P1 背景　变分推断 Variational Inference

频率角度 → 优化问题

线性回归:

① 模型: $f(w)=w^T x$

② 策略: loss function: $\begin{cases} L(w)=\sum_{i=1}^{N}\|w^T x_i-y_i\|^2 \\ \hat{w}=\arg\min L(w). \end{cases}$　无约束.

③ 算法: $\begin{cases} \text{解析解: } \frac{\partial L(w)}{\partial w} \triangleq 0 \Rightarrow w^*=(X^TX)^{-1}X^TY. \\ \text{数值解: GD} \to \text{Gradient Descent.} \end{cases}$

支持向量机(SVM)

① 模型: $f(w)=\text{sign}(w^T x+b).$

② 策略: Loss function: $\begin{cases} \min \frac{1}{2} w^T w \\ \text{s.t.}\quad y_i(w^T x_i+b)\geq 1,\ i=1,\cdots,N \end{cases}$　有约束、SGD优化

③ 算法: QP.

贝叶斯角度 → 积分问题

$$P(\theta|X)=\frac{P(X|\theta)\cdot P(\theta)}{P(X)}$$

贝叶斯 Inference.

贝叶斯决策　X: N个样本

$\tilde{x}$ 新的样本, 求 $P(\tilde{x}|X)$.

$$P(\tilde{x}|X)=\int_\theta P(\tilde{x},\theta|X)d\theta=\int_\theta P(\tilde{x}|\theta,X)\cdot P(\theta|X)\,d\theta.$$

$$=\int_\theta P(\tilde{x}|\theta)\cdot \underbrace{P(\theta|X)}_{\text{后验}}d\theta.\quad (\tilde{x}\perp X|\theta)$$

$$=E_{\theta|X}[P(\tilde{x}|\theta)]$$

Inference: $\begin{cases} \text{精确推断} \\ \text{近似推断: } \begin{cases} \text{确定性近似 — VI} \\ \text{随机近似 — MCMC.} \end{cases} \end{cases}$

P2 变分推断

X: Observed data. Z: Latent variable + parameter.

(X, Z): complete data.

$$\log P(X) = \log P(X,Z) - \log P(Z|X)$$

$$= \log \frac{P(X,Z)}{q(Z)} - \log \frac{P(Z|X)}{q(Z)}$$

$$左边 = \int_Z \log P(X)\, q(Z)\, dZ = \log P(X) \underbrace{\int_Z q(Z)\, dZ}_{=1} = \log P(X)$$

$$右边 = \underbrace{\int_Z q(Z) \log \frac{P(X,Z)}{q(Z)}\, dZ}_{\text{ELBO(evidence lower bound)}} - \underbrace{\int_Z q(Z) \log \frac{P(Z|X)}{q(Z)}\, dZ}_{KL(q\|p)}$$

$$= \underbrace{L(q)}_{变分} + \underbrace{KL(q\|p)}_{\geq 0}$$

$$\tilde{q}(Z) = \underset{q(Z)}{\operatorname{argmax}}\, L(q) \Rightarrow \tilde{q}(Z) \approx p(Z|X).$$

$$q(Z) = \prod_{i=1}^{M} q_i(Z_i) \longrightarrow \text{mean theory} \longrightarrow q_j(Z_j).$$

$$q_j(Z_j) = \hat{P}(X, Z_j).$$

$$L(q) = \underbrace{\int_Z q(Z) \log P(X,Z)\, dZ}_{①} - \underbrace{\int_Z q(Z) \log q(Z)\, dZ}_{②}.$$

$$① = \int_Z \prod_{i=1}^{M} q_i(Z_i) \log P(X,Z)\, dZ_1 \cdots dZ_M$$

$$= \int_{Z_j} q_j(Z_j) \left[\int_{\substack{Z_1 \cdots Z_m \\ (i \neq j)}} \prod_{i=1}^{M} q_i(Z_i) \cdot \log P(X,Z) \cdot dZ_1 \cdots dZ_M \right] dZ_j \quad (i \neq j)$$

$$= \int_{Z_j} q_j(Z_j) \left[\int_{\substack{Z_1 \cdots Z_m \\ (i \neq j)}} \log P(X,Z) \prod_{i=1}^{M} q_i(Z_i)\, dZ_i \right] dZ_j$$

$$= \int_{Z_j} q_j(Z_j) \cdot \underbrace{E_{\prod_{i=1, (i \neq j)}^{M} q_i(Z_i)} \left[\log P(X|Z) \right]}_{\log \hat{P}(X, Z_j)} dZ_j$$

$$② = \int_Z q(Z) \log q(Z)\, dZ = \int_Z \prod_{i=1}^{M} q_i(Z_i) \cdot \sum_{i=1}^{M} \log P_i(Z_i)\, dZ.$$

$$= \int_Z \prod_{i=1}^{M} q_i(Z_i) \left[\log q_1(Z_1) + \cdots + \log q_M(Z_M) \right] dZ.$$

$$= \sum_{i=1}^{M} \int_{Z_i} q_i(Z_i) \log q_i(Z_i)\, dZ_i = \int_{Z_j} q_j(Z_j) \log q_j(Z_j)\, dZ_j + C.$$

$$\int_Z \prod_{i=1}^{M} q_i(z_i) \cdot \log q_1(z_1)\, dZ = \int_Z q_1 q_2 \cdots q_M \cdot \log q_1 \, dz_1 \cdots dz_M.$$

$$= \int_{z_1} q_1 \log q_1 \, dz_1 \cdot \underbrace{\int_{z_2} q_2 \, dz_2}_{=1} \cdots \underbrace{\int_{z_M} q_M \, dz_M}_{=1}$$

$$= \int_{z_1} q_1 \log q_1 \, dz_1$$

$$① - ② = \int_{z_j} q_j(z_j) \cdot \log \frac{\hat{p}(X, z_j)}{q_j(z_j)} \, dz_j = -KL(q_j \| \hat{p}(X, z_j)) \leq 0$$