ACTL3182 Formula Sheet

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NOTE: This is a condensed version of the cheatsheet with emphasis on formulas and little to no theory. This cheatsheet may be revised throughout the term, please check https://github.com/BrownianNotion/ACTL3182_20T3.git for the latest version.

1 Modern Portfolio Theory

1.1 Utility Theory

1.1.1 Expected Utility Theorem

Individual prefers W_1 over W_2 iff $\mathbb{E}[U(W_1)] \geq \mathbb{E}[U(W_2)]$

1.1.2 Investor types

Type	Wealth Preference	Utility Preference	U''	Concavity
Risk-Averse	$\mathbb{E}[W] \succ W$	$U(\mathbb{E}[W]) > E[U(W)]$	$U'' < 0$	concave
Risk-Neutral	$\mathbb{E}[W] \sim W$	$U(\mathbb{E}[W]) = E[U(W)]$	$U'' = 0$	linear
Risk-Lover	$\mathbb{E}[W] \prec W$	$U(\mathbb{E}[W]) < E[U(W)]$	$U'' > 0$	convex

1.1.3 Risk Premium

The amount $\pi(W)$ that an individual will pay to give up risk:

$$\pi(W) = \mathbb{E}[W] - c(W)$$

where $c(W) := U^{-1}(\mathbb{E}[U(W)])$ is the **certainty wealth equivalent**.

1.1.4 Risk Aversion

Absolute risk aversion $A(w)$ and relative risk aversion $R(w)$

$$A(w) = -\frac{U''(w)}{U'(w)}, \quad R(w) = -w\frac{U''(w)}{U'(w)}.$$

1.2 Investment Risk Measures

1.2.1 Common Examples

1. Variance: $\int_{-\infty}^{\infty}(x-\mu)^2 f_X(x)dx$

2. Downside-Variance: $\int_{-\infty}^{\mu}(x-\mu)^2 f_X(x)dx$

3. Shortfall Probability: $\mathbb{P}(X \leq L)$
4. Expected Shortfall: $\int_{-\infty}^{L}(L-x)f_X(x)dx$
5. Shortfall Variance: $\int_{-\infty}^{L}(L-x)^2 f_X(x)dx$

1.2.2 Value-at-Risk:

The Value-at-risk **at level $\boldsymbol{\alpha}$** is the maximum possible loss from holding a portfolio over a **given time period** so that the **probability of a larger loss is $\mathbf{1-\alpha}$**.
In general,

$$\text{VaR}(\alpha) = \mu - X_{1-\alpha},$$

where $X_{1-\alpha}$ is the $1-\alpha$-quantile of X. If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $\text{VaR}(\alpha) = \sigma Z_\alpha$.

1.3 Portfolio Optimisation

1.3.1 Two assets

Global Minimum Variance Portfolio:

$$w_A = \frac{\sigma_B^2 - \sigma_{AB}}{\sigma_A^2 + \sigma_B^2 - 2\sigma_{AB}}, \quad w_B = 1 - w_A$$

Portfolio Variance:

$$\sigma_P^2 = w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2w_A w_B \rho_{AB}\sigma_A\sigma_B$$

1.3.2 N-risky assets

Minimise $\sigma_P^2 = \boldsymbol{w}^\top \Sigma \boldsymbol{w}$, subject to $\mathbf{1}^\top \boldsymbol{w} = 1$ and $\boldsymbol{z}^\top \boldsymbol{w} = \mu$.
Lagrangian: $\mathcal{L}(\boldsymbol{w}, \lambda, \gamma) = \frac{1}{2}\boldsymbol{w}^\top \Sigma \boldsymbol{w} + \lambda(1 - \mathbf{1}^\top \boldsymbol{w}) + \gamma(\mu - \boldsymbol{z}^\top \boldsymbol{w})$

Constants:

$$A = \mathbf{1}^\top \Sigma^{-1}\mathbf{1}, \quad B = \mathbf{1}^\top \Sigma^{-1}\boldsymbol{z} = \boldsymbol{z}^\top \Sigma^{-1}\mathbf{1}$$
$$C = \boldsymbol{z}^\top \Sigma^{-1}\boldsymbol{z}, \quad \Delta = AC - B^2$$

