Mathematical Programming 48 (1990) 161–220
North-Holland

FINITE-DIMENSIONAL VARIATIONAL INEQUALITY AND NONLINEAR COMPLEMENTARITY PROBLEMS: A SURVEY OF THEORY, ALGORITHMS AND APPLICATIONS

Patrick T. HARKER*

Decision Sciences Department, The Wharton School, University of Pennsylvania, Philadelphia, PA 19104-6366, USA

Jong-Shi PANG**

Department of Mathematical Sciences, The Whiting School of Engineering, The Johns Hopkins University, Baltimore, MD 21218, USA

Received 18 January 1988
Revised 17 February 1989

Over the past decade, the field of finite-dimensional variational inequality and complementarity problems has seen a rapid development in its theory of existence, uniqueness and sensitivity of solution(s), in the theory of algorithms, and in the application of these techniques to transportation planning, regional science, socio-economic analysis, energy modeling, and game theory. This paper provides a state-of-the-art review of these developments as well as a summary of some open research topics in this growing field.

Key words: Variational inequality, complementarity, fixed points, Walrasian equilibrium, traffic assignment, network equilibrium, spatial price equilibrium, Nash equilibrium.

1. Introduction

The computation of economic and game theoretic equilibria has been of great interest in the academic and professional communities ever since the path-breaking paper by Lemke and Howson [150] and the seminal work by Herbert Scarf [236] in the mid-1960's and early 1970's. The initial impetus for research on computing equilibria came from the need to empirically analyze general equilibrium theory and to apply this theory to study problems of taxation, unemployment, etc. In recent years, the growth of experimental economics and the use of sophisticated strategic planning models by industry has revitalized the need for efficient methods to analyze and numerically solve models of economic and game theoretic equilibria.

The initial methods which were used to compute economic equilibria were all based on the ingenious constructive proof by Lemke and Howson of the existence

* The research of this author was supported by the National Science Foundation Presidential Young Investigator Award ECE-8552773 and by the AT&T Program in Telecommunications Technology at the University of Pennsylvania.

** The research of this author was supported by the National Science Foundation under grant ECS-8644098.

of an equilibrium point for a bimatrix game, and have come to be known as fixed-point (or homotopy-based) methods. There were many well-known pioneers in this field: Scarf [235] introduced the notion of primitive sets and described the first algorithm to approximate a fixed point of a continuous mapping; Kuhn [139] and Hansen [89] were responsible for the introduction of simplexes; Eaves [50, 51] introduced piecewise-linear maps into the computational fixed-point literature. The classical reference by Todd [259] and the more recent text by Garcia and Zangwill [81] provide detailed discussions of these techniques. Many applications have been and continue to be made of these methods [90, 159, 173, 238, 239, 265]. While theoretically very powerful, homotopy-based methods have experienced difficulty in solving medium to large-scale equilibrium models. The PIES (Project Independence Evaluation System) energy model [5] which was developed at the U.S. Department of Energy in the late 1970's provided a useful piece of practical evidence demonstrating the inability of the fixed-point methods in handling real-life applications.

The other traditional approach for solving equilibrium models is nonlinear optimization (*equilibrium programming* in [81]). As discussed in Carey [27], this approach requires very restrictive assumptions on the model in order to work. Thus, solving equilibrium models as optimization problems does not provide a satisfactory alternative to the fixed point/homotopy methods.

In summary, the optimization and fixed point approaches each have their own advantages and disadvantages when applied to solve equilibrium models; separately, they either lack the generality or the computational efficiency which is necessary for solving large-scale equilibrium problems. In recent years, finite-dimensional variational inequality and nonlinear complementarity problems have emerged as very promising candidates for filling the gap created by the optimization and fixed point approaches. It is with this motivation that we undertake the current research.

To a large extent, the PIES model and the associated PIES algorithm have provided the impetus for the growth of the field of finite-dimensional variational inequality and nonlinear complementarity problems. Historically, the variational inequality problem was introduced by Philip Hartman and Guido Stampacchia in the seminal paper [107], and was subsequently expanded by Stampacchia in several classic papers [152, 155, 249]. For the most part, these early studies of the variational inequality problem were set in the context of calculus of variations/optimal control theory and in connection with the solution of boundary value problems posed in the form of partial differential equations; the books by Kinderlehrer and Stampacchia [130] and Baiocchi and Capelo [15] provide a thorough introduction to these applications of variational inequalities in infinite-dimensional metric spaces. The nonlinear complementarity problem, on the other hand, first appeared in Richard Cottle's Ph.D. dissertation [31], which was later published in [32]. The name "complementarity problem" was not used by Cottle in these two references, but was coined in a subsequent paper dealing with the linear case by Cottle, Habetler and Lemke [36]. In the early years of study on the nonlinear complementarity problem, most effort was concentrated on the question of the existence of solutions and to some extent, the development of algorithms; applications were rarely considered. During those years, much of the research on the nonlinear complementarity problem was overshadowed by the enthusiastic attention given to the special case of the linear complementarity problem. Understandably, this effort devoted to the linear case was due mainly to the ingenious discovery by Lemke and Howson [150] of the first constructive method for solving bimatrix games formulated as linear complementarity problems which, as mentioned above, led to the development of the entire family of fixed-point methods.

Since the beginning, it was recognized that the nonlinear complementarity problem is a special case of the variational inequality problem (see Karamardian [125]). Despite this recognition, the early developments of the two problems followed different paths. The most notable distinction is the setting in which the two problems were studied: variational inequalities in infinite-dimensional metric spaces and nonlinear complementarity in finite-dimensional Euclidean space. An attempt to bring the two areas closer together resulted in the conference held in Erice, Italy in 1978; the proceedings of this conference is available [35].

