

# CSCI 1515: Applied Cryptography

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These are lecture notes for CSCI 1515: Applied Cryptography taught at BROWN UNIVERSITY by Peihan Miao in the Spring of 2024.

These notes were originally taken by Jiahua Chen with gracious help and input from classmates and fellow TAs. Please direct any mistakes/errata to a thread on Ed, or feel free to pull request or submit an issue to the [notes repository](#).

Notes last updated February 21, 2024.

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## §1 January 24, 2024

### §1.1 Introduction

The course homepage is at <https://cs.brown.edu/courses/csci1515/spring-2024/>, where you can find information such as the [syllabus](#), projects, homeworks, calendar, lectures and more.

The course is offered in-person in *Bio Med 202*, as well as synchronously over Zoom and recorded asynchronously (lectures posted online). Lecture attendance and participation is highly encouraged!

**EdStem** will be used for course questions, and **Gradescope** is used for assignments.

#### §1.1.1 Staff

Our course staff have all taken or TA-ed the course before and are excited to help you learn!

Peihan has been at Brown for a couple of years and this was the second time she is teaching this course. Before Brown, she was at the University of Illinois Chicago. Before that, she finished her PhD at UC Berkeley in 2019 with a focus in cryptography. Afterwards, she worked in industry for a couple of years (Visa) before deciding to come back to academia. She still collaborates with industry to see what problems need to be solved in practice.

During her PhD, she started off doing more theoretical cryptography but also did internships and found applied cryptography fascinating as well. Now she works in both.

#### §1.1.2 Course Philosophy and Logistics

If look up other *applied* cryptography courses online or at other universities, you will find courses that have “applied” in their title. However, if you look at their syllabus or content, it’s still mostly theoretical crypto. This may (1) deter students from learning about crypto and (2) leave a gap between theoretical crypto and crypto in practice. (2) is bad because if someone makes a mistake in the crypto domain, the consequences are often significant.

As such, it’s helpful for students to get hands-on experience with cryptography:

- How cryptography has been used in practice and
- How cryptography will be used and implemented in the future.

The closest similar course is found at Stanford, which covers theoretical crypto in the first half and more applied crypto in the second. But even that course only covers very basic crypto that are very

well established. In the past 10 years or so, there are new and exciting topics in crypto that are gradually becoming more and more common which we will also cover in this course.

For this course, it will be *much less* about math and proofs, and much more about how you can use these tools to do something more fun. It will be coding heavy and all projects will be implemented in C++ using crypto libraries.

If, however, you are interested in the theoretical or mathematical side, you might consider other courses at Brown like CSCI 1510 and MATH 1580.

There is an option to capstone this course, contact Peihan about this. It would also be best to find a partner who is also capstoning this course.

The following is the grading policy:

Type	Percentage
Project 0	4%
Projects 1 & 2	20% (10% each)
Projects 3, 4 & 5	36% (12% each)
Homeworks	25% (5% each)
Final Project	25%

You have 4 total late days for *projects*, of which at most two can be used on a single project. Additionally, you have 3 total late days for *homeworks*, of which at most one can be used on a single homework.

All projects are independent, but collaboration is allowed and encouraged. However, you *must write up your own code*.

If you're sick, let Peihan know with a Dean's note.

## §1.2 What is cryptography?

At a high level, *cryptography is a set of techniques that protect sensitive or important information*.

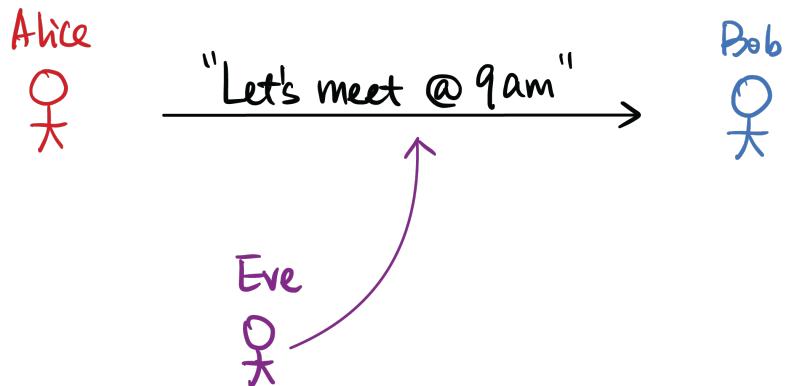
**Question.** Where is cryptography used in practice? What guarantees do we want in these scenarios?

- Online transactions
  - When you make a purchase, you might not want people to see your bank balance, what else you have purchased, etc.

- You also want to ensure that it was really *you* who purchased the item and not somebody else i.e. authentication
- Secure messaging
  - End-to-end texting, iMessage
  - We don't want anyone else to see our messages
- Online voting
  - Privacy of votes, validity of votes
- Databases
  - Secure storage

### §1.3 Secure Communication

We'll start with the most classic form of cryptography: *secure communication*.



Assume Alice wants to communicate to Bob “Let's meet at 9am”, what are some security guarantees we want?

- Eve cannot *see* the message from Alice to Bob.
- Eve cannot *alter* the message from Alice to Bob.

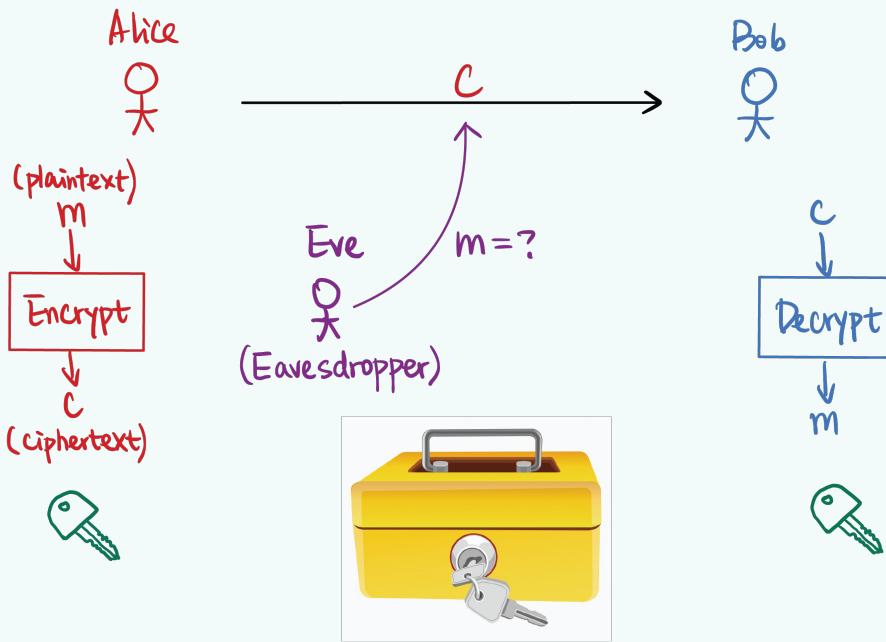
These two guarantees are the most important guarantees! The former is called message secrecy, the latter is called message integrity.

### §1.3.1 Message Secrecy

#### Definition 1.1 (Message Secrecy)

We want cryptography to allow Alice to *encrypt* the message  $m$  (which we call *plaintext*) by running an algorithm that produces a *ciphertext*  $c$ . We call this an *encryption scheme*.

Bob will be able to receive the ciphertext  $c$  and run a *decrypt* algorithm to produce the message  $m$  again. This is akin to a secure box that Alice locks up, and Bob unlocks, while Eve does not know the message. The easiest way is for Alice and Bob to agree on a shared secret key.

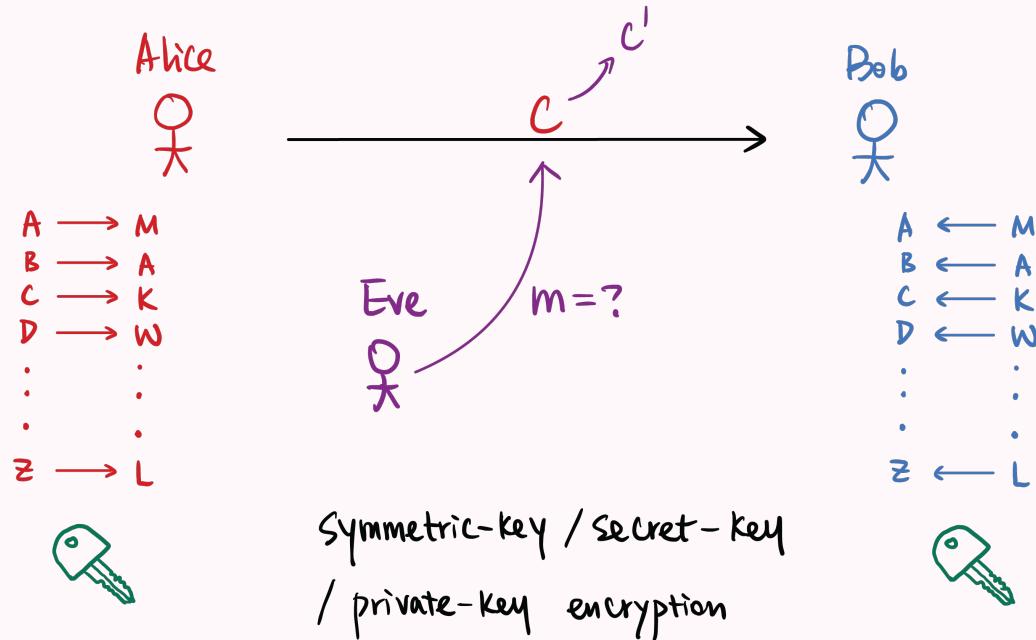


In this model, Eve is a weaker adversary, an *eavesdropper*. Eve can only see the message, not alter it.

#### Example 1.2 (Substitution Cipher)

The key that Alice and Bob jointly uses is a permutation mapping from  $\{A \dots Z\} \rightarrow \{A \dots Z\}$ . This mapping is the *secret key*.

Bob also has the mapping, and takes the inverse of the permutation to retrieve the message.



This scheme is not quite secure! *Why?*

Given a large enough text, you can apply frequency analysis and break the substitution cipher.

**Remark.** This encryption scheme also requires that Alice and Bob meet up in person to exchange this shared private key. Schemes like this are called *symmetric-key*, *secret-key*, or *private-key encryption*. They need to somehow agree first on the same secret key.

### Definition 1.3 (Public-key Encryption)

There is another primitive that is much stronger: public-key encryption. Bob publishes both a *public* key and a *private* key. You can consider a lock where you don't need a key to lock it<sup>1</sup>, and only Bob has the key to unlock it.



This is seemingly magic! Bob could publish a public key on his homepage, anyone can encrypt using a public key but only Bob can decrypt. *Stay tuned, we will see public-key encryption schemes next lecture!*

### §1.3.2 Message Integrity

Alice wants to send a message to Bob again, but Eve is stronger! Eve can now tamper with the message.

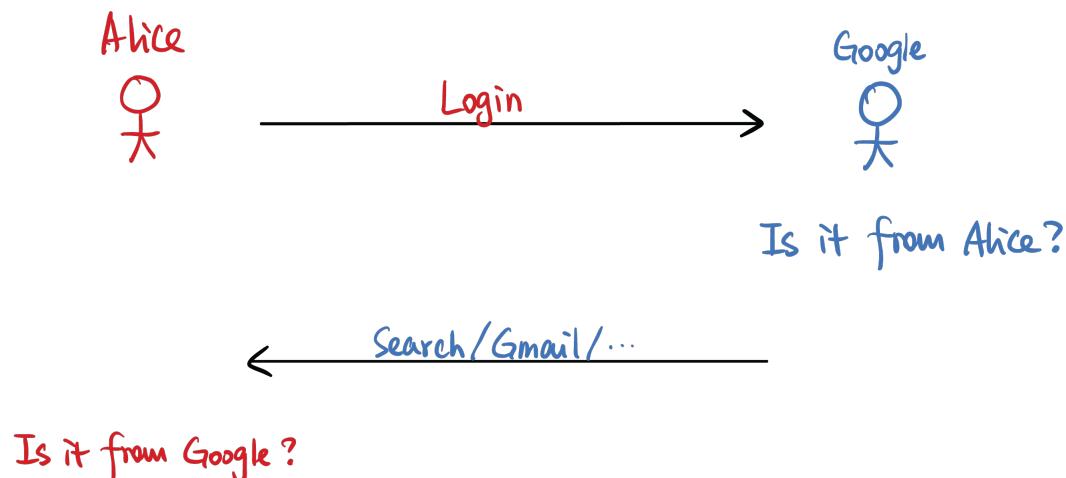
<sup>1</sup>You literally click it closed



Bob wants to ensure that the message *actually* comes from Alice. Does our previous scheme (of encrypting messages) solve this problem? Nope!

Eve can change the ciphertext to something else, they could pretend to be Alice. In secret-key schemes, if Eve figures out the secret-key, they can forge messages from Alice. Even if Eve doesn't know the underlying message, they could still change it to some other ciphertext which might be correlated to the original ciphertext, *without knowing the underlying message*. We'll see how Eve can meaningfully do this in some schemes. Alice could send a message "Let's meet at  $x$  AM" and Eve could tamper this to say "Let's meet at  $x + 1$  AM."

This is sort of an orthogonal problem to message secrecy. For example, when Alice logs in to Google, Google needs to verify that Alice actually is who she claims to be.



This property that we want is called message integrity.

### §1.3.3 Signal and Auth

The first two projects are Signal and Auth whose aim will be to cover secure messaging and secure authentication.

#### *Projects Overview*

0. Warm-up, you will implement some basic cryptographic schemes.
1. Secure Communication: how to communicate in secret.
2. Secure Authentication: how to authenticate yourself.
3. Zero-Knowledge Proofs: we'll use ZKPs to implement a secure voting scheme.
4. Secure Multiparty Computation: we'll implement a way to run any function securely between two parties.
5. Fully Homomorphic Encryption: a form of post-quantum cryptography.

We'll now introduce the latter three projects!

### §1.4 Zero-Knowledge Proofs

This is to prove something without *revealing* any additional knowledge.

For example, Alice may want to

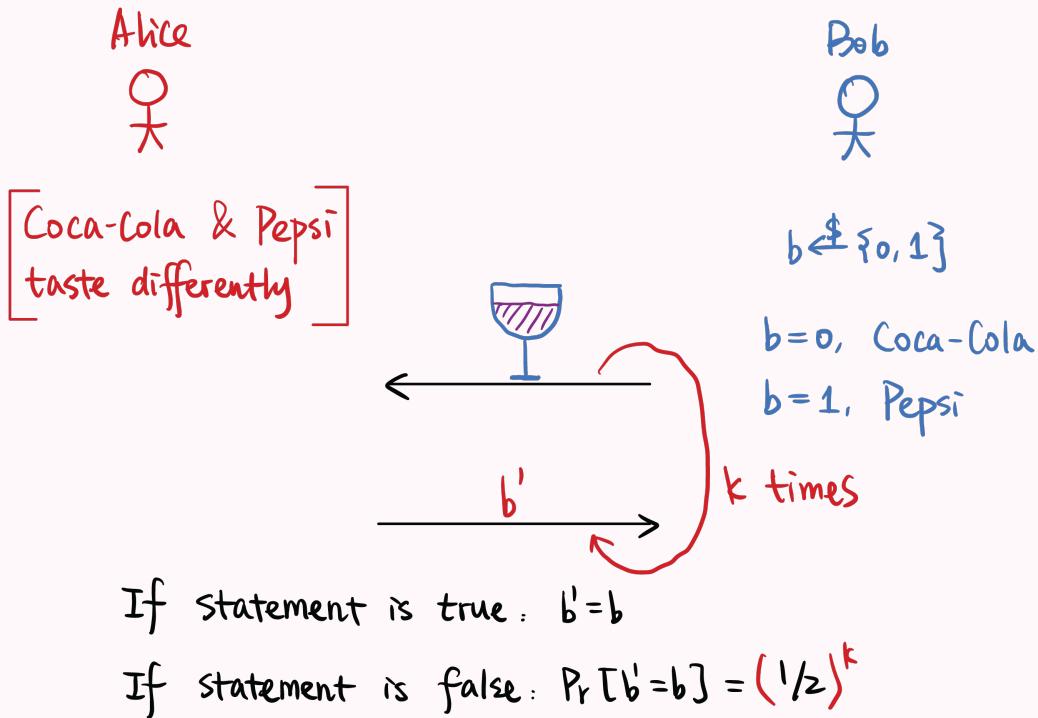
- Prove she knows the difference in taste between Coke and Pepsi without revealing how
- Prove that you have a bug in your code without revealing the bug
- She has the secret key for this ciphertext without revealing the plaintext

How is this possible?

#### Example

Alice claims to be able to differentiate between Coca-Cola and Pepsi! She wants to prove this to Bob without revealing her secrets.

Bob will randomly sample a bit  $b \xleftarrow{\$} \{0, 1\}$ , with  $b = 0$  being Coca-Cola and  $b = 1$  being Pepsi. Bob will let Alice taste this drink. Alice will give a guess  $b'$  of what drink it is.



If the statement is true,  $\Pr[b' = b] = 1$  (Alice always gives the correct prediction).

If the statement is false,  $\Pr[b' = b] = \frac{1}{2}$  (Alice is guessing with 0.5 probability).

To enhance this, we can run this a total of  $k$  times. If we run it enough times, Bob will be more and more confident in believing this. Alice getting this correct by chance has a  $\frac{1}{2^k}$  probability.

The key idea, however, is that Bob doesn't gain any knowledge of how Alice differentiates.

**Remark.** This is a similar strategy in proving graph non-isomorphism.

For people who have seen this before, generally speaking, any NP language can be proved in zero-knowledge. Alice has the *witness* to the membership in NP language.

## §1.5 Secure Multi-Party Computation

Alice   $x \in \{0, 1\}$  Second date?  $y \in \{0, 1\}$  Bob 

$$f(x, y) = x \wedge y$$

$x \in \{0, 1\}^{1000}$  Who is richer?  $y \in \{0, 1\}^{1000}$

$$f(x, y) = \begin{cases} \text{Alice if } x > y \\ \text{Bob otherwise} \end{cases}$$

$X = \left\{ \begin{array}{c} \text{friend}_A^1 \\ \vdots \\ \text{friend}_A^n \end{array} \right\}$  Common friends?  $Y = \left\{ \begin{array}{c} \text{friend}_B^1 \\ \vdots \\ \text{friend}_B^m \end{array} \right\}$

$$f(X, Y) = X \cap Y$$

$X = \left\{ \begin{array}{c} (\text{username}, \text{password}) \\ \vdots \end{array} \right\}$   $Y = \left\{ \begin{array}{c} (\text{usr}, \text{psw}) \\ \vdots \end{array} \right\}$

### Example (Secure AND)

Alice and Bob go on a first date, and they want to figure out whether they want to go on a second date. They will only go on a second date if and only if both agree to a second date.

How will they agree on this? They could tell each other, but this could be embarrassing. One way is for them to share with a third-party (this is what dating apps do!). However, there might not always be an appropriate third party (in healthcare examples, not everyone can be trusted with the data).

In this case, Alice has a choice bit  $x \in \{0, 1\}$  and Bob has a choice bit  $y \in \{0, 1\}$ . They are trying to jointly compute  $f(x, y) = x \wedge y$ .

**Remark 1.4.** Couldn't a party still figure out how the other party feels? For example, if Bob's bit was 1 and the joint result was 0, Bob can *infer* that Alice's bit was 0.

This is, in effect, the best we can do. The ideal guarantee is that each party only learns any information they can infer from the *output* and their input. However, they should not learn anything more.

What are we trying to achieve here? We want to jointly compute some function, where each party has private input, such that each party only learns the output. They should not learn anything about other parties' inputs.

#### Example (Yao's Millionaires' Problem)

Perhaps, Alice and Bob want to figure out who is richer. The inputs are  $x \in \{0, 1\}^{1000}$  and  $y \in \{0, 1\}^{1000}$  (for simplification, let's say they can express their wealth in 1000 bits). The output is the person who has the max.

$$f(x, y) = \begin{cases} \text{Alice} & \text{if } x > y \\ \text{Bob} & \text{otherwise} \end{cases}$$

#### Example (Private Set Intersection)

Alice and Bob meet for the first time and want to determine which of their friends they share. However, they do not want to reveal who specifically are their friends.

$X$  is a set of A's friends  $X = \{\text{friend}_A^1, \text{friend}_A^2, \dots, \text{friend}_A^n\}$  and Bob also has a set  $Y = \{\text{friend}_B^1, \text{friend}_B^2, \dots, \text{friend}_B^m\}$ . They want to jointly compute

$$f(X, Y) = X \cap Y.$$

You might need to reveal the cardinality of these sets, but you could also pad them up to a maximum number of friends.