Minimum Variance Portfolio:

$$\boldsymbol{w} = \lambda \Sigma^{-1}\mathbf{1} + \gamma \Sigma^{-1}\boldsymbol{z}$$

where

$$\lambda = \frac{C - \mu B}{\Delta}, \quad \gamma = \frac{\mu A - B}{\Delta}$$

Portfolio Variance:

$$\sigma_P^2 = \frac{A\mu^2 - 2B\mu + C}{\Delta}$$

Global Minimum Variance Portfolio:

$$\boldsymbol{w}_g = \frac{1}{A}\Sigma^{-1}\mathbf{1}$$

1.3.3 N-risky assets + risk-free

Minimise $\sigma_P^2 = \boldsymbol{w}^\top \Sigma \boldsymbol{w}$, subject to $(\boldsymbol{z} - r_f \mathbf{1})^\top \boldsymbol{w} = u - r_f$.
Lagrangian: $\mathcal{L}(\boldsymbol{w}, \lambda, \gamma) = \frac{1}{2}\boldsymbol{w}^\top \Sigma \boldsymbol{w} + \gamma(\mu - r_f - (\boldsymbol{z} - r_f \mathbf{1})^\top \boldsymbol{w})$

Minimum Variance Portfolio:

$$\boldsymbol{w} = \gamma \Sigma^{-1}(\boldsymbol{z} - r_f \mathbf{1})$$

where

$$\gamma = \frac{\mu - r_f}{A r_f^2 - 2B r_f + C}$$

Portfolio Variance:

$$\sigma_P^2 = \frac{(\mu - r_f)^2}{A r_f^2 - 2B r_f + C}$$

Tangency Portfolio:

$$\boldsymbol{w}_t = \gamma_t \Sigma^{-1}(\boldsymbol{z} - r_f \mathbf{1})$$

where

$$\gamma_t = \frac{1}{B - A r_f}$$

2 Asset Pricing Models

For all parts in section 2, i takes values $1, 2, \ldots N$ and indexes the assets in the market.

2.1 CAPM

2.1.1 Capital Market Line

$$\mu_e = r_f + \frac{\sigma_e}{\sigma_M}(\mu_M - r_f)$$

2.1.2 Security Market Line

$$\begin{aligned} \mu_i &= r_f + \beta_i(\mu_M - r_f), \quad \text{where} \\ \beta_i &= \frac{\sigma_{i,M}}{\sigma_M^2} \end{aligned}$$

2.1.3 Risk Decomposition

$$\begin{aligned} \sigma_i^2 &= \beta_i^2 \sigma_M^2 + \sigma_{\xi_i}^2 \\ &= \text{Systematic Risk} + \text{Non-systematic Risk} \end{aligned}$$

2.2 Factor Models

2.2.1 Single Factor Model (SFM)

$$r_i = \alpha_i + \beta_i f + \epsilon_i,$$

where α_i, β_i are stock-specific constants, f is the factor capturing market-wide price movement and ϵ_i is a noise term reflecting firm-specific risk.

2.2.2 SFM Assumptions

2.2.3 Risk Decomposition and Covariance

$$\begin{aligned}
\sigma_i^2 &= \beta_i^2 \sigma_f^2 + \sigma_{\epsilon_i}^2 = \text{Systematic Risk} + \text{Non-systematic Risk} \\
\sigma_{i,j} &= \beta_i \beta_j \sigma_f^2
\end{aligned}$$

2.2.4 Diversification

$$R_P^2 = \frac{\beta_P^2 \sigma_f^2}{\sigma_P^2} = \frac{\text{Systematic Risk}}{\text{Total Risk}}$$

Full diversification when $R_P^2 = 1$.

2.2.5 Multi-Factor Models

$$r_i = \alpha_i + \beta_{i,1} f_1 + \beta_{i,2} f_2 + \ldots + \beta_{i,K} f_K + \epsilon_i$$

Factors f_i may include inflation, economic growth, interest rates etc.

2.3 APT

2.3.1 Single-Factor APT Returns

$$\begin{aligned}
& r_i = a_i + b_i f + \epsilon_i \\
& \mathbb{E}[r_i] = a_i, \quad \sigma_i^2 = b_i^2 + \sigma_{\epsilon_i}^2, \quad \sigma_{i,j} = b_i b_j
\end{aligned}$$

Under no-arbitrage,

$$a_i = \lambda_0 + \lambda_1 b_i,$$

where $\lambda_0 = r_f$ if there is a risk-free asset.

