ACTL3182 Cheatsheet

Andrew Wu

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NOTE: Theory sections are written to simplify the text for easy revision. Yours answers to exam questions may need to be more detailed than what is written here. This cheatsheet may be revised throughout the term, please check https://github.com/BrownianNotion/ACTL3182_20T3.git for the latest version.

1 Modern Portfolio Theory

1.1 Utility Theory

1.1.1 Expected Utility Theorem

Individual prefers W_1 over W_2 iff $\mathbb{E}[U(W_1)] \geq \mathbb{E}[U(W_2)]$

1.1.2 Utility Axioms

The expected utility theorem is a consequence of the following axioms:

1. **Completeness (Comparability):** For all decisions A, B, either $A \prec B$, $A \succ B$ or $A \sim B$
2. **Transitivity:** If $A \succ B$ and $B \succ C$, then $A \succ C$
3. **Independence:** If $A \succ B$ then for any C,
$$G(A, C; \alpha) \sim G(B, C; \alpha)$$
4. **Measurability:** If $A \succ B \succ C$, there exists a unique α such that
$$B \sim G(A, C; \alpha)$$
5. **Ranking:** Suppose $A \succ B \succ D$, $A \succ C \succ D$, $B \sim G(A, D; \alpha_1)$ and $C \sim G(A, D; \alpha_2)$. Then, if $\alpha_1 \geq \alpha_2$, then $B \succ D$.
6. **Certainty Equivalent:** All gambles have a price.

1.1.3 Non-Satiation

Individuals always prefer more wealth to less: $U'(w) > 0$.

1.1.4 Investor types

Type	Wealth Preference	Utility Preference	U''	Concavity
Risk-Averse	$\mathbb{E}[W] \succ W$	$U(\mathbb{E}[W]) > E[U(W)]$	$U'' < 0$	concave
Risk-Neutral	$\mathbb{E}[W] \sim W$	$U(\mathbb{E}[W]) = E[U(W)]$	$U'' = 0$	linear
Risk-Lover	$\mathbb{E}[W] \prec W$	$U(\mathbb{E}[W]) < E[U(W)]$	$U'' > 0$	convex

1.1.5 Risk Premium

The amount $\pi(W)$ that an individual will pay to give up risk:

$$\pi(W) = \mathbb{E}[W] - c(W)$$

where $c(W) := U^{-1}(\mathbb{E}[U(W)])$ is the **certainty wealth equivalent**.

1.1.6 Risk Aversion

Absolute risk aversion $A(w)$ and relative risk aversion $R(w)$

$$A(w) = -\frac{U''(w)}{U'(w)}, \quad R(w) = -w\frac{U''(w)}{U'(w)}.$$

1.2 Investment Risk Measures

1.2.1 Definition

A function $\theta : X \to \mathbb{R}$ that summarises an investment's risk with a single number.

1.2.2 Common Examples

1. Variance: $\int_{-\infty}^{\infty}(x-\mu)^2 f_X(x)dx$
2. Downside-Variance: $\int_{-\infty}^{\mu}(x-\mu)^2 f_X(x)dx$
3. Shortfall Probability: $\mathbb{P}(X \leq L)$
4. Expected Shortfall: $\int_{-\infty}^{L}(L-x) f_X(x)dx$
5. Shortfall Variance: $\int_{-\infty}^{L}(L-x)^2 f_X(x)dx$

1.2.3 Value-at-Risk:

The Value-at-risk **at level $\boldsymbol{\alpha}$** is the maximum possible loss from holding a portfolio over a **given time period** so that the **probability of larger loss is $\boldsymbol{1-\alpha}$**.
In general,

$$\mathrm{VaR}(\alpha) = \mu - X_{1-\alpha},$$

where $X_{1-\alpha}$ is the $1-\alpha$-quantile of X. If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $\mathrm{VaR}(\alpha) = \sigma Z_{\alpha}$.

