UNSW Business School
School of Risk and Actuarial Studies

Asset-Liability and Derivative Models

Exercises

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Contents

Module 1

1.1 Utility Theory

1.1.1 Tips and recap

1. In this course, we always make decision to maximise the expected utility $E[U(\cdot)]$, which means in calculation we always plug in the utility function FIRST then calculate the expected value. The reverse order is wrong.

2. Note that utility function is unique up to an affine (linear) transform. Hence, we can always consider a nicer utility function in calculation, for example we can use $U(w) = -e^{-0.1w}$ and $U(w) = w$ in Exercise 1.5.

3. Sometime you can also pick your utility function by specifying the numerical values at 2 points, e.g. $U(0) = 0$ and $U(1) = 1$. This might be useful in some cases.

4. Risk premium for a risk-averse individual: Suppose a risk-averse individual is facing a risk. He/she would prefer the expected value $E[w - X]$ than the risk $w - X$ itself, i.e.

$$E(U(w - X)) < U(E(w - X)).$$

   Let's assume also the individual prefers more wealth than less such that the utility function is increasing. In this case, the certainty equivalent of the risk is

$$C = U^{-1}(E(U(w - X))) < E(w - X)$$

   by taking U^{-1} in the 2 sides of the first inequality. Hence, an individual is indifferent with have wealth C or facing risk $w - X$. The difference $E(w - X) - C$ is called the risk premium, i.e. the maximum amount the individual is willing to pay on top of $E[X]$ to get rid of the risk. Note insurance company is more or less risk neutral due to the law of large number. This is the profit for the insurance company!

5. For absolute and relative risk aversion, one can play with the exponential utility and the power utility.

1.1.2 Practice Questions

Exercise 1.1: [Solution] Luenberger (1ed: Chapter 9. Questions 1,2,5,6 / 2ed: Chapter 11. Questions 1,2,5,6). (Note: you will find L'Hopital's rule helpful in some of these exercises)

Exercise 1.2: [Solution] Explain the key factors that need to be considered when selecting a suitable utility function for certain individuals.

Exercise 1.3: [Solution] Explain the expected utility theorem and explain the four axioms that are required to derive the expected utility theorem.

Exercise 1.4: [Solution] An investment offers a rate of return of 5% with probability $\frac{1}{3}$ and 8% with probability $\frac{2}{3}$ within a time period.

1. Calculate the mean and standard deviation of the returns provided by this investment.
2. An investor has a utility function $u(w) = \log(w)$ where w denotes his terminal wealth from the investment. If he invests \$100 in the investment, what is his expected utility? If instead he invests \$200, what would be his expected utility? Briefly comment on the results.

Exercise 1.5: [Solution] Suppose we have a person that is being offered a gamble that has a 5% chance of losing everything, an 85% chance of losing nothing and a 10% chance of earning a 550% return. Would the person be willing to take this gamble if his utility function is:

1. $U(W) = 3 - 0.8e^{-0.1W}$
2. $U(W) = 5 + 0.7W$

Exercise 1.6: [Solution] A decision maker has utility function

$$u(w) = -e^{-3w}$$

and initial wealth w_0. This decision maker faces two random losses:

- loss X has a normal distribution with mean μ and standard deviation 4; and
- loss Y has a normal distribution with mean 10 and standard deviation 8.

1. Determine the range of values of μ for which the decision maker prefers X to Y.
2. Determine the range of values of μ for which the decision maker prefers Y to X.
3. Determine the range of values of μ for which the decision maker is indifferent between X and Y.

Exercise 1.7: [Solution] Suppose that a risk-averse investor also satisfies the principle of non-satiation. His utility function is given by

$$u(w) = a + be^{cw},$$

where a, b, and c are constants and w denotes wealth.

1. What can be said about the signs of the constants a, b, and c?
2. What are the properties of this utility function in terms of absolute and relative risk aversion?

Exercise 1.8: [Solution] Ed and Danny both have utility functions of the form

$$u(w) = w^a$$

and each has initial wealth of \$10.

1. Assuming that both Ed and Danny are risk-averse and prefer more to less, what can be deduced about the value of a?

2. Find the absolute and relative risk aversion measures for Ed and Danny's utility functions. What does this imply about their desire to invest in risky assets?

3. Now, suppose that Ed's $a = 0.4$ and Danny's $a = 0.6$. Suppose further that Danny owns a ticket for a lottery that pays out \$100 with a probability of 0.1 and \$0 with a probability of 0.9. Ed owns no lottery.

   i. Calculate the expected utilities for Ed and Danny.

   ii. Calculate the lowest price at which Danny is prepared to sell the lottery ticket to Ed. Is Ed willing to buy the ticket at this price? Comment briefly on your answer.

Exercise 1.9: [Solution] You are given the following information on investments A and B:

State of Nature	Probability	Return on Investment A	Return on Investment B
1	0.10	0%	5%
2	0.20	5%	10%
3	0.30	20%	15%
4	0.40	30%	20%

An investor can invest his wealth in either investment A or investment B, but not both.

1. Which investment is considered more risky? Explain your reasoning.

2. Which investment would be undertaken by a risk-neutral investor? Explain your reasoning.

Exercise 1.10: [Solution] Bob wants to invest a proportion of his money into a risky asset and the remainder in a riskless asset. If the riskless asset earns a certain return of r while the risky asset can either give a good return of u with probability p or a bad return of d with probability 1-p where $d < r < u$. We know that Bob has the following utility function: $U(W) = -e^{-0.15W}$ and invests the proportion ρ of his wealth w^* into the risky asset.

1. Find his expected utility after one period

2. Find ρ that maximises his expected utility

Exercise 1.11: [Solution] Suppose Alex is faced with two potential random losses X and Y:

1. Loss X is normally distributed with mean μ and variance 16.

2. Loss Y is Chi-square distributed with 8 degrees of freedom.

Find the range of μ values that would make Alex prefer Y over X if his utility function is given by $U(W) = 1 - 0.5e^{-0.3W}$.

1.1.3 Discussion Questions

Exercise 1.12: [Solution] Martha's preferences can be represented by a quadratic utility function of the form

$$u(w) = w - dw^2, \text{ for } d > 0$$

where $0 \leq w, w - L \leq \frac{1}{2d}$. Suppose that he faces a random loss L with mean $E(L) = \mu$ and variance $Var(L) = \sigma^2$.

1. Show that Martha's expected utility can be expressed as a function of μ and σ^2.
2. Show that Martha prefers more to less and is risk averse.
3. Show that she has an increasing absolute risk aversion.
4. Briefly explain why the quadratic utility function is sometimes considered inappropriate for modelling investor preferences.

Exercise 1.13: [Solution] An investor has the following utility function:

$$u(w) = 1 - e^{-aw},$$

where $w > 0$ is her wealth.

1. Show what constraints exist on the value of a if she prefers more wealth to less wealth (is non-satiated) and is risk averse.
2. Derive her absolute and relative risk aversion functions.
3. Explain the implications of the results in (a) and (b) above for the proportion of wealth she will invest in risky assets as her level of investable wealth changes.

Exercise 1.14: [Solution] Utility Theory can provide very powerful insights on the fundamental idea of investment. In this question we explore this idea in the following 2 period problem.

- Denote by w the wealth of an investor at time 0.
- At time 1, he will spend all his wealth, if there is any.
- He needs to make the following decisions: (1) how much to spend at time 0 (2) how to invest the rest of his wealth (at time 0) so he can spend more at time 1. Obviously, if he spend more at time 0, there is less wealth to spend at time 1 so he needs to do it wisely.
- His spending at time 0 cannot be negative.
- There are 2 choices of investment: (1) risky investment which costs p at time 0 and pays X at time 1, per unit of investment, and (2) risk-free investment with 0 interest, e.g. keep it as cash.
- Assume the individual choose to invest y units of the risky investment, and we assume further (for simplicity) there is no constraints on y, it can be any number - positive, negative, fraction, greater than 1, etc.

- Suppose, he invest s at time 0 in the risk-free investment. ($0 \leq s \leq w$)
- u_0 is the time 0 utility and $u_{0,1}$ is the time 0 utility of wealth at time 1. Generally speaking, there is no reason that the same utility (u_0) is used to evaluate the cash flow in the future.
- Hence the individual is facing the problem of choosing s (how much to invest in risk-free investment) and y (how much to invest in the risky investment) (and therefore spending the rest at time 0), i.e. the following problem:

$$\max_{s,y} E\left[u_0(w - s - yp) + u_{0,1}(s + yX)\right].$$

1. Assume the individual chooses the optimal level (s^*, y^*) and the first order condition holds for (s^*, y^*). Show that such choice satisfies

$$p = E[\frac{u'_{0,1}(s^* + y^*X)}{u'_0(w - s^* - y^*p)}X]$$

assuming we can interchange the expectation and the derivative operator (i.e. the functions u_0 and $u_{0,1}$ are "nice").

2. Interpret the term

$$m(X) = \frac{u'_{0,1}(s^* + y^*X)}{u'_0(w - s^* - y^*p)}.$$

3. For a risk-averse individual, discuss the magniture of m for 2 possible outcomes x_1 and x_2 with $x_1 < x_2$.

Exercise 1.15: [Solution] Recall that Taylor's expansion of G about $\mu = E(X)$, we have

$$G(X) = G(\mu) + (X - \mu) G'(\mu) + \frac{1}{2}(X - \mu)^2 G''(x^*),$$

for some x^* in (μ, X). (Note that the last term is the Lagrange Remainder that results from truncating at the first order)
Using the above, show that when $u''(w) \leq 0$ we have the utility function of a risk adverse individual .

Exercise 1.16: [Solution] Suppose an individual with an initial wealth of \$500 prefers to get a certain 10% return rather than an investment giving 0% return with a probability of 50% and 20% return with a probability of 50%. However, a friend advises her to split up her wealth and invest in both options in the ratio 8 : 2. Now with a wealth of \$1000 she would like to invest \$200 into the latter investment. Given this information, what can you infer about the risk behaviour of the individual.

Exercise 1.17: [Solution] Suppose an investor has the following utility function:

$$U(W) = -\frac{1}{\gamma}W^{-\gamma} \tag{1.1}$$

for some $0 < \gamma < 1$. He invests the proportion ρ of his wealth into a risky security which is Log-normally distributed with parameters (μ, σ^2). The remainder of his wealth is invested in a risk-free security that earns a return of r. Assume his initial wealth is 1.

1. Using his absolute and relative risk aversion, explain how his investment behaviour will change with wealth.
2. Find an expression for the expected utility.

Exercise 1.18: [Solution] [2018 FE] Suppose that the following holds for alternatives in the market: $P > Q > R > S$ and that the utilities of these alternatives satisfy

$$U(P) + U(S) = U(Q) + U(R).$$

Assuming that an individual is risk averse, is it true that

$$U\left(\frac{1}{2}Q + \frac{1}{2}R\right) > U\left(\frac{1}{2}P + \frac{1}{2}S\right)?$$

1.2 Investment Risk Measures

1.2.1 Practice Questions

Exercise 1.19: [Solution]

1. Define the following measures of investment risk:
   i. variance of return;
   ii. downside semi-variance of return; and
   iii. shortfall probability.
2. Give advantages of the variance of return as a measure of investment risk when compared to semi-variance and shortfall probabilities. Give one disadvantage of using the variance of return as a measure of investment risk.

Exercise 1.20: [Solution] Suppose that you are trying to choose between the investments whose distributions of returns are described below:

Investment A:	0.4 probability that it will return 10%
	0.2 probability that it will return 15%
	0.4 probability that it will return 20%
Investment B:	0.25 probability that it will return 10%
	0.70 probability that it will return 15%
	0.05 probability that it will return 40%
Investment C:	a uniform distribution on $(0.10, 0.20)$.

1. For each of these investments, calculate:
   i. Expected return;
   ii. Variance of the return;
   iii. Downside semi-variance;

iv. Expected shortfall below 12%; and

v. Shortfall probability below 15%.

2. Comment on what your calculations above reveal, and hence discuss how the investments should rank in order of attractiveness to a risk-averse investor.

Exercise 1.21: [Solution] You are given the following information on investments A and B:

State of Nature	Probability	Return on Investment A	Return on Investment B
1	0.10	0%	5%
2	0.20	5%	10%
3	0.30	20%	15%
4	0.40	30%	20%

For each of these investments, calculate the downside semi-variance.

1.2.2 Discussion Question

Exercise 1.22: [Solution] Suppose that you are working as an analyst in a superannuation company. Your manager has asked you what measure of risk should be used when considering the risk vs return performance of the fund. Consider potential measures, providing some advantages and disadvantages of each.

Exercise 1.23: [Solution] Suppose an investment gives $Y = 100X$ where X follows a log-normal distribution with parameters $\mu = 0.1$ and $\sigma^2 = 0.04^2$. Here the risk of the investment is defined as $E[Y] - Y$, i.e. the lost from its expectation.

1. What is the 5% Value-at-Risk of the investment?

2. What is the 5% Expected shortfall of the investment?

[Hint: You may want to look up P.18 of the formula booklet, or use the following result:

For any (nice) set B, we have the following identity

$$E[e^{\sqrt{v}Z}1(Z \in B)] = \exp(\frac{1}{2}v)P[\tilde{Z} + \sqrt{v} \in B],$$

where Z and $\tilde{Z}$ are $N(0,1)$ distributed.]

1.3 The Mean-Variance Portfolio Theory

1.3.1 Recap

Below are some useful information regarding matrix operation.

1. Definition: Matrix of dimension $m \times n$ is a table with m rows and n columns.

2. Transpose: Basically flipped the matrix, i.e. a_{ij} becomes a_{ji} and the dimension changes from $m \times n$ to $n \times m$. The transpose of A is denoted as A^T or sometimes A'.

3. A square matrix A is symmetric if $A^T = A$. For example, the variance-covariance matrix is symmetric.

4. 3 important shapes:
   - $n \times n$ matrix: square matrix, usually represents variance-covariance matrix. Often has inverse. Generally denoted by capital letters, e.g. A.
   - $n \times 1$ matrix: a column vecotr, usually represents the expected return of the individual assets in the portfolio. Generally denoted by lower case letters, e.g. z, w.
   - 1×1 matrix: this is a scalar/ number. It equals its transpose. Usually represents how good something is, i.e. a performance measure / criterion. For example, the variance of the portfolio. We usually differentiate it with respect to a vector to yeild an optimal solution.

5. Matrix multiplication: Shapes are important
   - Can only multiply when the shapes of the matrices are compatible, i.e. AB is well defined only when A is a $m \times p$ matrix and B is a $p \times n$ matrix. The p is in common. Therefore, generally speaking $BA \neq AB$ and may not be even well-defined.
   - Transpose: $(AB)^T = B^T A^T$. This basically means you need to flip the matrix when plugging in the transpose. More generally, $(ABC)^T = C^T B^T A^T$.
   - Az is a column vector, $z^T A^T = (Az)^T$ is a row vector.
   - $z^T y$ is a number and $z^T y = y^T z$, i.e. it equals to its transpose.
   - The above implies that $w^T A w$ is a number. In fact, this is sort of a quadratic term in w. Except we have to split the w due to the shape of the martices, i.e. there is no Aww or Aw^2 as w is now a vector.

6. Matrix differentiation: For a function of vector that gives a number, i.e. $f : \mathbb{R}^n \mapsto \mathbb{R}$, the derivative of $f(w)$ with respect to the vector w means the collection of the partial derivatives $\frac{\partial}{\partial w_i} f$ as a column vector, i.e.

$$\frac{\partial}{\partial w} f = [\frac{\partial f}{\partial w_1}, \frac{\partial f}{\partial w_2}, ..., \frac{\partial f}{\partial w_n}]^T.$$

   For the manipulation, we only need to remember 2 rules.
   - $\frac{\partial}{\partial w}(w^T A w) = 2Aw$. We usually scale it by half, i.e.
     $$\frac{\partial}{\partial w}(\frac{1}{2} w^T A w) = Aw$$
     so that we can simply remove everything on the left of A.
   - $\frac{\partial}{\partial w}(z^T w) = z$. We usually write w in front, i.e.
     $$\frac{\partial}{\partial w} w^T z = z$$
     so that the answer is the original expression without w^T (analogous to removing x from $(2x)' = 2$).

7. Breaking the brackets: It is the same as in the univariate case, EXCEPT we have to preserve the order, e.g. $(A+B)C = AC+BC$, $(A+B)(C+D) = AC+AD+BC+BD$, $A(B+C) = AB+AC$.

Lagragian: The idea of Lagragian method is to convert the (difficult) constrained optimisation problem to an (relatively easy) unconstrained optimisation problem, with slightly more variables. In fact, if we denote the (w^*, λ^*) the solution of

$$\min_{w,\lambda}(f(w) + \lambda g(w)),$$

then it can be shown that w^* is the solution to

$$\min_{w} f(w), \text{ subjected to } g(w) = 0.$$

This means we can consider the first optimization problem instead. Note the first optimization has an additional variable but no constrain, which is easier to compute than the original problem as we do not need to check the constrain all the time. Of course, with more constrains, you only need to have more Lagragian constants (variables).

1.3.2 Tips

1. Notations:
   (a) X is the collection of random returns of the assets under consideration.
   (b) $E[X] = z$ and $Var[X] = \Sigma$, where $Var[X]$ denotes the variance-covariane matrix.
   (c) Note $Cov[a'X, b'Y] = a'Cov[X, Y]b$. (Covariance is linear.)
   (d) Hence $E[w'X] = w'z$ (portfolio return),
   (e) $Var[w'X] = Cov[w'X, w'X] = w'Cov[X, X]w = w'Var[X]w = w'\Sigma w$ (portfolio variance) and similarly
   (f) $Cov[w_1'X, w_2'X] = w_1'\Sigma w_2$. (covariance between the return of 2 portfolios)
2. For the mean variance analysis, there are 4 important constants A, B, C and Δ. You may find the following trick helpful. We (1) first draw Σ^{-1} then (2) add 1 and 1 on two sides of it, which gives A. (3) We then subsequently substitute each 1 with z to get B and C. Finally (4) add a transpose on the first matrix to get the correct shape, i.e.

$$A : \Sigma^{-1} \to 1\Sigma^{-1}1 \to 1'\Sigma^{-1}1$$
$$B : \Sigma^{-1} \to 1\Sigma^{-1}z \to 1'\Sigma^{-1}z$$
$$C : \Sigma^{-1} \to z\Sigma^{-1}z \to z'\Sigma^{-1}z$$

   Note $B = 1'\Sigma^{-1}z = z'\Sigma^{-1}1$ as they are transpose of each other. Finally $\Delta = AC - B^2$.
3. Note from the solution of the optimal (stocks-only) problem, the solution $w = \lambda\Sigma^{-1}1 + \gamma\Sigma^{-1}z$ can be rewritten as

$$w = \lambda A(\frac{\Sigma^{-1}1}{A}) + \gamma B(\frac{\Sigma^{-1}z}{B})$$

   which is a linear combination of 2 funds $w_g = \frac{\Sigma^{-1}1}{A}$ and $w_d = \frac{\Sigma^{-1}z}{B}$ as $\lambda A + \gamma B = 1$, i.e. first step to 2 fund theorem.

4. There are typically 3 problems in the lecture
   (a) stocks only, want to find the portfolio that minimises the variance $\Longrightarrow$ 1 constrain $\Longrightarrow$ w_g.
   (b) stocks only, want to find the portfolio that minimises the variance WITH a target return $\Longrightarrow$ 2 constrains $\Longrightarrow$ linear combination of w_g and w_d
   (c) risk-free available, same problem with (2) $\Longrightarrow$ 1 constrain $\Longrightarrow$ $\tilde{\gamma} w_t$.
5. The mean and the variance of w_g and w_d can be calculated easily as follows:

   [Recall X is the collection of the random returns of all assets considered. The portfolio return is $w'X$ which is a weighted sum of the random return.]

   (a)
$$E[w_g'X] = z'w_g = \frac{z'\Sigma^{-1}1}{A} = \frac{B}{A},$$
$$Var[w_g'X] = w_g'\Sigma w_g = \frac{1}{A^2}1'\Sigma^{-1}\Sigma\Sigma^{-1}1 = \frac{1'\Sigma^{-1}1}{A^2} = \frac{1}{A}.$$
   (b) Similarly,
$$E[w_d'X] = \frac{C}{B},$$
$$Var[w_d'X] = \frac{C}{B^2}.$$

   Recall the variance of the solution to the MV problem without risk-free asset is a quadratic function of the mean. To find the formula without remembering it, we can proceed the following:

   (a) As w_g is the global minimal, the formula must take the form of $V(\mu) = a(\mu - E[w_g'X])^2 + Var[w_g'X]$, or explicitly
$$V(\mu) = a(\mu - \frac{B}{A})^2 + \frac{1}{A}$$
   for some positive number a to be determined.
   (b) To find a, we can simply substitute $E[w_d'X]$ and $Var[w_d'X]$ into the above formula as w_d is on the MV frontier, i.e.
$$\frac{C}{B^2} = a(\frac{C}{B} - \frac{B}{A})^2 + \frac{1}{A}$$
   from where you can determine a (once you know A, B and C).

   [Note some exam questions ask the variance for the MV portfolio for a given target return. You either have to remember the formula correctly, or have to be able to derive it within a short period of time.]

1.3.3 Practice Questions

Exercise 1.24: [Solution] Luenberge time contingent claim valuationr, Chapter 6 (both editions). Questions 3,6,7.

Exercise 1.25: [Solution] Explain two scenarios where the Markowitz Mean-Variance model can be applied.

Exercise 1.26: [Solution] Explain what is meant by the following terms, in the context of the mean-variance portfolio theory: contingent claim

1. efficient frontier; and
2. optimal portfolio.

Exercise 1.27: [Solution] An investor can invest in only two risky assets A and B. Asset A has an expected rate of return of 10% and a standard deviation of return of 20%. Asset B has an expected rate of return of 15% and a standard deviation of return of 30%. The correlation coefficient between the returns of Asset A and that of Asset B is 0.60. The investor invests 20% of his wealth in Asset A and 80% in Asset B.

1. Calculate the expected rate of return of the investor's portfolio.
2. Calculate the standard deviation of the return for this investor's portfolio.
3. Explain why the investor is considered risk-averse.

Exercise 1.28: [Solution] A portfolio of investments is 40% invested in Security A and 60% invested in Security B. The return from Security A is equally likely to be 8% or 13%. The return from Security B will be 10% with a 0.7 probability and 16% with a 0.3 probability.

1. For each security, calculate the expected return and the variance of return.
2. Calculate the portfolio's expected return and variance of return, assuming the correlation coefficient between the returns is:
   i. $+1$; and
   ii. -1.
3. Assuming the correlation coefficient is 0.90, can you find a different mix of Securities A and B that will yield the same expected return as the original portfolio? (If yes, identify that portfolio explicitly)

Exercise 1.29: [Solution] Prove that it is possible to create a portfolio that consists of two risky assets that are perfectly negatively correlated and that is completely risk-free.

Exercise 1.30: [Solution] An investor wishes to construct a portfolio consisting of a risk-free and a risky asset. His utility (depending on mean and variance, e.g. expected quadratic utility) is given by

$$U(\mu_p, \sigma_p) = \mu_p - \frac{1}{2}\sigma_p^2$$

where μ_p and σ_p are the mean and standard deviation of the portfolio rates of return. The risk-free asset has an expected rate of return of 5%. The risky asset has an expected rate of return of 8% and a variance of 4%. Determine the portfolio that will maximize the investor's utility.

Exercise 1.31: [Solution] An investor can invest in any combination of three risky assets, labeled 1, 2, and 3. The expected rate of return and the standard deviation of return for each of these securities are summarized below:

Security	Expected Return	Standard Deviation
1	2%	3%
2	4%	4%
3	6%	5%

You are given: $\rho_{12} = 0.25$, $\rho_{13} = 0.50$, and $\rho_{23} = -0.50$, where ρ_{ij} denotes the correlation coefficient between securities i and j. The investor has a portfolio that consists of 30% in security 1, 20% in security 2, and 50% in security 3.

1. Calculate the expected rate of return of the investor's portfolio.
2. Calculate the standard deviation of the return for this investor's portfolio.

Exercise 1.32: [Solution] Explain the two fund theorem and under what circumstances is this theorem true? How can this be applied in the efficient calculation of efficient frontiers?

Exercise 1.33: [Solution] Explain the one fund theorem and under what circumstances is this theorem true? Find the weights for the tangency portfolio.

Exercise 1.34: [Solution] Suppose we have n assets in the market and a risk-free asset. Assume we know all the characteristics of these $n+1$ assets.

1. Derive the weights for the tangency portfolio.
2. Find the mean and variance of the tangency portfolio.

1.3.4 Practice Questions - Implementation

The following questions involve practical implementations of the theory and techniques covered in this topic. The implementation will/should be done using Excel.

Being able to implement the techniques using software is important for your understanding of the material, and provides practice into some of the steps you will need to apply the models in practice. Note in particular that you may be asked to explain e.g. how you can implement models and techniques in the assessment tasks of the course.

Exercise 1.35: [Solution] (Practical Implementation using Excel) Find the portfolio (weight, mean and variance) with desired mean 0.0625 which has minimum variance. Assume that the risky assets satisfy:

$$\mathbf{z} = \begin{pmatrix} 0.07 \\ 0.06 \\ 0.08 \end{pmatrix}$$

and the variance-covariance matrix is

$$\mathbf{\Sigma} = \begin{pmatrix} 0.0004 & 0.0002121 & 0.003 \\ 0.0002121 & 0.0018 & 0.0006364 \\ 0.003 & 0.0006364 & 0.0036 \end{pmatrix}.$$

and the riskless asset is not available.

Exercise 1.36: [Solution] (Practical Implementation using Excel) You should create a spreadsheet from scratch that can enable you to compute the efficient frontier (and associated portfolios and properties) for a 4-risky asset setting. Doing so will provide you with a much better understanding and appreciation of the mathematics behind the techniques we used in this topic - Learning by doing!

1.3.5 Discussion Questions

Exercise 1.37: [Solution] Consider a two-security world in which the returns yielded by Assets 1 and 2 are perfectly correlated, though they have different expected returns.

1. Derive the equation of the efficient frontier in the expected return-standard deviation (μ-σ) space.
2. Use the result above to:
   i. Determine the gradient of the efficient frontier.
   ii. Show that the efficient frontier is a straight line in the μ-σ space through the points representing Assets 1 and 2.

Exercise 1.38: [Solution] Consider the n-asset Markowitz model without a riskless asset

1. Derive the Global Minimum Variance Portfolio.
2. Prove that the covariance between the Global Minimum Variance Portfolio and any other minimum variance portfolio is constant (and find this constant).
3. Derive the Covariance between any two minimum variance portfolio.

Exercise 1.39: [Solution] In specifying the Two Fund Theorem, we mentioned the global minimum variance portfolio and another minimum variance portfolio known as the "d" portfolio.

(a) State the "d" portfolio.

(b) Find the mean and variance of the "d" portfolio.

(c) Find the covariance between the global minimum variance portfolio and the "d" portfolio and confirm your answer with 3(b).

Exercise 1.40: [Solution] Prove the result given by the two fund theorem. Hint: The global minimum variance portfolio is given by $\mathbf{w_g} = \frac{\Sigma^{-1}\mathbf{1}}{\mathbf{A}}$ and another minimum variance portfolio the "d" portfolio is given by $\mathbf{w_d} = \frac{\Sigma^{-1}\mathbf{z}}{\mathbf{B}}$

Exercise 1.41: [Solution] We wish to make a passive investment portfolio r_p that can track the market portfolio r_m as close as possible. The measure used for evaluating the portfolio performance is the tracking error:

$$\sigma_{tracking} = Var(r_p - r_m)$$

and we wish to minimise this at all times whilst trying to achieve a mean return of μ. Assume we know the following information:

- $E(r_m) = \mu_m, Var(r_m) = \sigma_m^2$
- The portfolio consists of n uncorrelated stocks and the variance of each stock is given by: $\sigma_i^2 = i\sigma^2$
- The mean return of each stock is given by: $\mu_i = i\mu$
- The covariance of each stock with the market portfolio can be represented by: $Cov(r_i, r_m) = \beta_i\sigma_m^2$ i.e $Cov(r_p, r_m) = \mathbf{w}'\boldsymbol{\beta}\sigma_m^2$

Derive the weights for the optimal portfolio.

Exercise 1.42: [Solution] The market profolio is defined as the profolio that is proportional to the market. Specifically, if there are N assets in total in the market, with price of asset i being p_i, $i = 1, 2, ..., N$. Then the market portfolio is defined as $w = (w_1, ..., w_N)'$ with

$$w_i = \frac{p_i}{\sum_{i=1}^N p_i}, \quad i = 1, 2, ..., N.$$

(i) Suppose all investors in the market has the same estimation on the mean and the variance of each asset and they are mean-variance optimiser. What is the relationship between the market portfolio and the tangency portfolio? Why?

(ii) Suppose a new information is released which implies that the expected return of asset 2 should be higher than the current estimate (i.e. good news). Discuss what will happen if

(1) Everyone in the market can interpret this information in consensus.

(2) Only a handful "skilled" individuals can interpret the information correctly.

(iii) How can an average investor guard against the information asymmetry described in (iii)? Justify your answer and why this is not observed in practice?

[Note: This question is somewhat related to the Efficient Market Hypothesis which will be discussed in the next chapter.]

1.4 Solutions

1.4.1 Utility Theory

Solution 1.1: [Exercice] Luenberger, Chapter 9/11. Questions 1,2,5,6.

1. The possible outcomes and utility are:

Income	80000	90000	100000	110000	120000	130000	140000
Utility	16.82	17.32	17.78	18.21	18.61	18.99	19.34

The expected utility is the average (since each outcome is equally likely) which is 18.15. The certainty equivalent is the amount C such that

$$C^{0.25} = 18.15$$

Hence $C = 108610$.

2. Investment will be made if

$$E\left[U\left(W-w+x\right)\right]>E\left[U\left(W\right)\right]$$

for our case this is

$$E\left[-e^{-aW}e^{-a(x-w)}\right]>E\left[-e^{-aW}\right]$$

and by rearrangement:

$$e^{aw}E\left[e^{-ax}\right]<1$$

which is independent of W.

3. If results are consistent, we have $V\left(x\right)=aU\left(x\right)+b$. But since $V\left(A'\right)=A', V\left(B'\right)=B'$ we must have

$$\begin{aligned} A' &= aU\left(A'\right)+b \\ B' &= aU\left(B'\right)+b \end{aligned}$$

and hence

$$\begin{aligned} a &= \frac{A'-B'}{U\left(A'\right)-U\left(B'\right)} \\ b &= \frac{B'U\left(A'\right)-A'U\left(B'\right)}{U\left(A'\right)-U\left(B'\right)} \end{aligned}$$

4. [1] Linear: choose $\gamma=1, a=1$ and use L'Hopital's Rule

Quadratic: choose $\gamma=2, a>0, b=1/a>0$.Also can transfrom by adding $b^2/2$

Exponential: choose $b=1, \gamma=-\infty$ and use L'Hopital's Rule

Power: $b=0.\gamma<1$

Logarithmic: $b=0, a=1$.so

$$U\left(x\right)=\left(1-\gamma\right)^{1-\gamma}\frac{x^{\gamma}}{\gamma}$$

subtract constant $c=\frac{(1-\gamma)^{1-\gamma}}{\gamma}$ to obtain

$$U\left(x\right)=\left(1-\gamma\right)^{1-\gamma}\left(\frac{x^{\gamma}-1}{\gamma}\right)$$

Let $\gamma=0$ and use L'Hopital's Rule.

Arrow-Pratt risk aversion

$$\begin{aligned} a\left(x\right) &= -\frac{U''\left(x\right)}{U'\left(x\right)} \\ &= -\frac{-a^2\left(\frac{ax}{1-\gamma}+b\right)^{\gamma-2}}{a\left(\frac{ax}{1-\gamma}+b\right)^{\gamma-1}} \\ &= \frac{a}{\frac{ax}{1-\gamma}+b} \end{aligned}$$

as required.

[1] see end for additional details

Additional Info for Luenberger 11.6:

(a) Choose $\gamma = 1$ as the highest power of x is 1. Also choose $a = 1$ and $b = 0$. This gives

$$\begin{aligned}
\lim_{\gamma\to 1} U(x) &= \lim_{\gamma\to 1} \frac{1-\gamma}{\gamma}\left(\frac{x}{1-\gamma}\right)^{\gamma} \\
&= \lim_{\gamma\to 1} x^{\gamma}(1-\gamma)^{1-\gamma} \\
&= x \lim_{y\to 0} y^{y} \quad \text{(chagne of variable with } y = 1-\gamma)
\end{aligned}$$

To evaluate $\lim_{y\to 0} y^y$, we use L'Hopital's Rule as follows:

$$\begin{aligned}
\lim_{y\to 0} y^{y} &= \exp \lim_{y\to 0} \log(y^{y}) \quad \text{(as } e^{x} \text{ is a continuous function)} \\
&= \exp \lim_{y\to 0} y \log y \\
&= \exp \lim_{y\to 0} \frac{\log y}{1/y} \quad \left(\frac{\log y}{1/y} \text{ is } \frac{-\infty}{\infty}\right) \\
&= \exp \lim_{y\to 0} \frac{1/y}{-1/y^{2}} \quad \text{(L'Hopital's Rule, i.e. differentiate the top and the bottom)} \\
&= \exp \lim_{y\to 0}(-y) \\
&= 1
\end{aligned}$$

This gives $\lim_{\gamma\to} U(x) = x$.

(b) Choose $\gamma = 2$ as the highest power of x is 2.

$$\begin{aligned}
\lim_{\gamma\to 2} U(x) &= \lim_{\gamma\to 2} \frac{1-\gamma}{\gamma}\left(\frac{ax}{1-\gamma} + b\right)^{\gamma} \\
&= -\frac{1}{2}(-ax+b)^{2} \\
&= -\frac{1}{2}a^{2}x^{2} + abx - \frac{1}{2}b^{2} \\
&= x - \frac{1}{2}cx^{2} - \frac{1}{2}b^{2} \text{ (when } a^{2} = c,\ ab = 1)
\end{aligned}$$

Since $b > 0$ is defined in the function, then $a > 0$ and $b = \frac{1}{a} > 0$.

(c) Try $\gamma = -\infty$ as indicated in the question. Take $b = 1$, we have

$$\begin{aligned}
\lim_{\gamma\to-\infty} U(x) &= \lim_{\gamma\to-\infty} \frac{1-\gamma}{\gamma}\left(\frac{ax}{1-\gamma}+1\right)^{\gamma} \\
&= \lim_{\gamma\to-\infty} \frac{1-\gamma}{\gamma} \lim_{\gamma\to-\infty}((1+\frac{ax}{1-\gamma})^{-\gamma})^{-1} \\
&= -1\times\Big(\lim_{\gamma\to-\infty}(1+\frac{ax}{1-\gamma})^{-\gamma}\Big)^{-1} \quad \text{(as } x^{-1} \text{ is a continuous function)} \\
&= -1\times\Big(\frac{\lim_{\gamma\to-\infty}(1+\frac{ax}{1-\gamma})^{1-\gamma}}{\lim_{\gamma\to-\infty}(1+\frac{ax}{1-\gamma})}\Big)^{-1} \quad \text{(prepare to change the variable } 1-\gamma) \\
&= -1\times\Big(\frac{\lim_{y\to\infty}(1+\frac{ax}{y})^{y}}{1}\Big)^{-1} \quad (y = 1-\gamma \text{ and the denumerator is evaluated as 1)} \\
&= -1(e^{ax})^{-1} \quad \text{(Recall } e^x \text{ is defined as } \lim_{n\to\infty}(1+\frac{x}{n})^n) \\
&= -e^{-ax}.
\end{aligned}$$

Solution 1.2: [Exercice] The key factors that need to be considered include:

(i) The non-satiation principle - in general people are happier with more money (first derivative is positive)

(ii) The risk appetite of the individual - is the person risk averse, risk neutral or risk loving? (determines the sign of the second derivative)

(iii) The tractability of the chosen function - are calculations made using the utility function easy?

Solution 1.3: [Exercice] The expected utility theorem states that:

Under the assumption that all axioms on an individual's preferences holds, there is a utility function (unique up to affine transformation) such that the individual always choose the option with highest EXPECTED utility.

There are four axioms required to derive the expected utility theorem and they are:

(a) Complete/Comparable: It is assumed that risks can be compared and ranked. If X and Y are any two risks, then either $X \succ Y$, $Y \succ X$, or $X \sim Y$.

(b) Transitive: If $X \succcurlyeq Y$ and $Y \succcurlyeq Z$, then $X \succcurlyeq Z$, i.e. if X, Y and Z are three risks where X is preferred to Y and Y is preferred to Z, then X must be preferred to Z.

(c) Independent: If $X \succcurlyeq Y$, then a lottery/gamble that pays $\begin{cases} X, & \text{w.p. } p \\ Z, & \text{w.p. } 1-p \end{cases}$ will be preferred to a lottery/gamble that pays $\begin{cases} Y, & \text{w.p. } p \\ Z, & \text{w.p. } 1-p \end{cases}$. This means independence with respect to probability mixtures of uncertain outcomes.

(d) Measurability/continuity: If $X \succcurlyeq Y$ and $Y \succcurlyeq Z$, then there is a unique probability p such that investor is indifferent between Y and a gamble that pays $\begin{cases} X, & \text{w.p. } p \\ Z, & \text{w.p. } 1-p \end{cases}$.

Solution 1.4: [Exercice] Let R be the random rate of return

$$R = \begin{cases} 5\%, & \text{w.p. } 1/3 \\ 8\%, & \text{w.p. } 2/3 \end{cases}.$$

(a) Mean is $E(R) = 0.05\left(\frac{1}{3}\right) + 0.08\left(\frac{2}{3}\right) = 0.07 = 7\%$ and variance is $Var(R) = \left[0.05^2\left(\frac{1}{3}\right) + 0.08^2\left(\frac{2}{3}\right)\right] - (0.07)^2 = 0.0002$. The standard deviation is therefore $sd(R) = \sqrt{0.0002} = 0.0141421$.

(b) If \$100 is invested, his expected utility is

$$E\left[u\left(100\left(1+R\right)\right)\right] = E\left[\log 100\left(1+R\right)\right] = (\log 105)\left(\frac{1}{3}\right) + (\log 108)\left(\frac{2}{3}\right) = 4.6727$$

and if \$200 is invested (twice that of \$100) then his expected utility will be

$$E\left[u\left(200\left(1+R\right)\right)\right] = E\left[\log 200\left(1+R\right)\right] = (\log 210)\left(\frac{1}{3}\right) + (\log 216)\left(\frac{2}{3}\right) = 5.3659.$$

Note that his expected utility is greater with double the investment, but is less than double than what it was before. This is because the utility function is an increasing but concave (which indicates risk-averse) function of wealth. Hence, any increase in wealth produces a less than proportionate increase in utility.

[As opposed to a linear function, which indicates risk-neutral.]

Solution 1.5: [Exercice] First calculate the utility of wealth when he doesn't take the gamble and the expected utility of wealth when he does take the gamble, then compare the sizes of the two. Now the wealth outcomes of taking the gamble are as follows:

$$G = \begin{cases} 0 & w.p\ \ 0.05 \\ w & w.p\ \ 0.85 \\ 6.5w & w.p\ \ 0.10 \end{cases}$$

(a) Note the individual is making decision to maximise $E[U(\cdot)]$ with

$$U(w) = 3 - 0.8e^{-0.1w}$$

which is equivalent to making decision to maximise $E[\tilde{U}(\cdot)]$ with

$$\tilde{U}(w) = -e^{-0.1w}$$

as E is linear.

Now, if gamble, the expected utility is

$$\begin{aligned} E[\tilde{U}(G)] &= 0.05 \times (-1) + 0.85 \times (-e^{-0.1w}) + 0.1 \times (-e^{-0.1\times 6.5w}) \\ &= -0.05 - 0.85e^{-0.1w} - 0.1e^{-6.5w}. \end{aligned}$$

If not gamble, the expected utility is $E[\tilde{U}(w)] = -e^{-0.1w}$.

Hence, he prefers gamble over not gamble if

$$E[\tilde{U}(G)] > E[\tilde{U}(w)]$$

which is equivalent to

$$-0.05 - 0.85e^{-0.1w} - 0.1e^{-6.5w} > -e^{-0.1w}.$$

Rearranging gives

$$0.1e^{-6.5w} - 0.15e^{-0.1w} + 0.05 < 0.$$

By denoting $y = e^{-0.1w} \in [0,1]$ and divide both sides by 0.05, we have

$$f(y) = 2y^{6.5} - 3y + 1 < 0.$$

Obviously $f(1) = 0$ and $f(0) > 0$. To understand the behaviour of f, we take derivative to get

$$f'(y) = 13y^{5.5} - 3$$

which is an increasing function with $f'(0) < 0$ and $f'(1) > 0$. This implies that when y goes from 0 to 1, f decreases from a positive value to some negative values then increases to 0 at $y = 1$. In particular, it meanse that there is a point y_0 such that $f \geq 0$ on $[0, y_0]$ and $f \leq 0$ on $[y_0, 1]$.

Back to the original problem, we know that the individual prefers gamble when $y_0 < y = e^{-0.1w} < 1$ and we can determine y_0 numerically. Solving for w gives $0 \leq w \leq 10.9701$. Hence, we have:

1. He is indifferent between the two options when his wealth is 0 or \$10.97
2. He prefers the gamble when his wealth is between 0 and \$10.97
3. He prefers not taking the gamble when his wealth is greater than \$10.97

[Of course, with calculator, one does not need to study the derivatives of f. Use the "plot" function would be sufficient.]

(b) Note $U(w) = 5 + 0.7w$ is equivalent to $\tilde{U}(w) = w$, which implies that the individual is risk-neutral. Hence, it is sufficient to consider the expected values of the options.

If gamble,

$$E[G] = 0.85 \times w + 0.1 \times 6.5w = 1.5w.$$

Clearly, this is greater than $E[w] = w$. Hence, gamble is always preferred.

Solution 1.6: [Exercice] Note the M.G.F. of $Z \sim N(0,1)$ is given by

$$M_Z(t) = E[e^{tZ}] = \exp(\frac{1}{2}t^2).$$

For the option with loss X, the expected utility is

$$\begin{aligned}
E[U(w_0 - X)] &= E[-\exp(-3 \times (w_0 - X))] \\
&= E[-\exp(-3 \times (w_0 - N(\mu, 4^2)))] \\
&= E[-\exp(-3 \times (N(w_0 - \mu, 4^2)))] \\
&= E[-\exp(N(-3w_0 + 3\mu, (3 \times 4)^2))] \\
&= E[-\exp((-3w_0 + 3\mu) + (3 \times 4)Z)] \quad \text{(Note normal r.v. can be scaled)} \\
&= -\exp(-3w_0 + 3\mu)E[\exp((3 \times 4)Z)] \\
&= -\exp(-3w_0 + 3\mu)\exp(\frac{1}{2}(3 \times 4)^2) \quad \text{(Using } M_Z \text{ with } t = 3 \times 4)
\end{aligned}$$

Note the 3 comes from the utility function, μ and 4 are the mean and the standard deviation of the loss X. Hence, the expected utility for the option with loss Y is given by

$$E[U(w_0 - Y)] = -\exp(-3w_0 + 3 \times 10)\exp(\frac{1}{2}(3 \times 8)^2).$$

Therefore the option with loss X is preferred if $\frac{E[U(w_0-X)]}{E[U(w_0-Y)]} < 1$ (as $E[U(\cdot)]$ is negative), or $\log \frac{E[U(w_0-X)]}{E[U(w_0-Y)]} < 0$, where the left hand side is evaluated as

$$\begin{aligned}
\log \frac{E[U(w_0-X)]}{E[U(w_0-Y)]} &= \log \frac{-\exp(-3w_0+3\mu)\exp(\frac{1}{2}(3\times 4)^2)}{-\exp(-3w_0+3\times 10)\exp(\frac{1}{2}(3\times 8)^2)} \\
&= \log \exp(3\mu - 30 + \frac{1}{2}3^2(4^2-8^2)) \\
&= 3\mu - 30 + \frac{1}{2}3^2 4^2(1-2^2) \\
&= 3\mu - 30 - 3^2 \times 8 \times 3 \\
&= 3(\mu - 10 - 72) \\
&= 3(\mu - 82).
\end{aligned}$$

Therefore,

(a) $X \succcurlyeq Y$ if and only if $\mu \leq 82$.

(b) $Y \succcurlyeq X$ if and only if $\mu \geq 82$.

(c) $X \sim Y$ if and only if $\mu = 82$.

Solution 1.7: [Exercice] For a risk-averse non-satiated individual, we must have $u^{'}(w) > 0$ and $u^{''}(w) < 0$.

(a) If $u(w) = a + be^{cw}$, then $u^{'}(w) = bce^{cw} > 0$ and $u^{''}(w) = bc^2e^{cw} < 0$ if and only if $b < 0$ and $c < 0$. Therefore, we must have $b < 0$ and $c < 0$ for a risk-averse individual. a can be anything as utility functions are equivalent when adding a constant (and/or multiplying with a positive number), in a sense that the decision which maximises the expected utility is the same.

(b) $A(w) = -\dfrac{u^{''}(w)}{u^{'}(w)} = -\dfrac{bc^2e^{cw}}{bce^{cw}} = -c$ and therefore $R(w) = -cw$. Thus, since $c < 0$, the utility function exhibits constant absolute risk aversion and increasing relative risk aversion.

Solution 1.8: [Exercice] We are given that $u(w) = w^a$. Therefore, $u^{'}(w) = aw^{a-1}$ and $u^{''}(w) = a(a-1)w^{a-2}$.

(a) Prefer more to less implies $u^{'}(w) = aw^{a-1} > 0$ so that $a > 0$. Risk averse implies $u^{''}(w) = a(a-1)w^{a-2} < 0$. Now $a > 0$ implies $a-1 < 0$ or $a < 1$. Combining them, we must have $0 < a < 1$.

(b) $A(w) = \dfrac{(1-a)}{w}$ and $R(w) = 1-a$. Hence, for both Ed and Danny, the absolute risk aversion decreases with increasing wealth, so that their holdings of risky assets will increase with increasing wealth. Their relative risk aversion is constant, so that the proportion of their wealth that they hold as risky assets remains fixed.

(c) The lottery pays out $L = \begin{cases} 100, & \text{w.p. } 0.1 \\ 0, & \text{w.p. } 0.9 \end{cases}$.

i. Ed's expected utility is given by $u(10) = 10^{0.4} = 2.5119$ and Danny's expected utility is

$$E[u(10+L)] = 110^{0.6}(0.1) + 10^{0.6}(0.9) = 5.2611.$$

ii. Let P be the smallest price for which Danny is willing to sell the lottery. He will sell the lottery only if he can increase his expected utility by doing so. This will be the case if

$$u(10+P) \geq 5.2611 \text{ or equivalently } P \geq 5.9145.$$

If Ed purchases the ticket for $P = 5.9145$, then his expected utility will be

$$E[u(10+L-P)] = (110-5.9145)^{0.4}(0.1) + (10-5.9145)^{0.4}(0.9) = 2.2214.$$

Since this utility is smaller than his utility without the lottery ticket, then Ed will not buy the ticket. The price that Ed is willing to pay for the ticket is less than that for which Danny is willing to sell. This is because Ed is more risk-averse. This is indicated partly by: (1) the lower value of a in his utility function which is consequently more concave, and (2) the higher value of both the absolute risk aversion and relative risk aversion when a is lower.

[Note the market now is at "equilibrium" as the (relatively) risk-loving person is holding the ticket. In other words, there is no "market" for the ticket. Suppose alternatively Ed is holding the ticket, then there is a market for the ticket, i.e. the willing sell price is lower than the willing buy price. In this case, the actual price would be determined by the bargining power of the individuals.]

Solution 1.9: [Exercice] First, we compute the mean and variance of returns. For investment A, denote its random rate of return by $\widetilde{r}_A$ so that

$$E(\widetilde{r}_A) = 19 \text{ and } Var(\widetilde{r}_A) = 124.$$

For investment B, denote its random rate of return by $\widetilde{r}_B$ so that

$$E(\widetilde{r}_B) = 15 \text{ and } Var(\widetilde{r}_B) = 25.$$

(a) One possible answer is that Investment A is considered more 'risky' since it has a larger variance. - but it depends on how one defines 'risk'.

(b) A risk-neutral investor will always base his/her decision on the expected value. He will therefore choose A over B since it has a higher mean.

Solution 1.10: [Exercice]

1.

$$\begin{aligned}
E[U(W)] &= E[U(\rho w^*(1+r_{risky}) + (1-\rho)w^*(1+r))] \\
&= pU(\rho w^*(1+u) + (1-\rho)w^*(1+r)) + (1-p)U(\rho w^*(1+d) + (1-\rho)w^*(1+r)) \\
&= -pe^{-0.15(\rho w^*(1+u)+(1-\rho)w^*(1+r))} - (1-p)e^{-0.15(\rho w^*(1+d)+(1-\rho)w^*(1+r))} \\
&= -pe^{-0.15(w^*(1+r)+\rho w^*(u-r))} - (1-p)e^{-0.15(w^*(1+r)+\rho w^*(d-r))} \\
&= -e^{-0.15w^*(1+r)}\left(pe^{\rho(-0.15w^*(u-r))} + (1-p)e^{\rho(0.15w^*(r-d))}\right)
\end{aligned}$$

2. Denote the 3 positive numbers $A = e^{-0.15w^*(1+r)}$, $a = 0.15w^*(u-r)$ and $b = 0.15w^*(r-d)$, we have

$$f(\rho) = E[U(W)] = -A(pe^{-a\rho} + (1-p)e^{b\rho}).$$

Obviously we have $f'' < 0$ which means the function is concave. Now solving for $f' = 0$ we have

$$\begin{aligned} 0 = f'(\rho) = &\ -A(-ape^{-a\rho} + (1-p)be^{b\rho}) \\ &= Ab(1-p)e^{-a\rho}\Big(\frac{ap}{b(1-p)} - e^{(a+b)\rho}\Big) \end{aligned}$$

which gives

$$\begin{aligned} \rho^* &= \frac{1}{a+b}\log\frac{ap}{b(1-p)} \\ &= \frac{1}{a+b}\Big(\log\frac{p}{1-p} + \log\frac{a}{b}\Big) \\ &= \frac{1}{0.15w^*(u-d)}\Big(\log\frac{p}{1-p} + \log\frac{0.15w^*(u-r)}{0.15w^*(r-d)}\Big) \\ &= \frac{1}{0.15w^*(u-d)}\Big(\log\frac{p}{1-p} + \log\frac{u-r}{r-d}\Big) \end{aligned}$$

Solution 1.11: [Exercice] Assume that Alex has some arbitrary initial wealth W_0. Then his expected utility under each loss is:

$$\begin{aligned} E(U(W_X)) &= E\left(1 - 0.5e^{-0.3(W_0 - X)}\right) \\ &= 1 - 0.5e^{-0.3W_0}E\left(e^{0.3X}\right) \\ &= 1 - 0.5e^{-0.3W_0}e^{0.3\mu + \frac{16\cdot 0.3^2}{2}} \\ &= 1 - 0.5e^{-0.3W_0}e^{0.3\mu+0.72} \\ E(U(W_Y)) &= E\left(1 - 0.5e^{-0.3(W_0 - Y)}\right) \\ &= 1 - 0.5e^{-0.3W_0}E\left(e^{0.3Y}\right) \\ &= 1 - 0.5e^{-0.3W_0}(1 - 2\cdot 0.3)^{-\frac{8}{2}} \\ &= 1 - 0.5e^{-0.3W_0}(0.4)^{-4} \end{aligned}$$

We want to find μ such that $E(U(W_Y)) > E(U(W_X))$.

$$\begin{aligned} 1 - 0.5e^{-0.3W_0}(0.4)^{-4} &> 1 - 0.5e^{-0.3W_0}e^{0.3\mu+0.72} \\ (0.4)^{-4} &< e^{0.3\mu+0.72} \\ -4\ln(0.4) &< 0.3\mu + 0.72 \\ \mu &> 9.817 \end{aligned}$$

[Note one can also work with $\tilde{U}(w) = -e^{-0.3w}$]

Solution 1.12: [Exercice] Suppose the initial wealth is w (and $w - L \in [0, \frac{1}{2d}]$).

1. Thus, expected utility becomes

$$\begin{aligned} E[u(w-L)] &= E[(w-L) - d(w-L)^2] \\ &= E[w-L] - dE[(w-L)^2] \\ &= E[w-L] - d[Var(w-L) + (E[w-L])^2] \\ &= w - \mu - d(w-\mu)^2 - dVar(L) \\ &= w - \mu - d(w-\mu)^2 - d\sigma^2. \end{aligned}$$

2. $u(w) = w - dw^2$ implies $u'(w) = 1 - 2dw \geq 0$ as $w \in [0, \frac{1}{2d}]$ where the equality holds only when $w = \frac{1}{2d}$. Hence, Martha prefers more wealth than less for $0 \leq w < \frac{1}{2d}$. She cannot have more wealth when $w = \frac{1}{2d}$ by definition.

   $u''(w) = -2d < 0$ (i.e. risk-averse).

3. $A(w) = \dfrac{2d}{1-2dw}$. Since $A'(w) = \dfrac{4d^2}{(1-2dw)^2} > 0$, then absolute risk aversion is increasing. (We also treat $+\infty > 0$.)

4. The quadratic utility function is sometimes considered inappropriate for modelling investor preferences because:

   - utility increases with wealth only over a limited range of values, i.e. in this case, $0 \leq w \leq \frac{1}{2d}$. Martha cannot have wealth more than $\frac{1}{2d}$.
     Although we can set d to be a very large number, the curvature (i.e. $A(w)$ and $R(w)$) of the utility function also depends on d , which makes it difficult to use in practice.
   - it exhibits increasing absolute risk aversion which means that Martha holds fewer risky assets as her wealth increases, which does not seem likely in reality.

Solution 1.13: [Exercice] Since $u(w) = 1 - e^{-aw}$, then $u'(w) = ae^{-aw}$ and $u''(w) = -a^2e^{-aw}$.

1. For $u'(w) > 0$ and $u''(w) < 0$, then $a > 0$, i.e. the constant a must be strictly positive.
2. $A(w) = a$ and $R(w) = aw$.
3. From (b), investor has constant absloute risk aversion. This means that she will hold the same amount of her welath in risky assets as her level of wealth changes. Also, she has an increasing relative risk averison. This means that she will then hold a smaller proportion of her wealth in risky assets as her total wealth increases (and vice versa).

Solution 1.14: [Exercice]

1. With the given s^*, y^* must maximise
   $$\max_y E\left[u_0(w - s^* - yp) + u_{0,1}(s^* + yX)\right].$$
   Differentiate the above w.r.t y and set to 0 gives
   $$u_0'(w - s^* - yp)(-p) + E[u_{0,1}'(s^* + yX)X] = 0$$
   which is equivalent to
   $$p = E[\frac{u_{0,1}'(s^* + y^*X)}{u_0'(w - s^* - y^*p)}X]$$
   after rearranging and subtitute y with y^*.

2. The above equation states that the price p can be expressed as a expected "discounted" payoff X where
   $$m(X) = \frac{u_{0,1}'(s^* + y^*X)}{u_0'(w - s^* - y^*p)},$$
   which depends on the marginal utilities u_0' and $u_{0,1}'$, serves as a discounting factor.

   The stochastic discount factor m is also the marginal rate of substitution, as it represents the rate at which investors delay or bring forward their consumption.

3. From the formula
$$m(x) = \frac{u'_{0,1}(s^* + y^*x)}{u'_0(w - s^* - y^*p)},$$
by taking the derivative with respect to x (assume $u_{0,1}$ is nice), we have
$$m'(x) = \frac{y^*}{u'_0(w - s^* - y^*p)} u''_{0,1}(s^* + y^*x) < 0$$
as the individual is risk-averse. This implies that m is a decreasing function of x.

With $x_1 < x_2$, we have $m(x_1) > m(x_2)$. This means that favourable outcome (x_2) is being discounted more compared to the less favourable outcome (x_1).

[Note: This makes sense as

(i) A risk-averse individual is more concerned with the unfavourable outcomes than the favourable ones

(ii) Note by denoting $E_Q[\cdot] = E[m(\cdot)\cdot]$, we can express the price as an expected value! (i.e. $p = E_Q[X]$) This is essentially the argument for risk-neutral pricing in module 4, where one has to "remove" the risk premium by choosing a proper E_Q. (In particular we need m to be a "martingale".)

]

Solution 1.15: [Exercice] We have

$$\begin{aligned} E[G(X)] &= E\left[G(\mu) + (X-\mu)G'(\mu) + \frac{1}{2}(X-\mu)^2 G''(x^*)\right] \\ &= E[G(\mu)] + E\left[(X-\mu)G'(\mu)\right] + E\left[\frac{1}{2}(X-\mu)^2 G''(x^*)\right] \end{aligned}$$

Now notice that since $G(\mu)$ is a constant,
$$E[G(\mu)] = G(\mu).$$

Furthermore as $G'(\mu)$ is also a constant we have
$$\begin{aligned} E\left[(X-\mu)G'(\mu)\right] &= G'(\mu)E[(X-\mu)] \\ &= G'(\mu)(E[X]-\mu) \\ &= 0. \end{aligned}$$

Now note that with $(X-\mu)^2 > 0$ and $G''(x^*) < 0$ it follows that
$$E\left[\frac{1}{2}(X-\mu)^2 G''(x^*)\right] < 0$$

Hence in aggregate
$$\begin{aligned} E[G(X)] &= E[G(\mu)] + E\left[\frac{1}{2}(X-\mu)^2 G''(x^*)\right] \\ &< E[G(\mu)] = G[E(X)]. \end{aligned}$$

By replacing $G(X)$ with utility function $u(X)$, with $u"(X) < 0$ it follows that the utility function is that of a risk adverse individual, as $E[X]$ is preferred to X.