2.3.2 Multi-Factor APT Returns

$$r_i = a_i + b_{i,1} f_1 + b_{i,2} f_2 + \ldots + b_{i,K} f_K + \epsilon_i$$

where

$$\mathbb{E}[r_i] = a_i = \lambda_0 + \lambda_1 b_{i,1} + \lambda_2 b_{i,2} + \ldots + \lambda_K b_{i,K}$$

and $\lambda_0 = r_f$ if there is a risk-free asset.

3 Discrete Time Derivative Pricing

3.1 Options

Let: S_t denote the time t stock price, c_t, p_t denote the time t European call/put prices, T denote the option's maturity, K denote its strike price and r denote the risk-free rate.

3.1.1 Vanilla Options

Moneyness	**European Call**	**European Put**
In the money	$S_T > K$	$S_T < K$
At the money	$S_T = K$	$S_T = K$
Out of the money	$S_T < K$	$S_T > K$

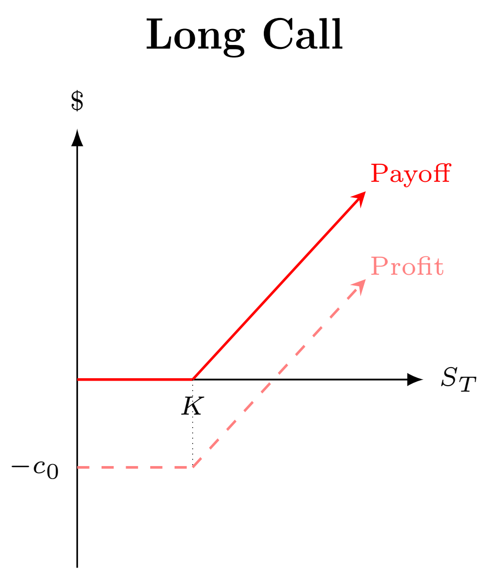
3.1.2 Payoff Functions and Price Bounds

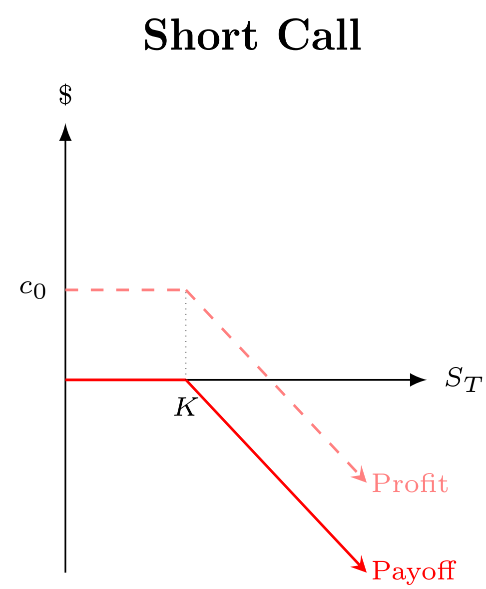
Define $0 \leq U \leq T$ as the exercise time for an American option. $U := \infty$ if option not exercised.
Note: $(X)_+ := \max\{0, X\}$. Option bounds are for option prices at **time $\boldsymbol{t}$**.

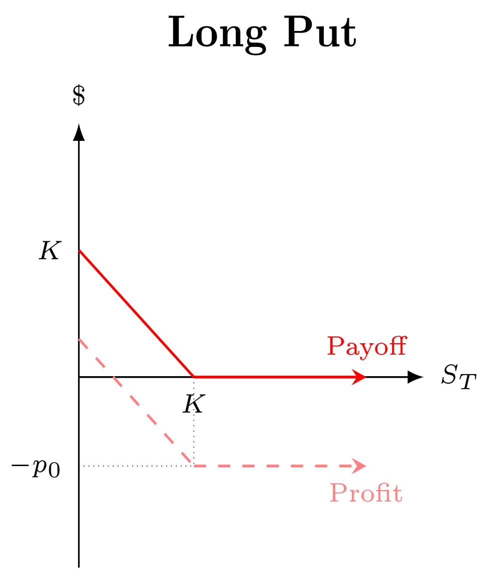
Option	**Payoff**	**Lower Bound**	**Upper Bound**
European Call	$(S_T - K)_+$	$S_t - Ke^{-r(T-t)}$	S_t
European Put	$(K - S_T)_+$	$Ke^{-r(T-t)} - S_t$	$Ke^{-r(T-t)}$
American Call	$(S_U - K)_+$	$S_t - Ke^{-r(T-t)}$	S_t
American Put	$(K - S_U)_+$	$K - S_t$	K