This has a lot of applications in practice! In Google Chrome, your browser will notify you that your password has been leaked on the internet, without having access to your passwords in the clear.  $X$  will be a set of *your* passwords, and Google will have a set  $Y$  of *leaked* passwords. The *intersection* of these sets are which passwords have been leaked over the internet, without revealing all passwords in the clear.

**Question.** Isn't the assumption that the size of input records is revealed weaker than using a trusted third-party?

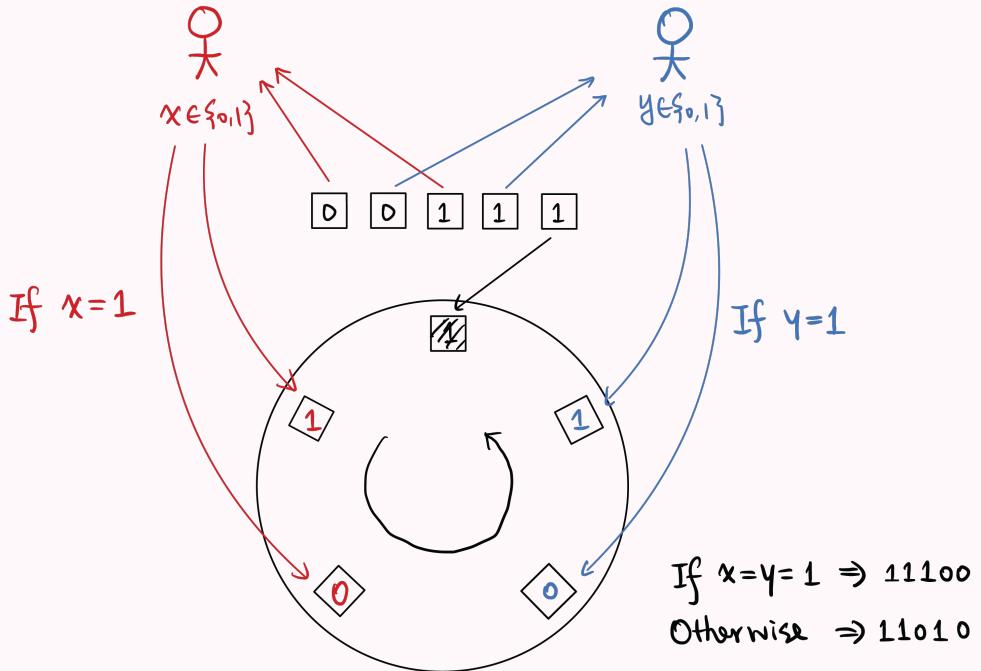
Yes, however in some cases (hospital health records), parties are legally obliged to keep data secure. We wish for security more than the secrecy of cardinality.

In the general case, Alice and Bob have some inputs  $x$  and  $y$  with bounded length, and they want to jointly compute some function  $f$  on these inputs. This is Secure Two-Party Computation. Furthermore, there could be multiple parties  $x_1, \dots, x_n$  that jointly compute  $f(x_1, \dots, x_n)$  that hides each input. This is Secure Multiparty Computation.

We'll explore a toy example with the bit-AND from the dating example.

### Example (Private Dating)

Alice and Bob have choice bits  $x \in \{0, 1\}$  and  $y \in \{0, 1\}$  respectively. There is a *physical* round table with 5 identical slots, one already filled in with a 1 facing down.



Alice and Bob each have identical 0, 1 cards (each of the 0 and 1 cards are indistinguishable from cards of the same value). Alice places her cards on the 2 slots in some order, and Bob does the same.

They then spin the table around and reveal all the cards, learning  $x \wedge y$ .

If  $x = 1$ , Alice places it as 1 on top of 0, and if  $y = 1$ , Bob places it as 1 on top of 0 as well. Otherwise, they flip them. If  $x = y = 1$ , then the 0's will be adjacent. If  $x \neq y$ , the order will be 1, 1, 0, 1, 0 (the 0's are not adjacent), regardless of which of Alice or Bob produced  $x = 0$  (or both!).

*This is a toy example! It doesn't use cryptography at all! Two parties have to sit in front of a table.*

This is called card-based cryptography. We will be using more secure primitives.

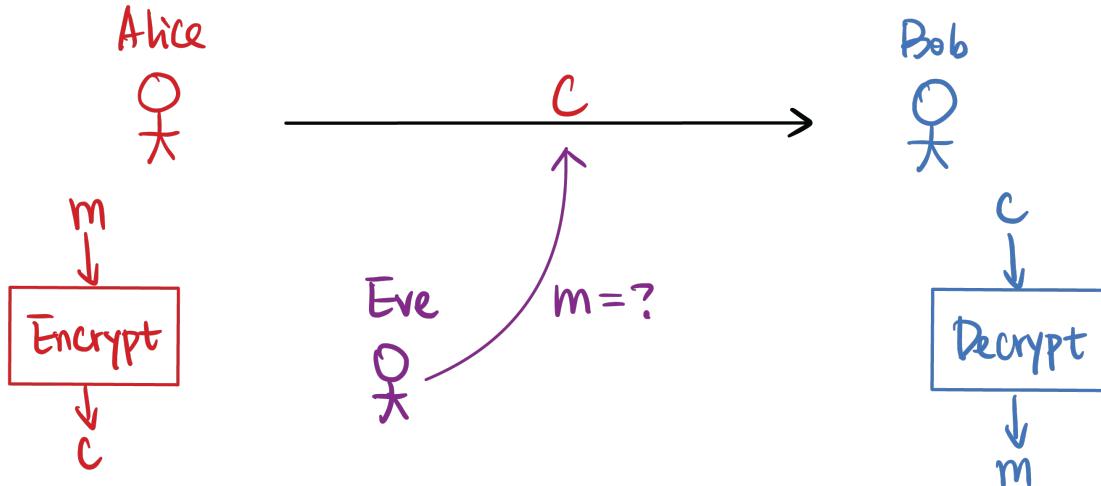
## §1.6 Fully Homomorphic Encryption

We'll come back to the secure messaging example.

Alice wants to send Bob a message. She encrypts it somehow and sends a ciphertext  $c_1 = \text{Enc}(m_1)$ . A nice feature for some encryption schemes is for Eve to do some computation homomorphically on the ciphertexts. Eve might possibly want to add ciphertexts (that leads to plaintext adding)

$$c_1 = \text{Enc}(m_1), c_2 = \text{Enc}(m_2) \Rightarrow c' = \text{Enc}(m_1 + m_2)$$

or perhaps  $c'' = \text{Enc}(m_1 \cdot m_2)$ , or compute arbitrary functions. *Sometimes*, this is simply adding  $c_1 + c_2$ , but usually not.

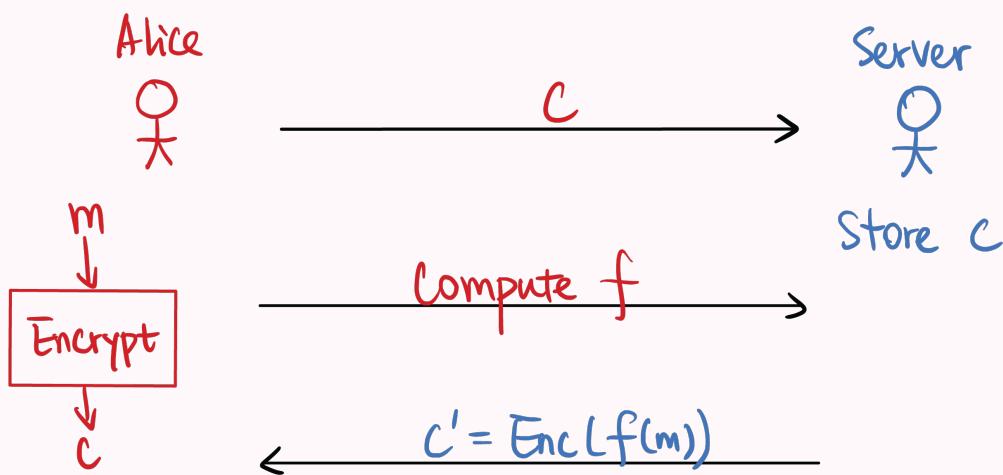


$$\begin{aligned} c_1 &= \text{Enc}(m_1) & \Rightarrow c' &= \text{Enc}(m_1 + m_2) \\ c_2 &= \text{Enc}(m_2) & c'' &= \text{Enc}(m_1 \cdot m_2) \end{aligned}$$

We want to hopefully compute any function in polynomial time!

### Example (Outsourced Computation)

Alice has some messages but doesn't have enough compute. There is a server that has *a lot* of compute!



Alice encrypts her data and stores it in the server. At some point, Alice might want to compute a function on the encrypted data on the server, without the server revealing the original data.

This is an example of how fully homomorphic encryption can be useful.

**Remark.** This problem was not solved until 2009 (when Peihan started her undergrad). Theoretically, it doesn't even seem that possible! Being able to compute functions on ciphertexts that correspond to functions on plaintexts.

To construct fully-homomorphic encryption, we'll be using lattice-based cryptography which is a post-quantum secure!

### §1.7 Further Topics

We might cover some other topics:

- Differential Privacy
- Crypto applications in machine learning
- Crypto techniques used in the blockchain<sup>2</sup>

*What else would you like to learn? What else do you want to understand? Do go through the semester with these in mind! How do I log into Google? How do I send messages to friends?*

Feel free to let us know on Ed!

<sup>2</sup>One important technique is Zero-Knowledge proofs, for example.

## §1.8 Q & A

- *What is the difference between CSCI 1515 and CSCI 1510 or MATH 1580?*

CSCI 1510 is essentially “theoretical cryptography.” It covers formal definitions and constructions and proofs. There is no coding, just proofs.

MATH 1580 considers crypto from the mathematical perspective. They try to understand some of the computational assumptions we assume from a mathematical standpoint. I.e. why is factoring hard to compute, and what is the best algorithm to compute it? In CS, we simply assume factoring is hard. MATH 1580 is more similar to number theory and group theory.

- *If you've taken MATH 1580 and CSCI 1515, would you still recommend taking CSCI 1510?*

There are still more things to learn from CSCI 1510. There is still more theory and it is a much more challenging course from the theoretical perspective. There are more rigorous proofs and reductions in CSCI 1510 that we do not cover in CSCI 1515.

- *Why C++*

Existing crypto libraries are mostly in C++ and most students have seen C/C++ in either cs33 or cs300. We did, however, consider implementing everything in Rust!

## §2 January 29, 2024

### §2.1 Q & A (continued)

Before we move on to content, we received a few more questions!

- *Is a crypto background necessary?*

Nope! No background is required. The *purpose* of this course is to teach you crypto!

- *Readings before/after lecture?*

It is not expected to go over the readings before lecture. The expectation is that these readings would help you learn more stuff after lecture if needed.

- *Another course with conflicting time schedule?*

Although lectures are recorded, we highly recommend participation and attending class in-person! As such, we don't recommend taking another course with a conflicting time schedule. However, if this is your last semester or you have an extenuating circumstance, Peihan will consider approving your ASK request. Moreover, the course will most likely be offered next Spring!

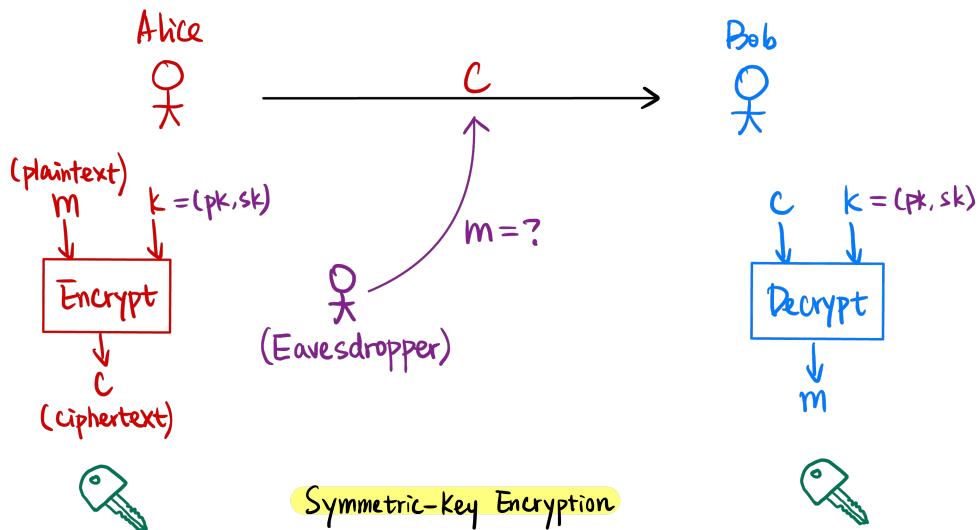
- *Call me Peihan!*

Peihan would highly prefer you call her "Peihan," not professor!

### §2.2 Encryption Scheme Basics

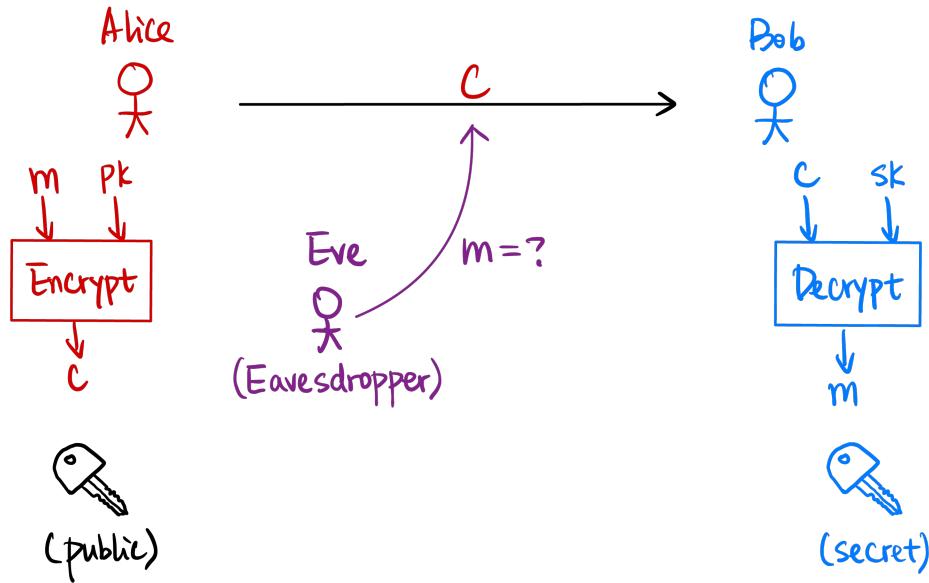
This lecture we'll cover encryption schemes. We briefly mentioned what encryption schemes were last class, we'll dive into the technical content: how we construct them, assumptions, RSA, ElGamal, etc.

Fundamentally, an encryption scheme protects message secrecy. If Alice wants to communicate to Bob, Alice will encrypt a message (plaintext) using some key which gives her a ciphertext. Sending the ciphertext through Bob using a public channel, Bob can use the key to decrypt the ciphertext and recover the message. An eavesdropper in the middle will have no idea what message has been transmitted.



In this case, they are using a shared key, which we called secret-key encryption or symmetric-key encryption.

A stronger version of private-key encryption is called public-key encryption. Alice and Bob do not need to agree on a shared secret key beforehand. There is a keypair  $(pk, sk)$ , a *public* and *private* key.



### §2.2.1 Syntax

**Definition 2.1 (Symmetric-Key Encryption)**

A symmetric-key encryption (SKE) scheme contains 3 algorithms,  $\Pi = (\text{Gen}, \text{Enc}, \text{Dec})$ .

**Generation.** Generates key  $k \leftarrow \text{Gen}$ .

**Encryption.** Encrypts message  $m$  with key  $k$ ,  $c \leftarrow \text{Enc}(k, m)$ , which we sometimes write as  $\text{Enc}_k(m)$ .

**Decryption.** Decrypts using key  $k$  to retrieve message  $m$ ,  $m := \text{Dec}(k, c)$ , or written as  $\text{Dec}_k(c)$ .

Note the notation  $\leftarrow$  and  $:=$  is different. In the case of generation and encryption, the produced key  $k$  or  $c$  follows a *distribution* (is randomly sampled), while we had better want decryption to be deterministic in producing the message.

In other words, we use  $\leftarrow$  when the algorithm might involve randomness and  $:=$  when the algorithm is deterministic.

**Definition 2.2 (Public-Key Encryption)**

A public-key encryption (PKE) scheme  $\Pi = (\text{Gen}, \text{Enc}, \text{Dec})$  contains the same 3 algorithms,

**Generation.** Generate keys  $(pk, sk) \leftarrow \text{Gen}$ .

**Encryption.** Use the public key to encrypt,  $c \leftarrow \text{Enc}(pk, m)$  or  $\text{Enc}_{pk}(m)$ .

**Decryption.** Use the secret key to decrypt,  $m := \text{Dec}(pk, c)$  or  $\text{Dec}_{sk}(c)$ .

**Remark 2.3.** Note that all these algorithms are public knowledge. This is known as Kirchoff's principle.

Intuitively, one key reason is because if the security of our scheme relies on hiding the *algorithm*, then if it is leaked we need to construct an entirely new algorithm. However, if the security of our scheme relies on, for example, a secret key, then we simply need to generate a new key.

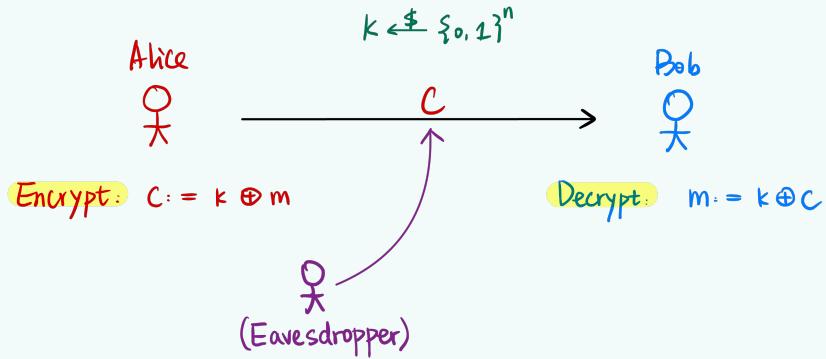
**Question.** If we can construct public-key encryption, why do we even bother with secret-key encryption? We could just use the  $(pk, sk)$  pair for our secret key, and this does the same thing.

1. First of all, public-key encryption is almost always *more expensive*. Symmetric-key encryption schemes give us efficiency.
2. Public-key encryption relies on much stronger computational assumptions, whereas symmetric key encryption are still post-quantum secure.

### §2.2.2 Symmetric-Key Encryption Schemes

#### Definition 2.4 (One-Time Pad)

Secret key is a uniformly randomly sampled  $n$  bit string  $k \xleftarrow{\$} \{0, 1\}^n$ .



**Encryption.** Alice uses the secret key and bitwise-XOR with the plaintext.

$$\begin{array}{rcl} \text{secret key } & k = 0100101 \\ \oplus \text{ plaintext } & m = 1001001 \\ \hline \text{ciphertext } & c = 1101100 \end{array}$$

**Decryption.** Bob uses the secret key and again bitwise-XOR with the ciphertext

$$\begin{array}{rcl} \text{secret key } & k = 0100101 \\ \oplus \text{ ciphertext } & c = 1101100 \\ \hline \text{plaintext } & m = 1001001 \end{array}$$

This is widely used in cryptography, called *masking* or *unmasking*.

**Question.** Why is this correct?

An XOR done twice with the same choice bit  $b$  is the identity. Or, an element is its own inverse with the XOR operator.

**Question.** Why is this secure?

We can think about this as the distribution of  $c$ .  $\forall m \in \{0, 1\}^n$ , the encryption of  $m$  is uniform over  $\{0, 1\}^n$  (since  $k$  was uniform).

Another way to think about this is that for any two messages  $m_0, m_1 \in \{0, 1\}^n$ ,  $\text{Enc}_k(m_0) \equiv \text{Enc}_k(m_1)$ . That is, the encryptions follow the *exact same* distribution. In this case, they are both uniform, but this is not always the case.

**Question.** Can we reuse  $k$ ? Should we use the same key again to encrypt another message? Or, it is possible for the eavesdropper to extract information.

For example,  $\text{Enc}_k(m)$  is  $c := k \oplus m$ , and  $\text{Enc}_k(m')$  is  $c' := k \oplus m'$ . If the two messages are the same, the ciphertexts are the same.

