ACTL3182 Module 4 Extra Questions: Radon-Nikodym Theorem, Martingale and Itô's Lemma

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1. Consider two independent coin flips; that is let the sample space be $\Omega := \{HH, HT, TH, TT\}$. Under the $\mathbb{P}$ measure, the coin is fair. Under the $\mathcal{Q}$ measure, the coin has a probability of $\frac{1}{3}$ of landing on heads on every flip.
   (a) Find the probability of each scenario under $\mathbb{P}$ and $\mathcal{Q}$.
   (b) Find the Radon-Nikodym Derivative $\frac{d\mathcal{Q}}{d\mathbb{P}}$.
   (c) Show that for the event A, where A represents getting a head on the first flip, that
   $$\mathcal{Q}(A) = \mathbb{E}_P\left[\frac{d\mathcal{Q}}{d\mathbb{P}} 1_A\right]$$

2. Consider the stochastic process
   $$X(n) = \sum_{k=1}^{n} Z_k, \quad X(0) = 0$$
   where Z_k, $k = 1, 2, ...n$ are iid with distribution
   $$Z_k = \begin{cases} +1 & \text{w.p. } \frac{1}{2} \\ -1 & \text{w.p. } \frac{1}{2}. \end{cases}$$
   (a) What is $\mathbb{E}[X(n+1)|\mathcal{F}_n]$?
   (b) Briefly explain why $X(n)$ is a martingale.
   (c) Now consider a time interval $[0, T]$ and divide the interval into n equal subintervals of length T/n. Redefine Z_k so that the jumps are now of size $\sqrt{T/n}$:
   $$Z_k = \begin{cases} +\sqrt{\frac{T}{n}} & \text{w.p. } \frac{1}{2} \\ -\sqrt{\frac{T}{n}} & \text{w.p. } \frac{1}{2}. \end{cases}$$
   Draw a possible graph of $X(t)$ for $n = 1, 2$. What happens as $n \to \infty$?
   (d) Calculate $\mathbb{E}[X(t)]$ and $\text{Var}[X(t)]$.
   (e) Using the Central Limit Theorem, determine the limiting distribution of $X(t)$.
   (f) What continuous-time process does $X(t)$ converge to?

3. Consider the continuous-time Radon-Nikodym derivative $\zeta(t)$:
   $$\zeta(t) = \exp\left(-\int_0^t \gamma_s dW(s) - \frac{1}{2}\int_0^t \gamma_s^2 ds\right)$$
   Prove that $\zeta(t)$ is a martingale.

4. (Challenge) Consider the stochastic differential equation

$$dX(t) = \alpha(\mu - X(t)) + \sigma dW(t), \quad X(0) = x_0.$$

The solution to this equation is called the Ornstein-Uhlenbeck process and can be used to model a massive particle moving under Brownian Motion under friction.

(a) Using Itô's Lemma, show that the solution to the following stochastic equation is the stochastic process $X(t)$ where

$$X(t) = e^{-\alpha t}x_0 + \mu(1 - e^{-\alpha t}) + \sigma \int_0^t e^{-\alpha(t-s)} dW(s).$$

(b) Explain why

$$\mathbb{E}[X(t)] = e^{-\alpha t}x_0 + \mu(1 - e^{-\alpha t}).$$

Hint: What type of stochastic process is $\int_0^t e^{-\alpha(t-s)} dW(s)$?

(c) The Itô isometry is a special property that allows the expected value of the product of Itô integrals to be calculated using Riemann integrals instead. That is, for two suitable stochastic processes $X(t)$, $Y(t)$, we have:

$$\mathbb{E}\left[\int_0^t X(s)dW(s) \int_0^t Y(s)dW(s)\right] = \mathbb{E}\left[\int_0^t X(s)Y(s)ds\right]$$

Assuming this property holds for $X(t)$, show that

$$\mathrm{Var}[X(t)] = \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t}).$$

(d) What is it the distribution of $X(t)$? Explain this intuitively.

(e) (Extra challenge!) Show that for any $s, t \geq 0$:

$$\mathrm{Cov}(X(s), X(t)) = \frac{\sigma^2}{2\alpha}(e^{-\alpha|t-s|} - e^{-\alpha(s+t)})$$