Solution 1.16: [Exercice] Since the gamble yields an expected return of 10% and the individual prefers that compared to the gamble, we can infer that she is risk averse.
Next, she invests \$100 into the gamble initially and then increases this amount to \$200 when her wealth increases, this is an example of Decreasing Absolute Risk Aversion (recall that Risk Aversion and the money invested in risky assets move inversely).
Finally, if you consider the proportion of wealth she invests in the gamble, it stays at a constant 20%, meaning this is a Constant Relative Risk Aversion.

Solution 1.17: [Exercice]

1.

$$\begin{aligned}
u'(w) &= w^{-(\gamma+1)} \\
u''(w) &= -(\gamma+1)w^{-(\gamma+2)} \\
A(w) &= \frac{(\gamma+1)w^{-(\gamma+2)}}{w^{-(\gamma+1)}} \\
&= \frac{\gamma+1}{w} \\
A'(w) &= -(\gamma+1)w^{-2} \\
R(w) &= (\gamma+1)
\end{aligned}$$

We can see his absolute risk aversion is decreasing with wealth and his relative risk aversion is constant. This means that he will invest more absolute dollars into the risky asset, but maintains the same proportion ρ of his wealth.

2. So his end of period wealth is given by: $W = (1-\rho)(1+r) + \rho(1+X)$.

$$\begin{aligned}
E(U(W)) &= E\left[-\frac{1}{\gamma}\left((1-\rho)(1+r)+\rho(1+X)\right)^{-\gamma}\right] \\
&= -\frac{1}{\gamma}E\left[((1-\rho)(1+r)+\rho+\rho X)^{-\gamma}\right] \\
&= -\frac{1}{\gamma}E\left[((1-\rho)(1+r)+\rho+\rho e^{\mu+\sigma Z})^{-\gamma}\right], \quad Z \sim N(0,1)
\end{aligned}$$

which has to be calculated numerically.

Solution 1.18: [Exercice] We are given that $P > Q > R > S$ and we know that $U(P)+U(S) = U(Q)+U(R)$. Rearranging we have

$$U(P) - U(Q) = U(R) - U(S). \tag{1.2}$$

As the investor is risk averse, this implies that

$$\frac{\partial U}{\partial W} > 0 \quad \text{and} \quad \frac{\partial^2 U}{\partial W^2} < 0. \tag{1.3}$$

In the following, we will use the mean-value theorem for differentiable functions: For $x < y$, we have

$$f(x) - f(y) = f'(z)(x-y),$$

for some z such that $x \le z \le y$.

This yields

$$\frac{U(P)-U(Q)}{P-Q}=U'(Z_1) \tag{1.4}$$

and

$$\frac{U(R)-U(S)}{R-S}=U'(Z_2), \tag{1.5}$$

with $P > Z_1 > Q > R > Z_2 > S$.

Now, using (1.3), we get

$$U'(Z_1) < U'(Z_2)$$

as U' is decreasing.

Substituting the above using equations (1.4) and (1.5), we get

$$\frac{U(P)-U(Q)}{P-Q} < \frac{U(R)-U(S)}{R-S}. \tag{1.6}$$

Using equations (1.2), (1.6) becomes

$$\begin{aligned} \frac{1}{P-Q} &< \frac{1}{R-S} \\ P+S &> R+Q \\ \frac{1}{2}P+\frac{1}{2}S &> \frac{1}{2}R+\frac{1}{2}Q \end{aligned}$$

Hence

$$U\left(\frac{1}{2}P+\frac{1}{2}S\right) > U\left(\frac{1}{2}Q+\frac{1}{2}R\right),$$

proving that the proposition is wrong.

[Alternative 1 (intuitive):

Consider the following 2 options:

$$A=\begin{cases} P, & w.p.\frac{1}{2} \\ S, & w.p.\frac{1}{2} \end{cases} \quad \text{and} \quad B=\begin{cases} Q, & w.p.\frac{1}{2} \\ R, & w.p.\frac{1}{2} \end{cases} \quad .$$

The given condition $U(P)+U(S)=U(Q)+U(R)$ implies that $E[U(A)]=E[U(B)]=U(C)$, where C is the certainty equivalent for both options. Note that risk premium θ is given by the expected value subtracting the risk premium, i.e.

$$\begin{aligned} \theta_A &= E[A]-C, \\ \theta_B &= E[B]-C. \end{aligned}$$

Finally, note that A is more volatile than B (e.g. larger variance) and hence the risk-averse individual should be willing to pay a higher risk premium, i.e. $\theta_A > \theta_B$. This gives $E[A] > E[B]$, or equivalently

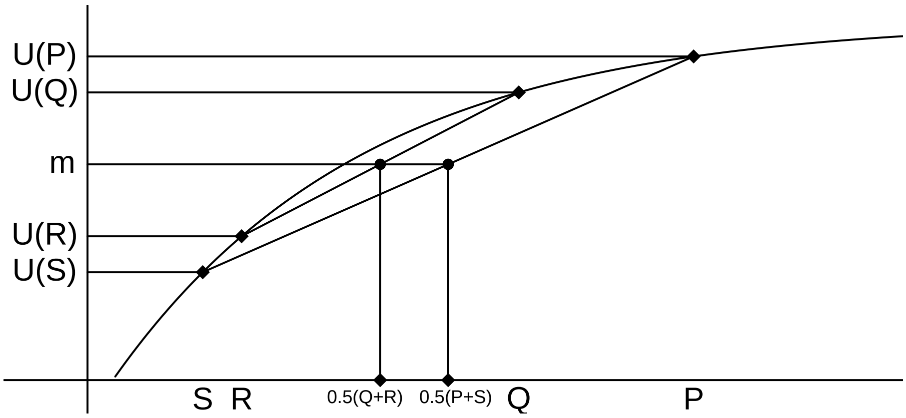
$$\frac{Q+R}{2} < \frac{P+S}{2}$$

which is also equivalent to

$$U\left(\frac{Q+R}{2}\right) < U\left(\frac{P+S}{2}\right)$$

as U is increasing. Hence, the answer to the question is negative.]

[Alternative 2 (graphical):



The graph above shows the answer to the question is negative.

An explanation is as follows. Since the individual is risk averse, the utility function U is concave. We can also assume that he/she prefers more wealth than less, which implies that U is increasing.

Note the graph of U together with l_{QR}, the straight line joining with $(Q, U(Q))$ and $(R, U(R))$, is also concave. This implies that l_{PS}, the straight line joining $(P, U(P))$ and $(S, U(S))$, is on the right of l_{QR}.

Now, the given condition is equivalent to

$$\frac{U(P)+U(S)}{2} = \frac{U(Q)+U(R)}{2} = m$$

and therefore $\frac{Q+R}{2}$ is the pre-image of m on l_{QR} while $\frac{P+S}{2}$ is the pre-image of m on l_{PS}. As the line l_{QR} is on the left of the line l_{PS}, we have

$$\frac{Q+R}{2} < \frac{P+S}{2}$$

which is equivalent to

$$U(\frac{Q+R}{2}) < U(\frac{P+S}{2})$$

as U is increasing.]

[Alternative 3 (Rigorous version of Alternative 2):

Denote

$$G(x) = \begin{cases} U(x), & x \notin [R, Q] \\ U(R) + \frac{x-U(R)}{U(Q)-U(R)}, & x \in [R, Q] \end{cases}$$

and

$$L(x) = U(S) + \frac{x - U(S)}{U(P) - U(S)},$$

we have that G is concave, which implies $G > L$ on (S, P). Now, we have $L(\frac{P+S}{2}) = m = G(\frac{Q+R}{2}) > L(\frac{Q+R}{2})$ which gives $\frac{P+S}{2} > \frac{Q+R}{2}$ (as L is increasing) and therefore by the increasing property of U

$$U(\frac{P+S}{2}) > U(\frac{Q+R}{2}).$$

Hence, the answer to the question is negative.]

1.4.2 Investment Risk Measures

Solution 1.19: [Exercice]

1. Define the following measures of investment risk:

   i. Variance of return: $V = E[(x-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 f_X(x)\,dx$ where $\mu = E[X] = \int_{-\infty}^{\infty} x f_X(x) dx$ is the mean return and $f_X(x)$ is the p.d.f. of the random return X.

   ii. Downside semi-variance of return: $D = E[(x-\mu)^2 1(x<\mu)] = \int_{-\infty}^{\mu} (x-\mu)^2 f_X(x)\,dx$.

   iii. Shortfall probability: $P[X < L] = \int_{-\infty}^{L} f_X(x)\,dx$, where L is the shortfall.

2. Some of the advantages of using the variance as a measure of investment risk include:

   i. Variance is mathematically easier to manipulate and compute.

   ii. Variance of return gives rise to more ”elegant” solutions.

   iii. It has not generally been shown that other measures of risk give far ”better” results.

   iv. if investors have quadratic utility functions, or if the returns are normally distributed, then variance of returns gives rise to optimum portfolios.

   However, some would argue that in some circumstances, it is the downside risk, and not the total uncertainty, that may be the more appropriate measure of risk. In this case, the semi-variance or shortfall probability may be more appropriate.

3. Neither measures are coherent. Variance measure does not satisfy positive homogeneity. Standard deviation does not satisfy translation invariance.

Solution 1.20: [Exercice] Suppose that you are trying to choose between the investments whose distributions of returns are described below:

Investment A:	0.4 probability that it will return 10%
	0.2 probability that it will return 15%
	0.4 probability that it will return 20%
Investment B:	0.25 probability that it will return 10%
	0.70 probability that it will return 15%
	0.05 probability that it will return 40%
Investment C:	a uniform distribution on $(0.10, 0.20)$.

1. Calculation of some risk measures:

i. Expected return:
Investment A: $(0.1)(0.4) + (0.15)(0.2) + (0.2)(0.4) = 0.15$
Investment B: $(0.1)(0.25) + (0.15)(0.7) + (0.4)(0.05) = 0.15$
Investment C: $\int_{0.1}^{0.2} 10x dx = 5\left(0.2^2 + 0.1^2\right) = 0.15$

ii. Variance of the return:
Investment A: $(0.1 - 0.15)^2(0.4) + (0.15 - 0.15)^2(0.2) + (0.2 - 0.15)^2(0.4) = 0.002$
Investment B: $(0.1-0.15)^2(0.25)+(0.15-0.15)^2(0.7)+(0.4-0.15)^2(0.05) = 0.00375$
Investment C: $\int_{0.1}^{0.2} 10x^2 dx - (0.15)^2 = \frac{10}{3}\left(0.2^3 - 0.1^3\right) - (0.15)^2 = 0.000833$

iii. Downside semi-variance:
Investment A: As $\mu_A = 0.15$, the only downside possibility is $X = 0.1$ which occurs at a probability of 0.4. Therefore, $(0.1 - 0.15)^2(0.4) = 0.001$
Investment B: As $\mu_B = 0.15$, the only downside possibility is $X = 0.1$ which occurs at a probability of 0.25. Therefore, $(0.1 - 0.15)^2(0.25) = 0.000625$
Investment C: As $\mu_C = 0.15$, the only downside possibility occurs when $X < 0.15$. Therefore,$\int_{0.1}^{0.15} 10\left(x - 0.15\right)^2 dx = \frac{10}{3}\left(0 + 0.05^3\right) = 0.000417$

iv. Expected shortfall below 12%; $E[(0.12 - X)_+] = E[(0.12 - X)1(0.12 > X)]$:
Investment A: $(0.12 - 0.1)(0.4) = 0.008$
Investment B: $(0.12 - 0.1)(0.25) = 0.005$
Investment C: $\int_{0.1}^{0.12} 10\left(0.12 - x\right) dx = 0.002$
[Note in some text, the expected shortfall is defined as $E[X|X < 0.12]$ which is different from the notation here]

v. Shortfall probability below 15% :
Investment A: $p\left(0.15\right) = \text{Prob}(X < 0.15) = 0.40$
Investment B: $p\left(0.15\right) = \text{Prob}(X < 0.15) = 0.25$
Investment C: $p\left(0.15\right) = \int_{0.1}^{0.15} 10dx = 0.5$

2. Comments: All 3 investments give the same expected return. Investment C appears to be the least risky on all measures except for the last risk measure of shortfall probability. Investment A has a lower variance than B; however, B has a lower semi-variance, lower expected shortfall and lower shortfall probability than Investment A.
Discussion: A risk-averse investor is likely to prefer Investment C. However, since C has the highest shortfall probability, it is considered the riskiest for investor who attaches a lot of utility to the return being at least 15%. It might be argued that B should be ranked second because it has less downside risk than Investment A, as revealed by its lower semi-variance and expected shortfall. However, we cannot be certain without knowing more details of the investor's utility function.

Solution 1.21: [Exercice] Downside semi-variance: $\int_{-\infty}^{\mu} \left(x - \mu\right)^2 f_X\left(x\right) dx$.

First, we compute the mean return for each investment:
Investment A: $(0)(0.1) + (0.05)(0.2) + (0.2)(0.3) + (0.3)(0.4) = 0.19$
Investment B: $(0.05)(0.1) + (0.1)(0.2) + (0.15)(0.3) + (0.2)(0.4) = 0.15$

The semi-variance is therefore computed as follows:
Investment A: $(0-0.19)^2(0.1)+(0.05-0.19)^2(0.2)=0.0075$
Investment B: $(0.05-0.15)^2(0.1)+(0.10-0.15)^2(0.2)=0.0015$

Solution 1.22: [Exercice] Depends on what you want to measure. For example, if they want to measure the amount of tail risk (very large losses) in the fund they could use the following:

(i) Value at Risk - the amount of loss at a specified confidence level i.e 5% VaR means that there is a 5% chance of losing at least X (easy to calculate but it doesn't consider the losses beyond the confidence level so the information in the tail is lost)

(ii) Expected shortfall - the average amount of loss beyond the specified confidence level i.e what is the average loss if we go past the specified confidence level (considers the losses beyong the confidence level but this can be difficult to calculate if the distribution is complicated)

If you want to measure the overall variability in returns you could use the usual variance (it includes information from the entire distribution as opposed to the above two measure, but the measure isn't as specific i.e it looks at good and bad risks)

Solution 1.23: [Exercice] Note the expected value is given by

$$E[X]=E[\exp(\mu+\sigma Z)]=e^{\mu}E[\exp(\sigma Z)]=\exp(\mu+\frac{1}{2}\sigma^2) \quad \text{(using the hint with } B=(-\infty,\infty))$$
$$= \exp(0.1+0.5\times 0.04^2)=\exp(0.1008).$$

1. Note the 5% quantile L is given by
$$0.05=P[X<L]=P[\exp(\mu+\sigma Z)<L]=P[\mu+\sigma Z<\log L]=P[Z<\frac{\log L-\mu}{\sigma}].$$
This implies that $\frac{\log L-\mu}{\sigma}=-1.644854$, or
$$L=\exp(-1.644854\times\sigma+\mu)=\exp(0.03420585)=1.034798.$$
Therefore, $VaR_{0.05}(E[X]-X)=E[X]-L=e^{0.1008}-e^{0.03420585}=0.07125781$. Thus,
$$VaR_{0.05}(E[Y]-Y)=7.125781.$$

2. With the 5% quantile L, we have
$$\begin{aligned}
E[X|X<L] &= \frac{E[X1(X<L)]}{P[X<L]}\\
&= \frac{E[\exp(\mu+\sigma Z)1(\exp(\mu+\sigma Z)<L)]}{0.05}\\
&= \frac{1}{0.05}e^{\mu}E[e^{\sigma Z}1(Z<\frac{\log L-\mu}{\sigma})]\\
&= \frac{1}{0.05}e^{\mu+\frac{1}{2}\sigma^2}P[\tilde{Z}+\sigma<\frac{\log L-\mu}{\sigma}] \quad \text{(from hint)}\\
&= \frac{1}{0.05}E[X]P[\tilde{Z}+0.04<-1.644854]\\
&= \frac{1}{0.05}e^{0.1008}P[\tilde{Z}<-1.684854]\\
&= e^{0.1008}\frac{0.04600837}{0.05}\\
&= 1.017756.
\end{aligned}$$

Therefore, the expected shortfall at 0.05 level is given by

$$ES_{0.05}(E[X]-X) = E[X] - E[X|X<L] = e^{0.1008}(1-\frac{0.04600837}{0.05}) = 0.08829928.$$

Thus,

$$ES_{0.05}(E[Y]-Y) = 8.829928.$$

1.4.3 Mean Variance Analysis

Solution 1.24: [Exercice] (L6.3) Using the equation of the variance of the portfolio (square the second equation on page 154) we get

$$Var(P) = 0.15^2(\alpha-2)^2,$$

which attains its minimum at $\alpha = 2$. This yield a portfolio variance of 0. Note that a correlation of 1 means that the 2 assets are in a linear relationship, implying perfectly dependent. Hence, by hedging (choosing the correct weights), we can obtain a portfolio with no risk.

(L6.6)

a) Plot will be a Horizontal Straight Line. Efficient Set will be the leftmost point of the line, which is to the left of the leftmost of the n assets by itself.

b) The sigma (overline) is defined in this question as follows:

$$\overline{\sigma}^2 = \left(\sum_{i=1}^{n} \frac{1}{\sigma_i^2}\right)^{-1}$$

Note the variance of the portfolio is $\sum_{i=1}^{n} w_i^2\sigma_i^2$ and the constrain is $\sum_{i=1}^{n} = 1$. Setting up Lagrangian (scaled by half for convenience) gives

$$L(w_i,\lambda) = \frac{1}{2}\sum_{i=1}^{n} w_i^2\sigma_i^2 - \lambda\left(\sum_{i=1}^{n} w_i - 1\right)$$

and we are interested in

$$\min_{w_i,\lambda} L(w_i,\lambda).$$

Taking partial derivatives give

$$\frac{\partial L}{\partial w_i} = w_i\sigma_i^2 - \lambda = 0 \qquad or \qquad w_i = \frac{\lambda}{\sigma_i^2}, \quad i = 1,2,...,n$$

$$\sum_{i=1}^{n} w_i = 1$$

Solving the above gives

$$w_i = \frac{\frac{1}{\sigma_i^2}}{\sum_{i=1}^{n}\frac{1}{\sigma_i^2}} = \frac{\overline{\sigma}^2}{\sigma_i^2}$$

Note that L is "quadratic" in (w_i,λ) guarantees the above solution is the minimum of L, which is also the solution to the constrained minimisation problem. The minimum variance is therefore

$$\sum_{i=1}^{n} w_i^2\sigma_i^2 = \sum_{i=1}^{n}\frac{\overline{\sigma}^4}{\sigma_i^2} = \overline{\sigma}^4\sum_{i=1}^{n}\frac{1}{\sigma_i^2} = \frac{\overline{\sigma}^4}{\overline{\sigma}^2} = \overline{\sigma}^2.$$

[Despite all assets are having the same expected value and uncorrelated, i.e. some assets are strictly outperformed by others, we do invest positive weights to those seemingly unpleasant assets to achieve minimal variance.]

(L6.7)

(a) The (global) minimum variance portfolio can be found by setting the Lagrangian

$$L(w_i, \lambda) = \frac{1}{2}w'\Sigma w - \lambda(1'w - 1)$$

where we are minimising $Var(w'X) = w'\Sigma w$ subjected to $1'w = 1$. Taking derivative w.r.t. w and setting to 0 gives

$$\Sigma w = \lambda 1 \iff w = \lambda\Sigma^{-1}1.$$

Using the constraint $1'w = 1$ gives $\lambda = \frac{1}{1'\Sigma^{-1}1} = 1/A$. Hence, we have

$$w_g = \frac{\Sigma^{-1}1}{1'\Sigma^{-1}1} = \begin{bmatrix} 0.5 \\ 0 \\ 0.5 \end{bmatrix}.$$

(b) Remember that the d portfolio is given by

$$w_d = \frac{\Sigma^{-1}\overline{r}}{1'\Sigma^{-1}\overline{r}} = \begin{bmatrix} \frac{1}{6} \\ \frac{1}{3} \\ \frac{1}{2} \end{bmatrix}.$$

(c) The constants A and B are evaluated as

$$\begin{aligned} A &= 1'\Sigma^{-1}1 = 1 \\ B &= 1'\Sigma^{-1}\overline{r} = 0.6 \end{aligned}$$

The solution to the MV problem with a risk free asset is the solution to the Lagragian

$$L(w_i, \gamma) = \frac{1}{2}w'\Sigma w - \gamma(w'(\overline{r} - r_f 1) - (\mu - r_f))$$

where we are minimising the portfolio variance $w'\Sigma w$ subjected to the return of the portfolio $w'\overline{r} + r_f(1 - w'1)$ equals μ, which gives the constraint.

Taking derivative w.r.t. w and setting to 0 gives

$$\Sigma w = \gamma(\overline{r} - r_f 1) \iff w = \gamma\Sigma^{-1}(\overline{r} - r_f 1)$$

Noting that the tangency portfolio is the above with an additional condition that the portfolio is comprised of only risky assets, i.e. $1'w = 1$. This gives

$$1 = 1'w = \gamma(1'\Sigma^{-1}\overline{r} - r_f 1'\Sigma^{-1}1) = \gamma(B - r_f A) \iff \gamma = \frac{1}{B - r_f A}.$$

Hence, the tangency portfolip is given by

$$\begin{aligned} w_t &= \frac{1}{B - Ar_f}\Sigma^{-1}(\overline{r} - r_f 1) \\ &= \frac{1}{B - Ar_f}\left(B\left(\frac{1}{B}\Sigma^{-1}\overline{r}\right) - r_f A\left(\frac{1}{A}\Sigma^{-1}1\right)\right) \\ &= \frac{1}{B - Ar_f}(Bw_d - r_f A w_g) \\ &= \frac{1}{0.6 - 1 \times 0.2}\left(0.6 \times \begin{bmatrix} \frac{1}{6} \\ \frac{1}{3} \\ \frac{1}{2} \end{bmatrix} - 0.2 \times 1 \times \begin{bmatrix} 0.5 \\ 0 \\ 0.5 \end{bmatrix}\right) \\ &= \begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}. \end{aligned}$$

(See also textbook examples 6.11,6.12 for similar reasoning)

Solution 1.25: [Exercice] If we consider some arbitrary utility function $u(\cdot)$ and take the Taylor Series expansion we get:

$$E(u(W)) = u(E(W)) + \frac{1}{2}u''(E(W))\sigma_W^2 + E\left[\sum_{n=3}^{\infty}\frac{1}{n!}u^{(n)}E(W)(W - E(W))^2\right]$$

So we will be left with the mean and variance components if and only if:

(i) We have a quadratic utility function (where all the high degree terms u^n, $n \geq 3$ are zero) or

(ii) We have normally distributed wealth (where the only parameters are the mean and the variance).

Solution 1.26: [Exercice]

(a) A portfolio is efficient if the investor cannot find an alternative portfolio that has higher expected return for the same variance. The *efficient frontier* is the set of all efficient portfolios.

(b) For a given utility function, an optimal portfolio is a portfolio w which maximises

$$E[U(W\exp(w'X))]$$

where W is the initial wealth and X is the collection of the (random continuously-compounded) return of the available assets.

Solution 1.27: [Exercice] Let the return on the portfolio be denoted by R_p so that

$$R_p = 0.20\widetilde{r}_A + 0.80\widetilde{r}_B.$$

(a) $E(R_p) = 0.20E(\widetilde{r}_A) + 0.80E(\widetilde{r}_B) = 0.20(0.1) + 0.80(0.15) = 0.14$ or 14%.

(b) The variance of the portfolio can be derived as

$$\begin{aligned} Var\left(R_p\right) &= (0.20)^2 Var\left(\widetilde{r}_A\right) + (0.80)^2 Var\left(\widetilde{r}_B\right) + 2\,(0.2)\,(0.8)\, Cov\left(\widetilde{r}_A, \widetilde{r}_B\right) \\ &= (0.20)^2 (0.2)^2 + (0.80)^2 (0.3)^2 + 2\,(0.2)\,(0.8)\, \rho_{AB} \sqrt{Var\left(\widetilde{r}_A\right) Var\left(\widetilde{r}_B\right)} \\ &= (0.20)^2 (0.2)^2 + (0.80)^2 (0.3)^2 + 2\,(0.2)\,(0.8)\,(0.6)\,(0.2)\,(0.3) \\ &= 0.07072. \end{aligned}$$

The standard deviation is therefore $\sigma_p = SD\left(R_p\right) = \sqrt{Var\left(R_p\right)} = \sqrt{0.07072} = 0.26593$ or 26.6%.

(c) In modern portfolio theory, risk is measured by the standard deviation of returns. Since the returns on the investments are not perfectly correlated, the investor can achieve an expected return that is proportionate to the proportions invested in each security, but with a standard deviation of returns (or risk) that is lower than the arithmetical average of the standard deviations of the individual risky assets. An investor who does this rather than investing 100% in Asset B is demonstrating risk aversion, because he/she is choosing to reduce the level of risk at the cost of attaining a lower expected return.

Solution 1.28: [Exercice] Let the return on the portfolio be denoted by R_p so that

$$R_p = 0.40\widetilde{r}_A + 0.60\widetilde{r}_B.$$

(a) The mean and variance of Security A are

$$E\left(\widetilde{r}_A\right) = (0.08)\,\frac{1}{2} + (0.13)\,\frac{1}{2} = 0.105 \text{ or } 10.5\%$$

and

$$Var\left(\widetilde{r}_A\right) = (0.08 - 0.105)^2\,\frac{1}{2} + (0.13 - 0.105)^2\,\frac{1}{2} = 0.000625,$$

respectively, and that of Security B are

$$E\left(\widetilde{r}_B\right) = (0.10)\,(0.7) + (0.16)\,(0.3) = 0.118 \text{ or } 11.8\%$$

and

$$Var\left(\widetilde{r}_B\right) = (0.10 - 0.118)^2\,(0.7) + (0.16 - 0.118)^2\,(0.3) = 0.000756,$$

respectively.

(b) The mean of the portfolio is given by

$$E\left(R_p\right) = 0.40(0.105) + 0.60(0.18) = 0.1128 \text{ or } 11.28\%.$$

i. If the correlation is +1, then variance of the portfolio is

$$\begin{aligned} Var\left(R_p\right) &= (0.40)^2\,(0.000625) + (0.60)^2\,(0.000756) \\ &\quad +2\,(0.4)\,(0.6)\,(1)\left(\sqrt{(0.000625)\,(0.000756)}\right) \\ &= 0.0007021. \end{aligned}$$

ii. If the correlation is −1, then variance of the portfolio is

$$\begin{aligned} Var\left(R_p\right) &= (0.40)^2\,(0.000625) + (0.60)^2\,(0.000756) \\ &\quad +2\,(0.4)\,(0.6)\,(-1)\left(\sqrt{(0.000625)\,(0.000756)}\right) \\ &= 0.0000422. \end{aligned}$$

(c) It is impossible to construct a different mix of securities A and B that will yield the same expected return as the original portfolio as
   (1) The expected return of A and B are different, and
   (2) Expected value is linear.

Solution 1.29: [Exercice] If there are only two risky assets, then the portfolio variance can be expressed as:

$$\sigma_p^2 = a_1^2\sigma_1^2 + a_2^2\sigma_2^2 + 2a_1a_2\rho_{12}\sigma_1\sigma_2, \quad a_1 + a_2 = 1.$$

But assets are perfectly negatively correlated, thus $\rho_{12} = -1$. The variance becomes

$$\sigma_p^2 = a_1^2\sigma_1^2 + a_2^2\sigma_2^2 - 2a_1a_2\sigma_1\sigma_2 = (a_1\sigma_1 - a_2\sigma_2)^2.$$

This is obviously 0 (lowest) when $a_1\sigma_1 = a_2\sigma_2$, or $a_2 = a_1\frac{\sigma_1}{\sigma_2}$, or

$$(a_1, a_2) = (\frac{\sigma_2}{\sigma_1 + \sigma_2}, \frac{\sigma_1}{\sigma_1 + \sigma_2}).$$

This implies that risks are completely eliminated.

Solution 1.30: [Exercice] Let a be the proportion invested in the risky asset whose return is denoted by $\widetilde{r}$. The return on the portfolio is therefore

$$R_p = a\widetilde{r} + (1-a)\, r_f$$

where r_f is the risk-free rate of return. The mean of the portfolio can be expressed as

$$E(R_p) = \mu_p = aE(\widetilde{r}) + (1-a)\, r_f = 0.08a + 0.05\,(1-a) = 0.03a + 0.05$$

and the variance of the portfolio can be expressed as

$$Var(R_p) = \sigma_p^2 = a^2 Var(\widetilde{r}) = 0.04a^2.$$

Now, the utility given by

$$U(\mu_p, \sigma_p) = \mu_p - \frac{1}{2}\sigma_p^2 = 0.03a + 0.05 - 0.02a^2.$$

The first-order condition leads us to

$$\frac{\partial U}{\partial a} = 0.03 - 0.04a = 0$$

which implies $a = 0.75$. The second derivative

$$\frac{\partial^2 U}{\partial a^2} = -0.04 < 0$$

ensures this gives the maximum. Thus, the investor should invest 75% in risky asset and 25% in the risk-free asset.

Solution 1.31: [Exercice] This is straightforward calculation. The student should be able to show: (a) $\mu_p = 4.40\%$ and (b) $\sigma_p = 2.8827\%$.

Solution 1.32: [Exercice] The two fund theorem states that the entire minimum variance frontier can be spanned by taking different proportions of two different minimum variance portfolios and only works when there is no riskless asset. This result implies that to calculate the efficient frontier, all we need are the expected returns and the variances of ANY 2 different MV portfolios. For example, we can use the global minimum variance portfolio and the "d" portfolio, both of which are extremely easy to find.

This theorem is meaningful in itself because it states that despite (a large number of assets (n of them) available, in the space of MV portfolio, there are only 2 portfolios that are fundamentally different and we can achieve MV when investing in ANY 2 different MV portfolios.

Solution 1.33: [Exercice] The one fund theorem states that when there is a riskless asset available, the efficient frontier is comprised of portfolios formed by the tangency portfolio together with the risk-free asset.

The solution to the MV problem with a risk free asset is the solution to the Lagragian

$$L(w_i, \gamma) = \frac{1}{2}w'\Sigma w - \gamma(w'(z - r_f 1) - (\mu - r_f))$$

where we are minimising the portfolio variance $w'\Sigma w$ subjected to the return of the portfolio $w'z + r_f(1 - w'1)$ equals μ, which gives the constraint.

Taking derivative w.r.t. w and setting to 0 gives

$$\Sigma w = \gamma(z - r_f 1) \iff w = \gamma\Sigma^{-1}(z - r_f 1)$$

Noting that the tangency portfolio is the above with an additional condition that the portfolio is comprised of only risky assets, i.e. $1'w = 1$. This gives

$$1 = 1'w = \gamma(1'\Sigma^{-1}z - r_f 1'\Sigma^{-1}1) = \gamma(B - r_f A) \iff \gamma = \frac{1}{B - r_f A}.$$

Therefore, the tangency portfolio is

$$w_t = \frac{1}{B - r_f A}\Sigma^{-1}(z - r_f 1).$$

Solution 1.34: [Exercice]

1. Same as the previous question.

2. We have

$$\begin{aligned} E[w_t X] &= z'w_t \\ &= \frac{1}{B - r_f A} z'\Sigma^{-1}(z - r_f 1) \\ &= \frac{C - r_f B}{B - r_f A} \end{aligned}$$

and

$$\begin{aligned}
Var(w_t X) &= w'\Sigma w \\
&= \frac{1}{(B-r_f A)^2}(z-r_f 1)'\Sigma^{-1}\Sigma\Sigma^{-1}(z-r_f 1) \\
&= \frac{1}{(B-r_f A)^2}(z-r_f 1)'\Sigma^{-1}(z-r_f 1) \\
&= \frac{1}{(B-r_f A)^2}(C-2r_f B+r_f^2 A) \\
&= \frac{C-2r_f B+r_f^2 A}{(B-r_f A)^2}.
\end{aligned}$$

Solution 1.35: [Exercice] Weights are: (-0.673, 1.212, 0.462) and stdev is 0.046.

All you have to do is to find the constants A, B, C, Δ. From there you can solve the constants λ and γ, which gives the portfolio.

The useful commands are

- MMULT(X, Y): matrix multiplication in Excel
- Minverse(X): inverse
- Transpose(X): transpose

and you need to use "Ctrl+Shift+Enter" to execute the commands.

Solution 1.36: [Exercice] Try it yourself! You can learn by doing!

Solution 1.37: [Exercice]

We show this applying basic principles to further illustrate the techniques we used (Note: It is possible to do this in a faster way by showing that both mean and standard deviation of a portfolio is a linear combination of the two stocks). We adopt the usual notations for means, variances, covariance, and correlations in a portfolio of two securities. In order to find the efficient frontier, we set up the minimization problem as follows: Minimize the variance of the portfolio

$$\begin{aligned}
\sigma_p^2 &= a_1^2\sigma_1^2 + a_2^2\sigma_2^2 + 2a_1a_2\sigma_{12} = a_1^2\sigma_1^2 + a_2^2\sigma_2^2 + 2a_1a_2\rho_{12}\sigma_1\sigma_2 \\
&= a_1^2\sigma_1^2 + a_2^2\sigma_2^2 + 2a_1a_2\sigma_1\sigma_2
\end{aligned}$$

subject to:

$$a_1\mu_1 + a_2\mu_2 = \mu_p$$

and

$$a_1 + a_2 = 1.$$

The Lagrangian function is therefore

$$L = a_1^2\sigma_1^2 + a_2^2\sigma_2^2 + 2a_1a_2\sigma_1\sigma_2 - \lambda\left(a_1\mu_1 + a_2\mu_2 - \mu_p\right) - \gamma\left(a_1 + a_2 - 1\right).$$

The first-order conditions are therefore:

$$\frac{\partial L}{\partial a_1} = 2a_1\sigma_1^2 + 2a_2\sigma_1\sigma_2 - \lambda\mu_1 - \gamma = 0,$$

$$\frac{\partial L}{\partial a_2} = 2a_2\sigma_2^2 + 2a_1\sigma_1\sigma_2 - \lambda\mu_2 - \gamma = 0,$$

$$\frac{\partial L}{\partial \lambda} = a_1\mu_1 + a_2\mu_2 - \mu_p = 0,$$

and

$$\frac{\partial L}{\partial \gamma} = a_1 + a_2 - 1 = 0.$$

1. Combining the last two equations, we have

$$a_1\mu_1 + (1 - a_1)\,\mu_2 - \mu_p = 0 \Rightarrow a_1\,(\mu_1 - \mu_2) = \mu_p - \mu_2.$$

Thus,

$$a_1 = \frac{\mu_p - \mu_2}{\mu_1 - \mu_2} \text{ and } a_2 = \frac{\mu_1 - \mu_p}{\mu_1 - \mu_2}.$$

Now, we substitute these terms back to the variance of the portfolio, we get

$$\begin{aligned}
\sigma_p^2 &= a_1^2\sigma_1^2 + a_2^2\sigma_2^2 + 2a_1a_2\sigma_1\sigma_2 = (a_1\sigma_1 + a_2\sigma_2)^2 \\
&= \left(\frac{\mu_p - \mu_2}{\mu_1 - \mu_2}\sigma_1 + \frac{\mu_1 - \mu_p}{\mu_1 - \mu_2}\sigma_2\right)^2 \\
&= \frac{1}{(\mu_1 - \mu_2)^2}\left[(\mu_p - \mu_2)\,\sigma_1 + (\mu_1 - \mu_p)\,\sigma_2\right]^2.
\end{aligned}$$

Thus, the standard deviation is

$$\begin{aligned}
\sigma_p &= \frac{1}{(\mu_1 - \mu_2)}\left[(\mu_p - \mu_2)\,\sigma_1 + (\mu_1 - \mu_p)\,\sigma_2\right] \\
&= \frac{1}{(\mu_1 - \mu_2)}\left[(\sigma_1 - \sigma_2)\,\mu_p + (\mu_1\sigma_2 - \mu_2\sigma_1)\right].
\end{aligned}$$

Therefore, finally the equation of the efficient frontier on the $(\mu - \sigma)$ space can be expressed as

$$\sigma_p = a\mu_p + b,$$

where $a = \dfrac{\sigma_1 - \sigma_2}{\mu_1 - \mu_2}$ and $b = \dfrac{\mu_1\sigma_2 - \mu_2\sigma_1}{\mu_1 - \mu_2}$.

2. The efficient frontier is normally plotted with μ_p on the vertical axis and σ_p on the horizontal axis. The equation can be re-written then as

$$\mu_p = \frac{1}{a}\sigma_p - \frac{b}{a}.$$

(a) Its gradient will therefore be

$$\frac{\partial \mu_p}{\partial \sigma_p} = \frac{1}{a} = \frac{\mu_1 - \mu_2}{\sigma_1 - \sigma_2}.$$

(b) Note that when $\mu_p = \mu_1$, then

$$\sigma_p = \frac{1}{(\mu_1 - \mu_2)}\left[(\sigma_1 - \sigma_2)\,\mu_1 + (\mu_1\sigma_2 - \mu_2\sigma_1)\right] = \frac{(\mu_1 - \mu_2)\,\sigma_1}{\mu_1 - \mu_2} = \sigma_1.$$

Thus, the point (σ_1, μ_1) lies on the efficient frontier. Similarly, when $\mu_p = \mu_2$, then $\sigma_p = \sigma_2$, so that (σ_2, μ_2) also lies on the efficient frontier. The efficient frontier is therefore a straight line that passes through the points (σ_1, μ_1) and (σ_2, μ_2).

Solution 1.38: [Exercice]

1. The global minimum variance portfolio can be found by setting the Lagrangian
$$L(w_i, \lambda) = \frac{1}{2} w'\Sigma w - \lambda(1'w - 1)$$
where we are minimising $Var(w'X) = w'\Sigma w$ subjected to $1'w = 1$. Taking derivative w.r.t. w and setting to 0 gives
$$\Sigma w = \lambda 1 \iff w = \lambda \Sigma^{-1} 1.$$
Using the constraint $1'w = 1$ gives $\lambda = \frac{1}{1'\Sigma^{-1}1} = 1/A$. Hence, we have
$$w_g = \frac{\Sigma^{-1}1}{A}.$$

2.
$$\begin{aligned} Cov(w_p'X, w_g'X) &= w_p'\Sigma w_g \quad \text{(the covariance formula for any 2 portfolios)} \\ &= w_p'\Sigma\frac{\Sigma^{-1}1}{A} \\ &= \frac{w_p'1}{A} \\ &= \frac{1}{A} \quad (\text{as } w_p'1 = 1) \end{aligned}$$

3. Consider two minimum variance portfolios,
$$\begin{aligned} w_a &= (1-a)\, w_g + a w_d \\ w_b &= (1-b)\, w_g + b w_d \end{aligned}$$
Note from
$$\begin{aligned} \sigma_d^2 &= w_d'\Sigma w_d \\ &= w_d'\Sigma\Sigma^{-1} z \frac{1}{B} \\ &= z'\Sigma^{-1} z \frac{1}{B^2} \\ &= \frac{C}{B^2} \end{aligned}$$
we have
$$\begin{aligned} & Cov\left(w_a'X, w_b'X\right) \\ = \;& Cov((1-a)w_g'X + a w_d'X, (1-b)w_g'X + b w_d'X) \\ = \;& (1-a)(1-b)\sigma_g^2 + ab\sigma_d^2 + [a(1-b) + b(1-a)]\sigma_{g,d} \\ = \;& (1-a-b+ab)\frac{1}{A} + ab\frac{C}{B^2} + (a - ab + b - ab)\frac{1}{A} \\ = \;& \frac{1}{A} - ab\frac{1}{A} + ab\frac{C}{B^2} \\ = \;& \frac{1}{A} + \frac{ab\Delta}{AB^2} \end{aligned}$$
using results from previous parts: $\sigma_g^2 = \frac{1}{A}$ (from part a), $\sigma_d^2 = \frac{C}{B^2}$, and $\sigma_{g,d} = Cov(Z_g, Z_d) = \frac{1}{A}$ (from part b).

Solution 1.39: [Exercice] The two fund theorem is given by $aw_g + (1-a)w_d = 1$, where a is a constant and w_g, w_d are the global minimum variance and d portfolios respectively.

1. $w_d := \frac{\Sigma^{-1}z}{B}$ by definition.
2. To get the mean we have $\mu_d = z'w_d = \frac{z\Sigma^{-1}z}{B} = \frac{C}{B}$.
   To get the variance we have $\sigma_d^2 = w_d'\Sigma w_d = \frac{z'\Sigma^{-1}\Sigma\Sigma^{-1}z}{B^2} = \frac{C}{B^2}$.
3. To get the covariance we have $w_d'\Sigma w_g = \frac{z'\Sigma^{-1}\Sigma\Sigma^{-1}1}{AB} = \frac{B}{AB} = \frac{1}{A}$, which is the same as the result in 3(b).

Solution 1.40: [Exercice] To get the minimum variance frontier we must minimise the Lagrangian:

$$\begin{aligned} L(w,\lambda,\gamma) &= \frac{1}{2}w'\Sigma w + \lambda(1-w'1) + \gamma(\mu - w'z) \\ \frac{dL}{dw} &= \Sigma w - \lambda 1 - \gamma z \end{aligned}$$

Then we need to solve the following system of equations (first order condition):

$$\begin{aligned} \Sigma w - \lambda 1 - \gamma z &= 0 \\ 1 - w'1 &= 0 \\ \mu - w'z &= 0 \end{aligned}$$

or equivalently

$$\begin{aligned} w &= \lambda\Sigma^{-1}1 + \gamma\Sigma^{-1}z \\ 1 &= w'1 \\ \mu &= w'z \end{aligned}$$

Solving these simultaneous equations yields:

$$\begin{aligned} \lambda &= \frac{C - B\mu}{\Delta} \\ \gamma &= \frac{A\mu - B}{\Delta} \end{aligned}$$

Hence, we can rewrite the solution to the minimisation problem as:

$$\begin{aligned} w &= A\lambda\frac{\Sigma^{-1}1}{A} + B\gamma\frac{\Sigma^{-1}z}{B} \\ &= A\lambda w_g + B\gamma w_d \end{aligned}$$

Now it is easy to see that $A\lambda + B\gamma = \frac{AC - AB\mu}{\Delta} + \frac{AB\mu - B^2}{\Delta} = \frac{AC - B^2}{\Delta} = \frac{\Delta}{\Delta} = 1$. This implies that any MV portfolio is of the form

$$w_c = cw_g + (1-c)w_d.$$

Suppose we have 2 other MV portfolios which takes form

$$\begin{aligned} w_1 &= aw_g + (1-a)w_d, \\ w_2 &= bw_g + (1-b)w_d. \end{aligned}$$

We want to form the portfolio w_c using only w_1 and w_2. This is possible if there is a ξ such that $\xi w_1 + \xi w_2 = w_c$, which is possible if

$$\begin{aligned} a\xi + b(1-\xi) &= c \\ (1-a)\xi + (1-b)(1-\xi) &= 1-c. \end{aligned}$$

With the help of the first equation, the second equation is

$$(1-a)\xi + (1-b)(1-\xi) = \xi + (1-\xi) - (a\xi + b(1-\xi)) = 1-c$$

which is always true. On the other hand, the first equation gives

$$\xi = \frac{c-b}{a-b}$$

which is well-defined as $a \neq b$ (as w_1 and w_2 are different portfolios). Hence, by investing $\xi w_1 + (1-\xi)w_2$ we have porfolio w_c, i.e. 2 fund theorem.

Solution 1.41: [Exercice]

$$\begin{aligned} Var(r_p - r_m) &= Var(r_p) - 2Cov(r_p, r_m) + Var(r_m) \\ &= w'\Sigma w - 2w'\beta\sigma_m^2 + \sigma_m^2 \end{aligned}$$

Note, we want to minimise the tracking error with the constraints that we have invest all assets that generates a target return μ. This problem can be solved by setting

$$L(w,\lambda,\gamma) = \frac{1}{2}w'\Sigma w - w'\beta\sigma_m^2 + \frac{1}{2}\sigma_m^2 + \lambda(1 - 1'w) + \gamma(\mu - w'z).$$

Taking derivative w.r.t. w and setting to 0 gives

$$\frac{dL}{dw} = \Sigma w - \sigma_m^2\beta - \lambda 1 - \gamma z = 0 \iff w = \sigma_m^2\Sigma^{-1}\beta + \lambda\Sigma^{-1}1 + \gamma\Sigma^{-1}z$$

Now using the constraints we get

$$\begin{aligned} &\begin{cases} \sigma_m^2 1'\Sigma^{-1}\beta + \lambda A + \gamma B = 1 \\ \sigma_m^2 z'\Sigma^{-1}\beta + \lambda B + \gamma C = \mu \end{cases} \\ \iff &\begin{cases} \lambda A + \gamma B = 1 - \sigma_m^2 1'\Sigma^{-1}\beta \\ \lambda B + \gamma C = \mu - \sigma_m^2 z'\Sigma^{-1}\beta \end{cases}. \end{aligned}$$

Denote $F = 1 - \sigma_m^2 1'\Sigma^{-1}\beta$ and $G = \mu - \sigma_m^2 z'\Sigma^{-1}\beta$ we have

$$\begin{aligned} &\begin{bmatrix} A & B \\ B & C \end{bmatrix}\begin{bmatrix} \lambda \\ \gamma \end{bmatrix} = \begin{bmatrix} F \\ G \end{bmatrix} \\ \iff &\begin{bmatrix} \lambda \\ \gamma \end{bmatrix} = \begin{bmatrix} A & B \\ B & C \end{bmatrix}^{-1}\begin{bmatrix} F \\ G \end{bmatrix} = \frac{1}{AC-B^2}\begin{bmatrix} C & -B \\ -B & A \end{bmatrix}\begin{bmatrix} F \\ G \end{bmatrix} \\ \iff &\begin{bmatrix} \lambda \\ \gamma \end{bmatrix} = \frac{1}{AC-B^2}\begin{bmatrix} CF - BG \\ AG - BF \end{bmatrix}. \end{aligned}$$

Now given we know the stock are uncorrelated and the variance/mean of each stock we have:

$$\Sigma = \sigma^2 \begin{bmatrix} 1 & 0 & 0\ldots & \\ 0 & 2 & 0\ldots & \\ \vdots & & \ddots & \\ \ldots & & & n \end{bmatrix}$$

$$\implies \Sigma^{-1} = \frac{1}{\sigma^2} \begin{bmatrix} 1 & 0 & 0\ldots & \\ 0 & \frac{1}{2} & 0\ldots & \\ \vdots & & \ddots & \\ \ldots & & & \frac{1}{n} \end{bmatrix}$$

$$\mathbf{z} = \mu \begin{bmatrix} 1 \\ 2 \\ \vdots \\ n \end{bmatrix}$$

We can then calculate the values of the matrices easily as follows:

$$A = 1'\Sigma^{-1}1 = \frac{1}{\sigma^2}\sum_{j=1}^{n}\frac{1}{j}$$

$$B = z'\Sigma^{-1}1 = \frac{n\mu}{\sigma^2}$$

$$C = z'\Sigma^{-1}z = \frac{\mu^2}{\sigma^2}\sum_{j=1}^{n} j$$

$$F = 1 - 1'\Sigma^{-1}\beta = 1 - \frac{1}{\sigma^2}\sum_{j=1}^{n}\frac{\beta_j}{j}$$

$$G = \mu - z'\Sigma^{-1}\beta = \mu - \frac{\mu}{\sigma^2}\sum_{j=1}^{n}\beta_j$$

Solution 1.42: [Exercice]

(i) The market portfolio would be the same as the tangency portfolio.

(ii) (1) In this case, the price of asset 2 immediately rises to equilibrium WITHOUT trading because noone would sell it learning such information. Upon equilibrium, the tangency profolio will change (include more asset 2) and is the same as the market portfolio (with now asset 2 having a higher price, consistent with the updated tangency portfolio).

(2) In this case, the skilled individuals would be able to improve their portfolio (and therefore potentially being able to beat the market) by buying asset 2. Average investors who sold asset 2 were being taken advantaged of. The demand for asset 2 eventually drive up the price and equilibrium is established. The end outcome is the same as in case (1) above but skilled investors were able to improve their portfolio in expense of the average investors.

(iii) Buy only the Market portfolio and the risk-free asset. There is a (net) demand for asset 2 but no (net) supply as the average investors are not interested in selling only asset 2. Eventually such demand will be met by an increase in the price of asset 2 WITHOUT trading. The average investors are having the same portfolio (in terms of w_i) which increases in value (due to the increase in p_i thanks to the good news). This is not observed in practice due to investor behaviours (which is the scoop of another course).

Module 2

2.1 The Capital Asset Pricing Model

2.1.1 Note

1. CAPM assumptions: The main assumptions in CAPM are

   (a) Mean-variance optimisers (unlikely but rational behaviour)

   (b) Homogeneous individuals, i.e. same estimation, same investment horizon, etc. (unlikely to be true but should not be a big problem as their interpretations come from the same source so estimations should be more or less the same)

   (c) Equilibrium, i.e. the investment market is at equilibrium. (hard to tell but should not be a big problem)

   There are other weaker assumptions which can be relaxed in some ways. However, you should remember a number of them as it may appear in the test/exam.

2. The market portfolio and the tangency portfolio. Note that the tangency portfolio is allowed to have negative weights but the market portfolio must have all strictly positive weights, since it does not make sense for the entire market to short-sell one stock.

3. Overpriced/underpriced:

   (a) The expected return implied by CAPM is the return at "equilibrium", which is based on the characteristics of the company. We should treat it as the "correct return".

   (b) The observed expected return is simply

   $$\frac{E[\text{Next period price}]}{\text{Current price}}$$

   which can be deviated from the "correct return" (i.e. the CAPM return).

   When the observed return is smaller, it means the current trading price is too high, i.e. overpriced. Likewise, when the observed return is higher, it means the current trading price is too low, i.e. underpriced. Generally speaking, we would like to buy the underpriced (high return) and sell the overpriced (low return).

4. Effifient portfolio and CAPM. Under CAPM, the ONLY efficient portfolios under CAPM are those investing in the market portfolio and the risk free asset (recall the 1 fund theorem). Therefore, the correlation between the return of the market portfolio and

any efficient portfolio is 1. To see why this is true, note that the efficient portfolio is a combination of the market portfolio and the risk free asset, hence the return is given by

$$r_p = \beta r_m + (1 - \beta) r_f,$$

which says r_p is perfectly linear in r_M. This means the correlation is 1. (Recall the definition of Pearson correlation)

Assumimg CAPM and we have an efficient portfolio with variacne σ_e, then the beta of the efficient portfolio can be calculated as

$$\beta = \frac{\sigma_p}{\sigma_m}$$

because we have $Var(r_p) = \beta^2 Var(r_m)$, or equivalently $\sigma_p^2 = \beta^2 \sigma_m^2$ in view of the linear equation above.

5. Final note on CAPM: All the calculations here are based on the assumption (conclusion) that the market portfolio is efficient. We did not use any characteristic of the market portfolio. In fact, all calculations still hold if we replace the market portfolio by an efficient portfolio. This motivates us to discuss in the next section the case when the market portfolio is not efficient (factor model).

2.1.2 Practice Questions

Exercise 2.1: [Solution] Consider an investment market in which the:

- risk-free rate of return on Treasury bills is 3.5%;
- expected return on the market as a whole is 10%;
- standard deviation of the return on the market as a whole is 20%; and
- assumptions of the Capital Asset Pricing Model (CAPM) hold.

1. Consider an efficient portfolio Z that consists entirely of Treasury bills and non-dividend paying shares, where there is no other types of investment. If Z yields an expected return of 8%, calculate its beta.
2. Calculate the standard deviation of returns for Portfolio Z.
3. Decompose the overall standard deviation of Portfolio Z into the amounts attributable to systematic and unsystematic risk.

Exercise 2.2: [Solution] State the key assumptions that the Capital Asset Pricing Model operates under and provide details where certain assumptions may not hold.

Exercise 2.3: [Solution] Explain the differences between the Capital Market Line and the Security Market Line.

[Note: Despite the 2 lines looks the same (a stright line going up from the risk free rate) they are 2 different concept.]

Exercise 2.4: [Solution]

1. State the equation of the Security Market Line (SML) relationship and, assuming that the market portfolio offers a return in excess of the risk-free rate, use it to derive the betas of the market portfolio and the risk-free asset.

2. Draw a diagram of the SML relationship and use it to derive the relationship itself.

3. What does the SML indicate about the relationship between risk and return?

Exercise 2.5: [Solution] Luenberger, Ch7. Q1,2,3,6,8,9. Note that in question 8 you should assume that c is independent of p and the market.

Exercise 2.6: [Solution] You are given the following information for returns on two stocks labeled A and B:

	Stock A	Stock B
α	0.04	0.09
β	1.20	1.50
σ_ε	0.25	0.40

You are also given that: $E[R_M] = 0.16$ and $\sigma_M = 0.20$. The returns generating process is assumed to be as follows:

$$R = \alpha + \beta R_M + \varepsilon$$

and

$$\sigma^2 = \beta^2 \sigma_M^2 + \sigma_\varepsilon^2$$

where R is a random variable representing the return on the stock, R_M is a random variable representing the return on a market index, ε is the residual term, σ^2, σ_M^2, and σ_ε^2 are the variances of the stock, index, and residual term, respectively. The residual terms of the returns generating processes for stocks A and B are assumed uncorrelated with each other and have a mean of zero. Now, calculate the following:

1. the mean and variance of the returns of each stock;

2. the covariance of returns between the stocks;

3. the beta of an equally-weighted portfolio of the two stocks; and

4. the expected return and variance of an equally-weighted portfolio of the two stocks.

Exercise 2.7: [Solution] Suppose we bought an asset for price p and we will sell the asset for a random price of X in one years time.

1. Using CAPM, find a deterministic expression for the price of the asset

2. Derive the certainty equivalent form of the CAPM pricing formula

3. Explain why it is essential for the CAPM pricing formula to exhibit linearity in prices

Exercise 2.8: [Solution] Consider a world where there are only 3 stocks with the following characteristics:

	Stock A	Stock B	Stock C
Shares Outstanding	200	80	160
Price per share	\$10	\$15	\$5
Mean Return	15%	10%	20%
Standard Deviation	20%	15%	10%

The stocks have the following correlations:

- $Corr(r_A, r_B) = 0.5$
- $Corr(r_A, r_C) = 0.75$
- $Corr(r_B, r_C) = 0.25$

If a risk-free asset exists and all the CAPM assumptions are satisfied:

1. Find the mean return of the market portfolio
2. Find the standard deviation of the market portfolio

Exercise 2.9: [Solution] Suppose we have a market portfolio where the mean return is 7% and the standard deviation is 20% in annual terms. In this market there also exists a risk-free asset that yields 3%. If we form a portfolio of two assets A and B (of equal weighting) and we have the following information:

- Asset A has a positive β
- Asset A and B have a covariance of 0.06
- The portfolio has a variance of 0.37
- The portfolio has idiosyncratic risk of 0.01

Find the mean return for each individual asset.

Exercise 2.10: [Solution] Consider the SML given by: $\mu_i = r_f + \beta_i(\mu_M - r_f)$. Now we have two assets A and B with the following characteristics:

	Asset A	Asset B
Mean	X	Y
β	β_A	β_B

with $\beta_A > \beta_B > 0$.

If we know that $X < r_f + \beta_A(\mu_M - r_f)$ and $Y > r_f + \beta_B(\mu_M - r_f)$, make a portfolio that will earn an arbitrage profit assuming that there are no idiosyncratic risks.

2.1.3 Discussion Questions

Exercise 2.11: [Solution] A security analyst forecasts that each of three scenarios for the next year is equally likely: (1) a boom, (2) controlled growth, and (3) a severe recession. Under these three states of the world, the analyst projects returns on a specific security, the market portfolio, and Treasury bills (proxy for risk-free securities) as shown on the table below:

	STATE OF THE WORLD		
	Optimistic (Boom)	Likely (Controlled Growth)	Pessimistic (Recession)
Probability of state	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
Return on market portfolio	20%	15%	-5%
Return on security A	25%	20%	-10%
Return on Treasury bills	7%	7%	7%

1. Find the equation for the Capital Market Line.
2. Calculate the expected return and variance for a portfolio of which 50% is invested in Treasury bills and 50% in the market portfolio.
3. Find the equation for the Security Market Line.
4. What is the expected return on security A? What expected return would be appropriate (based on the Security Market Line) for a security with the beta value of Security A?

Exercise 2.12: [Solution] Luenberger, Ch7, Question 8

Exercise 2.13: [Solution] Suppose that all the CAPM assumptions hold and that we know that the utility function of individuals (in terms of the mean and variance of the portfolio) are given by:

$$U\left(r_f + w'(z - r_f 1), w'\Sigma w\right) = 0.8e^{1.5(r_f + w'(z - r_f 1)) - \frac{w'\Sigma w}{2}}$$

What proportion of the individuals' portfolios will be allocated to the risk free asset, and what proportion will be allocated to the tangency portfolio? Hint: Maximise this utility function with respect to $\mathbf{w}$.