By XOR  $c$  and  $c'$ , we get

$$\begin{aligned} c \oplus c' &= (k \oplus m) \oplus (k \oplus m') \\ &= m \oplus m' \end{aligned}$$

This is why this is an *one-time pad*. This is a bit of an issue, to send an  $n$ -bit message, we need to agree on an  $n$ -bit message.

In fact, this is *the best* that we can do.

**Theorem 2.5 (Shannon's Theorem)**

*Informally*, for perfect (information-theoretic<sup>3</sup>) security, the key space must be at least as large as the message space.

$$|\mathcal{K}| \geq |\mathcal{M}|$$

where  $\mathcal{K}$  is the key space and  $\mathcal{M}$  is the message space.

**Question.** How can we circumvent this issue?

The high level idea is that we weaken our security guarantees *a little*. Instead of saying that they have to be *exactly the same* distribution, we say that they are *hard to distinguish* for an adversary with limited computational power. This is how modern cryptography gets around these lower bounds in classical cryptography. We can make *computational assumptions* about cryptography.

We can think about computational security,

**Definition 2.6 (Computational Security)**

We have computational security when two ciphertexts have distribution that cannot be distinguished using a polynomial-time algorithm.

---

<sup>3</sup>That the distributions of ciphertexts are identical, that  $\text{Enc}_k(m_0) \equiv \text{Enc}_k(m_1)$ .

**Definition 2.7 (Polynomial-Time Algorithm)**

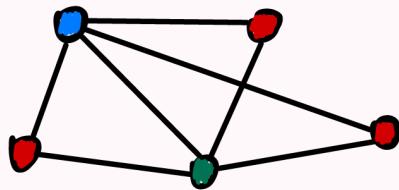
A polynomial time algorithm  $A(x)$  is one that takes input  $x$  of length  $n$ ,  $A$ 's running time is  $O(n^c)$  for a constant  $c$ .

**Definition 2.8 (NP Problem)**

A decision problem is in nondeterministic polynomial-time when its solution can be *verified* in polynomial time.

**Example 2.9 (Graph 3-Coloring)**

Given a graph, does it have a 3-coloring such that no two edges join the same color? For example,



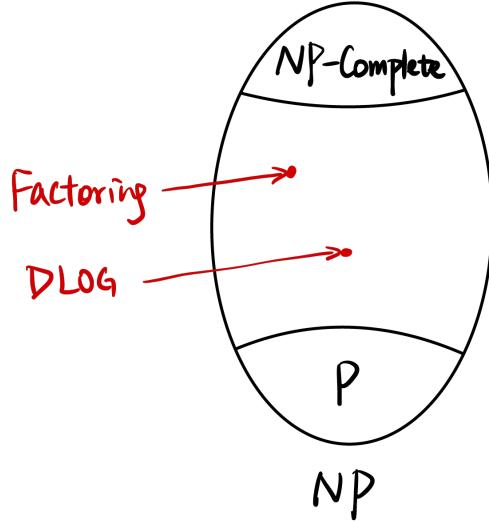
This can be *verified* in polynomial time (we can check if such a coloring is a valid 3-coloring), but it is computed in NP time.

**Definition 2.10**

An NP-complete problem is a “hardest” problem in NP. Every problem in NP is at least as hard as an NP-complete problem.

Right now, we assume  $P \neq NP$ . As of right now, there is no realistic algorithm that can solve any NP problem in polynomial-time.

Even further, we pick some problems not in NP-complete, not in P. We assume they are neither NP-complete nor in P (we don't yet have a reduction, but we don't know if one could exist) The reasoning behind using these problems is as we have no good cryptoscheme relying solely on NP-complete problems (we need something weaker).



Going back to our definition of computational security [definition 2.6](#),

#### Definition (Computational Security)

Let the adversary be computationally bounded (i.e. only a *polynomial-time algorithm*). Then  $\forall$  probabilistic poly-time algorithm  $\mathcal{A}$ ,

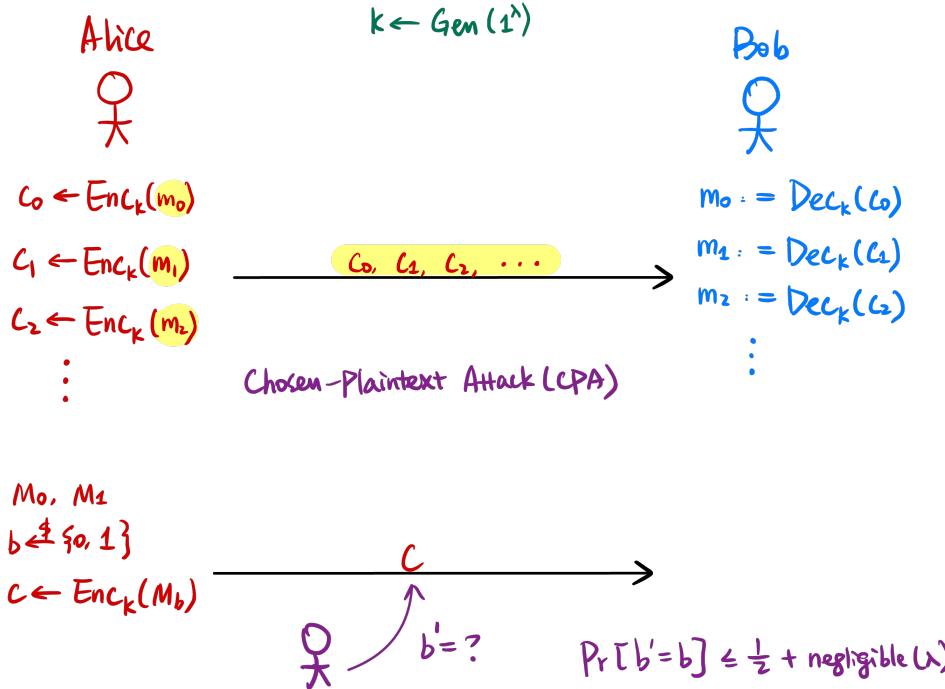
$$\text{Enc}_k(m_0) \stackrel{c}{\sim} \text{Enc}_k(m_1)$$

Where  $\stackrel{c}{\sim}$  is “computationally indistinguishable”.

What does it mean for distributions to be “computationally indistinguishable”? Let’s say Alice encrypts multiple messages  $m_0, m_1, \dots$  to Bob and produces  $c_0, c_1, \dots$ . Even if Eve can see all plaintexts  $m_i$  in the open and ciphertexts  $c_i$  in the open, between known  $m_0, m_1$  and randomly encrypting one of them  $c \leftarrow \text{Enc}_k(m_b)$  where  $b \xleftarrow{\$} \{0, 1\}$ , the adversary cannot determine what the random choice bit  $b$  is. That is,  $\Pr[b = b'] \leq \frac{1}{2} + \text{negligible}(\lambda)$ <sup>4</sup>. This is Chosen-Plaintext Attack (CPA) Security.

---

<sup>4</sup> $\lambda$  is the security parameter, roughly a measure of how secure the protocol is. If it were exactly equal  $\frac{1}{2}$ , we have information-theoretic security.



**Example 2.12**

If the best algorithm is a brute-force search for  $k$ , what should our key length be?

It can just be a  $\lambda$  bit string.

**Example**

What if the best algorithm is no longer a brute-force search, but instead for a key length  $l$  takes  $\sim \sqrt{2^l}$ ?

Our key length should be  $2\lambda$ . Doing the math, we want  $\sqrt{2^l} \equiv 2^\lambda$ , solving for  $l$  gives  $2\lambda$ .

Going back to the original problem of secret-key encryption, how can we use our newfound cryptographic constructions to improve this?

From a pseudorandom function/permuation (PRF/PRP), we can reuse our secret key by passing it through the pseudorandom function.

The current practical construction for PRD/PRP is called the block cipher, and the standardized implementation is AES<sup>5</sup>

It is a computational assumption<sup>6</sup> that the AES construction is secure, and the best attack is currently a brute-force search (in both classical and quantum computing realms).

### §2.2.3 Public-Key Encryption Schemes

Using computational assumptions, we explore some public-key encryption schemes.

**RSA Encryption.** This is based on factoring/RSA assumption, that factoring large numbers is hard.

**EIGamal Encryption.** This is based on the discrete logarithm/Diffie-Hellman Assumption, that finding discrete logs in  $\mathbb{Z}_p$  is hard.

**Lattice-Based Encryption.** The previous two schemes are not quantum secure. Quantum computation will break these schemes. Lattice-based encryption schemes are post-quantum secure. They are associated with the difficulty of finding ‘short’ vectors in lattices<sup>7</sup>.

Another thing worth mentioning is that

<sup>5</sup>Determined via a competition for such an algorithm in the early 2000s.

<sup>6</sup>Based on heuristic, not involving any number theory!

<sup>7</sup>Covered later in class, we focus on the first two now.

**Theorem 2.13**

(Very informally,) It is impossible to construct PKE from SKE in a black-box way. This is called “black-box separation”.

We first need to define a bit of number theory background.

**Definition 2.14**

We denote  $a \mid b$  as  $a$  divides  $b$ , that is, there is integer  $c$  such that  $b = a \cdot c$ .

**Definition 2.15**

The  $\gcd(a, b)$  is the greatest common divisor of  $a, b$ . If  $\gcd(a, b) = 1$ , then  $a, b$  are coprime.

**Question.** How do we compute gcd? What is its time complexity?

**Example**

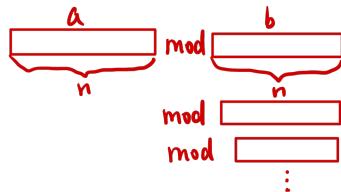
We use the Euclidean Algorithm. Take  $\gcd(12, 17)$ ,

$$\begin{aligned} 17 \mod 12 &= 5 \\ 12 \mod 5 &= 2 \\ 5 \mod 2 &= 1 \\ 2 \mod 1 &= 0 \end{aligned}$$

or take  $\gcd(12, 18)$

$$\begin{aligned} 18 \mod 12 &= 6 \\ 12 \mod 6 &= 0 \end{aligned}$$

If we have two bitstrings of length  $n$  bits, what is the running time of the Euclidian Algorithm?



Very informally, we see that every step, the length of  $a, b$  decrease by approximately 1 bit. Then, finding gcd is roughly order  $O(n)$ .

**Definition 2.16 (Mod)**

$a \bmod N$  is the remainder of  $a$  when divided by  $N$ .

$a \equiv b \pmod{N}$  means when  $a$  and  $b$  are congruent modulo  $N$ . That is,  $a \bmod N = b \bmod N$ .

**Question.** How might we compute  $a^b \bmod N$ ? What is the time complexity? Let  $a, b, N$  be  $n$ -bit strings.

Naïvely, we can repeatedly multiply. But this takes  $b$  steps ( $2^n$ ).

We can ‘repeatedly square’. For example, we can get to  $a^8$  faster by getting  $a^2$ , squaring to get  $a^4$ , and again to get  $a^8$ . We can take the bitstring of  $b$  and determine how to compute this.

**Example**

If  $b = 100101_2$ , we take  $a \cdot a^4 \cdot a^{32} \bmod N$  which can be calculated recursively (an example is given in the first assignment).

The time complexity of this is order  $O(n)$  for  $n$ -bit  $a, b, N$ <sup>8</sup>.

**Theorem 2.17 (Bezout’s Theorem, roughly)**

If  $\gcd(a, N) = 1$ , then  $\exists b$  such that

$$a \cdot b \equiv 1 \pmod{N}.$$

This is to say,  $a$  is invertible modulo  $N$ .  $b$  is its inverse, denoted as  $a^{-1}$ .

**Question.** How do we compute  $b$ ?

We can use the Extended Euclidean Algorithm!

**Example**

We write linear equations of  $a$  and  $N$  that sum to 1, using our previous Euclidean Algorithm.

---

<sup>8</sup>Not exactly order  $n$ , we should add the complexity of multiplication. However, this should be bounded by  $N$  since we can log at every step.

Take the previous example  $\gcd(12, 17)$ ,

$$\begin{aligned} 17 &\mod 12 = 5 \\ 12 &\mod 5 = 2 \\ 5 &\mod 2 = 1 \\ 2 &\mod 1 = 0 \end{aligned}$$

We write this as

$$\begin{aligned} 5 &= 17 - 12 \cdot 1 \\ 2 &= 12 - 5 \cdot 2 = 12 \cdot x + 17 \cdot y \\ 1 &= 5 - 2 \cdot 2 = 12 \cdot x' + 17 \cdot y' \end{aligned}$$

where we substitute the linear combination of 5 into 5 on line 2, substitute linear combination of 2 into 2 on line 1, each producing another linear combination of 12, 17.

If  $\gcd(a, N) = 1$ , we use the Extended Euclidean Algorithm to write  $1 = a \cdot x + N \cdot y$ , then  $1 \equiv a \cdot x \pmod{N}$ .

### Definition 2.18 (Group of Units mod $N$ )

We have set

$$\mathbb{Z}_N^\times := \{a \mid a \in [1, N-1], \gcd(a, N) = 1\}$$

which is the group of units modulo  $N$  (they are units since they all have an inverse by above).

### Definition 2.19 (Euler's Phi Function)

Euler's phi (or totient) function,  $\phi(N)$ , counts the number of elements in this set. That is,  $\phi(N) = |\mathbb{Z}_N^\times|$ .

### Theorem 2.20 (Euler's Theorem)

For all  $a, N$  where  $\gcd(a, N) = 1$ , we have that

$$a^{\phi(N)} \equiv 1 \pmod{N}.$$

With this, we can start talking about RSA.

#### §2.2.4 RSA

We first define the RSA assumption.

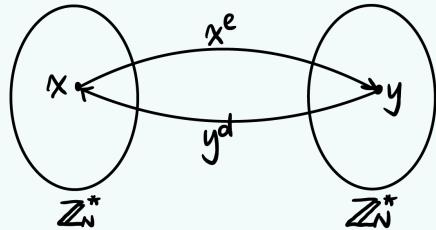
**Definition 2.21 (Factoring Assumption)**

Given two  $n$ -bit primes  $p, q$ , we compute  $N = p \cdot q$ . Given  $N$ , it's computationally hard to find  $p$  and  $q$  (classically).

**Definition 2.22 (RSA Assumption)**

Given two  $n$ -bit primes, we again compute  $N = p \cdot q$ , where  $\phi(N) = (p - 1)(q - 1)$ . We choose an  $e$  such that  $\gcd(e, \phi(N)) = 1$  and compute  $d \equiv e^{-1} \pmod{\phi(N)}$ .

Given  $N$  and a random  $y \xleftarrow{\$} \mathbb{Z}_N^\times$ , it's computationally hard to find  $x$  such that  $x^e \equiv y \pmod{N}$ .



However, given  $p, q$ , it's easy to find  $d$ . We know  $\phi(N) = (p - 1)(q - 1)$ , so we can compute  $d$  from  $e$  by running the Extended Euclidean Algorithm. Then, taking  $(x^e)^d \equiv x^{ed} \equiv x$  which allows us to extract  $x$  again.

Encrypting is exactly raising by power  $d$ , and decrypting is raising again by power  $e$ .

Remaining questions:

- How can we generate primes  $p, q$ ?
- How can we pick  $e$  such that  $\gcd(e, \phi(N)) = 1$ ?
- What security issues can you see?

We'll continue next class.

## §3 January 31, 2024

### §3.1 Basic Number Theory, *continued*

We recall a couple more definitions.

#### Definition 3.1

We first define the multiplicative group of integers modulo n as

$$\mathbb{Z}_N^* = \{a \in [1, N - 1] \mid \gcd(a, N) = 1\}$$

#### Definition 3.2

We define the Euler's totient function as

$$\phi(N) = |\mathbb{Z}_N^*|$$

#### Example 3.3

If  $N = p \cdot q$  where  $p, q$  are prime, then  $\phi(N) = (p - 1)(q - 1)$ .

#### Theorem 3.4 (Euler's Theorem)

$\forall a, N$  where  $\gcd(a, N) = 1$ , we have that  $a^{\phi(N)} \equiv 1 \pmod{N}$ .

#### Corollary 3.5

If

$$d \equiv e^{-1} \pmod{\phi(N)}$$

, then

$$\forall a \in \mathbb{Z}_N^*, (a^d)^e \equiv a \pmod{N}$$

.

### §3.2 RSA Encryption, *continued*

*Recall:* that the RSA encryption algorithm contains 3 components:

**Gen**( $1^\lambda$ ): Generate two  $n$ -bit primes  $p, q$ . We compute  $N = p \cdot q$  and  $\phi(N) = (p - 1)(q - 1)$ . Choose  $e$  such that  $\gcd(e, \phi(N)) = 1$ . We compute  $d = e^{-1} \pmod{\phi(N)}$ . Our public key  $pk = (N, e)$ , our secret key is  $sk = d$ .

$\text{Enc}_{pk}(m)$ :  $c = m^e \pmod{N}$ .

$\text{Dec}_{sk}(c)$ :  $m = c^d \pmod{N}$ .

We have a few remaining questions:

1. How do we generate 2 primes  $p, q$ ?
2. How do we choose such an  $e$ ?
3. How do we compute  $d = e^{-1} \pmod{\phi(N)}$ ?
4. How do we efficiently compute  $m^e \pmod{N}$  and  $c^d \pmod{N}$ .

How do we resolve these issues to ensure the `Gen` step is efficient (polynomial time).

1. We pick an arbitrary number  $p$  and check for primality efficiently (using Miller Rabin, a probabilistic primality test). We pick random numbers until they are prime. Since primes are ‘pretty dense’ in the integers, this can be done efficiently.
2. We can also guess! Since we’re unsure whether coprime numbers are dense, we can pick small prime  $e$ .
3. We can compute  $d$  using the Extended Euclidean Algorithm.
4. We can repeatedly square (using fast power algorithm).

**Question.** What happens if we can factor  $N$ ?

Then we can find  $p$  and  $q$  and calculate  $\phi(N) = (p - 1)(q - 1)$ , and then we can compute  $d = e^{-1} \pmod{\phi(N)}$ . Thus, RSA relies crucially on the factoring problem being hard.

**Question.** The above scheme is known as “plain” RSA. Are there any security issues?

- It relies on factoring being difficult (this is the computational assumption). Post-quantum, Shor’s Algorithm will break RSA.
- Recall last lecture that CPA (Chosen-Plaintext Attack) security was defined as an adversary not being able to discern between an encryption of  $m_0$  and  $m_1$ , *knowing*  $m_0$  and  $m_1$  in the clear.

Eve could just encrypt  $m_0$  and  $m_1$  themselves using public  $e$ , and discern which of the plaintexts the ciphertext corresponds to. Using RSA, you *really* have to be careful. For RSA, this is a very concrete attack.

The concrete reason is that the encryption algorithm  $\text{Enc}$  is *deterministic*. If you encrypt the same message twice, it will be the same ciphertext. We really want to be sure that  $m \xleftarrow{\$} \mathbb{Z}_N^\times$  (that it has enough entropy).

Returning on the RSA assumption. It's crucial that the  $y \xleftarrow{\$} \mathbb{Z}_N^\times$  is randomly sampled.

**Question.** In practice, how can RSA be useful with these limitations?

As long as we pick the plaintext which is randomly sampled, security for RSA holds. There is also a more involved way of using RSA that is CPA-secure, but we will not go in detail of it.

**Remark.** In practice, we usually set length of  $p$  and  $q$  to be 1024 bits, and the key length is 2048 bits.

Moreover, note that although exponentiation can be done in polynomial time, it's still a very expensive operation. This is why we use RSA and public key encryption is, in general, more expensive.

### §3.3 Intro to Group Theory

#### Definition 3.6 (Group)

A group is a set  $\mathbb{G}$  along with a binary operation  $\circ$  with properties:

**Closure.**  $\forall g, h \in \mathbb{G}, g \circ h \in \mathbb{G}$ .

**Existence of an identity.**  $\exists e \in \mathbb{G}$  such that  $\forall g \in \mathbb{G}, e \circ g = g \circ e = g$ .

**Existence of inverse.**  $\forall g \in \mathbb{G}, \exists h \in \mathbb{G}$  such that  $g \circ h = h \circ g = e$ . We denote the inverse of  $g$  as  $g^{-1}$ .

**Associativity.**  $\forall g_1, g_2, g_3 \in \mathbb{G}, (g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$ .

We say a group is additionally *Abelian* if it satisfies

**Commutativity.**  $\forall g, h \in \mathbb{G}, g \circ h = h \circ g$ .

For a finite group, we use  $|\mathbb{G}|$  to denote its *order*.

#### Example 3.7

$(\mathbb{Z}, +)$  is an Abelian group.