The finite-dimensional variational inequality problem (and in particular, the nonlinear complementarity problem) achieves its present-day status as a lively and fruitful area of research through the evolution of three major events, none of which are related to the infinite-dimensional version of this problem. First is the experience of the PIES model cited above. The second event is the publication of a paper by Michael J. Smith [241] which formulated the traffic assignment problem as a finite-dimensional variational inequality. (Actually, Smith set up the traffic assignment problem without realizing that his formulation was a variational inequality problem. It was S. Dafermos [39] who recognized Smith's formulation as a variational inequality problem. The related papers [58, 59] by S.C. Fang were initially written around the same time as Dafermos' paper [39], but were published a few years later.) Since the appearance of these papers, numerous models of spatial and aspatial equilibrium have appeared in the literature and have been used in practice: the prediction of interregional commodity flows [76, 200]; the solution of Nash equilibria [80, 91]; the prediction of freight transport prices and service [97]. The last event, which is a part of the renewed interest in computational methods in economics, was initiated by Lars Mathiesen who attempted to solve the Walrasian or general equilibrium model of economic activities with some of the recent techniques developed for the nonlinear complementarity problem. While achieving very encouraging numerical results [168, 169], Mathiesen's findings are puzzling in that the methods are not proven convergent in such applications. Mathiesen was not alone in his discovery; Preckel [210], Rutherford [231] and Stone [252] all obtained similar results which further support the benefits of the variational inequality/nonlinear complementarity problem approach for solving large-scale equilibrium problems. Furthermore, Preckel's [210] paper contains an interesting comparison between the homotopy and nonlinear complementarity techniques which provides another piece of evidence of the weakness of the former approach in practice. There already exist several articles that attempt to provide an understanding of these empirical results [55, 211, 232]; more work is sure to emerge.

Despite the surge of interest in and importance of finite-dimensional variational inequality and complementarity problems, there exists no single reference which gives a combined treatment of the theory, algorithms and applications of these two problems. In this regard, there are two related papers. Robinson [220] gives a brief review of "generalized equations", and Nagurney [191] provides a very limited discussion of the application of variational inequalities to regional science. It is the primary objective of this paper to provide a comprehensive, state-of-the-art review of finite-dimensional variational inequality and nonlinear complementarity problems. Our plan is to begin with the very fundamental results and to proceed gradually toward the frontier. Due to space limitations, much of our discussion is brief; the omitted details (proof, specialized results, examples, etc.) can be found in our forthcoming book [106].

The organization of the remainder of the paper is as follows. In Section 2, we formally state the finite-dimensional variational inequality and complementarity problems and present some fundamental facts concerning these problems. Section 3 develops the theory on the existence and uniqueness of solutions for the two problems, and various iterative algorithms for solving the problems are presented in Section 4. Sensitivity and stability results will be discussed in Section 5, and Section 6 will present the generalizations of the two problems to include point-to-set maps as well as the extension of mathematical programs to include variational inequality and/or complementarity constraints. Applications of the problems and their generalizations to various real-life situations will be surveyed in Section 7, and a list of research topics in this area which we feel deserve further investigation is given in Section 8. Finally, the Reference section contains the most comprehensive bibliography in this area to date.

2. Problem definitions and basic facts

In this section, we define the variational inequality and nonlinear complementarity problems, state their interrelationships, and describe their connections with other well-known problems in mathematical programming. Since this paper is restricted to problems in finite-dimensional Euclidean space $\mathbb{R}^n$, the term *finite-dimensional* will be dropped in the following discussion.

Definition 2.1. Let X be a nonempty subset of $\mathbb{R}^n$ and let F be a mapping from $\mathbb{R}^n$ into itself. The *variational inequality problem*, denoted by VI(X, F), is to find a vector $x^* \in X$ such that

$$F(x^*)^{\mathrm{T}}(y - x^*) \geqslant 0 \quad \text{for all } y \in X. \tag{1}$$

One typically assumes that the set X is closed and convex; in fact, X is often polyhedral in applications.

Loosely speaking, the variational inequality (1) states that the vector $F(x^*)$ must be at an acute angle with all *feasible* vectors emanating from x^*. Formally, $x^* \in X$ is a solution to VI(X, F) if and only if $F(x^*)$ is inward normal to X at x^*; i.e., $-F(x^*)$ belongs to the normal cone $N_X(x^*)$ where:

$$N_X(x) = \begin{cases} \{z\colon (y-x)^{\mathrm{T}} z \leq 0 \ \forall y \in X\} & \text{if } x \in X, \\ \emptyset & \text{otherwise.} \end{cases} \tag{2}$$

The simplest example of a variational inequality problem is that of solving a system of equations

$$F(x) = 0 \tag{3}$$

where F maps $\mathbb{R}^n$ into itself. It is easy to show that if $X = \mathbb{R}^n$, then x^* is a solution to VI(X, F) if and only if x^* solves (3) since the only vector $F(x^*)$ which is at an acute angle with all vectors in $\mathbb{R}^n$ is the zero vector.

The second example of a variational inequality problem is that of differentiable optimization [225, 155]. If $F(x)$ is the gradient of a real-valued differentiable function $f\colon \mathbb{R}^n \to \mathbb{R}$ and if X is convex, then VI(X, F) is just a restatement of the first-order necessary conditions of optimality for the following optimization problem:

$$\text{minimize } f(x) \quad \text{subject to } x \in X. \tag{4}$$

More specifically, x^* solves the problem VI(X, F) if and only if there is no feasible descent direction at x^*. Furthermore, if $f(x)$ is a *pseudo-convex* function (i.e., $\nabla f(x)^{\mathrm{T}}(y-x) \geq 0$ implies $f(y) \geq f(x)$ for all $x, y \in X$), then any solution to VI(X, F) is a global optimal solution of (4).