[Note: Generally speaking, we assume utility is a function of wealth w. However, in this question, we assume that it is a function of the expected value and the variance, which is a bit unusual. Nonetheless, we also want to maximise it.]

Exercise 2.14: [Solution] Assume that we are in a hypothetical world where there are only 3 assets as follows:

Asset	Mean Return	Beta
Risk-free	5%	0
Security X	17%	0.7
Security Y	24%	1.3

and the mean market rate of return is 20%.

1. Using the Capital Asset Pricing Model determine whether these securities are priced correctly, if not determine whether they are over or under priced. You may assume that the CAPM assumptions hold.
2. Now suppose that the prices of Security X and Security Y has been adjusted to match the values obtained from CAPM in part (a). A trader with insider information tells you

that the mean return for Security X and Security Y is in fact 16% and 24.3% respectively. What can you conclude about these two securities with respect to CAPM? How can you use this information to your advantage?

Exercise 2.15: [Solution] You are an analyst investigating the applications of the Capital Asset Pricing Model in the real world. You have noticed two particular stocks:

	Stock A	Stock B
1 yr Return	50%	35%
Beta	1.2	3

From this information are you able to deduce that CAPM is not applicable to the real world? Why?

2.2 Factor Models

2.2.1 Note

When the market portfolio is not efficient, we shall replace the market portfolio in the CAPM by an efficient portfolio. Therefore, the real problem is how to identify an efficient portfolio. Note that this is a difficult problem if we cannot bypass it by considering the market portfolio. Indeed, we will see later that direct estimation on (z, Σ) is not viable.

Our hope here is that the efficient portfolio must be well-diversified. Therefore, by combining a lot of well-diversified portfolio and the risk-free asset, we have hope to recover an efficient portfolio.

Now, each portfolio is representing a "factor" and we have N of them (which hopefully are enough to recover an efficient portfolio).

In terms of technicalities, this section is essentially the same as regression model. Note the factors are allowed to correlate. You should recall your knowledge from ACTL2131.

2.2.2 Practice Questions

Exercise 2.16: [Solution] Using a Single Factor model with the market return as the factor,

1. Explain what is meant by diversifiable and non-diversifiable risks of a security.
2. Explain how the expected return and the variance of the return depend on each of these two types of risk.
3. Discuss the nature of well-diversified portfolios with (i)$\beta > 1$ and (ii)$0 < \beta < 1$.

Exercise 2.17: [Solution] Assume that the return of a particular security follows the single index model with all the usual assumptions that go with it. Prove the following:

1. $E[R_i] = \alpha_i + \beta_i \mu_M$
2. $Var(R_i) = \beta_i^2 \mu_M^2 + \sigma_\varepsilon^2$

3. $Cov(R_i, R_j) = \beta_i \beta_j \mu_M^2$ for $i \neq j$.

Exercise 2.18: [Solution] Consider a portfolio of n securities which consists of returns from security i as $R_i,\ i = 1, 2, ..., n$. The return from each security is assumed to follow the single index model. Prove that the variance of the portfolio can be decomposed into diversifiable and non-diversifiable risks.

Exercise 2.19: [Solution] Suppose you are given the following observed relationship between the return on a security and that of the market:

Time	R_i	R_M
1	15	14
2	-3	2
3	10	3
4	5	6
5	-1	-2
6	6	7
7	-5	-8
8	3	12
9	5	5
10	14	4

The numbers shown are in percentages (%).

1. Estimate the parameters in the single index model.
2. Decompose the variance of the security into its diversifiable and non-diversifiable components.

Exercise 2.20: [Solution] Luenberger, Ch8 (both Ed). Q1.

Exercise 2.21: [Solution] Discuss the advantages and disadvantages of using factor models against the Capital Asset Pricing Model.

2.2.3 Discussion Questions

Exercise 2.22: [Solution] You are given the following summary statistics for monthly returns on 3 different stocks (labeled A, B, and C) and the $S\&P$ Index for a 12-month period:

	Security/Market			
	A	B	C	$S\&P$
	S_x			
	35.35	72.37	42.65	36.06
	S_{xy}			
A	613.8439	221.5418	739.4184	296.9104
B		559.2715	649.0168	256.0504
C			3179.835	582.4529
$S\&P$				250.8953

where $S_x = \sum x$ is the sum of the 12 monthly returns for each security and $S_{xy} = \sum (x - \overline{x})(y - \overline{y})$ as defined in the Linear Regression section of the *Formulae and Tables for Actuarial Examinations.* Assuming the single-index model holds:

1. Compute the mean return and variance of return for each stock.
2. Compute the expected return and standard deviation of a portfolio constructed by placing one-third of your funds in each stock.

Exercise 2.23: [Solution] Prove that the solution to the least squares estimate for a factor model is given by:

$$\hat{\gamma}_i = (f'f)^{-1}(f'r_i),$$

where $\hat{\gamma}_i = [\alpha_i, \beta_{i,1}, ..., \beta_{i,K}]'$ and

$$f = \begin{bmatrix} 1 & f_{1,1} & \dots & f_{K,1} \\ 1 & f_{1,2} & \dots & f_{K,2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & f_{1,T} & \dots & f_{K,T} \end{bmatrix}$$

[Note: This is a classical question on matrix differentiation.]

Exercise 2.24: [Solution] Given we have N assets, find the number of parameters that need to be estimated in order to use the mean-variance portfolio theory. Compare this to the number of parameters needed for a K-factor model. Are factor models always more efficient than mean-variance approach? Note: the factors are correlated.

2.3 Arbitrage Pricing Theory

2.3.1 Recap

Below are some important concepts.

1. We assume a factor model for individual stocks i, $i = 1, 2, 3, ...$ This implies that a return of a PORTFOLIO is
$$E[r_p] = \alpha_p + \beta_{1p} f_1 + ... + \beta_{Kp} f_K + \varepsilon_p$$
2. We assume a large asset universe, implying that the error term is negligle for WELL-DIVERSIFIED portfolio, i.e.
$$E[r_p] = \alpha_p + \beta_{1p} f_1 + ... + \beta_{Kp} f_K$$
[Note a common mistake is to treat the above as the APT equation. This is not correct because (1) the α_p are different for different portfolio and (2) $E[f_j]$ are assumed to be zero.]
3. We assume that there is no arbitrage (and we are free to buy and sell etc.) which further implies that there are λ_j, $j = 0, 1, ..., K$ such that
$$E[r_p] = \lambda_0 + \beta_{1p}\lambda_1 + ... + \beta_{Kp}\lambda_K$$
The λ's are called the market price of risk (per unit of factor exposure). THIS IS the APT equation.
4. Other mistakes includes

(a) $r_p = \lambda_0 + \beta_{1p}\lambda_1 + ... + \beta_{Kp}\lambda_K + \varepsilon_i$. This is non-sense.

(b) $r_p = \lambda_0 + \beta_{1p}\lambda_1 + ... + \beta_{Kp}\lambda_K$. You need to have the E (Remember r_p is random).

5. Although in the intemediate step we need to look at replicating portfolio (which replicates the random return), ultimately APT concerns the EXPECTED return as in the APT equation.

2.3.2 Finding the replicating portfolio

In this module, often a table consisting of the β's (exposure) on the factors are given to you (e.g. Ex2.25) and you are required to determined the λ's (price per exposure). Once you find the λ's, a new portfolio with different exposure (say $\beta_{1,n}$ and $\beta_{2,n}$) are then given to you are you are asked whether there is any arbitrage.

From the point of linear algebra, (there is only one such portfolio and) all you need to do is to invert the exposure matrix (usually 3 by 3 for 2 factors).

The exposure matrix takes form

$$B = \begin{bmatrix} 1 & b_1 & c_1 \\ 1 & b_2 & c_2 \\ 1 & b_3 & c_3 \end{bmatrix}$$

where the exposure to the factors f_1 and f_2 are b_i and c_i respectively for the $i-th$ portfolio (security). To find the λ's, we need to solve

$$B \begin{bmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix} \iff \begin{bmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \end{bmatrix} = B^{-1} \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix}.$$

To find the replicating portfolio for given exposures b and c, we need to solve

$$\begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix} B = \begin{bmatrix} 1 & b & c \end{bmatrix} \iff \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix} = \begin{bmatrix} 1 & b & c \end{bmatrix} B^{-1}.$$

[You can also solve (for (w_1, w_2, w_3)) the following 3 by 3 linear equation if you do not like the idea of inverting matrix.

$$\begin{cases} w_1 + w_2 + w_3 & = 1 \\ w_1 b_1 + w_2 b_2 + w_3 b_3 & = \beta_{1,n} \\ w_1 c_1 + w_2 c_2 + w_3 c_3 & = \beta_{2,n} \end{cases}$$

Essentially both method are equivalent.]

This means we need to find the inverse of the B. The inverse of B is actually quite simple, if you know the inverse formula. Anyways, it is given by

$$\frac{1}{\Delta} \begin{bmatrix} b_2 c_3 - b_3 c_2 & b_3 c_1 - b_1 c_3 & b_1 c_2 - b_2 c_1 \\ c_2 - c_3 & c_3 - c_1 & c_1 - c_2 \\ -(b_2 - b_3) & -(b_3 - b_1) & -(b_1 - b_2) \end{bmatrix} = \begin{bmatrix} w_0 \\ e_1 \\ e_2 \end{bmatrix}$$

where Δ is the sum of the first row, w_0, e_1 and e_2 are row vectors.

Note $1'w_0' = 1$, $1'e_1' = 0$, $1'e_2' = 0$ (sum of weights) and we shall see that w_0 correspond to the portfolio with no exposure to any betas (risk-free), while e_1 and e_2 are the "unit exposure" to the 2 factors f_1 and f_2 respectively. For example, if you want to replicate a portfolio with $\beta_1 = 2$ and $\beta_3 = -1$, you should use $w = w_0 + 2e_1 - e_2$.

Note the above formula is very symmetric. To remember it, for example you can memorize the second row, which goes $2 \to 3 \to 1 \to 2$ from left to right. Then the third row is also $2 \to 3 \to 1 \to 2$ with a negative sign. The first row can then be remembered as $2 \to 3 \to 1 \to 2$ for the b's and reversing the order of c in row 2.

2.3.3 Practice Questions

Exercise 2.25: [Solution] The expected returns of Portfolios A,B,C have been estimated using the APT model as shown below:

	$E(r_i)$	$\beta_{i,1}$	$\beta_{i,2}$
A	0.1	1	2.5
B	0.075	0.9	3
C	0.13	1.2	1.7

1. Find the values of λ_0, λ_1 and λ_2 for this APT model.
2. Suppose a new Portfolio "Z" is introduced into the market with the following characteristics:

$$E(r_Z) = 0.1, \quad \beta_{Z,1} = 1.5, \quad \beta_{Z,2} = 2.$$

   How can this information be used to construct an arbitrage portfolio?

Exercise 2.26: [Solution] Suppose we have a two factor model with residual risk that explains security returns consistently. If we know the risk-free rate is 5% and that the two factors are Momentum and Market return, find the expected return for an arbitrary well-diversified portfolio (in terms of the β's) under APT. We are given the following information about the APT model:

Portfolio	Expected Return (%)	$\beta_{momentum}$	β_{market}
A	10	1.2	0.5
B	15	3	-0.2

Exercise 2.27: [Solution] In a hypothetical market we have three well-diversified portfolios that follow the two factor model. We know that APT holds and it is given that $E[r_1] = 12\%, E[r_2] = 15\%, E[r_3] = 7\%$, where r_i's are the returns of the three portfolios, which have the following characteristics:

$$\begin{aligned} r_1 &= a_1 + 3f_1 + f_2 \\ r_2 &= a_2 + f_1 + 2f_2 \\ r_3 &= a_3 \end{aligned}$$

If a new well-diversified portfolio is introduced with the following characteristics:

$$r_4 = a_4 + 5f_1 - 2f_2$$

find $E[r_4]$.

Exercise 2.28: [Solution] Suppose we have securities that follow the single factor model without residual risk i.e $r_i = \alpha_i + \beta_i f$ where $E[f] = 0$. Form a risk-less portfolio with two assets under the single factor model and hence prove that under APT we have $E[r_i] = \lambda_0 + \beta_i \lambda_1$ for some λ_0 and λ_1.

2.3.4 Discussion Questions

Exercise 2.29: [Solution] Luenberger, Ch8. (both Ed)Q2.

Exercise 2.30: [Solution] It is believed that three entirely uncorrelated factors give a satisfactory explanation of investment returns. The risk premium on the three indices are 3% p.a., 5% p.a., and 9% p.a. respectively. The sensitivity of Portfolio A to these factors are 0, 0, and 2 respectively. The sensitivity of Portfolio B to all three factors is 0.75. The risk free rate of return is 6% p.a.

1. Calculate the expected returns on each portfolio assuming that the arbitrage pricing theory holds.
2. Calculate the characteristics of a portfolio consisting of 75% Portfolio A and 25% Portfolio B.

Exercise 2.31: [Solution] Discuss how you could select suitable factors for using APT to model stock returns. What things need to be considered?

Exercise 2.32: [Solution] [Proof of APT] This exercise aims to prove the APT step by step for interested reader. Essentially it is a simple exercise in linear algebra. It is not examable.

Suppose there are in total $n-1$ factor. Pick n linear independent portfolios and denote the exposure matrix B, i.e.

$$B = \begin{bmatrix} 1 & \beta_{1,1} & \beta_{1,2} & \dots & \beta_{1,n-1} \\ 1 & \beta_{2,1} & \beta_{2,2} & \dots & \beta_{2,n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \beta_{n,1} & \beta_{n,2} & \dots & \beta_{n,n-1} \end{bmatrix}$$

where the j-th portfolio has exposure $\beta_{j,k}$ to the k-th factor. (Linear independent simply means if I have chosen w and z, then for example $2w$ and $w+z$ are not allowed to be chosen.)

By denoting

$$B^{-1} = \begin{bmatrix} w_0 \\ e_1 \\ \vdots \\ e_{n-1} \end{bmatrix}, \quad \mu = \begin{bmatrix} E[r_1] \\ \vdots \\ E[r_n] \end{bmatrix}$$

where w_0, e_k are row vectors and $E[r_j]$ is the expected return of our j-th portfolio,

(i) Show $1'w_0' = 1$ and $1'e_k' = 0$, $k = 1, 2, ..., n-1$.

(ii) Show that portfolio $w_b = w_0 + b_1 e_1 + ... + b_{n-1}e_{n-1}$ has precisely b_k exposure to the k-th factor (for all k) and $1'w_b' = 1$ (sum of weights equals 1). (w_b is a portfolio on the n portfolio we have chosen.)

[From this result, we shall see that w_0 has no exposure to any of the factors and e_k has 1 unit of exposure to k-th factor and no exposure to any other factors. Essentially the inverse matrix B^{-1} untangle the mixing of the factors in the originally chosen n portfolios.]

(iii) Find the expected return of w_b?

(iv) Now suppose there is a portfolio y_b with exposure b_k to the k-th factor (for all k). What should its expected return be under the "no-arbitrage" assumption?

2.4 Model Fitting and Efficient Market Hypothesis

2.4.1 Recap on mean blur

In M1 part 3, we perform analysis on the formation of portfolio assuming we know the mean and variance. However, in practice, we only have historical data. Therefore, we need to estimate the mean and variance (covariance) of the stocks from the data. Suppose we are interested in the annual return of a particular stock, i.e. we are interested in estimating the mean and the variance of the annual return, we need to come up with an estimator for both of them.

The first method is a lazy method. Since we are interested in estimating the mean and variance of the ANNUAL return, we can just look up the price once a year, (e.g. Jan 3rd every year) and denote them as Y_j, $j = 1, 2, 3, ..., M$ which in total has M years of data (calculated from $M + 1$ observation). In this sense, the corresponding estimators are

$$\hat{\mu_a} = \frac{\sum_{j=1}^{M} Y_j}{M} = \bar{Y}$$
$$\hat{\sigma_a^2} = \frac{\sum_{j=1}^{M}(Y_j - \bar{Y})^2}{M-1}$$

which are the standard unbiased estimators used in Statistics.

You may suspect whether it is efficient to have only one data point per year. Hence, the question here is whether we can do better by perhaps having more observations (maybe using high frequency data). Let's assume within one year, we have n datapoints (from $n + 1$ observations with the same inter-observation time) denoted as X_i, $i = 1, 2, ..., n$. Then our (standard unbiased) estimators are

$$\hat{\mu}_{(n)} = \frac{\sum_{i=1}^{n} X_i}{n} = \bar{X}_{(n)}$$
$$\hat{\sigma^2}_{(n)} = \frac{\sum_{i=1}^{n}(X_i - \bar{X}_{(n)})^2}{n-1}$$

To calculate the ANNUAL return, we simply scale it by n with an assumption that returns from different period are i.i.d., i.e. $R_a = \sum_{i=1}^{n} R_{(n)}$. This implies that $E[R_a] = nE[R_{(n)}]$ and $Var[R_a] = nVar[R_{(n)}]$. Therefore, our estimators for the annual return and variance are

$$\hat{\mu_a}_{(n)} = \sum_{i=1}^{n} X_i = n\bar{X}_{(n)}$$
$$\hat{\sigma_a^2}_{(n)} = \frac{n}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_{(n)})^2$$

Since the estimators are unbiased, the goodness of the estimators are the variance (the lower the better). For the estimator of the mean, we have

$$Var[\hat{\mu_a}_{(n)}] = Var[\sum_{i=1}^{n} X_i] = nVar[X_i] = nVar[R_{(n)}] = Var[R_a] = \sigma_a^2$$

which does not depend on n! This means that we cannot improve the performance of the estimator by simply observing more frequently. It may appear conter-intuitive at first sight. However, if we recall that return is calculated only using the observation at the beginning and the end of the interval, it is perhaps not very surprising.

This means that to have a reliable estimate of the mean of the annual return, we really have to wait one year to get a new datapoint, which is impossible in practice because (1) we may need to wait 100 years depending on the precision required (2) returns of company typically are not stationary for long period because of the changes in the business environment, operation, etc. This is why "mean blur".

However, things are much nicer for variance. By further assuming that $R_i \sim N(\mu_{(n)}, \sigma^2_{(n)})$, we have

$$Var[\hat{\sigma^2_{a(n)}}] = Var[\frac{n}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_{(n)})^2] = (\frac{n}{n-1})^2 Var(\sum_{i=1}^{n}(X_i - \bar{X}_{(n)})^2),$$

where it is well known from Statistics that $\sum_{i=1}^{n}(X_i - \bar{X}_{(n)})^2 \sim \chi^2_{n-1}\sigma^2_{(n)}$ (χ^2_{n-1} is chi-sq distribution with degree of freedom $n-1$) which has variance $2(n-1)(\sigma^2_{(n)})^2$. Therefore, we have

$$Var[\hat{\sigma^2_{a(n)}}] = (\frac{n}{n-1})^2 2(n-1)(\sigma^2_{(n)})^2 = \frac{n}{n-1}2n(\sigma^2_{(n)})^2 = \frac{n}{n-1}2n(\frac{\sigma^2_a}{n})^2 = \frac{2}{n-1}\sigma^4_a$$

which is decreasing in n. Hence, we can improve our estimator of variance if we sample more observations within a year. In other words, the estimator for the variance is reliable.

2.4.2 Practice Questions

Exercise 2.33: [Solution] You have been doing some research into the stock picking skills of an investment manager 'ABC Investments'. Looking at the historical data over the past 5 years, you find that its average return is 12%pa while the market as a whole averaged 9%pa. Is this a sign of market inefficiency?

Exercise 2.34: [Solution] An estimate is reliable if the difference between the estimate and its true value is within 10% of its standard deviation 95% of the time. Assuming stationary stock price distribution over time, how many years of data is necessary to get a reliable estimate for the annual expected return?

[Note: Generally the standard deviation of annual stock returns are about 20%, 10% of standard deviation is then about 2%, which is comparable with the returns itself.]

Exercise 2.35: [Solution] Explain the Efficient Market Hypothesis and identify under which form of the hypthesis the following trading strategies will not work:

1. Active Trading
2. Passive Trading
3. Technical Trading
4. Fundamental Trading
5. Insider Trading

Exercise 2.36: [Solution] Luenberger, (1, 2Ed) Q 8.4

Exercise 2.37: [Solution] Luenberger, (1Ed) Q 8.6 (2Ed) 9.2

2.4.3 Discussion Questions

Exercise 2.38: [Solution] Luenberger, Q(1Ed) 8.5 (2Ed) 9.1

Exercise 2.39: [Solution] Luenberger, Q(1Ed) 8.7 (2Ed) 9.3

Exercise 2.40: [Solution] Assume that the samples are taken from $N(\mu, \sigma^2)$, prove the variance of the estimator of the variance is given by

$$Var[\hat{\sigma}^2] = \frac{2\sigma^4}{n-1}.$$

(Assume the samples are normally distributed)

2.5 Solutions

2.5.1 Capital Asset Pricing Model

Solution 2.1: [Exercice] We are given: $r_F = 0.035$, $E(R_m) = 0.10$, $\sigma_m = 0.20$, and the CAPM holds where:

$$E(R_z) = r_F + \beta_z [E(R_m) - r_F].$$

(a) Thus, the beta of portfolio Z is

$$\beta_z = \frac{E(R_z) - r_F}{E(R_m) - r_F} = \frac{0.045}{0.065} = 0.6923077.$$

(b) Since the portfolio is efficient, then

$$\beta_z = \frac{\sigma_z}{\sigma_m}$$

so that

$$\sigma_z = \beta_z \sigma_m = 0.6923077 \times 0.20 = 0.1384615 \text{ or } 13.85\%.$$

Note that the only efficient portfolios are those consists of market portfolio and the risk free asset (1 fund theorem).

(c) The returns on any efficient portfolio are perfectly correlated with those of the market, hence, $\rho_{z,m} = 1$. Consequently, portfolio Z cannot have unsystematic risks, this having been diversified away. Hence, the entire standard deviation of 13.85% is attributable to systematic risk. You can also check this by noting that

$$\sigma_z^2 = \beta_z^2 \sigma_m^2 + \text{"left-over"},$$

and since $\sigma_z = \beta_z \sigma_m$, this implies that there is zero "left-over" or no unsystematic or specific risks at all.

Solution 2.2: [Exercice]

1. Firstly we have the individual assumptions such as:

(a) Homogeneous: Investors have the same estimate for mean, standard deviation and covariances for securities. Investors have the same one-period horizon. They also have access to the same amount of information

(b) They can borrow or lend an any amount at the risk-free rate. There is no restriction on short-selling assets as well. (Assets can be bought and sold in any quantity)

(c) Investors are rational, risk-averse and utility maximisers

2. We also have the market assumptions such as:

(a) Equilibrium: The market is at equilibrium, i.e. the demand for the assets is exactly matched with the supply.

(b) Prices cannot be influenced by a single individual

(c) There are no market frictions such as taxes or transaction costs

Some reasons why the CAPM assumptions fail include:

- Homogenous and equilibrium assumption do not hold.
- Market prices can be influenced by fund managers of large investment funds
- Not all investors can access a risk-free rate and it is hard to specify what is considered as a risk-free rate
- Information is clearly not equally accessible to all investors as corporate insiders have more information about their own corporations
- There are taxes and transaction costs

Solution 2.3: [Exercice]

1. Capital Market Line - Is essentially the efficient frontier formed from mixing the risk-free asset with the tangency portfolio. As such, the CML only looks at the mean return and total risk of EFFICIENT portfolios, other things are not plotted.

2. Security Market Line - The SML shows the relationship between mean return and systematic risk so it is applicable to all assets whether they are efficient or not. The SML can be used to evaluate whether a security should be included in a portfolio by looking at over/underpricing. Essentially when assets plot above the SML they are underpriced (vice versa).

Solution 2.4: [Exercice]

(a) According to the "security market line", the expected return on any security is

$$E(R_i) = r_F + \beta_i \left[E(R_m) - r_F\right]$$

where r_F is the risk-free rate of return, β_i is the security beta, and $E(R_m)$ is the expected return in the market.
Note the beta of the market must be 1.

$$\beta_m = \frac{E(R_m) - r_F}{E(R_m) - r_F} = 1.$$

The market has a unit beta. For the risk-free asset, we would have

$$r_F = r_F + \beta_F \left[E(R_m) - r_F\right]$$

which implies that

$$\beta_F \left[E\left(R_m\right) - r_F\right] = 0.$$

However, since $E\left(R_m\right) \neq r_F$, it must be that $\beta_F = 0$. Thus, a risk-free asset has a zero beta.

[Note by the definition of β, without performing any calculation one can tell that $\beta_M = 1$ and $\beta_{r_f} = 0$.]

(b) The diagram which illustrates the SML relationship is the curve (line) that relates the beta of the security to the expected return, the beta being a measure of the risk. The gradient of the security market line must be given by:

$$\frac{E\left(R_m\right) - r_F}{\beta_m - 0} = \frac{E\left(R_m\right) - r_F}{1} = E\left(R_m\right) - r_F$$

and the intercept is at the risk-free rate r_F. Thus, the security market relationship must therefore be given by

$$E\left(R_i\right) = r_F + \beta_i \left[E\left(R_m\right) - r_F\right]$$

as required.

(c) The security market line tells us that the expected return of any asset is a linear function of its systematic risk, as measured by the beta factor. The expected return does not depend on any other factors and, in particular, the specific risk of an asset that can be eliminated by diversification.

[Note: Diversification is desired from a mean-variance point of view as it does not give any extra expected return with an extra risk.]

Solution 2.5: [Exercice] (L7.1)

1. $$E\left[r_i\right] = 0.07 + \frac{0.23 - 0.07}{0.32}\sigma$$

2. $\sigma = 0.64$. Solve
$$0.07w + 0.23\left(1 - w\right) = 0.39$$
hence $w = -1$. Borrow 1000 at the risk free rate and invest 2000 in the market

3. 1182

(L7.2)

1. $$\sigma_M^2 = Var(0.5(A + B)) = 0.25Var(A + B) = 0.25 \times (\sigma_A^2 + \sigma_B^2 + 2\sigma_{AB})$$
$$Cov(A, M) = Cov(A, 0.5(A + B)) = 0.5Var(A) + 0.5Cov(A, B) = 0.5\sigma_A^2 + 0.5\sigma_{AB}$$
Using the previous quantities, we obtain $\beta_A = \frac{Cov(A,M)}{\sigma_M^2}$. The quantities for asset B can be derived similarly.

$$\begin{aligned}
\sigma_M^2 &= \frac{1}{4}\left(\sigma_A^2 + 2\sigma_{AB} + \sigma_B^2\right) \\
\beta_A &= \frac{\frac{1}{2}\left(\sigma_A^2 + \sigma_{AB}\right)}{\sigma_M^2} \\
\beta_B &= \frac{\frac{1}{2}\left(\sigma_B^2 + \sigma_{AB}\right)}{\sigma_M^2}
\end{aligned}$$

2.

$$
\begin{aligned}
E\left[r_A\right] &= 0.10+\frac{5}{4}(0.18-0.10)=20\% \\
E\left[r_B\right] &= 0.10+\frac{3}{4}(0.18-0.10)=16\%
\end{aligned}
$$

(L7.3)

1. By the 2 fund theorem any portfolio on the minimum variance frontier can be formed by combining the w and v portfolios. Now if α is placed into first portfolio and $1-\alpha$ in the second portfolio, we have the weights for the combined portfolio as

$$
\left[\begin{array}{c}
(0.6\alpha+(1-\alpha)0.8 \\
0.2\alpha-0.2(1-\alpha) \\
0.2\alpha+0.4(1-\alpha)
\end{array}\right]
$$

The market portfolio must also satisfy the above. However by definition the market portfolio cannot contain assets in negative amounts, and so all three components must be non-negative. Hence from the first component we have $\alpha \leq 4$, the second $\alpha \geq 0.5$, and third $\alpha \leq 2$. In aggregate we must have $0.5 \leq \alpha \leq 2$.

Now observe that the expected returns for the three underlying securities are 10%, 20% and 10%. This means that the only change in any resulting portfolio mean (from 10%) will be due to differencs in the allocation to the second security (in particular, the higher the weight placed, the higher the resulting portfolio expected return.)

Looking at the weight in the second security in the combined portfolio above, we see that the weight placed is

$$
0.2\alpha-0.2(1-\alpha)=0.4\alpha-0.2
$$

which is increasing in α. This means that the minimum expected return for a portfolio on the minimum variance set will be given at the minimum possible α, i.e. $\alpha=0.5$, which gives a portfolio with weights

$$
\frac{1}{2}w+\frac{1}{2}v=\left[\begin{array}{c}
0.7 \\
0 \\
0.3
\end{array}\right]
$$

and the maximum expected return will be given when $\alpha=2$, i.e. a portfolio with weights

$$
2w-v=\left[\begin{array}{c}
0.4 \\
0.6 \\
0
\end{array}\right]
$$

These 2 cases have associated expected rates of return 0.10 and 0.16, and represent the bounds on the expected return on the market portfolio.

2. w has expected return 0.12. Since the market portfolio is efficient, it should have a higher expected return then that of the global minimum variance portfolio, the bounds are (0.12,0.16)

(L7.6)

1. The market consists of \$150 in A and \$300 in B. Hence

$$\begin{aligned} E[r_M] &= \left(\frac{150}{450}\right)E[r_A] + \left(\frac{300}{450}\right)E[r_B] \\ &= 0.13 \end{aligned}$$

2.

$$\begin{aligned} \sigma_M &= \left[\left(\frac{1}{3}\right)^2 (0.15)^2 + 2\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)(0.15)(0.09)\left(\frac{1}{3}\right) + \left(\frac{2}{3}\right)^2 (0.09)^2\right]^{0.5} \\ &= 0.09 \end{aligned}$$

3.

$$\begin{aligned} \beta_A &= \frac{\frac{1}{3}0.15^2 + \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)(0.15)(0.09)}{0.09^2} \\ &= 1.2963 \end{aligned}$$

(L7.9)Notice that

$$\begin{aligned} \sigma_{\alpha M} &= Cov\left(\alpha r_f + (1-\alpha) r_M, r_M\right) \\ &= (1-\alpha)\sigma_M^2 \\ E[x] &= p\left(1 + \alpha r_f + (1-\alpha) E[r_M]\right) \\ Cov(x, r_M) &= Cov\left(p\left(1 + \alpha r_f + (1-\alpha) r_M\right), r_M\right) \\ &= p(1-\alpha)\sigma_M^2 \end{aligned}$$

and hence method 1 gives

$$\begin{aligned} & \frac{E[x]}{1 + r_f + \beta\left(E[r_M] - r_f\right)} \\ =\ & \frac{p\left(1 + \alpha r_f + (1-\alpha) E[r_M]\right)}{1 + r_f + \frac{\sigma_{\alpha M}}{\sigma_M^2}\left(E[r_M] - r_f\right)} \\ =\ & p \end{aligned}$$

and method 2 gives

$$\begin{aligned} & \left(\frac{1}{1+r_f}\right)\left[E[x] - Cov(x, r_M)\frac{E[r_M] - r_f}{\sigma_M^2}\right] \\ =\ & \left(\frac{1}{1+r_f}\right)\left[\begin{array}{c} p\left(1 + \alpha r_f + (1-\alpha) E[r_M]\right) \\ -p(1-\alpha)\sigma_M^2 \frac{E[r_M] - r_f}{\sigma_M^2} \end{array}\right] \\ =\ & P \end{aligned}$$

as required.

Solution 2.6: [Exercice] We know that for security $i = A, B$, we have

$$E(R_i) = \alpha_i + \beta_i E(R_M)$$

and

$$Var(R_i) = \beta_i^2 \sigma_M^2 + \sigma_\varepsilon^2.$$

(a) You should be able to verify that: the mean of the returns are

$$E\left(R_A\right) = 0.232$$

and

$$E\left(R_B\right) = 0.33.$$

The variance of the returns are

$$Var\left(R_A\right) = 0.1201$$

and

$$Var\left(R_B\right) = 0.25.$$

(b) The covariance of the returns of the stocks is

$$Cov\left(R_A, R_B\right) = \beta_A \beta_B \sigma_M^2 = 0.072.$$

(c) An equally-weighted portfolio of the stocks will have a return $R_p = \frac{1}{2}\left(R_A + R_B\right)$ so that the beta of the portfolio will be

$$\beta_p = \frac{1}{2}\left(\beta_A + \beta_B\right) = \frac{1}{2}\left(1.2 + 1.5\right) = 1.35.$$

(d) The expected return of the portfolio is

$$E\left(R_p\right) = 0.281$$

and its variance is

$$Var\left(R_p\right) = 0.128525.$$

Solution 2.7: [Exercice]

1. The return for the asset is given by:

$$R = \frac{X - p}{p}$$

Applying CAPM to this equation we get:

$$\begin{aligned}
\frac{E[X] - p}{p} &= r_f + \beta(\mu_m - r_f) \\
E[X] - p &= p(r_f + \beta(\mu_m - r_f)) \\
E[X] &= p\left(1 + r_f + \beta(\mu_m - r_f)\right) \\
p &= \frac{E[X]}{1 + r_f + \beta(\mu_m - r_f)}
\end{aligned}$$

2.

$$\begin{aligned}
\beta &= \frac{Cov(R, r_m)}{\sigma_m^2} \\
&= \frac{Cov(\frac{X-p}{p}, r_m)}{\sigma_m^2} \\
&= \frac{Cov(X, r_m)}{p\sigma_m^2} \\
\implies p &= \frac{E[x]}{1 + r_f + \frac{Cov(X, r_m)}{p\sigma_m^2}(\mu_m - r_f)} \\
p(1 + r_f) + \frac{Cov(X, r_m)}{\sigma_m^2}(\mu_m - r_f) &= E[X] \\
p &= \frac{E[X] - \frac{Cov(X, r_m)}{\sigma_m^2}(\mu_m - r_f)}{1 + r_f}
\end{aligned}$$

3. Linearity means that the price of a portfolio of assets is equal to the sum of the individual prices of each asset, this condition is essential because it prevents arbitrage.

Solution 2.8: [Exercice] First we need to calculate the market caps to get the market portfolio.

$$\begin{aligned}
Cap_A &= \$2000 \\
Cap_B &= \$1200 \\
Cap_C &= \$800 \\
Total &= \$4000
\end{aligned}$$

Then our market portfolio is constructed as: $r_m = 0.5r_A + 0.3r_B + 0.2r_C$.

1.

$$\begin{aligned}
\mu_m &= 0.5\mu_A + 0.3\mu_B + 0.2\mu_C \\
&= 14.5\%
\end{aligned}$$

2.

$$\begin{aligned}
\sigma_m^2 &= w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + w_C^2\sigma_C^2 + 2(w_Aw_B\rho_{A,B}\sigma_A\sigma_B + w_Aw_C\rho_{A,C}\sigma_A\sigma_C + w_Bw_C\rho_{B,C}\sigma_B\sigma_C) \\
&= 0.012425 + 2(0.00225 + 0.0015 + 0.000225) \\
&= 0.020375 \\
\sigma_m &= 14.27\%
\end{aligned}$$

Solution 2.9: [Exercice] From the covariance we get the following:

$$\begin{aligned}
Cov(r_i, r_j) &= Cov(r_f + \beta_i(r_M - r_f) + \varepsilon_i, r_f + \beta_j(r_M - r_f) + \varepsilon_j) \\
&= \beta_i\beta_j\sigma_M^2 \\
&= 0.06, \\
\therefore \beta_i\beta_j &= \frac{0.06}{0.04} \\
&= 1.5.
\end{aligned}$$

This also implies that both β are positive. Now the portfolio variance gives us:

$$\begin{aligned} Var(r_p) &= \left(\frac{\beta_i+\beta_j}{2}\right)^2 \sigma_M^2 + \sigma_{\varepsilon_p}^2 \\ &= \frac{(\beta_i+\beta_j)^2}{4} 0.04 + 0.01 \\ &= 0.37 \\ \therefore \frac{(\beta_i+\beta_j)^2}{4} &= 9 \\ (\beta_i+\beta_j)^2 &= 36 \\ \beta_i+\beta_j &= 6 \end{aligned}$$

as the sum is positive.

Now recall this is the sum and products of roots so we need to solve a quadratic to get the betas.

$$\begin{aligned} \beta^2 - 6\beta + 1.5 = 0 \\ \beta &= \frac{6 \pm \sqrt{36 - 4 \cdot 1.5}}{2} \\ &= 5.738612788, 0.2613872125 \end{aligned}$$

Then substituting these values into CAPM we get:

$$\begin{aligned} \mu_A &= 4.04554885\% \\ \mu_B &= 25.95445115\% \end{aligned}$$

Solution 2.10: [Exercice] From the information we can deduce that Asset A is overpriced and Asset B is underpriced. Now what we can do is make a portfolio such that the portfolio beta is 0 (i.e risk-free).

$$\begin{aligned} r_p &= w \cdot r_A + (1-w) \cdot r_B \\ &= w(r_f + \beta_A(r_M - r_f)) + (1-w)(r_f + \beta_B(r_M - r_f)) \\ &= r_f + (w \cdot \beta_A + (1-w)\beta_B)(r_M - r_f) \\ \therefore w \cdot \beta_A + (1-w)\beta_B &= 0 \\ w &= \frac{-\beta_B}{\beta_A - \beta_B} \end{aligned}$$

Under CAPM this portfolio would give us: $E[r_p] = r_f$, however we are actually getting $E[r_p] = \frac{-\beta_B}{\beta_A-\beta_B} X + \frac{\beta_A}{\beta_A-\beta_B} Y$.
You can check that this is in fact greater than the risk-free rate:

$$\begin{aligned} E[r_p] &= \frac{\beta_A}{\beta_A - \beta_B} Y - \frac{\beta_B}{\beta_A - \beta_B} X \\ &> \frac{\beta_A}{\beta_A - \beta_B}(r_f + \beta_B(\mu_M - r_f)) + \frac{\beta_B}{\beta_A - \beta_B}(-r_f - \beta_A(\mu_M - r_f)) \\ &> \left(\frac{\beta_A}{\beta_A - \beta_B} - \frac{\beta_B}{\beta_A - \beta_B}\right) r_f + \left(\frac{\beta_A\beta_B}{\beta_A - \beta_B} - \frac{\beta_A\beta_B}{\beta_A - \beta_B}\right)(\mu_M - r_f) \\ &> r_f \end{aligned}$$

as required.

Solution 2.11: [Exercice] Note, to plot the CML and SML, it is sufficient to find the expected return and the variance of the market portfolio. For the market, the mean is

$$E(R_m) = (0.20 + 0.15 - 0.05) \times \frac{1}{3} = 0.1$$

and variance is

$$Var(R_m) = \left(0.2^2 + 0.15^2 + 0.05^2\right) \times \frac{1}{3} - (0.1)^2 = 0.01167.$$

We also find the means and variances of security A. For security A, its mean is

$$E(R_A) = (0.25 + 0.20 - 0.10) \times \frac{1}{3} = 0.1167$$

and variance is

$$Var(R_A) = \left(0.25^2 + 0.20^2 + 0.10^2\right) \times \frac{1}{3} - (0.1167)^2 = 0.02389.$$

(a) The CML equation is given by

$$E(R_p) = r_F + \frac{\sigma_p}{\sigma_m}\left[E(R_m) - r_F\right]$$

for any efficient portfolio p (i.e. combinations of market and risk free). Thus, the CML equation is

$$\begin{aligned} E(R_p) &= 0.07 + \frac{\sigma_p}{0.1080}\left[0.10 - 0.07\right] \\ &= 0.07 + 0.2778 \cdot \sigma_p, \end{aligned}$$

written as a function of the portfolio standard deviation.

(b) For a portfolio with 50% in Treasury bills and 50% in the market portfolio, its return is

$$R_q = 0.50 r_F + 0.50 R_m$$

after denoting this portfolio by q. Thus, its mean is

$$E(R_q) = 0.50 r_F + 0.50 E(R_m) = 0.50\,(0.07 + 0.10) = 0.085$$

and its variance is

$$Var(R_q) = 0.50^2 Var(R_m) = 0.0029175.$$

(c) The SML equation is given by

$$E(R_p) = r_F + \beta_p\left[E(R_m) - r_F\right]$$

for any (not necessarily efficient) portfolio p. Thus, the SML equation is

$$\begin{aligned} E(R_p) &= 0.07 + \beta_p\left[0.1 - 0.07\right] \\ &= 0.07 + 0.03 \cdot \beta_p, \end{aligned}$$

written as a function of the security beta.

(d) Since

$$Cov\left(R_A, R_m\right) = E\left[\left(R_A - E\left(R_A\right)\right)\left(R_m - E\left(R_m\right)\right)\right] = 0.01667,$$

then, the beta of security A is given by

$$\beta_A = \frac{Cov\left(R_A, R_m\right)}{\sigma_m^2} = \frac{0.01667}{0.01167} = 1.428.$$

The expected return for security A is $E\left(R_A\right) = 0.1167$. According to the Security Market Line, the expected return for security A would be

$$E\left(R_A\right) = -0.030 + 1.428 \cdot E\left(R_m\right) = -0.030 + 1.428\left(0.1\right) = 0.1128.$$

which illustrates that the analyst's prediction are not consistent with CAPM.

Solution 2.12: [Exercice] (L7.8) Note - in the following we assume that c is independent of p and the market. (the textbook is a little unclear as it states that 'uncertainty is uncorrelated')

1.

$$\begin{aligned} E\left[r\right] &= E\left[\frac{p-c}{c}\right] \\ &= E\left[\frac{1}{c}\right]E\left[p\right] - 1 \\ &= \left[0.5\frac{1}{20} + 0.5\frac{1}{16}\right]\left[24\right] - 1 \\ &= 35\% \end{aligned}$$

2.

$$\begin{aligned} \sigma_{PM} &= E\left[\left(\frac{p-c}{c} - E\left(\frac{p-c}{c}\right)\right)\left(r_M - \bar{r}_M\right)\right] \\ &= E\left[\left(\frac{p}{c} - E\left(\frac{p}{c}\right)\right)\left(r_M - \bar{r}_M\right)\right] \\ &= E\left(\frac{1}{c}\right)E[(p - E[P])(r_M - \bar{r}_M)] \\ &= \left(\frac{9}{160}\right)20\sigma_M^2 \end{aligned}$$

(notice where the independence of c was used) and hence

$$\beta_M = \frac{9}{8}$$

3.

$$E\left[r\right] = 0.09 + \frac{9}{8}\left(0.33 - 0.09\right) = 0.36$$

so the project is not acceptable.

Solution 2.13: [Exercice] Note to maximise

$$U\left(r_f + w'(z - r_f 1), w'\Sigma w\right) = 0.8e^{1.5(r_f + w'(z - r_f 1)) - \frac{w'\Sigma w}{2}}$$

is the same as to maximise

$$\tilde{U}(w) = 1.5(r_f + w'(z - r_f 1)) - \frac{w'\Sigma w}{2}$$

as $f(x) = 0.8e^x$ is an increasing function.

[Note this is valid ONLY when we are maximising a deterministic function U. For example, maximising $E[e^{f(x)}]$ is not the same as maximising $E[f(x)]$.]

It can be seen that $\tilde{U}$ is (concave) quadratic and therefore has a maximum which occurs when $\frac{\partial \tilde{U}}{\partial w} = 0$. Note

$$\frac{\partial}{\partial w}\tilde{U}(w) = 1.5(z - r_f 1) - \Sigma w = 0$$

is equivalent to

$$w = 1.5\Sigma^{-1}(z - r_f 1).$$

Moreover,

$$1'w = 1.5 \times 1'\Sigma^{-1}(z - r_f 1) = 1.5(B - r_f A)$$

and therefore the Therefore, individuals will allocate $1.5(B - r_f A)$ into the tangency portfolio and $1 - 1'w = 1 - 1.5(B - r_f A)$ into the risk-free asset.

Solution 2.14: [Exercice]

1. The CAPM equation is given by:

$$\mu = r_f + \beta(\mu_M - r_f)$$

   So substituting the respective beta values into the equation we get 15.5% and 24.5% respectively.
   Since the CAPM assumptions hold, we can see that the market is giving Security X a higher return than it should, meaning it is underpriced. On the other hand, Security Y is given a lower return than it should so it is overpriced. (Recall that price and return is inversely related i.e treat return as the discounting factor)

2. Here the whole market estimated the returns incorrectly, based on the current price. Hence, the answer is the same as part 1. We should buy the underpriced and sell the overpriced. Once the market realised the mis-estimation, the market will act such that the price will go to the equilibrium (the CAPM price) which gives us a profit.

   Note that at equilibrium, the mean returns of an asset is driven by its characteristics, i.e. the correlation with the market, while the (estimated) “observed” return is the ratio of the estimated next period price divided by the current price.

Solution 2.15: [Exercice] No this information doesn’t say CAPM is not applicable. The first thing you may notice is the return on the stock with lower beta is much higher than the stock with a higher beta even though CAPM implies higher beta should give you higher return. This doesn’t go against CAPM for the following reasons:

- CAPM implies higher beta would give a higher EXPECTED return, it doesn’t mean the return has to be higher, this could have been a bad year for stock B.
- We also need to know the alpha, stock A could have a significantly higher alpha than B so if the market doesn’t perform phenomenally then stock A can still beat stock B.
- There could be stock specific risk that contributes to the return

2.5.2 Factor Models

Solution 2.16: [Exercice]

1. Under the Single-Index model (i.e.single factor model with market return on index), there are two distinct risks of a security, where risk is defined to be the standard deviation of returns over successive periods of time. Diversifiable (or specific) risk is the risk unique or peculiar to the security (e.g. industry, company) and generally can be eliminated by having a suitably diversified portfolio of shares. Non-diversifiable (or systematic) risk is related to the market as a whole and cannot be further reduced by diversification.

2. Under the Single-Index model, the expected return of a security is given by
$$E(R_i) = \alpha_i + \beta_i E(R_M).$$
Thus, one can see that the expected return from a share depends on the non-diversifiable risk, but not on the diversifiable risk. However, the variance does depend on both the level of non-diversifiable as well as diversifiable risk according to
$$Var(R_i) = \beta_i^2 \sigma_M^2 + \sigma_{\varepsilon_i}^2.$$

3. For well-diversified portfolios, there ought to be very little market or non-systematic risk. Thus, a portfolio with beta:
   - greater than 1 is said to be aggressive - the return on the portfolio is likely to be more risky than the return on the market, and
   - less than 1 is said to be defensive - the return on the portfolio is likely to be less risky than the return on the market.

Solution 2.17: [Exercice] According to the Single Index model, the return on security i is given by
$$R_i = \alpha_i + \beta_i R_M + \varepsilon_i$$
where α_i, β_i are constants, R_M is the return in the market, and ε_i is a random component with $E(\varepsilon_i) = 0$ and $Var(\varepsilon_i) = \sigma_\varepsilon^2$.

1. $E(R_i) = E(\alpha_i + \beta_i R_M + \varepsilon_i) = \alpha_i + \beta_i E(R_M) + E(\varepsilon_i) = \alpha_i + \beta_i E(R_M)$.

2. For the variance,
$$\begin{aligned} Var(R_i) &= Var(\alpha_i + \beta_i R_M + \varepsilon_i) = Var(\beta_i R_M + \varepsilon_i) \\ &= \beta_i^2 Var(R_M) + Var(\varepsilon_i) + 2\beta_i Cov(R_M, \varepsilon_i) \\ &= \beta_i^2 \sigma_M^2 + \sigma_\varepsilon^2, \end{aligned}$$
since $Cov(R_M, \varepsilon_i) = 0$ by assumption.

3. For $i \neq j$, the covariance is
$$\begin{aligned} Cov(R_i, R_j) &= Cov(\alpha_i + \beta_i R_M + \varepsilon_i, \alpha_j + \beta_j R_M + \varepsilon_j) \\ &= Cov(\beta_i R_M + \varepsilon_i, \beta_j R_M + \varepsilon_j), \text{ since } \alpha_i, \alpha_j \text{ are constants} \\ &= Cov(\beta_i R_M, \beta_j R_M) + Cov(\beta_i R_M, \varepsilon_j) + Cov(\varepsilon_i, \beta_j R_M) + Cov(\varepsilon_i, \varepsilon_j) \\ &= \beta_i \beta_j Cov(R_M, R_M) + \beta_i Cov(R_M, \varepsilon_j) + \beta_j Cov(\varepsilon_i, R_M) + Cov(\varepsilon_i, \varepsilon_j) \\ &= \beta_i \beta_j Cov(R_M, R_M) \\ &= \beta_i \beta_j \sigma_M^2, \end{aligned}$$
since by assumption, we have $Cov(R_M, \varepsilon_j) = Cov(\varepsilon_i, R_M) = Cov(\varepsilon_i, \varepsilon_j) = 0$.

Solution 2.18: [Exercice] Consider a portfolio of n securities which consists of returns from security i as R_i, $i = 1, 2, ..., n$, where we assume that each security follows the single-index model: $R_i = \alpha_i + \beta_i R_M + \varepsilon_i$. Let the weights for each security be denoted by a_i. Hence, the return on the portfolio is given by

$$R_p = w_1 R_1 + w_2 R_2 + ... + w_n R_n = \sum_{i=1}^{n} w_i R_i.$$

Denote the portfolio alpha, beta and the error by

$$\alpha_p = \sum_{i=1}^{n} w_i \alpha_i, \quad \beta_p = \sum_{i=1}^{n} w_i \beta_i, \quad \varepsilon_p = \sum_{i=1}^{n} w_i \varepsilon_i,$$

we have

$$R_p = \alpha_p + \beta_p R_M + \varepsilon_p.$$

Hence, the variance of the portfolio can be expressed as

$$\begin{aligned} Var[R_p] &= Var[\alpha_p + \beta_p R_M + \varepsilon_p] = Var[\beta_p R_M + \varepsilon_p] \\ &= Var[\beta_p R_M] + Var[\varepsilon_p] + 2Cov[\beta_p R_M, \varepsilon_p] \\ &= \beta_p^2 Var[R_M] + Var[\varepsilon_p] + 2\beta_p Cov[R_M, \sum_{i=1}^{n} w_i \varepsilon_i] \\ &= \beta_p^2 \sigma_M^2 + \sigma_{\varepsilon_p}^2 + 2\beta_p \sum_{i=1}^{n} w_i Cov[R_M, \varepsilon_i] \\ &= \beta_p^2 \sigma_M^2 + \sigma_{\varepsilon_p}^2 \end{aligned}$$

as $Cov[R_M, \varepsilon_i] = 0$ by assumption, where

$$\begin{aligned} \sigma_{\varepsilon_p}^2 &= Var[\varepsilon_p] = Var[\sum_{i=1}^{n} w_i \varepsilon] = \sum_{i=1}^{n} Var[w_i \varepsilon_i] \quad \text{(by assumption that } Cov[\varepsilon_i, \varepsilon_j] = 0 \text{ for } i \neq j) \\ &= \sum_{i=1}^{n} w_i^2 Var[\varepsilon_i] = \sum_{i=1}^{n} w_i^2 \sigma_{\varepsilon_i}^2 \end{aligned}$$

Therefore, the variance can be decomposed into two types of risks:

$$\sigma_p^2 = \beta_p^2 \sigma_M^2 + \sum_{i=1}^{n} w_i^2 \sigma_{\varepsilon_i}^2,$$

where the first term $\beta_p^2 \sigma_M^2$ consists of that attributable to *non-diversifiable risk* while the second term $\sum_{i=1}^{n} w_i^2 \sigma_{\varepsilon_i}^2$ consists of the component attributable to *diversifiable risk*.

Solution 2.19: [Exercice] First, it can be shown that

$$\sum_{t=1}^{10} \left(R_{i,t} - \overline{R}_i\right)\left(R_{M,t} - \overline{R}_M\right) = 254.30 \text{ and } \sum_{t=1}^{10} \left(R_{M,t} - \overline{R}_M\right)^2 = 362.10.$$

(a) The least squares parameter estimates are

$$\widehat{\beta}_i = \frac{s_{i,M}}{s_{M,M}} = \frac{\sum_{t=1}^{10} \left(R_{i,t} - \overline{R}_i\right)\left(R_{M,t} - \overline{R}_M\right)}{\sum_{t=1}^{10} \left(R_{M,t} - \overline{R}_M\right)^2} = \frac{254.30}{362.10} = 0.7023$$

and

$$\widehat{\alpha}_i = \overline{R}_i - \widehat{\beta}_i \overline{R}_M = 4.9 - 0.7023\,(4.3) = 1.8801.$$

(b) Recall from regression analysis (note: you are not required to memorise this formula for the class test), the variance of the security can be decomposed into

$$s_i^2 = \widehat{\beta}_i^2 s_M^2 + s^2 \left(\frac{N-2}{N-1} \right)$$

where the first term represents the component that is non-diversifiable and the last term that is diversifiable. Note that

$$s_M^2 = \frac{\sum_{t=1}^{10} \left(R_{M,t} - \overline{R}_M \right)^2}{9} = \frac{362.10}{9} = 40.2333$$

and

$$s_i^2 = \frac{\sum_{t=1}^{10} \left(R_{i,t} - \overline{R}_i \right)^2}{9} = \frac{410.90}{9} = 45.6556.$$

Since

$$s^2 \left(\frac{N-2}{N-1} \right) = s_i^2 - \widehat{\beta}_i^2 s_M^2 = 45.6556 - (0.7023)^2 \, 40.2333 = 45.6556 - 19.8441 = 25.8115,$$

the variance of the security ($= 45.6556$) can be decomposed into diversifiable ($= 25.8115$) and non-diversifiable ($= 19.8441$).

Proof of the regression formula:

1. Note $SST = SSR + SSE$.
2. Divide both sides by $N-1$ yields $\frac{SST}{N-1} = \frac{SSR}{N-1} + \frac{SSE}{N-1}$
3. Recall $SSR = \sum (\hat{y} - \bar{y})^2$ and $SSE = (N-2)MSE$, we have $\frac{SST}{N-1} = \frac{\sum(\hat{y}-\bar{y})^2}{N-1} + MSE\frac{N-2}{N-1}$
4. Recall $\hat{y} - \bar{y} = \beta(X - \bar{X})$, we have
$$\frac{SST}{N-1} = \frac{\sum(\beta(X-\bar{X}))^2}{N-1} + MSE\frac{N-2}{N-1}$$
which is precisely what the formula says.

Solution 2.20: [Exercice]

(L8.1)

The beta of a portfolio is a weighted combination of the individual betas

$$\beta = 0.2 \times 1.1 + 0.5 \times 0.8 + 0.3 \times 1 = 0.92$$

Hence applying CAPM to the portfolio we find

$$\mu_p = 0.05 + 0.92 \, (0.12 - 0.05) = 11.44\%$$

Using the single factor model we have

$$\begin{aligned} \sigma_e^2 &= \sum_{i=A}^{C} w_i^2 \sigma_{e_i}^2 = 0.00033725 \\ \sigma_p^2 &= b^2 \sigma_M^2 + \sigma_e^2 = 0.92^2 0.18^2 + 0.00033725 = 0.2776 \\ \sigma_p &= 16.7\% \end{aligned}$$

Solution 2.21: [Exercice]

1. Advantages for factor model
   - Factor models require less parameters to be estimated given there aren't too many factors, this also means the parameter estimates will have less variance
   - Doesn't require the assumption of market equilibrium nor homogeneous individuals and therefore it is less assumption-heavy.
   - Doesn't rely on utility assumptions
2. Disadvantages for factor model
   - Doesn't specify how many factors are suitable
   - Doesn't specify what factors are suitable so this can be a very difficult decision, CAPM specifies the market portfolio is optimal

Solution 2.22: [Exercice] According to the Single-Index model, the expected return on security i is given by

$$E(R_i) = \alpha_i + \beta_i E(R_M),$$

and the variance is given by

$$Var(R_i) = \beta_i^2 \sigma_M^2 + \sigma_\varepsilon^2$$

Here $i = A$, B, C and D and the market M is the S&P index. Note that the problem uses y and/or x to identify the security and the market. The least squares estimates for the parameters are given by

$$\widehat{\beta}_i = r_{i,M} \frac{s_i}{s_M} = \frac{S_{xy}}{S_{xx}}, \text{ with } y \text{ to denote the security and } x \text{ the market}$$

and

$$\widehat{\alpha}_i = \overline{R}_i - \widehat{\beta}_i \overline{R}_M.$$

An estimate of the variance for security i is

$$s_i^2 = \frac{S_{yy}}{11}.$$

1. It can be verified that the mean and variance of the return for each security are as summarized below:

Security	$\widehat{\beta}_i$	$\widehat{\alpha}_i$	Mean $E(R_i) = \overline{R}_i$	Variance $s_i^2 = \frac{S_{yy}}{11}$
A	$\frac{296.9104}{250.8953} = 1.1834$	-0.6103	2.946	$\frac{613.8439}{11} = 55.804$
B	$\frac{256.0504}{250.8953} = 1.02055$	2.9641	6.031	$\frac{559.2715}{11} = 50.843$
C	$\frac{582.4529}{250.8953} = 2.32150$	-3.4219	3.554	$\frac{3179.835}{11} = 289.076$

2. First, we need to estimate the covariances of the returns. Recall that the formula for the covariance in the Single-Index model is given by

$$Cov\left(R_i, R_j\right) = \beta_i \beta_j \sigma_M^2.$$

Thus, the estimated covariances are

$$Cov\left(R_A, R_B\right) = \widehat{\beta}_A \widehat{\beta}_B \widehat{\sigma}_M^2 = (1.1834)\,(1.02055)\left(\frac{250.8953}{11}\right) = 27.5465,$$

$$Cov\left(R_A, R_C\right) = \widehat{\beta}_A \widehat{\beta}_C \widehat{\sigma}_M^2 = (1.1834)\,(2.32150)\left(\frac{250.8953}{11}\right) = 62.6614,$$

and

$$Cov\left(R_B, R_C\right) = \widehat{\beta}_B \widehat{\beta}_C \widehat{\sigma}_M^2 = (1.02055)\,(2.32150)\left(\frac{250.8953}{11}\right) = 54.0384.$$

(Observe that these are the variance implied by the single factor model, which are not simple sample covariances)

The portfolio return can be written as

$$R_p = \frac{1}{3}\left(R_A + R_B + R_C\right)$$

so that its expected return is

$$E\left(R_p\right) = \frac{1}{3}\left[E\left(R_A\right) + E\left(R_B\right) + E\left(R_C\right)\right] = \frac{1}{3}\left(2.946 + 6.031 + 3.554\right) = 4.177$$

and its variance is

$$\begin{aligned} Var\left(R_p\right) &= = \frac{1}{9} Var\left(R_A + R_B + R_C\right) \\ &= \frac{1}{9}\left[\begin{array}{c} Var\left(R_A\right) + Var\left(R_B\right) + Var\left(R_C\right) \\ +2Cov\left(R_A, R_B\right) + 2Cov\left(R_A, R_C\right) + 2Cov\left(R_B, R_C\right) \end{array}\right] \\ &= \frac{1}{9}\left[\begin{array}{c} 55.804 + 50.843 + 289.076 + 2\,(27.5465) \\ +2\,(62.6614) + 2\,(54.0384) \end{array}\right] = 76.024. \end{aligned}$$

Thus, $\sigma_p = \sqrt{76.024} = 8.7192$.