We can check so: two integers sum to an integer, identity is 0, the inverse of  $a$  is  $-a$ , addition is associative and commutative.

$(\mathbb{Z}, \cdot)$  is not a group.

$(\mathbb{Z}_N^\times, \cdot)$  is an Abelian group ( $\cdot$  is multiplication mod  $N$ ).

### Definition 3.8 (Cyclic Group)

Let  $\mathbb{G}$  be a group of order  $m$ . We denote

$$\langle g \rangle := \{e = g^0, g^1, g^2, \dots, g^{m-1}\}.$$

$\mathbb{G}$  is a cyclic group if  $\exists g \in \mathbb{G}$  such that  $\langle g \rangle = \mathbb{G}$ .  $g$  is called a generator of  $\mathbb{G}$ .

### Example

$\mathbb{Z}_p^\times$  (for prime  $p$ ) is a cyclic group of order  $p - 1$ <sup>9</sup>.

$$\mathbb{Z}_7^\times = \{3^0 = 1, 3^1, 3^2 = 2, 3^3 = 6, 3^4 = 5, 3^5 = 5\}.$$

**Question.** How do we find a generator?

For every element, we can continue taking powers until  $g^\alpha = 1$  for some  $\alpha$ . We hope that  $\alpha = p - 1$  (the order of  $g$  is the order of the group), but we know at least  $\alpha \mid p - 1$ .

## §3.4 Computational Assumptions

We have a few assumptions we make called the Diffie-Hellman Assumptions, in order of **weakest to strongest**<sup>10</sup> assumptions.

Let  $(\mathbb{G}, q, g) \leftarrow \mathcal{G}(1^\lambda)$  be a cyclic group  $\mathbb{G}$  of order  $q$  (a  $\theta(\lambda)$ -bit integer) with generator  $g$ . For integer groups, keys are usually 2048-bits. For elliptic curve groups, keys are usually 256-bits.

### Definition 3.9 (Discrete Logarithm (DLOG) Assumption)

Let  $x \xleftarrow{\$} \mathbb{Z}_q$ . We compute  $h = g^x$ .

Given  $(\mathbb{G}, q, g, h)$ , it's computationally hard to find the exponent  $x$  (classically).

<sup>9</sup>A proof of this extends beyond the scope of this course, but you are recommended to check out Math 1560 (Number Theory) or Math 1580 (Cryptography). You can take this on good faith.

<sup>10</sup>If one can solve DLOG, we can solve CDH. Given CDH, we can solve DDH. This is why CDH is *stronger* than DDH, and DDH is *stronger* than DLOG. It's not necessarily true the other way around (similar to factoring and DSA assumptions).

**Definition 3.10 (Computational Diffie-Hellman (CDH) Assumption)**

$x, y \xleftarrow{\$} \mathbb{Z}_q$ , compute  $h_1 = g^x, h_2 = g^y$ .

Given  $(\mathbb{G}, q, g, h_1, h_2)$ , it's computationally hard to find  $g^{xy}$ .

**Definition 3.11 (Decisional Diffie-Hellman (DDH) Assumption)**

$x, y, z \xleftarrow{\$} \mathbb{Z}_q$ . Compute  $h_1 = g^x, h_2 = g^y$ .

Given  $(\mathbb{G}, q, g, h_1, h_2)$ , it's computationally hard to distinguish between  $g^{xy}$  and  $g^z$ .

$$(g^x, g^y, g^{xy}) \stackrel{c}{\simeq} (g^x, g^y, g^z).$$

### §3.5 ElGamal Encryption

The ElGamal encryption scheme involves the following:

**Gen**( $1^\lambda$ ): We generate a group  $(\mathbb{G}, q, g) \leftarrow \mathcal{G}(1^\lambda)$ . We sample  $x \xleftarrow{\$} \mathbb{Z}_q$ , compute  $h = g^x$ . Our public key is  $pk = (\mathbb{G}, q, g, h)$ , secret key  $sk = x$ .

**Enc** <sub>$pk$</sub> ( $m$ ): We have  $m \in \mathbb{G}$ . We sample  $y \xleftarrow{\$} \mathbb{Z}_q$ . Our ciphertext is  $c = \langle g^y, h^y \cdot m \rangle$ . Note that  $h = g^x$ , so  $g^{xy} \stackrel{c}{\simeq} g^z$  is a one-time pad for our message  $m$ .

**Dec** <sub>$sk$</sub> ( $c$ ): To decrypt  $c = \langle c_1, c_2 \rangle$ , we raise

$$\begin{aligned} c_1^x &= (g^y)^x = g^{xy} \\ m &= \frac{g^{xy} \cdot m}{g^{xy}} = c_2 \cdot (c_1^x)^{-1}. \end{aligned}$$

Notes about ElGamal:

- Our group can be reused! We can use a public group that is fixed. In fact, there are *popular* groups out there used in practice. Some of these are Elliptic Curve groups which are much more efficient than integer groups. You don't need to use the details, yet you can use it! You can use any group, so long as the group satisfies the DDH assumption.
- Similar to RSA, this is breakable post-quantum. Given Shor's Algorithm, we can break discrete log.

### §3.6 Secure Key Exchange

Using DDH, we can construct something very important, *secure key exchange*.

**Definition 3.12 (Secure Key Exchange)**

Alice and Bob sends messages back and forth, and at the end of the protocol, can agree on a shared key.

An eavesdropper looking at said communications cannot figure out what shared key they came up with.

**Theorem 3.13**

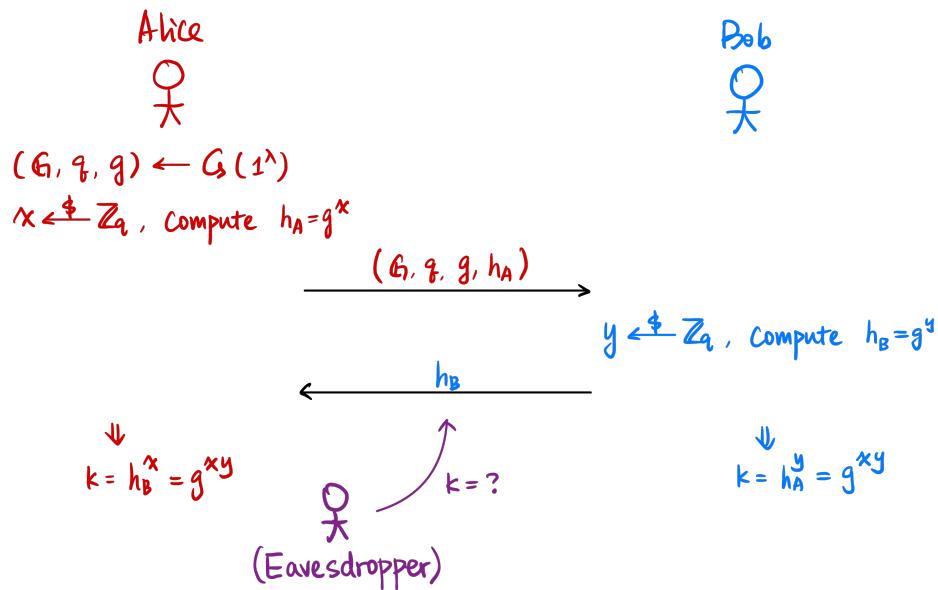
*Informally*, It's impossible to construct secure key exchange from secret-key encryption in a black-box way.

**Question.** How do we build a key exchange from public-key encryption?

Bob generates a keypair  $(pk, sk)$ . Alice generates a shared key  $k \xleftarrow{\$} \{0, 1\}^\lambda$ , and sends  $\text{Enc}_{pk}(k)$  to Bob.

Using Diffie-Hellman, it's very easy. We have group  $(\mathbb{G}, q, g) \leftarrow \mathcal{G}(1^\lambda)$ . Alice samples  $x \xleftarrow{\$} \mathbb{Z}_q$  and sends  $g^x$ . Bob also samples  $y \xleftarrow{\$} \mathbb{Z}_q$  and sends  $g^y$ . Both Alice and Bob compute  $g^{xy} = (g^x)^y = (g^y)^x$ .

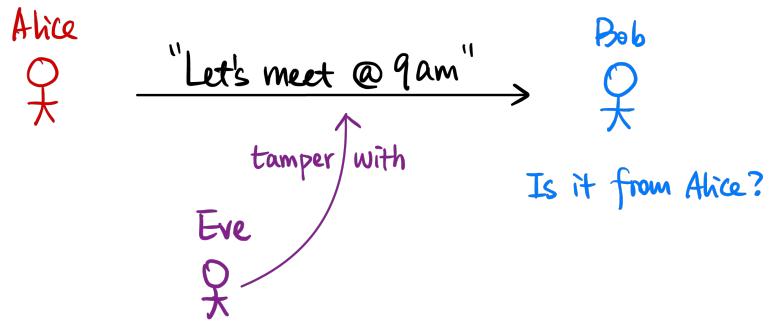
Diffie-Hellman Key Exchange



What happens in practice is that parties run Diffie-Hellman key exchange to agree on a shared key. Using that shared key, they run symmetric-key encryption. This gives us efficiency. Additionally, private-key encryptions don't rely on heavy assumptions on the security of protocols (such as the DDH, RSA assumptions).

### §3.7 Message Integrity

Alice sends a message to Bob, how does Bob ensure that the message came from Alice?



We can build up another line of protocols to ensure message integrity. It's similar to encryption, but the parties run 2 algorithms: *Authenticate* and *Verify*.

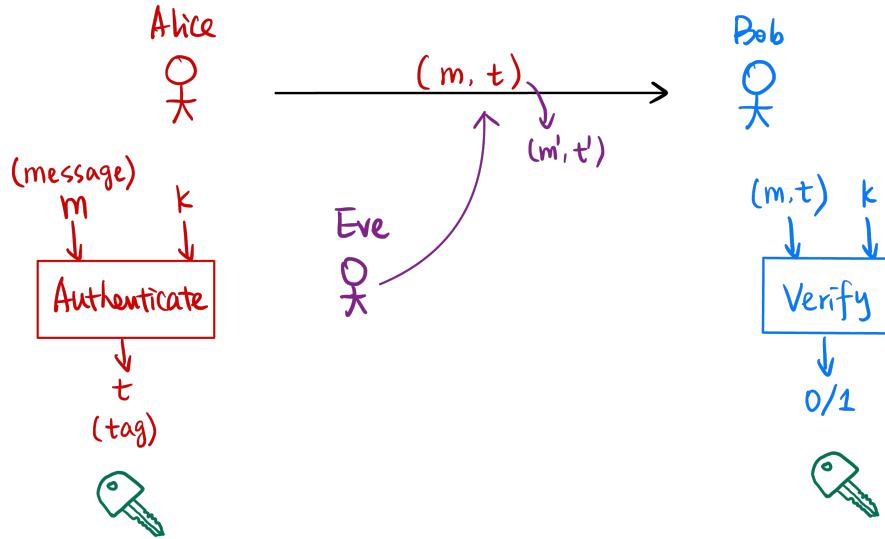
Using a message  $m$ , Alice can generate a *tag* or *signature*, and Bob can verify  $(m, t)$  is either valid or invalid.

Our adversary has been upgraded to an Eve who can now tamper with messages.

Similarly to encryption, we have symmetric-key and public-key encryption.

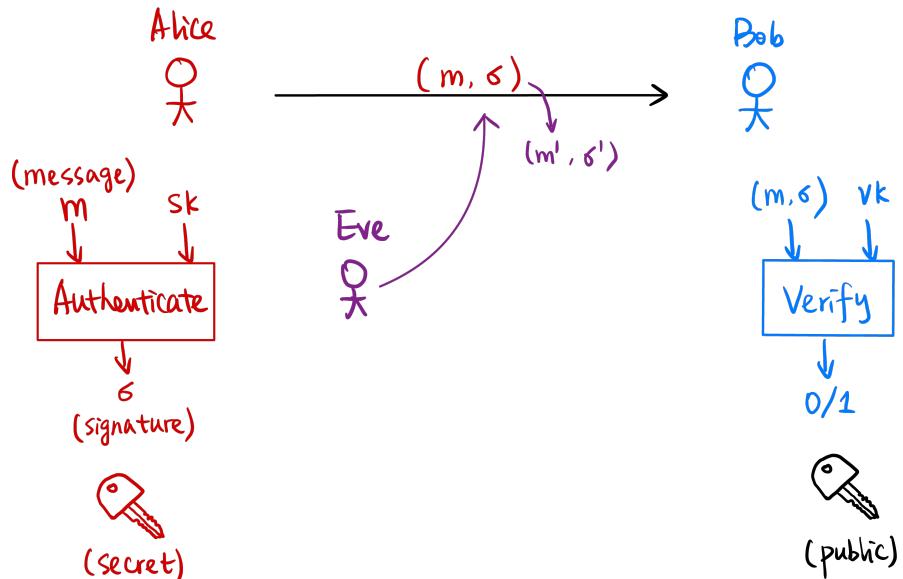
Using a shared key  $k$ , Alice can authenticate  $m$  using  $k$  to get a tag  $t$ . Similarly, Bob can verify whether  $(m, t)$  is valid using  $k$ . This is called a Message Authentication Code.

### Message Authentication Code (MAC)



Using a public key  $vk$  (verification key) and private key  $sk$  (secret/signing key), Alice can sign a message  $m$  using signing key  $sk$  to get a *signature*  $\sigma$ . Bob verifies  $(m, \sigma)$  is valid using  $vk$ . This is called a Digital Signature.

### Digital Signature



### §3.7.1 Syntax

The following is the syntax we use for MACs and digital signatures.

A message authentication code (MAC) scheme consists of  $\Pi = (\text{Gen}, \text{Mac}, \text{Verify})$ .

**Generation.**  $k \leftarrow \text{Gen}(1^\lambda)$ .

**Authentication.**  $t \leftarrow \text{Mac}_k(m)$ .

**Verification**  $0/1 := \text{Verify}_k(m, t)$ .

A digital signature scheme consists of  $\Pi = (\text{Gen}, \text{Sign}, \text{Verify})$ .

**Generation.**  $(sk, vk) \leftarrow \text{Gen}(1^\lambda)$ .

**Authentication.**  $\sigma \leftarrow \text{Sign}_{sk}(m)$ .

**Verification**  $0/1 := \text{Verify}_{vk}(m, \sigma)$ .

## §4 February 5, 2024

### §4.1 Announcements

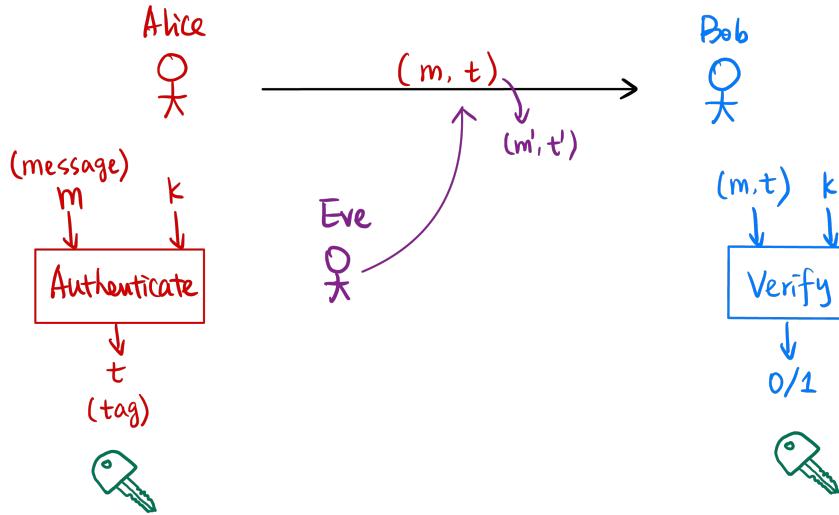
- New room! Due to more enrollment, starting next lecture we will be in Smith-Buonanno 106.
- Hiring more UTAs! If you know anyone who is qualified and interested, please let us know.

### §4.2 Message Integrity

Last lecture, we touched upon methods of authenticating a message. The symmetric-key version is called a MAC (message authentication code), the public-key version is called a digital signature. Let's review what we covered last time.

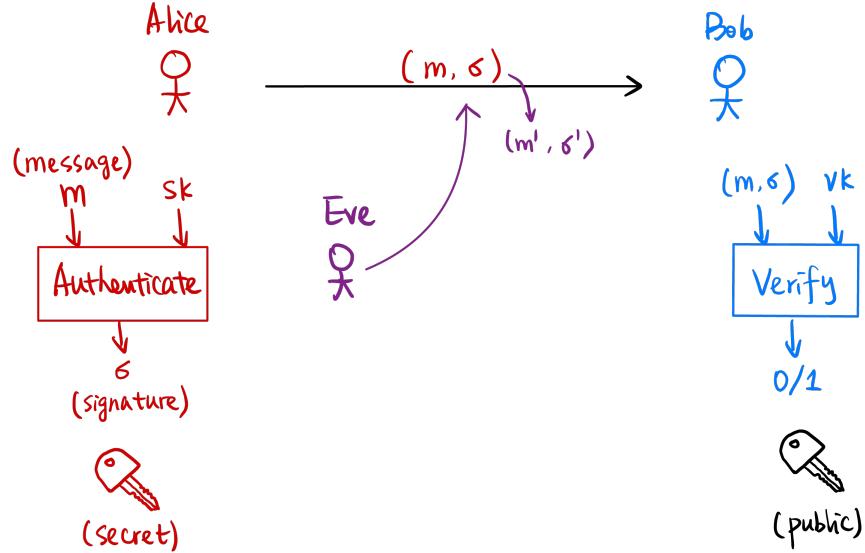
#### §4.2.1 Message Authentication Code

To authenticate a message, Alice will use the private key  $k$  to tag a message  $m$  with a tag  $t$ . Bob will verify that  $(m, t)$  is valid with key  $k$ .



#### §4.2.2 Digital Signature

Similarly we have the public-key version. Alice has a secret key  $sk$  to sign message  $m$  with signature  $\sigma$ . Bob (or anyone) can verify with the public key  $pk$  that  $(m, \sigma)$  is a valid signature.



### §4.2.3 Syntax

Recall the syntax of MAC and digital signatures (see [section 3.7.1](#)).

A message authentication code (MAC) scheme consists of  $\Pi = (\text{Gen}, \text{Mac}, \text{Verify})$ .

**Generation.**  $k \leftarrow \text{Gen}(1^\lambda)$ .

**Authentication.**  $t \leftarrow \text{Mac}_k(m)$ .

**Verification**  $0/1 := \text{Verify}_k(m, t)$ .

A digital signature scheme consists of  $\Pi = (\text{Gen}, \text{Sign}, \text{Verify})$ .

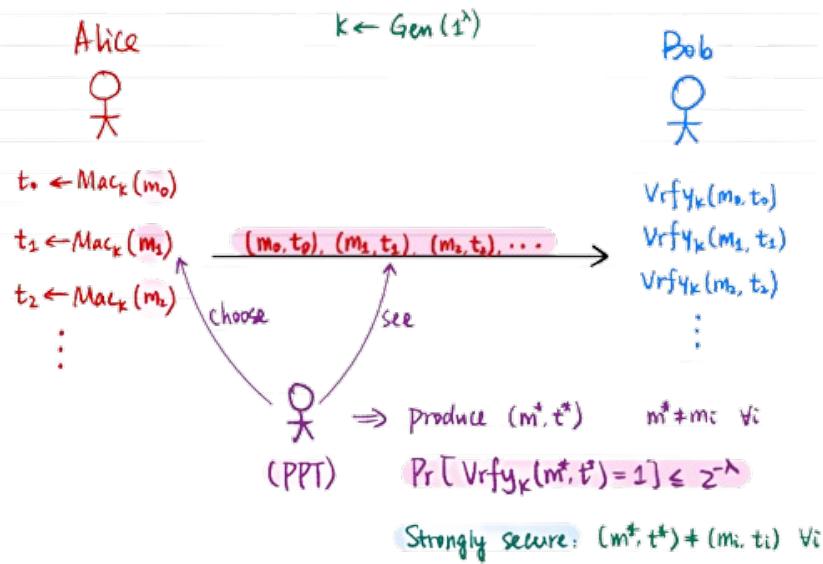
**Generation.**  $(sk, vk) \leftarrow \text{Gen}(1^\lambda)$ .

