

# ACTL3182 Formula Sheet

Andrew Wu

September 2020

**NOTE:** This is a condensed version of the cheatsheet with emphasis on formulas and little to no theory. This cheatsheet may be revised throughout the term, please check [https://github.com/BrownianNotion/ACTL3182\\_21T3.git](https://github.com/BrownianNotion/ACTL3182_21T3.git) for the latest version.

## 1 Modern Portfolio Theory

### 1.1 Utility Theory

#### 1.1.1 Expected Utility Theorem

Assuming the utility axioms,  $W_1$  is preferred over  $W_2$  iff  $\mathbb{E}[U(W_1)] \geq \mathbb{E}[U(W_2)]$ .

#### 1.1.2 Investor types

Type	Wealth Preference	Utility Preference	$U''$	Concavity
Risk-Averse	$\mathbb{E}[W] \succ W$	$U(\mathbb{E}[W]) > \mathbb{E}[U(W)]$	$U'' < 0$	concave
Risk-Neutral	$\mathbb{E}[W] \sim W$	$U(\mathbb{E}[W]) = \mathbb{E}[U(W)]$	$U'' = 0$	linear
Risk-Lover	$\mathbb{E}[W] \prec W$	$U(\mathbb{E}[W]) < \mathbb{E}[U(W)]$	$U'' > 0$	convex

#### 1.1.3 Risk Premium

The amount  $\pi(W)$  that an individual will pay to give up risk:

$$\pi(W) = \mathbb{E}[W] - c(W)$$

where  $c(W) := U^{-1}(\mathbb{E}[U(W)])$  is the **certainty wealth equivalent**.

#### 1.1.4 Risk Aversion

Absolute risk aversion  $A(w)$  and relative risk aversion  $R(w)$

$$A(w) = -\frac{U''(w)}{U'(w)}, \quad R(w) = -w \frac{U''(w)}{U'(w)}.$$

## 1.2 Investment Risk Measures

### 1.2.1 Common Examples

1. Variance:  $\int_{-\infty}^{\infty} (x - \mu)^2 f_X(x) dx$
2. Downside-Variance:  $\int_{-\infty}^{\mu} (x - \mu)^2 f_X(x) dx$

3. Shortfall Probability:  $\mathbb{P}(X \leq L)$
4. Expected Shortfall:  $\int_{-\infty}^L (L - x)f_X(x)dx$
5. Shortfall Variance:  $\int_{-\infty}^L (L - x)^2 f_X(x)dx$

### 1.2.2 Value-at-Risk:

The Value-at-risk at level  $\alpha$  is the maximum possible loss from holding a portfolio over a **given time period** so that the **probability of a larger loss is  $1 - \alpha$** .

In general,

$$\text{VaR}(\alpha) = \mu - X_{1-\alpha},$$

where  $X_{1-\alpha}$  is the  $1 - \alpha$ -quantile of  $X$ . If  $X \sim \mathcal{N}(\mu, \sigma^2)$ , then  $\text{VaR}(\alpha) = \sigma Z_\alpha$ .

## 1.3 Portfolio Optimisation

### 1.3.1 Two assets

Global Minimum Variance Portfolio:

$$w_A = \frac{\sigma_B^2 - \sigma_{AB}}{\sigma_A^2 + \sigma_B^2 - 2\sigma_{AB}}, \quad w_B = 1 - w_A$$

Portfolio Variance:

$$\sigma_P^2 = w_A^2 \sigma_A^2 + w_B^2 \sigma_B^2 + 2w_A w_B \rho_{AB} \sigma_A \sigma_B$$

### 1.3.2 N-risky assets

Minimise  $\sigma_P^2 = \mathbf{w}^\top \Sigma \mathbf{w}$ , subject to  $\mathbf{1}^\top \mathbf{w} = 1$  and  $\mathbf{z}^\top \mathbf{w} = \mu$ .

Lagrangian:  $\mathcal{L}(\mathbf{w}, \lambda, \gamma) = \frac{1}{2} \mathbf{w}^\top \Sigma \mathbf{w} + \lambda(1 - \mathbf{1}^\top \mathbf{w}) + \gamma(\mu - \mathbf{z}^\top \mathbf{w})$

Constants:

$$A = \mathbf{1}^\top \Sigma^{-1} \mathbf{1}, \quad B = \mathbf{1}^\top \Sigma^{-1} \mathbf{z} = \mathbf{z}^\top \Sigma^{-1} \mathbf{1} \\ C = \mathbf{z}^\top \Sigma^{-1} \mathbf{z}, \quad \Delta = AC - B^2$$