The above connection between the variational inequality problem and an optimization problem breaks down if F is not a gradient mapping. According to the *symmetry principle* [194, Theorem 4.1.6], if F is continuously differentiable, then a necessary and sufficient condition for F to be the gradient mapping of the function $f\colon \mathbb{R}^n \to \mathbb{R}$ is that the Jacobian matrix $\nabla F(x)$ is symmetric for all $x \in \mathbb{R}^n$. In such a case, the real-valued mapping $f\colon \mathbb{R}^n \to \mathbb{R}$ is defined by

$$f(x) = \int_0^1 F(x^0 + t(x - x^0))^{\mathrm{T}}(x - x^0)\,\mathrm{d}t \tag{5}$$

where x^0 is an arbitrary fixed vector in $\mathbb{R}^n$. Consequently, the symmetry of ∇F provides the key property under which the problem VI(X, F) is related to the differentiable mathematical program (4). Indeed, the general question of whether or not one can formulate an arbitrary variational inequality problem as a differentiable mathematical program in the absence of the symmetry assumption remains unanswered. In Carey [27], the implications of the symmetry assumption (also called the *integrability property* due to the integral formula (5)) are more fully discussed in the context of economic equilibria.

An important special case of VI(X, F) is the nonlinear complementarity problem NCP(F):

Definition 2.2. Let F be a mapping from $\mathbb{R}^n$ into itself. The *nonlinear complementarity problem*, denoted by NCP(F), is to find a vector $x^* \in \mathbb{R}^n_+$ such that

$$F(x^*) \in \mathbb{R}^n_+ \quad \text{and} \quad F(x^*)^{\mathrm{T}} x^* = 0. \tag{6}$$

In Definition 2.2 and throughout the paper, $\mathbb{R}^n_+$ denotes the nonnegative orthant of $\mathbb{R}^n$. When $F(x)$ is an affine function of x, say $F(x) = q + Mx$ for some given vector $q \in \mathbb{R}^n$ and matrix $M \in \mathbb{R}^{n \times n}$, the problem NCP($F$) reduces to the *linear complementarity problem*, which is denoted by LCP(q, M); the reader is referred to Murty [187] for an extensive treatment of the latter problem.

Geometrically, the nonlinear complementarity problem involves finding a nonnegative vector x^* such that the image $F(x^*)$ is also nonnegative and is orthogonal to x^*. By replacing $\mathbb{R}^n_+$ with an arbitrary convex cone, one can define the *generalized complementarity problem*:

Definition 2.3. Let X be a convex cone in $\mathbb{R}^n$ and let F be a mapping from $\mathbb{R}^n$ into itself. The *generalized complementarity problem*, denoted by GCP(X, F), is to find a vector $x^* \in X$ such that

$$F(x^*) \in X^* \quad \text{and} \quad F(x^*)^{\mathrm{T}} x^* = 0, \tag{7}$$

where X^* denotes the *dual cone* of X, i.e.

$$X^* = \{y \in \mathbb{R}^n : y^{\mathrm{T}} x \geq 0 \ \forall x \in X\}.$$

Karamardian [125] was the first to establish the following relationship between the generalized (and hence the nonlinear) complementarity problem and the variational inequality problem.

Proposition 2.1 [125]. *Let X be a convex cone. Then $x^* \in X$ solves the problem* VI(X, F) *if and only if x^* solves the* GCP(X, F).

Therefore, every generalized (and in particular, every nonlinear) complementarity problem is a variational inequality problem, but the converse is not true in general. Nevertheless, in the case where the set X is defined by inequalities of the form

$$X = \{x \in \mathbb{R}^n : g_i(x) \leq 0, \ i = 1, 2, \ldots, m; \ h_j(x) = 0, j = 1, 2, \ldots, p\}, \tag{8}$$

then the variational inequality VI(X, F) can be converted into a certain generalized complementarity problem provided that the functions $g : \mathbb{R}^n \to \mathbb{R}^m$ and $h : \mathbb{R}^n \to \mathbb{R}^p$ in (8) satisfy some standard constraint qualifications of the type often imposed in nonlinear programming [13]. One such qualification is the requirement that the functions g and h be affine; this case of course corresponds to X being a polyhedral set. The following result which summarizes this conversion has been used in differing contexts by several authors, including Eaves [48, 52, 53], Tobin [258] and Qui and Magnanti [212, 213].

Proposition 2.2. *Let $g:\mathbb{R}^n \to \mathbb{R}^m$ and $h:\mathbb{R}^n \to \mathbb{R}^p$ be continuously differentiable and let X be defined by* (8).

(a) *If x^* solves the* VI(X, F) *and if a certain constraint qualification holds for the set X at the point x^*, then for some $\pi^* \in \mathbb{R}^m$ and $\mu^* \in \mathbb{R}^p$, (x^*, π^*, μ^*) solves the* GCP$(\mathbb{R}^n \times \mathbb{R}^m_+ \times \mathbb{R}^p, H)$ *where $H:\mathbb{R}^{n+m+p} \to \mathbb{R}^{n+m+p}$ is defined by*

$$H\begin{pmatrix} x \\ \pi \\ \mu \end{pmatrix} = \begin{pmatrix} F(x) + \sum_{i=1}^m \pi_i \nabla g_i(x) + \sum_{j=1}^p \mu_j \nabla h_j(x) \\ -g(x) \\ h(x) \end{pmatrix}. \tag{9}$$

(b) *Conversely, if g_i is convex for $i = 1, 2, \ldots, m$ and h_j is affine for $j = 1, 2, \ldots, p$, and if (x^*, π^*, μ^*) solves the* GCP$(\mathbb{R}^n \times \mathbb{R}^m_+ \times \mathbb{R}^p, H)$, *then x^* solves the* VI(X, F).

Because of the special form of the defining set, the GCP$(\mathbb{R}^n \times \mathbb{R}^m_+ \times \mathbb{R}^p, H)$ in the above proposition may be considered as a *mixed nonlinear complementarity problem* due to the facts that some variables (x and μ) are not restricted to be nonnegative and that the corresponding components of the H-mapping are restricted to be equal to zero. Notice that the only assumptions in Proposition 2.2 are on the functions g and h; in particular, no assumption on the mapping F is required. One drawback with the conversion of the variational inequality problem into the complementarity framework is the increase in the number of variables from n to $(n+m+p)$. From a computational viewpoint, this increase in the problem dimension could have an adverse effect on a solution scheme for solving the variational inequality problem. Nevertheless, we shall see later that the mixed nonlinear complementarity formulation of the variational inequality problem provides a very useful vehicle for analyzing the latter problem in many contexts.