**Authentication.**  $\sigma \leftarrow \text{Sign}_{sk}(m)$ .

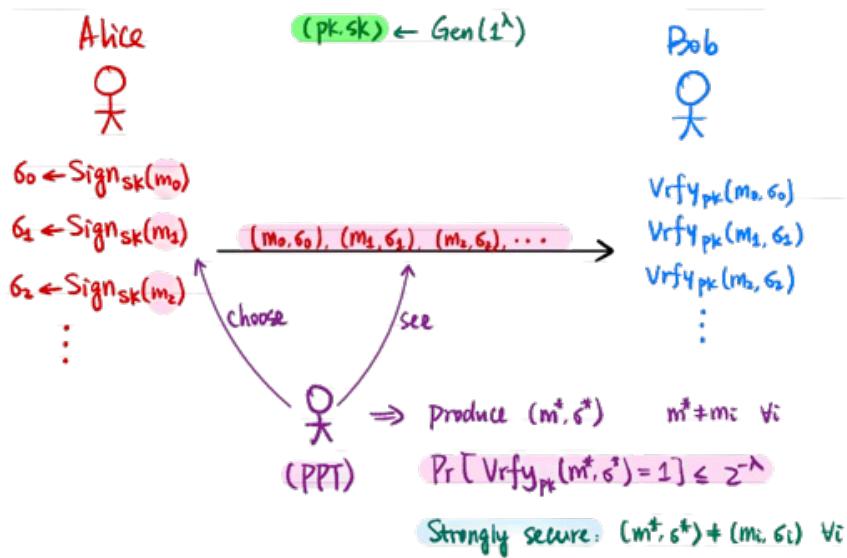
**Verification**  $0/1 := \text{Verify}_{vk}(m, \sigma)$ .

### §4.2.4 Chosen-Message Attack

Similar to chosen-plaintext attack from encryption, we have chosen-message attack security. An adversary chooses a number of messages to generate signatures or tags for. After that, the adversary will try to generate another valid pair of message and tag. We want to make sure that generating a new pair of message and tag is hard.

Chosen-Message Attack (CMA)

Does Mac have to be randomized?

Chosen-Message Attack (CMA)

Does Sign have to be randomized?

**Question.** Do the MAC and signature algorithms need to be randomized to be CMA secure?

No! Unlike CPA security, we don't need these algorithms to be randomized to be CMA secure.

### §4.3 RSA Signatures

Our RSA signatures algorithm works very similarly to RSA encryption.

We generate two  $n$ -bit primes  $p, q$ . Compute  $N := p \cdot q$  and  $\phi(N) = (p - 1)(q - 1)$ . Again choose  $e$  with  $\gcd(e, \phi(N)) = 1$  and invert  $d = e^{-1} \pmod{\phi(N)}$ . Given  $N$  and a random  $y \xleftarrow{\$} \mathbb{Z}_N^\times$ , it's computationally hard to find  $x$  such that  $x^e \equiv y \pmod{N}$ .

Similarly,  $sk := d$  and  $vk := (N, e)$ . To sign, we compute

$$\text{Sign}_{sk}(m) := m^d \pmod{N}.$$

To verify, we compute

$$\text{Verify}_{vk}(m, \sigma) := \sigma^e \stackrel{?}{\equiv} m \pmod{N}.$$

**Question.** Are there any security issues with RSA as we have constructed it so far?

Thinking back to our definition of chosen-message attack, if Eve knows many messages and signatures, she can compute another pair of valid message and keys. If we have messages

$$\begin{aligned} m_0, \sigma_0 &= m_0^d \pmod{N} \\ m_1, \sigma_1 &= m_1^d \pmod{N} \end{aligned}$$

We can compute  $m^* := m_0 \cdot m_1$  and  $\sigma^* := \sigma_0 \cdot \sigma_1 = (m_0 \cdot m_1)^d \pmod{N}$ .

We can do linear combinations of messages, as well as raising messages to arbitrary exponents, and we can get other messages with valid signatures.

There is an easy solution, however. We can hash our message  $m$  before we sign, like so

$$\begin{aligned} \text{Sign}_{sk}(m) &:= H(m)^d \pmod{N} \\ \text{Verify}_{vk}(m, \sigma) &:= \sigma^e \stackrel{?}{\equiv} H(m) \pmod{N} \end{aligned}$$

where  $H$  is a hash function<sup>11</sup>. This is a commonly known technique called ‘hash-and-sign’.

#### §4.3.1 Other Signature Schemes

There are also other signature schemes that rely on other hardness assumptions.

- RSA signatures rely on the RSA assumption.
- Schnorr/DSA signatures rely on the discrete log assumption.
- Lattice-based encryption schemes are post-quantum secure and rely on the hardness of certain lattice problems.

<sup>11</sup>A hash function is, briefly, a function that produces some random output that is hard to compute the inverse of.

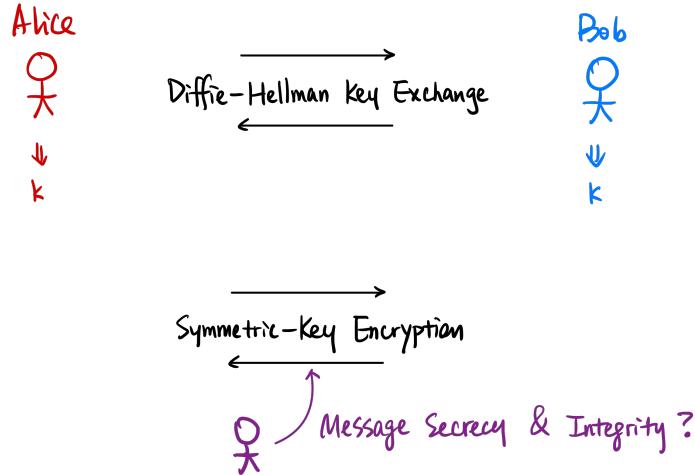
## §4.4 A Summary So Far

To summarize, here's all we've covered so far:

	Symmetric-Key	Public-Key
Message Secrecy	Primitive: SKE Construction: Block Cipher	Primitive: PKE Constructions: RSA/ElGamal
Message Integrity	Primitive: MAC Constructions: CBC-MAC/HMAC	Primitive: Signature Constructions: RSA/DSA
Secrecy & Integrity	Primitive: AE Construction: Encrypt-then-MAC	
Key Exchange		Construction: Diffie-Hellman
Important Tool	Primitive: Hash function Construction: SHA	

## §4.5 Authenticated Encryption

Generally, Alice and Bob will first perform a Diffie-Hellman key exchange, then use that shared key to conduct Symmetric-Key Encryption.

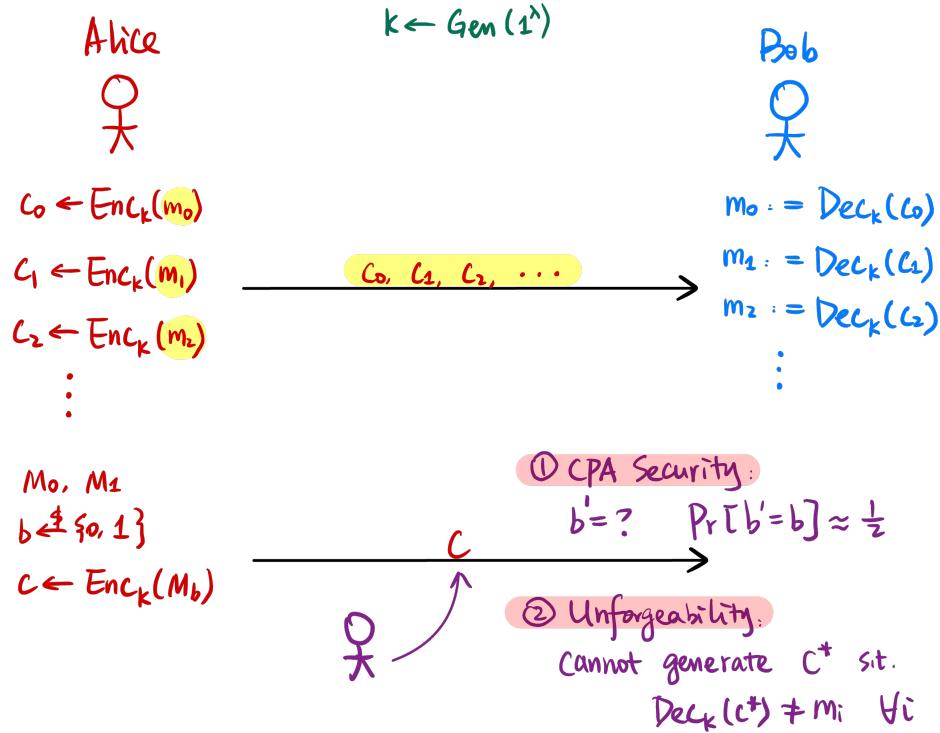


In reality, we want to achieve both message secrecy and integrity *at the same time*. For this, we can introduce Authenticated Encryption.

Our security definition is that our adversary can see the encryptions of many messages  $m_0, m_1, m_2$ . We want **CCA security**: that an adversary given previous ciphertexts and their decryptions cannot

distinguish between the encryptions of a fresh pair of messages  $m_0$  and  $m_1$ . Additionally, we want the property of **unforgeability**, that our adversary cannot generate a  $c^*$  that is a valid encryption, such that  $\text{Dec}_k(c^*) \neq m_i$  for any  $i$ .

### Authenticated Encryption (AE) ← Symmetric-Key Encryption Scheme

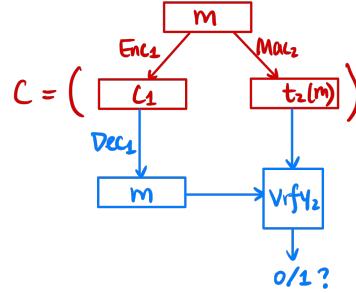


Now that we have two new primitives, we can construct Authenticated Encryption schemes.

#### §4.5.1 Encrypt-and-MAC?

Given a CPA-secure SKE scheme  $\Pi_1(\text{Gen}_1, \text{Enc}_1, \text{Dec}_1)$  and a CMA-secure MAC scheme  $\Pi_2 = (\text{Gen}_2, \text{Mac}_2, \text{Verify}_2)$ .

We construct an AE scheme  $\Pi = (\text{Gen}, \text{Enc}, \text{Dec})$  by composing encryption and MAC. We encrypt the plaintext and also compute the MAC the *plaintext*.



$\text{Gen}(1^\lambda)$ :  $k_1 := \text{Gen}_1(1^\lambda)$ ,  $k_2 := \text{Gen}_2(1^\lambda)$ , output  $(k_1, k_2)$ .

$\text{Enc}(m)$ : To encrypt, we first encrypt ciphertext  $c_1 := \text{Enc}_1(k_1, m)$  and sign message (in plaintext)  $t_2 := \text{Mac}_2(k_2, m)$  and output  $(c_1, t_2)$ .

$\text{Dec}(m)$ : We have ciphertext  $c = (c_1, t_2)$ . Our message is  $m := \text{Dec}_1(k_1, c_1)$ , and our verification bit  $b := \text{Verify}_1(k_2, (m, t_2))$ . If  $b = 1$ , output  $m$ , otherwise we output  $\perp$ .

**Question.** Is this scheme secure? Assuming the CPA-secure SKE scheme and CMA-secure MAC scheme, does this give us both CPA-security and unforgeability?

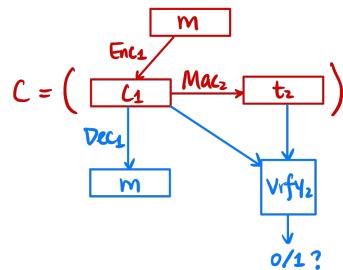
MAC gives you *unforgeability*—it doesn’t even try to hide the message at all. It’s possible that the MAC scheme reveals the message in the clear. For example, we might have a MAC scheme that includes the message in the signature *in the clear* (which is still secure)!

Since MAC doesn’t try to hide the message. If our MAC reveals something about our message, our composed scheme  $\Pi$  doesn’t give us CPA-security. You might still be able to infer something about the message.

We try something else...

### §4.5.2 Encrypt-then-MAC

We encrypt first, then we MAC on the *ciphertext*.



$\text{Gen}(1^\lambda)$ :  $k_1 := \text{Gen}_1(1^\lambda)$ ,  $k_2 := \text{Gen}_2(1^\lambda)$ , output  $(k_1, k_2)$ .

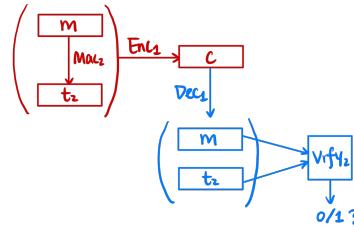
$\text{Enc}(m)$ : To encrypt, we first encrypt ciphertext  $c_1 := \text{Enc}_1(k_1, m)$  and sign message (in ciphertext)  $t_2 := \text{Mac}_2(k_2, \textcolor{red}{c_1})$  and output  $(c_1, t_2)$ .

$\text{Dec}(m)$ : We have ciphertext  $c = (c_1, t_2)$ . Our message is  $m := \text{Dec}_1(k_1, c_1)$ , and our verification bit  $b := \text{Verify}_1(k_2, (\textcolor{red}{c_1}, t_2))$ . If  $b = 1$ , output  $m$ , otherwise we output  $\perp$ .

You can prove that Encrypt-then-MAC schemes are CPA-secure and unforgeable.

### §4.5.3 MAC-then-Encrypt

Similarly, we can also MAC first, encrypt the entire ciphertext and tag concatenated.



$\text{Gen}(1^\lambda)$ :  $k_1 := \text{Gen}_1(1^\lambda)$ ,  $k_2 := \text{Gen}_2(1^\lambda)$ , output  $(k_1, k_2)$ .

$\text{Enc}(m)$ : To encrypt, we first encrypt ciphertext  $c_1 := \text{Enc}_1(k_1, m)$  and sign message (in plaintext)  $t_2 := \text{Mac}_2(k_2, \textcolor{red}{m} \parallel \textcolor{red}{t_2})$  and output  $(c_1, t_2)$ .

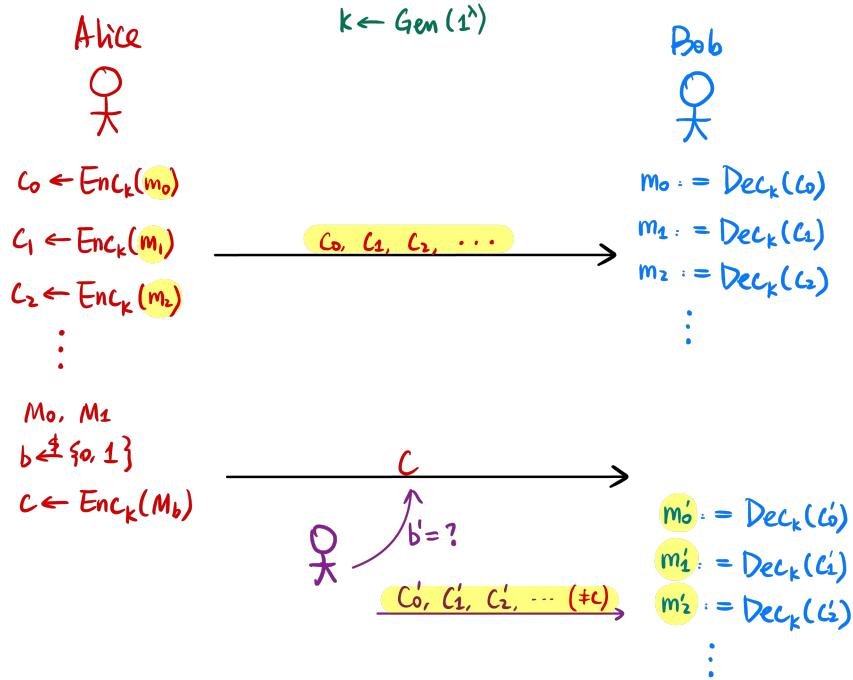
$\text{Dec}(m)$ : We have ciphertext  $c = (c_1, t_2)$ . Our message is  $\textcolor{red}{m} \parallel \textcolor{red}{t_2} := \text{Dec}_1(k_1, c_1)$ , and our verification bit  $b := \text{Verify}_1(k_2, (m, t_2))$ . If  $b = 1$ , output  $m$ , otherwise we output  $\perp$ .

**Question.** Is this secure?

This doesn't satisfy a stronger security definition called Chosen Ciphertext Attack (CCA) security. We might be able to forge ciphertexts that decrypt to valid message and tags.

### §4.5.4 Chosen Ciphertext Attack Security

On top of CMA security, the adversary can now request Alice to decrypt ciphertexts  $c_0, c_1, \dots$



We can prove that MAC-then-Encrypt is not CCA secure.

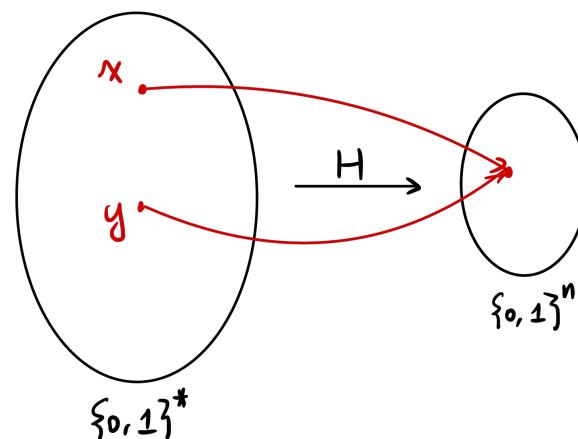
The moral of this is that **you should always use Encrypt-then-MAC**.

## §4.6 Hash Function

A hash function is a public function

$$H : 0, 1^* \rightarrow 0, 1^n$$

where  $n$  is order  $\Theta(\lambda)$ .



We want our hash function to be collision-resistant. That is, it's computationally hard to find  $x, y \in \{0, 1\}^*$  such that  $x \neq y$  yet  $H(x) = H(y)$  (which is called a collision).

How might one find a collision for function  $H : \{0, 1\}^* \rightarrow \{0, 1\}^n$ . We can try  $H(x_1), H(x_2), \dots, H(x_q)$ .

If  $H(x_1)$  outputs a random value,  $0, 1^n$ , what is the probability of finding a collision?

If  $q = 2^n + 1$ , our probability is exactly 1 (by pigeon-hole). If  $q = 2$ , our probability is  $\frac{1}{2^\lambda}$  (we have to get it right on the first try). What  $q$  do we need for a ‘reasonable’ probability?

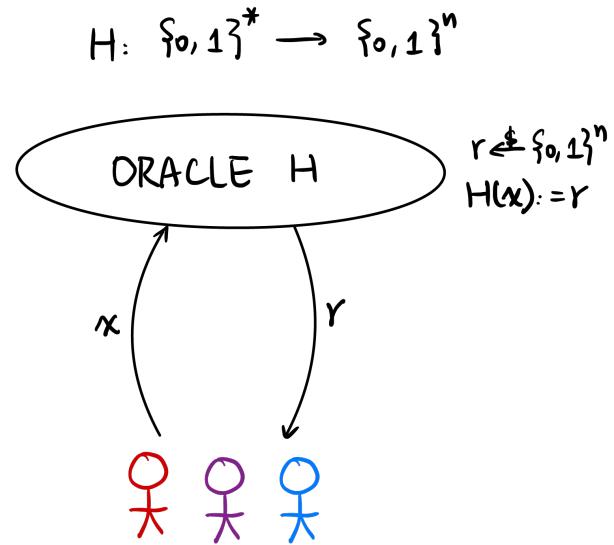
**Remark.** This is related to the birthday problem. If there are  $q$  students in a class, assume each student’s birthday is a random day  $y_i \xleftarrow{\$} [365]$ . What is the probability of a collision?  $q = 366$  gives 1,  $q = 23$  gives around 50%, and  $q = 70$  gives roughly 99.9%.

We can apply this trick to our hash function. If  $y_i \xleftarrow{\$} [N]$ , then  $q = N + 1$  gives us 100%, but  $q = \sqrt{N}$  gives 50% probability.

Knowing this, we want  $n = 2\lambda$  (output length of hash function). If  $\lambda = 128$ , we want  $n$  to be around 256.

#### §4.6.1 Random Oracle Model

Another way to model a hash function is the *Random Oracle Model*. We think of our hash function to be an oracle (in the sky) that can *only* take input and a random output (and if you give it the same input twice, the same output).



There are proofs that state that no hash functions can be a random oracle. There are schemes that can be secure in the random oracle model, but are not using hash functions<sup>12</sup>.

In reality, hash functions are *about as good as*<sup>13</sup> random oracles. Thinking of our hash functions as random oracles gives us a good intuitive understanding of how hash functions can be used in our schemes.

In this model, the best thing that an attacker can do is to try inputs and query for outputs.