1.3 Portfolio Optimisation

1.3.1 Two assets

Minimum Variance Portfolio:

$$w_A = \frac{\sigma_B^2 - \sigma_{AB}}{\sigma_A^2 + \sigma_B^2 - 2\sigma_{AB}}, \quad w_B = 1 - w_A$$

Portfolio Variance:

$$\sigma_P^2 = w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2w_Aw_B\rho_{AB}\sigma_A\sigma_B$$

1.3.2 N-risky assets

Minimise $\sigma_P^2 = \boldsymbol{w}^\top \Sigma \boldsymbol{w}$, subject to $\mathbf{1}^\top \boldsymbol{w} = 1$ and $\boldsymbol{z}^\top \boldsymbol{w} = \mu$.
Lagrangian: $\mathcal{L}(\boldsymbol{w}, \lambda, \gamma) = \frac{1}{2}\boldsymbol{w}^\top \Sigma \boldsymbol{w} + \lambda(1 - \mathbf{1}^\top \boldsymbol{w}) + \gamma(\mu - \boldsymbol{z}^\top \boldsymbol{w})$

Constants:

$$A = \mathbf{1}^\top \Sigma^{-1} \mathbf{1}, \quad B = \mathbf{1}^\top \Sigma^{-1} \boldsymbol{z} = \boldsymbol{z}^\top \Sigma^{-1} \mathbf{1}$$
$$C = \boldsymbol{z}^\top \Sigma^{-1} \boldsymbol{z}, \quad \Delta = AC - B^2$$

Minimum Variance Portfolio:

$$\boldsymbol{w} = \lambda \Sigma^{-1} \mathbf{1} + \gamma \Sigma^{-1} \boldsymbol{z}$$

where

$$\lambda = \frac{C - \mu B}{\Delta}, \quad \gamma = \frac{\mu A - B}{\Delta}$$

Portfolio Variance:

$$\sigma_P^2 = \frac{A\mu^2 - 2B\mu + C}{\Delta}$$

Global Minimum Variance Portfolio:

$$\boldsymbol{w}_g = \frac{1}{A} \Sigma^{-1} \mathbf{1}$$

1.3.3 N-risky assets + risk-free

Minimise $\sigma_P^2 = \boldsymbol{w}^\top \Sigma \boldsymbol{w}$, subject to $(\boldsymbol{z} - r_f \mathbf{1})^\top \boldsymbol{w} = u - r_f$.
Lagrangian: $\mathcal{L}(\boldsymbol{w}, \lambda, \gamma) = \frac{1}{2}\boldsymbol{w}^\top \Sigma \boldsymbol{w} + \gamma(\mu - r_f - (\boldsymbol{z} - r_f \mathbf{1})^\top \boldsymbol{w})$

Minimum Variance Portfolio:

$$\boldsymbol{w} = \gamma \Sigma^{-1} (\boldsymbol{z} - r_f \mathbf{1})$$

where

$$\gamma = \frac{\mu - r_f}{A r_f^2 - 2B r_f + C}$$

Portfolio Variance:

$$\sigma_P^2 = \frac{(\mu - r_f)^2}{A r_f^2 - 2B r_f + C}$$

Tangency Portfolio:

$$\boldsymbol{w}_t = \gamma_t \Sigma^{-1} (\boldsymbol{z} - r_f \mathbf{1})$$

where

$$\gamma_t = \frac{1}{B - A r_f}$$

1.3.4 Two Fund Theorem (N-risky assets)

All efficient portfolios are a linear combination of any two efficient portfolios.

1.3.5 One Fund Theorem (N-risk + risk-free)

All efficient portfolios are a linear combination of the tangent portfolio and the risk-free asset.

2 Asset Pricing Models

For all parts in section 2, i takes values $1, 2, ...N$ and indexes the assets in the market.

2.1 CAPM

2.2 CAPM Assumptions

1. Investors choose optimal portfolios based only on mean and variance of returns. Justified if returns are normally distributed returns or utility functions are quadratic.
2. Homogeneity: Investors agree on distribution of returns and plan over a single common period.
3. No Market Frictions (eg. taxes, short-selling restrictions, restricted information)
4. Investors can borrow or lend at the risk-free rate.
5. Market Portfolio consists of all publicly traded assets.

2.2.1 Capital Market Line

$$u_e = r_f + \frac{\sigma_e}{\sigma_M}(\mu_M - r_f)$$

2.2.2 Security Market Line

$$\begin{aligned} \mu_i &= r_f + \beta_i(\mu_M - r_f), \quad \text{where} \\ \beta_i &= \frac{\sigma_{i,M}}{\sigma_M^2} \end{aligned}$$

2.2.3 Risk Decomposition

$$\begin{aligned} \sigma_i^2 &= \beta_i^2 \sigma_M^2 + \sigma_{\xi_i}^2 \\ &= \text{Systematic Risk} + \text{Non-systematic Risk} \end{aligned}$$

2.3 Factor Models

2.3.1 Single Factor Model (SFM)

$$r_i = \alpha_i + \beta_i f + \epsilon_i,$$

where α_i, β_i are stock-specific constants, f is the factor capturing market-wide price movement and ϵ_i is a noise term reflecting firm-specific risk.

