Non-convex optimisation convex envelopes and Fenchel Conjugates

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1 Introduction

This note aims to extend propositions from **Bauschke2010** and **Tardella** to a class of optimization problems on Hilbert spaces with non-convex constraints.

2 Superadditivity of convex envelopes

Definition 1. *Let V be a convex subset of arbitrary space X and suppose $f : X \to [-\infty, \infty]$. The convex envelope of f, $\mathrm{conv}(f) : V \to [-\infty, \infty]$, is the largest convex minorant of f. That is, for any convex function $g : V \to [-\infty, \infty]$ such that $g \leq f$, $g \leq \mathrm{conv}(f)$.*

Definition 2. *Let V be a vector space. The functional $g : V \to \mathbb{R}$ is affine if there exists linear functional $l : V \to \mathbb{R}$ and $b \in V$ such that for all $v \in V$,*

$$g(v) = l(v) + b$$

If l is a continuous linear functional, then g is a continuous affine functional.

The following proposition extends **Proposition 2.20** of **Tardella** from convex subsets of $\mathbb{R}^n$ to general vector spaces.

Theorem 1. *Let V be a vector space. Suppose $f, g : V \to \mathbb{R}$ are arbitrary functionals. Then,*

$$\mathrm{conv}(f) + \mathrm{conv}(g) \leq \mathrm{conv}(f + g).$$

If g is affine, then

$$\mathrm{conv}(f) + \mathrm{conv}(g) = \mathrm{conv}(f + g)$$

Proof. Observe that $\mathrm{conv}(f) + \mathrm{conv}(g)$ is a convex underestimator of $f + g$. Hence, by definition,

$$\mathrm{conv}(f) + \mathrm{conv}(g) \leq \mathrm{conv}(f + g).$$

Suppose g is affine, then both g and $-g$ are convex. Thus, $\mathrm{conv}(g) = g$ and $\mathrm{conv}(-g) = g$. Hence,

$$\mathrm{conv}(f + g) - g = \mathrm{conv}(f + g) + \mathrm{conv}(-g) \leq \mathrm{conv}(f),$$

so

$$\mathrm{conv}(f) + \mathrm{conv}(g) \geq \mathrm{conv}(f + g).$$

Combining this with the first part of the proposition yields the desired equality. $\square$

3 Superadditivity of biconjugates

A similar proposition can be proved for biconjugates.

Definition 3. *Let H be a Hilbert Space and $f : H \to [-\infty, \infty]$. The Fenchel conjugate (or convex conjugate or Legendre Transform) is the function $f^* : H \to [-\infty, \infty]$*

$$f^*(x^*) = \sup_{x \in X} \{\langle x^*, x\rangle - f(x)\}$$

The biconjugate of f is the function $f^{**} := (f^*)^*$.

Lemma 1. *Let H be a Hilbert space and $f : H \to [-\infty, \infty]$. Then, f^{**} is the largest convex lower semicontinuous minorant of f. That is, for any convex lower semicontinuous function g such that $g \le f$, $g \le f^{**}$.*

Proof. See **Proposition 4.1** of **Ekeland**. □

Theorem 2. *Let H be a Hilbert space and $f, g : H \to \mathbb{R}$ be arbitrary functionals. Then,*

$$f^{**} + g^{**} \le (f + g)^{**}.$$

In particular, if g is affine continuous, then

$$f^{**} + g^{**} = (f + g)^{**}$$

Proof. By Lemma 1, f^{**}, g^{**} are convex and lower semicontinuous and $f \le f^{**}$, $g \le g^{**}$. Observe that $f^{**} + g^{**}$ is a convex minorant of $f + g$, and is also lower semicontinuous by the superadditivity of the infimum. Hence, by definition,

$$f^{**} + g^{**} \le (f + g)^{**}.$$

Now, if g is affine and continuous, both g and $-g$ are convex and lower semicontinuous. Hence, by Lemma 1, $g^{**} = g$ and $(-g)^{**} = -g$, so

$$(f + g)^{**} - g = (f + g)^{**} + (-g)^{**} \le f^{**},$$

which rearranges to

$$(f + g)^{**} \le f^{**} + g^{**}.$$

Combining this with the first inequality in the theorem yields the equality. □

4 Separable additivity of biconjugates

Definition 4. *Let X be some space. The function $f : X \to [-\infty, \infty]$ is proper if $f(x) \neq -\infty$ for all $x \in X$ and there exists $x \in X$ such that $f(x) < +\infty$.*

Definition 5. *Let $(X, \langle, \rangle)$ be a real Hilbert space. The functional $f : X \to [-\infty, \infty]$ has a continuous affine minorant if there exists $a \in X$ and $b \in \mathbb{R}$ such that for all $x \in X$:*

$$f(x) \ge \langle a, x\rangle + b$$

Lemma 2. *Let $(X, \langle , \rangle)$ be a real Hilbert space and suppose $f : X \to [-\infty, \infty]$ is proper and has a continuous affine minorant. Then, both $f^*, f^{**} : X \to [-\infty, \infty]$ are also proper and have affine minorants.*

Proof. See **Lemma 2.1** of **Klein** □

The following theorem is essentially a double application of **Proposition 13.30** of **Bauschke2010** where extra conditions are imposed to guarantee g^{**} is proper. The proof follows that of **Theorem 2.1** in **Klein** albeit with a modified definition of proper function to match **Bauschke2010**. As with **Bauschke2010**, we allow $\sup\{\cdot\}$ to take values $\pm\infty$.

