Non-convex optimisation convex envelopes and Fenchel Conjugates

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1 Introduction

This note aims to extend propositions from Bauschke and Combettes 2010 and Tardella 2004 to a class of optimization problems on Hilbert spaces with non-convex constraints.

2 Convex Envelopes

Definition 1. *Let V be a vector space. The functional $g : V \to \mathbb{R}$ is affine if there exists linear functional $l : V \to \mathbb{R}$ and $b \in V$ such that for all $v \in V$,*

$$g(v) = l(v) + b$$

The following proposition extends **Proposition 2.20** of Tardella 2004 from $\mathbb{R}^n$ to general vector spaces.

Theorem 1. *Let V be a vector space. Suppose $f, g : V \to \mathbb{R}$ are arbitrary functionals. Then,*

$$\operatorname{conv}(f) + \operatorname{conv}(g) \leq \operatorname{conv}(f + g).$$

If g is affine, then

$$\operatorname{conv}(f) + \operatorname{conv}(g) = \operatorname{conv}(f + g)$$

Proof. Observe that $\operatorname{conv}(f) + \operatorname{conv}(g)$ is a convex underestimator of $f + g$. Hence, by definition,

$$\operatorname{conv}(f) + \operatorname{conv}(g) \leq \operatorname{conv}(f + g).$$

Suppose g is affine, then both g and $-g$ are convex. Thus, $\operatorname{conv}(g) = g$ and $\operatorname{conv}(-g) = g$. Hence,

$$\operatorname{conv}(f + g) - g = \operatorname{conv}(f + g) + \operatorname{conv}(-g) \leq \operatorname{conv}(f),$$

so

$$\operatorname{conv}(f) + \operatorname{conv}(g) \geq \operatorname{conv}(f + g).$$

Combining this with the first part of the proposition yields the desired equality. □

3 Separable Additivity of Biconjugates

Definition 2. *Let X be some space. The function $f : X \to [-\infty, \infty]$ is proper if $f(x) \neq -\infty$ for all $x \in X$ and there exists $x \in X$ such that $f(x) < +\infty$.*

Definition 3. *Let $(X, \langle, \rangle)$ be a real Hilbert space. The functional $f : X \to [-\infty, \infty]$ has a continuous affine minorant if there exists $a \in X$ and $b \in \mathbb{R}$ such that for all $x \in X$:*

$$f(x) \geq \langle a, x \rangle + b$$

Lemma 1. *Let $(X, \langle, \rangle)$ be a real Hilbert space and suppose $f : X \to [-\infty, \infty]$ is proper and has a continuous affine minorant. Then, both $f^*, f^{**} : X \to [-\infty, \infty]$ are also proper and have affine minorants.*

Proof. See Klein Haneveld, Stougie, and Vlerk 1995 □

The following theorem is essentially a double application of **Proposition 13.30** of Bauschke and Combettes 2010 where extra conditions are imposed to guarantee g^{**} is proper. As with Bauschke and Combettes 2010, we allow $\sup\{\cdot\}$ to take values $\pm\infty$.

Theorem 2. *Let $X_1, X_2, \ldots X_n$ be real Hilbert spaces with inner products $\langle, \rangle_{i=1,2\ldots n}$ respectively and $X = X_1 \times X_2 \times \ldots X_n$. Suppose $g : X \to (-\infty, \infty]$ is a proper separable function defined by*

$$g(x) = \sum_{i=1}^{n} g_i(x_i), \quad \textit{for any } x = (x_1, x_2, \ldots, x_n) \in X,$$

where $g : X := g_i : X_i \to (-\infty, \infty]$, $i = 1, 2, \ldots n$ are proper functions with continuous affine minorants. Then, g^{**} is proper and for all $x \in X$,

$$g^{**}(x) = \sum_{i=1}^{n} g_i^{**}(x_i),$$

where g_i^{**} are proper functions.

Proof. Let $x = (x_1, \ldots x_n)$ and $y = (y_1, \ldots y_n)$ be elements of X. Observe that X equipped with the inner product

$$\langle x, y \rangle := \sum_{i=1}^{n} \langle x_i, y_i \rangle_i$$

defines a Hilbert space. Computing the Fenchel conjugate, we obtain:

$$\begin{aligned}
g^*(x^*) &= \sup_{x \in X} \{\langle x^*, x \rangle - g(x)\} \\
&= \sup_{x_1, x_2 \ldots x_n} \left\{ \sum_{i=1}^{n} \langle x_i^*, x_i \rangle - g_i(x_i) \right\} \\
&= \sum_{i=1}^{n} \sup_{x_i \in X_i} \{\langle x_i^*, x_i \rangle - g_i(x_i)\} \\
&= \sum_{i=1}^{n} g_i^*(x_i)
\end{aligned}$$

Observe that g^* is separable and $g_i^* : X_i \to [-\infty, \infty]$ are proper with continuous affine minorants by Lemma 1. Thus, reapplying the above line of reasoning yields

$$g^{**}(x) = \sum_{i=1}^{n} g_i^{**}(x_i),$$

where again, g_i^{**} are proper by Lemma 1. Finally, if g_i, $i = 1, 2 \ldots n$ have continuous affine minorants $\langle a_i, x\rangle + b_i$, then $\langle \sum_{i=1}^n a_i, x_i\rangle + \sum_{i=1}^n b_i$ is a continuous affine minorant of the proper function g, so g^{**} is proper by Lemma 1. □

References

[1] Heinz H. Bauschke and Patrick L. Combettes. *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*. Canadian Mathematical Society, Springer, 2010. ISBN: 9781441994660.

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[3] Fabio Tardella. “On the existence of polyhedral convex envelopes”. In: *Frontiers in global optimization*. 1st ed. Vol. 74. Springer US, 2004, pp. 563–566.