2.3.2 SFM Assumptions

1. (r_i, f) are jointly normal
2. $\epsilon_i \sim \mathcal{N}(0, \sigma_{\epsilon_i}^2)$
3. $\text{Cov}(\epsilon_i, f) = 0$
4. $\text{Cov}(\epsilon_i, \epsilon_j) = 0$ for $i \neq j$

2.3.3 Risk Decomposition and Covariance

$$\begin{aligned}\sigma_i^2 &= \beta_i^2\sigma_f^2 + \sigma_{\epsilon_i}^2 = \text{Systematic Risk} + \text{Non-systematic Risk}\\ \sigma_{i,j} &= \beta_i\beta_j\sigma_f^2\end{aligned}$$

2.3.4 Diversification

$$R_P^2 = \frac{\beta_P^2\sigma_f^2}{\sigma_P^2} = \frac{\text{Systematic Risk}}{\text{Total Risk}}$$

Full diversification when $R_P^2 = 1$.

2.3.5 Multi-Factor Models

$$r_i = \alpha_i + \beta_{i,1}f_1 + \beta_{i,2}f_2 + ... + \beta_{i,K}f_K + \epsilon_i$$

Factors f_i may include inflation, economic growth, interest rates etc.

2.4 APT

2.4.1 Assumptions

Replace rigid CAPM assumptions with more relaxed assumptions:

1. No arbitrage in the market
2. Large universe of securities
3. Returns follow a factor model

2.4.2 Single-Factor APT Assumptions

Assume returns follow the following factor model:

$$r_i = \alpha_i + \beta_i I + \epsilon_i$$

Define the standardised factor $f := (I - \mu_I)/\sigma_I$ ($\sim \mathcal{N}(0,1)$) and assume:

1. $\mathbb{E}[\epsilon_i] = 0$
2. $\text{Cov}(\epsilon_i, \epsilon_j) = 0$ if $i \neq j$
3. $\text{Cov}(\epsilon_i, I) = 0$
4. Unlimited short-selling

2.4.3 Single-Factor APT Returns

$$r_i = a_i + b_i f + \epsilon_i$$

$$\mathbb{E}[r_i] = a_i, \quad \sigma_i^2 = b_i^2 + \sigma_{\epsilon_i}^2, \quad \sigma_{i,j} = b_i b_j$$

Under no-arbitrage,

$$a_i = \lambda_0 + \lambda_1 b_i,$$

where $\lambda_0 = r_f$ if there is a risk-free asset.

2.4.4 Multi-Factor APT Returns

$$r_i = a_i + b_{i,1}f_1 + b_{i,2}f_2 + ... + b_{i,K}f_K + \epsilon_i$$

where

$$\mathbb{E}[r_i] = a_i = \lambda_0 + \lambda_1 b_{i,1} + \lambda_2 b_{i,2} + ... + \lambda_K b_{i,K}$$

and $\lambda_0 = r_f$ if there is a risk-free asset.

2.5 Market Efficiency

2.5.1 Investment approaches

1. Technical analysis: Analyse historical data (price and volume).
2. Fundamental analysis: Measure intrinsic value of stocks and compare with market price.
3. Efficient market selection: Define acceptable risk level and form broad portfolio.

2.5.2 Types of market efficiency

1. Strong form: Market prices reflect all information including insider information.
2. Semi-strong form: Market prices reflect all publicly available information
3. Weak form: Market prices reflect all historical data

3 Discrete Time Derivative Pricing

3.1 Options

Let: S_t denote the time t stock price, c_t, p_t denote the time t European call/put prices, T denote the option's maturity, K denote its strike price and r denote the risk-free rate.

3.1.1 Vanilla Options

A vanilla European call/put gives the holder the right **but not obligation** to buy/sell the asset at maturity T for strike price K. American options are defined the same way except they can be exercised at **any time up to maturity**.

