ACTL3182 Module 3 and 4 Extra Questions: Radon-Nikodym Theorem, Martingales and Itô's Lemma

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Unless stated otherwise, all processes should use their natural filtration.

1. Let $X := (X_n)_{n=1,2,...}$ be a (discrete-time) martingale with respect to filtration $(\mathcal{F}_n)_{n=1,2,...}$. Show that X is a martingale **if and only if**

$$\mathbb{E}[X_n|\mathcal{F}_n] = \mathbb{E}[X_{n+1}|\mathcal{F}_n] = X_n.$$

2. Consider the stochastic process

$$X_n = \sum_{k=1}^{n} Z_k, \quad X(0) = 0$$

where Z_k, $k = 1, 2, ...n$ are iid with distribution

$$Z_k = \begin{cases} +1 & \text{w.p. } \frac{1}{2} \\ -1 & \text{w.p. } \frac{1}{2}. \end{cases}$$

Show that X is a martingale.

3. Consider the stochastic process

$$X_n = \prod_{k=1}^{n} Z_k, \quad X(0) = 1$$

where Z_k, $k = 1, 2, ...n$ are iid with distribution

$$Z_k = \begin{cases} 2 & \text{w.p. } p \\ \frac{1}{2} & \text{w.p. } 1-p. \end{cases}$$

Find the probability p such that X is a martingale.

4. Consider the stochastic processes X, Y defined by:

$$X_n = \sum_{k=1}^{n} Z_k, \quad X(0) = 0$$

where Z_k, $k = 1, 2, ...n$ are iid with distribution

$$Z_k = \begin{cases} +1 & \text{w.p. } 0.6 \\ -1 & \text{w.p. } 0.4, \end{cases}$$

and

$$Y_n = Y_{n-1} + IY_{n-2}, \qquad Y_0 = Y_1 = 0,$$

where I is independent of Y_n for every $n = 0, 1, \dots$ and

$$I = \begin{cases} +1 & \text{w.p. } \frac{1}{2} \\ -1 & \text{w.p. } \frac{1}{2}. \end{cases}$$

Recall that $X := (X_n)_{n=0,1,2\dots}$ is a Markov process if for any n

$$\mathbb{P}(X_{n+1} = x | X_0, X_1, \dots, X_n) = \mathbb{P}(X_{n+1} = x | X_n),$$

or equivalently, for any nice (Borel-meausrable) function,

$$\mathbb{E}[f(X_{n+1}) | \mathcal{F}_n] = \mathbb{E}[f(X_{n+1}) | X_n].$$

(a) Explain why X is a Markov process but not a martingale.

(b) Explain why Y is a martingale but not a Markov process.

5. Consider two independent coin flips; that is let the sample space be $\Omega := \{HH, HT, TH, TT\}$. Under the $\mathbb{P}$ measure, the coin is fair. Under the $\mathcal{Q}$ measure, the coin has a probability of $\frac{1}{3}$ of landing on heads on each flip.

   (a) Find the probability of each scenario under $\mathbb{P}$ and $\mathcal{Q}$.

   (b) Find the Radon-Nikodym Derivative $\frac{d\mathcal{Q}}{d\mathbb{P}}$.

   (c) Show that
   $$\mathbb{E}\left[\frac{d\mathcal{Q}}{d\mathbb{P}}\right] = 1.$$

   (d) Show that for the event A, where A represents getting a head on the first flip, that
   $$\mathcal{Q}(A) = \mathbb{E}_P\left[\frac{d\mathcal{Q}}{d\mathbb{P}} 1_A\right].$$

6. The following question works with a scaled version of the random walk process in question 2:

   (a) Consider a time interval $[0, T]$ and divide the interval into n equal subintervals of length $\Delta = T/n$. Let $(X_t^n)_{t\in[0,T]}$ be the process
   $$X_t^n = \sqrt{\Delta} \sum_{k=1}^{\lfloor t/\Delta \rfloor} Z_k, \quad \text{for } 0 \le t \le T,$$
   where Z_k, $k = 1, 2, \dots n$ are iid with distribution
   $$Z_k = \begin{cases} +1 & \text{w.p. } \frac{1}{2} \\ -1 & \text{w.p. } \frac{1}{2}. \end{cases}$$
   Draw a possible graph of X_t^n for $n = 1, 2$. What happens as $n \to \infty$?