Theorem 3. *Let $X_1, X_2, \ldots X_n$ be real Hilbert spaces with inner products $\langle , \rangle_{i=1,2\ldots n}$ respectively and $X = X_1 \times X_2 \times \ldots X_n$. Suppose $g : X \to (-\infty, \infty]$ is a proper separable function defined by*

$$g(x) = \sum_{i=1}^{n} g_i(x_i), \quad \textit{for any } x = (x_1, x_2, \ldots, x_n) \in X,$$

where $g : X := g_i : X_i \to (-\infty, \infty]$, $i = 1, 2, \ldots n$ are proper functions with continuous affine minorants. Then, g^{**} is proper and for all $x \in X$,

$$g^{**}(x) = \sum_{i=1}^{n} g_i^{**}(x_i),$$

where g_i^{**} are proper functions.

Proof. Let $x = (x_1, \ldots x_n)$ and $y = (y_1, \ldots y_n)$ be elements of X. Observe that X equipped with the inner product

$$\langle x, y \rangle := \sum_{i=1}^{n} \langle x_i, y_i \rangle_i$$

defines a Hilbert space. Computing the Fenchel conjugate, we obtain:

$$\begin{aligned} g^*(x^*) &= \sup_{x \in X} \{\langle x^*, x \rangle - g(x)\} \\ &= \sup_{x_1, x_2 \ldots x_n} \left\{ \sum_{i=1}^{n} \langle x_i^*, x_i \rangle - g_i(x_i) \right\} \\ &= \sum_{i=1}^{n} \sup_{x_i \in X_i} \{\langle x_i^*, x_i \rangle - g_i(x_i)\} \\ &= \sum_{i=1}^{n} g_i^*(x_i) \end{aligned}$$

Observe that g^* is separable and $g_i^* : X_i \to [-\infty, \infty]$ are proper with continuous affine minorants by Lemma 2. Thus, reapplying the above line of reasoning yields

$$g^{**}(x) = \sum_{i=1}^{n} g_i^{**}(x_i),$$

where again, g_i^{**} are proper by Lemma 2. Finally, if g_i, $i = 1, 2 \ldots n$ have continuous affine minorants $\langle a_i, x \rangle + b_i$, then $\langle \sum_{i=1}^{n} a_i, x_i \rangle + \sum_{i=1}^{n} b_i$ is a continuous affine minorant of the proper function g, so g^{**} is proper by Lemma 2. □

Lemma 3. *Suppose X_i, $i = 1, 2 \dots n$ is a family of real Hilbert spaces with inner products $\langle \cdot, \cdot \rangle_i$ and denote the product Hilbert space by $\boldsymbol{X} = \bigoplus_{i=1}^n X_i$ (that is $\prod_{i=1}^n X_i$ with inner product $\sum_{i=1}^n \langle \cdot, \cdot \rangle_i$). Let $f : \boldsymbol{X} \to \mathbb{R}$ be an additively separable function;*

$$f(\boldsymbol{x}) = \sum_{i=1}^n f_i(x_i),$$

where $f_i : X_i \to \mathbb{R}$. Let $\boldsymbol{x} = (x_1, \dots x_n) \in dom\ f$. As with **Bauschke2010**, *we define $R_i : X_i \to \boldsymbol{X}$ such that the j^{th} component of $R_i y$ is y if $j = i$ and x_j otherwise. Then, if $u_i \in \partial_i f(\boldsymbol{x})$ for $i = 1, 2, \dots n$, then $\boldsymbol{u} = (u_1, \dots, u_n) \in \partial f(\boldsymbol{x})$.*

Proof. By definition, for all $i = 1, 2, \dots n$ and $y_i \in X_i$,

$$\begin{aligned}
&\langle y_i - x_i, u_i \rangle + (f \circ R_i)(x_i) \leq (f \circ R_i)(y_i) \\
&\langle y_i - x_i, u_i \rangle + \sum_{j=1}^n f(x_j) \leq f(y_i) + \sum_{j \neq i}^n f(x_j) \\
&\langle y_i - x_i, u_i \rangle + f(x_i) \leq f(y_i).
\end{aligned}$$

Adding these inequalities together, we have that for all $\boldsymbol{y} \in \boldsymbol{X}$,

$$\begin{aligned}
&\sum_{i=1}^n \langle y_i - x_i, u_i \rangle + \sum_{i=1}^n f(x_i) \leq \sum_{i=1}^n f(y_i) \\
&\langle \boldsymbol{y} - \boldsymbol{x}, \boldsymbol{u} \rangle + f(\boldsymbol{x}) \leq f(\boldsymbol{y}),
\end{aligned}$$

thus, $\boldsymbol{u} \in \partial f(\boldsymbol{x})$. $\square$