

ACTL3182 Module 4 Extra Questions: Radon-Nikodym Theorem, Martingale and Itô's Lemma

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1. Consider two independent coin flips; that is let the sample space be $\Omega := \{HH, HT, TH, TT\}$. Under the \mathbb{P} measure, the coin is fair. Under the \mathcal{Q} measure, the coin has a probability of $\frac{1}{3}$ of landing on heads on each flip.

- (a) Find the probability of each scenario under \mathbb{P} and \mathcal{Q} .
- (b) Find the Radon-Nikodym Derivative $\frac{d\mathcal{Q}}{d\mathbb{P}}$.
- (c) Show that for the event A , where A represents getting a head on the first flip, that

$$\mathcal{Q}(A) = \mathbb{E}_P \left[\frac{d\mathcal{Q}}{d\mathbb{P}} 1_A \right]$$

2. Consider the stochastic process

$$X(t) = \sum_{k=1}^t Z_k, \quad X(0) = 0$$

where Z_k , $k = 1, 2, \dots, t$ are iid with distribution

$$Z_k = \begin{cases} +1 & \text{w.p. } \frac{1}{2} \\ -1 & \text{w.p. } \frac{1}{2}. \end{cases}$$

- (a) What is $\mathbb{E}[X(t+1)|\mathcal{F}_t]$?
- (b) Briefly explain why $X(t)$ is a martingale.
- (c) Now consider a time interval $[0, t]$ and divide the interval into n equal subintervals of length t/n . Let $X_n(t)$ be the process

$$X_n(t) = \sum_{k=1}^n Z_{k,n}$$

where $Z_{k,n}$ are now jumps of size $\sqrt{t/n}$:

$$Z_{k,n} = \begin{cases} +\sqrt{\frac{t}{n}} & \text{w.p. } \frac{1}{2} \\ -\sqrt{\frac{t}{n}} & \text{w.p. } \frac{1}{2}. \end{cases}$$

Draw a possible graph of $X_n(t)$ for $n = 1, 2$. What happens as $n \rightarrow \infty$?

- (d) Calculate $\mathbb{E}[X_n(t)]$ and $\text{Var}[X_n(t)]$.
- (e) Using the Central Limit Theorem, determine the limiting distribution of $X_n(t)$.
- (f) What continuous-time process does $X_n(t)$ converge to?

3. Consider the continuous-time Radon-Nikodym derivative $\zeta(t)$:

$$\zeta(t) = \exp \left(- \int_0^t \gamma_s dW(s) - \frac{1}{2} \int_0^t \gamma_s^2 ds \right)$$

Prove that $\zeta(t)$ is a martingale.

4. (Challenge) Consider the stochastic differential equation (SDE)

$$dX(t) = \alpha(\mu - X(t)) + \sigma dW(t), \quad X(0) = x_0.$$

The solution to this equation is called the Ornstein-Uhlenbeck process and can be used to model a massive particle moving under Brownian Motion when friction is accounted for.

- (a) Using Itô's Lemma, show that the solution to the SDE above is

$$X(t) = e^{-\alpha t} x_0 + \mu(1 - e^{-\alpha t}) + \sigma \int_0^t e^{-\alpha(t-s)} dW(s).$$

- (b) Explain why

$$\mathbb{E}[X(t)] = e^{-\alpha t} x_0 + \mu(1 - e^{-\alpha t}).$$

Hint: What type of stochastic process is $\int_0^t e^{-\alpha(t-s)} dW(s)$?

- (c) The Itô isometry is a special property that allows the expected value of the product of Itô integrals to be calculated using Riemann integrals instead. That is, for two suitable stochastic processes $X(t)$, $Y(t)$, we have:

$$\mathbb{E} \left[\int_0^t X(s) dW(s) \int_0^t Y(s) dW(s) \right] = \mathbb{E} \left[\int_0^t X(s) Y(s) ds \right]$$

Assuming this property holds for $X(t)$, show that

$$\text{Var}[X(t)] = \frac{\sigma^2}{2\alpha} (1 - e^{-2\alpha t}).$$

- (d) What is the distribution of $X(t)$? Explain this intuitively.
 (e) Show that for any $s, t \geq 0$:

$$\text{Cov}(X(s), X(t)) = \frac{\sigma^2}{2\alpha} (e^{-\alpha|t-s|} - e^{-\alpha(s+t)})$$

Select Answers:

1. (b) The Radon-Nikodym derivative is the ratio of the two probabilities;

$$\frac{d\mathcal{Q}}{d\mathbb{P}} = \begin{cases} 4/9 & \text{if } HH \\ 8/9 & \text{if } HT \\ 8/9 & \text{if } TH \\ 16/9 & \text{if } TT \end{cases}$$

2. (a) $X(t)$ (d) $\mathbb{E}[X_n(t)] = 0$, $\text{Var}[X_n(t)] = t$ (e) $\mathcal{N}(0, t)$ (e) Wiener Process/Brownian Motion.

4. (d) $\mathcal{N} \left(e^{-\alpha t} x_0 + \mu(1 - e^{-\alpha t}), \frac{\sigma^2}{2\alpha} (1 - e^{-2\alpha t}) \right)$