Option Valuation with the Fast Fourier Transform Summary

Andrew Wu

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1 Motivation

1.1 Deficiencies of Black-Scholes

- GBM assumption in the standard Black-Scholes model is insufficient for the market as returns tend to be more peaked with longer tails than the normal distribution (leptokurtic) (and volatility smile).
- Want to use more complicated underlying processes that can include jumps and account for leptokurtic returns.
- Complicated underlying processes may have complicated densities (with no closed form), so the standard way of pricing a call via integration is difficult:

$$C_T(k) = e^{-rT} \int_k^\infty (e^s - e^k) q_T(s) ds$$

  where $q_T(s)$ is the density of the terminal log-stock price.

1.2 Fourier methods

- Characteristic functions usually more well behaved than the density and may have closed form.
- Goal: Express Fourier transform of call price (or components of it) in terms of the characteristic function of terminal log-stock ϕ_T, then use the inverse Fourier transform to recover the call price.
- For example: Scott 1997, Bakshi/Madan 1999 give the formulas

$$\text{Delta} := \Pi_1 = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re \left(\frac{e^{-iu \ln K} \phi_T(u-i)}{iu\phi_T(-i)} \right) du.$$
$$\mathbb{P}\left(S_T > K\right) := \Pi_2 = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \operatorname{Re} \left(\frac{e^{-iu \ln K} \phi_T(u)}{iu} \right) du.$$
$$\text{Price} = S_0 \Pi_1 - K e^{-rT} \Pi_2.$$

- Directly computing the integrals above is slow and doesn't take advantage of FFT algorithms.

2 Carr and Madan's Fourier method

2.1 Notation and definitions

Let:

- $s_T = \ln(S_T)$, $k = \ln(K)$ be the log terminal stock and log-strike prices respectively.
- ϕ_T be the characteristic function of the log-stock.
- $C_T(k)$ be the price of a call option with maturity T and log-strike k.

2.2 Modified call

- Call price $C_T(k)$ is not integrable, multiply by factor $e^{\alpha k}$ where $\alpha > 0$ is a parameter to be specified. Define the modified call option $c_T(k)$:

$$c_T(k) = e^{\alpha k} C_T(k).$$

- Express the Fourier transform $\psi_T(v)$ of modified call in terms of characteristic function.

$$\psi_T(v) = \frac{e^{-rT}\phi_T(v - (\alpha+1)i)}{\alpha^2 + \alpha - v^2 + i(2\alpha+1)v}$$

  Notice that if $\alpha = 0$ we would have a singularity at $v = 0$. Also, DFT needs to evaluate ψ_T at $v = 0$.

- The call price can be recovered by taking the inverse Fourier Transform and dividing by the modification factor and some symmetry arguments since $C_T(k)$ is real:

$$C_T(k) = \frac{e^{-\alpha k}}{\pi} \int_0^\infty e^{-ivk} \psi_T(v) dv.$$

2.3 Fast Fourier Transform

- Definition of DFT: The Discrete Fourier Transform (DFT) transforms a sequence of complex numbers $\{x_j\}_{j=0,1,\ldots N-1}$ into the sequence $\{X_k\}_{k=0,1,\ldots N-1}$, where

$$X_k = \sum_{j=0}^{N-1} e^{-2\pi i kj/N} x_j. \tag{1}$$

- Suppose we use N partition points $v_j = \eta j$, $j = 0, 1, \ldots N$, where η is the spacing size and $a = N\eta$ is the upper integration limit (need to truncate the integral as infinite upper bound):
- Then we have the following DFT approximation

$$C_T(k) \approx \frac{e^{-\alpha k}}{\pi} \int_0^a e^{-ikv} \psi_T(v) dv \approx \sum_{j=0}^{N-1} e^{-ikv_j} \psi_T(v_j) \eta.$$

- Now define a series of log-strikes in order to use FFT. Let

$$k_u = -\frac{N\lambda}{2} + u\lambda$$

  where $u = 0, 1, 2 \ldots N-1$, yielding log-strikes uniformly spaced in $[-\frac{N\lambda}{2}, \frac{N\lambda}{2})$.

- To use DFT, set

$$\lambda = \frac{2\pi}{N\eta}$$

  which unfortunately yields a tradeoff between integral accuracy (smaller η) and increasing strike spacing (larger η).

- Use Simpsons rule weights to achieve more accurate integral estimate:

$$C(k_u) \approx \frac{e^{-\alpha k_u}}{\pi} \sum_{j=0}^{N-1} e^{-2\pi i j u/N} e^{\pi j} \psi_T(v_j) \cdot \frac{\eta}{3} \left[3 + (-1)^{j+1} - \delta_j\right].$$

3 Application and Implementation

3.1 Variance gamma process

- The stock price is assumed to be driven by a Variance-Gamma process, a 1-D pure-jump Markov process:

$$S_t = S_0 \exp\{(r + \omega)t + X_t(\sigma, \theta, \nu)\},$$

- By default, we set

$$\omega = \frac{1}{\nu} \ln(1 - \theta\nu - \frac{1}{2}\sigma^2\nu),$$

  so that the mean rate of return on the stock is r.

- X_t is calculated by evaluating an arithmetic Brownian motion with drift θ and volatility σ at a random time $\Gamma(t; 1, \nu)$ which is a Gamma process, a pure jump process with independent increments that follow a gamma distribution.

$$X_t(\theta, \sigma, \nu) = \theta\Gamma(t; 1, \nu) + \sigma W(\Gamma(t; 1, \nu)).$$

- Variance Gamma is supported in general equilibrium model and addresses volatility smile/no jumps in the traditional GBM assumption of the Black-Scholes Model.

- Variance Gamma also has complicated density (not closed form but analytic due to Bessel function), however characteristic function is relatively simple, so it is ideal for testing this approach.

- Characteristic function of terminal log-stock s_T is:

$$\phi_T(u) = (1 - i\theta\nu u + \frac{1}{2}\nu\sigma^2 u^2)^{-T/\nu} \exp\{(\ln(S_0) + (r + \omega)T) iu\}$$

3.2 Code

- Used parameter combination 4 in Carr Madan paper:

$$r = 0.05, S_0 = 100, \sigma = 0.25, \nu = 2, \theta = -0.1.$$

- For GBM, Fast Fourier Transform often gave an error of between 1 - 2.

- Fast Fourier Transform gave a significant time saving but accuracy was poor across all Fourier methods for shorter maturities, making comparison difficult.

4 Conclusion

- FFT has potential to be useful, perhaps some better quadrature rules can be used than Simpson's rule to increase accuracy.
- More robust than some of the other Fourier methods, which resulted in large errors due to the pole at 0 in the integral.
- However, computes unnecessary stock prices that may be far out or in the money (although still faster) due to the requirement for no. of strikes = no. partition points in quadrature.
- Note: Call option when VG used as underlying has closed form in terms of confluent hypergeometric functions but I was unable to implement this in Python without errors in my integrals. Matsuda (2004) experienced the same issues.