In order to establish many of the existence results in the next section and to pave the way for the development of a large family of iterative methods in Section 4, it is useful to reformulate the variational inequality problem as a classical fixed-point problem. For this purpose, let us define the notion of projection:

Definition 2.4. Let X be a nonempty, closed and convex subet of $\mathbb{R}^n$ and let G be any $n \times n$ symmetric positive definite matrix. Then the *projection under the G-norm* of a point $y \in \mathbb{R}^n$ onto the set X, denoted as $\mathrm{pr}_{G,X}(y)$, is defined as the solution (which must exist and be unique) to the following mathematical program:

$$\underset{x \in X}{\text{minimize}} \quad \|y - x\|_G \tag{10}$$

where $\|x\|_G = (x^{\mathrm{T}} G x)^{1/2}$ denotes the G-norm of a vector $x \in \mathbb{R}^n$.

Using this definition, it is easy to establish the following:

Proposition 2.3 [48]. *Let X be a nonempty closed convex subset of $\mathbb{R}^n$ and let G be any $n \times n$ symmetric positive definite matrix. Then x^* solves the problem* VI(X, F) *if* *and only if*

$$x^* = \mathrm{pr}_{G,X}(x^* - G^{-1}F(x^*)); \tag{11}$$

i.e., if and only if x^* is a fixed point of the mapping $H : \mathbb{R}^n \to \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$H(x) = \mathrm{pr}_{G,X}(x - G^{-1}F(x)). \tag{12}$$

In the context of the nonlinear complementarity problem NCP(F) (where $X = \mathbb{R}^n_+$), the mapping (12) takes on a simple form:

$$H(x) = \max(0, x - F(x)) \tag{13}$$

with G taken to be the identity matrix. In general, if $H : \mathbb{R}^n \to \mathbb{R}^n$, then x^* is a fixed point of H if and only if x^* is a zero of the mapping $\tilde{H}(x) = H(x) - x$. Thus, both the variational inequality and nonlinear complementarity problems can be formulated as the classical problem of solving a system of nonlinear equations. Mangasarian [156] established the following generalized formulation of the nonlinear complementarity problem as that of solving a system of equations.

Proposition 2.4 [156]. *Let $\theta : \mathbb{R} \to \mathbb{R}$ be any strictly increasing function with $\theta(0) = 0$. Then a vector $x^* \in \mathbb{R}^n$ solves the* NCP(F) *if and only if $\tilde{H}(x^*) = 0$ where $\tilde{H} : \mathbb{R}^n \to \mathbb{R}^n$ is defined by*

$$\tilde{H}_i(x) = \theta(|F_i(x) - x_i|) - \theta(F_i(x)) - \theta(x_i) \quad \forall i = 1, 2, \ldots, n. \tag{14}$$

By taking $\theta(t) = t$, it is not difficult to show that the mapping $\tilde{H}(x)$ in (14) reduces to $H(x) - x$ with H being defined in (13). Subramanian [253, 254] has explored the direct solution of (14) via the classical Newton's method for nonlinear equations, and Garcia and Zangwill [81, Chapter 19] provide an alternative formulation of the complementarity problem as a system of nonlinear equations.

The next set of relationships to be discussed in this section involves the reformulation of the complementarity and variational inequality problems as optimization problems. As was mentioned previously, the symmetry assumption plays a crucial role in this regard for the variational inequality problem. However, this assumption is not needed for the nonlinear complementarity problem as the following result illustrates.

Proposition 2.5 [32]. *Let $F : \mathbb{R}^n \to \mathbb{R}^n$. A vector $x^* \in \mathbb{R}^n$ solves the* NCP(F) *if and only if x^* solves*

$$\text{minimize } F(x)^{\mathrm{T}}x \quad \text{subject to } F(x) \geqslant 0, \; x \geqslant 0, \tag{15}$$

and $F(x^*)^{\mathrm{T}}x^* = 0$.

In general, the feasible region of (15) is not convex; however, in the special case of F being affine (i.e., in the case of the linear complementarity problem), problem (15) is a quadratic program. Indeed, in this case, program (15) has been used extensively for the study of the LCP [37].

For the variational inequality problem VI(X, F), the *gap function* provides an optimization problem formulation:

Definition 2.5 [111]. The *gap function* $g(x)$ associated with problem VI(X, F) is defined by

$$g(x) = \underset{y \in X}{\text{maximum}} \; F(x)^{\mathrm{T}}(x-y). \tag{16}$$

Note that $g(x) \geqslant 0$ for all $x \in X$ and that $g(x) = 0$ if and only if x is a solution to the VI(X, F). Thus, the following mathematical programming formulation of any variational inequality problem can be established.

Proposition 2.6 [111]. *The vector $x^* \in X$ is a solution to the* VI(X, F) *if and only if x^* solves*

$$\underset{x \in X}{\text{minimize}} \; g(x) = \underset{x \in X}{\text{minimize}} \; \underset{y \in X}{\text{maximum}} \; F(x)^{\mathrm{T}}(x-y) \tag{17}$$

and $g(x^*) = 0$.

It is easy to see that the program (17) reduces to (15) in the case of the nonlinear complementarity problem with $X = \mathbb{R}^n_+$.

Thus, the gap function provides a minimax formulation of the variational inequality problem. Unfortunately, like almost all minimax problems of interest, the gap function leads to a nondifferentiable optimization problem. Other related optimization problem formulations of VI(X, F) exist [111, 112, 243]. In particular, the paper by Hearn et al. [112] provides a detailed discussion of the various formulations and their convexity properties, and the recent research by Marcotte [164, 165, 166, 167] exploits the gap function formulation of VI(X, F) in the design of globally convergent algorithms (see Section 4).