   (b) Calculate $\mathbb{E}[X_t^n]$ and $\text{Var}[X_t^n]$.

   (c) Using the Central Limit Theorem, determine the limiting distribution of X_t^n.

   (d) What continuous-time process does X_t^n converge to? (no formal justification required)

7. Consider the continuous-time Radon-Nikodym derivative ζ_t:
   $$\zeta_t = \exp\left(-\int_0^t \gamma_s dW_s - \frac{1}{2}\int_0^t \gamma_s^2 ds\right)$$
   Prove that ζ_t is a martingale.

8. (Challenge) Consider the stochastic differential equation (SDE)

$$dX(t) = \alpha(\mu - X(t)) + \sigma dW(t), \quad X(0) = x_0.$$

The solution to this equation is called the Ornstein-Uhlenbeck process and can be used to model a massive particle moving under Brownian Motion when friction is accounted for.

(a) Using Itô's Lemma, show that the solution to the SDE above is

$$X(t) = e^{-\alpha t}x_0 + \mu(1 - e^{-\alpha t}) + \sigma \int_0^t e^{-\alpha(t-s)} dW(s).$$

(b) Explain why

$$\mathbb{E}[X(t)] = e^{-\alpha t}x_0 + \mu(1 - e^{-\alpha t}).$$

Hint: What type of stochastic process is $\int_0^t e^{-\alpha(t-s)} dW(s)$?

(c) The Itô isometry is a special property that allows the expected value of the product of Itô integrals to be calculated using Riemann integrals instead. That is, for two suitable stochastic processes $X(t)$, $Y(t)$, we have:

$$\mathbb{E}\left[\int_0^t X(s)dW(s) \int_0^t Y(s)dW(s)\right] = \mathbb{E}\left[\int_0^t X(s)Y(s)ds\right]$$

Assuming this property holds for $X(t)$, show that

$$\text{Var}[X(t)] = \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t}).$$

(d) What is it the distribution of $X(t)$? Explain this intuitively.

(e) Show that for any $s, t \geq 0$:

$$\text{Cov}(X(s), X(t)) = \frac{\sigma^2}{2\alpha}(e^{-\alpha|t-s|} - e^{-\alpha(s+t)})$$

9. The price at time t of a put option is given by

$$p(t, S_t) = Ke^{-r(T-t)}N(-d_2) - S_t N(-d_1),$$

where

$$d_1 = \frac{\ln(S_t/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \qquad d_2 = d_1 - \sigma\sqrt{T-t}.$$

Derive the expressions for the replicating portfolio $(\phi(t), \psi(t))$ given that

$$\phi(t) = \frac{\partial p}{\partial S}(t, S_t), \qquad \psi(t) = \frac{p(t, S_t) - S_t\phi(t)}{B(t)}.$$

Select Answers:

3. $p = \frac{1}{3}$

5. (b) The Radon-Nikodym derivative is the ratio of the two probabilities;

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \begin{cases} 4/9 & \text{if } HH \\ 8/9 & \text{if } HT \\ 8/9 & \text{if } TH \\ 16/9 & \text{if } TT \end{cases}$$

6. (b) $\mathbb{E}[X_t^n] = 0$, $\text{Var}[X_t^n] = \Delta\lfloor t/\Delta \rfloor \approx t$ for n large (c) $\mathcal{N}(0, t)$ (d) Brownian Motion.

8. (d) $\mathcal{N}\left(e^{-\alpha t}x_0 + \mu(1 - e^{-\alpha t}), \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t})\right)$