Moneyness	European Call	European Put
In the money	$S_T > K$	$S_T < K$
At the money	$S_T = K$	$S_T = K$
Out of the money	$S_T < K$	$S_T > K$

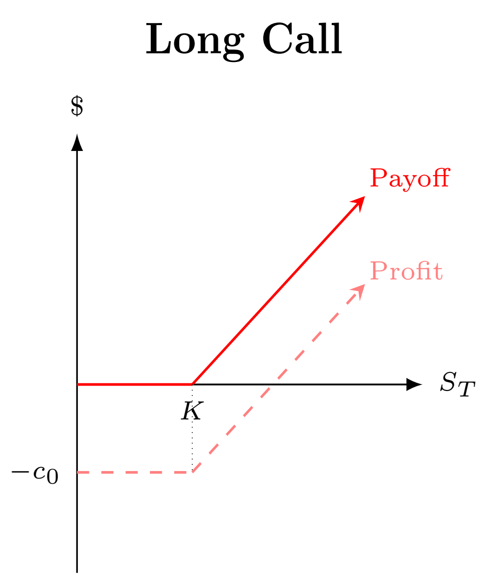
3.1.2 Payoff Functions and Price Bounds

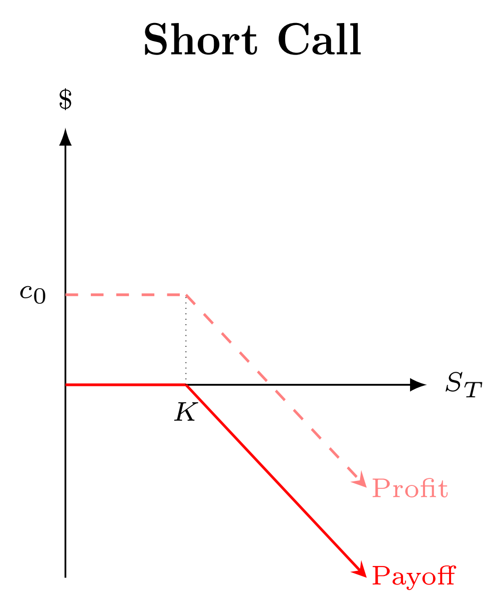
Define $0 \le U \le T$ as the exercise time for an American option. $U := \infty$ if option not exercised.
Note: $(X)_+ := \max\{0, X\}$. Option bounds are for option prices at **time t**.

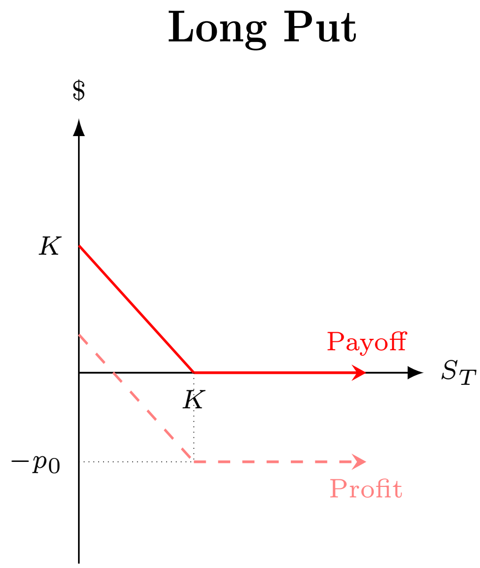
Option	Payoff	Lower Bound	Upper Bound
European Call	$(S_T - K)_+$	$S_t - Ke^{-r(T-t)}$	S_t
European Put	$(K - S_T)_+$	$Ke^{-r(T-t)} - S_t$	$Ke^{-r(T-t)}$
American Call	$(S_U - K)_+$	$S_t - Ke^{-r(T-t)}$	S_t
American Put	$(K - S_U)_+$	$K - S_t$	K