3. Existence and uniqueness theory

Over the past two decades, a large body of literature has developed on the existence and uniqueness of solutions to variational inequalities and nonlinear complementarity problems. Due to the diversity of results, it is not possible to list them all. In this section, we have chosen to provide the most fundamental results and some of their consequences; many other results can be found in the references listed at the end of the paper.

The most basic result on the existence of a solution to the variational inequality problem VI(X, F) requires the set X to be compact and convex, and the mapping F to be continuous. From this basic result, many corollaries can be derived by replacing the compactness of X with additional conditions on F. The basic existence result is stated in the following theorem which can be proved by applying Brouwer's fixed-point theorem to the mapping H defined in equation (12).

Theorem 3.1 [48, 107]. *Let X be a nonempty, compact and convex subset of $\mathbb{R}^n$ and let F be a continuous mapping from X into $\mathbb{R}^n$. Then there exists a solution to the problem* $\mathrm{VI}(X, F)$.

In the case where the set X is not bounded (as in the nonlinear complementarity problem), one must introduce some notion of the boundedness of the solution in order to establish the existence of at least one solution. This boundedness is accomplished in several ways. For example, if one can exhibit a bounded set D within X such that no point outside D is a candidate for solution, then one can establish the existence of a solution. Placing this argument into a rigorous statement, we have the following result.

Corollary 3.1 [48, 126, 181]. *Let X be a nonempty, closed and convex subset of $\mathbb{R}^n$ and let F be a continuous mapping from $\mathbb{R}^n$ into itself. If there exists a nonempty bounded subset D of X such that for every $x \in X \setminus D$ there is a $y \in D$ with*

$$F(x)^{\mathrm{T}}(x-y) \geqslant 0, \tag{18}$$

then the problem $\mathrm{VI}(X, F)$ *has a solution.*

In general, the a priori identification of the set D is difficult. However, if the mapping F possesses some additional properties, then the existence of D becomes a straightforward matter. To explain these properties, we introduce the following definitions:

Definition 3.1 [194]. The mapping $F: \mathbb{R}^n \to \mathbb{R}^n$ is said to be

(a) *monotone over a set X* if

$$[F(x)-F(y)]^{\mathrm{T}}(x-y) \geqslant 0 \quad \forall x, y \in X; \tag{19}$$

(b) *pseudo-monotone over X* if

$$F(y)^{\mathrm{T}}(x-y) \geqslant 0 \text{ implies } F(x)^{\mathrm{T}}(x-y) \geqslant 0 \quad \forall x, y \in X; \tag{20}$$

(c) *strictly monotone over X* if

$$[F(x)-F(y)]^{\mathrm{T}}(x-y) > 0 \quad \forall x, y \in X,\ x \neq y; \tag{21}$$

(d) *strongly monotone over X* if there exists an $\alpha > 0$ such that

$$[F(x)-F(y)]^{\mathrm{T}}(x-y) \geqslant \alpha \|x-y\|^2 \quad \forall x, y \in X; \tag{22}$$

(e) *coercive with respect to X* if there exists a vector $x^0 \in X$ such that

$$\lim_{x \in X, \|x\| \to \infty} F(x)^{\mathrm{T}}(x-x^0)/\|x\| = \infty \tag{23}$$

where $\|\cdot\|$ denotes any vector norm in $\mathbb{R}^n$.

Note that if F is strongly monotone over X, then it is coercive with respect to X. If $F(x)=q+Mx$ is affine and $X=\mathbb{R}^n$, then F is coercive if and only if it is strongly monotone, which in turn is equivalent to the positive definiteness of the matrix M. More generally, if F is continuously differentiable, then the various monotonicity properties of F are related to the positive semi-definiteness or positive definiteness of the Jacobian matrix $\nabla F(x)$ [194, Chapter 5.4]. Among these properties, it is clear that pseudo-monotonicity is the weakest, followed in order by monotonicity, strict monotonicity and strong monotonicity.

It is well-known that the solution set of a convex mathematical program is convex. The following generalizes this result to the case of a variational inequality problem of the pseudo-monotone type. The result was originally proven by Minty for the case of monotone variational inequality problems (in Hilbert space), and was later generalized slightly by Karamardian to the form stated below (for pseudo-monotone problems).

Proposition 3.1 [128, 180]. *Let X be a nonempty, closed and convex subset of* $\mathbb{R}^n$ and let *F be a continuous, pseudo-monotone mapping from X into $\mathbb{R}^n$. Then x^* solves the problem* VI(X, F) if and only if *$x^* \in X$ and*

$$F(y)^{\mathrm{T}}(y-x^*) \geqslant 0 \quad \forall y \in X. \tag{24}$$

In particular, the solution set of VI(X, F) *is convex if it is nonempty.*

In general, the variational inequality problem VI(X, F) can have more than one solution. If F is strictly monotone, then the problem VI(X, F) can have at most one solution:

Proposition 3.2. *If F is strictly monotone on X, then the problem* VI(X, F) *has at most one solution.*

Neither Proposition 3.1 nor 3.2 guarantees the existence of a solution to problem VI(X, F); a common approach to establish existence is to employ the coercivity property of F. Under this property, it is clear that the bounded set D required in Corollary 3.1 can be taken to be a sufficiently large ball intersecting with X. Thus, the existence of a solution follows from this corollary and moreover, the solution set can be shown to be compact:

Theorem 3.2 [107, 183]. *Let X be a nonempty, closed, convex subset of $\mathbb{R}^n$ and let F be a continuous mapping from X into $\mathbb{R}^n$. If F is coercive with respect to X, then the problem* VI(X, F) *has a nonempty, compact solution set.*

Coercivity can be slightly weakened by extending the result of Smith [245] for nonlinear complementarity problems to the more general variational inequality setting:

Theorem 3.3 [106]. *Let X be a nonempty, closed, convex subset of $\mathbb{R}^n$ and let F be a continuous mapping from X into $\mathbb{R}^n$. If there exists a vector $x^0 \in X$ such that the set*

$$\{x \in X: F(x)^{\mathrm{T}}(x - x^0) < 0\}$$

if nonempty, is bounded, then the problem VI(X, F) *has a solution. Moreover, if the closed set*

$$\{x \in X: F(x)^{\mathrm{T}}(x - x^0) \leqslant 0\}$$

is bounded, then the solution set is compact.

