

Non-convex optimisation convex envelopes and Fenchel Conjugates

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1 Introduction

This note aims to extend propositions from Bauschke and Combettes [2010](#) and Tardella [2004](#) to a class of optimization problems on Hilbert spaces with non-convex constraints.

2 Superadditivity of convex envelopes

Definition 1. Let V be a convex subset of arbitrary space X and suppose $f : X \rightarrow [-\infty, \infty]$. The convex envelope of f , $\text{conv}(f) : V \rightarrow [-\infty, \infty]$, is the largest convex minorant of f . That is, for any convex function $g : V \rightarrow [-\infty, \infty]$ such that $g \leq f$, $g \leq \text{conv}(f)$.

Definition 2. Let V be a vector space. The functional $g : V \rightarrow \mathbb{R}$ is affine if there exists linear functional $l : V \rightarrow \mathbb{R}$ and $b \in \mathbb{R}$ such that for all $v \in V$,

$$g(v) = l(v) + b$$

If l is a continuous linear functional, then g is a continuous affine functional.

The following proposition extends **Proposition 2.20** of Tardella [2004](#) from convex subsets of \mathbb{R}^n to general vector spaces.

Theorem 1. Let V be a vector space. Suppose $f, g : V \rightarrow \mathbb{R}$ are arbitrary functionals. Then,

$$\text{conv}(f) + \text{conv}(g) \leq \text{conv}(f + g).$$

If g is affine, then

$$\text{conv}(f) + \text{conv}(g) = \text{conv}(f + g)$$

Proof. Observe that $\text{conv}(f) + \text{conv}(g)$ is a convex underestimator of $f + g$. Hence, by definition,

$$\text{conv}(f) + \text{conv}(g) \leq \text{conv}(f + g).$$

Suppose g is affine, then both g and $-g$ are convex. Thus, $\text{conv}(g) = g$ and $\text{conv}(-g) = g$. Hence,

$$\text{conv}(f + g) - g = \text{conv}(f + g) + \text{conv}(-g) \leq \text{conv}(f),$$

so

$$\text{conv}(f) + \text{conv}(g) \geq \text{conv}(f + g).$$

Combining this with the first part of the proposition yields the desired equality. \square

3 Superadditivity of biconjugates

A similar proposition can be proved for biconjugates.

Definition 3. Let H be a Hilbert Space and $f : H \rightarrow [-\infty, \infty]$. The Fenchel conjugate (or convex conjugate or Legendre Transform) is the function $f^* : H \rightarrow [-\infty, \infty]$

$$f^*(x^*) = \sup_{x \in X} \{\langle x^*, x \rangle - f(x)\}$$

The biconjugate of f is the function $f^{**} := (f^*)^*$.

Lemma 1. Let H be a Hilbert space and $f : H \rightarrow [-\infty, \infty]$. Then, f^{**} is the largest convex lower semicontinuous minorant of f . That is, for any convex lower semicontinuous function g such that $g \leq f$, $g \leq f^{**}$.

Proof. See **Proposition 4.1** of Ekeland and Teman 1999. □

Theorem 2. Let H be a Hilbert space and $f, g : H \rightarrow \mathbb{R}$ be arbitrary functionals. Then,

$$f^{**} + g^{**} \leq (f + g)^{**}.$$

In particular, if g is affine continuous, then

$$f^{**} + g^{**} = (f + g)^{**}$$

Proof. By **Lemma 1**, f^{**}, g^{**} are convex and lower semicontinuous and $f \leq f^{**}, g \leq g^{**}$. Observe that $f^{**} + g^{**}$ is a convex minorant of $f + g$, and is also lower semicontinuous by the superadditivity of the infimum. Hence, by definition,

$$f^{**} + g^{**} \leq (f + g)^{**}.$$

Now, if g is affine and continuous, both g and $-g$ are convex and lower semicontinuous. Hence, by **Lemma 1**, $g^{**} = g$ and $(-g)^{**} = -g$, so

$$(f + g)^{**} - g = (f + g)^{**} + (-g)^{**} \leq f^{**},$$

which rearranges to

$$(f + g)^{**} \leq f^{**} + g^{**}.$$

Combining this with the first inequality in the theorem yields the equality. □

4 Separable additivity of biconjugates

Definition 4. Let X be some space. The function $f : X \rightarrow [-\infty, \infty]$ is proper if $f(x) \neq -\infty$ for all $x \in X$ and there exists $x \in X$ such that $f(x) < +\infty$.

Definition 5. Let $(X, \langle \cdot, \cdot \rangle)$ be a real Hilbert space. The functional $f : X \rightarrow [-\infty, \infty]$ has a continuous affine minorant if there exists $a \in X$ and $b \in \mathbb{R}$ such that for all $x \in X$:

$$f(x) \geq \langle a, x \rangle + b$$

Lemma 2. *Let $(X, \langle \cdot, \cdot \rangle)$ be a real Hilbert space and suppose $f : X \rightarrow [-\infty, \infty]$ is proper and has a continuous affine minorant. Then, both $f^*, f^{**} : X \rightarrow [-\infty, \infty]$ are also proper and have affine minorants.*

Proof. See **Lemma 2.1** of Klein Haneveld, Stougie, and Vlerk 1995 □

The following theorem is essentially a double application of **Proposition 13.30** of Bauschke and Combettes 2010 where extra conditions are imposed to guarantee g^{**} is proper. The proof follows that of **Theorem 2.1** in Klein Haneveld, Stougie, and Vlerk 1995 albeit with a modified definition of proper function to match Bauschke and Combettes 2010. As with Bauschke and Combettes 2010, we allow $\sup\{\cdot\}$ to take values $\pm\infty$.

