messages, this gives us no additional information.

- Arbitrary-length messages? This is where problems arise—the adversary could first query for a message of 1 block, then 2 blocks, then 3 blocks, etc. By combining this information, they could produce new valid signatures.

  A concrete attack is an adversary querying for $\mathsf{Mac}(m)$ to produce tag, then querying for $\mathsf{Mac}(\mathsf{tag}) = \mathsf{Mac}(m||0) = \mathsf{tag}'$ which allows the adversary to forge a new message.

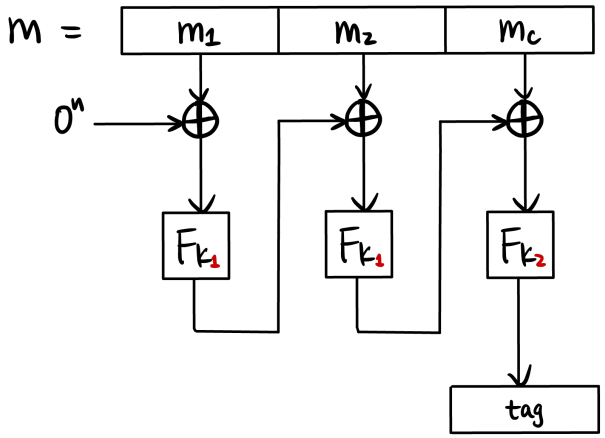
> **Remark.** Our constructions of authenticated encryption calls for an encryption scheme and MAC scheme. It's crucial that the two schemes have *different keys*. Using the same key k for both encryption and MAC can cuase issues (information from one could reveal something about the other).

We have a fix for the CMA-vulnerability in arbitrary-length messages:

§6.1.3 Encrypt-last-block CBC-MAC (ECBC-MAC)

The vulnerability earlier was due to our encryption being *associative*, so to speak.

We can fix this is to use a different key for the last block:



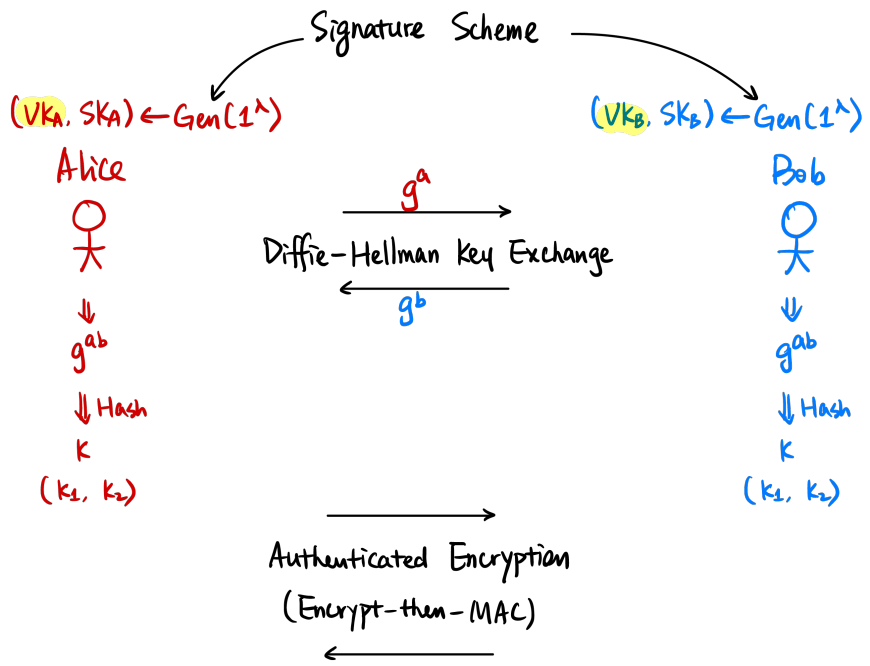
We could also attach length of messages to the first block, or other techniques.

The nuance in CBC-MAC means that realistically, we almost always use HMAC.