Solution 2.23: [Exercice] With

$$\begin{aligned} r_{i,t} &= \alpha_i + \beta_{i,1} f_{1,t} + \ldots + \beta_{i,K} f_{K,t} + \varepsilon_t \\ &= [1, f_{1,t}, \ldots, f_{K,t}]\gamma_i + \varepsilon_t, \end{aligned}$$

where $r_{i,t}$ is the observation (data) of the return at time t and $f_{j,t}$ is the obsercation (data) of the factor f_j at time t.

Write out all the $r_{i,t}$, $t = 1, \ldots, T$ in a column, we have

$$r_i = f\gamma_i + \varepsilon$$

where $r_i = [r_{i,1}, \ldots, r_{i,T}]'$ and $\varepsilon = [\varepsilon_1, \ldots, \varepsilon_T]'$.

We wish to choose (the best) γ_i such that the squared error $\varepsilon'\varepsilon$ is minimised. Note

$$\begin{aligned}\varepsilon'\varepsilon &= (r_i - f\gamma_i)'(r_i - f\gamma_i)\\ &= r_i'r_i - \gamma_i'f'r_i - r_i'f\gamma_i + \gamma_i'f'f\gamma_i \quad \text{(by the transpose law)}\\ &= \gamma_i'f'f\gamma_i - 2\gamma_i'f'r_i + r_i'r_i \quad \text{(as } \gamma_i'f'r_i \text{ is a scalar)}\end{aligned}$$

Hence, to minimise $\varepsilon'\varepsilon$ is the same as to minimise w.r.t. γ_i the following function

$$F(\gamma_i) = \frac{1}{2}\gamma_i'f'f\gamma_i - \gamma_i'f'r_i$$

as $r_i'r_i$ is independent of γ_i and we scale the objective by half. Note that F is quadratic in γ_i and hence the minimum is achieved when $\frac{\partial F}{\partial \gamma_i} = 0$, i.e.

$$\begin{aligned} 0 &= \frac{\partial}{\partial \gamma_i}F(\gamma_i) = f'f\gamma_i - f'r_i\\ \iff & f'f\gamma_i = f'r_i\\ \iff & \gamma_i = (f'f)^{-1}f'r_i\end{aligned}$$

Hence, we have $\hat{\gamma}_i = (f'f)^{-1}f'r_i$.

Solution 2.24: [Exercice] For mean-variance approach if we have N assets then we require:

1. N means
2. N variances
3. $\frac{N(N-1)}{2}$ covariances - consider the N by N variance-covariance matrix, subtract the N variances and you will be left with twice the covariances so just half this

So in total this approach will require $\frac{N^2}{2} + \frac{3N}{2}$ parameters.

For a K-factor model we require:

1. the N α values - one for each asset
2. the NK β values - one for each asset and factor
3. the N $\sigma^2_{\varepsilon_i}$ values - one for each asset
4. the K μ_{f_i} factor means - one for each factor
5. the K $\sigma^2_{f_i}$ factor variances - one for each factor
6. the $\frac{K(K-1)}{2}$ factor covariances

So in total the K-factor model will require $\frac{K^2}{2} + 2N + NK + \frac{3K}{2}$ parameters.
Comparing with the number of parameters of the mean variance, the order here is $K^2 + NK$ instead of N^2, which is generally much smaller, as K is small and N is large.

2.5.3 Arbitrage Pricing Theory

Solution 2.25: [Exercice]

1. To find $\lambda_0, \lambda_1, \lambda_2$ you need to setup a system of three equations to get the following:

$$\begin{aligned} \lambda_0 + \lambda_1 + 2.5\lambda_2 &= 0.1 \\ \lambda_0 + 0.9\lambda_1 + 3\lambda_2 &= 0.075 \\ \lambda_0 + 1.2\lambda_1 + 1.7\lambda_2 &= 0.13 \end{aligned}$$

   Denote

$$B = \begin{bmatrix} 1 & 1 & 2.5 \\ 1 & 0.9 & 3 \\ 1 & 1.2 & 1.7 \end{bmatrix}, \text{ we have } B^{-1} = \frac{1}{\Delta}\begin{bmatrix} -2.07 & 1.3 & 0.75 \\ 1.3 & -0.8 & -0.5 \\ 0.3 & -0.2 & -0.1 \end{bmatrix}$$

   with $\Delta = -0.02$. we get $[\lambda_0, \lambda_1, \lambda_2]^T = B^{-1}[0.1, 0.075, 0.13]^T = [0.6, -0.25, -0.1]^T$.

2. Now according to (a), the APT estimate for $E(r_Z)$ is:

$$\begin{aligned} E(r_Z) &= 0.6 - 0.25 \cdot 1.5 - 0.1 \cdot 2 \\ &= 0.025 \end{aligned}$$

   Since the market return is much higher than that predicted by the APT model, we know that Security Z is being underpriced.

   Denote

$$\begin{bmatrix} w_0 \\ e_1 \\ e_2 \end{bmatrix} = B^{-1} = \frac{1}{-0.02}\begin{bmatrix} -2.07 & 1.3 & 0.75 \\ 1.3 & -0.8 & -0.5 \\ 0.3 & -0.2 & -0.1 \end{bmatrix},$$

   the replicate the portfolio is given by

$$\begin{aligned} Z_2 &= w_0 + 1.5e_1 + 2e_2 \\ &= \frac{1}{-0.02}[0.48, -0.3, -0.2] \\ &= [-24, 15, 10]. \end{aligned}$$

   Note

   (i) the exposure of f_1 is $\beta_{Z_2,1} = -24 + 0.9 \times 15 + 1.2 \times 10 = 1.5$,

   (ii) the exposure of f_2 is $\beta_{Z_2,2} = 2.5 \times -24 + 3 \times 15 + 1.7 \times 10 = 2$,

   (iii) and the alpha for Z_2 is $\alpha_{Z_2} = E[Z_2] = 0.1 \times -24 + 0.075 \times 15 + 0.13 \times 10 = 0.025$.

   Hence, by buying Z and selling Z_2, with probability 1, we get a return of

$$\alpha_Z - \alpha_{Z_2} = E[Z] - E[Z_1] = 0.1 - 0.025 > 0$$

   on zero investment, i.e. an arbitrage!

Solution 2.26: [Exercice] So under APT we know that the expected return of any portfolio is given by:

$$
\begin{aligned}
E[r_A] &= r_f + \beta_{A,1}\lambda_1 + \beta_{A,2}\lambda_2 \\
0.1 &= 0.05 + 1.2\lambda_1 + 0.5\lambda_2 \\
E[r_B] &= r_f + \beta_{B,1}\lambda_1 + \beta_{B,2}\lambda_2 \\
0.15 &= 0.05 + 3\lambda_1 - 0.2\lambda_2
\end{aligned}
$$

Now solving these two simultaneous equations for λ_1, λ_2 we get:

$$
\begin{aligned}
\lambda_1 &= \frac{1}{29} \\
\lambda_2 &= \frac{1}{58}
\end{aligned}
$$

Then the expected return for an arbitrary portfolio under APT is:

$$E[r_X] = 0.05 + \beta_{X,1}\frac{1}{29} + \beta_{X,2}\frac{1}{58}$$

Solution 2.27: [Exercice] Using APT we have:

$$
\begin{aligned}
E[r_1] &= \lambda_0 + 3\lambda_1 + \lambda_2 \\
E[r_2] &= \lambda_0 + \lambda_1 + 2\lambda_2 \\
E[r_3] &= \lambda_0 \\
\implies \lambda_0 &= 0.07 \\
0.12 &= 0.07 + 3\lambda_1 + \lambda_2 \\
0.15 &= 0.07 + \lambda_1 + 2\lambda_2
\end{aligned}
$$

Hence, solving the above two simultaneous equations we get:

$$
\begin{aligned}
\lambda_1 &= 0.004 \\
\lambda_2 &= 0.038 \\
E[r_4] &= 0.07 + 5 \cdot 0.004 - 2 \cdot 0.038 \\
&= 0.014
\end{aligned}
$$

Solution 2.28: [Exercice] Let us first form some portfolio with asset i and j:

$$
\begin{aligned}
r_p &= wr_i + (1-w)r_j \\
&= w\alpha_i + w\beta_i f + (1-w)\alpha_j + (1-w)\beta_j f \\
&= \alpha_j + w(\alpha_i - \alpha_j) + (w\beta_i + (1-w)\beta_j)f
\end{aligned}
$$

Now to get a risk-less portfolio we need there to be no random factor exposure, so find w such that f disappears

$$
\begin{aligned}
w\beta_i + (1-w)\beta_j &= 0 \\
w(\beta_i - \beta_j) + \beta_j &= 0 \\
w &= -\frac{\beta_j}{\beta_i - \beta_j}
\end{aligned}
$$

Now since this portfolio is risk-less, the return of this portfolio must the the same for any i, j. Denote this number as λ_0 we have

$$\begin{aligned}
\lambda_0 &= -\frac{\beta_j}{\beta_i - \beta_j}(\alpha_i - \alpha_j) + \alpha_j \\
\lambda_0\beta_i - \lambda_0\beta_j &= -\beta_j\alpha_i + \beta_j\alpha_j + \beta_i\alpha_j - \beta_j\alpha_j \\
\beta_j\alpha_i - \lambda_0\beta_j &= \beta_i\alpha_j - \beta_i\lambda_0 \\
\frac{\alpha_i - \lambda_0}{\beta_i} &= \frac{\alpha_j - \lambda_0}{\beta_j} \text{ (for any i and j)}
\end{aligned}$$

Denote this as λ_1, we have

$$\begin{aligned}
\lambda_1 &= \frac{\alpha_i - \lambda_0}{\beta_i} \\
\implies \alpha_i &= \lambda_0 + \beta_i\lambda_1.
\end{aligned}$$

Finally, $E[f] = 0 \implies E[r_i] = \alpha_i$ gives

$$E[r_i] = \lambda_0 + \beta_i\lambda_1$$

as required.

Solution 2.29: [Exercice] (L8.2)

By the APT we have $\lambda_0 = r_f = 10\%$ and

$$\begin{aligned}
0.15 &= 0.10 + 2\lambda_1 + \lambda_2 \\
0.2 &= 0.10 + 3\lambda_1 + 4\lambda_2
\end{aligned}$$

so

$$\begin{aligned}
\lambda_1 &= 0.02 \\
\lambda_2 &= 0.01
\end{aligned}$$

Solution 2.30: [Exercice] APT (or Arbitrage Pricing Theory) gives

$$E(R_i) = \lambda_0 + \sum_{k=1}^{n} \beta_{i,k}\lambda_k,$$

where $\beta_{i,k}$ refers to the sensitivity of the factor k, λ_k is the risk premium, and λ_0 is generally the return when risk premiums are zero (i.e. the risk-free rate of return).

1. Thus, for security A,

   $$E(R_A) = \lambda_0 + \beta_{A,1}\lambda_1 + \beta_{A,22}\lambda_2 + \beta_{A,3}\lambda_3 = .06 + (0 \times .03) + (0 \times .05) + (2 \times .09) = 24\%,$$

   and for security B, we have

   $$\begin{aligned}
   E(R_B) &= \lambda_0 + \beta_{B,1}\lambda_1 + \beta_{B,2}\lambda_2 + \beta_{B,3}\lambda_3 \\
   &= 0.06 + (0.75 \times 0.03) + (0.75 \times 0.05) + (0.75 \times 0.09) \\
   &= 18.75\%.
   \end{aligned}$$

2. For a portfolio of 75% security A and 25% security B, the return is

$$R_p = 0.75R_A + 0.25R_B$$

so that its expected return is

$$\begin{aligned}
E(R_p) &= 0.75E(R_A) + 0.25E(R_B) \\
&= 0.75\left(\lambda_0 + \sum_{k=1}^{n} \beta_{A,k}\lambda_k\right) + 0.25\left(\lambda_0 + \sum_{k=1}^{n} \beta_{A,k}\lambda_k\right) \\
&= (0.75\lambda_0 + 0.25\lambda_0) + \left(0.75\sum_{k=1}^{n} \beta_{A,k}\lambda_k + 0.25\sum_{k=1}^{n} \beta_{A,k}\lambda_k\right) \\
&= (0.75\lambda_0 + 0.25\lambda_0) + \sum_{k=1}^{n} (0.75\beta_{A,k} + 0.25\beta_{B,k})\lambda_k \\
&= \lambda_0 + \sum_{k=1}^{n} \beta_{p,k}\lambda_k
\end{aligned}$$

where the sensitivity to the factor is $\beta_{p,k} = 0.75\beta_{A,k} + 0.25\beta_{B,k}$ for $k = 1, 2, 3$. Thus, these sensitivities are

$$(0.75\beta_{A,1} + 0.25\beta_{B,1}) = 0.75(0) + 0.25(0.75) = 0.1875$$

for Factor 1,

$$(0.75\beta_{A,2} + 0.25\beta_{B,2}) = 0.75(0) + 0.25(0.75) = 0.1875$$

for Factor 2, and

$$(0.75\beta_{A,3} + 0.25\beta_{B,3}) = 0.75(2) + 0.25(0.75) = 1.6875$$

for Factor 3.

Solution 2.31: [Exercice] You can:

- Select factors based on economic reasoning and intuition
- Regress stock returns against a large number of factors and use those that are statistically significant, factor analysis, principal component analysis
- Select factors with high correlation with the dependent variable (not among themselves)
- Select factors related to the underlying drivers of stock returns

Things that need to be considered:

- Statistically significant factors may not make any economic or intuitive sense
- Factors selected on economic reasoning/intuition may not be reflected very well i.e perhaps there are other factors that we do not consider/know that has a greater influence on returns
- Correlation and causation are not the same, factors with high correlation doesn't necessarily provide a good explanation of the returns (it may just be a coincidence)

- Factors related to the underlying drivers may not be available

Solution 2.32: [Exercice]

(i) By the definition of B^{-1}, we have

$$\begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} = I = B^{-1}B = \begin{bmatrix} w_0 \\ e_1 \\ \vdots \\ e_{n-1} \end{bmatrix} \begin{bmatrix} 1 & * & * & \dots & * \\ 1 & * & * & \dots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & * & * & \dots & * \end{bmatrix} = \begin{bmatrix} 1'w_0' & * & * & \dots & * \\ 1'e_1' & * & * & \dots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1'e_{n-1}' & * & * & \dots & * \end{bmatrix},$$

with $*$ denotes some numbers. Compare the first column, we get the result.

(ii) Using the result in (i), we have

$$1'w_b' = 1'w_0' + \sum_k 1'e_k' = 1 + \sum_k 0 = 1.$$

For the exposure, note that the exposure vector of a portfolio $w = (w_1, ..., w_n)$ (with $\sum_i w_i = 1$) is wB by linearity (you can play with some numbers to convince yourself). Moreover, we can write the portfolio w_b as

$$[1, b_1, ..., b_{n-1}] \begin{bmatrix} w_0 \\ e_1 \\ \vdots \\ e_{n-1} \end{bmatrix} = [1, b_1, ..., b_{n-1}]B^{-1}.$$

Hence, the exposure vector of w_b is

$$w_b B = [1, b_1, ..., b_{n-1}]B^{-1}B = [1, b_1, ..., b_{n-1}],$$

as required.

(iii) The expected return of w_b is given by

$$w_b\mu = [1, b_1, ..., b_{n-1}]B^{-1}\mu = [1, b_1, ..., b_{n-1}]\lambda$$

with $\lambda = B^{-1}\mu$. If we write out λ explicity as $\lambda = [\lambda_0, \lambda_1, ..., \lambda_{n-1}]'$ we have

$$E[r_{w_b}] = \lambda_0 + b_1\lambda_1 + ... + b_{n-1}\lambda_{n-1}$$

where r_{w_b} is the (random) return of the portfolio w_b. This is the APT equation.

(iv) Using the factor model expression of y_b we have

$$\begin{cases} r_{y_b} = \alpha_{y_b} + b_1 f_1 + ... + b_{n-1} f_{n-1} \\ r_{w_b} = \alpha_{w_b} + b_1 f_1 + ... + b_{n-1} f_{n-1} \end{cases}.$$

We now form a portfolio, say X, by longing y_b and shorting w_b. Then the return on X is

$$r_X = r_{y_b} - r_{w_b} = \alpha_{y_b} - \alpha_{w_b}.$$

Note we do not have any cash flow at inception ($t = 0$) as we buy y_b and sell w_b at the same time. Unwinding the portfolio at the end of the period (say $t = 1$), the cash flow would be $\alpha_{y_b} - \alpha_{w_b}$. If $\alpha_{y_b} > \alpha_{w_b}$, by longing X we have a positive return with probability 1. Similarly, if $\alpha_{y_b} < \alpha_{w_b}$, by shorting X, we again have a positive return with probability 1. This is arbitrage. Hence, no-arbitrage assumption implies that $\alpha_{y_b} = \alpha_{w_b}$, which implies

$$E[r_{y_b}] = E[r_{w_b}] = \lambda_0 + b_1\lambda_1 + ... + b_{n-1}\lambda_{n-1}$$

with λ_k's given in (iii).

[Note the whole agrument is on the random return r, not the expected return $E[r]$.]

[Note also the assumption that $E[f_k] = 0$ is not needed.]

2.5.4 Model Fitting and Efficient Market Hypothesis

Solution 2.33: [Exercice] Not necessarily, as we need to compare risk adjusted returns. The extra average return could be due to higher risks taken. (recall the β in CAPM.)

Solution 2.34: [Exercice]

Assuming the observed annually returns X_i, $i = 1, 2, ..., n$ are normally distributed, we have

$$P(|\bar{X} - \mu| < 0.1\sigma) = P(|\frac{\bar{X} - \mu}{\sigma}| < 0.1) = 0.95.$$

Note we have $\frac{\bar{X}-\mu}{\sigma} \sim N(0, \frac{1}{n})$, implies that $P(|\bar{X} - \mu| < 0.1\sigma) = P(|Z| < 0.1\sqrt{n}) = 0.95$ which further implies that

$$0.1\sqrt{n} = 1.96.$$

This yields $n = (1.96/0.1)^2 \approx 400$ years, much longer than a human lifetime!

Solution 2.35: [Exercice] There are three forms of the Efficient Market Hypothesis:

- Weak form efficient - all the past price history of a stock is already incorporated into the current stock price
- Semi-strong form efficient - all the relevant public information is already incorporated into the current stock price
- Strong form efficient - all relevant information is already incorporated into the current stock price

1. EMH goes against this
2. EMH supports this so passive trading will work
3. Weak form efficiency goes against this
4. Semi-strong form efficiency goes against this
5. Strong form efficient goes against this

Solution 2.36: [Exercice] Note by direct expansion we have

$$\sum_{i=1}^{n}(X_i - \bar{X})^2 = \sum_{i=1}^{n} X_i^2 - n\bar{X}^2.$$

Next, we replace X_i by $X_i - \mu$ ($\mu = E[X]$) in the above to obtain

$$\sum_{i=1}^{n}(X_i - \bar{X})^2 = \sum_{i=1}^{n}((X_i - \mu) - (\bar{X} - \mu))^2 = \sum_{i=1}^{n}(X_i - \mu)^2 - n(\bar{X} - \mu)^2.$$

Now, by taking expectation, we get

$$\begin{aligned} E[\sum_{i=1}^{n}(X_i - \bar{X})^2] &= \sum_{i=1}^{n}(X_i - \mu)^2 - nE[(\bar{X} - \mu)^2] \\ &= nVar(X_i) - nVar(\bar{X}) \\ &= n\sigma^2 - n\frac{\sigma^2}{n} \\ &= (n-1)\sigma^2 \end{aligned}$$

Therefore, we have $E[s^2] = \frac{1}{n-1}E[\sum_{i=1}^{n}(X_i - \bar{X})^2] = \sigma^2$, i.e. unbiased.

Solution 2.37: [Exercice]

1.
$$\begin{aligned} \hat{\bar{\gamma}}_m &= \frac{1}{n}\sum_{i=1}^{n}\gamma_i = 1\% \\ \hat{\bar{\gamma}}_{yr} &= 12\hat{\bar{\gamma}}_m = 12\%. \end{aligned}$$

2.
$$\begin{aligned} \hat{\sigma}_m^2 &= \frac{1}{n-1}\sum_{i=1}^{n}\left(\gamma_i - \hat{\bar{\gamma}}_m\right)^2 = 0.00072 \\ \hat{\sigma}_{yr} &= \sqrt{12}\hat{\sigma}_m = 9.29\%. \end{aligned}$$

3.
$$\begin{aligned} se\left(\hat{\bar{\gamma}}_m\right) &= \sqrt{\widehat{Var[\hat{\bar{\gamma}}_m]}} = \sqrt{\frac{\hat{\sigma}_m^2}{n}} = \frac{\hat{\sigma}_m}{\sqrt{n}} = 0.55\% \\ se\left(\hat{\bar{\gamma}}_{yr}\right) &= \sqrt{\widehat{Var[\hat{\bar{\gamma}}_{yr}]}} = \sqrt{Var[12\hat{\bar{\gamma}}_m]} = 12\sqrt{Var[\hat{\bar{\gamma}}_m]} = 12 \times 0.55\% = 6.6\% \\ se\left(\hat{\sigma}_m^2\right) &= \sqrt{Var[\widehat{Var[\hat{\bar{\gamma}}_m]}]} = \sqrt{\frac{2\hat{\sigma}_m^4}{n-1}} = \frac{\sqrt{2}\hat{\sigma}_m^2}{\sqrt{n-1}} = \frac{\sqrt{2}\times 0.00072}{\sqrt{23}} = 0.00021 \\ se\left(\hat{\sigma}_{yr}^2\right) &= se\left(12 \times \hat{\sigma}_m^2\right) = 12 \times se\left(\hat{\sigma}_m^2\right) = 0.0025. \end{aligned}$$

[Note standard error is the estimate of the standard deviation as we replace the variance σ_m^2 by $\hat{\sigma}_m^2$ estimated from data.]

4. From the previous exercise we know that the estimate of $\bar{\gamma}$ will not be improved by having weekly, rather than monthly samples. All that matters is the total length of the period that is observed. However, the estimate of σ^2 can be improved. In fact, by letting $se^{week}\left(\sigma_{yr}^2\right)$ denote the standard error in $\hat{\sigma}_{yr}^2$ based on weekly data, we expect that $se^{week}\left(\hat{\sigma}_{yr}^2\right) = \sqrt{\frac{23}{104}}se\left(\hat{\sigma}_{yr}^2\right) = 0.47\hat{\sigma}_{yr}^2$ =0.0012.

Solution 2.38: [Exercice]

1.
$$\sigma\left(\hat{\bar{\gamma}}\right) = \sqrt{Var[\hat{\bar{\gamma}}]} = \sqrt{Var[n\hat{\bar{\gamma}}_n]} = n\sqrt{Var[\hat{\bar{\gamma}}_n]} = \sqrt{n}\sigma_n = \sqrt{n}\frac{\sigma}{\sqrt{n}} = \sigma.$$

   Hence $\sigma\left(\hat{\bar{\gamma}}\right)$ is independent of n.

2. Assuming normality,
$$\sigma\left(\hat{\sigma}^2\right) = \sigma\left(n\hat{\sigma}_n^2\right) = n\frac{\sqrt{2}\sigma_n^2}{\sqrt{n-1}} = n\frac{\sqrt{2}}{\sqrt{n-1}}\frac{\sigma^2}{n} = \frac{\sqrt{2}\sigma^2}{\sqrt{n-1}}.$$

   Part (a) shows that by using smaller periods to get more samples does not improve the estimate of $\bar{\gamma}$. Part (b) shows that using smaller peridos to get more samples does improve the estimate of σ^2.

Solution 2.39: [Exercice] First divide the year into half-month intervals and index these time points by i. Let γ_i be the return over the i-th full month (but some will start midway through the month). We let $\overline{\gamma} = \mu_{\frac{1}{12}}$ and $\sigma^2_{\frac{1}{12}}$ denote the monthly expected return and variance of that return.

Now let ρ_i be the return over the i-th half-month period. Assume that these returns are uncorrelated. Then $\bar{\rho}_i = \mu_{\frac{1}{24}} = \frac{\mu_{\frac{1}{12}}}{2}$ and $Var[\rho_i] = \sigma^2_{\frac{1}{24}} = \frac{\sigma^2_{\frac{1}{12}}}{2}$. The return over any monthly period is a sum of two half-month returns; that is, the monthly return γ_i is $\gamma_i = \rho_i + \rho_{i+1}$. It is easy to see that $cov\left(\gamma_i, \gamma_{i+1}\right) = \frac{1}{2}\sigma^2_{\frac{1}{12}}$ and $cov\left(\gamma_i, \gamma_j\right) = 0$ for $|i-j| > 1$.

Now for Gavin's scheme we form the estimate

$$\hat{\mu}_g = \frac{1}{24}\sum_{i=1}^{24}\gamma_i.$$

We need to evaluate

$$\begin{aligned}
Var[\hat{\mu}_g] &= E\left[\left(\hat{\mu}_g - \mu\right)^2\right] \\
&= E\left[\left(\frac{\sum_{i=1}^{24}\gamma_i}{24} - \mu\right)^2\right] \\
&= \frac{1}{24^2}E\left[\left(\sum_{i=1}^{24}\left(\gamma_i - \mu\right)\right)^2\right] \\
&= \frac{1}{24^2}\sum_{i,j=1}^{24} cov\left(\gamma_i, \gamma_j\right) \\
&= \frac{1}{24^2}\sum_{i=1}^{24}\left[cov\left(\gamma_{i-1}, \gamma_i\right) + cov\left(\gamma_i, \gamma_i\right) + cov\left(\gamma_i, \gamma_{i+1}\right)\right].
\end{aligned}$$

Except at the end periods, each i will give three terms as shown. We will ignore the slight discrepancy at the ends and assume that every i gives the three terms as shown in the summation.

The terms are $\frac{1}{2}\sigma^2_{\frac{1}{12}}$, $\sigma^2_{\frac{1}{12}}$, and $\frac{1}{2}\sigma^2_{\frac{1}{12}}$, respectively. Hence we have

$$Var[\hat{\mu}_g] = \frac{1}{24^2} \times 24 \left(\frac{1}{2} + 1 + \frac{1}{2} \right) \sigma^2_{\frac{1}{12}}$$

which is identical to the result for twelve nonoverlapping months of data. (2nd equation on page 215)

Solution 2.40: [Exercice] If we assume that the samples are taken from $N(\mu, \sigma^2)$, then using a well known result in statistics we have

$$\begin{aligned} \frac{(n-1)\hat{\sigma}^2}{\sigma^2} &\sim \chi^2_{n-1} \\ Var(\frac{(n-1)\hat{\sigma}^2}{\sigma^2}) &= Var(\chi^2_{n-1}) \\ \frac{(n-1)^2}{\sigma^4} Var(\hat{\sigma}^2) &= 2(n-1) \\ \implies Var(\hat{\sigma}^2) &= \frac{2\sigma^4}{n-1} \end{aligned}$$

Module 3

3.1 Derivative Securities

3.1.1 Tips

This idea of this module is to introduce the "no-arbitrage" argument. In particular, we have the "ranking principle":

If payoff A is always greater than payoff B, then the time 0 price of payoff A has to be greater than that of payoff B.

Therefore, we need to understand the payoff diagram. For example, we should think of how an asset evolves through time, e.g.

1. $S(0) \to S(T)$
2. $K \to Ke^{rT}$

Therefore, we should "discount" K to Ke^{-rT} and $S(T)$ to $S(0)$.

["Discount" in a sense of the amount in the portfolio needed to replicate the payoff]

Below are some examples:

- Put-Call parity: Note by holding call and selling put at same strike and maturity, the payoff is $S(T) - K$, hence by "discounting" we have

$$C(K,T) - P(K,T) = S(0) - Ke^{-rT}$$

- Bounds for European call: Note the payoff of European call is $(S(T) - K)_+$, which is obviously sandwiched by $S(T) - K$ and $S(T)$. Hence by "discounting" we have

$$S(T) - K \leq (S(T) - K)_+ \leq S(T) \to S(0) - Ke^{-rT} \leq C(K,T) \leq S(0).$$

  In case the inequality does not hold, we buy the cheap and sell the expensive, which guarantee us to have a profit at time 0. By holding such portfolio, it is clear that our payoff is always non-negative. This is an arbitrage. The same logic applies to put.

- European call is the same as American call for non-dividend-paying stocks. This means early exercise is not desired. Suppose an individual exercises at time $t < T$, then the payoff at t is $S(t) - K$. However, if exercise at time T, the payoff would have been

$S(T) - K$ at time T. By "discounting", $S(T) - K$ at time T worths $S(t) - Ke^{-r(T-t)}$ at time t, which is greater than $S(t) - K$. Hence, one will never exercise the American call earlier. If the stock price is high, he/she will borrow (short sell) a stock and pay the stock later from exercising the call at T.

- Bounds for American Put: Note the payoff at time $t \le T$ for American put option is $(K - S(T))_+ \le K$. Upon discounting, we have $p(K,T) \le Ke^{-rt} \le K$, which is the upper bound. When exercising at time 0, the payoff is $K - S(0)$ which is the low bound for the price. Therefore, we have
$$K - S(0) \le p(K,T) \le K.$$

3.1.2 Practice Questions

Exercise 3.1: [Solution]

1. Explain the difference between a "European" and an "American" option.
2. Give reasons why an individual might wish to purchase a call option.

Exercise 3.2: [Solution]

1. Explain what is meant by the "intrinsic value" of an option.
2. Briefly explain what is meant by in-the-money, at-the-money, and out-of-the-money.
3. Sketch the position diagram for a European call option with premium c and exercise price K. In this diagram, mark clearly the regions where the option is in-the-money, at-the-money, and out-of-the-money.

Exercise 3.3: [Solution] Identify the profit or loss, ignoring of course any trading costs, in each of the following scenarios:

1. A call option with an exercise price of 480 is bought for a premium of 37. The price of the underlying share is 495 at the expiry date.
2. A put option with an exercise price of 180 is bought for a premium of 12. The price of the underlying share is 150 at the expiry date.
3. A put option with an exercise price of 250 is written for a premium of 22. The price of the underlying share is 272 at the expiry date.

Exercise 3.4: [Solution] A European call option and a European put option each have a strike price of $105.

1. Suppose share price at expiry is assumed to be uniformly distributed between 90 and 110. Calculate the expected payoffs for both these options.
2. Suppose share price at expiry is assumed to be uniformly distributed between 80 and 120. Calculate the expected payoffs for both these options
3. Comment on the difference between the results above.

4. Calculate the difference between the expected payoff of the put and the expected payoff of the call in (a) and (b) above. Comment on your anwers.

Exercise 3.5: [Solution] A three-month European call option with an exercise price of 500 on a share whose current price is 480 is currently priced at 42.

1. What would you expect the price to be for a 3-month put option with the same exercise price if the risk-free interest rate is 6% compounded continuously and no dividends are payable during the life of the option?

2. What assumptions did you have to make to come up with the answer above?

Exercise 3.6: [Solution] How would you expect the price of a European put option on a non-dividend paying share to change (justify your answers) if:

1. the current share price fell, and

2. there was a sudden increase in the risk-free rate of interest.

Exercise 3.7: [Solution] Luenberger (1Ed) Q12.1. (2Ed)14.1

Exercise 3.8: [Solution] Sketch the payoff diagram and explain when an investor would want this payoff for each of the following strategies:

1. Butterfly spread (long call at strike K_1, short two calls at K_2 and long a call at K_3, $K_1 < K_2 < K_3$).

2. Bull spread (long call at strike K_1 short call at K_2, $K_1 < K_2$)

3. Long straddle (long call and put at strike K_1)

Exercise 3.9: [Solution] Suppose we have an European call option that is currently outside of its bounds i.e $c(K, t) < S_t - Ke^{-r(T-t)}$. Explain how you can use this to make an arbitrage profit, clearly describe the portfolio you will need to make.

3.1.3 Discussion Questions

Exercise 3.10: [Solution] Luenberger, (1Ed) Q12.4 (2Ed) 14.4

Exercise 3.11: [Solution] Put-Call Parity:

1. State what is meant by put-call parity.

2. Derive an expression for the put-call parity of a European option that has a dividend payable prior to the exercise date.

3. If the equality in (b) does not hold, explain how an arbitrageur can make a riskless profit.

Exercise 3.12: [Solution] The table below shows the closing prices (symbolized by letters) on a particular day for a series of American[1] call options with different strike prices and expiry dates on a particular risky non-dividend paying security.

Call Option Prices		
	strike price	
expiry date	\$125	\$150
3 months	w	y
6 months	x	z

1. Write down, with justification, the strictest inequalities that can be deduced for the relative values of w, x, y, and z, assuming that the market is arbitrage-free. Your inequalities should not involve any other quantities.
2. Write down the numerical values for a lower and an upper bound for x given that the current share price is \$120 and the continuously-compounded annual risk-free interest rate is 6%.

Exercise 3.13: [Solution] Derive the option boundaries for the European put:

Option	Lower Bound	Upper Bound
European put	$p_(K,t) \geq Ke^{-r(T-t)} - S_t$	$p_{K,t} \leq Ke^{-r(T-t)}$

Exercise 3.14: [Solution] Recall that the prices of call and put options are related by the put-call parity: $c_t + Ke^{-r(T-t)} = p_t + S_t$ and should be true at all times to avoid arbitrage. Now suppose that for some reason one of the options (we do not know if its the put or the call) is mispriced so the put-call parity is not true. Explain how you could use this situation to earn an arbitrage profit.

Exercise 3.15: [Solution] Suppose we have an European option that has the following payoff at maturity:

$$X = max(min(100 - max(S_T - 20, 0), 50), 0)$$

1. Is this option in/at/out of the money if the underlying is \$70 at maturity?
2. Simplify the payoff function?

3.2 Discrete Time Financial Modelling

3.2.1 Tips

1. This module concerns replicating portfolio in binomial tree. Generally speaking, we buy ϕ stock and ψ bond to hedge the payoff. It suffices to consider only 1 step as we can

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work backwards. Consider $S \to \{uS, dS\}$ with payoff $X = \{p_u, p_d\}$ and $B \to e^r B$ then by considering $q = \frac{e^r - d}{u-d}$ as the "up" Q-probability, the price of the payoff is $P_X = e^{-r} E_Q[X]$. To replicate the portfolio, we always have $\phi = \frac{p_u - p_d}{uS - dS}$, (i.e. the option delta) and $\psi = \frac{p_u - \phi uS}{e^r B}$ (or equivalently $\psi = \frac{p_d - \phi dS}{e^r B}$). Note however in each step the stock and bond prices may change! It is always a good habit to write out the S an B at each time for replicating.

2. In general, we do not need to go backward step by step, unless the payoff function is not at maturity (e.g. American option when you can exercise early). Note the randomness in this module is the path, which can be for example (u, u, u) or (u, d, u). If the payoff is at maturity, to take the Q expectation, we can simply collect all paths and do the E at one go.

3.2.2 Practice Questions

Exercise 3.16: [Solution] The price of a non-dividend paying stock at time 1, S_1, is related to the price at time 0, S_0, as follows:

$$S_1 = \begin{cases} uS_0, & \text{w.p. } p \\ dS_0, & \text{w.p. } 1-p \end{cases}.$$

The continuously compounded rate of return on a risk-free asset per period is r.

1. Derive an expression for the replicating portfolio for a European call option written on the stock that expires at time 1 and has a strike price of K, where $dS_0 < K < uS_0$.

2. Show that the price of the option in (a) can be written as the discounted expected payoff under a probability measure Q. Hence, find an expression for the probability q of an upward move in the stock price under Q.

Exercise 3.17: [Solution] Luenberger, Section 12/14.6 describes a binomial tree that is 're-combining', i.e., an up move followed by a down move arrives at a stock price that is the same as if the order of up and down were reversed.

1. Find the probability mass function for the stock price after N periods.

2. Discuss why such a tree can be numerically more attractive than that of a general non-recombining tree (as presented in the lectures)

3. Find the q probabilities for this model.

Exercise 3.18: [Solution] Suppose that the stock price is currently \$120 and every month the stock price can either go up by 2% with a probability of 60% or down by 2% with a probability of 40%. If the effective annual interest rate is 10%. Calculate the price of a European put option (with a strike of \$118) that expires in 2 months time.

Exercise 3.19: [Solution] We have a stock that is currently priced at \$10 and it can increase by 10% or decrease by 10% each year over the next three-year period and there is a continuous compounding risk-free rate of 5%. Calculate the price of a European call with strike price \$12 which matures after three year.

Exercise 3.20: [Solution] In a one-period binomial model, it is assumed that the current share price of 260 will either increase to 285 or decrease to 250 at the end of one year. The annual risk-free interest rate is 5% compounded continuously and assume that this share pays no dividends.

1. Calculate the price of a one-year European call option with a strike price of 275.
2. Calculate the price of a one-year European put option with a strike price of 275.
3. Verify numerically that the put-call parity relationship holds in this case.

Exercise 3.21: [Solution] Consider a three year derivative that has the following payoff:

$$X = \max(S_T, 9)$$

if and only if

$$\max_{t=0,1,2,3} S_t < 11,$$

otherwise it pays nothing. If the current stock price is \$10 and it can increase/decrease by 5%, find the value of this option at time 0 if the risk-free rate is 3% continuous compounding.

Exercise 3.22: [Solution] Suppose we have a two year derivative that pays:

$$X = \begin{cases} S_T - 11 & \text{if } S_T > 11 \\ -0.50 & \text{if } S_T \leq 11 \end{cases}$$

i.e, you will have to pay \$0.50 if the outcome is unfavourable at maturity.
If the current stock price is \$10 and for each period it can either increase by a factor of 1.1 or decrease by a factor of 0.9 and the risk-free rate is 5% continuous compounding. Find the price of the payoff X.

Exercise 3.23: [Solution] Consider a two year derivative with the following payoff:

$$X = \begin{cases} (S(T) - 10)^3 & \text{if } S(T) > 10 \\ -(10 - S(T))^2 & \text{if } S(T) \leq 10 \end{cases}$$

If the current stock price is \$10 and it can increase by a factor of 1.15 or decrease by a factor of 0.8. Find the price of this derivative if the continuous compounding risk-free rate is 3%.

Exercise 3.24: [Solution] Luenberger, Chapter 12 (1Ed); 14 (2Ed). Questions 10,15
[For Q15, use $u = \exp(\sigma\sqrt{t})$ and $d = 1/u$.]

Exercise 3.25: [Solution] Consider a one period binomial model,

1. Explain briefly why it must be assumed that $d < e^r < u$.
2. Derive a formula for θ, the expected one-period rate of return on the share based on the real-world probability p of an up movement.
3. Show that the real-world variance of this one-period rate of return on the share is

$$p\,(1-p)\,(u-d)^2\,.$$

4. Show that $p > q$ if and only if $(1+\theta) > e^r$. Verbally interpret this result.

Exercise 3.26: [Solution] Show that the discounted stock price process $Z(t) = \frac{S(t)}{B(t)}$ is a Q-martingale under the usual setting shown in the lecture slides. You may assume the continuously compounded rate of return is r. Suppose the stock price can go up by a factor of u and go down by a factor of d. Note: $B(t)$ is the value of the bond process at time t.

Exercise 3.27: [Solution] Suppose we have a T month European call option with a strike price of K and the stock price is currently S_0. Now the stock price can either go up by a factor of u or go down by a factor of d every month. (Assume the continuous compounded rate of return is $r\%$).

1. Identify the Q probability measure that makes $Z(t) = \frac{S(t)}{B(t)}$ a Q martingale. Hint: Refer to the result in question 5
2. Now given that $Y(t) = E_Q\left[B(T)^{-1}X|F(t)\right]$ is a Q martingale, the discrete Martingale Representation Theorem states that there exists a pre-visible process $\phi(t)$ such that $Y(t) = Y(0) + \sum_{k=1}^{t} \phi(k)\left(Z(k) - Z(k-1)\right)$. Show that if we construct a portfolio of $\phi(t+1)$ stock and $\psi(t+1) = Y(t) - \phi(t+1)\frac{S(t)}{B(t)}$ bond then this portfolio is replicating and self-financing.
3. Since this portfolio is replicating and self-financing, the price of the derivative must equal the price of this portfolio under no arbitrage so we have $V(0) = E_Q\left[B(T)^{-1}X|F(0)\right]$. Hence, find the price of a 3 month European call option with a strike price of \$95 where the stock price is currently \$90. Now the stock price can either go up by a factor of 1.03 or go down by a factor of 0.98 every month. (Assume the continuous compounded rate of return is 12%).

Exercise 3.28: [Solution] A binomial lattice is used to model the price of a non-dividend paying share up to time T. The time interval $(0, T)$ is subdivided into a large number of intervals of lengths $\delta t = T/n$. It is assumed that, at each node in the binomial lattice, the share price is equally likely to increase by a factor

$$u = e^{\mu\delta t + \sigma\sqrt{\delta t}}$$

or decrease by a factor

$$d = e^{\mu\delta t - \sigma\sqrt{\delta t}}.$$

The movements at each period are assumed to be independent.

1. Show that, if the share price makes a total of X_n "up jumps", the share price at time T will be
$$S_T = S_0 \exp\left[\mu T + \sigma\sqrt{T}\left(\frac{2X_n - n}{\sqrt{n}}\right)\right]$$
where S_0 denotes the initial share price.
2. Write down the distribution of X_n and state how this distribution can be approximated when n is large.
3. Hence, determine the asymptotic distribution of $\dfrac{S_T}{S_0}$ for large n.

3.2.3 Discussion Questions

Exercise 3.29: [Solution] A non-dividend paying stock has a current price of 100. In any unit of time, the price of the stock is expected to increase by 10% or decrease by 5%. The continuously compounded risk-free interest rate is 4% per period. A European call option is written with a strike price of 103 and is exercisable after two units of time, i.e. at $t = 2$. The bond price at time 0 is 1.

1. Establish, using a binomial tree, the replicating portfolio for the option at each point in time.
2. Calculate the value of the option at $t = 0$.

Exercise 3.30: [Solution] Luenberger, Chapter 12 (1Ed); 14 (2Ed). Question 9

Exercise 3.31: [Solution] Suppose we have a \$100 stock that can either increase by 10% or decrease by 5% over a year. Your trading company wishes to hedge against stock price falls by purchasing a two year American put option on the stock with a strike price of \$100. If the effective annual rate of interest is 8%, find the value you should pay per option and whether the option should be exercised early
Hint: When pricing an European option, the value at each node is the holding value, however to value an American option you need to consider the larger of the intrinsic value and the holding value at each node.

Exercise 3.32: [Solution] A trading firm has recently released a new "exotic" option which pays you \$100 if the price of the stock EVER goes above \$25 or \$0 if the stock price never goes above \$25 over a two year period. Using a two period-binomial tree calculate the price of this exotic option if the continuous rate of return is 10% p.a.. The stock has the following features:

- The current price is \$20
- The stock price can either rise by 30% or fall by 5%

Exercise 3.33: [Solution] Suppose we have a one year option where the underlying is a forward contract on a stock. The option pays out $X = max(F(1,3) - 10, 0)$ where $F(1,3)$ is the forward price from time 1 to time 3. We know that the current stock price is \$10 and can either increase by a factor of 1.3 or decrease by a factor of 0.7 over any one year. Using a continuous compounding risk-free rate of 5%, find the price of this option.

Exercise 3.34: [Solution] We have a particular option that pays the following contingent claim at maturity:

$$X = \max_{t=0,1,2} S_t - \min_{t=0,1,2} S_t$$

The underlying is a non-dividend paying stock and is currently worth \$10 which can either increase by a factor of 1.25 or decrease by a factor of 0.8 over any one year. Using a continuously compounded risk-free rate of 4% and a two period binomial tree find the price of the option.

Exercise 3.35: [Solution] In a two period binomial tree we have a stock that is currently priced at \$200 and we know that the stochastic component can go up by 5% or down by 3%

each year. We are informed that this is a dividend paying stock with a single dividend of \$20 to be paid out in one years time. Calculate the price of a 2 year European call option on this stock with a strike of \$195. (Assume the effective annual rate of interst is 4%).
Hint: The changes in stock price should only be due to the stochastic component since the dividend is known in advance, so the stock price at each node should be broken down into a stochastic and a dividend component i.e $S^*(t) = S_{stochastic}(t) + PV_{dividend}(t)$.

Exercise 3.36: [Solution] A client has approached you to enquire about how much a particular exotic option should cost. The exotic option matures in three months time and pays out $max(20S(T) - S^2(T), 0)$ at maturity only if the stock price never drops below \$7.50 over the three month period, otherwise the option pays nothing. Calculate the cost of this exotic option if the underlying stock has the following characteristics:

- The current price is \$10
- The stock price is equally likely to rise/fall by 30% every month

Assume the effective monthly interest rate is 2%.

Exercise 3.37: [Solution] Consider a discrete time binomial tree with two steps, with $u = 1.2, d = 0.9$, and $e^r = 1.05$ per period. The current stock price is 10. Suppose we are interested in valuing a special type of Asian option where the payoff is the arithmetic average of the stock price at times 0, 1, and 2.

1. Discuss why we need to use a non-recombining tree to value this derivative, even though the stock price process recombines.
2. Find the value of this option.

Exercise 3.38: [Solution] Consider a discrete three step binomial tree with $u = 1.3, d = 0.7$ and $e^r = 1.1$. The current stock price is \$10. We want to price a three year put option that has a knockout condition. This derivative is a normal put option with strike price \$7.50 but as soon as the stock price touches the knockout price of \$12 the option becomes worthless. Find the price of this option at time 0. Why would an investor choose to buy such a derivative over a regular put option?

3.3 Solutions

3.3.1 Derivative Securities

Solution 3.1: [Exercice]

1. A "European" option allows the holder to exercise the option only at the expiry of the option. In contrast, an "American" option allows the holder to exercise the option at any time during the life of the option contract.
2. An individual may wish to purchase a call option for a number of reasons. For example, the purchaser may:

(a) wish (or need) to acquire a holding of this security at some time in the future but may not wish to (or be able to) purchase the shares outright at the current market price.

(b) wish to hedge an existing negative holding of this security against possible future price increases, assuming that he or she has already sold the security short.

(c) an investor who believes the security will appreciate in the future and therefore can increase the leverage (and return) of the investment.

(d) be an investor who believes the option (or the underlying security) is underpriced and expects to make a profit by selling the option at a higher price.

Solution 3.2: [Exercice]

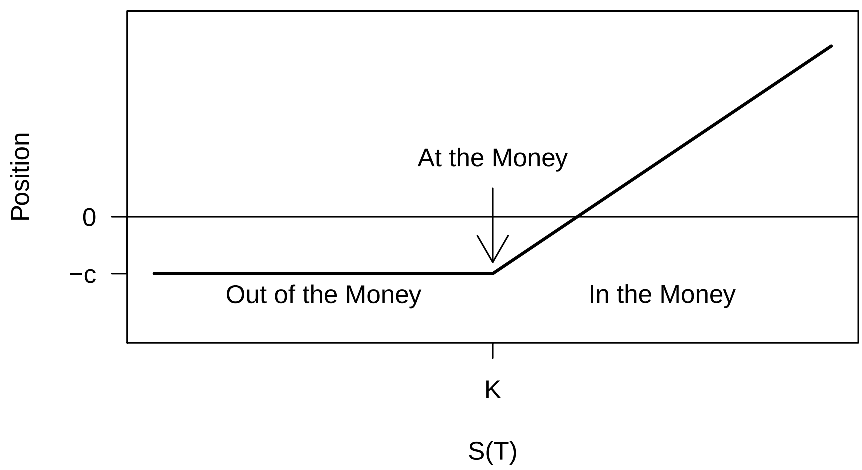
1. The "intrinsic value" of an option is the value if the option was due to expire now. For example:

   (a) for a call option, its intrinsic value is equal to $S_t - K$;

   (b) for a put option, it is equal to $K - S_t$.

2. If the intrinsic value is positive, the option is said to be in-the-money. If the intrinsic value is equal to zero, then it is said to be at-the-money. And if the intrinsic value is negative, it is out-of-the-money.

3. The position diagram for the European call option is as follows:



Solution 3.3: [Exercice]

1. Profit $= -37 + (495 - 480) = -22$, a loss of 22 per contract purchased.
2. Profit $= -12 + (180 - 150) = +18$, a gain of 18 per contract purchased.
3. Profit $= +22 + 0 = +22$, a gain of 22 per contract purchased. Here, you have collected the premium and the option has expired worthless.

Solution 3.4: [Exercice]

1. Let S denote the share price at expiry so that $S \sim U(90, 110)$. Its density is therefore
$$f_S(s) = \frac{1}{20} \text{ for } 90 \leq s \leq 110.$$
For the call option, the payoff function is $\max(s - 105, 0)$. Its expected payoff therefore is
$$\int_{105}^{110} (s - 105) \frac{1}{20} ds = \frac{5}{8} = 0.625.$$
For the put option, the payoff function is $\max(105 - s, 0)$. Its expected payoff therefore is
$$\int_{90}^{105} (105 - s) \frac{1}{20} ds = \frac{45}{8} = 5.625.$$

2. Repeating these calculations assuming this time that $S \sim U(80, 120)$. Its density is
$$f_S(s) = \frac{1}{40} \text{ for } 80 \leq s \leq 120.$$
For the call option, the payoff function is $\max(s - 105, 0)$. Its expected payoff therefore is
$$\int_{105}^{120} (s - 105) \frac{1}{40} ds = \frac{45}{16} = 2.8125.$$
For the put option, the payoff function is $\max(105 - s, 0)$. Its expected payoff therefore is
$$\int_{80}^{105} (105 - s) \frac{1}{20} ds = \frac{125}{16} = 7.8125.$$

3. Both distributions used for the share price have a mean of 100, but the volatility (i.e. standard deviation) has been doubled in part (b). Comparing the anwers, we see that the expected payoffs have increased for both options. Their corresponding option values will also increase. This is exactly what we would expect it to be if the volatility increased.

4. In each case, the difference between the expected payoffs is 5. This is exactly consistent with the put-call parity relationsip because we are effectively finding the value of these options assuming they are about to expire. So the difference between the value of the put and the call should equal the difference between the strike price (equal to 105) and the "average" share price at expiry (equal to 100).

Solution 3.5: [Exercice]

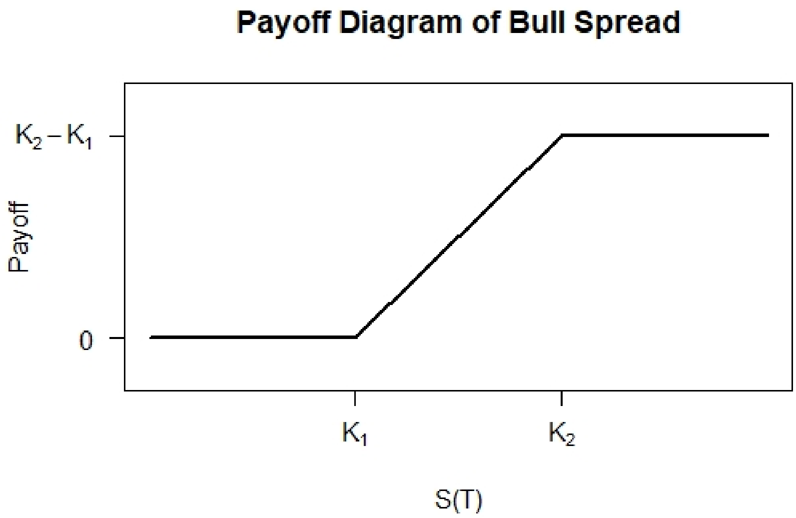
1. The put-call parity theorem states that the values of the call and the put options are related as follows:
$$c_t + Ke^{-r(T-t)} = p_t + S_t.$$
Using the given parameter values, we get:
$$42 + 500e^{-0.06(0.25)} = p_t + 480$$
which gives the value of the put option:
$$p_t = 54.56.$$

2. This calculation assumes that the market in which the share and the options are traded is arbitrage-free and that no dividends are payable. In addition, there is no transaction fee as well as restrictions on buying and selling.

Solution 3.6: [Exercice]

1. A European put option gives you the option to sell a share on the expiry date for a specified price. If the current share price fell, you would expect future share prices to be lower as well. Your option to sell the share at the specified price would now become more valuable and the value of the put would therefore increase.
2. If there was a sudden increase in the risk-free rate of interest, owning the put option (which you could sell and convert to cash) becoms relatively more expensive since you are losing more interest on the cash you effectively have tied up in it. So the value of the put option would go down.

Solution 3.7: [Exercice] (L12.1) The payoff diagram (note this is different from the position diagram by a vertical shift due to the effect of the cost of the option(s)) is

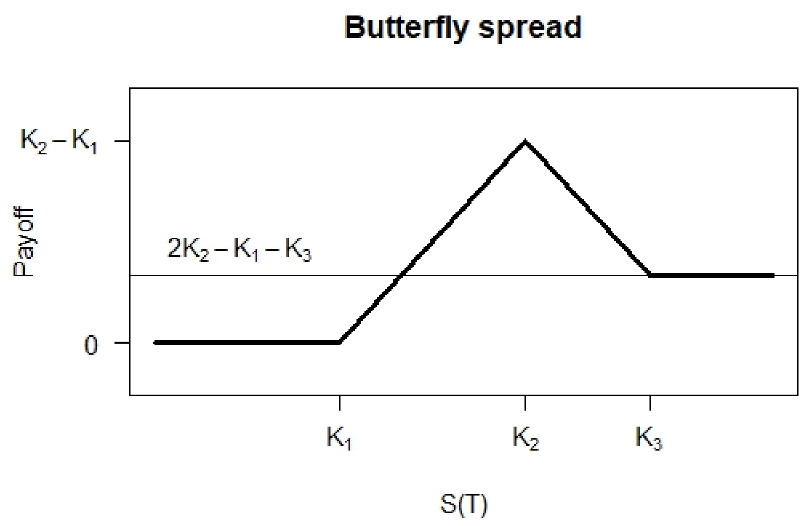


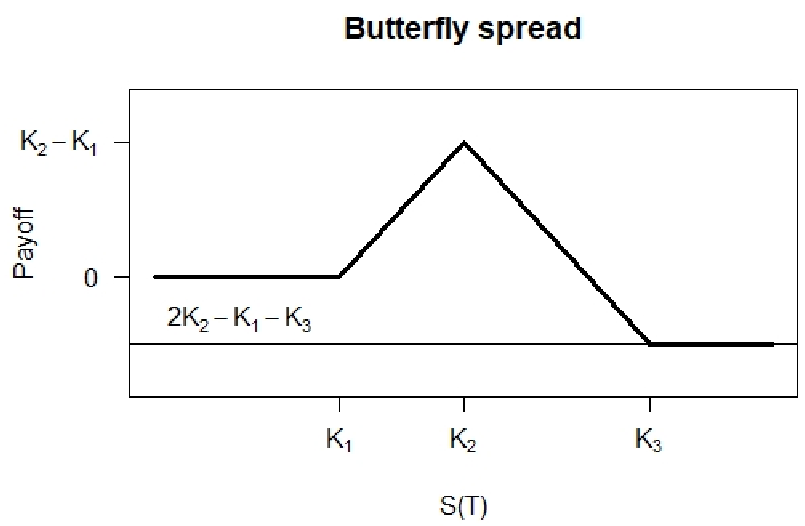
The initial cost is nonnegative since $C(K_1) \geq C(K_2)$

Solution 3.8: [Exercice]

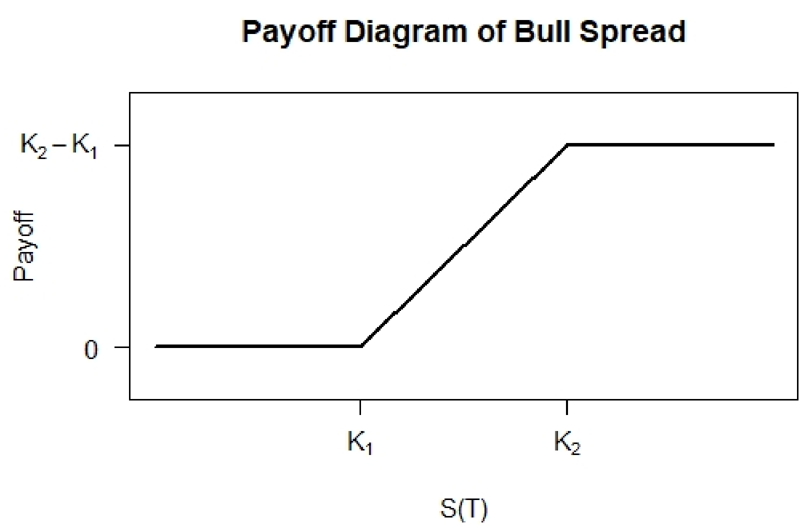
1. An investor would want this payoff if they expect low stock volatility around K_2.