#### §4.6.2 Constructions for Hash Function

**MDS.** Output length 128-bit. Best known attack is in  $2^{16}$ . A collision was found in 2004.

And we also have Secure Hash Functions (SHA), founded by NIST.

**SHA-0.** Standardized in 1993. Output length is 160-bit. Best known attack is in  $2^{39}$ .

**SHA-1.** Standardized in 1995. Output length is 160-bit. Best known attack is in  $2^{63}$ , and a collision was found in 2017.

**SHA-2.** Standardized in 2001. Output length of 224, 256, 284, 512-bit. The most commonly used is SHA-256.

**SHA-3.** There was a competition from 2007-2012 for new hash functions. SHA-3 was released in 2015, and has output length 224, 256, 2384, 512-bit. This is *completely different* from SHA-2.

**Remark.** The folklore is that during a session at a cryptography conference, a mathematician, Xiaoyun Wang, presented slide-after-slide of attacks on MDS and SHA-0, astounding the audience.

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<sup>12</sup>Some constructions don't rely on this model.

<sup>13</sup>But can never be...

## §5 February 07, 2024

### §5.1 Announcements

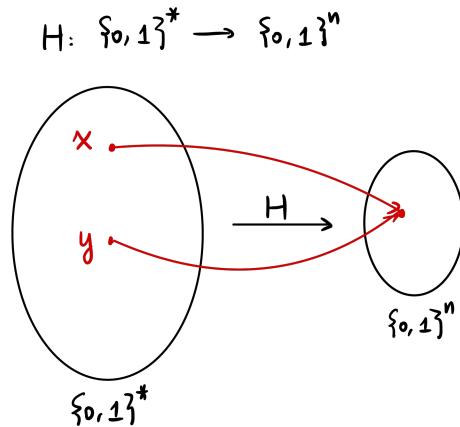
**Capstone:** 1515 is right now in the google form.

**Security Pathway:** 1515 is one of the course for security pathway even it is not listed in the CAB system.

### §5.2 Cryptographic Hash Function, *continued*

Recall that we defined a hash function to be a public and deterministic function for which it is computationally hard to find a collision. That is, finding two distinct strings  $x, y$  such that  $H(x) = H(y)$  is computationally difficult.

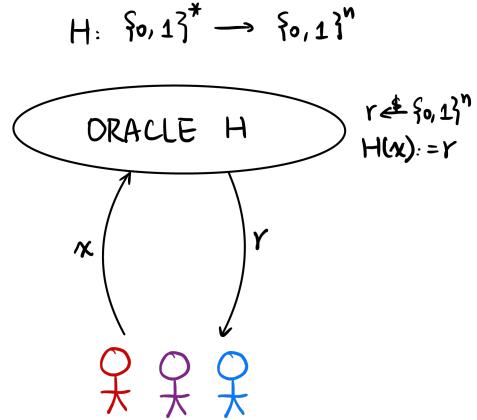
For the hash function, the input domain is arbitrary-length string, and the output is a fixed-length string - 256 bits. The security guarantee is Collision-Resistant Hash Function(CRHF).



One model that we sometimes use to facilitate our analysis is the random oracle model. We assume our hash function is a random oracle ‘in the sky’, and the only thing you can do is to send a query to it. The oracle will produce random hashes and send it back to the queries.

This is a very strong assumption, thus it’s difficult to achieve the model theoretically. However, in reality it kind of makes sense. Since you can only query it, it’s way more difficult to find the collision.

If you are given an arbitrary output, it’s extremely hard to find its input. But it is not the case for CRHF, since CRHF only assumes you can not find a collision, thus this model is stronger than CRHF.



### §5.2.1 Constructions

Apart from MD5, we had Secure Hash Functions(SHA), including SHA-0, SHA-1, SHA-2 and SHA-3.

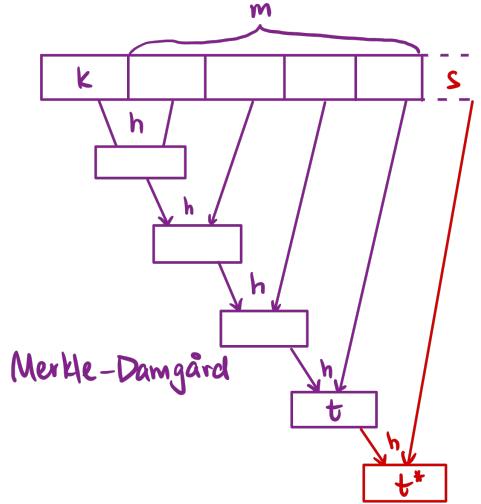
The hash function constructions that are still in practice (and unbroken) are SHA-2 and SHA-3.

### §5.2.2 Applications

**HMAC.** We can use a hash function to conduct a MAC. Computing a tag involves computing the hash function on the key appended to the message  $(k||m)$ . It is computationally difficult to find another  $k||m'$  that produces the same hash. This is a scheme that looks like

$$\text{Mac}_k(m) = H(k||m).$$

However, an adversary could potentially attach some additional  $s$  to  $m$  to produce  $m' = m||s$  such that they can easily compute  $\text{tag}' = H(\text{tag}||s)$ . This is due to the Merkle-Damgård construction of SHA-2, which associatively tags blocks of the message one-by-one.



Therefore, in practice, we use a nested MAC like

$$\text{Mac}_k(m) = H(k \parallel H(k \parallel m))$$

and just to be sure (that we're not reusing the key), we produce  $k_1, k_2$  as such

$$\text{HMAC}_k(m) = H(k_1 \parallel H(k_2 \parallel m))$$

such that  $k_1 = k \oplus \text{opad}$  and  $k_2 = k \oplus \text{ipad}$ , some one-time pads.

**Hash-and-Sign.** There are some other applications of a hash function. We've seen before with RSA that we want to Hash-and-Sign, removing any homomorphism that an adversary could exploit. Additionally, this allows us to sign larger messages since they are constant size after hashing.

**Password Authentication.** Another application is password authentication. Instead of storing plaintext passwords on servers, websites can store a hash of the password instead. This means that the passwords are not compromised even if the server is compromised.

**Deduplicate Files.** We can also use hash functions to deduplicate files. We can hash two files to produce identifiers  $h_1$  and  $h_2$ . If  $h_1 \neq h_2$ , this implies  $D_1 \neq D_2$ . If  $h_1 = h_2$ , it almost always<sup>14</sup> implies that  $D_1 = D_2$ .

**HKDF (Key Derivation Function).** We can derive more keys from a shared key, essentially using a hash function as a pseudorandom generator (PRG).

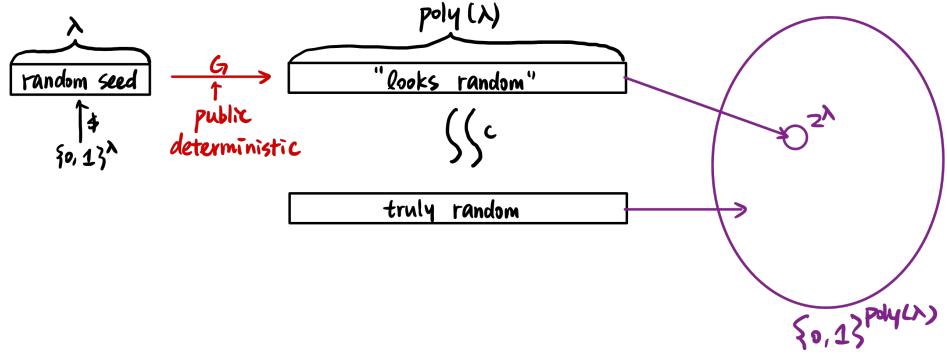
For example, if there is  $g^{ab}$  shared key, we can do

$$\text{HMAC}(g^{ab}, \text{salt})$$

Using a random seed, and concatenating a public deterministic salt  $G$ , we can generate a random<sup>15</sup> string.

<sup>14</sup>If they are not equal, we've found a collision for our hash function, which is extremely unlikely.

<sup>15</sup>Computationally random, because if our computational power were to be unbounded, we can try all strings.



**Pseudorandom Generator (PRG).** Given a hash function  $H$ , we can generate a PRG easily for any length string by generating

$$\begin{aligned} \text{seed} &\xleftarrow{\$} \{0,1\}^\lambda \longrightarrow H(\text{seed}||00\cdots 00) \\ &\quad H(\text{seed}||00\cdots 01) \\ &\quad H(\text{seed}||00\cdots 10) \\ &\quad \vdots \end{aligned}$$

We can take a bit of randomness (like the way we move our mouse, type keyboard, system properties) and generate our seed.

**Fast Membership Proof (Merkle Tree).** Using hash functions, we can generate Merkle Trees to prove membership. In blockchains, this is equivalent to checking if a transaction occurred.

**SKE Scheme?** Could we use this to encrypt? If we have a secret key  $k \xleftarrow{\$} \{0,1\}^\lambda$ , can we just encrypt by

$$\text{Enc}_k(m) = H(k||m)$$

Well, we can't decrypt for one without having unbounded computational power. If our plaintext  $m$  comes from a small set, like  $\{0, \dots, 10\}$ , we could decrypt properly. However, this is not CPA-secure, since the adversary could just query for all the messages.

**Remark 5.1.** In general, all deterministic encryption schemes are not CPA-secure.

### §5.3 Putting it Together: Secure Communication

This is essentially what we want to do in the second project.

We use Diffie-Hellman Key Exchange between Alice and Bob to get shared  $g^{ab}$ . Hashing the shared key using an HKDF, we can get shared key  $k = (k_1, k_2)$  (one for AES encryption, one for HMAC). Then, they perform authenticated encryption, namely Encrypt-then-MAC.

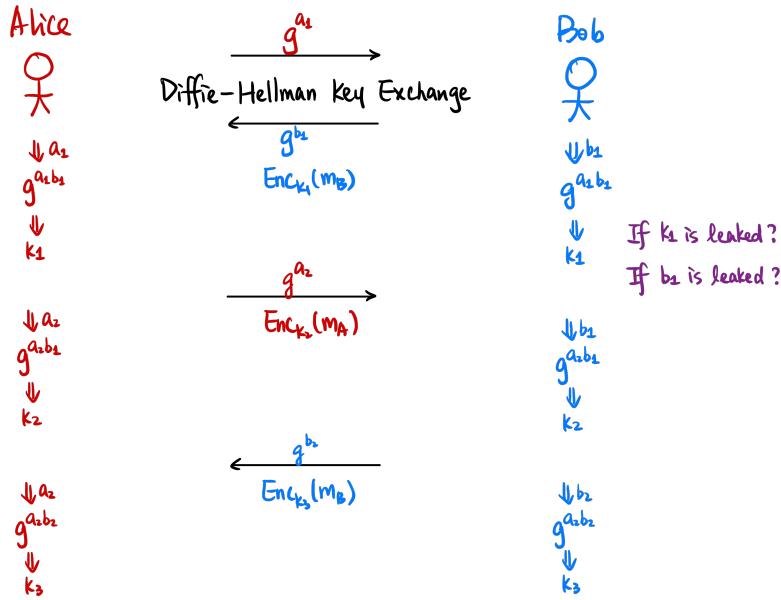
**Question.** Are there any issues with this scheme?

An Eve could pretend to be Alice to Bob and Bob to Alice, fudging up their shared keys. This is called a *Man-in-the-Middle* attack.

### §5.3.1 Diffie-Hellman Ratchet

What if a secret key gets leaked, or cracked? One simple way to fix this is to perform a Diffie-Hellman key exchange on every message. However, this incurs additional communications costs.

Here's another idea: with every new message (when the direction of communications shifts), the party sending the message sends a new Diffie-Hellman public key for themselves. For example, if Bob is sending a message to Alice and he knows Alice's public key  $g^{a_1}$  and his previous secret was  $b_1$  (hence shared  $g^{a_1 b_1}$ ), Bob will generate new key  $b_2, g^{b_2}$  and encrypt using  $g^{a_1 b_2}$ , sending  $g^{b_2}$  as public to Alice. Alice can recompute the shared key before decrypting.



This is the protocol used in the Signal messaging app, and is what you will implement for Project 1.

**Question.** What if  $k_1$  is leaked?

We might have leaked one key, but the other keys are still computationally hard to compute.  $k_1 = g^{a_1 b_1}$  is known, but it's equivalent to DDH to compute  $g^{a_1 b_2}$  or other keys.

**Question.** What if  $b_1$  is leaked?

We can compute key  $k_1 = g^{a_1 b_1}$  and  $k_2 = g^{a_2 b_1}$ , but no further keys are leaked, and the next round of communications (after Bob refreshes his private key  $b_2$ ) is still secure.

## §5.4 Block Cipher

To summarize, here's what we've seen so far (this table should be familiar):

	Symmetric-Key	Public-Key
Message Secrecy	Primitive: SKE Construction: <b>Block Cipher</b>	Primitive: PKE Constructions: RSA/ElGamal
Message Integrity	Primitive: MAC Constructions: CBC-MAC/HMAC	Primitive: Signature Constructions: RSA/DSA
Secrecy & Integrity	Primitive: AE Construction: Encrypt-then-MAC	
Key Exchange		Construction: Diffie-Hellman
Important Tool	Primitive: Hash function Construction: SHA	

The only thing we haven't seen thus far is a block cipher. We first start with the definitions.

We saw earlier that a Pseudorandom Generator (PRG) produces a string that looks random. We also have Pseudorandom Functions (PRF), which are 'random-looking' functions.

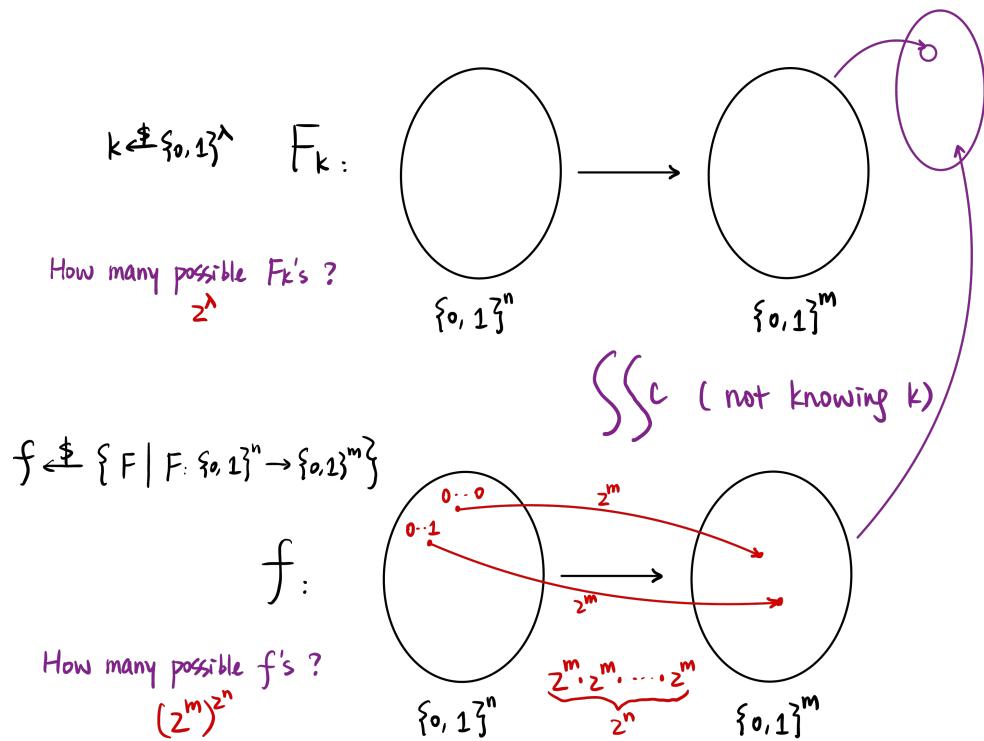
### §5.4.1 Pseudorandom Function (PRF)

Our Pseudorandom Function  $F$  is a keyed function<sup>16</sup>  $F : \{0, 1\}^\lambda \times \{0, 1\}^n \rightarrow \{0, 1\}^m$ ,  $F$  will take key  $k$  and input  $x$  to produce output  $y$ ,  $F(k, x) = y$ .

Without knowing our key  $k$ ,  $F_k$  is computationally indistinguishable from some random  $f \xleftarrow{\$} \{F \mid F : \{0, 1\}^n \rightarrow \{0, 1\}^m\}$ .

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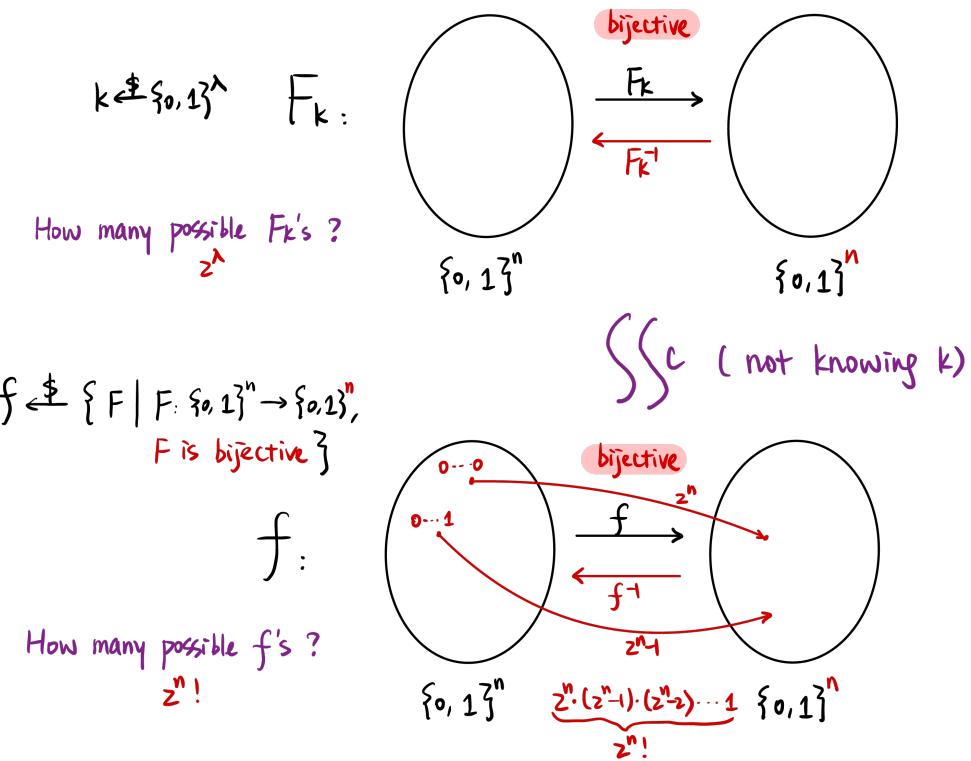
<sup>16</sup>In deterministic polynomial-time.



We have  $2^\lambda$  possible  $F_k$ 's, and we have  $(2^m)^{2^n}$  possible functions  $f$ . A computationally unbounded adversary could try all possible functions and distinguish our function, since  $F_k$  lives in a subset of the space of  $f$ . However, in reality, we can assume that  $F_k$  is computationally indistinguishable from any generic function.

### §5.4.2 Pseudorandom Permutation (PRP)

A further assumption is that our function is a bijection.  $F_k$  is a keyed function from  $F_k : \{0,1\}^n \rightarrow \{0,1\}^n$ . We still have  $2^\lambda$  possible  $F_k$ 's since there are  $2^\lambda$



**Question.** Again, how many possible  $f$ 's are there?

Our first string has  $2^n$  choices to map to, our second choice has  $2^n - 1$ , so there are

$$(2^n)(2^n - 1)(2^n - 2) \cdots 1 = 2^n!$$

Still, this is a much larger number than  $2^\lambda$ , so we still make a computational assumption that our keyed function  $F_k$  is still computationally indistinguishable from a random function  $f$ .