§6.2 Putting it Together

Looking back at section 4.5, we've collected everything we need so far for secure communication.

For Alice and Bob to communicate, they first exchange keys using a Diffie-Hellman key exchange, then perform authenticated encryption.



However, this still does not mitigate against a man-in-the-middle attack. Thus, before exchanging keys, Alice and Bob should publish verification keys (to a digital signature scheme, see section 4.1.2). Using this digital signature, Alice and Bob will each sign their Diffie-Hellman public values g^a, g^b using their signing key, which will be attached to the message. They can respectively verify that these values came from each other, and not some Eve in the middle.

§7 February 16, 2023

We've now learned all the cryptographic primitives we need in this course.

Recall that we had a way for Alice and Bob to communicate securely, first exchanging a shared Diffie-Hellman key and then performing AES encryption.

However, this is prone to a man-in-the-middle attack. One way to solve this is for parties to *sign* their Diffie-Hellman publics, and verify using a publicly known verification key.

Is this now secure against an adversary in the middle? Yes, because the public values are guaranteed (via our digital signature scheme) by Alice and Bob's signing key. The man in the middle does not have access to the signing key, and cannot sign a phony public value.

§7.1 SSH

This is *exactly* how the SSH algorithm works. Let's work through the steps of GitHub's SSH setup to see how it works.

The instructions are given for the EDDSA-25519 algorithm, which relies on elliptic curves.

1. We first generate a signing keypair $(vk_A, sk_A) \leftarrow \mathsf{Gen}(1^\lambda)$ via

   ```
   $ ssh-keygen -t ed25519 -C "your_email@example.com"
   ```

   vk_A is the `id_ed25519.pub` (the public key) sk_A is `id_ed25519` (the private key).

2. We upload our public key to our account on GitHub. This is equivalent of communicating our vk_A to GitHub.

3. When we're connecting via SSH to GitHub for the first time, our terminal will prompt us that this is a new server with a new verification key.

   ```
   > The authenticity of host 'github.com (IP ADDRESS)' can't be established.
   > RSA key fingerprint is SHA256:nThbg6kXUpJWGl7E1IGOCspRomTxdCARLviKw6E5SY8.
   > Are you sure you want to continue connecting (yes/no)?
   ```

   which we can verify against GitHub's known verification keys[28]. This is the equivalent of receiving a vk_B from GitHub.

[28] The security of our web upload to GitHub, or GitHub's site which publishes the verification key, relies on the security of the website, likely through TLS. But you could also imagine exchanging keys in person, etc.

§7.2 One-Sided Secure Authentication

In some circumstances, it's more difficult for a client to communicate their verification key to a server than it is for a server to do so. A server might publish their verification key, and trust that all clients are not compromised.

What could an adversary potentially do? The adversary could not pretend to be the server since they have no access to the server's signing key. The adversary *can* pretend to be the client and talk to the server. The adversary could forward all messages sent to the server, and can also communicate g^b, σ_b back to the client (it's a valid signature since it has not been modified).

At the end of this protocol, the client has Diffie-Hellman private g^{ab} and the adversary and server will have g^{eb} (where g^e, e is a Diffie-Hellman keypair the adversary provided to the client). Whatever the client sends to the server cannot be decrypted by the adversary, since it is encrypted with g^{ab}, however, the server's communications *could* be decrypted by the adversary.

This can be easily circumvented by requiring the server and user complete their handshake—the server could request a hash or encryption of the shared secret, and realize that they are communicating to an adversary when this cannot be forged by the man-in-the-middle.

§7.3 Public Key Infrastructure

How can we know who has which public keys on the internet? We can rely on a Public Key Infrastructure (PKI) to know each other's public keys.

If Bob purports to be `bob.com` and wants to prove that vk_B belongs to him, Bob will send a certificate signing request (CSR) to a Certificate Authority (CA)[29].