Minimum Variance Portfolio:

$$\mathbf{w} = \lambda \Sigma^{-1} \mathbf{1} + \gamma \Sigma^{-1} \mathbf{z}$$

where

$$\lambda = \frac{C - \mu B}{\Delta}, \quad \gamma = \frac{\mu A - B}{\Delta}$$

Portfolio Variance:

$$\sigma_P^2 = \frac{A\mu^2 - 2B\mu + C}{\Delta}$$

Global Minimum Variance Portfolio:

$$\mathbf{w}_g = \frac{1}{A} \Sigma^{-1} \mathbf{1}$$

### 1.3.3 N-risky assets + risk-free

Minimise  $\sigma_P^2 = \mathbf{w}^\top \Sigma \mathbf{w}$ , subject to  $(\mathbf{z} - r_f \mathbf{1})^\top \mathbf{w} = u - r_f$ .

Lagrangian:  $\mathcal{L}(\mathbf{w}, \lambda, \gamma) = \frac{1}{2} \mathbf{w}^\top \Sigma \mathbf{w} + \gamma(\mu - r_f - (\mathbf{z} - r_f \mathbf{1})^\top \mathbf{w})$

Minimum Variance Portfolio:

$$\mathbf{w} = \gamma \Sigma^{-1} (\mathbf{z} - r_f \mathbf{1})$$

where

$$\gamma = \frac{\mu - r_f}{Ar_f^2 - 2Br_f + C}$$

Portfolio Variance:

$$\sigma_P^2 = \frac{(\mu - r_f)^2}{Ar_f^2 - 2Br_f + C}$$

Tangency Portfolio:

$$\mathbf{w}_t = \gamma_t \Sigma^{-1} (\mathbf{z} - r_f \mathbf{1})$$

where

$$\gamma_t = \frac{1}{B - Ar_f}$$

## 2 Asset Pricing Models

For all parts in section 2,  $i$  takes values  $1, 2, \dots, N$  and indexes the assets in the market.

### 2.1 CAPM

#### 2.1.1 Capital Market Line

$$\mu_e = r_f + \frac{\sigma_e}{\sigma_M} (\mu_M - r_f)$$

#### 2.1.2 Security Market Line

$$\mu_i = r_f + \beta_i (\mu_M - r_f), \quad \text{where}$$

$$\beta_i = \frac{\sigma_{i,M}}{\sigma_M^2}$$

#### 2.1.3 Risk Decomposition

$$\begin{aligned} \sigma_i^2 &= \beta_i^2 \sigma_M^2 + \sigma_{\xi_i}^2 \\ &= \text{Systematic Risk} + \text{Non-systematic Risk} \end{aligned}$$

## 2.2 Factor Models

### 2.2.1 Single Factor Model (SFM)

$$r_i = \alpha_i + \beta_i f + \epsilon_i,$$

where  $\alpha_i, \beta_i$  are stock-specific constants,  $f$  is the factor capturing market-wide price movement and  $\epsilon_i$  is a noise term reflecting firm-specific risk.

### 2.2.2 SFM Assumptions

### 2.2.3 Risk Decomposition and Covariance

$$\begin{aligned}\sigma_i^2 &= \beta_i^2 \sigma_f^2 + \sigma_{\epsilon_i}^2 = \text{Systematic Risk} + \text{Non-systematic Risk} \\ \sigma_{i,j} &= \beta_i \beta_j \sigma_f^2\end{aligned}$$

### 2.2.4 Diversification

$$R_P^2 = \frac{\beta_P^2 \sigma_f^2}{\sigma_P^2} = \frac{\text{Systematic Risk}}{\text{Total Risk}}$$

Full diversification when  $R_P^2 = 1$ .

### 2.2.5 Multi-Factor Models

$$r_i = \alpha_i + \beta_{i,1}f_1 + \beta_{i,2}f_2 + \dots + \beta_{i,K}f_K + \epsilon_i$$

Factors  $f_i$  may include inflation, economic growth, interest rates etc.

## 2.3 APT

### 2.3.1 Single-Factor APT Returns

$$\begin{aligned}r_i &= a_i + b_i f + \epsilon_i \\ \mathbb{E}[r_i] &= a_i, \quad \sigma_i^2 = b_i^2 + \sigma_{\epsilon_i}^2, \quad \sigma_{i,j} = b_i b_j\end{aligned}$$

Under no-arbitrage,

$$a_i = \lambda_0 + \lambda_1 b_i,$$

where  $\lambda_0 = r_f$  if there is a risk-free asset.