Theorem 3. *Let X_1, X_2, \dots, X_n be real Hilbert spaces with inner products $\langle \cdot, \cdot \rangle_{i=1,2,\dots,n}$ respectively and $X = X_1 \times X_2 \times \dots \times X_n$. Suppose $g : X \rightarrow (-\infty, \infty]$ is a proper separable function defined by*

$$g(x) = \sum_{i=1}^n g_i(x_i), \quad \text{for any } x = (x_1, x_2, \dots, x_n) \in X,$$

where $g : X := g_i : X_i \rightarrow (-\infty, \infty]$, $i = 1, 2, \dots, n$ are proper functions with continuous affine minorants. Then, g^{**} is proper and for all $x \in X$,

$$g^{**}(x) = \sum_{i=1}^n g_i^{**}(x_i),$$

where g_i^{**} are proper functions.

Proof. Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ be elements of X . Observe that X equipped with the inner product

$$\langle x, y \rangle := \sum_{i=1}^n \langle x_i, y_i \rangle_i$$

defines a Hilbert space. Computing the Fenchel conjugate, we obtain:

$$\begin{aligned} g^*(x^*) &= \sup_{x \in X} \{ \langle x^*, x \rangle - g(x) \} \\ &= \sup_{x_1, x_2, \dots, x_n} \left\{ \sum_{i=1}^n \langle x_i^*, x_i \rangle - g_i(x_i) \right\} \\ &= \sum_{i=1}^n \sup_{x_i \in X_i} \{ \langle x_i^*, x_i \rangle - g_i(x_i) \} \\ &= \sum_{i=1}^n g_i^*(x_i^*) \end{aligned}$$

Observe that g^* is separable and $g_i^* : X_i \rightarrow [-\infty, \infty]$ are proper with continuous affine minorants by **Lemma 2**. Thus, reapplying the above line of reasoning yields

$$g^{**}(x) = \sum_{i=1}^n g_i^{**}(x_i),$$

where again, g_i^{**} are proper by **Lemma 2**. Finally, if $g_i, i = 1, 2, \dots, n$ have continuous affine minorants $\langle a_i, x \rangle + b_i$, then $\langle \sum_{i=1}^n a_i, x \rangle + \sum_{i=1}^n b_i$ is a continuous affine minorant of the proper function g , so g^{**} is proper by **Lemma 2**. □

5 Subdifferentials of separable functions

This section gives a simple lemma connecting the total and partial subdifferentials of additively separable functions.

Lemma 3. *Suppose $X_i, i = 1, 2, \dots, n$ is a family of real Hilbert spaces with inner products $\langle \cdot, \cdot \rangle_i$ and denote the product Hilbert space by $\mathbf{X} = \bigoplus_{i=1}^n X_i$ (that is $\prod_{i=1}^n X_i$ with inner product $\sum_{i=1}^n \langle \cdot, \cdot \rangle_i$). Let $f : \mathbf{X} \rightarrow \mathbb{R}$ be an additively separable function;*

$$f(\mathbf{x}) = \sum_{i=1}^n f_i(x_i),$$

where $f_i : X_i \rightarrow \mathbb{R}$. Let $\mathbf{x} = (x_1, \dots, x_n) \in \text{dom } f$. As with Bauschke and Combettes [2010](#), we define $R_i : X_i \rightarrow \mathbf{X}$ such that the j^{th} component of $R_i y$ is y if $j = i$ and x_j otherwise. Then, if $u_i \in \partial_i f(\mathbf{x})$ for $i = 1, 2, \dots, n$, then $\mathbf{u} = (u_1, \dots, u_n) \in \partial f(\mathbf{x})$.

Proof. By definition, for all $i = 1, 2, \dots, n$ and $y_i \in X_i$,

$$\begin{aligned} \langle y_i - x_i, u_i \rangle_i + (f \circ R_i)(x_i) &\leq (f \circ R_i)(y_i) \\ \langle y_i - x_i, u_i \rangle_i + \sum_{j=1}^n f(x_j) &\leq f(y_i) + \sum_{j \neq i}^n f(x_j) \\ \langle y_i - x_i, u_i \rangle_i + f(x_i) &\leq f(y_i). \end{aligned}$$

Adding these inequalities together, we have that for all $\mathbf{y} \in \mathbf{X}$,

$$\begin{aligned} \sum_{i=1}^n \langle y_i - x_i, u_i \rangle_i + \sum_{i=1}^n f(x_i) &\leq \sum_{i=1}^n f(y_i) \\ \langle \mathbf{y} - \mathbf{x}, \mathbf{u} \rangle + f(\mathbf{x}) &\leq f(\mathbf{y}), \end{aligned}$$

thus, $\mathbf{u} \in \partial f(\mathbf{x})$. □

References

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