ACTL3182 Module 4 Extra Questions: Radon-Nikodym Theorem, Martingale and Itô's Lemma

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1. Let $X := (X_n)_{n=1,2,...}$ be a (discrete-time) martingale with respect to filtration $(\mathcal{F}_n)_{n=1,2,...}$. Show that X is a martingale **if and only if**

$$\mathbb{E}[X_n|\mathcal{F}_n] = X_n, \quad \text{and} \quad \mathbb{E}[X_{n+1}|\mathcal{F}_n].$$

2. Consider the stochastic process

$$X_n = \sum_{k=1}^{n} Z_k, \quad X(0) = 0$$

where Z_k, $k = 1, 2, ...n$ are iid with distribution

$$Z_k = \begin{cases} +1 & \text{w.p. } \frac{1}{2} \\ -1 & \text{w.p. } \frac{1}{2}. \end{cases}$$

Show that X is a martingale.

3. Consider the stochastic process

$$X_n = \prod_{k=1}^{n} Z_k, \quad X(0) = 1$$

where Z_k, $k = 1, 2, ...n$ are iid with distribution

$$Z_k = \begin{cases} 2 & \text{w.p. } p \\ \frac{1}{2} & \text{w.p. } 1-p. \end{cases}$$

Find the probability p such that X is a martingale.

4. Consider n stocks with iid returns r_i, $i = 1, \ldots, n$ such that $\mathbb{E}[r_i] = \mu$ and $\mathrm{Var}(r_i) = \sigma^2$.

   (a) Let a_i, b_i, $i = 1, \ldots, n$ be real numbers. Prove the Cauchy-Schwarz inequality:

   $$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right),$$

   and that equality holds if and only if $b_i = \lambda a_i$ for some constant λ.
   Hint: Consider the quadratic $Q(\lambda) = \sum_{i=1}^{n} (\lambda a_i - b_i)^2$.

   (b) Deduce that if $\sum_{i=1}^{n} a_i = 1$, then

   $$\sum_{i=1}^{n} a_i^2 \geq \frac{1}{n}.$$

(c) Derive the weights of the global minimum variance portfolio in this market.

5. Consider two independent coin flips; that is let the sample space be $\Omega := \{HH, HT, TH, TT\}$. Under the $\mathbb{P}$ measure, the coin is fair. Under the $\mathcal{Q}$ measure, the coin has a probability of $\frac{1}{3}$ of landing on heads on each flip.

   (a) Find the probability of each scenario under $\mathbb{P}$ and $\mathcal{Q}$.

   (b) Find the Radon-Nikodym Derivative $\frac{d\mathcal{Q}}{d\mathbb{P}}$.

   (c) Show that for the event A, where A represents getting a head on the first flip, that

$$\mathcal{Q}(A) = \mathbb{E}_P\left[\frac{d\mathcal{Q}}{d\mathbb{P}}1_A\right]$$

6. Consider the stochastic process

$$X_n = \sum_{k=1}^{n} Z_k, \quad X(0) = 0$$

   where Z_k, $k = 1, 2, ...n$ are iid with distribution

$$Z_k = \begin{cases} +1 & \text{w.p. } \frac{1}{2} \\ -1 & \text{w.p. } \frac{1}{2}. \end{cases}$$

   (a) What is $\mathbb{E}[X_{n+1}|\mathcal{F}_n]$?

   (b) Briefly explain why X_n is a martingale.

   (c) Now consider a time interval $[0, T]$ and divide the interval into n equal subintervals of length $\Delta = T/n$. Let X_t^n be the process

$$X_t^n = \sqrt{\Delta} \sum_{k=1}^{\lfloor t/\Delta \rfloor} Z_k, \quad \text{for } 0 \leq t \leq T.$$

   Draw a possible graph of X_t^n for $n = 1, 2$. What happens as $n \to \infty$?

   (d) Calculate $\mathbb{E}[X_t^n]$ and $\text{Var}[X_t^n]$.

   (e) Using the Central Limit Theorem, determine the limiting distribution of X_t^n.

   (f) What continuous-time process does X_t^n converge to? (no formal justification required)

7. Consider the continuous-time Radon-Nikodym derivative ζ_t:

$$\zeta_t = \exp\left(-\int_0^t \gamma_s dW_s - \frac{1}{2}\int_0^t \gamma_s^2 ds\right)$$

   Prove that ζ_t is a martingale.

8. (Challenge) Consider the stochastic differential equation (SDE)

$$dX(t) = \alpha(\mu - X(t)) + \sigma dW(t), \quad X(0) = x_0.$$

   The solution to this equation is called the Ornstein-Uhlenbeck process and can be used to model a massive particle moving under Brownian Motion when friction is accounted for.

   (a) Using Itô's Lemma, show that the solution to the SDE above is

$$X(t) = e^{-\alpha t}x_0 + \mu(1 - e^{-\alpha t}) + \sigma \int_0^t e^{-\alpha(t-s)} dW(s).$$

(b) Explain why
$$\mathbb{E}[X(t)] = e^{-\alpha t}x_0 + \mu(1 - e^{-\alpha t}).$$
Hint: What type of stochastic process is $\int_0^t e^{-\alpha(t-s)}dW(s)$?

(c) The Itô isometry is a special property that allows the expected value of the product of Itô integrals to be calculated using Riemann integrals instead. That is, for two suitable stochastic processes $X(t)$, $Y(t)$, we have:
$$\mathbb{E}\left[\int_0^t X(s)dW(s)\int_0^t Y(s)dW(s)\right] = \mathbb{E}\left[\int_0^t X(s)Y(s)ds\right]$$
Assuming this property holds for $X(t)$, show that
$$\text{Var}[X(t)] = \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t}).$$

(d) What is it the distribution of $X(t)$? Explain this intuitively.

(e) Show that for any $s, t \geq 0$:
$$\text{Cov}(X(s), X(t)) = \frac{\sigma^2}{2\alpha}(e^{-\alpha|t-s|} - e^{-\alpha(s+t)})$$

9. The price at time t of a put option is given by
$$p(t, S_t) = Ke^{-r(T-t)}N(-d_2) - S_tN(-d_1),$$
where
$$d_1 = \frac{\ln(S_t/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \qquad d_2 = d_1 - \sigma\sqrt{T-t}.$$
Derive the expressions for the replicating portfolio $(\phi(t), \psi(t))$ given that
$$\phi(t) = \frac{\partial p}{\partial S}(t, S_t), \qquad \psi(t) = \frac{p(t, S_t) - S_t\phi(t)}{B(t)}.$$

Select Answers:

1. (b) The Radon-Nikodym derivative is the ratio of the two probabilities;
$$\frac{dQ}{d\mathbb{P}} = \begin{cases} 4/9 & \text{if } HH \\ 8/9 & \text{if } HT \\ 8/9 & \text{if } TH \\ 16/9 & \text{if } TT \end{cases}$$

2. (a) $X(t)$ (d) $\mathbb{E}[X_t^n] = 0$, $\text{Var}[X_t^n] = \Delta\lfloor t/\Delta\rfloor \approx t$ for n large (e) $\mathcal{N}(0, t)$ (e) Brownian Motion.

4. (d) $\mathcal{N}\left(e^{-\alpha t}x_0 + \mu(1 - e^{-\alpha t}), \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t})\right)$