## §6 February 12, 2023

### §6.0.1 Pseudorandom Function (PRF), *continued*

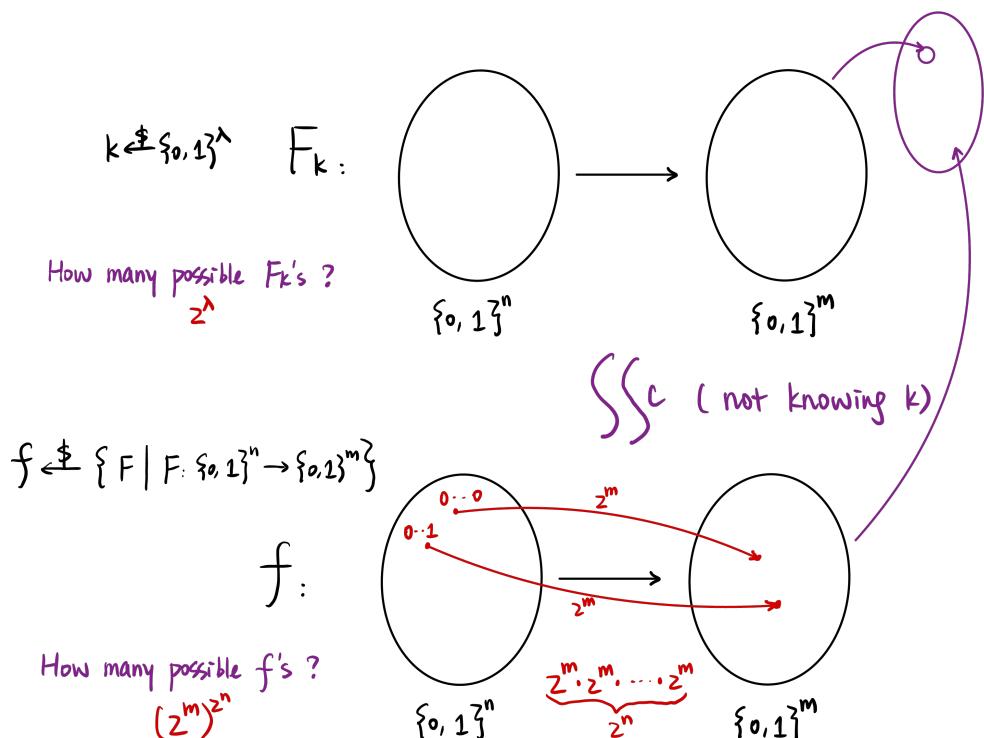
A block cipher, at a very high level, is a pseudo random function.

Recall last time that we talked about pseudorandom *generators* (PRG), which takes a seed and expands it into a long string of pseudorandom bits. This “random-looking” **string** is computationally indistinguishable from a truly random string.

A pseudorandom *function* (PRF) is a “random-looking” **function** that takes a key and an input and produces an output. This function is computationally indistinguishable from a truly random function.

More formally, our pseudorandom function  $F$  is a keyed function<sup>17</sup>  $F : \{0, 1\}^\lambda \times \{0, 1\}^n \rightarrow \{0, 1\}^m$ ,  $F$  will take key  $k$  and input  $x$  to produce output  $y$ ,  $F(k, x) = y$ .

Without knowing our key  $k$ ,  $F_k$  is computationally indistinguishable from some random  $f \xleftarrow{\$} \{F \mid F : \{0, 1\}^n \rightarrow \{0, 1\}^m\}$ .



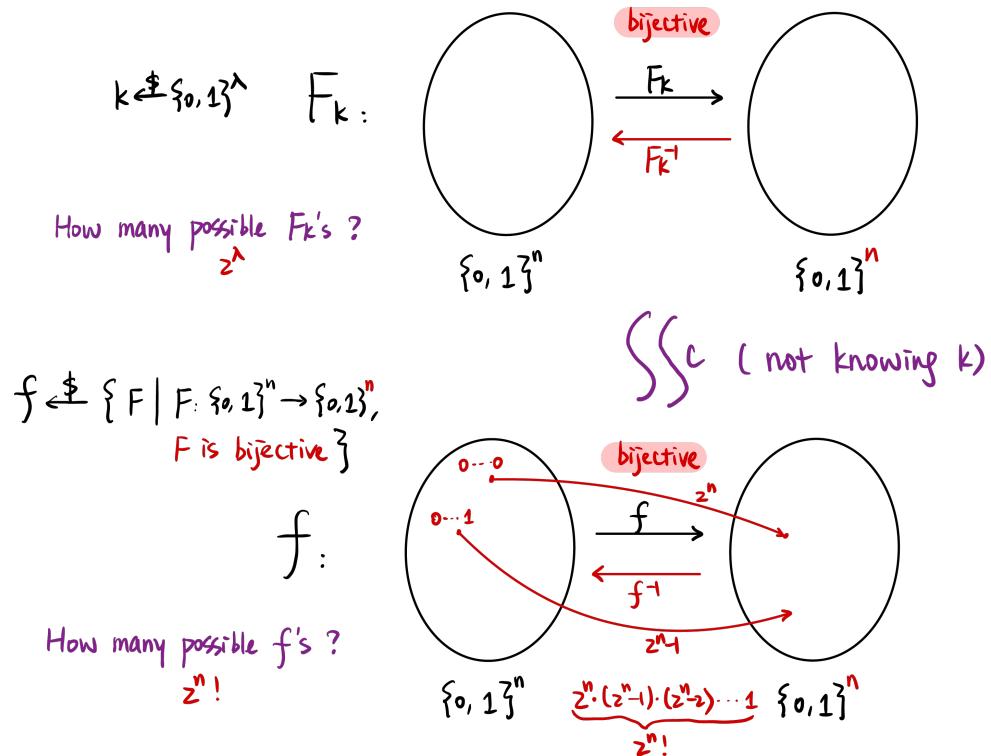
We have  $2^\lambda$  possible  $F_k$ 's, and we have  $(2^m)^{2^n}$  possible functions  $f$ . A computationally unbounded adversary could try all possible functions and distinguish our function, since  $F_k$  lives in a subset of

<sup>17</sup>In deterministic polynomial-time.

the space of  $f$ . However, in reality, we can assume that  $F_k$  is computationally indistinguishable from any generic function.

### §6.0.2 Pseudorandom Permutation (PRP)

A further assumption is that our function is a bijection.  $F_k$  is a keyed function from  $F_k : \{0, 1\}^n \rightarrow \{0, 1\}^n$ . We still have  $2^\lambda$  possible  $F_k$ 's since there are  $2^\lambda$



**Question.** Again, how many possible  $f$ 's are there?

Our first string has  $2^n$  choices to map to, our second choice has  $2^n - 1$ , so there are

$$(2^n)(2^n - 1)(2^n - 2) \cdots 1 = 2^n!$$

Still, this is a much larger number than  $2^\lambda$ , so we still make a computational assumption that our keyed function  $F_k$  is still computationally indistinguishable from a random function  $f$ .

## §6.1 Block Ciphers

Looking back on [section 4.4](#), the last outstanding primitive was the block cipher. We saw this last lecture, we'll continue discussing the block cipher.

Recall that we had seen pseudorandom functions which are keyed functions that are computationally indistinguishable from *all* random functions from  $\{0, 1\}^n \rightarrow \{0, 1\}^m$ . A stronger form of pseudorandom functions are pseudorandom permutations: a keyed bijective map between  $\{0, 1\}^n \rightarrow \{0, 1\}^n$  that is computational indistinguishable from pseudorandom permutations.

Block ciphers are a special form of pseudorandom permutation. It is a keyed function

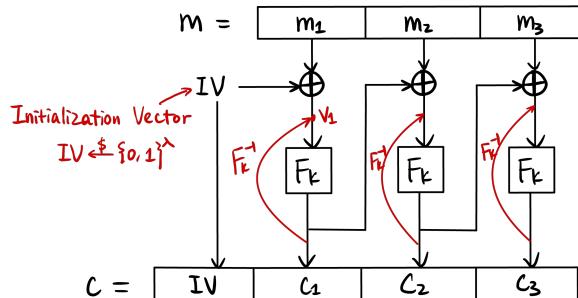
$$F : \{0, 1\}^\lambda \times \{0, 1\}^n \rightarrow \{0, 1\}^n$$

where  $\lambda$  is the key length and  $n$  is the block length. The practical construction of which is AES, which takes blocks of  $n = 128$  and key length  $\lambda = 128, 192, 256$  as choices.

### §6.1.1 Modes of Operation

**Electronic Code Book (ECB) Mode:** We will run our block cipher on each block of our message individually. However, this is not CPA secure, since encryptions are deterministic. We need to ‘seed’ our encryption with some random value.

**Cipher Block Chaining (CBC) Mode:** Instead of running on our block cipher on each block individually, every block will get an additional *initialization vector* IV, which is XORed onto each message before running the block cipher.



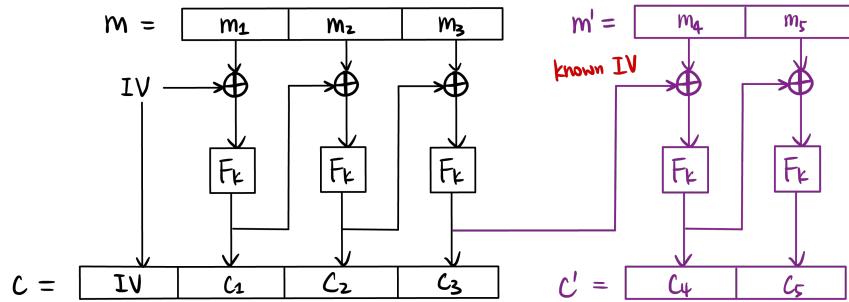
We waved our hand over the fact that this is CPA secure—but it relies on the initialization vector being random.

*What if our IV is not randomly sampled?* Consider an IV that is *different* but not randomly sampled. For example, the IV is  $0 \dots 00$  for the first message,  $0 \dots 01$  for the second message, and so on. Do we still have security?

Unfortunately not. Say  $m_1$  is XORed onto  $0 \cdots 01$ , an adversary under CPA can choose plaintext that is  $m_1$  with its last bit flipped, such that  $v_1$  is manipulated and the block cipher is again deterministic.

It is crucial that IV is randomly selected, and that the next IVs for future blocks (of the same message) are also pseudorandom (that are the previous ciphertext, which is okay).

**Chained Cipher Block Chaining (Chained-CBC) Mode:** We touched on this earlier, but there is a mode of operation of CBC that feeds the last cipher block as the new IV for the next message.



Similar to the case earlier, an adversary here can select a next message *based on* their knowledge of the previous ciphertext and hence the upcoming IV.

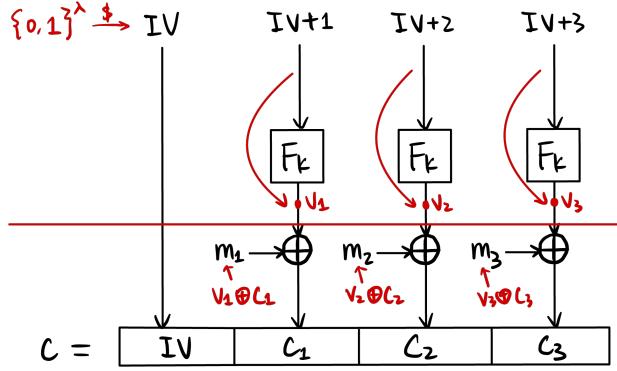
This makes chained-CBC *very subtly* different than CBC. If we squint our eyes enough, it just looks like sending a single message using CBC mode. The key difference is that between rounds of communication  $m$  and  $m'$ , an adversary could influence  $m'$  given the knowledge of the previous round.

**Remark.** Another note that this is *very subtle!* To the extent that when *Signal* was being developed, the course staff initially wrote the solution using Chained-CBC mode. This highlights the difficulty in creating real-world cryptographic systems!

The following will be new modes not covered last lecture:

**Counter (CTR) Mode:** Instead of chaining each successive IV from the previous block ciphertexts, we'll encrypt *only* the IV  $\xleftarrow{\$} \{0, 1\}^\lambda$ , and XOR the encrypted  $F_k(\text{IV} + i)$  to mask  $m_i$ , like a one-time pad.

Another way to think about the CTR mode is that we're using  $F_k$  and a random IV to generate a long enough one-time pad to pad the entire message.



*How do we decrypt?* Since we know the first IV, we can compute the one-time pads  $F_k(IV + i)$  and XOR with  $m_i$ s. This scheme is valid.

*Is this CPA secure?* The XOR after  $F_k$  might throw you off and cast doubt in your mind. However, this mode of operation is (!) CPA-secure. Even if we know  $IV, IV + 1, IV + 2, \dots$ , we can't figure out the output of  $F_k$  that becomes our one-time pad (to do so contradicts the CPA security of our block cipher). The CPA security of each  $F_k$  being pseudorandom guarantees the CPA security of this scheme.

*What about a “stateful CTR mode” which just increments IV every successive time?* Instead of sending a new IV for the next message, we'll just increment the IV from before. Similar to Chained-CBC mode, the adversary will know the IV that is going into the next message. However, this doesn't *really* help the adversary. They've never seen those encrypted IV values before, and hence cannot modify the message given this information.

This is a distinction from last time, where the IV was XORed onto the message directly, which could be tampered with by an adversary who knows the IV.

*What if IV is not randomly sampled?* Nothing really breaks down, unlike the previous case. We just want to make sure that two IVs are not reused and don't collide. If IVs collide, two blocks will have the same one-time pad, which is potentially a problem. This doesn't prevent us from using  $0 \dots 00, 0 \dots 01, 0 \dots 10, \dots$  as our IV values at all. In practice, however, they are still randomly sampled to prevent collisions.

*Can we parallelize this?* Yes, we can compute  $F_k(IV + i)$  in parallel and XOR onto each block. Similar for encryption and decryption.

*Can we construct a PRG from a PRF?* Using a seed  $(IV, k)$ , we can generate an  $n\lambda$  bit string

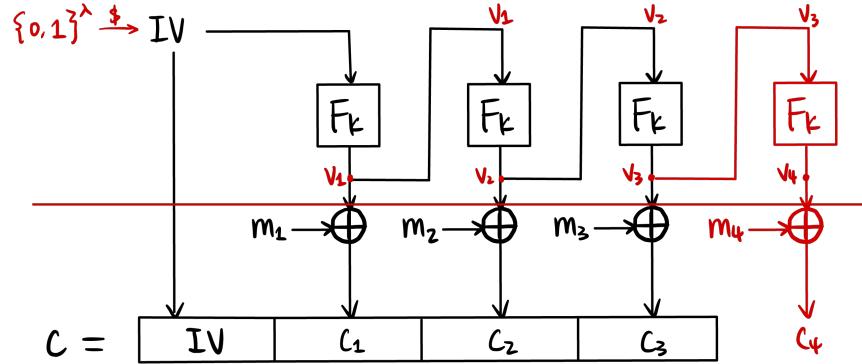
$$G(k||IV) = F_k(IV)||F_k(IV + 1)||F_k(IV + 2)||\dots$$

In fact, we can get rid of IV entirely and start at 0,

$$G(k) = F_k(0)||F_k(1)||F_k(2)||\dots$$

Counter mode essentially uses this PRG with private  $k$  to generate a long one-time pad which is used to pad the message. Another note is that in this mode, we don't even require a pseudorandom permutation, since we don't need to invert the function at any point.

**Output Feedback (OFB) Mode:** This is a mix of CBC and CTR modes. Successive one-time pad blocks are fed into the next  $F_k$  as IV, and they are XORed with the message after encryption.



We have the same questions. *How do we decrypt? Is this CPA secure? Is a “stateful” version of OFB secure? Can we use this to construct a PRG?*

We can decrypt similarly: we decrypt the first block, get the IV for the next block and continue on. All security is guaranteed by the same reasoning as in counter mode: we know IV but still cannot compute  $F_k(\text{IV})$ . Similar to counter mode, this is another form of PRG (which chains successive blocks instead of using IVs in series) that generates a long one-time pad. Again, our IV doesn't need to be randomly sampled, but it should not collide with previous IV values.

A difference to counter mode is that we cannot parallelize this scheme. However, in both CTR and OFB modes, we can precompute the entire one-time pad in both encryption and decryption to happen in the offline phase. The online phase (when parties are communicating) is limited to cheap XOR operations.

**Question.** We've listed *a lot* of benefits to counter mode or output feedback mode. Why do people use CBC mode at all?

We've seen how things can go wrong catastrophically<sup>18</sup>. This is more true for counter mode than CBC mode. If our IV is reused in counter mode, our entire one-time pad has been exposed previously<sup>19</sup>. However, if our IV is reused in CBC mode, the worst that could happen is something akin to ECB mode, and no messages are compromised.

At the end of the day, *engineers are quite oblivious to cryptographic schemes!* Libraries only specify for *some key* and *some IV*, so it is exceedingly easy to screw up your cryptographic scheme by

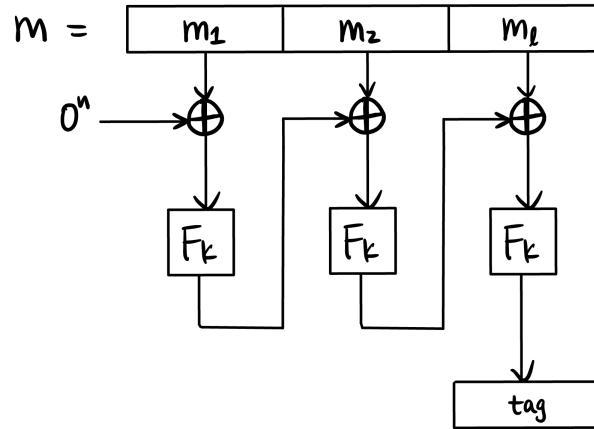
<sup>18</sup>We nearly made mistakes in this course!

<sup>19</sup>XORing our ciphertexts will give  $m \oplus m'$ .

reusing IVs, etc. CBC mode is simply more foolproof and incurs better outcomes in case it is used incorrectly<sup>20</sup>.

### §6.1.2 CBC-MAC

We can use block ciphers to construct a MAC scheme. Splitting up our message into blocks, we feed blocks into  $F_k$  and chain to next blocks. In the end, the final cipher output is our tag.



*How do we verify?* We can just Mac the message again and check that the tag matches. If  $F_k$  is invertible, we can also go the other way.

*Is this CMA secure?*

- Fixed-length messages of length  $l \cdot n$ ? Yes, since we can only query for fixed-length messages, this gives us no additional information.
- Arbitrary-length messages? This is where problems arise—the adversary could first query for a message of 1 block, then 2 blocks, then 3 blocks, etc. By combining this information, they could produce new valid signatures.

A concrete attack is an adversary querying for  $\text{Mac}(m)$  to produce  $\text{tag}$ , then querying for  $\text{Mac}(\text{tag}) = \text{Mac}(m||0) = \text{tag}'$  which allows the adversary to forge a new message.

**Remark.** Our constructions of authenticated encryption calls for an encryption scheme and MAC scheme. It's crucial that the two schemes have *different keys*. Using the same key  $k$  for both encryption and MAC can cause issues (information from one could reveal something about the other).

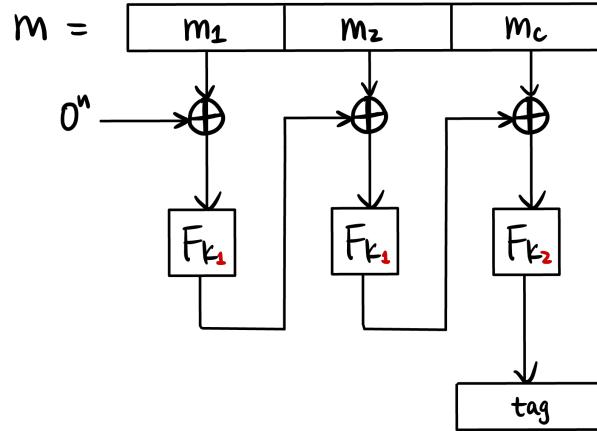
We have a fix for the CMA-vulnerability in arbitrary-length messages:

<sup>20</sup>However, if Peihan were to implement a block cipher scheme herself, would opt for counter mode.

### §6.1.3 Encrypt-last-block CBC-MAC (ECBC-MAC)

The vulnerability earlier was due to our encryption being *associative*, so to speak.

We can fix this is to use a different key for the last block:



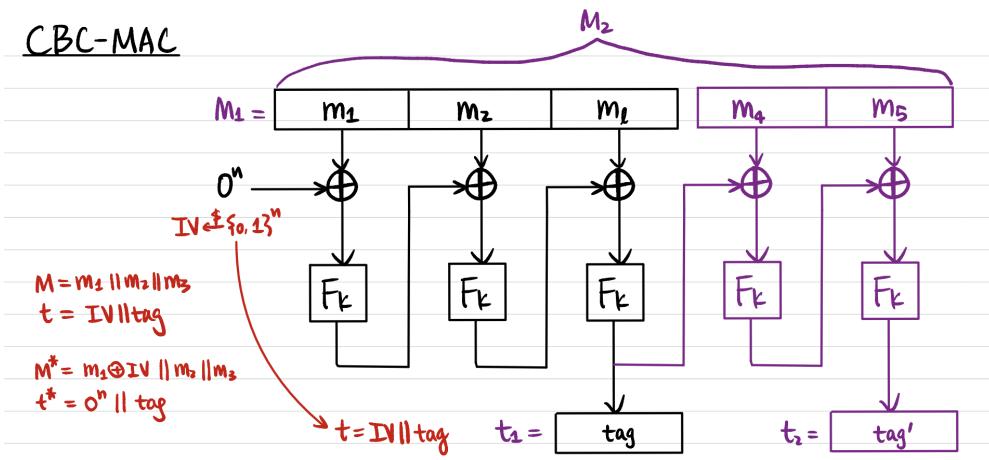
We could also attach length of messages to the first block, or other techniques.