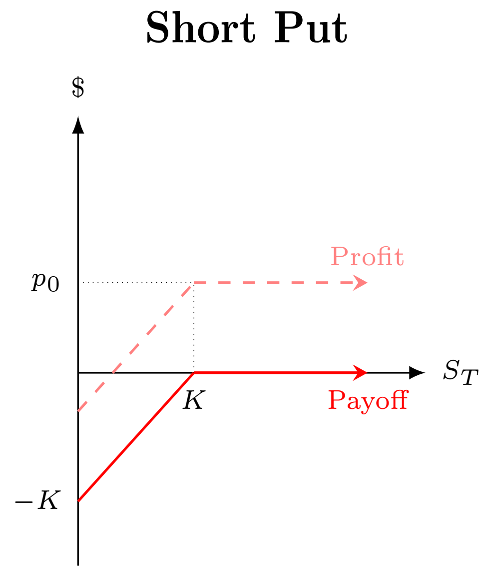
3.1.3 Position Diagrams

Long position = buy, short position = sell. Position diagrams graph the option payoff vs terminal asset price for a given position. Profit diagrams shift the position curves up/down as they account for the price of the option. Below are some position diagrams for European options.









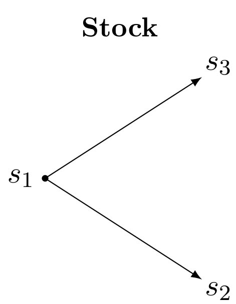
3.1.4 Put-Call Parity

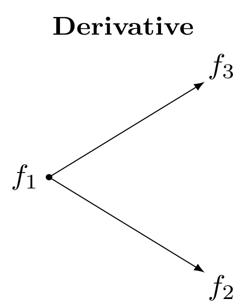
Under no arbitrage, for $0 \leq t \leq T$:

$$c_t + Ke^{-r(T-t)} = p_t + S_t$$

3.2 Discrete Time Pricing

3.2.1 One Period Binomial Model





The replicating portfolio (stocks, bonds) is:

$$\phi = \frac{f_3 - f_2}{s_3 - s_2}, \quad \psi = \frac{1}{B(0)} e^{-r\delta t}(f_3 - \phi s_3)$$

The derivative price today is:

$$V_0 = \phi s_1 + \psi B(0)$$

Rewriting with risk-neutral probabilities:

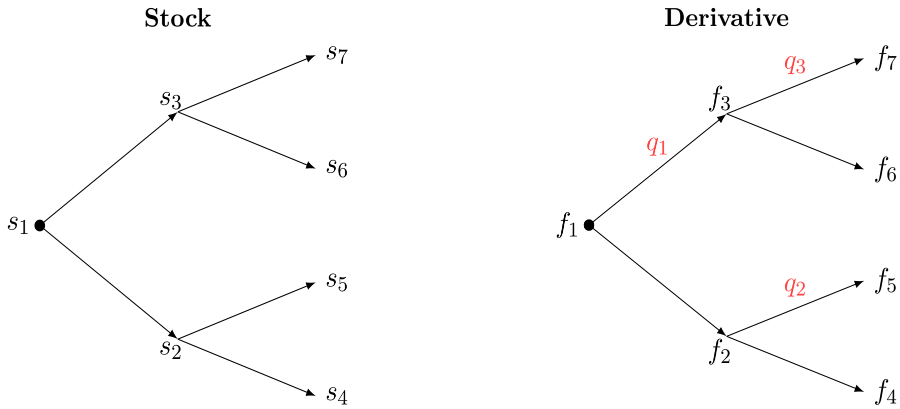
$$V_0 = e^{-r\delta t}(qf_3 + (1-q)f_2) = e^{-r\delta t}\mathbb{E}^{\mathcal{Q}}[X]$$

where

$$q = \frac{s_1 e^{r\delta t} - s_2}{s_3 - s_2}.$$

3.2.2 Multi-Period Binomial Model

Apply the one-period Binomial model to each internal node and recurse work backwards to find price. As an example, below is the two-period binomial model price:



For $j = 1, 2, 3$:

$$q_j = \frac{s_j e^{r\delta t} - s_{2j}}{s_{2j+1} - s_{2j}}, \quad f_j = e^{-r\delta t}\left[q_j f_{2j+1} + (1 - q_j) f_{2j}\right]$$

Substituting f_2, f_3 into f_1,

$$f_1 = e^{-2\delta t}\left[q_1 q_3 f_7 + q_1(1 - q_3) f_6 + (1 - q_1) q_2 f_5 + (1 - q_1)(1 - q_2) f_4\right]$$

3.2.3 Risk-Neutral Probabilities

- Under risk neutral world, price = discounted expected value of payoff.
- Risk preference of investor no longer needs to be considered.
- The real world probabilities of the stock going up or down are no longer needed for pricing.