The CA will sign the message $(\texttt{bob.com}, vk_B)$ and send that signature σ back to Bob. This verifies that the user of `bob.com` holds signing key sk_B with public key vk_B.

The standard of which is the X.509 certificate.

For example, when we try to access `facebook.com`, we can check that the certificate is valid[30]

This pivots on the fact that *everyone* must know vk of the certificate authority. We shift our trust from individual sites and users to the certificate authorities. Most devices have the vks of trusted authorities built in.

[29] *The higher beings that be*...this is companies like *DigiCert*, *Let's Encrypt*, etc.
[30] In browsers, this is represented by the lock symbol—clicking on that will allow you to verify that certificate.

§7.3.1 Certificate Chain

In reality, there are several certificate authorities, and they also form *chains* of certificate authorities.

A Root CA[31] with a known (vk, sk) $\mathsf{Gen}(1^\lambda)$ can first sign the vk_1 of an Intermediate CA1, producing cert $\mathsf{cert}_1 = \sigma \leftarrow \mathsf{Sign}_{sk}(vk_1)$.

Then, the Intermediate CA1 can sign a certificate for Intermediate CA2, but we'll have to preserve this chain. Intermediate CA1 could produce cert $\sigma_1 \leftarrow \mathsf{Sign}_{sk_1}(vk_2)$, but how do we know that sk_1 is valid? So, we'll need to include vk_1 and vk_1's signature signed by sk. That is,

$$\begin{aligned}\mathsf{cert}_2 =& vk_1, \sigma \leftarrow \mathsf{Sign}_{sk}(vk_1),\\ & vk_2, \sigma_2 \leftarrow \mathsf{Sign}_{sk_1}(vk_2)\end{aligned}$$

Finally, Intermediate CA2 can sign Bob's verification key using their chain. Bob's certificate will contain

$$\begin{aligned}\mathsf{cert}_B =& vk_1, \sigma \leftarrow \mathsf{Sign}_{sk}(vk_1),\\ & vk_2, \sigma_2 \leftarrow \mathsf{Sign}_{sk_1}(vk_2)\\ & vk_B, \sigma_B \leftarrow \mathsf{Sign}_{sk_2}(vk_B)\end{aligned}$$

How can an Intermediate CA restrict Bob's use of these certificates? What if Bob will then go on and start signing his own certificates for people? We can concatenate information in each certificate that restricts its use. It could specify whether it is being issued to an *end user*, or even additional information like validity time.

To protect against CAs that get compromised, certificates are short-lived and have set validity times. Additionally, certificate authorities can publish revocation lists that browsers check against when validating a certificate.

§7.4 Password-Based Authentication

Sometimes, you also want to *authenticate* with a server using a password. The naïve implementation is that a user with an ID sends a hash of the password $h = H(\text{password})$ to the server. The server stores (ID, h).

In this case, an adversary could launch an *Online Dictionary Attack* and try a lot of passwords with the server.

If the server were to be compromised, and its database compromised, the adversary can conduct an *Offline Dictionary Attack* on the database. Additionally, the adversary can precompute all hashes and check against the database.

[31] We mentioned earlier that CAs are built into devices. For example, here is a list of all root certificates that are built-in for Apple devices. This can go wrong too! CAs have been misused which causes implications on the security of the internet.

How can we prevent this?

§7.4.1 Salting

One way of ensuring that the hashing is non-deterministic is for servers to generate a salt $\xleftarrow{\$} \{0,1\}^s$ and send it to the user. The user will hash $H(\text{password}||\text{salt})$ and send that to the server. The server stores a database of $(\text{ID}, \text{salt}, \text{h})$.

When logging in, the user first sends their ID to the server, the server will send the salt back, the user hashes their password, and the hash is sent to the server for verification.

Does this allow the user to use a weak password? Nope! The adversary can always brute-force the password.

However, there are still issues with this scheme...we'll discuss another technique next time, *peppering*, that will make the adversary's life even harder.