The nuance in CBC-MAC means that realistically, we almost always use HMAC.

**Question 6.1.** For CBC-MAC, if we randomly sample the IV and include it in the tag, will this be CMA secure?

No! Consider if the adversary queries for the tag of  $m := m_1||m_2||m_3$  and receive the tag  $t := (\text{IV}, \text{tag})$ .

The adversary can generate a new valid tag for  $m^* := m_1 \oplus \text{IV}||m_2||m_3$  and  $t^* := (0^n, \text{tag})$ .



How to verify?

CMA (Chosen Message Attack) Secure?

- Fixed-length messages of length  $\leq n$  Yes!
- Arbitrary-length messages No!

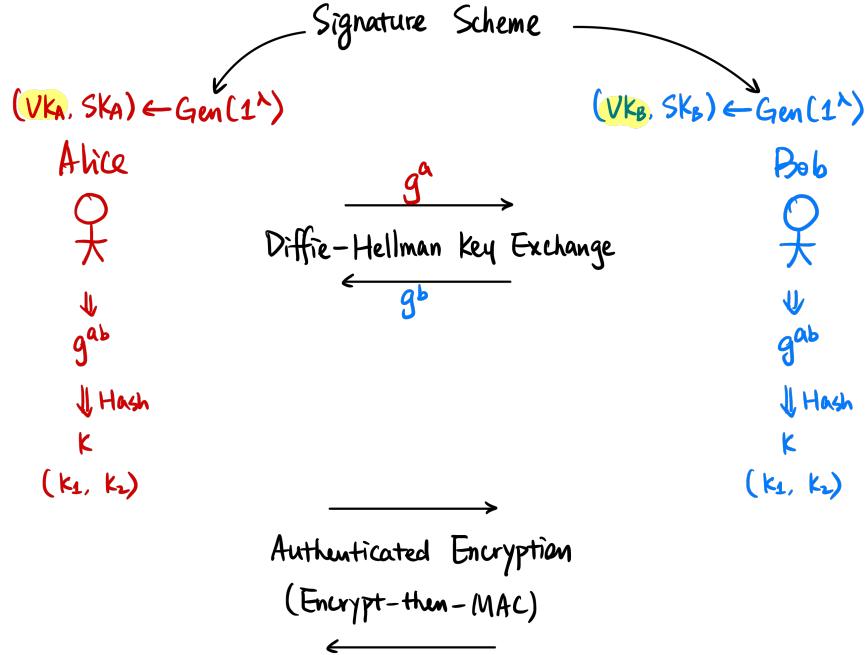
$$M^* = m_4 \oplus t_1 \parallel m_5$$

$$t^* = t_2$$

## §6.2 Putting it Together

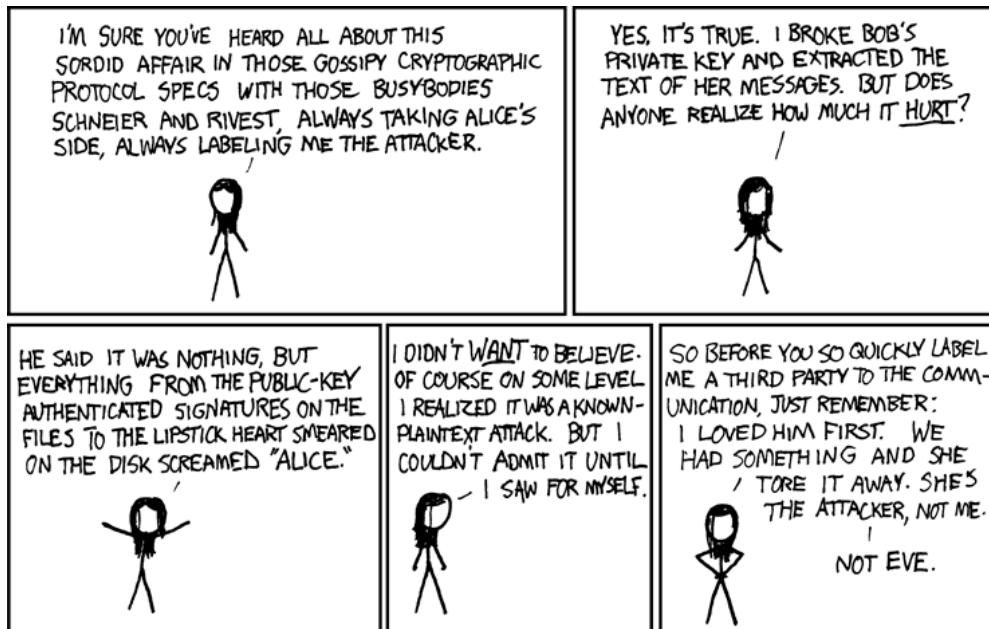
Looking back at section 4.4, we've collected everything we need so far for secure communication.

For Alice and Bob to communicate, they first exchange keys using a Diffie-Hellman key exchange, then perform authenticated encryption.



However, this still does not mitigate against a man-in-the-middle attack. Thus, before exchanging keys, Alice and Bob should publish verification keys (to a digital signature scheme, see [section 4.2.2](#)). Using this digital signature, Alice and Bob will each sign their Diffie-Hellman public values  $g^a, g^b$  using their signing key, which will be attached to the message. They can respectively verify that these values came from each other, and not some Eve in the middle.

## §7 February 14, 2024

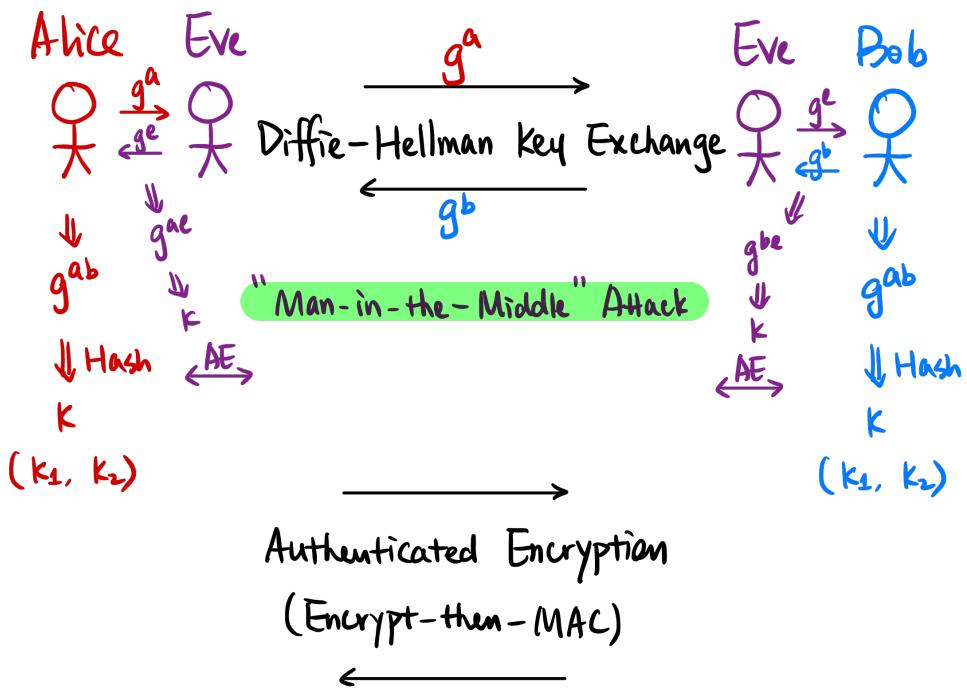


*xkcd 177. Happy Valentine's Day! Stay safe by selecting authenticated encryption schemes (see section 4.5) that are CPA secure and unforgeable to communicate with your date.*

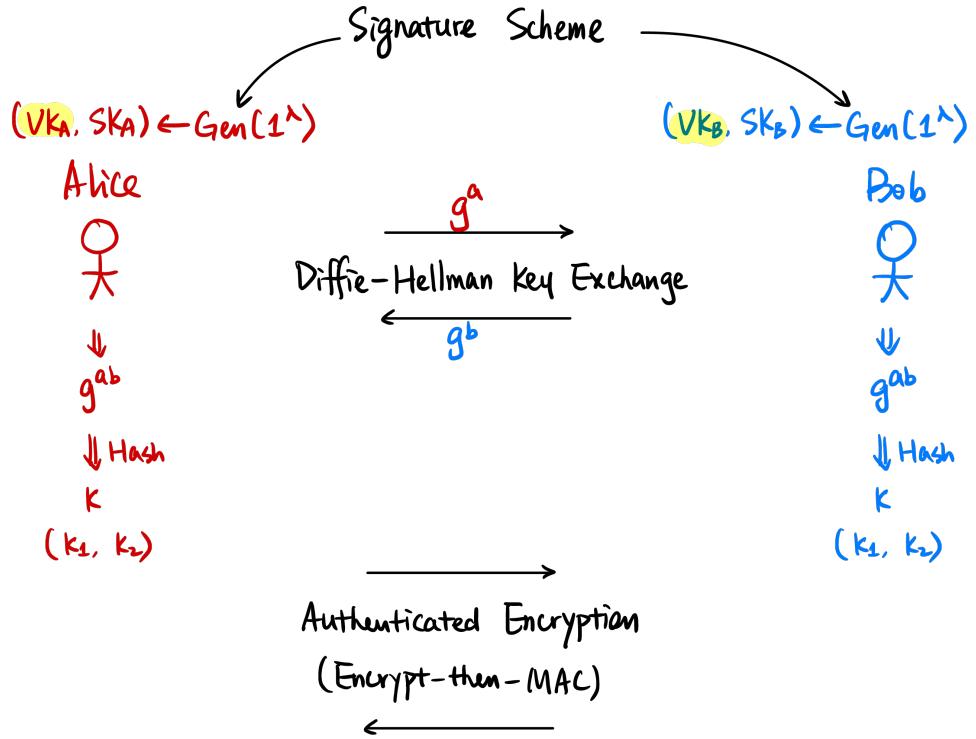
This lecture we will go over the topics needed for the next project, Auth. Namely,

- One-Sided Secure Authentication
- Password-Based Authentication
- Two-Factor Authentication (2FA)
- Putting it All Together: Secure Authentication
- Public Key Infrastructure (PKI)

Recall that we had a way for Alice and Bob to communicate securely, first exchanging a shared Diffie-Hellman key and then performing AES encryption.



However, this is prone to a man-in-the-middle attack. One way to solve this is for parties to *sign* their own Diffie-Hellman public values before sending, and then verify the other party's public value by using their public verification key.

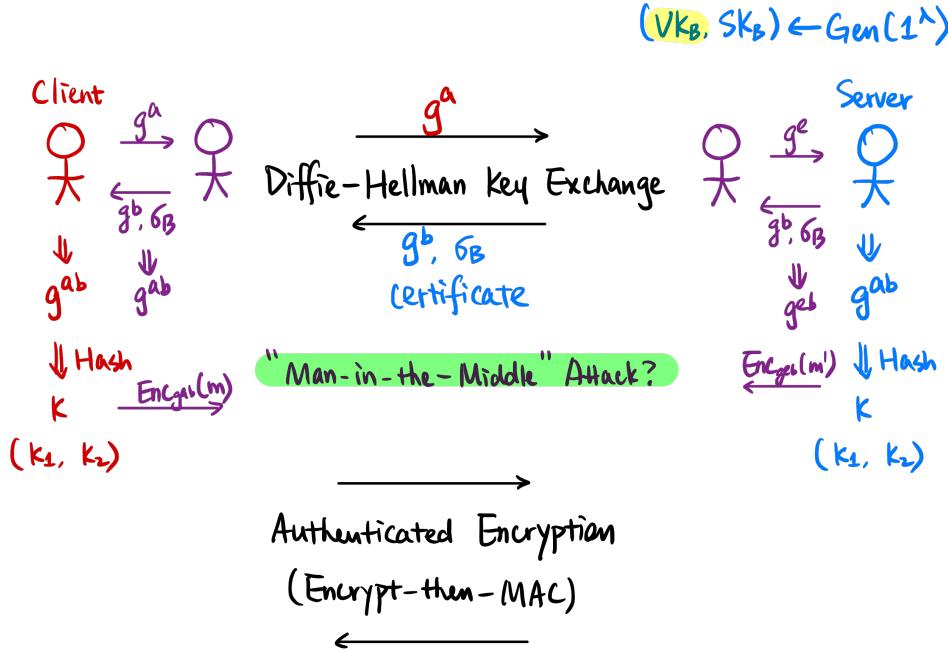


*Is this now secure against an adversary in the middle?* Yes, because the public values are guaranteed (via our digital signature scheme) by Alice and Bob's signing key. The man in the middle does not have access to the signing key, and cannot sign a phony public value.

However, how do we know the verification keys really belong to who we think they belong to? This is the problem of *authentication*. We are sort of in a chicken and egg problem...

### §7.1 One-Sided Secure Authentication

In some circumstances, it's more difficult for a client to communicate their verification key to a server than it is for a server to do so. A server might publish their verification key, and trust that all clients are not compromised.



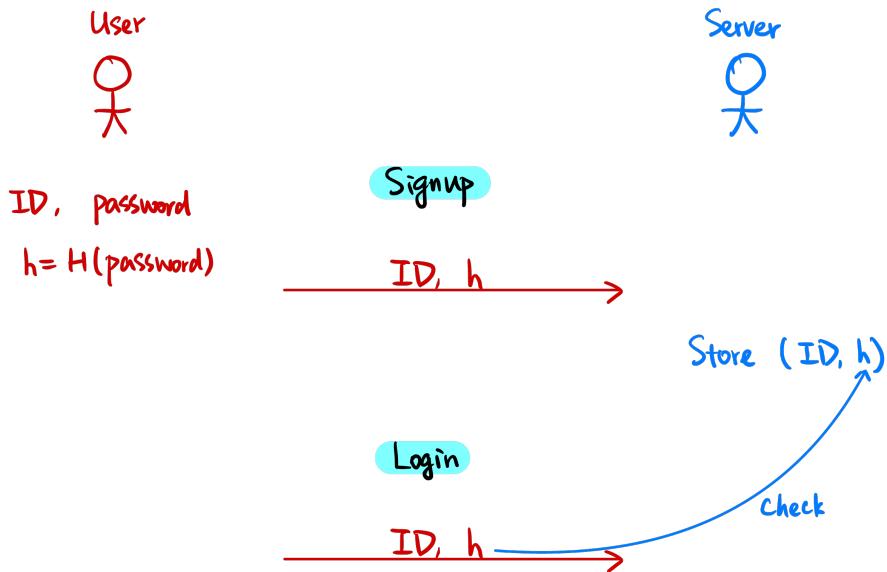
*What could an adversary potentially do?* The adversary could not pretend to be the server since they have no access to the server's signing key. The adversary *can* pretend to be the client and talk to the server. The adversary could forward all messages sent to the server, and can also communicate  $g^b, \sigma_b$  back to the client (it's a valid signature since it has not been modified).

At the end of this protocol, the client has Diffie-Hellman private  $g^{ab}$  and the adversary and server will have  $g^{eb}$  (where  $g^e, e$  is a Diffie-Hellman keypair the adversary provided to the client). Whatever the client sends to the server cannot be decrypted by the adversary, since it is encrypted with  $g^{ab}$ , however, the server's communications *could* be decrypted by the adversary.

This can be easily circumvented by requiring the server and user complete their handshake—the server could request a hash or encryption of the shared secret, and realize that they are communicating to an adversary when this cannot be forged by the man-in-the-middle.

## §7.2 Password-Based Authentication

Sometimes, you also want to *authenticate* with a server using a password. The naïve implementation is that a user with an ID sends a hash of the password  $h = H(\text{password})$  to the server. The server stores  $(ID, h)$ .

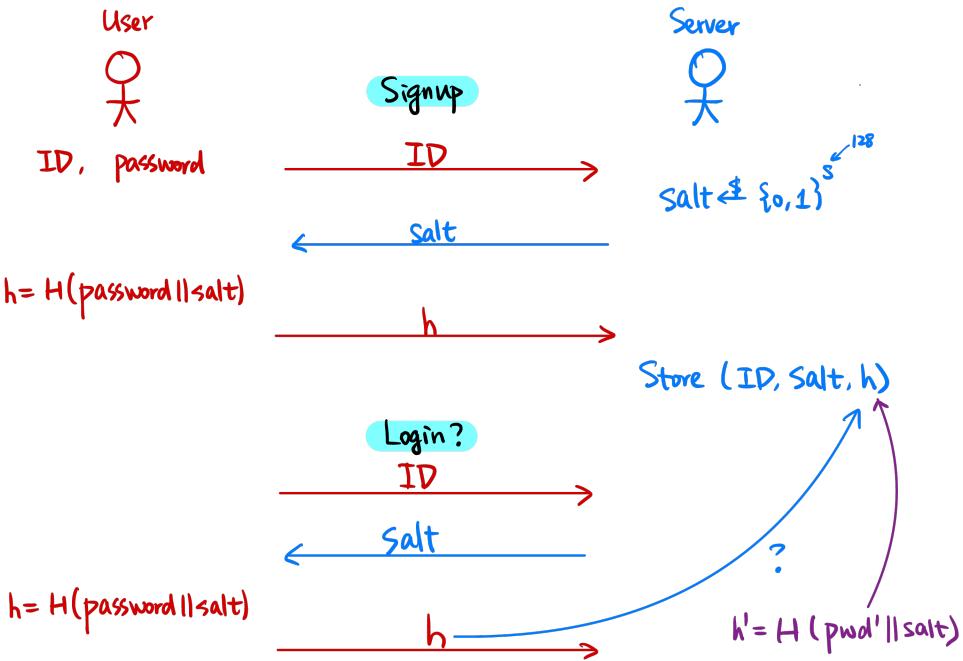


In this case, an adversary could launch an *Online Dictionary Attack* and try a lot of passwords with the server.

If the server were to be compromised, and its database compromised, the adversary can conduct an *Offline Dictionary Attack* on the database. Additionally, the adversary can precompute all hashes and check against the database.

*How can we prevent this?*

### §7.2.1 Salting



One way of ensuring that the hashing is non-deterministic is for servers to generate a salt  $\leftarrow \{0,1\}^s$  and send it to the user. The user will hash  $H(\text{password} \parallel \text{salt})$  and send that to the server. The server stores a database of  $(\text{ID}, \text{salt}, h)$ .

When logging in, the user first sends their ID to the server, the server will send the salt back, the user hashes their password, and the hash is sent to the server for verification.

*Does this allow the user to use a weak password?* Nope! The adversary can always brute-force the password.

To further solve these problems, we can use *peppering*. We send the salt to the user and the user computes hash  $h = H(\text{password} \parallel \text{salt})$ . Then, we pick a random pepper and hash  $h^* = H(h \parallel \text{pepper})$  and stores  $h^*$ . Now, even if the server is compromised, there is no way to find the preimage of  $h^*$ , so adversaries knowing  $h^*$  will still have to do try all  $2^p$  possible peppers for each dictionary guess. We still can't log into the server since the server hashes our login hash again.

Additionally, one strategy to make it *even harder* for an adversary is to make hashing more difficult (time-consuming). For example, we can compose SHA256 in certain ways<sup>21</sup>. There are also memory-hard hash functions, like scrypt.

*Even with all this, is it still safe to use a weak password?* Nope! A dictionary attack is still possible, and with weak passwords will be hard to crack.

<sup>21</sup>The natural way is to hash multiple times, say 100. However, this is actually not more secure in the case of SHA256 but there are specific ways of composition. For example, there are application-specific integrated circuits (ASIC) that can compute hash functions very efficiently.

### §7.2.2 Two-Factor Authentication

Now we'll discuss how servers implement two-factor authentication.

For phone number verification, on signing up, the user sends a phone number with their password hash. The server stores their phone number. Every time, the server will generate challenge  $r \xleftarrow{\$} \{0,1\}^k$

For app-generated codes, the user and server will first share a seed **seed** and use a pseudorandom function  $F_{\text{seed}}(\text{time})$ . The server and the user can input the same time, and the outputs will be the same. Generally, the server will test the last 30/60 seconds of values.