### 2.3.2 Multi-Factor APT Returns

$$r_i = a_i + b_{i,1}f_1 + b_{i,2}f_2 + \dots + b_{i,K}f_K + \epsilon_i$$

where

$$\mathbb{E}[r_i] = a_i = \lambda_0 + \lambda_1 b_{i,1} + \lambda_2 b_{i,2} + \dots + \lambda_K b_{i,K}$$

and  $\lambda_0 = r_f$  if there is a risk-free asset.

### 3 Discrete Time Derivative Pricing

#### 3.1 Options

Let:  $S_t$  denote the time  $t$  stock price,  $c_t, p_t$  denote the time  $t$  European call/put prices,  $T$  denote the option's maturity,  $K$  denote its strike price and  $r$  denote the risk-free rate.

##### 3.1.1 Vanilla Options

Moneyiness	European Call	European Put
In the money	$S_T > K$	$S_T < K$
At the money	$S_T = K$	$S_T = K$
Out of the money	$S_T < K$	$S_T > K$

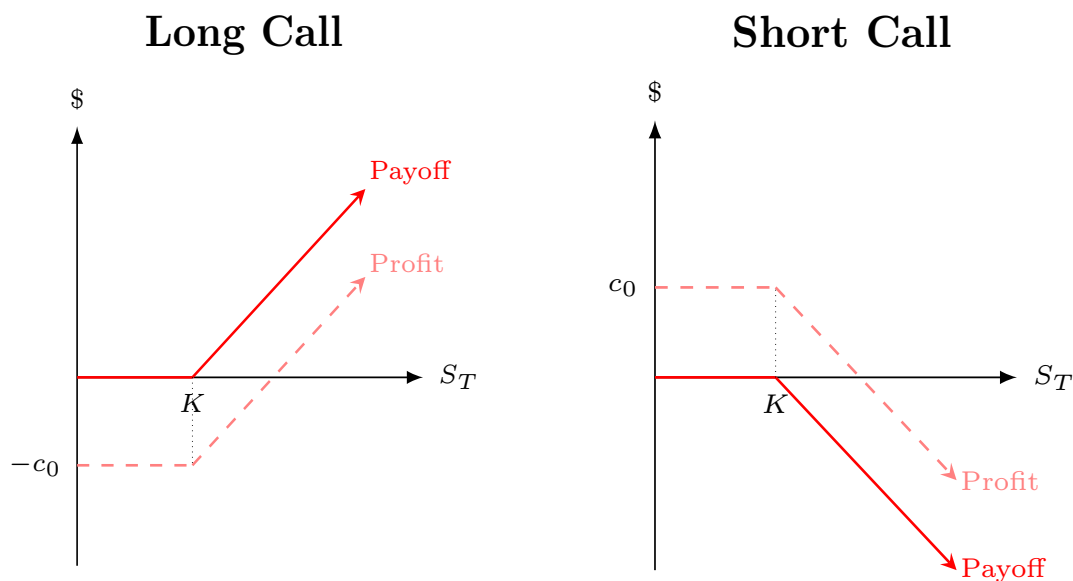
##### 3.1.2 Payoff Functions and Price Bounds

Define  $0 \leq U \leq T$  as the exercise time for an American option.  $U := \infty$  if option not exercised. Note:  $(X)_+ := \max\{0, X\}$ . Option bounds are for option prices at **time  $t$** .

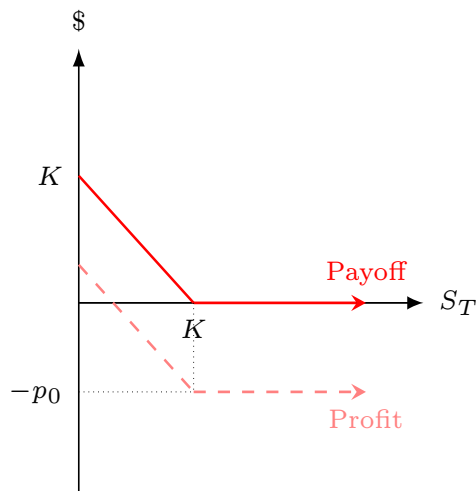
Option	Payoff	Lower Bound	Upper Bound
European Call	$(S_T - K)_+$	$S_t - Ke^{-r(T-t)}$	$S_t$
European Put	$(K - S_T)_+$	$Ke^{-r(T-t)} - S_t$	$Ke^{-r(T-t)}$
American Call	$(S_U - K)_+$	$S_t - Ke^{-r(T-t)}$	$S_t$
American Put	$(K - S_U)_+$	$K - S_t$	$K$