4 Continuous Time Derivative Pricing

4.1 Measure Theory

4.1.1 Equivalent Probability Measures

Measures $\mathbb{P}$ and $\mathcal{Q}$ are equivalent iff for all events A, $\qquad \mathbb{P}(A) > 0 \quad \text{iff} \quad \mathcal{Q}(A) > 0.$

4.1.2 Radon-Nikodym Derivative

The Radon-Nikodym Derivative (denoted by ζ or $\frac{d\mathcal{Q}}{d\mathbb{P}}$) is a function that allows the interchange between equivalent probability measures. If $\mathbb{P}$ and $\mathcal{Q}$ are equivalent, then for all events A:

$$\mathcal{Q}(A) = \mathbb{E}^{\mathcal{Q}}[1_A] = \mathbb{E}^{\mathbb{P}}[\zeta \cdot 1_A]$$

4.2 Stochastic Processes

4.2.1 Definition

A stochastic process is a set of random variables $\{X(t) : t \in \mathcal{I}\}$ where $\mathcal{I}$ is a usually a time-interval (eg. $[0, T]$). Usually, we refer to the process as $X(t)$ for short.

4.2.2 Filtration

A filtration $\mathcal{F}_t$ contains the history of the stochastic process $X(t)$ up to time t.

4.2.3 Martingale

A stochastic process $M(t)$ is a martingale iff for $s \leq t$,

$$\mathbb{E}[M(t)|\mathcal{F}_s] = M(s)$$

4.2.4 Brownian Motion

A stochastic process $W(t)$ is a $\mathbb{P}$-Wiener Process (Brownian Motion) iff under measure $\mathbb{P}$:

1. $W_0 = 0$ almost surely
2. $W(t)$ is continuous almost surely
3. $W(t) - W(s) \sim \mathcal{N}(0, t - s)$
4. $W(t) - W(s)$ is independent of $\mathcal{F}_s$

4.3 Stochastic Calculus

4.3.1 Stochastic Differential Equation

Suppose the stochastic process $X(t)$ can be written as

$$X(t) = x_0 + \int_0^t a(s)ds + \int_0^t b(s)dW(s)$$

where $a(\cdot), b(\cdot)$ are appropriate functions (possibly stochastic processes) and x_0 is a constant. This is abbreviated as

$$dX(t) = a(t)dt + b(t)dW(t), \qquad X(0) = x_0.$$

4.3.2 Itô's Lemma

Itô's Lemma tells us how to differentiate functions of stochastic processes.

Let $f(x), F(t,x)$ be deterministic functions with continuous (partial) derivatives up to the second order. Then,

$$df(X(t)) = f'(X_t)dt + \frac{1}{2}f''(X(t))dX(t)^2$$
$$dF(t,X(t)) = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial x}dX(t) + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}dX(t)^2$$

In the second equation, all partial derivatives of F are evaluated at $(t, X(t))$.
These equations can be simplified using the following multiplication table:

$\times$	dt	$dW(t)$
dt	0	0
$dW(t)$	0	dt

4.3.3 Radon-Nikodym Derivative

For stochastic processes, the Radon-Nikodym Derivative becomes a stochastic process ζ_t (specifically, a martingale). For all events $A \in \mathcal{F}_t$,

$$\mathcal{Q}(A) = \mathbb{E}^{\mathcal{Q}}[1_A] = \mathbb{E}^{\mathbb{P}}[\zeta_t \cdot 1_A]$$

4.3.4 Girsanov-Theorem

Girsanov's Theorem will help us obtain the stochastic differential equation for a stock under a different measure.

Suppose $W(t)$ is a $\mathbb{P}$-Brownian Motion and $\gamma(\cdot)$ is a pre-visible process. Then there exists an equivalent measure $\mathcal{Q}$ with Radon-Nikodym derivative

$$\zeta_t = \exp\left[-\int_0^t \gamma(s)dW(s) - \int_0^t \gamma(s)^2 ds\right]$$

such that $W^{\mathcal{Q}}$, where

$$W^{\mathcal{Q}}(t) = W(t) - \int_0^t \gamma(s)ds,$$

is a $\mathcal{Q}$-Brownian Motion.

4.4 Pricing Framework

4.4.1 Fundamental Theorem of Asset Pricing

The market has **no arbitrage** iff there exists an **equivalent martingale measure** $\mathcal{Q}$ under which **discounted stock processes are martingales**. $\mathcal{Q}$ is also called the risk-neutral measure.