   The graph is one of the following depending on the relative sizes of $K_3 - K_2$ and $K_2 - K_1$:

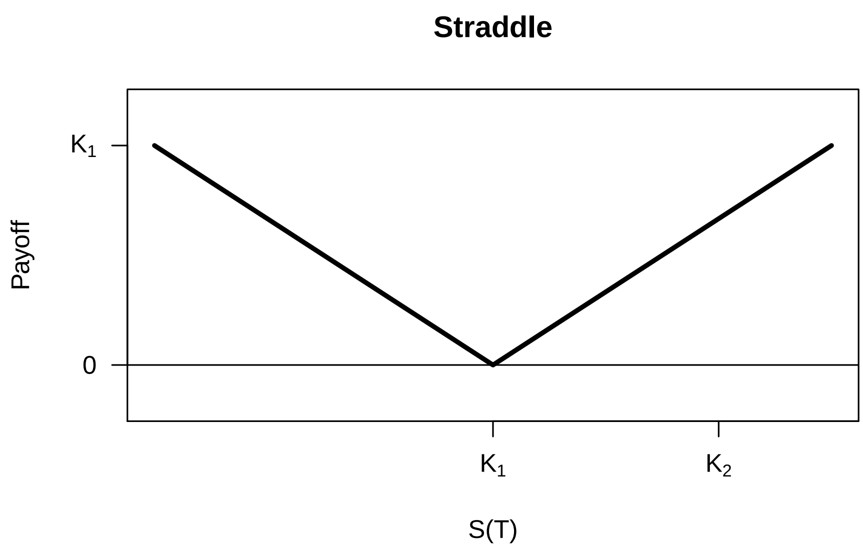




2. An investor would want this payoff if they expect the stock to rise within some range (this is usually cheaper than a normal call)



3. An investor would want this payoff if they expect really high stock volatility around K_1 i.e major announcement/events



Solution 3.9: [Exercice] Note $c(K,t) \to (S_T - K)_+$ and the portfolio $X = (S_t - Ke^{-r(T-t)}) \to S_T - K$ from time t to time T. Hence, we shall buy the (cheap) call option and sell the (expensive) portfolio X.

This is done as follows:

- Long a call option $c(K,t)$
- Short a stock S_t
- Long $Ke^{-r(T-t)}$ worth of bond

Now notice our cashflows at time t and at maturity:

Time	Cashflow
t	$S_t - Ke^{-r(T-t)} - c(K,t) > 0$
T	$\begin{cases} 0 & S_T \geq K \\ K - S_T > 0 & S_T < K \end{cases}$

So this portfolio gives us a strictly postive profit at time t and non-zero cash outflow at any time with probability 1, i.e. arbitrage.

Solution 3.10: [Exercice]

1. Assume $K_2 > K_1$ and suppose to the contrary that $C(K_1) < C(K_2)$. Buy option 1 and short option 2. Use option 1 to cover the obligations of option 2. Keep profit of $C(K_2) - C(K_1)$. This can be done since

$$(S_T - K_1)^+ \geq (S_T - K_2)^+$$

for all S_T.

2. Suppose to the contrary that $K_2 - K_1 < C(K_1) - C(K_2)$. Buy option 2 and short option 1 to obtain $K_2 - K_1 + \varepsilon$. Use option 2 and $K_2 - K_1$ to cover option 1 since

$$\begin{aligned} & (S_T - K_2)^+ + K_2 - K_1 \\ = & \max(K_2 - K_1, S_T - K_1) \\ \geq & \max(S_T - K_1, 0) \end{aligned}$$

and you can keep ε as profit. (We have ignored interest rates)

3. Assume $K_3 > K_2 > K_1$ and suppose to the contrary that

$$C(K_2) > \frac{K_3 - K_2}{K_3 - K_1} C(K_1) + \frac{K_2 - K_1}{K_3 - K_1} C(K_3)$$

Then buy $\frac{K_3 - K_2}{K_3 - K_1}$ of option 1 and $\frac{K_2 - K_1}{K_3 - K_1}$ of option 3. Short 1 of option 2. The left over profit is $\varepsilon > 0$.

Notice that

$$\begin{aligned} & \frac{K_3 - K_2}{K_3 - K_1} C(K_1) + \frac{K_2 - K_1}{K_3 - K_1} C(K_3) \\ \geq & \frac{K_3 - K_2}{K_3 - K_1}(S - K_1) + \frac{K_2 - K_1}{K_3 - K_1}(S - K_3) \\ = & \frac{K_3 - K_2}{K_3 - K_1}(S - K_2 + K_2 - K_1) + \frac{K_2 - K_1}{K_3 - K_1}(S - K_2 + K_2 - K_3) \\ = & S - K_2 \end{aligned}$$

and also

$$\frac{K_3 - K_2}{K_3 - K_1} C(K_1) + \frac{K_2 - K_1}{K_3 - K_1} C(K_3) > 0$$

so option 2's payments are covered.

Solution 3.11: [Exercice]

1. A put-call parity expresses a relationship between the price of a put option and the price of a call option on a stock where the options have the same exercise dates and the same strike prices. In the case of a non-dividend paying stock, we have

$$c_{K,t} + Ke^{-r(T-t)} = p_{K,t} + S_t$$

where $c_{K,t}$ is the value of the call option, $p_{K,t}$ is the value of the put option, K is the exercise/strike price, r is the risk-free rate compounded continuously, S_t is the current value of the stock, and T is the exercise date.

2. We shall denote the present value of the dividends payable by D. Now, consider the following two investment portfolios:

- Portfolio A: one European call and cash amount of $D + Ke^{-r(T-t)}$.
- Portfolio B: one European put and a unit of the underlying share.

The following is easy to show:

	Then the value of	
At exercise date, if	Portfolio A	Portfolio B
$S_T \geq K$	$De^{r(T-t)} + S_T$	$S_T + De^{r(T-t)}$
$S_T < K$	$De^{r(T-t)} + K$	$K + De^{r(T-t)}$

Clearly, portfolios A and B have the same value in all circumstances at the exercise date T. Hence, they must be equivalent at all earlier times. The portfolios at time t are of equal values, therefore, we have the following put-call parity relationship:

$$c_{K,t} + D + Ke^{-r(T-t)} = p_{K,t} + S_t.$$

3. Consider the case where

$$c_{K,t} + D + Ke^{-r(T-t)} < p_{K,t} + S_t,$$

then for some positive amount A, we have

$$A + c_{K,t} + D + Ke^{-r(T-t)} = p_{K,t} + S_t.$$

Hence, we can short one share and sell a put and receive $p_t + S_t$. At the exercise date, we know the value of this portfolio will be

$$\max\left[De^{r(T-t)} + S_T, De^{r(T-t)} + K\right].$$

However, we know that the value of a portfolio invested in a European call and a cash of $D + Ke^{-r(T-t)}$ have the same worth

$$\max\left[S_T + De^{r(T-t)}, K + De^{r(T-t)}\right].$$

Hence, we are left with a cash A which would have accumulated to $Ae^{r(T-t)}$ at the exercise date. The strategy therefore is:

- short one share and sell a put
- buy one call and put on cash deposit of $A + D + Ke^{-r(T-t)}$.

If the inequality is reversed, alse reverse the investment. That is, swap long positions for short positions and vice versa.

Solution 3.12: [Exercice]

1. As the option is American, the value of an option is greater if the remaining life is longer (since the shorter maturity option can be seen as a subset of the longer maturity option). So, we expect $x > w$ and $z > y$.

The value of a call option is smaller if the exercise price is greater. So, we expect $y < w$ and $z < x$.

The value of an option on a risky asset will be strictly positive. Combining these results, we have

$$0 < y < (w, z) < x.$$

That is, w and z must have values between y and x. Based on the information given, we cannot determine the order of w and z. They could go either directions.

2. The lower bound for a American call option is given by the inequality:
$$c_t \geq S_t - Ke^{-r(T-t)}.$$
Using the parameter values for x, this gives
$$c_t \geq S_t - Ke^{-r(T-t)} = 120 - 125e^{-0.06(0.5)} = -1.3057.$$
Since this is negative, we can take $c_t \geq 0$ instead.

The upper bound for a American call option is given by the inequality:
$$c_t \leq S_t.$$
In which case, $c_t \leq 120$.

Solution 3.13: [Exercice] Note the value at maturity of the European put is
$$K - S_T \leq (K - S_T)_+ \leq K.$$
Hene at time t, we shall have
$$Ke^{-r(T-t)} - S_t \leq p_{K,t} \leq Ke^{-r(T-t)}.$$
For the lower bound, we shall consider the two portfolios A and B as follows

1. Poartfolio A: long a put and long a stock
2. Portfolio B: cash amount of $Ke^{-r(T-t)}$

Then we have the following cashflows at time t and maturity (T):

	A: Put + Stock (Lower bound)	B: Cash (Upper bound)
t	$p_{K,t} + S_t$	$Ke^{-r(T-t)}$
T	$\begin{cases} K & \text{if } S_t \leq K \\ S_t \geq K & \text{if } S_t \geq K \end{cases}$	$\begin{cases} K & S_t \leq K \\ K & S_t \geq K \end{cases}$

Then with portfolio A we always get at least K, so the time t value of portfolio A must have a higher value than $Ke^{-r(T-t)}$ and therefore $p_{K,t} \geq Ke^{-r(T-t)} - S_t$.
Now the payoff of the put at maturity is $\max(K - S_T, 0) \leq K$ therefore the time t value of the put must be lower than that of portfolio B, i.e. $p_{K,t} \leq Ke^{-r(T-t)}$.

Solution 3.14: [Exercice] The relationship was derived by making two portfolios that had the exact same payoffs at maturity, which then implied their prices must be the same to prevent arbitrage opportunities.
In the case that one of the options is mispriced all we need to do is consider two portfolios such as:

1. Long call $c_{K,t}$ and short $p_{K,t}$
2. Long S_t stock and short $Ke^{-r(T-t)}$ cash

Both portfolio gives payoff $S_T - K$ at time T. You can then long the cheap portfolio and short the expensive portfolio to get a time t profit.

Solution 3.15: [Exercice]

1. It is in the money as you will get a payoff of 50.

2. Assume $0 < S_T < 20$, then $X = \max(\min(100, 50), 0) = 50$. Next, assume $20 < S_T < 70$, then $X = \max(\min(120 - S_T, 50), 0) = 50$. Assume $70 < S_T < 120$, then $X = \max(\min(120 - S_T, 50), 0) = 120 - S_T$. Finally, assume $S_T > 120$, then $X = \max(\min(120 - S_T, 50), 0) = \max(120 - S_T, 0) = 0$. Hence, the payoff function is

$$X = \begin{cases} 50, \ S_T \in [0, 70], \\ 120 - S_T, \ S_T \in [70, 120] \\ 0, \ S_T \in (120, \infty) \end{cases} .$$

3.3.2 Discrete Time Financial Modelling

Solution 3.16: [Exercice]

1. Let the replicating portfolio be (ϕ, ψ) which consists of ϕ units of stocks and ψ units of cash. Thus, we have two equations

$$\phi S_0 u + \psi e^r = u S_0 - K$$

and

$$\phi S_0 d + \psi e^r = 0$$

with two unknowns ϕ and ψ. It is straightforward to show that these solutions are

$$\phi = \frac{u S_0 - K}{S_0 u - S_0 d} = \frac{u - (K/S_0)}{u - d}$$

and

$$\psi = -\phi S_0 d e^{-r} = \left[\frac{u S_0 - K}{(d - u)}\right] d e^{-r}.$$

2. The value of the replicating portfolio at time $t = 0$ is

$$\begin{aligned} V_0 &= \phi S_0 + \psi \\ &= \left(\frac{u - (K/S_0)}{u - d}\right) S_0 + \left[\frac{u S_0 - K}{(d - u)}\right] d e^{-r} \\ &= \left(\frac{u S_0 - K}{u - d}\right) + \left[\frac{u S_0 - K}{(d - u)}\right] d e^{-r} \\ &= e^{-r} (u S_0 - K) \left[\frac{e^r - d}{u - d}\right] \\ &= e^{-r} \left[q \times (u S_0 - K) + (1 - q) \times 0\right] \end{aligned}$$

where

$$q = \frac{e^r - d}{u - d}$$

is the Q probability of an up-movement. Thus, in this case, we have

$$V_0 = E_Q \left(C_1 e^{-r}\right).$$

Solution 3.17: [Exercice] Under such a setup, the binomial model is simplified such that

$$S(t+1)\,|(S(t)=S_j) = \begin{cases} S_j u_j, & \text{if the price is up} \\ S_j d_j, & \text{if the price is down} \end{cases}.$$

(this is called a multiplicative model)

Suppose we assume that the sizes of the up and down movements are the same in all states, that is to say,

$$u_j = u \text{ and } d_j = d \text{ for all } j$$

where u and d are fixed constants. Observe that the stock price after an up then down move is the same as it it went down then up. Thus the tree 'recombines'.

1. The price of the underlying stock at time t is

$$S(t) = s_1 u^{N_t} d^{t-N_t}$$

where N_t denotes the number of up-movements between time 0 to t. Observe that N_t is clearly a binomial random variable

$$N_t \sim \text{binomial}(t, q).$$

Furthermore, for $0 < t < n$, we have:

(a) N_t is independent of $N_n - N_t$ and

(b) $(N_n - N_t) \sim$binomial$(n-t, q)$.

2. In a non-recombining tree, at time $t = n$, we would, in general, have 2^n states, but when the binomial trees are re-combined, we have only $n+1$ possible states. This means that computationally a recombining tree is much simpler.

3. The q 'probabilities' become

$$q_j = \frac{S_j e^{r\delta t} - S_j d_j}{S_j u_j - S_j d_j} = \frac{e^{r\delta t} - d}{u - d} = q$$

which are constant all throughout.

Solution 3.18: [Exercice] Refer to the following two period binomial tree.
Since we are interested in monthly periods we calculate the effective rate of interest to be: $(1.1)^{\frac{1}{12}} = 1.00797414$ and the Q probability to be $q = \frac{(1.1)^{\frac{1}{12}} - 0.98}{1.02 - 0.98} = 0.6993535107$.

The paths are as follows:

Path	Stock Price process	Payoff	Q-probabilities
(u,u) :	$120 \to 122.4 \to 124.848$	(0)	w.p. q^2
(u,d) :	$120 \to 122.4 \to 119.952$	(0)	w.p. $q(1-q)$
(d,u) :	$120 \to 117.6 \to 119.952$	(0)	w.p. $q(1-q)$
(d,d) :	$120 \to 117.6 \to 115.248$	$(118 - 115.248)$	w.p. $(1-q)^2$

Hence, the price is given by

$$(1.1)^{-2/12}(1-q)^2 \times (118 - 115.248) = 0.2448285.$$

Solution 3.19: [Exercice] As usual calculate the q probability: $q = \frac{e^{0.05}-0.9}{1.1-0.9} = 0.7563554819$. Note only the (u,u,u) path gives a non-zero payoff, being $10 \times 1.1^3 - 12 = 1.31$. The payoff is given by

$$e^{-0.05\times 3} q^3 \times 1.31 = 0.487871.$$

Solution 3.20: [Exercice] It can be verified that the q probability of an up-movement is given by:

$$q = \frac{e^r - d}{u-d} = \frac{e^{0.05} - (250/260)}{(285/260) - (250/260)} = \frac{260e^{0.05} - 250}{35} = 0.66659.$$

1. For the "European" call option with exercise price 275, the payoff function is

$$C_1 = \begin{cases} \max(285-275, 0) = 10, & \text{if price is up} \\ \max(250-275, 0) = 0, & \text{if price is down} \end{cases}.$$

   Thus, the value of the call is the expectation of the present value of this under Q, which is

$$E_Q\left(C_1 e^{-r}\right) = (10 \times 0.66659 + 0 \times 0.33341)\, e^{-0.05} = 6.34.$$

2. For the "European" put option with exercise price 275, the payoff function is

$$C_1 = \begin{cases} \max(275-285, 0) = 0, & \text{if price is up} \\ \max(275-250, 0) = 25, & \text{if price is down} \end{cases}.$$

   Thus, the value of the put is the expectation of the present value of this under Q, which

$$E_Q\left(C_1 e^{-r}\right) = (0 \times 0.66659 + 25 \times 0.33341)\, e^{-0.05} = 7.93.$$

3. The put-call parity relationship states that (in words):

$$\text{call value} + \text{discounted strike price} = \text{put value} + \text{share price}.$$

   The LHS is: $6.34 + 275e^{-0.05} = 267.93$.

   The RHS is: $7.93 + 260 = 267.93$.

   Since these are equal, the put-call parity relationship holds in this case.

Solution 3.21: [Exercice] There are in total 8 paths as follows:

Path	Stock Price process	Payoff	Q-probabilities
(u,u,u):	$10 \to 10.5 \to 11.025 \to 11.57625$	(0)	
(u,u,d):	$10 \to 10.5 \to 11.025 \to 10.47375$	(0)	
(u,d,u):	$10 \to 10.5 \to 9.975 \to 10.47375$	(10.47375)	w.p. $q^2(1-q)$
(u,d,d):	$10 \to 10.5 \to 9.975 \to 9.47625$	(9.47625)	w.p. $q(1-q)^2$
(d,u,u):	$10 \to 9.5 \to 9.975 \to 10.47375$	(10.47375)	w.p. $q^2(1-q)$
(d,u,d):	$10 \to 9.5 \to 9.975 \to 9.47625$	(9.47625)	w.p. $q(1-q)^2$
(d,d,u):	$10 \to 9.5 \to 9.025 \to 9.47625$	(9.47625)	w.p. $q(1-q)^2$
(d,d,d):	$10 \to 0.5 \to 9.025 \to 8.57375$	(9)	w.p. $(1-q)^3$

The q probability is $q = \frac{e^{0.03}-0.95}{1.05-0.95} = 0.8045453395$. Hence, the price is given by

$$e^{-0.03\times 3}\Big(2q^2(1-q) \times 10.47375 + 3q(1-q)^2 \times 9.47625 + (1-q)^3 \times 9\Big) = 3.282093.$$

[In exam, you can draw a tree diagram instead of the table.]

Solution 3.22: [Exercice] The q probability is: $q = \frac{e^{0.05}-0.9}{1.1-0.9} = 0.7563554819$. The paths are as follows:

Path	Stock Price process	Payoff	Q-probabilities
(u,u) :	$10 \to 11 \to 12.1$	(1.1)	w.p. q^2
(u,d) :	$10 \to 11 \to 9.9$	(-0.5)	w.p. $q(1-q)$
(d,u) :	$10 \to 9 \to 9.9$	(-0.5)	w.p. $q(1-q)$
(d,d) :	$10 \to 9 \to 8.1$	(-0.5)	w.p. $(1-q)^2$

The price is given by

$$e^{-0.05\times 2}\Big(q^2 \times 1.1 + (1-q^2) \times -0.5\Big) = 0.3757951.$$

Solution 3.23: [Exercice] The q probability is give by $q = \frac{e^{0.03}-0.8}{1.15-0.8} = 0.6584415256$. The paths are as follows:

Path	Stock Price process	Payoff	Q-probabilities
(u,u) :	$10 \to 11.5 \to 13.225$	(3.225^3)	w.p. q^2
(u,d) :	$10 \to 11.5 \to 9.2$	(-0.8^2)	w.p. $q(1-q)$
(d,u) :	$10 \to 8 \to 9.2$	(-0.8^2)	w.p. $q(1-q)$
(d,d) :	$10 \to 8 \to 6.4$	(-3.6^2)	w.p. $(1-q)^2$

The price is given by

$$e^{-0.03\times 2}\Big(q^2 \times (3.225^3 + 2q(1-q) \times -0.8^2 + (1-q)^2 \times -3.6^2\Big) = 12.00013.$$

Solution 3.24: [Exercice] (L12/14.10)

Payoff is

$$\begin{aligned} \max(0.5S, S-K) &= 0.5S + \max(0, 0.5S-K) \\ &= 0.5S + 0.5\max(0, S-2K) \end{aligned}$$

so by linear pricing

$$C_H = 0.5P + 0.5C_2$$

thus $\alpha = 0.5, \beta = 0, \gamma = 0.5$

(L12/14.15)

Use the solutions for the call and put in the examples in the text. At the end of the third year, the value will be the maximum of the call or put values. Then just work backwards till time 0. This gives 6.73 as the option value.

Solution 3.25: [Exercice]

1. The model requires that the market must be arbitrage-free. This condition translates to the requirement that

   $$d < e^r < u.$$

   Suppose $e^r < d$, one can always borrow to buy stock to get a positive return with 0 investment. This is arbitrage. A similar argument applies when $e^r > u$. Our crucial argument for replicating the portfolio is the “law of one price” which assumes no-arbitrage. When this assumption does not hold, our argument does not work. Therefore, we do need $d < e^r < u$.

   [Note this condition guarantees that the q probability is legitimate, i.e. $0 < q < 1$.]

2. The expected rate of return on the underlying share based on the real-world probabilities is:
$$\theta = \frac{pS_0u + (1-p)\,S_0d - S_0}{S_0} = pu + (1-p)\,d - 1.$$

3. Similarly, the variance of the rate of return is
$$\begin{aligned}\sigma^2 &= pu^2 + (1-p)\,d^2 - (1+\theta)^2 \\ &= pu^2 + (1-p)\,d^2 - [pu + (1-p)\,d]^2.\end{aligned}$$
If we collect the p's together, expand and then simplify, we get
$$\begin{aligned}\sigma^2 &= p\left(u^2 - d^2\right) + d^2 - [p\,(u-d) + d]^2 \\ &= p\left(u^2 - d^2\right) + d^2 - p^2\,(u-d)^2 - 2pd\,(u-d) - d^2 \\ &= p\,(u-d)\,(u+d) - p^2\,(u-d)^2 - 2pd\,(u-d) \\ &= p\,(u-d)\,[(u+d) - p\,(u-d) - 2d] \\ &= p\,(u-d)\,[(u-d) - p\,(u-d)] \\ &= p\,(1-p)\,(u-d)^2\end{aligned}$$
which gives the desired result.

4. Since $u > d$ by definition, the inequality $p > q$ is equivalent to
$$p\,(u-d) + d > q\,(u-d) + d.$$
This implies that
$$1 + \theta > \frac{e^r - d}{u-d}\,(u-d) + d = e^r.$$
In words, this says that the real-world probability of an up movement is greater than the risk-neutral probability whenever the expected increase in the value of the underlying share exceeds the risk-free interest rate, that is, whenever the underlying share is risky.

Solution 3.26: [Exercice] We need to show Z is a Q-martingale, or equivalently
$$E_Q\left[Z(t)|\mathcal{F}(s)\right] = Z(s),\ \ s \le t$$
We first show that it is the case for 1 time step, i.e.
$$E_Q[Z(t+1)|\mathcal{F}(t)] = Z(t).$$
At any time $B(t+1) = B(t)e^r$, hence we have
$$\begin{aligned}E_Q[Z(t+1)|\mathcal{F}(t)] &= E_Q[\frac{S(t+1)}{B(t+1)}|\mathcal{F}(t)] \\ &= E_Q[\frac{S(t)}{B(t+1)}\frac{S(t+1)}{S(t)}|\mathcal{F}(t)] \\ &= E_Q[\frac{S(t)}{B(t)e^r}\frac{S(t+1)}{S(t)}|\mathcal{F}(t)]\end{aligned}$$
Note that $S(t+1)/S(t)$ is the future movement of the stock price which is independent of $\mathcal{F}(t)$ and $S(t)/B(t)$ is "known" given $\mathcal{F}(t)$. Hence, we can simplify the above to

$$\begin{aligned}
E_Q[Z(t+1)|\mathcal{F}(t)] &= E_Q[\frac{S(t)}{B(t)e^r}\frac{S(t+1)}{S(t)}|\mathcal{F}(t)] \\
&= E_Q[\frac{S(t+1)}{S(t)}]\frac{S(t)}{B(t)}e^{-r} \\
&= \Big(q \times u + (1-q) \times d\Big)Z(t)e^{-r} \\
&= \Big(\frac{(e^r - d) \times u + (u - d - (e^r - d)) \times d}{u-d}\Big)Z(t)e^{-r} \\
&= \Big(\frac{ue^r - ud + (u - e^r)d}{u-d}\Big)Z(t)e^{-r} \\
&= \Big(\frac{ue^r - ud + ud - e^r d}{u-d}\Big)Z(t)e^{-r} \\
&= Z(t)
\end{aligned}$$

as required.

We now have established the result when $t-s \leq 1$. We proceed by induction. Suppose we have the result for $t-s \leq k$, i.e.

$$E_Q[Z(t)|\mathcal{F}(s)] = Z(s)$$

for $t-s \leq k$, then for $t-s = k+1$, we have

$$\begin{aligned}
E_Q[Z(t+k+1)|\mathcal{F}(t)] &= E_Q[E_Q[Z(t+k+1)|\mathcal{F}(t+k)]|\mathcal{F}(t)] \text{ (Tower property)} \\
&= E_Q[Z(t+k)|\mathcal{F}(t)] \text{ (result for 1 step)} \\
&= Z(t) \text{ (result for } k \text{ step by hypothesis).}
\end{aligned}$$

This completes the proof, i.e. Z is a martingale.

Solution 3.27: [Exercice]

1. Using the result of Question 5 directly, the Q measure we need is $q = \frac{e^r - d}{u-d}$.

2. Note we have
   (a) $V(t) = Y(t)B(t)$
   (b) $\phi(t+1)$ stock
   (c) $\psi(t+1) = Y(t) - \phi(t+1)Z(t)$ bond, such that the portfolio value is $V(t) = Y(t)B(t)$
   (d) Martingale representation theorem, i.e. $Y(t+1) - Y(t) = \phi(t+1)(Z(t+1) - Z(t))$, which is the same as $Y(t+1) - \phi(t+1)Z(t+1) = Y(t) - \phi(t+1)Z(t) = \psi(t+1)$, or

   $$Y(t+1) = \psi(t+1) + \phi(t+1)Z(t+1).$$

   Therefore, we have

   $$\begin{aligned}
   V(t+1) &= Y(t+1)B(t+1) \\
   &= (\psi(t+1) + \phi(t+1)Z(t+1))B(t+1), \quad \text{by the above equation} \\
   &= \psi(t+1)B(t+1) + \phi(t+1)S(t+1),
   \end{aligned}$$

i.e. Self-financing (as holding ϕ stocks and ψ bonds would give us $V(t+1)$ in the next period without giving in or taking out extra cash).

Of course, by noting $V(t) = \psi(t+1)B(t) + \phi(t+1)S(t)$, taking difference, we have

$$V(t+1) - V(t) = \psi(t+1)(B(t+1) - B(t)) + \phi(t+1)(S(t+1) - S(t)).$$

which is an alternative view on self-financing, i.e. difference in the portfolio value is solely driven by the change in the assets in the portfolio (the stock and the bond) so that no extra cash is needed to maintain the portfolio.

Now to show this portfolio is replicating, consider a random payoff X at maturity T. From above we can conclude that at maturity our portfolio is worth:

$$\begin{aligned}
V(T) &= B(T)Y(T) \\
&= B(T)E_Q\left[\frac{X}{B(T)}|\mathcal{F}(T)\right] \quad \text{(Definition of Y(t))} \\
&= E_Q\left[X|\mathcal{F}(T)\right] \quad \text{(Multiply by } B(T)) \\
&= X \quad \text{(We know } X \text{ given } \mathcal{F}(t))
\end{aligned}$$

So the value of the portfolio at maturity is exactly the same as the derivative payoff. Hence, we have a replicating and self-financing portfolio.

3. The first step now is to find the possible outcomes of the random payoff X i.e what are the payoffs for our call option at maturity.

   To simplify things we can find out the possible outcomes for the stock at maturity first:

$$\begin{cases}
\$90 \cdot 1.03^3 & w.p \quad q^3 \\
\$90 \cdot 1.03^2 \cdot 0.98 & w.p \quad 3q^2(1-q) \\
\$90 \cdot 1.03 \cdot 0.98^2 & w.p \quad 3q(1-q)^2 \\
\$90 \cdot 0.98^3 & w.p \quad (1-q)^3
\end{cases}$$

   where $q = \frac{e^{0.01}-0.98}{1.03-0.98} = 0.6010033417$

   Then we know X has the following possible outcomes using $max(S(T) - 95, 0)$:

$$\begin{cases}
\$3.34543 & w.p \quad 0.2170854221 \\
\$0 & w.p \quad 0.432358784 \\
\$0 & w.p \quad 0.287036191 \\
\$0 & w.p \quad 0.06351960302
\end{cases}$$

   Then we know $V(0) = \frac{3.34543 \cdot 0.2170854221+0+0+0}{B(3)} = e^{-0.03}0.7262440837 = \$0.7047803273$

Solution 3.28: [Exercice]

1. If the share price makes X_n up jumps, then there must be $n - X_n$ down jumps. Its value therefore at time T is

$$S_T = S_0 \times u^{X_n} \times d^{n-X_n}.$$

   Using the expressions given for u and d, we have

$$\begin{aligned}
S_T &= S_0 \times e^{\left(\mu\delta t+\sigma\sqrt{\delta t}\right)X_n} \times e^{\left(\mu\delta t-\sigma\sqrt{\delta t}\right)(n-X_n)} \\
&= S_0 \times \exp\left[n\mu\delta t + (2X_n - n)\,\sigma\sqrt{\delta t}\right].
\end{aligned}$$

Using the fact that $\delta t = T/n$, we get the required result:

$$S_T = S_0 \exp\left[\mu T + \sigma\sqrt{T}\left(\frac{2X_n - n}{\sqrt{n}}\right)\right].$$

2. Since there are n independent price movements, each equally likely to go up or down, then X_n has a binomial distribution with parameters n and $\frac{1}{2}$. If n is large enough, this can be approximated by a normal distribution with the same mean and variance. That is $X_n \sim N\left(\frac{1}{2}n, \frac{1}{4}n\right)$.

3. Using the result in Part 2, we then know that asymptotically, i.e. when n is large, then

$$\frac{X_n - (n/2)}{\sqrt{n/4}} = \frac{2X_n - n}{\sqrt{n}} \sim N(0,1)$$

is standard normal. From part 1, if we take logs of both sides we get

$$\log\frac{S_T}{S_0} = \mu T + \sigma\sqrt{T}\left(\frac{2X_n - n}{\sqrt{n}}\right).$$

Note by part 2, $X_n \sim N(\frac{n}{2}, \frac{n}{4})$. Therefore,

$$E[\mu T + \sigma\sqrt{T}(\frac{2X_n - n}{\sqrt{n}})] = \mu T + \sigma\sqrt{T}(\frac{2(n/2) - n}{\sqrt{n}}) = \mu T$$

and

$$Var(\mu T + \sigma\sqrt{T}(\frac{2X_n - n}{\sqrt{n}})) = Var(\sigma\sqrt{T}(\frac{2X_n}{\sqrt{n}})) = \frac{4\sigma^2 T}{n}Var(X_n) = \sigma^2 T,$$

which implies

$$\log\frac{S_T}{S_0} \sim N\left(\mu T, \sigma^2 T\right).$$

Therefore, $\frac{S_T}{S_0}$ has a log-normal distribution with parameters μT and $\sigma^2 T$.

Solution 3.29: [Exercice]

1. At time $t = 1$, the replicating portfolios are

$$\phi_{t=1}(up) = 1 \text{ and } \psi_{t=1}(up) = -95.081$$

given the stock price is up and

$$\phi_{t=1}(down) = 0.105 \text{ and } \psi_{t=1}(down) = -8.7696$$

given the stock price is down. At time $t = 0$, the replicating portfolio is

$$\phi_{t=0} = 0.678 \text{ and } \psi_{t=0} = -61.023.$$

(be careful how you index the model - as long as the intuition is correct the answer should be the same)

2. At time $t = 1$, the value of the replicating portfolios are

$$1 \times 110 - 98.96 = 11.039$$

given the stock price is up and

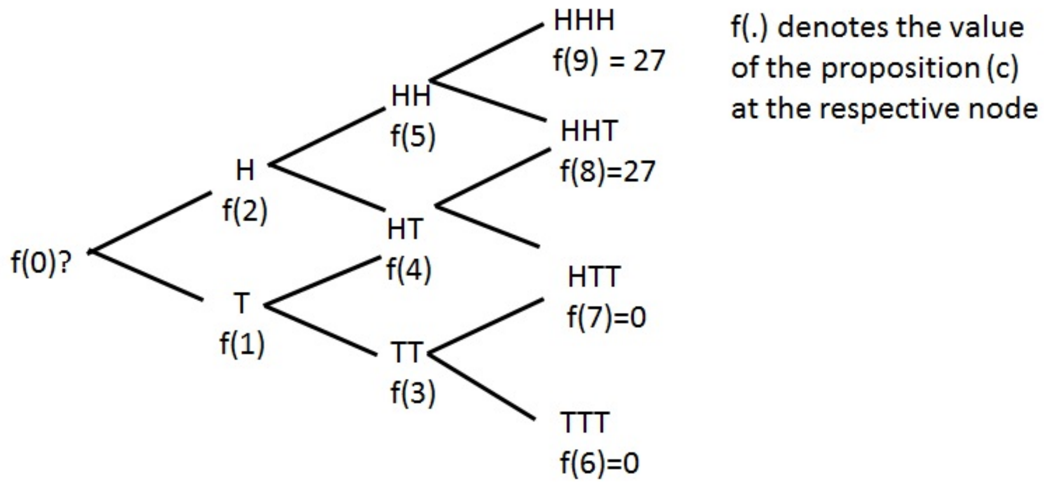
$$0.105 \times 95 - 9.128 = 0.873$$

At time $t = 0$, the value of the replicating portfolio (and hence the value of the call) is

$$0.678 \times 100 - 61.023 = 6.752.$$

Solution 3.30: [Exercice] (L12/14.9)

This question can be treated as a binomial option pricing model.To price the proposition (c), we replicate its payoffs using another "portfolio" consisting of ϕ units of gamble G (proposition (a)), and ψ unit of cash (proposition (b)). Note that each unit of G costs \$1, and it can either provide \$3 or \$0 depending on whether the outcome is a head or a tail. Consider the binomial tree below:



We work backward from the third coin flip. We need to match the payoff of this portfolio with the payoff of proposition (c), meaning:

$$\begin{aligned} \phi(5) * 3 + \psi(5) &= 27, && \text{if H in the last flip,} \\ \phi(5) * 0 + \psi(5) &= 27, && \text{if T in the last flip,} \end{aligned}$$

This results in

$$\psi(5) = 27,\ \phi(5) = 0$$

This portfolio replicates the proposition (c), hence its value at node 5 is the same as the value of proposition (c) at node 5. Hence the value of the proposition at the previous node is

$$f(5) = \phi(5) * G + \psi(5) = \psi(5) = 27$$

Similarly to find the price of the proposition (c) at node 4, we replicate its outcome using $\phi(4)$ units of gamble G and $\psi(4)$ units of cash. Matching the payoff of this portfolio with the payoff of proposition (c) we we have

$$\begin{aligned} \phi(4) * 3 + \psi(4) &= 27, && \text{if H in the last flip,} \\ \phi(4) * 0 + \psi(4) &= 0, && \text{if T in the last flip,} \end{aligned}$$

This results in

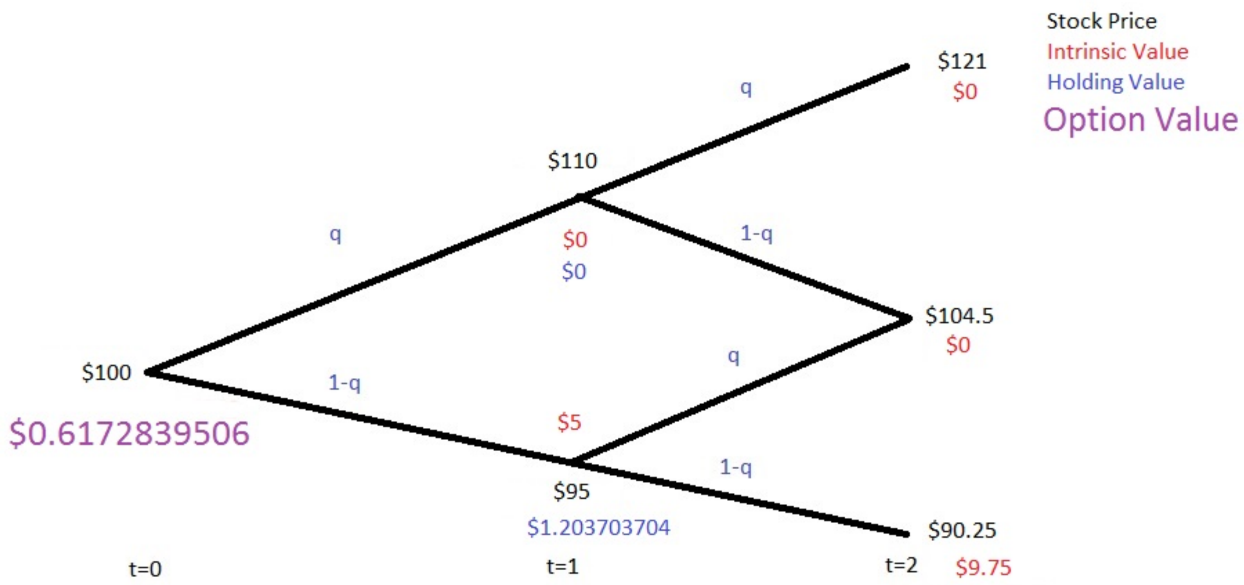
$$\psi(4) = 0,\ \phi(4) = 9$$

Hence the value of the proposition at the previous node is

$$f(4) = \phi(4) * G + \psi(4) = \psi(4) = 9$$

Similarly we can show that $f(3) = 0,\ f(2) = 15,\ f(1) = 3,\ f(0) = 7$.

Solution 3.31: [Exercice] Pricing an American option is very similar to European, but you also need to consider the intrinsic values at each node.



As shown in the diagram, first we calculate the derivative payoff at maturity and then we can calculate the holding values backwards.

$$\begin{aligned} f_d &= \frac{q \cdot 0 + (1-q) \cdot 9.75}{1.08} \\ &= \$1.203703704 \\ f_u &= 0 \end{aligned}$$

Now we need to calculate the intrinsic value (exercise value) at nodes 2 and 3 as well:

$$\begin{aligned} f_d^* &= max(100 - 95, 0) \\ &= \$5 \\ f_u^* &= max(100 - 110, 0) \\ &= \$0 \end{aligned}$$

Then we use the larger of the two values to calculate f_1:

$$\begin{aligned} f &= \frac{q \cdot 0 + (1-q) \cdot 5}{1.08} \\ &= \$0.6172839506 \end{aligned}$$

So ideally the option should be exercised early at time 1 if the stock price goes down.

[Note in this case we need to evaluate node by node as the payoff is not at maturity. Please make sure you are familiar with this method as well as the usual method.]

Solution 3.32: [Exercice] The q probability is give by $q = \frac{e^{0.1}-0.95}{1.3-0.95} = 0.4433455$. The paths are as follows:

Path	Stock Price process	Payoff	Q-probabilities
(u,u) :	$20 \to 26 \to 33.8$	(100)	w.p. q^2
(u,d) :	$20 \to 26 \to 24.7$	(100)	w.p. $q(1-q)$
(d,u) :	$20 \to 19 \to 24.7$	(0)	w.p. $q(1-q)$
(d,d) :	$20 \to 19 \to 18.05$	(0)	w.p. $(1-q)^2$

The price is given by

$$e^{-0.1\times 2}\Big(q^2 \times 100 + q(1-q)\times 100\Big) = e^{-0.2} \times q \times 100 = 36.29806.$$

Solution 3.33: [Exercice] Recall that the forward price of a forward contract is simply the future value of the underlying asset evaluated at the risk-free interest rate. i.e $F(t,T) = S_t e^{r(T-t)}$. To calculate the forward prices at time 1 we only need one year of the stock price, refer to the following diagram.

$$
\begin{aligned}
q &= \frac{e^{0.05} - 0.7}{1.3 - 0.7} \\
&= 0.5854518273 \\
f_u &= 13e^{0.1} - 10 \\
&= \$4.367221935 \\
f_d &= max(7e^{0.1} - 10, 0) \\
&= 0 \\
f &= \frac{q \cdot 4.367221935}{e^{0.05}} \\
&= \$2.432101549.
\end{aligned}
$$

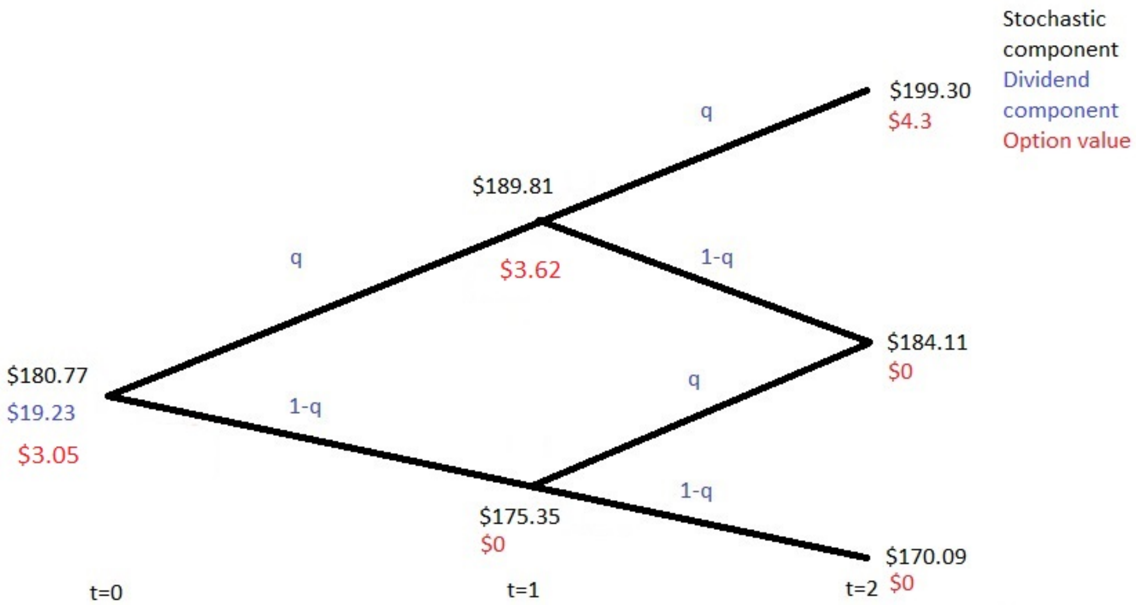
Solution 3.34: [Exercice]
The q probability is give by $q = \frac{e^{0.04} - 0.8}{1.25 - 0.8} = 0.5351351$. The paths are as follows:

Path	Stock Price process	Payoff	Q-probabilities
(u, u) :	$10 \to 12.5 \to 15.625$	(5.625)	w.p. q^2
(u, d) :	$10 \to 12.5 \to 10$	(2.5)	w.p. $q(1-q)$
(d, u) :	$10 \to 8 \to 10$	(2)	w.p. $q(1-q)$
(d, d) :	$10 \to 8 \to 6.4$	(3.6)	w.p. $(1-q)^2$

The price is given by

$$
e^{-0.04 \times 2}\Big(q^2 \times 5.625 + q(1-q) \times 2.5 + (1-q)q \times 2 + (1-q)^2 \times 3.6\Big) = 3.238506.
$$

Solution 3.35: [Exercice] First we need to calculate the stochastic component of the stock price. The present value of the dividend is $\frac{20}{1.04}$ = \$19.23076923 so we can work out the stochastic component to be $200 - 19.23 = \$180.77$.
Now it is this stochastic component that changes so our binomial tree is as follows:



Note that immediately after time 1 the dividend is paid out, so there is no longer a dividend component after t=1.
Here, $q = \frac{1.04-0.97}{1.05-0.97} = 0.875$, therefore price is given by

$$(1.04)^{-2} \times q^2 \times 4.3 = 3.042451.$$

Solution 3.36: [Exercice]
The payoff is given by $X = (20S(T) - S(T)^2)_+ 1(\min_{t \le T} S(t) \ge 7.5)$, which is equivalent to

$$X = S(T) \times (20 - S(T)) 1(\min_{t \le T} S(t) \ge 7.5) 1(S(T) \le 20).$$

There are in total 8 paths as follows:

Path	Stock Price process	Payoff	Q-probabilities
(u,u,u) :	$10 \to 13 \to 16.9 \to 21.97$	(0)	
(u,u,d) :	$10 \to 13 \to 16.9 \to 11.83$	$(11.83 \times (20 - 11.83))$	w.p. $q^2(1-q)$
(u,d,u) :	$10 \to 13 \to 9.1 \to 11.83$	$(11.83 \times (20 - 11.83))$	w.p. $q^2(1-q)$
(u,d,d) :	$10 \to 13 \to 9.1 \to 6.37$	(0)	
(d,u,u) :	$10 \to 7 \to 9.1 \to 11.83$	(0)	
(d,u,d) :	$10 \to 7 \to 9.1 \to 6.37$	(0)	
(d,d,u) :	$10 \to 7 \to 4.9 \to 6.37$	(0)	
(d,d,d) :	$10 \to 7 \to 4.9 \to 3.43$	(0)	

The q probability is $q = \frac{1.02-0.7}{1.3-0.7} = 0.5333333$. Hence, the price is given by

$$\begin{aligned} &(1.02)^{-3} \times 2q^2(1-q) \times 11.83 \times (20 - 11.83) \\ = &\ 24.17912. \end{aligned}$$

Solution 3.37: [Exercice]

1. The payoff is path dependent - in particular, the payoff if the stock price first rose then fell may be different to what happens if it fell then rose (even though the stock price itself will be the same). Hence to keep track of all possible outcomes we need to use a non-recombining tree.

2. The payoff is given by
$$X = \frac{S(0) + S(1) + S(2)}{3}.$$
The q probability is give by $q = \frac{1.05-0.9}{1.2-0.9} = 0.5$. The paths are as follows:

Path	Stock Price process	Payoff	Q-probabilities
(u,u) :	$10 \to 12 \to 14.4$	(12.1333)	w.p. q^2
(u,d) :	$10 \to 12 \to 10.8$	(10.9333)	w.p. $q(1-q)$
(d,u) :	$10 \to 9 \to 10.8$	(9.9333)	w.p. $q(1-q)$
(d,d) :	$10 \to 9 \to 8.1$	(9.0333)	w.p. $(1-q)^2$

The price is given by

$$\begin{aligned} &(1.05)^{-2} \times \Big(q^2 \times 12.1333 + q(1-q) \times 10.9333 + (1-q)q \times 9.9333 + (1-q)^2 \times 9.0333\Big) \\ &= 9.531368. \end{aligned}$$

Solution 3.38: [Exercice] The payoff is given by

$$X = (7.5 - S(T))_{+}1(\max_{t \leq T} S(t) < 12).$$

The q probability is give by $q = \frac{1.1-0.7}{1.3-0.7} = \frac{2}{3}$.
There are in total 8 paths as follows:

Path	Stock Price process	Payoff	Q-probabilities
(u,u,u) :	$10 \to 13 \to 16.9 \to 21.97$	(0)	
(u,u,d) :	$10 \to 13 \to 16.9 \to 11.83$	(0)	
(u,d,u) :	$10 \to 13 \to 9.1 \to 11.83$	(0)	
(u,d,d) :	$10 \to 13 \to 9.1 \to 6.37$	(0)	
(d,u,u) :	$10 \to 7 \to 9.1 \to 11.83$	(0)	
(d,u,d) :	$10 \to 7 \to 9.1 \to 6.37$	$(7.5 - 6.37)$	w.p. $q(1-q)^2$
(d,d,u) :	$10 \to 7 \to 4.9 \to 6.37$	$(7.5 - 6.37)$	w.p. $q(1-q)^2$
(d,d,d) :	$10 \to 7 \to 4.9 \to 3.43$	$(7.5 - 3.43)$	w.p. $(1-q)^3$

The price is given by

$$\begin{aligned}&(1.1)^{-3} \times \Big(2q(1-q)^2 \times (7.5 - 6.37) + (1-q)^3 \times (7.5 - 3.43)\Big)\\&= 0.2390294.\end{aligned}$$

The price of regular put options are always greater than equal to that of these knockouts, therefore investors may want to purchase these types of options to increase their leverage.

Alternatively, when the stock price is too high, there is a little chance that the put option can be exercised. By early surrendering, the price of the option can be reduced.

Module 4

4.1 Continuous Time Derivative Valuation

4.1.1 Some technical results

The following technical knowledge are helpful to understand the course material:

1. The filtration $\mathbb{F} = \{\mathcal{F}_t; t \geq 0\}$. Filtration can be thought of the evolution of observations, i.e. you have more observations from time to time. For example, we may have $\mathcal{F}_0$ representing the stock price at time 0. Likewise, $\mathcal{F}_1$ represents the stock price from time 0 to time 1. $\mathcal{F}_2$ represents the stock price from time 0 to time 2. In particular, it also contains the observation from time 0 to time 1, i.e. $\mathcal{F}_1 \subset \mathcal{F}_2$. Generally, we have $\mathcal{F}_s \subset \mathcal{F}_t$ for $s \leq t$.

2. Conditional expectation $E[\cdot|\mathcal{F}_t]$. Intuitively, this is the same as the usual conditional expectation $E[\cdot|Y]$ for a random variable Y. Note $E[\cdot|\mathcal{F}_t]$ itself is a random variable (analogous to $E[\cdot|Y]$). Similar to the usual expectation, $E[X|\mathcal{F}_t]$ means "Our best guess (estimate) of (the value of) X restricted to (or using only) the observation up to time t". For the operations of the conditional expectations, we have the following 3 rules:

   (a) $E[X|\mathcal{F}_0] = E[X]$ because we do not have any useful observations (yet). Our best guess would be the expected value itself.

   (b) $E[X|\mathcal{F}_T] = X$ if X is the payoff which is known at time T (technically we say X is *adapted to* $\mathcal{F}_T$), because once we have all observations up to time T, we are able to tell what X is.

   (c) $E[E[X|\mathcal{F}_t]|\mathcal{F}_s] = E[X|\mathcal{F}_s]$ for $s \leq t$. The left hand side means we are guessing X by first restricting our observations up to time t then further restricting our observations up to time s. This is essentially the same as restricting our observations up to time s which is the right hand side. This property is called the "tower property". In fact, it also works for $t < s$ with a similar argument, although we will not need it in this course.

3. Martingale. Suppose the observations $\mathbb{F}$ is given. $M = \{M(t); t \geq 0\}$ is a martingale if $E[M(t)|\mathcal{F}_s] = M(s)$ for $s \leq t$. This means that M "on average" does not move, or our "best estimate" is the currenct value. This stream of mathematics originates from some French Mathematicians who studied gambling. If the gamble is a martingale, neither party can win. The objective is to study/find a strategy which is a sub-martingale, something that is increasing on average, i.e. profit!! Examples of martingales include:

   (a) $M_t = E[X|\mathcal{F}_t]$ for any X. This is simply the result of the tower property.

(b) $\frac{S(t)}{B(t)}$ under Q. You need to verify this (using Itô's lemma).

(c) Brownian motion $W = \{W(t); t \geq 0\}$. This is the result of stationary and independent increment which is normal distributed with mean 0.

4.1.2 Recap, Tips and Examples

The following are some useful tips for doing exercises.

1. Prove or disprove Brownian motion. There is no shortcut, you really need to check the definitions, i.e. the following
   (a) $W(0) = 0$,
   (b) (The sample path of) W is continuous,
   (c) W has stationary and independent increment,
   (d) $W(t)$ is distributed as $N(0,t)$, or equivalently verify the increment $W(t) - W(s) \sim N(0, t-s)$ for $s < t$ which can be done simutaneously with (c).
2. Change of measure. In terms of the operation, you can safely write

$$\begin{aligned}\sigma(X(t),t)dW(t) &= \sigma(X(t),t)(dW(t) + \gamma(t)dt) - \sigma(X(t),t)\gamma(t)dt \\ &= \sigma(S(t),t)dW^Q(t) - \sigma(X(t),t)\gamma(t)dt\end{aligned}$$

3. Verifying martingales
   (a) In discrete time, it suffices to check only one step, i.e. $E[M(t+1)|\mathcal{F}_t] = M(t)$ thanks to the tower property.
   (b) In continuous time, most of the time, it suffices to express M as an integral of a Brownian motion, i.e. $M(t) = \int_0^t Y(s)dW(s)$ for some Y. In practice, typically all you need to show is

$$dM(t) = 0dt + Y(t)dW(t),$$

   i.e. no drift term. Otherwise, you have to stick with the definition, i.e. check $E[M(t)|\mathcal{F}_s] = M(s)$.
   (c) To show something is NOT a martingale, one way is to compute its expected value. If the expected value is not a constant, it is not a martingale.
4. Itô's lemma. The reason we need to use Itô's lemma is that the Brownian motion fluctuates too much in a sense that the quadratic variation is non-zero. Therefore, we cannot take limits as we would like to in standard calculus. For example $d\log W(t) \neq \frac{dW(t)}{W(t)}$. The idea of Itô's lemma is to look at the *function* of a Brownian motion / or stochastic processes. Of course, it depends on the function f. Let's assume f has only 1 argument. Itô's lemma then says for a process X, we have (roughly speaking) $df(X(t)) = f_x(X(t))dX(t) + \frac{1}{2}f_{xx}(X(t))(dX(t))^2$, where $f_x(y) = \frac{\partial}{\partial x}f(x)|_{x=y}$ is the partial derivative of f w.r.t. the x argument. Similarly f_{xx} is the second derivative of f w.r.t. the x argument. We usually write the shorthand notation

$$df = f_x dX + \frac{1}{2}f_{xx}(dX)^2.$$

   This means in order to compute the df, we need to (1) compute the partial derivatives f_x and f_{xx}, (2) then replace x by $X(t)$ (3) compute $dX(t)$ and $(dX(t))^2$ using the "box rule" (i.e. $dt \times dt = 0$, $dt \times dW = 0$ and $dW \times dW = dt$) and (4) finally putting everything in the formula. Below are some examples.

(a) $f(x) = \log(x)$, $dS(t) = S(t)(\mu dt + \sigma dW(t))$ and we are interested in $d\log S(t) = df(S(t))$. In this case, we have (i) $f_x = \frac{1}{x}$ and $f_{xx} = \frac{-1}{x^2}$ (ii) $(dS(t))^2 = (S(t))^2\sigma^2 dt$, which altogether yields (iii)

$$\begin{aligned} d\log S(t) = df(S(t)) &= \frac{1}{S(t)}S(t)(\mu dt + \sigma dW(t)) + \frac{1}{2}\frac{-1}{(S(t))^2}(S(t))^2\sigma^2 dt \\ &= (\mu - \frac{1}{2}\sigma^2)dt + \sigma dW(t), \end{aligned}$$

which is not the same as $\frac{dS(t)}{S(t)}$.

Of course, in an exam situation, you will be asked to find $d\log(S(t))$ without telling you what the function f is. In this case, it is also one of your tasks to figure out what f is.

(b) $f(t,x) = tx$, with the same $dS(t)$ and we are interested in $d(tS(t)) = df(t, S(t))$. In this case, our function has an extra t argument and therefore the Itô's lemma should also have a t argument, i.e.

$$df = f_t dt + f_x dX + \frac{1}{2}f_{xx}(dX)^2.$$

To answer this question we compute (i)$f_t = x$, $f_x = t$ and $f_{xx} = 0$ (ii) $(dS(t))^2 = (S(t))^2\sigma^2 dt$ (although not necessary here because the partial derivative f_{xx} is zero) which altogether yields (iii)

$$\begin{aligned} d(tS(t)) = df(t, S(t)) &= S(t)dt + tS(t)(\mu dt + \sigma dW(t)) \\ &= S(t)((\mu t + 1)dt + \sigma t dW(t)) \end{aligned}$$

(c) $f(x,y) = \frac{x}{y}$ with $\frac{dX(t)}{X(t)} = \mu_x dt + \sigma_x dW(t)$, $\frac{dY(t)}{Y(t)} = \mu_y + \sigma_y d\tilde{W}(t)$ and $d(W(t)\tilde{W}(t)) = \rho dt$ and we are interested in $d\frac{X(t)}{Y(t)} = df(X(t), Y(t))$. As we are having 2 stochastic processes, the second degree terms matter. Therefore, the Itô's lemma is

$$df = f_x dX + \frac{1}{2}f_{xx}(dX)^2 + f_y dY + \frac{1}{2}f_{yy}(dY)^2 + f_{xy}(dX)(dY).$$

Note the cross term does not have the half because there are 2 of them, namely $f_{xy}(dX)(dY)$ and $f_{yx}(dY)(dX)$. The answer to this problem is left as an exercise.