##### 3.1.3 Position Diagrams

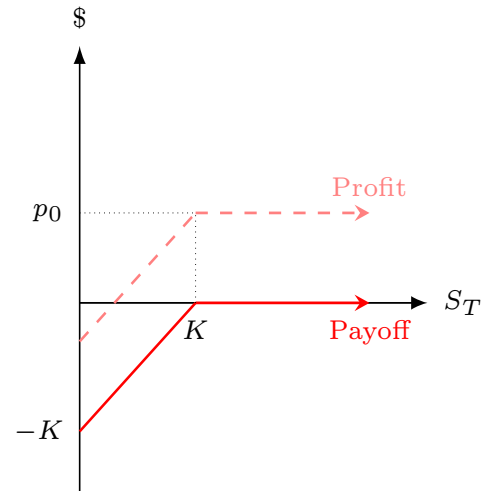
Long position = buy, short position = sell.



## Long Put



## Short Put



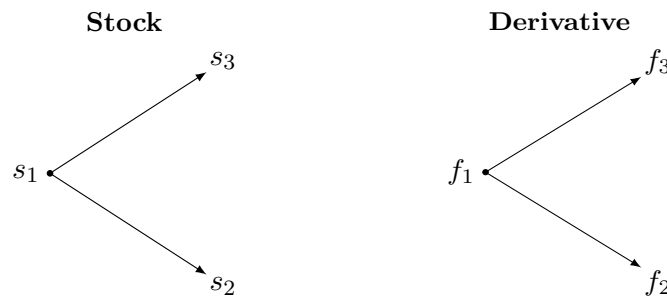
### 3.1.4 Put-Call Parity

Under no arbitrage, for  $0 \leq t \leq T$ :

$$c_t + Ke^{-r(T-t)} = p_t + S_t$$

## 3.2 Discrete Time Pricing

### 3.2.1 One Period Binomial Model



The replicating portfolio (stocks, bonds) is:

$$\phi = \frac{f_3 - f_2}{s_3 - s_2}, \quad \psi = \frac{1}{B(0)} e^{-r\delta t} (f_3 - \phi s_3)$$

The derivative price today is:

$$V_0 = \phi s_1 + \psi B(0)$$

Rewriting with risk-neutral probabilities:

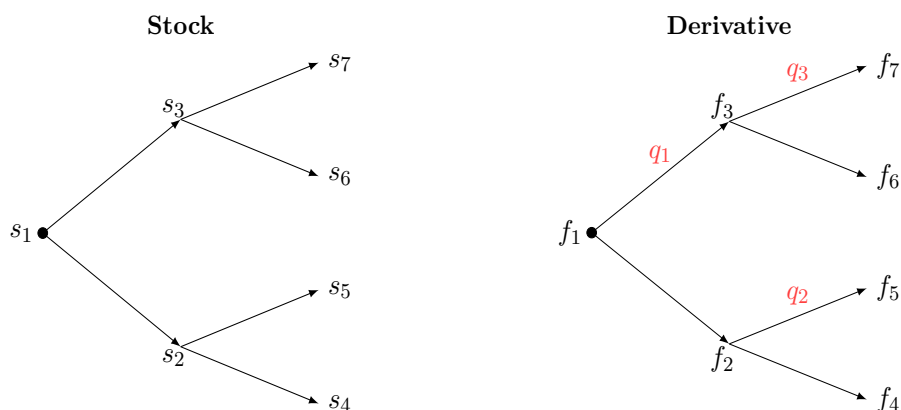
$$V_0 = e^{-r\delta t} (qf_3 + (1-q)f_2) = e^{-r\delta t} \mathbb{E}^Q[X]$$

where

$$q = \frac{s_1 e^{r\delta t} - s_2}{s_3 - s_2}.$$

### 3.2.2 Multi-Period Binomial Model

Apply the one-period Binomial model to each internal node and recurse work backwards to find price. As an example, below is the two-period binomial model price:



For  $j = 1, 2, 3$ :

$$q_j = \frac{s_j e^{r\delta t} - s_{2j}}{s_{2j+1} - s_{2j}}, \quad f_j = e^{-r\delta t} [q_j f_{2j+1} + (1 - q_j) f_{2j}]$$

Substituting  $f_2, f_3$  into  $f_1$ ,

$$f_1 = e^{-2r\delta t} [q_1 q_3 f_7 + q_1 (1 - q_3) f_6 + (1 - q_1) q_2 f_5 + (1 - q_1) (1 - q_2) f_4]$$

## 4 Continuous Time Derivative Pricing

### 4.1 Stochastic Calculus

#### 4.1.1 Stochastic Differential Equation

Suppose the stochastic process  $X_t$  can be written as

$$X_t = x_0 + \int_0^t a_s ds + \int_0^t b_s dW_s$$

where  $a(\cdot), b(\cdot)$  are appropriate functions (possibly stochastic processes) and  $x_0$  is a constant. This is abbreviated as

$$dX_t = a_t dt + b_t dW_t, \quad X(0) = x_0.$$

#### 4.1.2 Itô's Lemma

Itô's Lemma tells us how to differentiate functions of stochastic processes.