4.4.2 Pricing Framework Summary

The following steps give a rigorous justification for derivative pricing under $\mathcal{Q}$.

1. Identify a measure $\mathcal{Q}$ where the discounted stock process $Z(t) = S(t)/B(t)$ is a $\mathcal{Q}$-martingale.
2. Let $Y(t) = \mathbb{E}^{\mathcal{Q}}[X/B(T)|\mathcal{F}_t]$. Under $\mathcal{Q}$, $Y(t)$ and $Z(t)$ are martingales so by the martingale representation theorem, there exists a pre-visible process ϕ such that
$$Y(t) = Y(0) + \int_0^t \phi(s)dZ(s)$$
3. Construct a portfolio $\phi(t)$ units of Stock and $\psi(t) := Y(t) - \phi(t)Z(t)$ dollars of Bond.
4. The portfolio $(\phi(t), \psi(t))$ is self-financing and has the same payoff as the derivative at maturity.
5. By the Law of One Price, the prices must match at $t = 0$ so
$$\text{Price} = \mathbb{E}^{\mathcal{Q}}[X/B(T)]$$

4.5 Black-Scholes-Merton Model

4.5.1 Assumptions

1. No arbitrage
2. Unlimited borrowing at the risk-free rate r
3. No Market frictions (eg. short selling restrictions, transaction fees)
4. The underlying stock does not pay dividends.
5. The stock process follows a Geometric Brownian Motion
$$\begin{aligned} S(t) &= S_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + W(t)} \\ \iff dS(t) &= \mu S(t)dt + \sigma S(t)dW(t) \end{aligned}$$
6. The option is European

4.5.2 Put and Call Formulas

Define:

$$d_1 = \frac{\ln\left(S(t)/K\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}, \qquad d_2 = d_1 - \sigma\sqrt{T-t}$$

Then,

$$\begin{aligned} c_t &= S(t)N(d_1) - Ke^{-r(T-t)}N(d_2) \\ p_t &= Ke^{-r(T-t)}N(-d_2) - S_tN(-d_1), \end{aligned}$$

where $N(\cdot)$ is the cdf of the standard normal distribution.

5 Term Structure Modelling and Asset-Liability Management

5.1 Short-Rate Models

5.1.1 Merton Model:

$$dr(t) = \alpha dt + \sigma dW(t)$$

- Tractable
- $r(t)$ and $\int_0^t r(t)dt$ are normally distributed.
- Unrealistic (no mean-reversion and allows negative values)

5.1.2 Hull White Model:

$$dr(t) = \alpha(t)(\mu(t) - r(t))dt + \sigma(t)dW(t)$$

- Generalised version of Merton-Model
- α, σ no longer constant

5.1.3 Vasicek Model:

$$dr(t) = \alpha(\mu - r(t))dt + \sigma dW(t)$$

- Mean-Reverting
- $r(t)$ and $\int_0^t r(t)dt$ are normally distributed.
- Still allows negative values.

5.1.4 CIR Model:

$$dr(t) = \alpha(\mu - r(t))dt + \sigma\sqrt{r(t)}dW(t)$$

- Mean-Reverting
- Square root term keeps $r(t)$ positive

5.2 Ho-Lee Model

5.2.1 Definition

The Ho-Lee model models the short rate using a binomial tree.

$$r(k, s) = a(k) + b(k)s$$

where k =time and s = number of jumps.

$$\begin{array}{ccc} & & r(2,2) \\ & r(1,1) & r(2,1) \\ r(0,0) & r(1,0) & r(2,0) \end{array} \qquad \begin{array}{ccc} & & a(2)+2b(2) \\ & a(1)+b(1) & a(2)+b(2) \\ a(0) & a(1) & a(2) \end{array}$$

5.2.2 Calibration

In general,

$$b(k) = 2 \cdot \text{st.dev}(r(k))$$

The parameters $a(k)$ are found by equating with price of Zero-Coupon Bonds. Take $q = 1/2$ in most situations.

$$B(0,1) = \frac{1}{1+a(0)}$$

$$B(0,2) = \frac{1}{1+a(0)} \left((1-q)\frac{1}{1+a(1)} + q\frac{1}{1+a(1)+b(1)} \right)$$