(d) $f(t,x,y) = e^t xy$. At this point, we should be able to see that the Itô's lemma should take a form of

$$df = f_t dt + f_x dX + \frac{1}{2}f_{xx}(dX)^2 + f_y dY + \frac{1}{2}f_{yy}(dY)^2 + f_{xy}(dX)(dY).$$

[Note: for correctness, you are strongly encouraged to write out f explicitly for each calculation. Sometimes people write $\frac{\partial}{\partial t}f(t, S(t))$ which is confusing because there is also a t inside the $S(t)$. By writing out f, you can avoid this confusion. To summarise, the version of Itô's lemma depends on what your function is, in particular the number of arguments.]

5. Given $\frac{dS(t)}{S(t)} = \mu(S(t),t)dt + \sigma(S(t),t)dW(t)$ and $\frac{dB(t)}{B(t)} = rdt$. In order to have $\frac{S(t)}{B(t)}$ being a martingale, we should change the measure such that

$$\frac{dS(t)}{S(t)} = rdt + \sigma(S(t),t)dW^Q(t),$$

i.e. replacing $\mu(S(t),t)$ by r and replacing "W" with W^Q. The intuition is that in the risk neutral world, investors only care about the expected return and they are indifference with investing the stock or the bond. Therefore, the bond and the stock must appreciate at the same rate rdt.

6. Hedging portfolio. Recall that we have 2 Q-martingales

$$Y(t) = E_Q[\frac{X}{B(T)}|\mathcal{F}_t],$$
$$Z(t) = \frac{S(t)}{B(t)}.$$

Note for $Y(t)$, the term inside the expectation is always $X/B(T)$ with capital T, not the time t. To form a hedging portfolio, we always invest $\phi(t)$ of the stocks at time t, (where ϕ is given by the Martingale representation theorem) and make sure that the portfolio always have amount $V(t) = Y(t)B(t)$ at time t. This implies that the number of bond invested is given by $\psi(t) = Y(t) - \phi(t)Z(t)$. You should check that with such ψ you have $\phi(t)S(t) + \psi(t)B(t) = V(t) = Y(t)B(t)$.

Now, with such portfolio, we have $V(T) = Y(T)B(T) = X$ and $V(0) = Y(0) = E_Q[X/B(T)]$ (recall the rules for conditional expectations). Therefore, our strategy is consistent with the derivative payoff X. IF furthermore this strategy is self-financing, by a no-arbitrage argument, the price of the derivative MUST be $Y(0)$. In fact, it can be shown this strategy is self-financing with a few steps of algebra. (again via Itô's lemma)

7. Calculating expected values of call/put option payoffs. You can also refer to P.18 of the formula booklet. Here we should discuss the general case when $S(T) \sim S(0)\exp(N(m,v)) \sim S(0)\exp(m + \sqrt{v}Z)$ which is a log-normal distribution. (v stands for variance) Note

(a) $E[S(T)] = S(0)\exp(m + \frac{1}{2}v)$ which can be shown by MGF or using the result in (d).

(b) We shall express everything in terms of Z (the sole randomness) if possible. For example,

$$E[1(S(T) > K)] = P[S(T) > K] = P[Z > -d_2(K)]$$

where $d_2(K) = \frac{\log\frac{S(0)}{K} + m}{\sqrt{v}}$. Thus, it is easy to calculate the expected value of an indicator function (which appears very often in the payoff function of options).

(c) Another trick is to note that Z has the same distribution with $Z' = -Z$ (both being $N(0,1)$), hence the above expression can be computed as

$$P[Z > -d_2(K)] = P[-Z < d_2(K)] = P(Z' < d_1(K)) = \Phi(d_2(K)).$$

(d) For any (nice) set B, we have the following identity

$$E[e^{\sqrt{v}Z}1(Z \in B)] = \exp(\frac{1}{2}v)P[\tilde{Z} + \sqrt{v} \in B],$$

where Z and $\tilde{Z}$ are $N(0,1)$ distributed.
[The adjustments are (i) $e^{\sqrt{v}Z} \to \exp(\frac{1}{2}v)$ and (ii) $Z \in B \to \tilde{Z} + \sqrt{v} \in B$!]

(e) Sometimes the payoff function is linear in $S(T)$ and thus the expected payoff takes a form of

$$E[S(T)1(S(T) \in A)]$$

where A is some (nice) set. We shall also express everything in terms of Z. Note the event $(S(T) \in A)$ is the same as the event $(Z \in B)$ for some set B. Therefore, the above is evaluated as

$$\begin{aligned} E[S(T)1(S(T) \in A)] &= E[S(0)\exp(m + \sqrt{v}Z)1(Z \in B)] \\ &= S(0)e^m E[e^{\sqrt{v}Z} 1(Z \in B)] \\ &= S(0)e^m e^{\frac{1}{2}v} P[\tilde{Z} + \sqrt{v} \in B] \quad \text{(using (d))} \\ &= E[S(T)]P[\tilde{Z} + \sqrt{v} \in B], \quad \text{(using (a))} \end{aligned}$$

or simply

$$E[S(T)1(Z \in B)] = E[S(T)]P[\tilde{Z} + \sqrt{v} \in B].$$

[As a result, we can "pull out" the $S(T)$ and substitute Z with $\tilde{Z} + \sqrt{v}$. Hence, the problem reduces to identify the set B (the range of the integration) which is an interval related to $-d_2$.]
[For example, when $S(T) \to Z$, we have $K \to -d_2(K)$.]

(f) Examples:

i. The linear part of the call option. We have

$$\begin{aligned} E[S(T)1(S(T) > K)] &= E[S(T)1(Z > -d_2(K))] \quad \text{(Identify } B) \\ &= E[S(T)]P[\tilde{Z} + \sqrt{v} > -d_2(K)]. \quad \text{(using the trick)} \end{aligned}$$

Normally, for this expression, one would further proceed to yield

$$\begin{aligned} E[S(T)1(S(T) > K)] &= E[S(T)]P[\tilde{Z} + \sqrt{v} > -d_2(K)] \\ &= E[S(T)]P[-\tilde{Z} < d_2(K) + \sqrt{v}] \\ &= E[S(T)]\Phi(d_2(K) + \sqrt{v}) \end{aligned}$$

and the term $d_2(K) + \sqrt{v}$ is often denoted as $d_1(K)$.

ii.

$$\begin{aligned} E[1(K_2 > S_T > K_1)] &= P[-d_2(K_2) > Z > -d_2(K_1)] = P[d_2(K_2) < -Z < d_2(K_1)] \\ &= \Phi(d_2(K_1)) - \Phi(d_2(K_2)) \end{aligned}$$

and

$$\begin{aligned} E[S(T)1(K_2 > S_T > K_1)] &= E[S(T)1(-d_2(K_2) > Z > -d_2(K_1))] \quad \text{(Identify } B) \\ &= E[S(T)]P[-d_2(K_2) > \tilde{Z} + \sqrt{v} > -d_2(K_1)] \quad \text{(using the trick)} \\ &= E[S(T)]P[d_1(K_2) < -\tilde{Z} < d_1(K_1)] \\ &= S(0)\exp(m + \frac{1}{2}v)(\Phi(d_1(K_1)) - \Phi(d_1(K_2))), \end{aligned}$$

where the interval of integration $B = (-d_2(K_1), -d_2(K_2))$.

iii. Suppose we have $\frac{dS(t)}{S(t)} = rdt + \sigma dW^Q(t)$ and $\frac{dB(t)}{B(t)} = rdt$, and we are interested in the price of the payoff $X = S(T)^2 1(S(T)^2 > K)$. In this case, we need to calculate $E[S(T)^2 1(S(T)^2 > K)]$. Note

$$S(T)^2 = S(0)^2 \exp(2(r - \frac{1}{2}\sigma^2)T + 2\sigma W^Q(T)) \sim S(0)^2 \exp(N(m, v))$$

with $m = 2(r - \frac{1}{2}\sigma^2)T$ and $v = 4\sigma^2 T$. Hence, from the above analysis, we have

$$\begin{aligned}
E_Q[X] &= E[S(T)^2 1(S(T)^2 > K)] \\
&= E[S(T)^2 1(Z > -d_2(K))],\ d_2(K) = \frac{\log \frac{S(0)^2}{K} + m}{\sqrt{v}} \quad \text{(Identify } B\text{)} \\
&= E[S(T)^2] P[\tilde{Z} + \sqrt{v} > -d_2(K)] \quad \text{(using the trick)} \\
&= S(0)^2 \exp(m + \frac{1}{2}v)\Phi(d_2(K) + \sqrt{v}) \\
&= S(0)^2 \exp(2(r - \frac{1}{2}\sigma^2)T + \frac{1}{2}4\sigma^2 T)\Phi(\frac{\log \frac{S(0)^2}{K} + 2(r - \frac{1}{2}\sigma^2)T}{\sqrt{4\sigma^2 T}} + \sqrt{4\sigma^2 T}) \\
&= S(0)^2 \exp(2rT + \sigma^2 T)\Phi(\frac{\log \frac{S(0)^2}{K} + (2r + 3\sigma^2)T}{\sqrt{4\sigma^2 T}})
\end{aligned}$$

which gives

$$Price = e^{-rT} E_Q[X] = S(0)^2 \exp((r + \sigma^2)T)\Phi(\frac{\log \frac{S(0)^2}{K} + (2r + 3\sigma^2)T}{\sqrt{4\sigma^2 T}}).$$

iv. Calculating $E[S(T)^{3/4} 1(S(T) > K)]$. In this case, we need to transform the problem to $E[S(T)^{3/4} 1(S(T)^{3/4} > K^{3/4})]$, where we can proceed by identifying the interval of integration $B = (-d_2(K^{3/4}), \infty)$.

8. Generally speaking, when given only stock and bond, we would like to tune the Q-measure such that $\frac{S(t)}{B(t)}$ is a martingale.

9. Proof of the formula in 7(d). We use the "completing the square" approach as follows:

$$\begin{aligned}
E[e^{\sqrt{v}Z} 1(Z \in B)] &= \int_B e^{\sqrt{v}z} f_Z(z) dz \\
&= \int_B \frac{1}{\sqrt{2\pi}} \exp(\sqrt{v}z) \exp(\frac{-1}{2} z^2) dz \\
&= \int_B \frac{1}{\sqrt{2\pi}} \exp(\frac{-1}{2}(z^2 - 2\sqrt{v}z)) dz \\
&= \int_B \frac{1}{\sqrt{2\pi}} \exp(\frac{-1}{2}(z^2 - 2\sqrt{v}z + v)) \exp(\frac{1}{2}v) dz \\
&= \exp(\frac{1}{2}v) \int_B \frac{1}{\sqrt{2\pi}} \exp(\frac{-1}{2}(z - \sqrt{v})^2) dz \\
&= \exp(\frac{1}{2}v) P[\tilde{Z} + \sqrt{v} \in B]
\end{aligned}$$

where $\tilde{Z} \sim N(0, 1)$ and the last equality follows because the integral in the second to last line is $P[N(\sqrt{v}, 1) \in B]$ and we take out the mean $\sqrt{v}$ to get a $N(0, 1)$ distributed $\tilde{Z}$).

10. Sometimes it may be useful to recognize the payoff function (by drawing it). For example

$$P(S) = \min\{(S(T) - K_1)1_{S(T)>K_1}, K_2 - K_1\}$$

is a bull-spread which can be expressed as a difference of 2 call options with maturity K_1 and K_2.

4.1.3 Practice Questions

Exercise 4.1: [Solution] Consider the 2 step binomial tree used in the lectures to illustrate the Radon Nikodym derivative. Following the notation of the lecture notes $\varsigma(t)$ is the Radon Nikodym derivate but only following paths up to time t, and only looking at the ration of probabilities up to that time.

1. Draw the process corresponding to the process $\varsigma(t)$ on the tree.
2. Show that another representation for $\varsigma(t)$ is

$$\varsigma(t) = E_P\left[\frac{dQ}{dP}|F_t\right]$$

for t=0,1,2.

Exercise 4.2: [Solution] Suppose we have 2 sets of measure P and Q (defined on the same probability space $(\Omega, \mathcal{F})$) which are equivalent, i.e. $\mathbb{P}_P[A] = 0 \iff \mathbb{P}_Q[A] = 0$ (for any set $A \in \mathcal{F}$). Recall the Radon-Nikodym Derivative is defined by

$$\frac{dQ}{dP}(\omega) = \frac{\mathbb{P}_Q[\omega]}{\mathbb{P}_P[\omega]}$$

for any set $\omega \in \Omega$. Prove:

1. $E_p\left[\frac{dQ}{dP}\right] = 1$
2. $E_p\left[\frac{dQ}{dP}X\right] = E_Q[X]$
3. $E_p\left[\frac{dQ}{dP}1_A\right] = \mathbb{P}_Q[A]$.

Exercise 4.3: [Solution] (Baxter and Rennie, Chapter 3) If Z is normal (0,1), then the process $X(t) = \sqrt{t}Z$ is continuous and is marginally distributed as a normal N(0,t). Is $X(t)$ a Brownian motion?

Exercise 4.4: [Solution] (Baxter and Rennie, Chapter 3) If $W(t)$ and $Z(t)$ are two independent Brownian motions and ρ is a constant between -1 and 1, then is the process $X(t) = \rho W(t) + \sqrt{1-\rho^2}Z(t)$ a Brownian motion?.

Exercise 4.5: [Solution] If $W(\cdot)$ is a standard brownian motion, then is the process $\frac{1}{\sqrt{c}}W(ct)$ a brownian motion? (c is some arbitrary constant)

Exercise 4.6: [Solution] If $W(\cdot)$ is a standard brownian motion, then is the process $W(s) - W(s-t)$ where $0 \le t \le s$ a brownian motion?

Exercise 4.7: [Solution] Let $X(t)$ be the standard brownian motion, find the following for $u < s < t$:

1. $E(X(t)|X(s) = B)$
2. $Var(X(t)|X(s) = B)$
3. $Cov(X(s), X(t))$
4. The distribution of $X(t) + X(s)|X(u) = A$

Exercise 4.8: [Solution]

Discuss two methods that can be used to build a continuous time financial model using Brownian Motion.

Exercise 4.9: [Solution] Luenberger, Ex 11.6

Exercise 4.10: [Solution] Luenberger, Ex 11.7

Exercise 4.11: [Solution] Luenberger, Ex 11.8

Exercise 4.12: [Solution] Suppose we have the following asset dynamics:

$$dS(t) = \mu S^{\frac{3}{2}}(t)dt + \sigma S^{\frac{3}{2}}(t)dW(t)$$

Now we have a derivative that depends on the process $Y(t) = S^2(t)$. Find the stochastic differential equation of this process.

Exercise 4.13: [Solution] Let $W(\cdot)$ be a P brownian motion, determine whether the following processes are P martingales and provide justification for your answer.

1. $X(t) = W^2(t)$
2. $Y(t) = \frac{1}{t}W(t)$

Exercise 4.14: [Solution] Given that $W(t)$ is a P-Brownian Motion and we have a process $Y(t) = e^{-\gamma W(t) - \frac{1}{2}\gamma^2 t}$ where γ is a constant. Now show that we can define a Q probability by

$$Q[A] = E_P[Y(T)1_A].$$

[Hint: P is a probability if (1) $P[A] > 0$ for any event A (2) $P[\Omega] = 1$ and (3) for disjoint sets (events) E_k, it holds $P[\cup_k E_k] = \sum_k P[E_k]$.]

Exercise 4.15: [Solution] [This question is for your own interest, it is not examable.]

Given that $W(t)$ is a P-Brownian Motion and we have a process $Y(t) = e^{-\gamma W(t) - \frac{1}{2}\gamma^2 t}$ where γ is a constant. Define a Q probability by

$$Q[A] = E_P[Y(T)1_A].$$

Proof that $W_Q = \{W(t) + \gamma t; 0 \leq t \leq T\}$ is a Q-Brownian Motion.

[Hint: To check the stationary and independent increments, consider $E_Q[1_A e^{\theta(W_Q(t) - W_Q(s))}]$ where $\theta \in \mathbb{R}$, $A \in \mathcal{F}_u$ and $0 \leq u \leq s \leq t \leq T$.]

Exercise 4.16: [Solution] Suppose the stock price can be modelled by geometric Brownian motion

$$\begin{aligned} dS(t) &= \mu S(t)\,dt + \sigma S(t)\,dW(t) \\ S(0) &= s \end{aligned}$$

and that the bond can be represented as

$$\begin{aligned} dB(t) &= rB(t)\,dt \\ B(0) &= 1 \end{aligned}$$

1. what is the solution to the stock price SDE?
2. what is the solution to the bond price differential equation?
3. What are the dynamics of the discounted stock price $\frac{S(t)}{B(t)}$
4. Change $Z(t)$ from the P measure to the Q measure and find the market price of risk γ that ensures $Z(t)$ is a Q-martingale.

Exercise 4.17: [Solution] Suppose that instead of using the usual GBM for stock prices, we find that the new dynamics of the stock price is given by:

$$dS(t) = \mu S^2(t)dt + \sigma S^2(t)dW(t)$$

while the bond process is the same as usual:

$$dB(t) = rB(t)dt$$

1. Find the dynamics of the discounted stock price $\frac{S(t)}{B(t)}$
2. Find the market price of risk under no arbitrage conditions

Exercise 4.18: [Solution] Suppose that we have a particular stock that has the following dynamic:

$$dS(t) = \mu S^{\frac{3}{2}}(t)dt + \sigma S^{\frac{3}{2}}(t)dW(t)$$

while the bond process is the usual $dB(t) = rB(t)dt$.

1. Find the dynamics of the discounted stock price $\frac{S(t)}{B(t)}$
2. Find the market price of risk under no arbitrage conditions

Exercise 4.19: [Solution]

Consider a model where

$$\begin{aligned} dS(t) &= \mu S(t)dt + \sigma S(t)dW(t) \\ S(0) &= s \end{aligned}$$

and a bond $B(t)$ is available for investment. The interest rate $r = 0$.

Working from first principles (no arbitrage, replicating portfolios, martingales etc) derive a formula that can be used to value derivatives (e.g a call option).

Exercise 4.20: [Solution] As alternative to the standard GBM model some authors have proposed a model where

$$\begin{aligned} dS(t) &= \mu S(t)dt + \sigma S^{\frac{3}{4}}(t)dW(t) \\ S(0) &= s \end{aligned}$$

and a bond $B(t)$ is available for investment with some constant interest rate $r > 0$..

1. From general reasoning provide a guess as to the time 0 value of a derivative that pays X at time T?
2. Working from first principles (no arbitrage, replicating portfolios, martingales etc), discuss how one can value derivatives in this model.

Exercise 4.21: [Solution] Check whether the following portfolios are self-financing if $dS(t) = \mu dt + \sigma dW(t)$ and $B(t) = 1$:

1. $\phi(t) = 1, \psi(t) = 1$
2. $\phi(t) = 2W(t), \psi(t) = -2\mu tW(t) - W^2(t)$
3. $\phi(t) = \frac{1}{2}S(t), \psi(t) = -\mu\sigma tW(t)$

Exercise 4.22: [Solution] Suppose there is a non-dividend paying asset whose price follows the process

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t).$$

as well as a bond satisfying

$$dB(t) = rB(t)dt$$

Under the Q measure it is known that

$$dS = rS(t)dt + \sigma S(t)dW^Q(t)$$

Consider a derivative with a payoff (at time T) which only depends on the price of the stock at T. It is known that the price of a such a derivative at time t can be represented as $V(S(t), t)$, a function of $S(t)$ and t.

1. Apply Ito's lemma to $V(S(t), t)$ to find its dynamics under Q.
2. Using the self financing condition and the equation for $dB(t)$, show that (by matching coefficients of $dW^Q(t)$ and dt) we have

   (i)
   $$\phi(t) = \frac{\partial V(S(t), t)}{\partial S(t)}$$

   (ii)
   $$\frac{1}{2}\sigma^2 S^2(t)\frac{\partial^2 V(S,t)}{\partial S^2}\Big|_{S=S(t)} + rS(t)\frac{\partial V(S,t)}{\partial S}\Big|_{S=S(t)} - rV(S(t),t) + \frac{\partial V(S,t)}{\partial t}\Big|_{S=S(t)} = 0$$

   (Remark: the above is called the Black-Scholes Partial Differential Equation. Historically Black and Scholes (and Merton) solved this equation (along with the requirement that $V(S,T) = (S-K)^+$) to arrive at the call option formula. Besides being of historical interest, this link between PDEs and option pricing opens up a lot of useful numerical techniques that people use in practice to price options.)

Exercise 4.23: [Solution] A researcher has recently proposed that the stock price process should follow:

$$dS(t) = \mu S^{-1}(t)dt + \sigma S(t)dW(t)$$

If the bond process is the same as the usual Black-Scholes assumption, find the equivalent process in the Q measure.

Exercise 4.24: [Solution] Suppose the stock price is modelled by

$$\begin{aligned} dS(t) &= \mu S(t)\,dt + \sigma S(t)\,dW(t) \\ S(0) &= s \end{aligned}$$

in the P measure. Show that the probability that a European call option with a strike price of K will be exercised is given by

$$\Phi(d).$$

Give a formula for d.

Exercise 4.25: [Solution] Suppose the stock price is modelled by

$$\begin{aligned} dS(t) &= \mu S(t)\,dt + \sigma S(t)\,dW(t) \\ S(0) &= s \end{aligned}$$

in the P measure. Find the probability of an option being exercised if the exotic option makes a positive payout if the stock price lies between K_1 and K_2 at maturity.

Exercise 4.26: [Solution] You are given the following information about a European put option on a non-dividend paying stock:

current stock price:	S
risk-free rate, compounded continuously p.a.:	5%
volatility parameter:	0.2
put option exercise price:	750
time to expiration:	3 months

1. Using the Black-Scholes equation, derive an expression of the value of the option in terms of the current price of the stock.
2. Calculate the price for this option if the current stock price is 775

Exercise 4.27: [Solution] The Black-Scholes formula for the value of a European call option on a non-dividend paying stock at time t can be expressed as:

$$c(K,t) = S(t)\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2)$$

where

$$d_1 = \frac{\log\left(\frac{S(t)}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

and

$$d_2 = d_1 - \sigma\sqrt{T-t}.$$

1. Define all the symbols used in the formula.
2. Using the formula, show that the call price $c(K,t)$ is the maximum of $S(0) - Ke^{-r(T-t)}$ or zero, depending on the strike price, when σ tends to zero.

Exercise 4.28: [Solution] Suppose we have the following asset dynamics:

$$\begin{aligned} dS(t) &= \mu S(t)dt + \sigma S(t)dW^Q(t) \\ dB(t) &= rB(t)dt \end{aligned}$$

together with the standard "nice" assumptions.

Find the time 0 value of a binary option that pays at time T \$100 if $S(T) \geq K$ or \$0 otherwise.

Exercise 4.29: [Solution] Assume that we have a particular option that has the following payoff:

$$X = \begin{cases} 0 & \text{if } S(T) \leq K_1 \\ S(T) - K_1 & \text{if } K_1 < S(T) \leq K_2 \\ K_2 - K_1 & \text{if } S(T) > K_2 \end{cases}$$

Assuming Black-Scholes assumptions are true, find the price of this particular product.

Exercise 4.30: [Solution] Suppose the stock has the following P-dynamics:

$$dS(t) = \mu S^{\frac{3}{2}}(t)dt + \sigma S(t)dW(t)$$

and the market price of risk is given by: $\gamma(t) = \frac{\mu S^{\frac{1}{2}}(t) - r}{\sigma}$. Find the price of an option that pays $\max(S(T) - K, 0)$ at maturity when the bond process is the usual $dB(t) = rB(t)dt$.

Exercise 4.31: [Solution] If we have the usual Black-Scholes assumptions:

$$\begin{aligned} dS(t) &= \mu S(t)dt + \sigma S(t)dW(t) \\ dB(t) &= rB(t)dt \end{aligned}$$

Find the price of an option that has the following payoff at maturity:

$$X = \begin{cases} \$25 & \text{if } S(T) > K_1 \\ \$50 & \text{if } S(T) > K_2 \\ \$100 & \text{if } S(T) > K_3 \end{cases}$$

Exercise 4.32: [Solution] Define the delta, gamma, and theta of an option.

Exercise 4.33: [Solution] Luenberger, Q13.5 (1Ed) 15.5 (2Ed)

4.1.4 Discussion Questions

Exercise 4.34: [Solution] Derive (intuitively) Ito's lemma using the Taylor expansion and Box rule.

1. $Z(t) = f(W(t))$, where f is twice differentiable.

2. $Z(t) = f(t, W(t))$, where f is differentiable in the first argument and twice differentiable in the second argument.

3. $Z(t) = f(W_1(t), W_2(t))$, where W_1 and W_2 are standard Brownian motions with correalation $dW_1(t)dW_2(t) = \rho dt$ and f is twice differentiable in both arguments.

4. $Z(t) = f(t, W_1(t), W_2(t))$, where W_1 and W_2 are standard Brownian motions with correalation $dW_1(t)dW_2(t) = \rho dt$ and f is differentiable in the first argument and twice differentiable in the last two arguments.

Exercise 4.35: [Solution] Suppose

$$\begin{aligned} dX(t) &= (\alpha + \beta X(t))\,dt + dW(t) \\ X(0) &= x \end{aligned}$$

what are the dynamics of

1. $Y(t) = e^{X(t)}$
2. $Y(t) = e^{X(t)-\frac{1}{2}t}$
3. $Y(t) = X^2(t)$

Exercise 4.36: [Solution] Suppose

$$\begin{aligned} dX(t) &= (\alpha + \beta X(t))\,dt + \sqrt{X(t)}dW(t) \\ X(0) &= x \end{aligned}$$

what are the dynamics of

1. $Y(t) = e^{X(t)}$
2. $Y(t) = e^{X(t)-\frac{1}{2}t}$
3. $Y(t) = X^2(t)$

Exercise 4.37: [Solution] Find the closed form solution of the following SDE:

$$dY(t) = \left((\mu - \alpha)tdt + \sigma^2 dW(t)\right)Y(t)$$

Exercise 4.38: [Solution] Find the SDE of the following equation:

$$e^{3t^2 - \mu t + \sigma W(t) - 2\sigma W^2(t)}$$

Exercise 4.39: [Solution] Find the Radon-Nikodym derivative such that we have

$$dX(t) = X(t)(\sigma dW(t) + \mu dt)$$

and

$$dX(t) = X(t)\,\sigma dW^Q(t)$$

Exercise 4.40: [Solution] Do you think it is possible to find a Radon-Nikodym derivative such that we have

$$dX(t) = X(t)(\sigma dW(t) + \mu dt)$$

and

$$dX(t) = X^2(t)\,\sigma dW^Q(t)?$$

Briefly (and heuristically) justify your answer.

Exercise 4.41: [Solution] Suppose

$$\begin{aligned} dX(t) &= (\alpha + \beta X(t))\,dt + dW(t) \\ X(0) &= x \end{aligned}$$

Consider a measure Q where the Radon-Nikodym derivative is defined via

$$\gamma(t) = \kappa$$

i.e. a constant κ over time. Find the dynamics of $X(\cdot)$ under the measure Q.

Exercise 4.42: [Solution] If we know a stock has the following process:

$$dS(t) = \mu S^{\frac{3}{2}}(t)dt + \sigma S(t)dW(t)$$

and the bond process is given by $dB(t) = rB(t)dt$.

1. Find the market price of risk under no arbitrage
2. Find the stock process under the Q measure

Exercise 4.43: [Solution] If we know a stock has the following process:

$$dS(t) = \mu S^2(t)dt + \sigma S^2(t)dW(t)$$

and the bond process is given by $dB(t) = rB(t)dt$.

1. Find the market price of risk under no arbitrage
2. Find the stock process under the Q measure

Exercise 4.44: [Solution] Consider constants θ, γ, with $\gamma > 0$, and let $f(x;\theta,\gamma^2)$ denote the probability density function (evaluated at x) of a normal distribution with mean θ and variance γ^2. Show that we have the following identity:

$$e^x f(x;\theta,\gamma^2) = e^{\theta+\frac{1}{2}\gamma^2} f(x;\theta+\gamma^2,\gamma^2).$$

Exercise 4.45: [Solution] Assume the Black Scholes assumptions for the dynamics of the stock and bond. No dividends are payable. Find the price of an option that pays, at time T, $S(T)$ if it is between two constants K_1 and K_2, or 0 otherwise.

Exercise 4.46: [Solution] Consider a Brownian motion $W_1(t)$ under some measure P_1, it is known that, for constant ρ, and $y > 0$, we have

$$P_1\left(max_{0\le s\le t}(W_1(s) + \rho s) > y\right) = \Phi\left(\frac{-y+\rho t}{\sqrt{t}}\right) + e^{2\rho y}\Phi\left(\frac{-y-\rho t}{\sqrt{t}}\right)$$

Assume that the stock price satisfies

$$dS(t) = rS(t)dt + \sigma S(t)dW^Q(t)$$

under the pricing measure Q (it is also called a risk neutral measure). A simple form of Barrier Option pays \$1 at time T provided the maximum of the stock price from 0 to T is greater then $KS(0)$ for some constant K. Find the price of this option at time 0.

Exercise 4.47: [Solution] Suppose we have another barrier option that has the same stock price process as the previous question, however the option pays out in different levels. The option pays out \$0.5 at time T provided the maximum stock price up to time T is greater than $K_1S(0)$ and it pays out \$1 at time T provided the maximum stock price up to T is greater than $K_2S(0)$, with $K_1 < K_2$. Find the price of this option.

Exercise 4.48: [Solution] Suppose the stock price follows a GBM, and pays a continuous dividend of δSdt. Find the dynamics of the stock price under the measure Q. (Extension: Derive the price of a call option in this setting).

Exercise 4.49: [Solution] Suppose we have an option that has the following payoff:

$$\begin{cases} S(T) & \text{if } S(T) > K \\ M & \text{if } S(T) \leq K \end{cases}$$

where M is a constant and $M < K$. Assuming Black-Scholes assumptions are true, find the price of this option at time 0. Is there any reason why someone would buy this type of option?

Exercise 4.50: [Solution] Suppose we know that the Q dynamics of a particular stock is:

$$dS(t) = rS(t)dt + \sigma S(t)dW^Q(t)$$

and $S(0) = s$. If $r > 0, \sigma > 0$

1. Find the SDE for the process $X(t) = S^2(t)$ and hence deduce its distribution
2. Find the price of an option that pays $\max(S(T)^2 - K, 0)$ at maturity

4.2 Solutions

4.2.1 Continuous Time Derivative Valuation

Solution 4.1: [Exercice]

1. The Radon Nikodym derivative $\varsigma(t)$ is the ratio of the p and q probabilities.

$t=0$	$t=1$	$t=2$
		$\frac{q_1q_2}{p_1p_2}$
	$\frac{q_1}{p_1}$	$\frac{q_1(1-q_2)}{p_1(1-p_2)}$
1		
	$\frac{1-q_1}{1-p_1}$	$\frac{(1-q_1)q_3}{(1-p_1)p3}$
		$\frac{(1-q_1)(1-q_3)}{(1-p_1)(1-p_3)}$

2. Recall the randomness here is the path and by definition of dQ/dP we have

Path	$\frac{dQ}{dP}$
(u,u) :	$\frac{q_1q_2}{p_1p_2}$
(u,d) :	$\frac{q_1(1-q_2)}{p_1(1-p_2)}$
(d,u) :	$\frac{(1-q_1)q_2}{(1-p_1)p_2}$
(d,d) :	$\frac{(1-q_1)(1-q_2)}{(1-p_1)(1-p_2)}$

At time 2, we know $\frac{dQ}{dP}$ and therefore

$$\varsigma(2) = E_P\left[\frac{dQ}{dP}\middle|\mathcal{F}_2\right] = \frac{dQ}{dP}$$

which gives the column for $t = 2$ in part 1 (depending on which path it takes).

For $t = 1$, there are 2 cases depending on whether it goes up (which correspond to the set $(u) = \{(u,u),(u,d)\}$) or goes down (correspond to the set $(d) = \{(d,u),(d,d)\}$).

When it goes up, (we are restricted to the set $(u) \in \mathcal{F}_1$ and) we have

$$\begin{aligned}
\varsigma(1)(u) &= E_P\left[\frac{dQ}{dP}\middle|(u)\right] \\
&= \frac{\frac{dQ}{dP}(u,u)\mathbb{P}_P[(u,u)] + \frac{dQ}{dP}(u,d)\mathbb{P}_P[(u,d)]}{\mathbb{P}_P[(u,u)]+\mathbb{P}_P[(u,d)]} \quad \text{(Conditional expection formula)} \\
&= \frac{q_1q_2 + q_1(1-q_2)}{p_1p_2 + p_1(1-p_2)} \\
&= \frac{q_1}{p_1}
\end{aligned}$$

Likewise, for $(d) \in \mathcal{F}_2$ (when it goes down), we have

$$\begin{aligned}
\varsigma(1)(d) &= E_P\left[\frac{dQ}{dP}\middle|(d)\right] \\
&= \frac{\frac{dQ}{dP}(d,u)\mathbb{P}_P[(d,u)] + \frac{dQ}{dP}(d,d)\mathbb{P}_P[(d,d)]}{\mathbb{P}_P[(d,u)]+\mathbb{P}_P[(d,d)]} \quad \text{(Conditional expection formula)} \\
&= \frac{(1-q_1)q_2 + (1-q_1)(1-q_2)}{(1-p_1)p_2 + (1-p_1)(1-p_2)} \\
&= \frac{1-q_1}{1-p_1}
\end{aligned}$$

Therefore, we recover the column for $t = 1$ in part 1. Finally, for $t = 0$, we have (with P-probability 1)

$$\begin{aligned}
\varsigma(0) &= E_P\left[\frac{dQ}{dP}\middle|\mathcal{F}_0\right] \\
&= E_P\left[\frac{dQ}{dP}\right] \\
&= \frac{dQ}{dP}(u,u))\mathbb{P}_P[(u,u)] + \frac{dQ}{dP}(u,d))\mathbb{P}_P[(u,d)] + \frac{dQ}{dP}(d,u))\mathbb{P}_P[(d,u)] + \frac{dQ}{dP}(d,d))\mathbb{P}_P[(d,d)] \\
&= q_1q_2 + q_1(1-q_2) + (1-q_1)q_1 + (1-q_1)(1-q_2) \\
&= 1
\end{aligned}$$

which recovers the column for $t = 0$ in part 1.

[Remember $\varsigma(t)$ as a conditonal expection is a random variable.]

Solution 4.2: [Exercice]

1.
$$E_P\left[\frac{dQ}{dP}\right] = \sum_{\omega\in\Omega}\frac{\mathbb{P}_Q[\omega]}{\mathbb{P}_P[\omega]}\mathbb{P}_P[\omega] = \sum_{\omega\in\Omega}\mathbb{P}_Q[\omega] = 1$$

as Q is a probability.

2.
$$E_P\left[\frac{dQ}{dP}X\right] = \sum_{\omega\in\Omega}\frac{\mathbb{P}_Q[\omega]}{\mathbb{P}_P[\omega]}X(\omega)\mathbb{P}_P[\omega] = \sum_{\omega\in\Omega}\mathbb{P}_Q[\omega]X(\omega) = E_Q[X]$$

[Note: A random variable X takes different values depending on the drawn sample ω, i.e. it is a function of the elements in the sample space Ω.]

3. Taking $X = 1_A$ in part 2 we have
$$E_P\left[\frac{dQ}{dP}1_A\right] = E_Q[1_A] = \mathbb{P}_Q[A].$$

[The whole proof can be adapted to continuous setting by changing the sum to an integral and the probability to a density if necessary.]

Solution 4.3: [Exercice] We have

1. $X(0) = 0$, $X(t)$ is continuous.
2. The increment $X(t) - X(s) = (\sqrt{t} - \sqrt{s})Z$ which is not stationary nor independent.
3. The increment is not $N(0, t - s)$ in general.

Therefore, it is not a Brownian motion.

Solution 4.4: [Exercice]

1. $X(0) = 0; X(t)$ is continuous since it is a linear combination of 2 continuous processes.
2. We have
$$\begin{aligned} & X(t+s) - X(s) \\ = \; & \rho\left(W(t+s) - W(s)\right) + \sqrt{1-\rho^2}\left(Z(t+s) - Z(s)\right). \end{aligned}$$

(a) Note $W(t+s) - W(s) \sim N(0,t)$ and $Z(t+s) - Z(t) \sim N(0,t)$. Since W and Z are independent, we have $X(t+s) - X(s) \sim N(0, \rho^2 t + (1-\rho^2)t) \sim N(0,t)$

(b) The increment is stationary as it does not depend on s. The increment is also independent of $\mathcal{F}_s$ as both increments of W and Z are.

As X satisfies all conditions, it is a BM.

Solution 4.5: [Exercice] Same as the above questions.

1. $X(0) = \frac{1}{\sqrt{c}}W(0) = 0$. It is clear that X is continuous as W is.

2. We have

$$\begin{aligned} X(t+s) - X(s) &= \frac{1}{\sqrt{c}}W(c(t+s)) - \frac{1}{\sqrt{c}}W(cs) \\ &= \frac{1}{\sqrt{c}}\left(W(ct+cs) - W(cs)\right) \\ &= \frac{1}{\sqrt{c}}W(ct). \end{aligned}$$

(a) This implies $X(t+s) - X(s) = \frac{1}{\sqrt{c}}W(c(t)) \sim N(0, \frac{1}{c}ct) \sim N(0,t)$.

(b) Note the increcment does not depend on s, thus stationary. The increment is also independent of $\mathcal{F}_s$.

As X satisfies all conditions, it is a BM.

Solution 4.6: [Exercice] Same as above questions.

1. $X(0) = W(s) - W(s) = 0$. X is continuous as W is.

2. We have

$$\begin{aligned} X(t+h) - X(h) &= W(s) - W(s-(t+h)) - W(s) + W(s-h) \\ &= W(s-h) - W(s-t-h) \\ &= W(s-h-(s-t-h)) - W(s-t-h-(s-t-h)) \\ &= W(t) \end{aligned}$$

(a) This implies $X(t+h) - X(h) = W(t) \sim N(0,t)$.

(b) Note the increment does not depend on s and hence is stationary. Note $\mathcal{F}_h$ now is $\sigma(W(u); u \in [s-h, s])$, the observation of W from time $s-h$ to s. Moreover, we have $s-(t+h) > 0$ by definition, which gives $t < s-h$. Now, note that $X(t+h)-X(h) = W(t)$ is independent of its future observations $\mathcal{F}_h = \sigma(W(u); u \in [s-h, s])$, we can conclude that X has independent increments.

Therefore X is a BM.

Solution 4.7: [Exercice] $X(t)$ is standard brownian motion so $X(t) \sim N(0,t)$ and has independent, stationary increments. Note we always consider (artifically create) the increment term.

1.

$$\begin{aligned}E(X(t)|X(s)=B) &= E(X(t)-X(s)+X(s)|X(s)=B)\\ &= E(X(t)-X(s)|X(s)=B)+E(X(s)|X(s)=B)\\ &= E(X(t)-X(s))+B \quad \text{(Independent increments)}\\ &= B\end{aligned}$$

2.

$$\begin{aligned}Var(X(t)|X(s)=B) &= Var(X(t)-X(s)+X(s)|X(s)=B)\\ &= Var(X(t)-X(s)+B|X(s)=B)\\ &= Var(X(t)-X(s)|X(s)=B)\\ &= Var(X(t)-X(s)) \quad \text{(Independent increments)}\\ &= Var(N(0,t-s))\\ &= t-s\end{aligned}$$

3.

$$\begin{aligned}Cov(X(s),X(t)) &= Cov(X(s),X(t)-X(s)+X(s))\\ &= Cov(X(s),X(t)-X(s))+Cov(X(s),X(s))\\ &= 0+Var(X(s)) \quad \text{(Independent increments)}\\ &= s\end{aligned}$$

4.

$$\begin{aligned}(X(t)+X(s)|X(u)=A) &= ([X(t)-X(s)]+2\,[X(s)-X(u)]+2X(u)|X(u)=A)\\ &= ([X(t)-X(s)]+2\,[X(s)-X(u)]+2A) \quad \text{(Ind. increments)}\end{aligned}$$

This is just the sum of independent normal variables, hence:

$$\begin{aligned}X(t)+X(s)|X(u)=A &\sim N(0,t-s)+2\times N(0,s-u)+2A\\ &\sim N(2A,t-s+4s-4u)\\ &\sim N(2A,t+3s-4u)\end{aligned}$$

Solution 4.8: [Exercice] In a Brownian motion framework, one can either specify the stock price as a function of a Brownian motion, e.g.

$$S(t)=S(0)\,e^{\left(\mu-\frac{1}{2}\sigma^2\right)t+\sigma W(t)}$$

or alternatively one can specify its dynamics via a SDE e.g.

$$\begin{aligned}dS(t) &= \mu S(t)dt+\sigma S(t)dW(t)\\ S(0) &= s.\end{aligned}$$

If the function form of $S(t)$ is given, we can derive the SDE using Itô's lemma. In this case, the 2 formulation are equivalent.

However from a SDE it is usually impossible to get a “closed form solution” which express $S(t)$ as an explicit function of $W(t)$. In general, there is no reason why there is a “solution” to the SDE!!

Nevertheless it is up to the modeller to choose which to use. In general the SDE form is often chosen as it allows for more generalities easily. For example, the coefficient can depend on the history of the process.

[Of course, one also need to verify that the solution of the SDE exists. This is generally the case when the coefficients are not too large (explode).]

Solution 4.9: [Exercice] $S(1)$ is lognormal with parameters $\nu = \left(\mu - \frac{1}{2}\sigma^2\right) = 0.12, \sigma = 0.40$. Hence

$$\begin{aligned} E\left[\ln\left(S\left(1\right)\right)\right] &= 0.12 \\ \sqrt{Var\left[\ln\left(S\left(1\right)\right)\right]} &= 0.4 \\ E\left[S\left(1\right)\right] &= e^{0.12+\frac{1}{2}(0.4)^2} \\ &= e^{\mu} = 1.22 \end{aligned}$$

(notice that the $-\frac{1}{2}\sigma^2$ term is cancelled out - so Ito's lemma in this case is just telling you the difference between normal and lognormal distributions).

and finally

$$\sqrt{Var\left[S\left(1\right)\right]} = e^{0.2}\left(e^{0.16} - 1\right)^{\frac{1}{2}} = 0.51$$

(the last two calculations are derived from the MGF of a normal, or just by definition of standard deviation of lognormal distributions).

Solution 4.10: [Exercice] We have

$$f\left(x\right) = x^{\frac{1}{2}}$$

so

$$\begin{aligned} f_x &= \frac{1}{2}x^{-\frac{1}{2}} \\ \frac{1}{2}f_{xx} &= -\frac{1}{8}x^{-\frac{3}{2}} \end{aligned}$$

and hence by Ito

$$\begin{aligned} dG(t) = df(S(t)) &= f_x dS + \frac{1}{2}f_{xx}dS^2 \\ &= \frac{1}{2}S^{-\frac{1}{2}}(t)\left(aS(t)dt + bS(t)dW(t)\right) - \frac{1}{8}S^{-\frac{3}{2}}(t)\left(bS(t)dW(t)\right)^2 \\ &= \frac{1}{2}S^{-\frac{1}{2}}(t)\left(aS(t)dt + bS(t)dW(t)\right) - \frac{1}{8}S^{\frac{1}{2}}(t)b^2dt \\ &= \frac{1}{2}S^{\frac{1}{2}}(t)\left(adt + bdW(t)\right) - \frac{1}{8}S^{\frac{1}{2}}(t)b^2dt \\ &= \frac{1}{2}S^{\frac{1}{2}}(t)\left((a - \frac{1}{4}b^2)dt + bdW(t)\right) \\ &= \frac{1}{2}G(t)\left((a - \frac{1}{4}b^2)dt + bdW(t)\right) \end{aligned}$$

Solution 4.11: [Exercice] Since

$$f\left(x\right) = e^x$$

we have

$$\begin{aligned} f_x &= e^x \\ \frac{1}{2} f_{xx} &= \frac{1}{2} e^x \end{aligned}$$

and hence by Ito

$$\begin{aligned} dS(t) = df(Q(t)) &= f_x dQ + \frac{1}{2} f_{xx} (dQ)^2 \\ &= e^{Q(t)} dQ(t) + \frac{1}{2} e^{Q(t)} (dQ(t))^2 \\ &= e^{Q(t)} ((\mu - \frac{1}{2}\sigma^2) dt + \sigma dW(t)) + \frac{1}{2} e^{Q(t)} \sigma^2 dt \\ &= e^{Q(t)} ((\mu - \frac{1}{2}\sigma^2) dt + \frac{1}{2}\sigma^2 dt + \sigma dW(t)) \\ &= e^{Q(t)} (\mu dt + \sigma dW(t)) \\ &= S(t)(\mu dt + \sigma dW(t)) \end{aligned}$$

as expected.

Solution 4.12: [Exercice] Using Ito's lemma on $f(s) = s^2$ we get $f_s = 2s$ and $\frac{1}{2} f_{ss} = 1$ and therefore

$$\begin{aligned} dY(t) = df(S(t)) &= f_s dS + \frac{1}{2} f_{ss} (dS)^2 \\ &= 2S(t) dS(t) + (dS(t))^2 \\ &= 2S(t)(\mu S^{\frac{3}{2}}(t) dt + \sigma S^{\frac{3}{2}}(t) dW(t)) + (\mu S^{\frac{3}{2}}(t) dt + \sigma S^{\frac{3}{2}}(t) dW(t))^2 \\ &= 2\mu S^{\frac{5}{2}}(t) dt + 2\sigma S^{\frac{5}{2}}(t) dW(t) + \sigma^2 S^3(t) dt \\ &= (2\mu + \sigma^2 S^{\frac{1}{2}}(t)) S^{\frac{5}{2}}(t) dt + 2\sigma S^{\frac{5}{2}}(t) dW(t). \end{aligned}$$

Solution 4.13: [Exercice]

1. Using Ito on $f(x) = x^2$, we have

$$\begin{aligned} dW^2(t) &= 2W(t) dW(t) + \frac{1}{2} 2 (dW(t))^2 \\ &= 2W(t) dW(t) + dt \end{aligned}$$

   Since there is a non-zero drift term, so this is not a martingale.

   Alternative solution:

   Taking expectation, we have

$$E[W^2(t)] = E[N(0,t)^2] = Var[N(0,t)] = t,$$

   which is not a constant. Hence W^2 is not a martingale.

2. Using Ito on $f(t,x) = x/t$, we have

$$dY(t) = df(t, W(t)) = -\frac{W(t)}{t^2} dt + \frac{1}{t} dW(t)$$

There is a non-zero drift term, $Y(t) = W(t)/t$ is not a martingale.

Alternative solution:

For $s \le t$,

$$\begin{aligned}
E[Y(t) - Y(s)|\mathcal{F}(s)] &= E[\frac{W(t)}{t} - \frac{W(s)}{s}|\mathcal{F}(s)] \\
&= E[\frac{sW(t) - tW(s)}{st}|\mathcal{F}(s)] \\
&= E[\frac{s}{st}(W(t) - W(s)) - \frac{(t-s)W(s)}{ts}|\mathcal{F}(s)] \\
&= -\frac{t-s}{t}\frac{W(s)}{s}
\end{aligned}$$

which is not identical to zero. This shows that Y is not a martingale.

Solution 4.14: [Exercice] We simply need to show that $Q(A) := E_P[Y(T)1(A)]$ is a legitimate probability, i.e. it satisfies all axioms.

1. The first axiom is $Q \ge 0$. By the definition of $Y(T) = \exp(-\gamma W(T) - \frac{1}{2}\gamma^2 T)$, we have

   $$Q[A] = E_P[\exp(-\gamma W(T) - \frac{1}{2}\gamma^2 T)1(A)] \ge 0$$

   as all terms are positive.

2. The second axiom is $Q[\Omega] = 1$. By direct computation, we have

   $$\begin{aligned}
   Q[\Omega] &= E_P[Y(T)1(\Omega)] \\
   &= E_P[\exp(-\gamma W(T) - \frac{1}{2}\gamma^2 T)] \\
   &= \exp(-\frac{1}{2}\gamma^2 T)E_P[\exp(-\gamma W(T))] \\
   &= \exp(-\frac{1}{2}\gamma^2 T)\exp(\frac{1}{2}\gamma^2 T) \\
   &= 1
   \end{aligned}$$

   as $W(T) \sim N(0, T)$.

3. The last axiom is $Q[\cup_k E_k] = \sum_k Q[E_k]$ for disjoint sets E_k. This can be verified easily as follows:

   $$\begin{aligned}
   Q[\cup_k E_k] &= E_P[Y(T)1(\cup_k E_k)] \\
   &= E_P[Y(T)\sum_k 1(E_k)] \text{ as } E_k\text{'s are disjoint} \\
   &= \sum_k E_P[Y(T)1(E_k)] \\
   &= \sum_k Q[E_k].
   \end{aligned}$$

Solution 4.15: [Exercice]

1. Clearly, we have $W_Q(0) = W(0) + \gamma \times 0 = 0$ and W_Q is continuous as W is.

2. For $A \in \mathcal{F}_u$, $u \leq s \leq t \leq T$, $\theta \in \mathbb{R}$, we have

$$\begin{aligned}
&E_Q[1_A e^{\theta(W_Q(t)-W_Q(s))}] \\
&= E_Q[1_A Y(T) e^{\theta(W_Q(t)-W_Q(s))}] \\
&= E_P[1_A Y(u) \frac{Y(T)}{Y(u)} e^{\theta(W_Q(t)-W_Q(s))}] \\
&= E_P[1_A Y(u) e^{-\gamma(W(T)-W(u))-\frac{1}{2}\gamma^2(T-u)} e^{\theta(W_Q(t)-W_Q(s))}] \\
&= E_P[1_A Y(u) e^{-\gamma(W(T)-W(t))-\frac{1}{2}\gamma^2(T-t)} e^{-\gamma(W(t)-W(s))-\frac{1}{2}\gamma^2(t-s)} e^{-\gamma(W(s)-W(u))-\frac{1}{2}\gamma^2(s-u)} e^{\theta(W_Q(t)-W_Q(s))}]
\end{aligned}$$

Note $W_Q(t) - W_Q(s) = W_Q(t-s) + \gamma(t-s)$ and by the independent increment property with $E_P[e^{-\gamma(W(s)-W(u))-\frac{1}{2}\gamma^2(s-u)}] = E_P[e^{-\gamma(W(T)-W(t))-\frac{1}{2}\gamma^2(T-t)}] = 1$, we have

$$\begin{aligned}
&E_Q[1_A e^{\theta(W_Q(t)-W_Q(s))}] \\
&= E_P[1_A Y(u) e^{-\gamma(W(t)-W(s))-\frac{1}{2}\gamma^2(t-s)} e^{\theta(W_Q(t)-W_Q(s))}] \\
&= E_P[1_A Y(u) e^{-\gamma(W(t)-W(s))} e^{\theta(W(t)-W(s))}] e^{\frac{1}{2}\gamma^2(t-s)} e^{\theta\gamma(t-s)} \\
&= E_P[1_A Y(u) e^{(\theta-\gamma)(W(t)-W(s))}] e^{\frac{1}{2}\gamma^2(t-s)} e^{\theta\gamma(t-s)}
\end{aligned}$$

Again, by independent increment, we know that $1_A Y(u)$ and $W(t)-W(s)$ are independent and hence we have

$$\begin{aligned}
&E_Q[1_A e^{\theta(W_Q(t)-W_Q(s))}] \\
&= E_P[1_A Y(u)] E_P[e^{(\theta-\gamma)(W(t)-W(s))}] e^{\frac{1}{2}\gamma^2(t-s)} e^{\theta\gamma(t-s)} \\
&= E_P[1_A Y(u)] e^{\frac{1}{2}(\theta-\gamma)^2(t-s)} e^{\frac{1}{2}\gamma^2(t-s)} e^{\theta\gamma(t-s)} \\
&= E_P[1_A Y(u)] e^{\frac{1}{2}(t-s)((\theta-\gamma)^2-\gamma^2+2\theta\gamma)} \\
&= E_P[1_A Y(u)] e^{\frac{1}{2}(t-s)(\theta(\theta-2\gamma)+2\theta\gamma)} \\
&= E_P[1_A Y(u)] e^{\frac{1}{2}\theta^2(t-s)} \\
&= E_Q[1_A] e^{\frac{1}{2}\theta^2(t-s)}
\end{aligned}$$

(a) By taking $A = \Omega$ we have

$$E_Q[e^{\theta(W_Q(t)-W_Q(s))}] = e^{\frac{1}{2}\theta^2(t-s)}$$

which shows that $W_Q(t) - W_Q(s)$ is distributed as $N(0, t-s)$. (recall the MGF of $N(0, t-s)$.)

(b) As the distribution of $W_Q(t) - W_Q(s)$ only depends on $t - s$, it is stationary.

(c) Observe

$$E_Q[1_A e^{\theta(W_Q(t)-W_Q(s))}] = E_Q[1_A] e^{\frac{1}{2}\theta^2(t-s)} = E_Q[1_A] E_Q[e^{\theta(W_Q(t)-W_Q(s))}],$$

we can conclude that the increment $W_Q(t) - W_Q(s)$ and the filtration $\mathcal{F}_u$ are independent (as it is independent of any event $A \in \mathcal{F}_u$).

Therefore, W_Q is a Q-BM for $0 \leq t \leq T$.

Solution 4.16: [Exercice]

1. The solution is
$$S\left(t\right)=S\left(0\right)e^{\left(\mu-\frac{1}{2}\sigma^{2}\right)t+\sigma W(t)}$$
This is called a Geometric Brownian motion. You can verify it via Itô's lemma on $f(s) = \log(s)$.

2. This is a standard differential equation, with solution
$$B\left(t\right)=e^{rt}$$
(you can differentiate this to check that this is true)

3. Note B is smooth and in fact, by denoting $f(t,x)=e^{-rt}x$, we have
$$Z(t)=f(t,S(t))=e^{-rt}S(t)=\frac{S(t)}{B(t)}.$$
Note
$$\begin{aligned} f_t &= -r\times f \\ f_x &= e^{-rt} \\ f_{xx} &= 0, \end{aligned}$$
hence by Ito we have
$$\begin{aligned} dZ(t)=df(t,S(t)) &= -rZ(t)dt+e^{-rt}dS(t) \\ &= -rZ(t)dt+e^{-rT}S(t)(\mu dt+\sigma dW(t)) \\ &= -rZ(t)dt+Z(t)(\mu dt+\sigma dW(t)) \\ &= Z(t)((\mu-r)dt+\sigma dW(t)). \end{aligned}$$

4. Using the Girsanov Theorem we know that:
$$\begin{aligned} dZ(t)/Z(t) &= \left(\mu-r\right)dt+\sigma dW(t) \\ &= \left(\mu-r\right)dt+\sigma(dW^{Q}(t)-\gamma(t)dt) \\ &= \left(\mu-r-\sigma\gamma(t)\right)dt+\sigma dW^{Q} \end{aligned}$$
To make sure $Z(t)$ is a Q martingale we need the drift term to be zero, i.e $\gamma(t)=\frac{\mu-r}{\sigma}$.

Solution 4.17: [Exercice]

1. Denote $f(x,y) = x/y$, and note that $dB(t)^2 = 0$ and $dS(t)dB(t) = 0$ as B is smooth, with
$$\begin{aligned} f_x &= 1/y \\ f_{xx} &= 0 \\ f_y &= -x/y^2, \end{aligned}$$
we have from Ito
$$\begin{aligned} dZ(t)=df(S(t),B(t)) &= \frac{1}{B}dS(t)+\frac{-S(t)}{B(t)^2}dB(t) \\ &= \frac{S(t)}{B(t)}(\mu dt+\sigma dW(t))S(t)-\frac{S(t)}{B(t)}rdt \\ &= Z(t)S(t)(\mu dt+\sigma dW(t))-rZ(t)dt \\ &= Z(t)\Big((\mu S(t)-r)dt+\sigma S(t)dW(t)\Big). \end{aligned}$$

2. Using Girsanov's Theorem

$$\begin{aligned} dZ(t) &= (\mu S(t) - r)\, Z(t)dt + \sigma Z(t)S(t)(dW^Q - \gamma(t)dt) \\ &= (\mu S(t) - r - \sigma\gamma(t)S(t))\, Z(t)dt + \sigma Z(t)S(t)dW^Q(t) \end{aligned}$$

We shall tune the γ such that $Z(t)$ is a Q-martingale, i.e. the drift term is 0. This implies that we choose

$$\gamma(t) = \frac{\mu S(t) - r}{\sigma S(t)}.$$

[Note γ is "predictable" and therefore such choice is legitimate. "Predictable" does not mean you can predict the stock price in the future, but in a sense that you can predict the stock price at the very next instant. One example is when the stock price is continuous (which is the case in thie course as Brownian motion is continuous).]

Solution 4.18: [Exercice] Same as above.