Since strong monotonicity implies both coercivity and strict monotonicity we can, by combining Theorem 3.2 and Proposition 3.2, derive the following existence and uniqueness result for variational inequality problems of the strongly monotone type.

Corollary 3.2. *Let X be a nonempty, closed, convex subset of $\mathbb{R}^n$ and let F be a continuous mapping from X into $\mathbb{R}^n$. If F is strongly monotone with respect to X, then there exists a unique solution to the problem* VI(X, F).

When the mapping F is pseudo-monotone or monotone, the variational inequality problem need not have a solution. However, if a certain Slater-type constraint qualification holds, then pseudomonotonicity is sufficient in order to establish existence; this result is simply an extension of the work by Karamardian [127] on the generalized complementarity problem. In stating the result, we need the notion of the dual cone X^* of an arbitrary set X; this is defined exactly as in the case where X is a convex cone (cf. Definition 2.3):

Theorem 3.4 [106]. *Let X be a nonempty, closed, convex subset of $\mathbb{R}^n$ and let F be a continuous mapping from $\mathbb{R}^n$ into itself. Suppose that F is pseudo-monotone with respect to X and that there exists a vector $x^0 \in X$ such that $F(x^0) \in \operatorname{int}(X^*)$, where* $\operatorname{int}(\cdot)$ *denotes the interior of the set. Then the problem* VI(X, F) *has a nonempty, compact, convex solution set.*

Karamardian [127] provides various characterizations for vectors in the interior of the dual cone X^*. We simply mention here that the existence of the vector $x^0 \in X$ such that $F(x^0) \in \operatorname{int}(X^*)$ allows one to easily construct the set D required in Corollary 3.1, thereby establishing Theorem 3.4.

In the case of the generalized complementarity problem GCP(X, F), the above results can be sharpened due to the extra structure imposed on the problem via X being a cone. First, we recall two properties of a cone:

Definition 3.2. A cone C is *pointed* if $C \cap (-C) = \{0\}$. A cone C is *solid* if it has a nonempty topological interior.

Definition 3.3. The *feasible set* of a generalized complementarity problem GCP(X, F) is defined as

$$\Omega(X, F) = \{x \in X: F(x) \in X^*\}. \tag{25}$$

Vectors in $\Omega(X, F)$ are said to be feasible, and the problem GCP(X, F) is said to be feasible if $\Omega(X, F)$ is nonempty.

Given these definitions, the following existence theorem for generalized complementarity problems can be stated.

Theorem 3.5 [128]. *Let X be a solid, pointed, closed, convex cone in $\mathbb{R}^n$. If F is continuous and strictly monotone with respect to X and if the* GCP(X, F) *is feasible, then the* GCP(X, F) *has a unique solution.*

Theorem 3.5 fails to hold if strict monotonicity is weakened to mere monotonicity; Megiddo [177] provides an example of a feasible, monotone nonlinear complementarity problem with no solution. Further results for the generalized complementarity problem, especially in the case of the linear complementarity problem (F being an affine mapping and $X = \mathbb{R}^n_+$) can be generated based upon Theorem 3.5; see [48, 49, 126, 128, 203].

Traditionally, the nonlinear complementarity problem NCP(F) was studied as a generalization of the linear complementarity problem. As such, many existence results for the former problem were derived as extensions of the corresponding results for the latter problem. In turn, much of the study of the linear complementarity problem is related to various classes of matrices; see [187]. By generalizing the properties of these matrix classes to general mappings, one obtains a host of special function classes. In what follows, we shall review some of these classes of mappings and their relationship to the nonlinear complementarity problem. Our first result of this type is an existence and uniqueness theorem by Cottle [32, 34]. It involves the notion of *positively bounded Jacobians*, and the original proof was constructive in the sense that an algorithm was employed to actually compute the unique solution.

Theorem 3.6 [32]. *Let $F : \mathbb{R}^n_+ \to \mathbb{R}^n$ be continuously differentiable and suppose that there exists a $\delta \in (0, 1)$ such that all principal minors of the Jacobian matrix $\nabla F(x)$ are bounded between δ and δ^{-1} for all $x \in \mathbb{R}^n_+$. Then the* NCP(F) *has a unique solution.*

Our next definition is motivated by the class of *copositive matrices* which in turn generalizes the class of nonnegative matrices.

Definition 3.4. A mapping $F : X \to \mathbb{R}^n$ is said to be

(a) *copositive with respect to X* if

$$[F(x) - F(0)]^{\mathrm{T}} x \geqslant 0 \quad \forall x \in X;$$

(b) *strictly copositive with respect to X* if

$$[F(x) - F(0)]^{\mathrm{T}} x > 0 \quad \forall x \in X,\ x \neq 0;$$

(c) *strongly copositive with respect to X* if there exists a scalar $\alpha > 0$ such that

$$[F(x) - F(0)]^{\mathrm{T}} x \geqslant \alpha \|x\|_2^2 \quad \forall x \in X.$$

The above copositivity properties are clearly the generalized versions of the corresponding monotonicity properties in Definition 3.1. Since a strongly copositive mapping is coercive with respect to $\mathbb{R}^n_+$, it follows from Theorem 3.2 that the NCP(F) must have a nonempty, compact solution set if F is strongly copositive with respect to $\mathbb{R}^n_+$. If F is strictly copositive, then the following result holds.