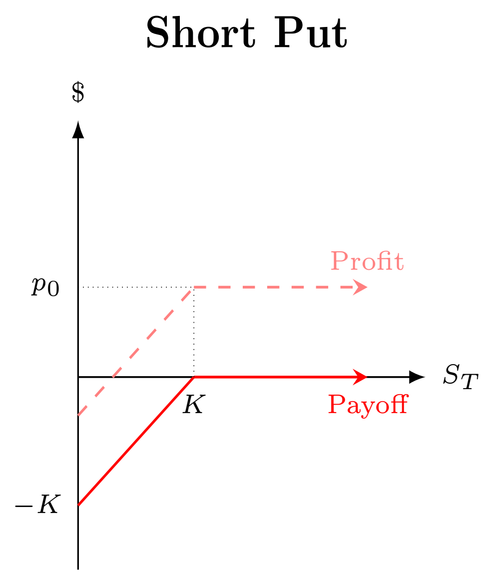
3.1.3 Position Diagrams

Long position = buy, short position = sell.









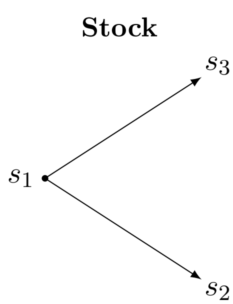
3.1.4 Put-Call Parity

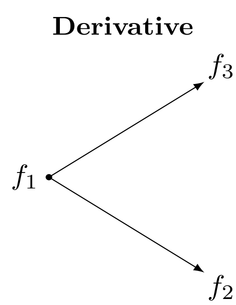
Under no arbitrage, for $0 \le t \le T$:

$$c_t + Ke^{-r(T-t)} = p_t + S_t$$

3.2 Discrete Time Pricing

3.2.1 One Period Binomial Model





The replicating portfolio (stocks, bonds) is:

$$\phi = \frac{f_3 - f_2}{s_3 - s_2}, \quad \psi = \frac{1}{B(0)} e^{-r\delta t}(f_3 - \phi s_3)$$

The derivative price today is:

$$V_0 = \phi s_1 + \psi B(0)$$

Rewriting with risk-neutral probabilities:

$$V_0 = e^{-r\delta t}(q f_3 + (1-q) f_2) = e^{-r\delta t}\mathbb{E}^{\mathcal{Q}}[X]$$

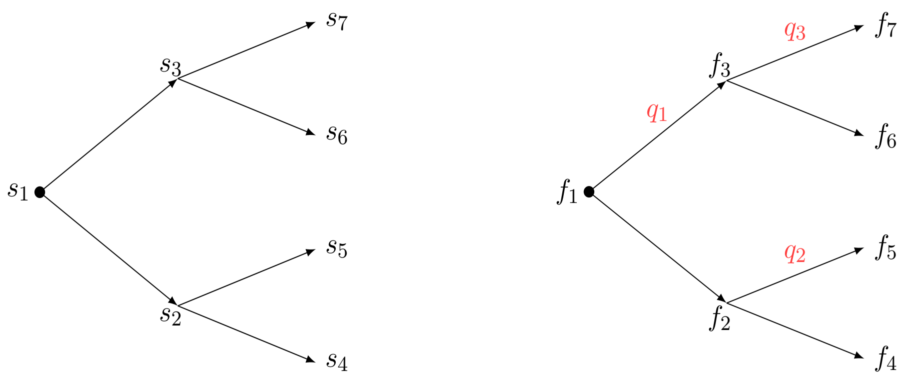
where

$$q = \frac{s_1 e^{r\delta t} - s_2}{s_3 - s_2}.$$

3.2.2 Multi-Period Binomial Model

Apply the one-period Binomial model to each internal node and recurse work backwards to find price. As an example, below is the two-period binomial model price:

Stock **Derivative**



For $j = 1, 2, 3$:

$$q_j = \frac{s_j e^{r\delta t} - s_{2j}}{s_{2j+1} - s_{2j}}, \quad f_j = e^{-r\delta t} \left[q_j f_{2j+1} + (1 - q_j) f_{2j} \right]$$

