Non convex optimisation convex envelopes and Fenchel Conjugates

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1 Convex Envelopes

Definition 1. *Let V be a vector space. The functional $g : V \to \mathbb{R}$ is affine if there exists linear functional $l : V \to \mathbb{R}$ and $b \in V$ such that for all $v \in V$,*

$$g(v) = l(v) + b$$

Proposition 1. *Let V be a vector space. Suppose $f, g : V \to \mathbb{R}$ are arbitrary functionals. Then,*

$$\operatorname{conv}(f) + \operatorname{conv}(g) \leq \operatorname{conv}(f + g).$$

If g is affine, then

$$\operatorname{conv}(f) + \operatorname{conv}(g) = \operatorname{conv}(f + g)$$

Proof. Observe that $\operatorname{conv}(f) + \operatorname{conv}(g)$ is a convex underestimator of $f + g$. Hence, by definition,

$$\operatorname{conv}(f) + \operatorname{conv}(g) \leq \operatorname{conv}(f + g).$$

Suppose g is affine, then both g and $-g$ are convex. Thus, $\operatorname{conv}(g) = g$ and $\operatorname{conv}(-g) = g$. Hence,

$$\operatorname{conv}(f + g) - g = \operatorname{conv}(f + g) + \operatorname{conv}(-g) \leq \operatorname{conv}(f),$$

so

$$\operatorname{conv}(f) + \operatorname{conv}(g) \geq \operatorname{conv}(f + g).$$

Combining this with the first part of the proposition yields the desired equality. □

Note: Ensure V is a vector space, not a subset of a vector space when applying proposition 1. This ensures V is a convex set, otherwise the domain of f needs to be extended to the convex hull of the set.

2 Separable Additivity of Biconjugates

Definition 2. *Let X be some space. The function $f : X \to [-\infty, \infty]$ is proper if $f(x) \neq -\infty$ for all $x \in X$ and there exists $x \in X$ such that $f(x) < +\infty$.*

Definition 3. *Let $(X, \langle , \rangle)$ be a real Hilbert space. The functional $f : X \to [-\infty, \infty]$ has a continuous affine minorant if there exists $a \in X$ and $b \in \mathbb{R}$ such that for all $x \in X$:*

$$f(x) \geq \langle a, x \rangle + b$$

Lemma 1. *Let $(X, \langle,\rangle)$ be a real Hilbert space and suppose $f : X \to [-\infty,\infty]$ is proper and has a continuous affine minorant. Then, both $f^*, f^{**} : X \to [-\infty,\infty]$ are also proper and have affine minorants.*

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Theorem 1. *Let $X_1, X_2, \ldots X_n$ be real Hilbert spaces with inner products $\langle,\rangle_{i=1,2\ldots n}$ respectively and $X = X_1 \times X_2 \times \ldots X_n$. Suppose $g : X \to (-\infty,\infty]$ is a proper separable function defined by*

$$g(x) = \sum_{i=1}^{n} g_i(x_i), \quad \text{for any } x = (x_1, x_2, \ldots, x_n) \in X,$$

where $g : X := g_i : X_i \to (-\infty,\infty]$, $i = 1, 2, \ldots n$ are proper functions with continuous affine minorants. Then, g^{**} is proper and for all $x \in X$,

$$g^{**}(x) = \sum_{i=1}^{n} g_i^{**}(x_i),$$

where g_i^{**} are proper functions.

Proof. Let $x = (x_1, \ldots x_n)$ and $y = (y_1, \ldots y_n)$ be elements of X. Observe that X equipped with the inner product

$$\langle x, y\rangle := \sum_{i=1}^{n} \langle x_i, y_i\rangle_i$$

defines a Hilbert space. Computing the Fenchel conjugate, we obtain:

$$\begin{aligned} g^*(x^*) &= \sup_{x\in X} \{\langle x^*, x\rangle - g(x)\} \\ &= \sup_{x_1,x_2\ldots x_n} \left\{ \sum_{i=1}^{n} \langle x_i^*, x_i\rangle - g_i(x_i) \right\} \\ &= \sum_{i=1}^{n} \sup_{x_i\in X_i} \{\langle x_i^*, x_i\rangle - g_i(x_i)\} \\ &= \sum_{i=1}^{n} g_i^*(x_i) \end{aligned}$$

Observe that g^* is separable and $g_i^* : X_i \to [-\infty,\infty]$ are proper with continuous affine minorants by 1. Thus, reapplying the above line of reasoning yields

$$g^{**}(x) = \sum_{i=1}^{n} g_i^{**}(x_i),$$

where again, g_i^{**} are proper by 1. Finally, if g_i, $i = 1, 2 \ldots n$ have continuous affine minorants $\langle a_i, x\rangle + b_i$, then $\langle \sum_{i=1}^{n} a_i, x_i\rangle + \sum_{i=1}^{n} b_i$ is a continuous affine minorant of the proper function g, so g^{**} is proper by 1.

Notes:

1. Proof of separability of g^* is adapted from Proposition 13.30 of Bauschke "Convex analysis and monotone operator theory in Hilbert Spaces."

2. We allow $\sup = +\infty$ since we are dealing with functions on the extended reals. This also means we do not need to worry about the sets $\{\langle x_i^*, x_i \rangle - g_i(x_i)\}$ being bounded above, hence additivity of the sup should always hold.

$\square$