Theorem 3.7 [182]. *Let $F:\mathbb{R}^n_+ \to \mathbb{R}^n$ be continuous and strictly copositive with respect to $\mathbb{R}^n_+$. If there exists a mapping $c:\mathbb{R}_+ \to \mathbb{R}$ such that $c(\lambda) \to \infty$ as $\lambda \to \infty$, and for all $\lambda \geq 1$, $x \geq 0$,*

$$[F(\lambda x) - F(0)]^{\mathrm{T}} x \geq c(\lambda)[F(x) - F(0)]^{\mathrm{T}} x, \tag{26}$$

then the problem NCP(F) *has a nonempty, compact solution set.*

As shown in Definition 3.4 and Theorem 3.7, the mapping

$$G(x) = F(x) - F(0) \tag{27}$$

plays an important role in the nonlinear complementarity problem; this again is motivated by the linear complementarity problem. For the latter problem, the mapping G is obviously linear and thus, condition (26) is satisfied with $c(\lambda) = \lambda$. More generally, the same condition will hold with $c(\lambda) = \lambda^{\alpha}$ if G is positively homogeneous of degree $\alpha > 0$; i.e., if $G(\lambda x) = \lambda^{\alpha} G(x)$ for $\lambda > 0$.

The strict copositivity assumption in Theorem 3.7 can be relaxed through the introduction of the class of *d-regular* mappings:

Definition 3.5. For any vector $x \in \mathbb{R}^n_+$, define the index sets

$$I_+(x) = \{i\colon x_i > 0\} \quad \text{and} \quad I_0(x) = \{i\colon x_i = 0\}.$$

Let $d > 0$ be an arbitrary vector in $\mathbb{R}^n$. A mapping $G:\mathbb{R}^n \to \mathbb{R}^n$ is said to be *d-regular* if the following system of equations has no solution in $(x, t) \in \mathbb{R}^n_+ \times \mathbb{R}_+$ with $x \neq 0$:

$$\begin{aligned} G_i(x) + td_i &= 0, \quad i \in I_+(x), \\ G_i(x) + td_i &\geq 0, \quad i \in I_0(x). \end{aligned} \tag{28}$$

Equivalently, G is d-regular if, for any scalar $r > 0$, the *augmented nonlinear complementarity problem* NCP(H) defined by $H:\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$,

$$H\begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} G(x) + td \\ r - d^{\mathrm{T}} x \end{pmatrix},$$

has no solution (x, t) with $x \neq 0$.

It is easy to see that the augmented NCP(H) with H defined as above always has a solution (x, t) since the problem is simply the complementarity formulation of the VI(X, G) with $X = \{x \geq 0\colon d^{\mathrm{T}} x \leq r\}$ (cf. Proposition 2.2). The above definition of d-regularity requires that zero be the only solution to NCP(H).

The notion of d-regularity was introduced by Karamardian [126] who was motivated by the Lemke algorithm for solving the linear complementarity problem. If F is strictly copositive with respect to $\mathbb{R}_+^n$, then the mapping G in (27) is d-regular for any $d>0$. The following theorem gives an existence result for the nonlinear complementarity problem with a d-regular mapping.

Theorem 3.8 [126]. *Let F be a continuous mapping from $\mathbb{R}^n$ into itself and let G be defined by* (27). *Suppose that G is positively homogeneous of degree $\alpha>0$ and that G is d-regular for some $d>0$. Then the problem* NCP(F) *has a nonempty, compact solution set.*

There exists a generalization of the notion of copositivity, called *semi-monotonicity*, which also implies the property of d-regularity; the reader is referred to [126] for details. In what follows, we shall discuss another generalization of the notion of monotonicity which is applicable to the variational inequality problem VI(X, F) where X is a rectangular set; this of course includes the case of the nonlinear complementarity problem where $X=\mathbb{R}_+^n$.

Definition 3.6. A subset of $\mathbb{R}^n$ is a *rectangle* if it is the Cartesian product of n one-dimensional closed intervals.

Definition 3.7. A mapping $F: X\to\mathbb{R}^n$ is said to be a

(a) *P-function on X* if

$$\max_{1\leqslant i\leqslant n}[F_i(x)-F_i(y)](x_i-y_i)>0 \quad \forall x, y\in X,\ x\neq y,$$

(b) *uniform P-function on X* if there exists a scalar $\alpha>0$ such that

$$\max_{1\leqslant i\leqslant n}[F_i(x)-F_i(y)](x_i-y_i)\geqslant \alpha\|x-y\|_2^2 \quad \forall x, y\in X,\ x\neq y.$$

The notion of a P-function was introduced in [184] as a generalization of a P-matrix for an affine F. Such a matrix has many interesting properties [66], one of which is used in the above definition of a P-function. If F is strictly monotone over X, then it is a P-function on X; and if F is strongly monotone, then it is a uniform P-function. Megiddo and Kojima [179], in their study of the existence and uniqueness of solutions to the nonlinear complementarity problem via the theory of global homeomorphism of mappings, have given examples to show that the condition of positively bounded Jacobians used in Theorem 3.6 and the uniform P-property are related but independent. More specifically, if F satisfies the former property, then it must be a P-function but not necessarily a uniform P-function; conversely, a uniform P-function need not have bounded Jacobians.

With the above definitions of P-functions, one has the following result.

Theorem 3.9 [182]. *Let X be a nonempty rectangle in $\mathbb{R}^n$ and let F be a continuous mapping from X into $\mathbb{R}^n$.*

(a) *If F is a P-function on X, then the problem* VI(X, F) *has at most one solution.*

(b) *If F is a uniform P-function, then the problem* VI(X, F) *has a unique solution.*

The next set of results concern the use of *least-element theory* in the analysis of the variational inequality problem VI(X, F) with a rectangular set X. The results in this theory were initiated in the paper by Cottle and Veinott [38], which was subsequently extended by Tamir [256], Moré [182] and Pang [196]. Central to this theory is the notion of a Z-function:

Definition 3.8. A mapping $F:\mathbb{R}^n \to \mathbb{R}^n$ is said to be a *Z-function* (or *off-diagonally antitone*) if for any $i \neq j$ and for any $x \in \mathbb{R}^n$, the one-dimensional scalar-valued function $g_{ij}:\mathbb{R} \to \mathbb{R}$ defined by

$$g_{ij}(t) = f_i(x + te^j), \quad t \in \mathbb{R},$$

where e^j is the jth unit vector, is antitone (i.e., nonincreasing).

