ACTL3182 Module 4 Extra Questions: Radon-Nikodym Theorem, Martingale and Itô's Lemma

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1. Consider two independent coin flips; that is let the sample space be $\Omega := \{HH, HT, TH, TT\}$. Under the $\mathbb{P}$ measure, the coin is fair. Under the $\mathcal{Q}$ measure, the coin has a probability of $\frac{1}{3}$ of landing on heads on each flip.

   (a) Find the probability of each scenario under $\mathbb{P}$ and $\mathcal{Q}$.

   (b) Find the Radon-Nikodym Derivative $\frac{d\mathcal{Q}}{d\mathbb{P}}$.

   (c) Show that for the event A, where A represents getting a head on the first flip, that

$$\mathcal{Q}(A) = \mathbb{E}_P\left[\frac{d\mathcal{Q}}{d\mathbb{P}} 1_A\right]$$

2. Consider the stochastic process

$$X(t) = \sum_{k=1}^{t} Z_k, \quad X(0) = 0$$

   where Z_k, $k = 1, 2, ...t$ are iid with distribution

$$Z_k = \begin{cases} +1 & \text{w.p. } \frac{1}{2} \\ -1 & \text{w.p. } \frac{1}{2}. \end{cases}$$

   (a) What is $\mathbb{E}[X(t+1)|\mathcal{F}_t]$?

   (b) Briefly explain why $X(t)$ is a martingale.

   (c) Now consider a time interval $[0, t]$ and divide the interval into n equal subintervals of length t/n. Let $X_n(t)$ be the process

$$X_n(t) = \sum_{k=1}^{n} Z_{k,n}$$

   where $Z_{k,n}$ are now jumps of size $\sqrt{t/n}$:

$$Z_{k,n} = \begin{cases} +\sqrt{\frac{t}{n}} & \text{w.p. } \frac{1}{2} \\ -\sqrt{\frac{t}{n}} & \text{w.p. } \frac{1}{2}. \end{cases}$$

   Draw a possible graph of $X_n(t)$ for $n = 1, 2$. What happens as $n \to \infty$?

   (d) Calculate $\mathbb{E}[X_n(t)]$ and $\text{Var}[X_n(t)]$.

   (e) Using the Central Limit Theorem, determine the limiting distribution of $X_n(t)$.

   (f) What continuous-time process does $X_n(t)$ converge to?

3. Consider the continuous-time Radon-Nikodym derivative $\zeta(t)$:

$$\zeta(t) = \exp\left(-\int_0^t \gamma_s dW(s) - \frac{1}{2}\int_0^t \gamma_s^2 ds\right)$$

Prove that $\zeta(t)$ is a martingale.

4. (Challenge) Consider the stochastic differential equation (SDE)

$$dX(t) = \alpha(\mu - X(t)) + \sigma dW(t), \quad X(0) = x_0.$$

The solution to this equation is called the Ornstein-Uhlenbeck process and can be used to model a massive particle moving under Brownian Motion when friction is accounted for.

(a) Using Itô's Lemma, show that the solution to the SDE above is

$$X(t) = e^{-\alpha t}x_0 + \mu(1 - e^{-\alpha t}) + \sigma\int_0^t e^{-\alpha(t-s)}dW(s).$$

(b) Explain why

$$\mathbb{E}[X(t)] = e^{-\alpha t}x_0 + \mu(1 - e^{-\alpha t}).$$

Hint: What type of stochastic process is $\int_0^t e^{-\alpha(t-s)}dW(s)$?

(c) The Itô isometry is a special property that allows the expected value of the product of Itô integrals to be calculated using Riemann integrals instead. That is, for two suitable stochastic processes $X(t)$, $Y(t)$, we have:

$$\mathbb{E}\left[\int_0^t X(s)dW(s)\int_0^t Y(s)dW(s)\right] = \mathbb{E}\left[\int_0^t X(s)Y(s)ds\right]$$

Assuming this property holds for $X(t)$, show that

$$\text{Var}[X(t)] = \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t}).$$

(d) What is it the distribution of $X(t)$? Explain this intuitively.

(e) Show that for any $s, t \geq 0$:

$$\text{Cov}(X(s), X(t)) = \frac{\sigma^2}{2\alpha}(e^{-\alpha|t-s|} - e^{-\alpha(s+t)})$$

5. The price at time t of a put option is given by

$$p(t, S_t) = Ke^{-r(T-t)}N(-d_2) - S_tN(d_1),$$

where

$$d_1 = \frac{\ln(S_t/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \qquad d_2 = d_1 - \sigma\sqrt{T-t}.$$

Derive the expressions for the replicating portfolio $(\phi(t), \psi(t))$ given that

$$\phi(t) = \frac{\partial p}{\partial S}(t, S_t), \qquad \psi(t) = \frac{p(t, S_t) - S_t\phi(t)}{B(t)}.$$

Select Answers:

1. (b) The Radon-Nikodym derivative is the ratio of the two probabilities;

$$\frac{d\mathcal{Q}}{d\mathbb{P}} = \begin{cases} 4/9 & \text{if } HH \\ 8/9 & \text{if } HT \\ 8/9 & \text{if } TH \\ 16/9 & \text{if } TT \end{cases}$$

2. (a) $X(t)$ (d) $\mathbb{E}[X_n(t)] = 0$, $\text{Var}[X_n(t)] = t$ (e) $\mathcal{N}(0, t)$ (e) Wiener Process/Brownian Motion.

4. (d) $\mathcal{N}\left(e^{-\alpha t}x_0 + \mu(1 - e^{-\alpha t}), \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t})\right)$