Substituting f_2, f_3 into f_1,

$$f_1 = e^{-2\delta t} \left[q_1 q_3 f_7 + q_1(1 - q_3) f_6 + (1 - q_1) q_2 f_5 + (1 - q_1)(1 - q_2) f_4 \right]$$

4 Continuous Time Derivative Pricing

4.1 Stochastic Calculus

4.1.1 Stochastic Differential Equation

Suppose the stochastic process $X(t)$ can be written as

$$X(t) = x_0 + \int_0^t a(s)ds + \int_0^t b(s)dW(s)$$

where $a(\cdot), b(\cdot)$ are appropriate functions (possibly stochastic processes) and x_0 is a constant.
This is abbreviated as

$$dX(t) = a(t)dt + b(t)dW(t), \qquad X(0) = x_0.$$

4.1.2 Itô's Lemma

Let $f(x), F(t, x)$ be deterministic functions with continuous (partial) derivatives up to the second order. Then,

$$df(X(t)) = f'(X(t))dt + \frac{1}{2} f''(X(t))dX(t)^2$$
$$dF(t, X(t)) = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial x}dX(t) + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}dX(t)^2$$

Itô multiplication table:

$\times$	dt	$dW(t)$
dt	0	0
$dW(t)$	0	dt

4.1.3 Girsanov-Theorem

Suppose $W(t)$ is a $\mathbb{P}$-Brownian Motion and $\gamma(\cdot)$ is a pre-visible process. Then there exists an equivalent measure $\mathcal{Q}$ with Radon-Nikodym derivative

$$\zeta_t = \exp\left[-\int_0^t \gamma(s)dW(s) - \int_0^t \gamma(s)^2 ds\right]$$

such that W^Q, where

$$W^{\mathcal{Q}}(t) = W(t) - \int_0^t \gamma(s)ds,$$

is a $\mathcal{Q}$-Brownian Motion.

4.2 Pricing Framework

4.3 Black-Scholes-Merton Model

4.3.1 Stock Process

The stock process follows a Geometric Brownian Motion

$$S(t) = S_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + W(t)}$$
$$\iff dS(t) = \mu S(t)dt + \sigma S(t)dW(t)$$

4.3.2 Put and Call Formulas

Define:

$$d_1 = \frac{\ln\left(S(t)/K\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}, \qquad d_2 = d_1 - \sigma\sqrt{T-t}$$

Then,

$$c_t = S(t)N(d_1) - Ke^{-r(T-t)}N(d_2)$$
$$p_t = Ke^{-r(T-t)}N(-d_2) - S_tN(-d_1),$$

where $N(\cdot)$ is the cdf of the standard normal distribution.

5 Term Structure Modelling and Asset-Liability Management

5.1 Short-Rate Models

5.1.1 Merton Model:

$$dr(t) = \alpha dt + \sigma dW(t)$$

5.1.2 Hull White Model:

$$dr(t) = \alpha(t)(\mu(t) - r(t))dt + \sigma(t)dW(t)$$

5.1.3 Vasicek Model:

$$dr(t) = \alpha(\mu - r(t))dt + \sigma dW(t)$$

5.1.4 CIR Model:

$$dr(t) = \alpha(\mu - r(t))dt + \sigma\sqrt{r(t)}dW(t)$$

5.2 Ho-Lee Model

5.2.1 Definition

$$r(k,s) = a(k) + b(k)s$$

where k =time and s = number of jumps.

$$\begin{array}{ccc} & & r(2,2) \\ & r(1,1) & r(2,1) \\ r(0,0) & r(1,0) & r(2,0) \end{array} \qquad \begin{array}{ccc} & & a(2)+2b(2) \\ & a(1)+b(1) & a(2)+b(2) \\ a(0) & a(1) & a(2) \end{array}$$

5.2.2 Calibration

In general,

$$b(k) = 2 \cdot \text{st.dev}(r(k))$$

The parameters $a(k)$ are found by equating with price of Zero-Coupon Bonds. Take $q = 1/2$ in most situations.

$$B(0,1) = \frac{1}{1+a(0)}$$

$$B(0,2) = \frac{1}{1+a(0)}\left((1-q)\frac{1}{1+a(1)} + q\frac{1}{1+a(1)+b(1)}\right)$$