1. Using Ito's lemma:

$$\begin{aligned} dZ(t) = d\frac{S(t)}{B(t)} &= \frac{dS(t)}{B(t)} - \frac{S(t)}{B^2(t)}dB(t) \\ &= \frac{\mu S^{\frac{3}{2}}(t)dt + \sigma S^{\frac{3}{2}}(t)dW(t)}{B(t)} - \frac{S(t)}{B^2(t)}rB(t)dt \\ &= \mu Z(t)S^{\frac{1}{2}}(t)dt + \sigma Z(t)S^{\frac{1}{2}}(t)dW(t) - rZ(t)dt \\ &= \left(\mu S^{\frac{1}{2}}(t) - r\right) Z(t)dt + \sigma Z(t)S^{\frac{1}{2}}(t)dW(t) \end{aligned}$$

2. Using Girsanov's theorem to find $\gamma(t)$:

$$\begin{aligned} \frac{dZ(t)}{Z(t)} &= \left(\mu S^{\frac{1}{2}}(t) - r\right) dt + \sigma S^{\frac{1}{2}}(t)dW(t) \\ &= \left(\mu S^{\frac{1}{2}}(t) - r\right) dt + \sigma S^{\frac{1}{2}}(t)\left(dW^Q(t) - \gamma dt\right) \\ &= \left(\mu S^{\frac{1}{2}}(t) - r - \sigma\gamma S^{\frac{1}{2}}(t)\right) dt + \sigma S^{\frac{1}{2}}(t)dW^Q(t) \end{aligned}$$

where we tune

$$\gamma(t) = \frac{\mu S^{\frac{1}{2}}(t) - r}{\sigma S^{\frac{1}{2}}(t)}$$

such that the drift is zero.

Solution 4.19: [Exercice] (NOTE: you should realize that the following questions are just a repetition of the lecture material - this is a key benefit of the 'general approach' we used.)

This should be similar to the lecture notes.:

Remember that the first steps of all derivative pricing can be thought of as

- Find a measure Q such that $Z(t)$ is a martingale
- Consider the process $Y(t) = E_Q[X/B(T)|\mathcal{F}(t)] = E_Q[X|\mathcal{F}(t)]$ which is a Q-martingale (as $r = 0$ in this case)

- Use the previsible process $\phi(t)$ such that $dY(t) = \phi(t)\,dZ(t) = \phi(t)dS(t)$ to form a self-financing replicating strategy which is consistent with the payoff X.

Step 1:

Tune Q such that $Z = S$ is a martingale. Since

$$dS(t) = \mu S(t)\,dt + \sigma S(t)\,dW(t)$$

by change of measure we have

$$\begin{aligned} dS(t) &= \mu S(t)\,dt + \sigma S(t)\left(dW^Q(t) - \gamma(t)dt\right) \\ &= (\mu - \gamma(t)\sigma)\,S(t)\,dt + \sigma S(t)\,dW^Q(t) \end{aligned}$$

where we shall choose

$$\gamma(t) = \frac{\mu}{\sigma}$$

which is predictable as it is a constant. Therefore, we find a measure Q such that $Z = S$ is a Q-martingale.

Step 2:

We have 2 Q martingales Z and Y, the Martingale representation theorem says that there exists a $\phi(t)$ such that

$$dY(t) = \phi(t)\,dZ(t) = \phi(t)dS(t).$$

Steps 3-4:

Construction strategy is to hold

$\phi(t)$ of the stock $S(t)$

$\psi(t) = Y(t) - \phi(t)\,Z(t) = Y(t) - \phi(t)S(t)$ of the bond (with value being 1)

To check that this is a replicating portfolio, notice that

$$\begin{aligned} V(t) &= \phi(t)\,S(t) + \psi(t) \\ &= \phi(t)\,S(t) + Y(t) - \phi(t)\,S(t) \\ &= Y(t) \end{aligned}$$

so

$$\begin{aligned} dV(t) = dY(t) &= \phi(t)\,dS(t) \\ &= \phi(t)\,dS(t) + \psi(t)\,dB(t) \end{aligned}$$

which is the self financing condition. (remember again that $dB = 0$ as B is constant)

On maturity T we have

$$\begin{aligned} V(T) &= Y(T) \\ &= E_Q[X|\mathcal{F}_T] \\ &= X \end{aligned}$$

so this is a replicating strategy.

So for there to be no arbitrage in the model, the time 0 price of our contingent claim must be

$$E_Q[X|\mathcal{F}_0] = E_Q[X]$$

the Q discounted expectation.

Solution 4.20: [Exercice]

1. The value should be
$$E_Q\left[X/B(T)\right]$$
where the stock price Q dynamics are
$$\begin{aligned} dS(t) &= rS(t)dt + \sigma S^{\frac{3}{4}}(t)dW^Q(t) \\ S(0) &= s \end{aligned}$$

2. Remember that the first steps of all derivative pricing can be thought of as: Find a measure Q such that $Z(t)$ is a martingale; Consider the process $Y(t) = E_Q\left[\frac{1}{B(T)}X|\mathcal{F}_t\right]$ which is also a Q-martingale; Use the previsible process $\phi(t)$ such that $dY(t) = \phi(t)\,dZ(t)$ to form a self-financing replicating strategy which is consistent with the payoff X.

   Step 1: Tune Q such that Z is a martingale.

   (Using function $f = x/y$) Ito gives us
$$\begin{aligned} dZ(t) = d(S(t)/B(t)) &= \frac{1}{B(t)}dS(t) - \frac{S(t)}{B(t)^2}dB(t) \\ &= \frac{S(t)}{B(t)}(\mu dt + S(t)^{\frac{-1}{4}}\sigma dW(t)) - \frac{S(t)}{B(t)}rdt \\ &= Z(t)((\mu - r)dt + \sigma S(t)^{\frac{-1}{4}}dW(t)). \end{aligned}$$

   Via change of measure we have
$$\begin{aligned} dZ(t)/Z(t) &= (\mu - r - \sigma S(t)^{\frac{-1}{4}}\gamma(t))dt + \sigma S(t)^{\frac{-1}{4}}(dW(t) + \gamma(t)dt) \\ &= (\mu - r - \sigma S(t)^{\frac{-1}{4}}\gamma(t))dt + \sigma S(t)^{\frac{-1}{4}}dW^Q(t) \end{aligned}$$
   where we choose the predictable process
$$\gamma(t) = \frac{\mu - r}{\sigma}S^{\frac{1}{4}}(t)$$
   such that
$$dZ(t)/Z(t) = \sigma S(t)^{\frac{-1}{4}}dW^Q(t),$$
   i.e. Z is a Q-martingale.

   Plugging back to $S(t)$, we have
$$\begin{aligned} dS(t)/S(t) &= \mu dt + \sigma S(t)^{\frac{-1}{4}}dW(t) \\ &= (\mu - \sigma S(t)^{\frac{-1}{4}}\gamma(t))dt + \sigma S(t)^{\frac{-1}{4}}(dW(t) + \gamma(t)dt) \\ &= rdt + \sigma S(t)^{\frac{-1}{4}}dW^Q(t). \end{aligned}$$

   Step 2:

   We have 2 Q martingales Z and Y, therefore from the Martingale Representation Theorem, there exists $\phi(t)$ such that
$$dY(t) = \phi(t)\,dZ(t)$$

Step 3:

Construction strategy is to hold

$\phi(t)$ of the stock $S(t)$

$\psi(t) = Y(t) - \phi(t) Z(t)$ of the bond

To check that this is a replicating portfolio. Notice that

$$\begin{aligned} V(t) &= \phi(t) S(t) + \psi(t) B(t) \\ &= \phi(t) S(t) + Y(t) B(t) - \phi(t) \frac{S(t)}{B(t)} B(t) \\ &= Y(t) B(t) \end{aligned}$$

so

$$\begin{aligned} dV(t) &= d(Y(t) B(t)) \\ &= B(t) dY(t) + Y(t) dB(t) \end{aligned}$$

as B is smooth. Recall from the definition of ψ, we have $Y(t) = \psi(t) + \phi(t)Z(t)$ and $B(t)Z(t) = S(t)$. Hence we can simplify the above to

$$\begin{aligned} dV(t) &= B(t)\phi(t)dZ(t) + (\psi(t) + \phi(t)Z(t))dB(t) \\ &= B(t)\phi(t)Z(t)\sigma S(t)^{\frac{-1}{4}} dW^Q(t) + \psi(t)dB(t) + \phi(t)Z(t)B(t)rdt \\ &= \phi(t)S(t)\Big(rdt + \sigma S(t)^{\frac{-1}{4}} dW^Q(t)\Big) + \psi(t)dB(t) \\ &= \phi(t)dS(t) + \psi(t)dB(t) \end{aligned}$$

which is the self financing condition.

At maturity T we have

$$\begin{aligned} V(T) &= Y(T) B(T) \\ &= E_Q\left[\frac{1}{B(T)} X|\mathcal{F}_T\right] B(T) \\ &= X \end{aligned}$$

so this is a replicating strategy.

So for there to be no arbitrage in the model, the time 0 price of our contingent payoff must be

$$E_Q\left[\frac{1}{B(T)} X|\mathcal{F}_0\right].$$

Solution 4.21: [Exercice] The self-financing condition is given by $dV(t) = \phi(t)dS(t) + \psi(t)dB(t)$ and we know the solution to the SDE is $S(t) = S(0) + \mu t + \sigma W(t)$.

1. So our portfolio is $V(t) = S(t) + B(t)$.

$$\begin{aligned} dV(t) &= dS(t) + dB(t) \\ &= \phi(t)dS(t) + \psi(t)dB(t) \end{aligned}$$

as required, therefore this portfolio is self-financing.

2. Our portfolio is $V(t) = 2W(t)S(t) - (2\mu t W(t) + W^2(t))B(t)$.

$$\begin{aligned} V(t) &= 2W(t)(S(0) + \mu t + \sigma W(t)) - (2\mu t W(t) + W^2(t)) \\ &= 2S(0)W(t) + 2\mu t W(t) + 2\sigma W^2(t) - 2\mu t W(t) - W^2(t) \\ &= 2S(0)W(t) + (2\sigma - 1)W^2(t) \end{aligned}$$

Applying Ito's lemma on this result we have

$$\begin{aligned} dV(t) &= 2S(0)dW(t) + (2\sigma - 1)(2W(t)dW(t) + \frac{1}{2}2(dW(t))^2) \\ &= 2S(0)dW(t) + (2\sigma - 1)(dt + 2W(t)dW(t)) \end{aligned}$$

Compare this with formula for portfolio dynamics (with $dB(t) = 0$ as $B(t) = 1$)

$$\begin{aligned} dV(t) &= \phi(t)dS(t) + \psi(t)dB(t) \\ &= 2W(t)(\mu dt + \sigma dW(t)) - (2\mu t W(t) + W^2(t))dB(t) \\ &= 2W(t)(\mu dt + \sigma dW(t)) \end{aligned}$$

by collecting the dt and $2dW(t)$ term, we have

$$\begin{cases} 2\sigma - 1 = 2\mu W(t) \\ S(0) + (2\sigma - 1)W(t) = \sigma W(t) \end{cases}.$$

Since $W(t)$ is random, the first equation can never be true for all t. Therefore, we conclude that such portfolio is not self-financing.

3. Our portfolio is $V(t) = \frac{1}{2}S^2(t) - \mu\sigma t W(t)B(t)$.

$$\begin{aligned} V(t) &= \frac{1}{2}(S(0)^2 + 2S(0)(\mu t + \sigma W(t)) + \mu^2 t^2 + 2\mu t\sigma W(t) + \sigma^2 W^2(t)) - \mu\sigma t W(t) \\ &= \frac{1}{2}(\mu^2 t^2 + \sigma^2 W^2(t)) + \frac{1}{2}S(0)(2S(t) - S(0)) \\ dV(t) &= \frac{1}{2}(2\mu^2 t dt + 2\sigma^2 W(t)dW(t) + \frac{1}{2}2\sigma^2(dW(t))^2) + S(0)dS(t) \\ &= \frac{1}{2}(2\mu^2 t dt + 2\sigma^2 W(t)dW(t) + \sigma^2 dt + 2\mu S(0)dt + 2\sigma S(0)dW(t)) \\ &= \frac{1}{2}((2\mu^2 t + 2\mu S(0) + \sigma^2)dt + (\sigma W(t) + S(0))2\sigma dW(t)) \end{aligned}$$

Now notice that $\phi(t)dS(t) = \frac{1}{2}S(t)(\mu dt + \sigma dW(t))$ which would have a $W(t)dt$ term, so this portfolio can't satisfy the self-financing condition.

Solution 4.22: [Exercice] From the lectures we see that , by the applying Ito's lemma on $V(S(t), t)$ we get

$$\begin{aligned} dV &= \frac{\partial V}{\partial s}\sigma S dW^Q \\ &\quad + \left(\frac{\partial V}{\partial s} rS + \frac{1}{2}\frac{\partial^2 V}{\partial s^2}\sigma^2 S^2 + \frac{\partial V}{\partial t}\right) dt \end{aligned}$$

BUT remember that the key financial idea is to find a self-financing replicating strategy. So in particular self financing tells us that

$$dV = \phi(t)\, dS + \varphi(t)\, dB$$

but $dB(t) = rB(t)\,dt$ and so

$$\begin{aligned} dV &= \phi\sigma S dW^Q \\ &\quad + (\phi r S + \varphi r B)\,dt \end{aligned}$$

now we know that SDE representations are unique. So we can match up the two equations. Matching $dW^Q(t)$ term we have $\phi(t)\sigma S(t) = \frac{\partial V(S(t),t)}{\partial S(t)}\sigma S(t)$, which gives

$$\phi(t) = \frac{\partial V(S(t),t)}{\partial S(t)}.$$

Matching dt we have

$$(\phi r S + \varphi r B) = \left(\frac{\partial V}{\partial s} r S + \frac{1}{2}\frac{\partial^2 V}{\partial s^2}\sigma^2 S^2 + \frac{\partial V}{\partial t}\right)$$

and substituting for ϕ and $\varphi = \frac{V - \phi S}{B}$ we have the left hand side

$$\begin{aligned} (\phi r S + \varphi r B) &= \frac{\partial V}{\partial s} r S + \frac{\left(V - \frac{\partial V}{\partial s} S\right)}{B} r B \\ &= rV \end{aligned}$$

and hence our equality becomes:

$$rV = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} + \frac{\partial V}{\partial t}$$

which is the PDE we were looking for.

(Note: This equation was useful as mathematicians have a lot of well developed techniques to solve equations of this form. Hence an important, practical, numerical technique for solving option prices arises from this equation.)

Solution 4.23: [Exercice] Recall to change measures we need to find the market price of risk $\gamma(t)$.

$$\begin{aligned} dZ(t) &= \frac{dS(t)}{B(t)} - \frac{S(t)}{B^2(t)} dB(t) \\ &= \frac{\mu S^{-1}(t)dt + \sigma S(t) dW(t)}{B(t)} - \frac{S(t)}{B^2(t)} r B(t) dt \\ &= \mu \frac{Z(t)}{S^2(t)} dt + \sigma Z(t) dW(t) - rZ(t)dt \\ &= Z(t)\left((\frac{\mu}{S^2(t)} - r)dt + \sigma dW(t)\right) \\ &= Z(t)\left((\frac{\mu}{S^2(t)} - r - \sigma\gamma(t))dt + \sigma(dW(t) + \gamma(t)dt)\right) \\ &= Z(t)\left((\frac{\mu}{S^2(t)} - r - \sigma\gamma(t))dt + \sigma dW^Q(t)\right) \\ \implies \gamma(t) &= \frac{\frac{\mu}{S^2(t)} - r}{\sigma}. \end{aligned}$$

Then the stock process in the Q measure is given by:

$$\begin{aligned} dS(t) &= \mu S^{-1}(t)dt + \sigma S(t)\left(dW^Q(t) - \frac{\frac{\mu}{S^2(t)} - r}{\sigma}dt\right) \\ &= \left(\mu S^{-1}(t) - \mu S^{-1}(t) + rS(t)\right)dt + \sigma S(t)dW^Q(t) \\ &= rS(t)dt + \sigma S(t)dW^Q(t) \end{aligned}$$

Solution 4.24: [Exercice] Assume that S_0 is the current price of the underlying share. Then the probability that the European call option will be exercised is

$$\begin{aligned} \mathbb{P}\left(S_T > K\,|S_0\right) &= \mathbb{P}\left(S_0 \exp\left[\left(\mu - \tfrac{1}{2}\sigma^2\right)T + \sigma Z_T\right] > K\right) \\ &= \mathbb{P}\left(Z_T > \frac{\log\left(K/S_0\right) - \left(\mu - \frac{1}{2}\sigma^2\right)T}{\sigma}\right) \\ &= \mathbb{P}\left(\frac{Z_T}{\sqrt{T}} > \frac{\log\left(K/S_0\right) - \left(\mu - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right) \\ &= \mathbb{P}\left(Z \leq \frac{\log\left(S_0/K\right) + \left(\mu - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right) \\ &= \Phi\left(d\right) \end{aligned}$$

where

$$d = \frac{\log\left(S_0/K\right) + \left(\mu - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}.$$

Note that we have used the fact that $Z_T \sim N\left(0, T\right)$.

Solution 4.25: [Exercice] We know from lectures that the solution to this SDE is:

$$S(T) = se^{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma W(T)} \sim s\exp((\mu - \frac{1}{2}\sigma^2)T + \sqrt{\sigma^2 T}Z)$$

Hence, the probability is given by

$$\begin{aligned} P(K_1 \leq S(T) \leq K_2) &= P(K_1 < s\exp((\mu - \frac{1}{2}\sigma^2)T + \sqrt{\sigma^2 T}Z) < K_2) \\ &= P(\frac{\log\frac{K_1}{s} - (\mu - \frac{1}{2}\sigma^2)T}{\sqrt{\sigma^2 T}} < Z < \frac{\log\frac{K_2}{s} - (\mu - \frac{1}{2}\sigma^2)T}{\sqrt{\sigma^2 T}}) \\ &= P(\frac{\log\frac{s}{K_1} + (\mu - \frac{1}{2}\sigma^2)T}{\sqrt{\sigma^2 T}} > -Z > \frac{\log\frac{s}{K_2} + (\mu - \frac{1}{2}\sigma^2)T}{\sqrt{\sigma^2 T}}) \\ &= P(d_2(K_1) > -Z > d_2(K_2)) \\ &= \Phi(d_2(K_1)) - \Phi(d_2(K_2)) \end{aligned}$$

where $-Z \sim N(0,1)$ and $d_2(K) = \frac{\log\frac{s}{K} + (\mu - \frac{1}{2}\sigma^2)T}{\sqrt{\sigma^2 T}}$.

Solution 4.26: [Exercice]

1. The price can be derived as

$$
\begin{aligned}
price &= e^{-rT}E_Q[(K-S(T))_+] \\
&= Ke^{-rT}E_Q[1(K>S(T))] - e^{-rT}E_Q[S(T)1(K>S(T))] \\
&= Ke^{-rT}P[-d_2(K)>Z] - e^{-rT}E_Q[S(T)1(-d_2(K)>Z)] \\
&= Ke^{-rT}\Phi(-d_2(K)) - e^{-rT}E_Q[S(T)]P[-d_2(K)>\tilde{Z}+\sqrt{\sigma^2T}] \\
&= Ke^{-rT}\Phi(-d_2(K)) - S(0)P[-d_1(K)>\tilde{Z}] \\
&= Ke^{-rT}\Phi(-d_2(K)) - S(0)\Phi(-d_1(K)),
\end{aligned}
$$

where $d_1(K) = d_2(K) + \sqrt{\sigma^2 T}$ and $d_2(K) = \frac{\log\frac{S(0)}{K}+(r-\frac{1}{2}\sigma^2)T}{\sqrt{\sigma^2T}}$.

2. With $S(0)=775$, $K=750$, $r=0.05$, $T=0.25$ and $\sigma=0.2$, we get

$$
\begin{aligned}
d_2(K) &= \frac{\log\frac{775}{750}+(0.05-\frac{1}{2}0.2^2)\times 0.25}{\sqrt{0.2^2\times 0.25}} = 0.4028982, \\
d_1(K) &= 0.4028982 + \sqrt{0.2^2\times 0.25} = 0.5028982
\end{aligned}
$$

and therefore

$$
price = 750\times\exp(-0.05\times 0.25)\times\Phi(-0.4028982) - 775\times\Phi(-0.5028982) = 16.1069.
$$

Solution 4.27: [Exercice]

1. In the Black-Scholes formula for the European call option, the symbols:
   - $S(t)$ is the current price of the underlying share;
   - K is the exercise or strike price of the option;
   - r is the risk-free interest rate compounded continuously;
   - T is the expiry date of the call option;
   - σ is the volatility of stock prices, and
   - $\Phi(\cdot)$ is the cumulative distribution function of a standard normal distribution.

2. First, note that the payoff for the call option is $\max(S(T)-K,0)$. when $\sigma=0$, $S(T)$ is simply $S(t)e^{r(T-t)}$. Discounting this at r as it is risk-free, we get

$$
\begin{aligned}
c(K,t) &= e^{-r(T-t)}\max\left(S(t)e^{r(T-t)}-K,0\right) \\
&= \max\left(S(t)-Ke^{-r(T-t)},0\right).
\end{aligned}
$$

The Black-Scholes pricing formula gives

$$
c(K,t) = S(t)\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2).
$$

If $S(t) > Ke^{-r(T-t)}$, then $\log\left(\frac{S(t)}{K}\right) + r(T-t) > 0$. As $\sigma\to 0$, then both $d_1\to\infty$ and $d_2\to\infty$. This implies both $\Phi(d_1)$ and $\Phi(d_2)$ tend to 1. Thus,

$$
c(K,t) = S(t)\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2) \to S(t) - Ke^{-r(T-t)}.
$$

On the other hand, if $S(t) < Ke^{-r(T-t)}$, then $\log\left(\frac{S(t)}{K}\right) + r\,(T-t) < 0$. As $\sigma \to 0$, then both $d_1 \to -\infty$ and $d_2 \to -\infty$. This implies both $\Phi\,(d_1)$ and $\Phi\,(d_2)$ tend to 0. Thus, $c(K,t) = S(t)\Phi\,(d_1) - Ke^{-r(T-t)}\Phi\,(d_2) \to 0$.

If $S(t) = Ke^{-r(T-t)}$, the formula of $d_1$and d_2 is

$$\frac{\log\frac{S(t)}{K} + (r \pm \frac{1}{2}\sigma^2)(T-t)}{\sqrt{\sigma^2(T-t)}} = \frac{\log\frac{S(t)}{K} + r(T-t)}{\sqrt{\sigma^2(T-t)}} \pm \frac{1}{2}\sqrt{T-t}\frac{\sigma^2}{\sqrt{\sigma^2}} \to 0$$

as $\sigma \downarrow 0$. This gives

$$c(K,t) = S(t)\phi(d_1) - Ke^{-r(T-t)}\Phi(d_2) \to \frac{1}{2}(S(t) - Ke^{-r(T-t)}) = 0,$$

which is consistent with the case $S(t) > Ke^{-r(T-t)}$.

Solution 4.28: [Exercice] First write out the payout at maturity:

$$\begin{cases} \$100 & \text{if } S(T) \geq K \\ \$0 & \text{if } S(T) < K \end{cases}$$

From the dynamic of the stock, we have

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dW^Q(t)$$

which is the same as

$$d\log S(t) = (\mu - \frac{1}{2}\sigma^2) + \sigma dW^Q(t)$$

via Ito's lemma. Integrating this from time 0 to T yields

$$S(T) = S(0)\exp((\mu - \frac{1}{2}\sigma^2)T + \sigma W(T)) \sim S(0)\exp(m + \sqrt{v}Z)$$

with $m = (\mu - \frac{1}{2}\sigma^2)T$ and $v = \sigma^2 T$.

The payoff is then given by

$$\begin{aligned} Price &= e^{-rT}E_Q[100 \times 1(S(T) \geq K)] \\ &= 100e^{-rT}P_Q[S(0)\exp(m + \sqrt{v}Z) \geq K] \\ &= 100e^{-rT}P_Q[-Z \leq \frac{\log\frac{S(0)}{K} + m}{\sqrt{v}}] \\ &= 100e^{-rT}\Phi(\frac{\log\frac{S(0)}{K} + m}{\sqrt{v}}) \\ &= 100e^{-rT}\Phi(\frac{\log\frac{S(0)}{K} + (\mu - \frac{1}{2}\sigma^2)T}{\sqrt{\sigma^2 T}}) \end{aligned}$$

where we recognise $-Z$ is $N(0,1)$ distributed.

Solution 4.29: [Exercice] If you look carefully, this option has the same payoff as a bullspread option strategy consisting of a long call with strike price $\$K_1$ and a short call with strike price $\$K_2$, $K_2 > K_1$, i.e.

$$X = (S(T) - K_1)_+ - (S(T) - K_2)_+.$$

So we can simply price this as two options using the Black-Scholes formula.

$$\begin{aligned}
price &= e^{-rT}E_Q\left[X\right] \\
&= e^{-rT}\left(E_Q\left[(S(T)-K_1)_+\right] - E_Q\left[(S(T)-K_2)_+\right]\right) \\
&= C_T(K_1) - C_T(K_2) \\
&= S(0)\Phi\left(d_1(K_1)\right) - K_1e^{-rT}\Phi(d_2(K_1)) - S(0)\Phi\left(d_1(K_2)\right) + K_2e^{-rT}\Phi\left(d_2(K_2)\right) \\
&= S(0)\left(\Phi\left(d_1(K_1)\right) - \Phi\left(d_1(K_2)\right)\right) - K_1e^{-rT}\Phi(d_2(K_1)) + K_2e^{-rT}\Phi\left(d_2(K_2)\right)
\end{aligned}$$

Alternative solution: Note the payoff function is

$$X = \begin{cases} 0, & S(T) \in (0, K_1) \\ S(T) - K_1, & S(T) \in (K_1, K_2) \\ K_2 - K_1, & S(T) \in (K_2, \infty) \end{cases}$$

and $S(T) \sim S(0)\exp(m + \sqrt{v}Z)$ with $m = (r - \frac{1}{2}\sigma^2)T$ and $v = \sigma^2 T$. Hence, the price is given by

$$\begin{aligned}
price &= e^{-rT}E_Q[X] \\
&= e^{-rT}E_Q[(S(T)-K_1)1(K_1 < S(T) < K_2) + (K_2 - K_1)1(S(T) > K_2)] \\
&= e^{-rT}E_Q[S(T)1(K_1 < S(T) < K_2)] \\
&\quad - K_1e^{-rT}E_Q[1(K_1 < S(T) < K_2)] \\
&\quad + (K_2 - K_1)e^{-rT}E_Q[1(S(T) > K_2)].
\end{aligned}$$

The first term is evaluated as

$$\begin{aligned}
e^{-rT}E_Q[S(T)1(K_1 < S(T) < K_2)] &= e^{-rT}E_Q[S(T)1(-d_2(K_1) < Z < -d_2(K_2)))] \\
&= e^{-rT}E_Q[S(T)]P[-d_2(K_1) < \tilde{Z} + \sqrt{v} < -d_2(K_2)] \\
&= e^{-rT}\exp(m + \frac{1}{2}v)P[d_1(K_2) < -\tilde{Z} < d_1(K_1)] \\
&= e^{-rT}S(0)\exp(rT - \frac{1}{2}\sigma^2T + \frac{1}{2}\sigma^2T)P[d_1(K_2) < -\tilde{Z} < d_1(K_1)] \\
&= S(0)P[d_1(K_2) < -\tilde{Z} < d_1(K_1)] \\
&= S(0)(\Phi(d_1(K_1)) - \Phi(d_1(K_2))).
\end{aligned}$$

The second term is evaluated as

$$\begin{aligned}
-K_1e^{-rT}E_Q[1(K_1 < S(T) < K_2)] &= -K_1e^{-rT}P_Q[-d_2(K_1) < Z < -d_2(K_2)] \\
&= -K_1e^{-rT}P_Q[d_2(K_2) < -Z < d_2(K_1)] \\
&= -K_1e^{-rT}\Phi(d_2(K_1)) + K_1e^{-rT}\Phi(d_2(K_2)).
\end{aligned}$$

The third term is evaluated as

$$\begin{aligned}
(K_2 - K_1)e^{-rT}E_Q[1(S(T) > K_2)] &= (K_2 - K_1)e^{-rT}P_Q[Z > -d_2(K_2)] \\
&= (K_2 - K_1)e^{-rT}P[-Z < d_2(K_2)] \\
&= K_2e^{-rT}\Phi(d_2(K_2)) - K_1e^{-rT}\Phi(d_2(K_2)).
\end{aligned}$$

Summing all of them gives

$$\begin{aligned}
price &= S(0)(\Phi(d_1(K_1)) - \Phi(d_1(K_2))) - K_1e^{-rT}\Phi(d_2(K_1)) + K_1e^{-rT}\Phi(d_2(K_2)) \\
&\quad + K_2e^{-rT}\Phi(d_2(K_2)) - K_1e^{-rT}\Phi(d_2(K_2)) \\
&= S(0)(\Phi(d_1(K_1)) - \Phi(d_1(K_2))) - K_1e^{-rT}\Phi(d_2(K_1)) + K_2e^{-rT}\Phi(d_2(K_2)).
\end{aligned}$$

[This is a good exercise to practice how to calculate the $E_Q[X]$.]

Solution 4.30: [Exercice] Notice that the Q-dynamics is exactly the same as that of the Black-Scholes assumptions:

$$\begin{aligned} dS^Q(t) &= \mu S^{\frac{3}{2}}(t)dt + \sigma S(t)\left(dW^Q(t) - \frac{\mu S^{\frac{1}{2}}(t) - r}{\sigma}dt\right) \\ &= (\mu S^{\frac{3}{2}}(t) - \mu S^{\frac{3}{2}}(t) + rS(t))dt + \sigma S(t)dW^Q(t) \\ &= rS(t)dt + \sigma S(t)dW^Q(t) \end{aligned}$$

Then in fact we can just use Black-Scholes equations to price this:

$$price = S(0)\Phi\left(d_1\right) - Ke^{-rT}\Phi\left(d_2\right).$$

Solution 4.31: [Exercice] If we consider the discounted Q expectations we have:

$$\begin{aligned} price &= e^{-rT}E_Q\left[X\right] \\ &= e^{-rT}\left(25P_Q(S(T) > K_1) + 25P_Q(S(T) > K_2) + 50P_Q(S(T) > K_3)\right) \end{aligned}$$

Now under Black-Scholes, notice these payments are the second terms in the formula.
Then our price is:

$$price = e^{-rT}\left(25\Phi(d_2(K_1)) + 25\Phi(d_2(K_2)) + 50\Phi(d_2(K_3))\right)$$

where $d_2(K) = \frac{\ln\left(\frac{S_0}{K} + \left(r - \frac{\sigma^2}{2}\right)T\right)}{\sigma\sqrt{T}}$.

Solution 4.32: [Exercice]

1. Delta is the change in the option price with respect to the change in the price of the underlying asset, i.e. $\Delta = \dfrac{\partial V}{\partial S}$. Gamma is the rate of change of the delta as the price of the underlying asset change, i.e. $\Gamma = \dfrac{\partial \Delta}{\partial S} = \dfrac{\partial^2 V}{\partial S^2}$. Theta measures how quickly the time value of the option changes as the option moves towards expiration, i.e. $\theta = \dfrac{\partial V}{\partial t}$.
2. Delta hedging is an attempt to set up a riskless portfolio consisting of a position in a derivative on a stock and a position in the stock. Assume that the delta (Δ) of a call option is 0.6. This means for a small change in the stock price, the option changes by about 60% of the change. Imagine the investor sold 20 option contracts, that is, options to buy $2,000$ shares, assuming one unit of option corresponds to 100 unit of shares. The investor's position could be hedged by buying $0.6 \times 2,000 = 1,200$ shares. If the stock price goes up by \$1, the investor will make a gain of $\$1,200$. However, he will also make a loss of $0.6 \times (-2,000) = -\$1,200$ on the options written. In general, the gain (or loss) on the option position would offset the loss (or gain) on the stock position.

Solution 4.33: [Exercice] The call at \$63 is \$6.557. Hence $\Delta \approx 6.557 - 5.798 = 0.759$.

Changing T to 5/12 to 5/12 +0.1 we get a call price of 6.49. Hence $\Theta \approx \frac{6.49-5.798}{0.1} = 6.92$.

Solution 4.34: [Exercice] Remember that $(dW)^2 \approx dt, (dW)(dt) \approx 0, (dt)^2 \approx 0$

[In your workout, you should omit the argument and write e.g. W for $W(t)$, H for $H(W(t), t)$]

1. Denote x the argument of f.

$$\begin{aligned} dZ(t) = df(W(t)) &= f_x(W(t))dW(t) + \frac{1}{2}f_{xx}(W(t))(dW(t))^2 + \text{higher order } dW \text{ terms} \\ &= f_x(W(t))dW(t) + \frac{1}{2}f_{xx}(W(t))dt \end{aligned}$$

   since by the box rule, all higher order terms are zero.

2. Denote (t, x) the arguments of f.

$$\begin{aligned} dZ(t) = df(t, W(t)) &= f_t(t, W(t))dt + f_x(t, W(t))dW(t) + \frac{1}{2}f_{xx}(t, W(t))(dW(t))^2 \\ &\quad + \text{higher order } dW, dt \text{ terms} \\ &= f_t(t, W(t))dt + f_x(W(t))dW(t) + \frac{1}{2}f_{xx}(W(t))dt \end{aligned}$$

   since by the box rule, all higher order terms are zero.

3. Denote (x, y) the arguments of f.

$$\begin{aligned} dZ(t) = df(W_1(t), W_2(t)) &= f_x(W_1(t), W_2(t))dW_1(t) + f_y(W_1(t), W_2(t))dW_2(t) \\ &\quad + \frac{1}{2}f_{xx}(W_1(t), W_2(t))(dW_1(t))^2 + \frac{1}{2}f_{yy}(W_1(t), W_2(t))(dW_2(t))^2 \\ &\quad + f_{xy}(W_1(t), W_2(t))dW_1(t)dW_2(t) + \text{higher order } dW_1, dW_2 \text{ terms} \\ &= f_x(W_1(t), W_2(t))dW_1(t) + f_y(W_1(t), W_2(t))dW_2(t) \\ &\quad + \frac{1}{2}f_{xx}(W_1(t), W_2(t))dt + \frac{1}{2}f_{yy}(W_1(t), W_2(t))dt \\ &\quad + f_{xy}(W_1(t), W_2(t))\rho dt \end{aligned}$$

   since by the box rule, all higher order terms are zero.

4. Similar to the above, skipped. An additional $f_t dt$ appears in addition to the answer in 3.

Solution 4.35: [Exercice]

1. Using

$$f(x) = e^x$$

   we have

$$\begin{aligned} f_x &= f \\ \frac{1}{2}f_{xx} &= \frac{1}{2}f \end{aligned}$$

   and hence by Ito

$$\begin{aligned} dY(t) = df(X(t)) &= f_x dX + \frac{1}{2}f_{xx}(dX)^2 \\ &= Y(t)dX(t) + \frac{1}{2}Y(t)\left(dX(t)\right)^2 \\ &= Y(t)\left((\alpha + \beta X(t))\,dt + dW(t)\right) + \frac{1}{2}Y(t)dt \\ &= Y(t)\Big((\frac{1}{2} + \alpha + \beta X(t))dt + dW(t)\Big) \\ &= Y(t)\Big((\frac{1}{2} + \alpha + \beta \log Y(t))dt + dW(t)\Big) \end{aligned}$$

where we use $X(t) = \log Y(t)$ such that the equation is solely in $Y(t)$. This is not necessary though.

2. Let

$$f(t,x) = e^{x-\frac{1}{2}t}$$

so

$$\begin{aligned} f_t &= -\frac{1}{2}f \\ f_x &= f \\ \frac{1}{2}f_{xx} &= \frac{1}{2}f \end{aligned}$$

and hence by Itô's lemma

$$\begin{aligned} dY(t) = df(t,X(t)) &= f_t dt + f_x dX + \frac{1}{2}f_{xx}(dX)^2 \\ &= Y(t)\Big(-\frac{1}{2}dt + dX(t) + \frac{1}{2}(dX(t))^2\Big) \\ &= Y(t)\Big(-\frac{1}{2}dt + (\alpha+\beta X(t))dt + dW(t) + \frac{1}{2}dt\Big) \\ &= Y(t)\Big((\alpha+\beta(\frac{1}{2}t+\log Y(t)))dt + dW(t)\Big) \end{aligned}$$

3. Let

$$f(x) = x^2$$

so

$$\begin{aligned} f_x &= 2x \\ \frac{1}{2}f_{xx} &= 1 \end{aligned}$$

and therefore by Itô's lemma we have

$$\begin{aligned} dY(t) = df(X(t)) &= f_x dX + \frac{1}{2}f_{xx}(dX)^2 \\ &= 2X(t)dX(t) + (dX(t))^2 \\ &= 2X(t)((\alpha+\beta X(t))dt + dW(t)) + dt \\ &= (2X(t)(\alpha+\beta X(t)) + 1)dt + 2X(t)dW(t) \\ &= (2\sqrt{Y(t)}(\alpha+\beta\sqrt{Y(t)}) + 1)dt + 2\sqrt{Y(t)}dW(t) \end{aligned}$$

Solution 4.36: [Exercice] This question the same as the previous question except the coefficient for $dW(t)$ term is now $\sqrt{X(t)}$.

1.

$$\begin{aligned} dY(t) = df(X(t)) &= f_x dX + \frac{1}{2}f_{xx}(dX)^2 \\ &= Y(t)dX(t) + \frac{1}{2}Y(t)\,(dX(t))^2 \\ &= Y(t)\,\Big((\alpha+\beta X(t))\,dt + \sqrt{X(t)}dW(t)\Big) + \frac{1}{2}Y(t)X(t)dt \\ &= Y(t)((\frac{1}{2}X(t)+\alpha+\beta X(t))dt + \sqrt{X(t)}dW(t)) \\ &= Y(t)\big((\alpha+(\beta+\frac{1}{2})\log Y(t))dt + \sqrt{\log Y(t)}dW(t)\big) \end{aligned}$$

2.

$$\begin{aligned} dY(t) = df(t, X(t)) &= f_t dt + f_x dX + \frac{1}{2} f_{xx} (dX)^2 \\ &= Y(t)\Big(-\frac{1}{2}dt + dX(t) + \frac{1}{2}(dX(t))^2 \Big) \\ &= Y(t)\Big(-\frac{1}{2}dt + (\alpha + \beta X(t))dt + \sqrt{X(t)}dW(t) + \frac{1}{2}X(t)dt \Big) \\ &= Y(t)\Big(((\alpha - \frac{1}{2}) + (\beta + \frac{1}{2})X(t))dt + \sqrt{X(t)}dW(t) \Big) \\ &= Y(t)\Big(((\alpha - \frac{1}{2}) + (\beta + \frac{1}{2})(\frac{1}{2}t + \log Y(t)))dt + \sqrt{\frac{1}{2}t + \log Y(t)}dW(t) \Big) \end{aligned}$$

3.

$$\begin{aligned} dY(t) = df(X(t)) &= f_x dX + \frac{1}{2} f_{xx} (dX)^2 \\ &= 2X(t)dX(t) + (dX(t))^2 \\ &= 2X(t)\Big((\alpha + \beta X(t))dt + \sqrt{X(t)}dW(t) \Big) + X(t)dt \\ &= 2X(t)\Big((\alpha + \frac{1}{2} + \beta X(t))dt + \sqrt{X(t)}dW(t) \Big) \\ &= 2\sqrt{Y(t)}\Big((\alpha + \frac{1}{2} + \beta\sqrt{Y(t)})dt + (Y(t))^{\frac{1}{4}}dW(t) \Big) \end{aligned}$$

Solution 4.37: [Exercice] From definition of Y, we have

$$\begin{aligned} \frac{dY(t)}{Y(t)} &= (\mu - \alpha)tdt + \sigma^2 dW(t) \\ \iff d\log Y(t) &= ((\mu - \alpha)t - \frac{\sigma^4}{2})dt + \sigma^2 dW(t) \end{aligned}$$

(Change the dummy variable t to s then) Integration from 0 to t gives

$$\log Y(t) = \frac{1}{2}(\mu - \alpha)t^2 - \frac{\sigma^4}{2}t + \sigma^2 W(t)$$

which gives

$$Y(t) = \exp\Big(\frac{1}{2}(\mu - \alpha)t^2 - \frac{\sigma^4}{2}t + \sigma^2 W(t) \Big).$$

Solution 4.38: [Exercice] Let

$$f(t, x) = e^{3t^2 - \mu t + \sigma x - 2\sigma x^2},$$

we have

$$\begin{aligned} f_t &= f \times (6t - \mu) \\ f_x &= f \times (\sigma - 4\sigma x) \\ \frac{1}{2} f_{xx} &= \frac{1}{2}\Big(f_x(\sigma - 4\sigma x) + f(-4\sigma) \Big) \\ &= f \times \frac{1}{2}(-4\sigma + (\sigma - 4\sigma x)^2) \end{aligned}$$

Hence, by Itô's lemma, we have

$$
\begin{aligned}
dS(t) = df(t, W(t)) &= f_t dt + f_x dW + \frac{1}{2} f_{xx} (dW)^2 \\
&= S(t)\Big((6t - \mu)dt + (\sigma - 4\sigma W(t))dW(t) + \frac{1}{2}(-4\sigma + (\sigma - 4\sigma W(t))^2)dt\Big) \\
&= S(t)\Big(\frac{1}{2}(12t - 2\mu - 4\sigma + \sigma^2(1 - 4W(t))^2)dt + (1 - 4W(t))\sigma dW(t)\Big)
\end{aligned}
$$

Solution 4.39: [Exercice] By the CMG theorem, we have

$$dW^Q(t) = dW(t) + \gamma(t)dt$$

being a Brownian motion under Q. Hence we have

$$dX(t) = \sigma X(t)dW^Q(t) = \sigma X(t)(dW(t) + \gamma(t)dt$$

which means that we need

$$\gamma(t) = \frac{\mu}{\sigma}$$

(which is a constant which does not depend on time) to match the dynamics of $X(\cdot)$ under P as required. The associated R-N derivative is then

$$\frac{dQ}{dP} = e^{-\int_0^T \gamma(t)dW(t) - \frac{1}{2}\int_0^T \gamma^2(t)dt}$$

with $\gamma(t) = \frac{\mu}{\sigma}$.

Solution 4.40: [Exercice] Generally speaking, when the process takes form of

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t,$$

upon change of measure, we have

$$dX_t = (\mu(X_t, t) - \sigma(X_t, t)\gamma(X_t, t))dt + \sigma(X_t, t)dW_t^Q.$$

Note that we can only change the term before the dt term but the term before the dW_t term remains unchanged (i.e. equal to $\sigma(X_t, t)$), when we move from W_t to W_t^Q.

Therefore the answer to the question is negative.

Solution 4.41: [Exercice] We have

$$
\begin{aligned}
dX(t) &= (\alpha + \beta X(t))\, dt + dW(t) \\
&= (\alpha + \beta X(t))dt + dW^Q(t) - \kappa dt \\
&= (\alpha - \kappa + \beta X(t))dt + dW^Q(t)
\end{aligned}
$$

(Note that this means that the distribution of $X(t)$ under P and Q are the same, except with a change in parameters. Remark: Be careful though as this type of result is actually not true in general, but tends to happen in many examples in financial modelling)

Solution 4.42: [Exercice]

1. We have done this so many times. Using Ito's lemma

$$\begin{aligned}
dZ(t) &= \frac{dS(t)}{B(t)} - \frac{S(t)}{B^2(t)}dB(t) \\
&= \frac{\mu S^{\frac{3}{2}}(t)dt + \sigma S(t)dW(t)}{B(t)} - \frac{S(t)}{B^2(t)}rB(t)dt \\
&= \mu Z(t)S^{\frac{1}{2}}(t)dt + \sigma Z(t)dW(t) - rZ(t)dt \\
&= \left(\mu S^{\frac{1}{2}}(t) - r\right) Z(t)dt + \sigma Z(t)dW(t) \\
&= \left(\mu S^{\frac{1}{2}}(t) - r\right) Z(t)dt + \sigma Z(t)\left(dW^Q(t) - \gamma dt\right) \text{ (Girsanov Theorem)} \\
&= \left(\mu S^{\frac{1}{2}}(t) - r - \sigma\gamma\right) Z(t)dt + \sigma Z(t)dW^Q(t)
\end{aligned}$$

So $\gamma(t) = \frac{\mu S^{\frac{1}{2}}(t) - r}{\sigma}$.

2.

$$\begin{aligned}
dS(t) &= \mu S^{\frac{3}{2}}(t)dt + \sigma S(t)\left(dW^Q(t) - \gamma dt\right) \\
&= \left(\mu S^{\frac{3}{2}}(t) - \sigma S(t)\gamma\right) dt + \sigma S(t)dW^Q(t) \\
&= \left(\mu S^{\frac{3}{2}}(t) - S(t)(\mu S^{\frac{1}{2}}(t) - r)\right) dt + \sigma S(t)dW^Q(t) \\
&= rS(t)dt + \sigma S(t)dW^Q(t)
\end{aligned}$$

Solution 4.43: [Exercice]

1. Same as the above. Using Ito's lemma

$$\begin{aligned}
dZ(t) &= \frac{dS(t)}{B(t)} - \frac{S(t)}{B^2(t)}dB(t) \\
&= \frac{\mu S^2(t)dt + \sigma S^2(t)dW(t)}{B(t)} - \frac{S(t)}{B^2(t)}rB(t)dt \\
&= \mu Z(t)S(t)dt + \sigma Z(t)S(t)dW(t) - rZ(t)dt \\
&= Z(t)\Big(\mu S(t)dt + \sigma S(t)dW(t) - rdt\Big) \\
&= Z(t)\Big((\mu S(t) - r)dt + \sigma S(t)dW(t)\Big) \\
&= Z(t)\Big((\mu S(t) - r - \sigma S(t)\gamma(t))dt + \sigma S(t)(W(t) + \gamma(t)dt)\Big) \\
&= Z(t)\Big((\mu S(t) - r - \sigma S(t)\gamma(t))dt + \sigma S(t)dW^Q(t)\Big)
\end{aligned}$$

Then we have $\gamma(t) = \frac{\mu S(t) - r}{\sigma S(t)}$

2.

$$\begin{aligned}
dS(t) &= \mu S^2(t)dt + \sigma S^2(t)\left(dW^Q(t) - \frac{\mu S(t) - r}{\sigma S(t)}dt\right) \\
&= \left(\mu S^2(t) - \mu S^2(t) + rS(t)\right) dt + \sigma S^2(t)dW^Q(t) \\
&= rS(t)dt + \sigma S^2(t)dW^Q(t)
\end{aligned}$$

Solution 4.44: [Exercice] This comes from completing the square. The LHS can be expanded as

$$
\begin{aligned}
e^x f(x;\theta,\gamma^2) &= \frac{1}{\gamma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\theta}{\gamma}\right)^2 + x \right] \\
&= \frac{1}{\gamma\sqrt{2\pi}} \exp\left[-\frac{1}{2\gamma^2}\left(x^2 - 2x\theta + \theta^2 - 2x\gamma^2\right) \right] \\
&= \exp\left(\theta + \frac{1}{2}\gamma^2\right)\frac{1}{\gamma\sqrt{2\pi}} \exp\left[-\frac{1}{2\gamma^2}\left(x - (\theta+\gamma^2)\right)^2 \right] \\
&= e^{\theta+\frac{1}{2}\gamma^2} f(x;\theta+\gamma^2,\gamma^2)
\end{aligned}
$$

as required.

This is very useful for computing e.g. call option prices in Black-Scholes. (Remark: The formula in the question will be provided to you in assessments if it is essential.)

Solution 4.45: [Exercice] The price (at time 0) is

$$e^{-rT} E_Q[S(T) 1(K_1 \le S(T) \le K_2)]$$

where under the Q-measure

$$S(T) \sim S(0) e^{N(r-\frac{1}{2}\sigma^2)T,\sigma^2 T)}.$$

Hence, we have (recall from the tip)

$$
\begin{aligned}
Price &= e^{-rT} E_Q[S(T) 1(K_1 \le S(T) \le K_2)] \\
&= e^{-rT} E_Q[S(T) 1(-d_2(K_1) \le Z \le -d_2(K_2))], \quad Z = \frac{\log\frac{S(T)}{S(0)} - (r-\frac{1}{2}\sigma^2)T}{\sqrt{\sigma^2 T}} \\
&= e^{-rT} E_Q[S(T)] E[1(-d_2(K_1) \le \tilde{Z} + \sqrt{\sigma^2 T} \le -d_2(K_2))] \\
&= e^{-rT} E_Q[S(T)] E[1((d_2(K_1) + \sqrt{\sigma^2 T}) \ge -\tilde{Z} \ge (d_2(K_2) + \sqrt{\sigma^2 T}))] \\
&= e^{-rT} E_Q[S(T)] E[1(d_1(K_1) \ge -\tilde{Z} \ge d_1(K_2))] \\
&= e^{-rT} S(0) \exp(rT - \frac{1}{2}\sigma^2 T + \frac{1}{2}\sigma^2 T) P[(d_1(K_1) \ge N(0,1) \ge d_1(K_2))] \\
&= S(0)(\Phi(d_1(K_1)) - \Phi(d_1(K_2))
\end{aligned}
$$

because $\tilde{Z}$ and $-\tilde{Z}$ are both distributed as $N(0,1)$.

Solution 4.46: [Exercice] The payoff function is

$$X = 1(\max_{0\le t\le T} S(t) > KS(0)).$$

Hence, the price is given by

$$e^{-rT} P_Q[\max_{0\le t\le T} S(t) > KS(0)].$$

From the given formula, we have

$$
\begin{aligned}
P_Q[\max_{0\le t\le T} S(t) > KS(0)] &= P_Q[\max_{0\le t\le T} S(0) e^{(r-\frac{1}{2}\sigma^2)t + \sigma W^Q(t)} > KS(0)] \\
&= P_Q[\max_{0\le t\le T} \left(\frac{r-\frac{1}{2}\sigma^2}{\sigma} t + W^Q(t)\right) > \frac{\log K}{\sigma}] \quad \text{(as } f(x) = S(0)e^{\sigma x} \text{ is increasing)} \\
&= \Phi\left(\frac{-y+\rho T}{\sqrt{T}}\right) + e^{2\rho y}\Phi\left(\frac{-y-\rho T}{\sqrt{T}}\right)
\end{aligned}
$$

with $\rho = \frac{r-\frac{1}{2}\sigma^2}{\sigma}$ and $y = \frac{\log K}{\sigma}$.
This gives

$$price = e^{-rT}\Phi\left(\frac{-y+\rho T}{\sqrt{T}}\right) + e^{2\rho y - rT}\Phi\left(\frac{-y-\rho T}{\sqrt{T}}\right).$$

Solution 4.47: [Exercice] First write out the payoff:

$$X = \begin{cases} \$0 & \text{else} \\ \$0.5 & \max_{0\le t\le T} S(t) > K_1 S(0) \\ \$1 & \max_{0\le t\le T} S(t) > K_2 S(0) \end{cases}$$

Note that the second event includes the last event. Therefore we can decompose the above as

$$X = 0.5 \times 1(\max_{0\le t\le T} S(t) > K_1 S(0)) + 0.5 \times 1(\max_{0\le t\le T} S(t) > K_2 S(0)).$$

Therefore, the price is given by

$$price = 0.5e^{-rT} P_Q[\max_{0\le t\le T} S(t) > K_1 S(0)] + 0.5e^{-rT} P_Q[\max_{0\le t\le T} S(t) > K_2 S(0)]$$

From the previous question, we know that

$$e^{-rT} P[\max_{0\le t\le T} S(t) > KS(0)] = e^{-rT}\Phi\left(\frac{-\frac{\log K}{\sigma}+\rho T}{\sqrt{T}}\right) + e^{2\rho\frac{\log K}{\sigma} - rT}\Phi\left(\frac{-\frac{\log K}{\sigma}-\rho T}{\sqrt{T}}\right)$$

with $\rho = \frac{r-\frac{1}{2}\sigma^2}{\sigma}$. Hence we have

$$\begin{aligned} price &= 0.5e^{-rT} P_Q[\max_{0\le t\le T} S(t) > K_1 S(0)] + 0.5e^{-rT} P_Q[\max_{0\le t\le T} S(t) > K_2 S(0)] \\ &= 0.5\Bigg(e^{-rT}\Phi\left(\frac{-\frac{\log K_1}{\sigma}+\rho T}{\sqrt{T}}\right) + e^{2\rho\frac{\log K_1}{\sigma} - rT}\Phi\left(\frac{-\frac{\log K_1}{\sigma}-\rho T}{\sqrt{T}}\right) \\ &\quad + e^{-rT}\Phi\left(\frac{-\frac{\log K_2}{\sigma}+\rho T}{\sqrt{T}}\right) + e^{2\rho\frac{\log K_2}{\sigma} - rT}\Phi\left(\frac{-\frac{\log K_2}{\sigma}-\rho T}{\sqrt{T}}\right)\Bigg). \end{aligned}$$

Solution 4.48: [Exercice] Note the asset as being the stock accumulated with reinvested dividends. (This is an important concept as when we replicate the portfolio we are holding the asset, not the price.) The dynamics of this asset (with value denoted by $\widetilde{S}$) is then

$$d\widetilde{S}(t) = (\mu+\delta)\widetilde{S}(t)dt + \sigma\widetilde{S}(t)dW(t)$$

We now tune the measure such that the discounted asset

$$Z(t) = \frac{\widetilde{S}(t)}{B(t)}$$

is a martingale.

Via Ito's lemma, we have

$$\begin{aligned} dZ(t) &= (\mu+\delta-r)Z(t)dt + \sigma Z(t)dW(t) \\ &= (\mu+\delta-r)Z(t)dt + \sigma Z(t)(dW^Q(t) - \gamma dt) \\ &= (\mu+\delta-r-\sigma\gamma)Z(t)dt + \sigma Z(t)dW^Q(t) \end{aligned}$$

Hence, we shall choose

$$\gamma = \frac{\mu + \delta - r}{\sigma}$$

which implies

$$\mu - \sigma\gamma = r - \delta.$$

Substituing into the dynamics of the plain stock price, we have

$$\begin{aligned}
dS(t) &= \mu S(t)dt + \sigma S(t)dW(t) \\
&= \mu S(t)dt + \sigma S(t)(dW^Q(t) - \gamma dt) \\
&= (\mu - \sigma\gamma)S(t)dt + \sigma S(t)dW^Q(t) \\
&= (r - \delta)S(t)dt + \sigma S(t)dW^Q(t).
\end{aligned}$$

Price of a call option. From the setting, we have

$$\begin{aligned}
price &= e^{-rT}E_Q[(S(T) - K)_+] \\
\frac{dS(t)}{S(t)} &= (r - \delta)dt + \sigma dW^Q(t)
\end{aligned}$$

(Using Ito's lemma on $\log S(t)$) The second equation gives $d\log S(t) = (r-\delta-\frac{1}{2}\sigma^2)dt+\sigma dW^Q(t)$, or

$$S(T) \sim S(0)\exp(m + \sqrt{v}Z)$$

where $m = (r - \delta - \frac{1}{2}\sigma^2)T$ and $v = \sigma^2 T$. Hence, the price is computed as

$$\begin{aligned}
price &= e^{-rT}E_Q[(S(T) - K)_+] \\
&= e^{-rT}E_Q[S(T)1(S(T) > K)] - Ke^{-rT}E_Q[1(S(T) > K)] \\
&= e^{-rT}E_Q[S(T)1(Z > -d_2(K))] - Ke^{-rT}E_Q[1(Z > -d_2(K))],\ d_2(K) = \frac{\log\frac{S(0)}{K} + m}{\sqrt{v}} \\
&= e^{-rT}E_Q[S(T)]P[\tilde{Z} + \sqrt{v} > -d_2(K)] - Ke^{-rT}P[-Z < d_2(K)] \\
&= e^{-rT}E_Q[S(T)]P[-\tilde{Z} < d_2(K) + \sqrt{v}] - Ke^{-rT}P[-Z < d_2(K)] \\
&= e^{-rT}\exp(m + \frac{1}{2}v)P[-\tilde{Z} < d_1(K)] - Ke^{-rT}P[-Z < d_2(K)],\ d_1(K) = d_2(K) + \sqrt{v} \\
&= e^{-rT}S(0)e^{rT-\delta T+\frac{1}{2}\sigma^2 T+\frac{1}{2}\sigma^2 T}\Phi(d_1(K)) - Ke^{-rT}\Phi(d_2(K)) \\
&= S(0)e^{-\delta T}\Phi(d_1(K)) - Ke^{-rT}\Phi(d_2(K)).
\end{aligned}$$

Note $Z, -Z, \tilde{Z}, -\tilde{Z}$ are all $N(0, 1)$ distributed.

Solution 4.49: [Exercice] Our payoff is:

$$\begin{cases} S(T) & \text{if } S(T) > K \\ M & \text{if } S(T) \leq K \end{cases}$$

Then our option price is:

$$\begin{aligned}
Price &= e^{-rT}E_Q[X] \\
&= e^{-rT}\Big(E_Q[S(T)1(S(T) > K)] + ME_Q[1(S(T) \leq K)]\Big) \\
&= e^{-rT}\Big(E_Q[S(T)1(S(T) > K)] + ME_Q[1 - 1(S(T) > K)]\Big) \\
&= e^{-rT}\Big(E_Q[S(T)1(S(T) > K)] + M(1 - E_Q[1(S(T) > K)])\Big) \\
&= e^{-rT}E_Q[S(T)1(S(T) > K)] + Me^{-rT}(1 - E_Q[1(S(T) > K)]) \\
&= S(0)\Phi(d_1) + Me^{-rT}(1 - \Phi(d_2))
\end{aligned}$$

where the last line uses the 2 terms in BS formula.