Let  $f(x), F(t, x)$  be deterministic functions with continuous (partial) derivatives up to the second order. Then,

$$df(X_t) = f'(X_t) dX_t + \frac{1}{2} f''(X_t) dX_t^2$$

$$dF(t, X_t) = \frac{\partial F}{\partial t} dt + \frac{\partial F}{\partial x} dX_t + \frac{1}{2} \frac{\partial^2 F}{\partial x^2} dX_t^2$$

In the second equation, all partial derivatives of  $F$  are evaluated at  $(t, X_t)$ . These equations can be simplified using the following multiplication table:

$$\begin{array}{ccc} \times & dt & dW_t \\ \hline dt & 0 & 0 \\ \hline dW_t & 0 & dt \\ \hline \end{array}$$

### 4.1.3 Girsanov-Theorem

Suppose  $W_t$  is a  $\mathbb{P}$ -Brownian Motion and  $\gamma(\cdot)$  is a pre-visible process. Then there exists an equivalent measure  $\mathcal{Q}$  with Radon-Nikodym derivative

$$\zeta_T = \exp \left[ - \int_0^T \gamma_s dW_s - \frac{1}{2} \int_0^T \gamma_s^2 ds \right]$$

such that  $W_t^{\mathcal{Q}}$ , where

$$W_t^{\mathcal{Q}} = W_t + \int_0^t \gamma_s ds,$$

is a  $\mathcal{Q}$ -Brownian Motion.

## 4.2 Black-Scholes-Merton Model

### 4.2.1 Stock Process

The stock process follows a Geometric Brownian Motion

$$\begin{aligned} S(t) &= S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + W(t)} \\ \iff dS(t) &= \mu S(t)dt + \sigma S(t)dW(t) \end{aligned}$$

### 4.2.2 Put and Call Formulas

Define:

$$d_1 = \frac{\ln(S(t)/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \quad d_2 = d_1 - \sigma\sqrt{T-t}$$

Then,

$$\begin{aligned} c_t &= S(t)N(d_1) - Ke^{-r(T-t)}N(d_2) \\ p_t &= Ke^{-r(T-t)}N(-d_2) - S_tN(-d_1), \end{aligned}$$

where  $N(\cdot)$  is the cdf of the standard normal distribution.



## 5 Term Structure Modelling and Asset-Liability Management

### 5.1 Short-Rate Models

#### 5.1.1 Merton Model:

$$dr(t) = \alpha dt + \sigma dW(t)$$

#### 5.1.2 Hull White Model:

$$dr(t) = \alpha(t)(\mu(t) - r(t))dt + \sigma(t)dW(t)$$

#### 5.1.3 Vasicek Model:

$$dr(t) = \alpha(\mu - r(t))dt + \sigma dW(t)$$

#### 5.1.4 CIR Model:

$$dr(t) = \alpha(\mu - r(t))dt + \sigma\sqrt{r(t)}dW(t)$$

### 5.2 Ho-Lee Model

#### 5.2.1 Definition

$$r(k, s) = a(k) + b(k)s$$

where  $k$  = time and  $s$  = number of jumps.

$$\begin{array}{ccccccc} & & r(2, 2) & & & & a(2) + 2b(2) \\ & r(1, 1) & r(2, 1) & & a(1) + b(1) & a(2) + b(2) & \\ r(0, 0) & r(1, 0) & r(2, 0) & a(0) & a(1) & a(2) & \end{array}$$

#### 5.2.2 Calibration

In general,

$$b(k) = 2 \cdot \text{st.dev}(r(k))$$

The parameters  $a(k)$  are found by equating with price of Zero-Coupon Bonds. Take  $q = 1/2$  in most situations.

$$B(0, 1) = \frac{1}{1 + a(0)}$$

$$B(0, 2) = \frac{1}{1 + a(0)} \left( (1 - q) \frac{1}{1 + a(1)} + q \frac{1}{1 + a(1) + b(1)} \right)$$