Solution 4.50: [Exercice]

1. Let $X(t) = S(t)^2$, By Itô's lemma we have

$$\begin{aligned}
dX(t) &= 2S(t)dS(t) + \frac{1}{2}2dS^2(t) \\
&= 2S(t)(rS(t)dt + \sigma S(t)dW(t)) + (r^2S^2(t)dt^2 + 2\sigma rS^2(t)dtdW(t) + \sigma^2S^2(t)dW^2(t)) \\
&= 2rS^2(t)dt + 2\sigma S^2(t)dW(t) + \sigma^2S^2(t)dt \\
&= (2r + \sigma^2)S^2(t)dt + 2\sigma S^2(t)dW(t) \\
&= (2r + \sigma^2)X(t)dt + 2\sigma X(t)dW(t)
\end{aligned}$$

2. We need to compute $e^{-rT}E_Q[(S(T)^2 - K)_+]$. To do that, we need to find the distribution of $S(T)^2$. There are 2 methods to do that:

   - Direct method: From part (1), we know

   $$\frac{d(S(t)^2)}{S(t)^2} = (2r + \sigma^2)dt + 2\sigma dW(t)$$

   which by Itô's lemma (on the log function) gives

   $$d\log(S(t)^2) = (2r + \sigma^2 - \frac{1}{2}(2\sigma)^2)dt + 2\sigma dW(t)$$

   (i.e. when converting from dX/X to $d\log X$ we have to adjust the drift term by subtracting half of the square of the volatility term) or

   $$d\log(S(t)^2) = (2r - \sigma^2)dt + 2\sigma dW(t).$$

   Via integration, we have

   $$S(T)^2 \sim S(0)^2 \exp N((2r - \sigma^2)T, 4\sigma^2T).$$

   - Compute from $S(T)$: From $dS(t)/S(t) = rdt + \sigma dW(t)$, via Itô's lemma using log function gives

   $$d\log S(t) = (r - \frac{1}{2}\sigma^2)dt + \sigma dW(t).$$

   Via integration, we have

   $$\log\frac{S(T)}{S(t)} = (r - \frac{1}{2}\sigma^2)T + \sigma W(T).$$

Now, by multiplying the above by 2, we have

$$\log \frac{S(T)^2}{S(0)^2} = 2\log\frac{S(T)}{S(t)} = (2r-\sigma^2)T + 2\sigma W(T)$$

which gives the same result (except we do not need part 1).

For $E_Q[S(T)^2 1(S(T)^2 > K)]$, using $d_2(K) = \frac{\log\frac{S(0)^2}{K}+(2r-\sigma^2)T}{\sqrt{r\sigma^2 T}}$ and $Z = \frac{\log\frac{S(T)^2}{S(0)^2}-(2r-\sigma^2)T}{\sqrt{4\sigma^2 T}}$, we have

$$\begin{aligned}
&E[S(T)^2 1(S(T)^2 > K)] \\
&= E[S(T)^2 1(Z > -d_2(K))] \\
&= E[S(T)^2]P[\tilde{Z} + \sqrt{4\sigma^2 T} > -d_2(K)] \\
&= E[S(T)^2]P[-\tilde{Z} < d_2(K) + \sqrt{4\sigma^2 T}] \\
&= E[S(T)^2]\Phi(d_2(K) + \sqrt{4\sigma^2 T}) \\
&= S(0)^2\exp(2(r-\frac{1}{2}\sigma^2)T + \frac{1}{2}4\sigma^2 T)\Phi(\frac{\log\frac{S(0)^2}{K} + 2(r-\frac{1}{2}\sigma^2)T}{\sqrt{4\sigma^2 T}} + \sqrt{4\sigma^2 T}) \\
&= S(0)^2\exp(2rT + \sigma^2 T)\Phi(\frac{\log\frac{S(0)^2}{K} + (2r+3\sigma^2)T}{\sqrt{4\sigma^2 T}}).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&E[1(S(T)^2 > K)] \\
&= P[Z > -d_2(K)] \\
&= \Phi(d_2(K)).
\end{aligned}$$

Therefore the price of the payoff is given by

$$\begin{aligned}
Price &= e^{-rT}E_Q[(S(T)^2 - K)_+] \\
&= e^{-rT}E_Q[S(T)^2 1(S(T)^2 > K)] - Ke^{-rT}E_Q[1(S(T)^2 > K)] \\
&= S(0)^2\exp((r+\sigma^2)T)\Phi(d_1(K)) - Ke^{-rT}\Phi(d_2(K)),
\end{aligned}$$

where $d_1(K) = d_2(K) + \sqrt{4\sigma^2 T} = \frac{\log\frac{S(0)^2}{K}+(2r+3\sigma^2)T}{\sqrt{4\sigma^2 T}}$.

Module 5

5.1 Term Structure Modelling

5.1.1 Practice Questions

Exercise 5.1: [Solution] Describe the main differences and similarities between the Vasicek and the Cox-Ingersoll-Ross (CIR) models.

Exercise 5.2: [Solution] Consider the Vasicek model. The P dynamics are

$$dr(t) = \alpha\left(\mu - r(t)\right)dt + \sigma dW(t)$$

1. Let
$$K(t) = \int_0^t \alpha ds$$
so that $dK = \alpha dt$. What is
$$d\left(e^{K(t)}r(t)\right)?$$

2. Use (a) to show that the solution to the above SDE is
$$r(t) = e^{-\alpha t}r(0) + \left(1 - e^{-\alpha t}\right)\mu + \sigma e^{-\alpha t}\int_0^t e^{\alpha u}dW(u)$$

3. A result from stochastic process is the following: (subject to technicalities) if $f(\cdot)$ is a deterministic function, then $\int_0^t f(u)dW(u)$ is Normally distributed with mean 0 and variance
$$\int_0^t (f(u))^2 du.$$
Find the distribution of $r(t)$.

Exercise 5.3: [Solution] The bond price under the Merton model is

$$B(t,T) = \exp\left\{-r(t)(T-t) - \frac{1}{2}\mu(T-t)^2 + \frac{1}{6}\sigma^2(T-t)^3\right\},$$

where

$$dr(t) = \mu dt + \sigma dW^Q(t).$$

Use Ito to find $dB(t,T)$.

Exercise 5.4: [Solution] Luenberger, 14.6(1Ed) ‘6.6 (2Ed)

Exercise 5.5: [Solution] Consider the following term structure of interest rates:

Term	Spot Rate
1	5%
2	6%

and volatility

Term	Volatility
1	na
2	1%

1. Find the Q interest rate lattice under the Ho-Lee model.
2. Calculate the value of a derivative which pays \$100 at time 2 if the interest rate for the year from 1 to 2 rises above 7% else you get \$0.

Exercise 5.6: [Solution] Suppose we have calculated the 3-period Q interest rate lattice using the market bond prices to be:

		0.07
	0.06	0.05
0.05	0.04	0.03

Suppose your firm has just entered into an interest rate swap with a bank, under the contract you are required to receive the floating Q-interest rate and pay a fixed Q-interest rate of 4%. If the swap payment is due after three periods time, find the value of the contract your firm is holding. (For simplicity assume a notional value of \$100).

Exercise 5.7: [Solution] (Computer Implementation) Consider a term structure with pa spot rates

Term	*Spot Rate*
1	4%
2	4.5%
3	4.5%

and volatility

Term	*Vol*
1	*na*
2	2%
3	2.5%

1. Using the Ho-Lee model, Setup a Q interest rate lattice.
2. Find the price of a call option on a ZCB (with maturity at time 3). The payoff of the call is at time 2 and the strike is $\frac{100}{1.06}$

Exercise 5.8: [Solution] Consider the discrete time Black-Derman-Toy model for setting up our interest rate lattice, represented by the following equation:

$$r(k, s) = a(k)e^{b(k)s}$$

where k=0,1 is the time and s = 0,1,2,...,k is the state. The spot rates are as follows:

Term(years)	Spot Rate (p.a.)
1	6%
2	5.5%

and we know that $b(1) = 0.02$.
Find the Q interest rate lattice.

Exercise 5.9: [Solution] Suppose we have decided to use the discrete time Black-Derman-Toy model for finding the Q interest rate lattice, which can be represented by:

$$r(k,s) = a(k)e^{b(k)s}$$

where k=0,1,2 is the time and s = 0,1,2,...,k is the state.
Now this lattice has already been set up but some corruption in the data has lead to some parts of the lattice being lost as seen below:

		6%
	X%	Y%
3%	4%	5%

Now we know $b(1) = 0.02$.

1. Find the missing values in the interest rate lattice
2. Find the price of a 2 year call option on a 3 year zero coupon bond if the strike price is $\frac{100}{1.055}$

Exercise 5.10: [Solution] Suppose we have developed a model that has predicted the following Q interest rate lattice:

			0.06
		0.04	0.045
	0.03	0.03	0.04
0.02	0.025	0.02	0.025

Now using this lattice calculate the predicted spot curve.

5.1.2 Discussion Questions

Exercise 5.11: [Solution] The CIR model of interest rates is

$$dr(t) = \alpha(\mu - r(t))dt + \sigma\sqrt{r(t)}dW(t)$$

(there are certain restrictions on the parameters to ensure that this process is well behaved as an interest rate)

1. Assuming that technical conditions are satisfied, find potential assumptions on the market price of risk such that the dynamics of $r(\cdot)$ under Q has the same form as that in P (although with different parameters)
2. Briefly discuss why this model may be preferred to the Merton model.

3. Suppose
$$dr(t) = \alpha(\mu' - r(t))dt + \sigma\sqrt{r(t)}dW^Q(t)$$
For some constant μ'.

   (a) Consider a small period of time Δt. Suggest how you can use a discrete approximation (on $dr(t)$) to arrive at an algorithm for simulating
   $$r(\Delta t), r(2\Delta t), r(3\Delta t), \cdots$$
   You can assume that you are able to simulate independent Normal random variables.
   (b) Suggest how you may then use simulation to approximate $\int_0^t r(u)du$.
   (c) It is known that, given information up to till t, the time t price of a zero coupon bond with maturity T can be presented in the following form:
   $$B(t,T) = e^{\lambda(t,T)r(t)+\nu(t,T)}$$
   for some deterministic functions $\lambda(t,T), \nu(t,T)$, and where $r(t)$ the short rate at time t. Suppose you known the close form representation of $\lambda(t,T), \nu(t,T)$. Briefly suggest how you can may be able to use simulation to price (at time 0) a call option, with exercise date t, on a zero coupon bond (with corresponding maturity T) in this framework.

Exercise 5.12: [Solution] Under the Merton Model we have that the short rate is $r(t) = r(0) + \alpha t + \sigma W(t)$. Now doing a change of measure we see that we obtain $r(t) = r(0) + \alpha' t + \sigma W^Q(t)$ where $\alpha' = \alpha - \sigma\gamma$.
Prove that the conditional mean and variance of the integrated short rate $I(t,T_1) = \int_t^{T_1} r(s)ds$ are:
$$E_Q\left[I(t,T_1)|F_t\right] = r(t)(T_1 - t) + \frac{\alpha'}{2}(T_1 - t)^2$$
$$Var_Q\left[I(t,T_1)|F_t\right] = \frac{\sigma^2}{3}(T_1 - t)^3$$

Hence, show that the price of a ZCB is given by:
$$B(t,T_1) = e^{-r(t)(T_1-t)-\frac{\alpha'}{2}(T_1-t)^2+\frac{1}{6}\sigma^2(T_1-t)^3}$$

Exercise 5.13: [Solution] Suppose that interest rates follow the Merton Model $r(t) = r(0) + \alpha t + \sigma W^Q(t)$. We have a particular binary option that puts \$1 into a savings account at the start of the option and accrues interest over the life of the option. If the interest rate at maturity is higher than K then the amount in the savings account is given to the option holder, otherwise the amount in the account is kept by the option writer. Calculate the price an investor should pay for this option.

Exercise 5.14: [Solution] The Q dynamics for the Vasicek model is given as:
$$dr(t) = \alpha(\mu - r(t))dt + \sigma dW^Q(t)$$
which has the following solution:
$$r(t) = e^{-\alpha t} r(0) + \left(1 - e^{-\alpha t}\right)\mu + \sigma e^{-\alpha t}\int_0^t e^{\alpha u} dW(u)$$

Find an expression for the price of a ZCB maturing in T years time. Hint: You may want to use the following identity.

$$\int_0^T \int_0^t \sigma e^{-\alpha(t-u)} dW(u) dt = \int_0^T \int_u^T \sigma e^{-\alpha(t-u)} dt dW(u)$$

Exercise 5.15: [Solution] Consider a term structure with pa spot rates

Term	*Spot Rate*
1	5%
2	5%
3	6%

and volatility

Term	*Vol*
1	*na*
2	2%
3	2%

1. Show that $a(0) = 0.05, a(1) = 0.03, a(2) = 0.041$.
2. Find the price of a derivative contract which pays the \$1000 times difference the short rate at time 2 and 6% if the amount is positive, or zero otherwise. (Extension question - find the price of a call option as per the lecture example)

5.2 Asset-Liability Management

5.2.1 Practice Questions

Exercise 5.16: [Solution] Let $P(t)$ denote the market price of a stock in year t. Assume that $P(t)$ follows a random walk model expressed as follows:

$$\log P(t) = \log P(t-1) + \mu + \sigma\varepsilon_t$$

where ε_t are i.i.d. normal random variables with $E(\varepsilon_t) = 0$ and $Var(\varepsilon_t) = 1$.

1. Denote by $Q(t) = \log\left[\dfrac{P(t)}{P(t-1)}\right]$. Show that $Q(t)$ has a normal distribution.
2. Find the mean and variance of $Q(t)$.

Exercise 5.17: [Solution] (Revision) You are given the following model for the market price of a stock:

$$\log P(t+1) = (1+\gamma)\log P(t) + \mu(1-\gamma t) + \sigma\varepsilon_{t+1},$$

where ε_{t+1} are i.i.d. random variables with zero mean and unit variance. Let $Q(t) = \log P(t) - \mu t$. Prove the following:

1. $E[Q(t)] = 0$.
2. $Var[Q(t)] = \dfrac{\sigma^2}{1-(1+\gamma)^2}$.

3. $Cov\left(Q\left(t\right), Q\left(t-1\right)\right) = Var\left[Q\left(t\right)\right] \times \left(1+\gamma\right).$

Exercise 5.18: [Solution] Suppose you are an actuarial assistant working in the product design team for a capital guaranteed contract. The contract guarantees that $y\%$ of the premium will be returned if the investment portfolio ends up being worth less then this amount. List some issues you may consider when designing the other details of this product, paying particular attention to the determination of y.

Exercise 5.19: [Solution] Discuss and explain some of the real world problems that can be solved by using the techniques learnt in Financial Economics.

Exercise 5.20: [Solution] (Computer Implementation) Suppose we want to value a three year Asian option which pays if the average of the stock prices at the end of each year is higher than the strike price at maturity i.e $X = (A_T - K)^+$ where A_T is the average stock price up to time T. Construct a spreadsheet that can simulate the stock price at three points in time (t=1,t=2,t=3) and calculate the price of such an option by taking the average. Assume we have the following information:

- Stock price follows geometric brownian motion
- $\mu = 5\%$
- $r = 4\%$
- $\sigma = 15\%$
- $S(0) = \$10$
- $K = \$11$

Extension: Calculate the long term average price of the option by taking the average of 1000 of these simulations. Hint: Simulate stock price scenarios under the Q measure, calculate the value of the option under each scenario and then take the average across scenarios.

Exercise 5.21: [Solution] (Computer Implementation) Following on from the previous exercise, suppose we now want to know the probability of the option writer defaulting (assume that he will default if the option can be exercised at maturity) at maturity. Calculate the probability of default and suggest things the writer can do to reduce this probability.
Hint: Simulate the stock price scenarios under the P measure.

Exercise 5.22: [Solution] (Computer Implementation) Following on from the previous exercise, suppose a company has the following balance sheet at time 0:

Assets	Liabilities
$\$700,000$ cash	$\$660,000$ of the above Asian options
$\$300,000$ shares in the underlying stock	3 year ZCBs with a face value of $\$200,000$
	Equity
	$\$140,000$ Shareholders equity

Now if the Asian options were sold for $0.66 each find what the probability of default is for the company after three years. How much cash reserve is required to ensure the company can still be solvent three years later with 95% probability. (Assume the cash earns risk-free interest)

5.2.2 Discussion Questions

Exercise 5.23: [Solution] In the Wilkie stochastic investment model, the force of inflation from $t-1$ to t, $I(t)$, is modelled as

$$I(t) = \mu_Q + \alpha_Q [I(t-1) - \mu_Q] + \sigma_Q Z_Q(t),$$

where

$$I(t) = \log [Q(t) / Q(t-1)],$$

$Q(t) =$ level of the inflation index, and $Z_Q(t)$ are i.i.d. $N(0,1)$ random variables, and μ_Q, α_Q and σ_Q are all parameters to be estimated. The parameter values recommended by Wilkie in his paper based on historical data are: $\widehat{\mu}_Q = 0.047$, $\widehat{\alpha}_Q = 0.58$ and $\widehat{\sigma}_Q = 0.0425$. Calculate, given that the annual rate of inflation is 2.5%, the 95% confidence interval for the rate of inflation over the following year. Comment on this result

Exercise 5.24: [Solution] A contract pays off according to gains of the stock index $S(t)$, with a guaranteed minimum payout and maximum payout. More precisely, it is a five year contract which pays out 90% times the ratio of the terminal and initial values of the index. Or it pays out 130% of the initial value if otherwise it would be less, or 180% of the initial value if otherwise it would be more. How much is this contract worth? Suppose that $\mu = 6\%, \sigma = 15\%, r = 6.5\%$, and dividend yield $\delta = 4\%$.

Exercise 5.25: [Solution] Suppose that you are investigating the probability of ruin of a capital guaranteed contract where the payout is interest sensitive. You setup a term structure model and decide to use simulation to check the probability of ruin. The parameters are derived by ensuring that the parameters match the current yield curve perfectly. Is this ok and why/why not?

Exercise 5.26: [Solution] A three year capital guarantee product promises to pay according to the gains in a particular fund and ensures that losses will not exceed 20% of the original investment. If losses exceed 20% then the company will cover the difference. How much is this contract worth? Suppose that $\mu = 4\%, \sigma = 20\%, r = 5\%$, dividend yield $\delta = 2\%$ and assume that the dynamics of the fund follows geometric brownian motion.

Exercise 5.27: [Solution] Suppose we have a 2 year capital product that gives another 10% return on top of the return on a stock index however it has a maximum payout of 50% return on investment. If the index follows a GBM with the following parameters: $\mu = 10\%, \sigma = 10\%, r = 3\%$, find the value of this product.

Exercise 5.28: [Solution] Suppose there is a one year capital guarantee product that has the following characteristics:

- The company charges $x\%$ in fees on purchase of the product and the remainder is invested in the company's special investment portfolio
- If the investment portfolio at the end of the year is worth more than 80% of the investor's initial capital, then the investor is paid the full amount of the investment portfolio
- If the investment portfolio at the end of the year is worth less than 80% of the investor's initial capital, then the investor is paid 80% of his initial capital

If we know the company's investment portfolio follows geometric brownian motion and has the following parameters: $\mu = 20\%, \sigma = 25\%$ and $r = 6\%$ (continuous compounding).

Find an expression that can find the value of x. (Notice that if we forgot about the dividends the price would be 1.0183 - which is 5% too high)

5.3 Solutions

5.3.1 Term Structure Modelling

Solution 5.1: [Exercice] Similarities: Both models are based upon an assumption that the short rate is generated by a stochastic process that is: mean-reverting; subject only to a single source of uncertainty or randomness. time.

Differences: The Vasicek model allows for negative yields, in which case, it becomes possible for the price of a zero-coupon bond to be larger than 1. In contrast, the CIR model does not permit negative interest rates and so the price of a zero-coupon bond must lie between 0 and 1. The variance of the short rate in the Vasicek model is constant over time, whereas in the CIR model, the variance increases with its level. This can be seen from the difference in the short rate processes between the two models. In the Vasicek model, this process is governed by

$$dR = \alpha\left(\mu - R\right)dt + \sigma dz$$

while in the CIR model, we have

$$dR = \alpha\left(\mu - R\right)dt + \sigma\sqrt{R}dz.$$

Solution 5.2: [Exercice]

1. Let $f(t,x) = e^{K(t)}x$, which gives

$$\begin{aligned} f_t &= f \times \alpha \\ f_x &= e^{K(t)} \\ f_{xx} &= 0 \end{aligned}$$

Via Ito's lemma, we have

$$\begin{aligned} df(t,r(t)) &= e^{K(t)}r(t)\alpha dt + e^{K(t)}dr(t) \\ &= e^{K(t)}r(t)\alpha dt + e^{K(t)}\Big(\alpha\left(\mu - r(t)\right)dt + \sigma dW(t)\Big) \\ &= e^{K(t)}\alpha\mu dt + e^{K(t)}r(t)\alpha dt - e^{K(t)}r(t)\alpha dt + e^{K(t)}\sigma dW(t) \\ &= e^{K(t)}\alpha\mu dt + e^{K(t)}\sigma dW(t). \end{aligned}$$

2. Integrate a from $t = 0$ to $t = T$ gives

$$\begin{aligned} e^{\alpha T}r(T) - r(0) &= \mu\int_0^T \alpha e^{\alpha t}dt + \int_0^T \sigma e^{\alpha t}dW(t) \\ &= \mu(e^{\alpha T} - 1) + \int_0^T \sigma e^{\alpha t}dW(t). \end{aligned}$$

By rearranging the terms, we have

$$r(T) = e^{-\alpha T}r(0) + \mu(1 - e^{-\alpha T}) + \int_0^T \sigma e^{-\alpha(T-t)}dW(t).$$

3. Since the only randomness comes from

$$\sigma\int_0^t e^{\alpha u}dW\left(u\right)$$

we see that $r(t)$ is normal with mean

$$e^{-\alpha t} r(0) + \left(1 - e^{-\alpha t}\right)\mu$$

and variance

$$\begin{aligned} & \sigma^2 e^{-2\alpha t} \int_0^t e^{2au} du \\ = \; & \sigma^2 \frac{(1 - e^{-2\alpha t})}{2\alpha}. \end{aligned}$$

Solution 5.3: [Exercice]

$$\begin{aligned} & B(t,T) \\ = \; & \exp\left\{-r(t)(T-t) - \frac{1}{2}\mu (T-t)^2 + \frac{1}{6}\sigma^2 (T-t)^3\right\} \end{aligned}$$

By denoting

$$f(t,x) = \exp(-x(T-t) - \frac{1}{2}\mu(T-t)^2 + \frac{1}{6}\sigma^2(T-t)^3)$$

we have

$$\begin{aligned} f_t &= f \times \left(x + \mu(T-t) - \frac{1}{2}\sigma^2(T-t)^2\right) \\ f_x &= f \times -(T-t) \\ \frac{1}{2} f_{xx} &= f \times \frac{1}{2}(T-t)^2 \end{aligned}$$

and hence under Q

$$\begin{aligned} dB(t,T) = df(t,r(t)) &= B(t,T)\Big((r(t) + \mu(T-t) - \frac{1}{2}\sigma^2(T-t)^2)dt - (T-t)dr(t) + \frac{1}{2}(T-t)^2(dr(t))^2\Big) \\ &= B(t,T)\Big((r(t) + \mu(T-t) - \frac{1}{2}\sigma^2(T-t)^2)dt - \mu(T-t)dt - (T-t)\sigma dW(t) + \frac{1}{2}(T-t)^2\sigma^2 dt\Big) \\ &= B(t,T)(r(t)dt - \sigma(T-t)dW(t)). \end{aligned}$$

which makes sense as

- the drift term is rB under Q
- the bond volatility coefficient approaches zero as $t \to T$

Solution 5.4: [Exercice] At node $r(k-1) = a(k-1) + sb(k-1)$, it can go either $r(k) = a(k) + sb(k)$ or $r(k) = a(k) + (s+1)b(k)$, with probability 0.5 for either case.

Therefore, the mean is given by

$$E[r(k)] = \frac{1}{2}(a(k) + sb(k) + a(k) + (s+1)b(k))$$

. On the other hand, the second moment

$$E[r(k)^2] = \frac{1}{2}((a(k) + sb(k))^2 + (a(k) + (s+1)b(k))^2)$$

.

Note (Q-)variance is given by $Var[r(k)] = E[r(k)^2] - E[r(k)]^2$, which is evaluated as

$$\begin{aligned} & \frac{1}{2}\left(a\left(k\right)+b\left(k\right)s\right)^{2}+\frac{1}{2}\left(a\left(k\right)+b\left(k\right)\left(s+1\right)\right)^{2} \\ & -\frac{1}{4}\left(a\left(k\right)+b\left(k\right)\left(s+1\right)+a\left(k\right)+b\left(k\right)s\right)^{2} \\ = & \frac{1}{4}b\left(k\right)^{2} \end{aligned}$$

as required. Note that this is the Q variance not the P variance. In practice we use many periods and approximate the Q variance by the P variance. (this is what we have done in the lectures and in the text)

Solution 5.5: [Exercice] Our two step interest rate lattice looks like:

$$\begin{array}{cc} & \text{a(1) + b(1)} \\ \text{a(0)} & \text{a(1)} \end{array}$$

1. It is clear that a(0) is the one year spot rate and b(1) is 2 times the volatility. Therefore, we have $a(0) = 0.05,\ b(1) = 0.02$. To calculate $a(1)$, we need to calculate the price of a 2 year Zero Coupon Bond using the market interest rates and equate that to the price using our interest rate lattice.

$$\begin{aligned} p_{market} &= \frac{1}{1.06^2} \\ p_Q &= E_Q[1/B(2)] = E_Q[\frac{1}{1+r(0)}\frac{1}{1+r(1)}] \\ &= \frac{1}{2}\left(\frac{1}{1.05}\frac{1}{1+a(1)} + \frac{1}{1.05}\frac{1}{1+a(1)+0.02}\right) \\ \frac{1}{1.06^2} &= \frac{0.5}{1.05}\left(\frac{1}{A(1)} + \frac{1}{A(1)+0.02}\right) \quad \text{(for simplicity use A(1) = 1+a(1))} \\ \frac{1.05}{0.5\cdot 1.06^2} &= \frac{1}{A(1)} + \frac{1}{A(1)+0.02} \\ \frac{1.05}{0.5\cdot 1.06^2} &= \frac{2A(1)+0.02}{A(1)^2+0.02A(1)} \\ 0 &= \frac{1.05}{0.5\cdot 1.06^2}A(1)^2 + \left(0.02\frac{1.05}{0.5\cdot 1.06^2} - 2\right)A(1) - 0.02 \\ &= 1.06018868 \quad \text{(Quadratic formula)} \end{aligned}$$

   Therefore we have $a(1) = 0.060189$.

2. The path of interest rates and the corresponding payoff at time 2 are given by

$$\begin{aligned} (u) &: 0.05 \to 0.080189 \quad (100) \\ (d) &: 0.05 \to 0.060189 \quad (0) \end{aligned}$$

   where the price is given by $E_Q[X/B(T)]$. This gives

$$\begin{aligned} Price &= E_Q[X/B(2)] \\ &= \frac{1}{2}\Big(\frac{1}{1.05}\frac{1}{1.0801890}\times 100 + \frac{1}{1.05}\frac{1}{1.0601890}\times 0\Big) \\ &= 44.084. \end{aligned}$$

Solution 5.6: [Exercice] The path of interest rates and the corresponding payoff at time 3 are given by

$$\begin{aligned}
(u,u) &: 0.05 \to 0.06 \to 0.07 \quad (100*(0.07-0.04)=3)\\
(u,d) &: 0.05 \to 0.06 \to 0.05 \quad (100*(0.05-0.04)=1)\\
(d,u) &: 0.05 \to 0.04 \to 0.05 \quad (100*(0.05-0.04)=1)\\
(d,d) &: 0.05 \to 0.04 \to 0.03 \quad (100*(0.03-0.04)=-1)
\end{aligned}$$

Hence, the price is given by

$$price = E_Q[X/B(3)] = \frac{1}{4}\begin{bmatrix} \frac{1}{1.05}\frac{1}{1.06}\frac{1}{1.07}\times 3 \\ +\frac{1}{1.05}\frac{1}{1.06}\frac{1}{1.05}\times 1 \\ +\frac{1}{1.05}\frac{1}{1.04}\frac{1}{1.05}\times 1 \\ +\frac{1}{1.05}\frac{1}{1.04}\frac{1}{1.03}\times -1 \end{bmatrix} = 0.8394589.$$

Solution 5.7: [Exercice] It is a straight-forward implementation following the same argument as Ex 5.5. You simply need to calibrate $a(1)$ then $a(2)$ from the data (Spot rate). A sample R solution is the following.

1. Step 0 (Preparation): define functions to create the paths and find the root of a given function.

```
Jp<-function(x,y){#x is a matrix, y is a vector #Attach y to x<- (x,y)
        #Dimensions
        xrow<-nrow(x)
        xcol<-ncol(x)
        #Modify y
        ym<-c()
        for (j in 1:(length(y)-1)){
                temp<-c(y[j],y[j+1])
                ym<-c(ym,rep(temp,choose(length(y)-2,j-1)))
        }
        ##
        z<-c() #the answer
        #Loop
        for (i in 1:xrow){ #each x
                z<-c(z,c(x[i,],ym[2*i-1]),c(x[i,],ym[2*i]))
        }
        ans<-matrix(z,ncol=xcol+1,byrow=TRUE)
        return(ans)
}
#Bisection method
Findroot_bi<-function(a,b,tol,fun){
        atemp<-a
        btemp<-b
        if (fun(a)*fun(b)>0){ans<-"error"}else{
                while(abs(atemp-btemp)>tol){
                        midtemp = 0.5*(atemp+btemp)
                        fa <-fun(atemp)
                        fb <-fun(btemp)
                        fm <-fun(midtemp)
```

```
                    if(fm*fb>0){btemp <- midtemp}else{atemp <- midtemp}
                    }
                    ans<-0.5*(atemp+btemp)
            }
            return(ans)
    }
```

2. Step 2: Model calibration - Find the $a(i)$'s.

```
Spot<-c(0.04,0.045,0.045)
vol<-c(0,0.02,0.025)
#Calibrate - find the a's
b<-2*vol
a<-c(Spot[1])
Z<-matrix(c(Spot[1]),nrow=1)
for (i in 2:(length(Spot))){ #How many parameters to calibrate
#Assume having calibrated (i-1) of the a's
f<-function(a){
        Zfun<-Jp(Z,a+0:i*b[i])
        ZDfun<-1/(1+Zfun)
        P<-rep(1,nrow(Zfun))
        for (j in 1:ncol(ZDfun)){
        P<-P*ZDfun[,j]
        }
  return(sum(P)/length(P)-1/(1+Spot[i])^i)
}
ai<-Findroot_bi(-0.5,1,10^-6,f)
a<-c(a,ai)
Z<-Jp(Z,ai+0:i*b[i])
}
```

3. Part 2: Find the price of the call option:

```
K<-1/1.06
P<-sapply(1/(1+Z[,3]),function(x){max(x-K,0)})
ZD<-1/(1+Z)
100*mean(ZD[,1]*ZD[,2]*P)
```

The price is 1.948459.

Solution 5.8: [Exercice] It is clear that $a(0) = 0.06$.
Now to get $a(1)$ we need to find the price of a 2 year zcb using the interest rate lattice.

$$p_Q = E_Q[\frac{1}{B(T)}] = \frac{1}{2}\left(\frac{1}{1.06}\frac{1}{1+a(1)} + \frac{1}{1.06}\frac{1}{1+a(1)e^{0.02}}\right)$$

$$p_{Spot} = \frac{1}{1.055^2}$$

$$\frac{0.5}{1.06}\left(\frac{1}{1+a(1)}+\frac{1}{1+a(1)e^{0.02}}\right)=\frac{1}{1.055^2}$$
$$\frac{0.5}{1.06}\left(\frac{2+a(1)(1+e^{0.02})}{(1+a(1))(1+a(1)e^{0.02})}\right)=\frac{1}{1.055^2}$$
$$\frac{0.5}{1.06}\left(2+a(1)(1+e^{0.02})\right)=\frac{(1+a(1))(1+a(1)e^{0.02})}{1.055^2}$$
$$\frac{1}{1.06}+a(1)\frac{0.5(1+e^{0.02})}{1.06}=\frac{1+a(1)(1+e^{0.02})+a(1)^2e^{0.02}}{1.055^2}$$

$$0=\frac{a(1)^2e^{0.02}}{1.055^2}+a(1)\left(1+e^{0.02}\right)\left(\frac{1}{1.055^2}-\frac{0.5}{1.06}\right)+\left(\frac{1}{1.055^2}-\frac{1}{1.06}\right)$$
$$a(1)=0.04952360165 \text{ (Quadratic Formula)}$$

Solution 5.9: [Exercice]

1. We know that X satisfies:

$$a(1)e^{b(1)\cdot 1}=0.04e^{0.02}$$

   Then we have $X=0.0408080536$.
   Now Y satisfies:

$$a(2)e^{b(2)}$$

   We know that $a(2)=0.05$ so $e^{2b(2)}=\frac{0.06}{0.05}$ so $b(2)=0.0911607784$.
   Then we have $Y=0.05e^{0.0911607784}=0.05477225575$.

2. Note the payoff at time 2 is given by $(\frac{100}{1+r(2)}-\frac{100}{1.055})_+$, because the interest rate from time 2 to 3, $r(2)$ is known at time 2, which implies that the ZCB price at time 2 must be $1/(1+r(2))$. Now, the path of interest rates and the corresponding payoff at time 2 are given by

$$\begin{aligned}
(u,u): 0.03\to 0.0408\to 0.06 &\quad ((\frac{100}{1.06}-\frac{100}{1.055})_+=0)\\
(u,d): 0.03\to 0.0408\to 0.0548 &\quad ((\frac{100}{1.0548}-\frac{100}{1.055})_+=0.02046616)\\
(d,u): 0.03\to 0.04\to 0.0548 &\quad ((\frac{100}{1.0548}-\frac{100}{1.055})_+=0.02046616)\\
(d,d): 0.03\to 0.04\to 0.05 &\quad ((\frac{100}{1.05}-\frac{100}{1.055})_+=0.45136538)
\end{aligned}$$

   Hence, the price is given by

$$price=E_Q[X/B(2)]=\frac{1}{4}\begin{bmatrix}\frac{1}{1.03}\frac{1}{1.0408}\times 0\\ +\frac{1}{1.03}\frac{1}{1.0408}\times 0.02046616\\ +\frac{1}{1.03}\frac{1}{1.04}\times 0.02046616\\ +\frac{1}{1.03}\frac{1}{1.04}\times 0.45136538\end{bmatrix}=0.1148903.$$

Solution 5.10: [Exercice] Now we can directly deduce that the one year spot rate is $S_1 = 2\%$. To get the other spot rates, we need to calculate the price of ZCB's of maturity 2,3,4 years.

$$\frac{1}{(1+S_2)^2} = \frac{1}{2}\left(\frac{1}{1.02}\frac{1}{1.03} + \frac{1}{1.02}\frac{1}{1.025}\right)$$
$$\therefore S_2 = 2.37401016\%$$
$$\frac{1}{(1+S_3)^3} = \frac{1}{4}\begin{bmatrix} \frac{1}{1.02}\frac{1}{1.03}\frac{1}{1.04} \\ +\frac{1}{1.02}\frac{1}{1.03}\frac{1}{1.03} \\ +\frac{1}{1.02}\frac{1}{1.025}\frac{1}{1.03} \\ +\frac{1}{1.02}\frac{1}{1.025}\frac{1}{1.02} \end{bmatrix}$$
$$\therefore S_3 = 2.580234031\%$$

You can check that $S_4 = 2.9917\%$.

Solution 5.11: [Exercice]

1. We have
$$dr(t) = \big(\alpha(\mu - r(t)) - \sigma\sqrt{r(t)}\gamma(t)\big)dt + \sigma\sqrt{r(t)}dW^Q(t)$$
hence to be of the same form as P we need either $\gamma(t) = k\sqrt{r(t)}$, or $\gamma(t) = \frac{k}{\sqrt{r(t)}}$, for some constant k. Note that technical conditions will need to be checked in practice to verify this.

2. CIR has two excellent features compared to Merton: It is mean reverting, and it is strictly positive. Hence this model may be preferred over Merton. (It is also still reasonably tractable, although no longer as simple as the Merton or Vasicek models)

3. NOTE: The aim of parts (a) and (b) below are just for students to come up with some ideas/develop some intuition about simulating the process - The precise method used is not important.

   (a) One simple approach would be via a discrete approximation of the SDE: with $dt = \frac{k}{n}t$ and $dW(t) = N(0, \frac{t}{n})$: staring with $r(0)$, we can progress with a time step dt as follows:
   $$r\left(\frac{k}{n}t\right) = r\left(\frac{k-1}{n}t\right) + \alpha\left(\mu' - r\left(\frac{k-1}{n}t\right)\right)\frac{t}{n} + \sigma\sqrt{r\left(\frac{k-1}{n}t\right)}Z_k$$
   where Z_k are iid $N(0, \frac{t}{n})$ random variables.

   (b) One approach will be
   $$\int_0^t r(u)du \approx \sum_{k=1}^{n} r\left(\frac{k-1}{n}t\right)\left(\frac{t}{n}\right)$$
   hence the integral can be approximately simulated using the approximation in part (i) above.

   (c) Given the results above, we can then note that the time 0 price of the Option is
   $$E_Q[\frac{1}{B(t)}(B(t,T) - K)^+]$$

Hence we want
$$E_Q[e^{\int_0^t r(u)du}(e^{\lambda(t,T)r(t)+\nu(t,T)} - K)^+]$$
Which can be simulated using the approximations above. This will then provide all the information we need to use simulation to price the derivative - run the above algorithm many times (till the standard error is small, then the average outcome will be our price).

Solution 5.12: [Exercice]

we simply need a couple of observations and integration by parts.

Conditional on $\mathcal{F}_t$ simply means the interest rate process r_s starts at $r(t)$ at time t, which implies that without loss of generality it is sufficient to consider only the case when time goes from 0 to $T(= T_1 - t)$.

Assuming starting at $s = 0$, by $dr_s = \mu ds + \sigma dW_s$, we have $r_s = r_0 + \mu s + \int_0^s \sigma dW_u$. By further integrating, we have
$$\int_0^T r_s ds = r_t T + \mu \int_0^T s ds + \int_0^T \int_0^s \sigma dW_u.$$
Using integration by parts, we have
$$\int_0^T \int_0^s \sigma dW_u = \int_0^T (T - u)\sigma dW_u,$$
which implies that
$$\int_0^T r_s ds = r_t T + \mu \frac{T^2}{2} + \int_0^T (T - s)\sigma dW_s.$$
[Note: Integration by parts only works for simple cases. In general, we need (stochastic) Fubini type of theorems.]

This implies that $\int_0^T r_s ds$ is normally distributed with mean $r_t T + \mu \frac{T^2}{2}$ and variance $\int_0^T (T - s)^2 \sigma^2 ds = \frac{\sigma^2 T^3}{3}$.

Finally, by shifting the time, i.e. replacing T by $T_1 - t$ and 0 by t on the R.H.S, we have $\int_t^{T_1} r_s ds$ is normally distributed with mean $r_t(T_1 - t) + \mu \frac{(T_1-t)^2}{2}$ and variance $\frac{\sigma^2 (T_1-t)^3}{3}$.

The price of a ZCB is given by the discounted Q expectation:
$$\begin{aligned}
B(t, T_1) &= E_Q\left[\$1 \cdot e^{-\int_t^{T_1} r(s)ds} | F_t\right] \\
&= M_{I(t,T_1)}(-1) \text{ (Using MGF of a normal variable: } E[e^{tY}] \text{ with } t = -1,\, Y = I(.)) \\
&= e^{-r(t)(T_1-t) - \frac{\alpha'}{2}(T_1-t)^2 + \frac{\sigma^2}{6}(T_1-t)^3}
\end{aligned}$$

Solution 5.13: [Exercice] First off our payoff is:
$$X = \begin{cases} e^{\int_0^T r(s)ds} & \text{if } r(T) > K \\ 0 & \text{if } r(T) \leq K \end{cases}$$

The price of any derivative is given by the discounted Q expectation:

$$\begin{aligned}
price &= E_Q\left[e^{-\int_0^T r(s)ds}X\right] \\
&= E_Q\left[e^{-\int_0^T r(s)ds}e^{\int_0^T r(s)ds}1_{r(T)>K}\right] = P_Q[r(T) > K] \\
&= P_Q[r(0) + \alpha T + \sigma W^Q(T) > K] \\
&= P_Q\left[Z > \frac{K - r(0) - \alpha T}{\sigma\sqrt{T}}\right] \\
&= P_Q[-Z < \frac{r(0) + \alpha T - K}{\sigma\sqrt{T}}] \\
&= \Phi(d)
\end{aligned}$$

where $d = \frac{r(0)+\alpha T-K}{\sigma\sqrt{T}}$.

Solution 5.14: [Exercice] First the price of a ZCB is given by the following discounted Q expectation:

$$B(0,T) = E_Q\left[e^{-\int_0^T r(t)dt}\right]$$

So we need to find the distribution of $\int_0^T r(t)dt$. From the question, we know that under Q,

$$r(t) = e^{-\alpha t}r(0) + (1 - e^{-\alpha t})\mu + \int_0^t \sigma e^{-\alpha(t-u)}dW(u).$$

Hence, via integration, we have

$$\begin{aligned}
&\int_0^T r(t)dt \\
&= r(0)\int_0^T e^{-\alpha t}dt + \mu\int_0^T (1 - e^{-\alpha t})dt + \int_0^T\int_0^t \sigma e^{-\alpha(t-u)}dW(u)dt \\
&= r(0)\frac{1}{\alpha}(1 - e^{-\alpha T}) + \mu T - \mu\frac{1}{\alpha}(1 - e^{-\alpha T}) + \int_0^T\int_0^t \sigma e^{-\alpha(t-u)}dW(u)dt
\end{aligned}$$

Now, from the hint, we have

$$\int_0^T\int_0^t \sigma e^{-\alpha(t-u)}dW(u)dt = \int_0^T\int_u^T \sigma e^{-\alpha(t-u)}dtdW(u)$$

and therefore we have

$$\begin{aligned}
\int_0^T r(t)dt &= r(0)\frac{1}{\alpha}(1 - e^{-\alpha T}) + \mu T - \mu\frac{1}{\alpha}(1 - e^{-\alpha T}) + \int_0^T\int_u^T \sigma e^{-\alpha(t-u)}dtdW(u) \\
&= \frac{r(0) - \mu}{\alpha}(1 - e^{-\alpha T}) + \mu T + \int_0^T \sigma e^{\alpha u}\int_u^T e^{-\alpha t}dtdW(u) \\
&= \frac{r(0) - \mu}{\alpha}(1 - e^{-\alpha T}) + \mu T + \int_0^T \sigma e^{\alpha u}\frac{e^{-\alpha u} - e^{-\alpha T}}{\alpha}dW(u) \\
&= \frac{r(0) - \mu}{\alpha}(1 - e^{-\alpha T}) + \mu T + \int_0^T \sigma\frac{1 - e^{-\alpha(T-u)}}{\alpha}dW(u)
\end{aligned}$$

which shows that $\int_0^T r(t)dt$ is normally distributed.

Note

$$E[\int_0^T r(t)dt] = \frac{r(0)-\mu}{\alpha}(1-e^{-\alpha T}) + \mu T$$

and

$$\begin{aligned}
Var[\int_0^T r(t)dt] &= \int_0^T \Big(\sigma \frac{1-e^{-\alpha(T-u)}}{\alpha}\Big)^2 du \\
&= \frac{\sigma^2}{\alpha^2}\int_0^T (1-e^{-\alpha(T-u)})^2 du \\
&= \frac{\sigma^2}{\alpha^2}\int_0^T (1-e^{-\alpha u})^2 du \quad \text{(change of variable)} \\
&= \frac{\sigma^2}{\alpha^2}\int_0^T (1-2e^{-\alpha u}+e^{-2\alpha u}) du \\
&= \frac{\sigma^2}{\alpha^2}\Big(T - 2\frac{1-e^{-\alpha T}}{\alpha} + \frac{1-e^{-2\alpha T}}{2\alpha}\Big).
\end{aligned}$$

Hence, the bond price is actually log normally distributed and we have:

$$\begin{aligned}
B(0,T) &= E_Q\left[e^{-\int_0^T r(t)dt}\right] \\
&= e^{-E_Q\left(\int_0^T r(t)dt\right)+\frac{1}{2}Var\left(\int_0^T r(t)dt\right)} \\
&= \exp\Big(-\mu T - \frac{r(0)-\mu}{\alpha}(1-e^{-\alpha T}) + \frac{\sigma^2}{2\alpha^2}\big(T - 2\frac{1-e^{-\alpha T}}{\alpha} + \frac{1-e^{-2\alpha T}}{2\alpha}\big)\Big).
\end{aligned}$$

Solution 5.15: [Exercice]

1. The bs are given by 2 times the volatility for the period, i.e. $b(1) = b(2) = 0.04$. $a(0)$ is the same as the interest rate from $t = 0$ to $t = 1$ as such rate is known at time 0, i.e. $a(0) = 0.05$.

   To caculate $a(1)$, we build a one-step tree and use the implied interest rate from $t = 0$ to $t = 2$ to calibrate $a(1)$, i.e.

Path	Interest rate process	Q-probabilities
(u) :	$a(0) = 0.05 \to a(1) + b(1) = a(1) + 0.04$	w.p. 0.5
(d) :	$0.05 \to a(1)$	w.p. 0.5

   This means the bond price with payoff 1 at the end of $t = 2$ evaluated at $t = 0$ is given by

   $$\begin{aligned}
   p &= E_Q[\frac{1}{(1+i(0))(1+i(1))}] \\
   &= \frac{1}{2}\Big(\frac{1}{1.05}\frac{1}{1+a(1)} + \frac{1}{1.05}\frac{1}{1+a(1)+b(1)}\Big) \\
   &= \frac{1}{2}\Big(\frac{1}{1.05}\frac{1}{1+a(1)} + \frac{1}{1.05}\frac{1}{1+a(1)+0.04}\Big)
   \end{aligned}$$

   Note that without arbitrage, this number has to be the same as the data given by the question, i.e. $p = 1/1.05^2$. On solving, we get $a(1) = 0.0304$. Knowing $a(1)$, we can now proceed to $a(2)$ using the same approach. This gives $a(2) = 0.04135$.

2. Using the numbers given by the question, i.e. $a(1) = 0.03$, $a(2) = 0.041$, and the payoff function $X = 1000(i(2) - 0.06)_+$, the corresponding tree in the Q-dynamic is given by

Path	Interest rate process	Payoff at $t = 2$	Q-probabilities
(u, u) :	$0.05 \to 0.07 \to 0.121$	(61)	w.p. 0.25
(u, d) :	$0.05 \to 0.07 \to 0.081$	(21)	w.p. 0.25
(d, u) :	$0.05 \to 0.03 \to 0.081$	(21)	w.p. 0.25
(d, d) :	$0.05 \to 0.03 \to 0.041$	(0)	w.p. 0.25

Using the same pricing formula, i.e. $p = E_Q[\frac{X}{(1+i(0))(1+i(1))}]$, we get $p = 23.1$.

5.3.2 Asset-Liability Management

Solution 5.16: [Exercice] Since

$$\log P(t) = \log P(t-1) + \mu + \sigma\varepsilon_t,$$

then

$$\log \frac{P(t)}{P(t-1)} = \mu + \sigma\varepsilon_t.$$

1. Since $\varepsilon_t \sim N(0,1)$, then $\mu + \sigma\varepsilon_t \sim N(\mu, \sigma^2)$. That is, $Q(t) = \mu + \sigma\varepsilon_t$ is normally distributed.

2. Its mean is μ and its variance is σ^2.

Solution 5.17: [Exercice] First, deduct $\mu(t+1)$ to both sides of the given price equation and we get:

$$\log P(t+1) - \mu(t+1) = (1+\gamma)\log P(t) + \mu(1-\gamma t) + \sigma\varepsilon_{t+1} - \mu(t+1)$$

and re-arranging, we get:

$$Q(t+1) = (1+\gamma) Q(t) + \sigma\varepsilon_{t+1}.$$

Now, we use the backward operator [1] B and substitute $s = t+1$. Therefore,

$$Q(s) = (1+\gamma) BQ(s) + \sigma\varepsilon_s.$$

Solving for $Q(t)$, we can show we have

$$Q(t) = \sum_{k=0}^{\infty} \sigma (1+\gamma)^k \varepsilon_{t-k}$$

.

1. Thus, the expectation

$$\begin{aligned} E[Q(t)] &= E\left[\sum_{k=0}^{\infty} \sigma (1+\gamma)^k \varepsilon_{t-k}\right] \\ &= \sum_{k=0}^{\infty} \sigma (1+\gamma)^k E(\varepsilon_{t-k}) = 0 \end{aligned}$$

since $E(\varepsilon_{t-k}) = 0$ for all $t-k$.

[1] note 1018: this question can be done by recursion without the use of the operator

2. The variance is

$$\begin{aligned} Var\left[Q\left(t\right)\right] &= Var\left[\sum_{k=0}^{\infty}\sigma\left(1+\gamma\right)^{k}\varepsilon_{t-k}\right] \\ &= \sum_{k=0}^{\infty}\sigma^{2}\left(1+\gamma\right)^{2k}Var\left(\varepsilon_{t-k}\right) \\ &= \sum_{k=0}^{\infty}\sigma^{2}\left(1+\gamma\right)^{2k}=\frac{\sigma^{2}}{1-\left(1+\gamma\right)^{2}}. \end{aligned}$$

3. First, note that

$$\begin{aligned} Q\left(t\right) &= \sum_{k=0}^{\infty}\sigma\left(1+\gamma\right)^{k}\varepsilon_{t-k}=\sigma\varepsilon_{t}+\sum_{k=1}^{\infty}\sigma\left(1+\gamma\right)^{k}\varepsilon_{t-k} \\ &= \sigma\varepsilon_{t}+\sum_{k=0}^{\infty}\sigma\left(1+\gamma\right)^{k+1}\varepsilon_{t-k-1}=\sigma\varepsilon_{t}+\left(1+\gamma\right)\sum_{k=0}^{\infty}\sigma\left(1+\gamma\right)^{k}\varepsilon_{t-k-1} \\ &= \sigma\varepsilon_{t}+\left(1+\gamma\right)Q\left(t-1\right). \end{aligned}$$

Thus, the covariance is

$$\begin{aligned} Cov\left[Q\left(t\right),Q\left(t-1\right)\right] &= Cov\left[\sigma\varepsilon_{t}+\left(1+\gamma\right)Q\left(t-1\right),Q\left(t-1\right)\right] \\ &= Cov\left[\sigma\varepsilon_{t},Q\left(t-1\right)\right]+\left(1+\gamma\right)Cov\left[Q\left(t-1\right),Q\left(t-1\right)\right] \\ &= \left(1+\gamma\right)\times Var\left[Q\left(t-1\right)\right]=\left(1+\gamma\right)\times Var\left[Q\left(t\right)\right]. \end{aligned}$$

Solution 5.18: [Exercice] Some possibilities include

- What will be in the investment portfolio? Fixed Interest? Domestic Equity?
- If say domestic equity, then is it going to be a portfolio that tracks the index?
- How does the guarantee apply? For example, is it just an end of year guarantee or does the guarantee apply throughout the year (ie European or American type?)
- What is the term of the investment? what if any are the early redemption penalties
- Can be hedge our risk via either portfolio insurance or purchasing an option explicitly
- Other concerns include expenses, fees

There are two main methods of determining y - use a stochastic asset return model as we discussed in the lectures, or we can investigate the cost of the option from option pricing theory.

Notice that this method may involve an iterative/numerical procedure as the guarantee (which is like a strike price) will turn up in the d_1, d_2 terms in the Black-Scholes formula, and (depending on the exact contract specifications) we may have to solve equations of the form

$$f\left(y\right)=Black-Scholes(y)$$

where f is some deterministic function.

Solution 5.19: [Exercice]

- Utiliy Theory - Can be used to help people select make decisions i.e stock/portfolio selection in particular and whether to buy insurance or not
- Mean Variance - Used for asset allocations i.e setting up passive investment funds
- CAPM and APT - Pricing of assets and asset allocation
- Brownian motion - Simulate the dynamics of asset prices and interest rates used for derivative pricing
- Aggregate - The pricing of specific assets, how to select assets optimally and ultimately the management of assets and liabilities

Solution 5.20: [Exercice] Check spreadsheet for solution.

Solution 5.21: [Exercice] Check spreadsheet for solution.

Solution 5.22: [Exercice] Check spreadsheet for solution.

Solution 5.23: [Exercice] We are given that the annual rate of inflation for year t is

$$\frac{Q(t) - Q(t-1)}{Q(t-1)} = .025$$

so that $I(t) = \log 1.025 = 0.0246926$. The forecast for next year's force of inflation is therefore

$$\widehat{I(t+1)} = \widehat{\mu}_Q + \widehat{\alpha}_Q \left[I(t) - \widehat{\mu}_Q\right] = 0.047 + 0.58\,(0.0246926 - 0.047) = 0.03461715.$$

A 95% confidence interval for this forecast is therefore

$$\widehat{I(t+1)} \pm \widehat{\sigma}_Q Z_{0.025} = 0.03461715 \pm (0.0425) \times 1.96.$$

This interval results in

$$(-0.04924, 0.11736).$$

Therefore, for the rate of inflation, the 95% confidence interval is given by

$$\left(e^{-0.04924} - 1, e^{0.11736} - 1\right) = (-4.8\%, 12.5\%).$$

This range appears to be very wide given that the previous year's inflation rate is 2.5%. However, note that the Wilkie model was designed as a long-term model for actuarial use. Therefore, a one year projection could be considered a mis-use of the model. The inflation rate in the previous year was below the long term mean of 4.7%. Hence, the expected inflation rate next year will be higher than that just experienced. This is because of the mean-reverting property of the model itself. The modelled symmetry of inflation does not seem realistic in this case.

Solution 5.24: [Exercice] This question is from Baxter and Rennie, Ch 4.

The claim is

$$X = \min\left(\max\left(1.3, 0.9 S_T\right), 1.8\right)$$

with maturity $T = 5$, and initial value $S(0) = 1$. Rewrite the claim as

$$X = 1.3 + 0.9\left((S_T - 1.444)_+ - (S_T - 2)_+\right)$$

and so we see that it is actually cash plus the difference of two calls. Using the Black Scholes formula with dividends (also sometimes called the Garman -Kolhagen formula) the price is then

$$\begin{aligned} V_0 &= 1.3e^{-rT} + 0.9\,(0.0422 - 0.0067) \\ &= 0.9712 \end{aligned}$$

(Notice that if we forgot about the dividends the price would be 1.0183 - which is 5% too high)

Solution 5.25: [Exercice] If we are looking at solvency then we should be looking at real world parameters / dynamics / probabilities (i.e. P). The parameters that price assets using discounted expectation are Q parameters / dynamics / probabilities and so will not be appropriate. On the other hand once you have projected it forward then the pricing of any securities in the future will require the Q parameters.

Remember that

- Real world projections - use P
- Pricing by discounted expectation - use Q (or P with a risk adjustment)

Solution 5.26: [Exercice] First off our payoff function is:

$$\begin{aligned} X &= \max(S(T), 0.8S(0)) \\ &= 0.8S(0) + (S(T) - 0.8S(0))_+ \end{aligned}$$

Then our price is:

$$\begin{aligned} p &= e^{-rT} E_Q\left[X\right] \\ &= e^{-rT}\left(0.8S(0) + E_Q\left[(S(T) - 0.8S(0))_+\right]\right) \text{ (Cash + Call with strike } 0.8S(0)) \\ &= 0.8S(0)e^{-rT} + C_0 \text{ (Black-Scholes Formula with dividends)} \\ &= 0.8S(0)e^{-rT} + S(0)e^{-rT}\left(e^{3(0.05-0.02)}\Phi(d_1) - 0.8\Phi(d_2)\right) \\ &= 0.8S(0)e^{-rT} + S(0)e^{-rT}\,(0.3261913736) \\ &= 0.9693218983S(0) \end{aligned}$$

where $d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{0.8S(0)}\right) + \left(r - \delta + \frac{\sigma^2}{2}\right)T\right], d_2 = d_1 - \sigma\sqrt{T}$.

Solution 5.27: [Exercice] So our payoff function is: Let S(0)=1

$$\begin{aligned} X &= \min(1.1S(T), 1.5) \\ &= 1.5 - (1.5 - 1.1S(T))_+ \\ &= 1.5 - 1.1\left(\frac{1.5}{1.1} - S(T)\right)_+ \end{aligned}$$

Which is cash subtract a put option, so the value is:

$$\begin{aligned} p &= e^{-rT} E_Q\left[1.5 - 1.1\left(\frac{1.5}{1.1} - S(T)\right)_+\right] \\ &= 1.5e^{-rT} - 1.1P_0 \text{ (Black-Scholes)} \\ &= \$1.097 \end{aligned}$$

Solution 5.28: [Exercice] Suppose we initially invest \$100, then the amount placed in the investment portfolio is $S_0 = 100 - x$.
The payment at the end of the year is:

$$\max(S_1, 80) = \max(S_1 - 80, 0) + 80$$

which is cash plus a call option.
Then x should be such that the price of this product is \$100. Then:

$$BS(100 - x, 80) + 80e^{-0.06} = 100$$

where BS$(100 - x, 80)$ is the Black-Scholes formula for a call option with initial price $100 - x$, strike price at 80, risk free rate of 6% continuous compounding and volatility of 25%. The solution will need to be solved numerically.