If $F(x) = q + Mx$ is affine, then F is a Z-function if and only if the matrix M has nonpositive off-diagonal entries; i.e., if M is a *Z-matrix*. The feasible region of the nonlinear complementarity problem with a Z-function satisfies a certain lattice property which is defined as follows:

Definition 3.9. A subset S of $\mathbb{R}^n$ is said to be a *meet* (*join*) *semi-sublattice* (with respect to the *usual ordering* of $\mathbb{R}^n$) if for any two vectors $x, y \in S$, the vector $z = \min(x, y)$ ($\max(x, y)$ respectively) which is called the *meet* (*join*) of x and y and defined as the componentwise minimum (maximum) of x and y, is also contained in S. A subset S of $\mathbb{R}^n$ is *bounded below* (*above*) if there exists a vector $u \in \mathbb{R}^n$ such that $x \geqslant (\leqslant) u$ $\forall x \in S$. If such a vector u happens to be in S, then u is called a *least* (*greatest*) *element* of S.

It is clear that if a least (greatest) element of a set exists, then it must be unique. A sufficient condition for a nonempty set S to possess a least (greatest) element is that S is a closed (in the tolopogical sense) meet (join) semi-sublattice that is also bounded below (above). In this case, the least (greatest) element can be obtained as the unique solution of the following mathematical program:

$$\text{minimize (maximize) } p^{\mathrm{T}}x \quad \text{subject to } x \in S \tag{29}$$

for any positive vector $p > 0$.

In order to apply the above results to the variational inequality problem VI(X, F) with X being a rectangle, one must define the set S. For this purpose, let us write $X = \prod_{i=1}^n X_i$ where each X_i is one of the following four types of closed intervals: $[a_i, \infty)$, $(-\infty, b_i]$, $[a_i, b_i]$ or (∞, ∞). With this representation, let us define the set

$$S = \bigcap_{i=1}^{n} \{x \in X\colon \text{either } x_i = b_i \text{ or } F_i(x) \geqslant 0\}. \tag{30}$$

The statement $x_i = b_i$ in the definition of S is interpreted as vacuously false if X_i is not bounded above. In particular, if $X = \mathbb{R}^n_+$ (as in the nonlinear complementarity problem), the set S becomes the feasible region of the problem NCP(F) given by $\Omega(\mathbb{R}^n_+, F)$. In general, the set S is not convex, even in the case where F is affine. However, one has the following result.

Theorem 3.10 [196]. *Let X be a nonempty rectangle in $\mathbb{R}^n$, $F: \mathbb{R}^n \to \mathbb{R}^n$, and S be defined in* (30). *Then*

(a) *S contains the solution set of the problem* VI(X, F);

(b) *if F is a Z-function, then S is a meet semi-sublattice*;

(c) *if F is a continuous and if S is nonempty and bounded below, then S contains a least element x^*; moreover, x^* solves the problem* VI(X, F).

When the rectangle X is bounded above, the vector b is an element of the set S; thus, S is nonempty in this case. On the other hand, the set S is always bounded below in the case of the nonlinear complementarity problem. Furthermore, S is nonempty if and only if the NCP(F) is feasible. Thus, specializing Theorem 3.10 yields:

Corollary 3.3 [256]. *Let $F: \mathbb{R}^n \to \mathbb{R}^n$ be a continuous Z-function. If the problem* NCP(F) *is feasible, then it has a solution x^* which is the least element of the feasible set $\Omega(\mathbb{R}^n_+, F)$.*

Finally, in the case of the problem VI(X, F) with X being an arbitrary rectangle, we mention that the nonemptiness and bounded-below property of the set S required in Theorem 3.10 can be verified by imposing certain *order-coercivity* and *surjectivity* assumptions on the mapping F; see [106] for details.

We close this section by reviewing some results concerning the local uniqueness of a solution to the variational inequality problem and specializationns of these results to the nonlinear complementarity problem. The two main uniqueness results that were mentioned above, Proposition 3.2 for the variational inequality problem and Theorem 3.9(a) for the nonlinear complementarity problem, assert when a solution is *globally unique*. The results below provide conditions under which a given solution is *isolated* in the sense that there exists a neighborhood of the solution within which no other solution to the problem exists. This notion is defined more precisely:

Definition 3.10. A solution x^* of the variational inequality problem VI(X, F) is *locally unique* (*or isolated*) if there exists a neighborhood N of x^* such that x^* is the only solution of the VI(X, F) in N.

The following result gives a sufficient condition for a solution of the problem VI(X, F) to be locally unique.

Proposition 3.3 [258]. *Let F be a once continuously differentiable mapping from $\mathbb{R}^n$ into itself, and let x^* be a solution to the problem* VI(X, F). *If the Jacobian matrix $\nabla F(x^*)$ is positive definite, then x^* is locally unique.*

The above proposition requires essentially no assumptions on the set X. If X is defined by differentiable inequalities of the form (8), then the positive definiteness of $\nabla F(x^*)$ can be weakened. The following result generalizes the second-order sufficiency conditions for an optimization problem.

Theorem 3.11 [142]. *Let $F:\mathbb{R}^n \to \mathbb{R}^n$ be once continuously differentiable, and let $g:\mathbb{R}^n \to \mathbb{R}^m$ and $h:\mathbb{R}^n \to \mathbb{R}^p$ be twice continuously differentiable. Let X be defined in* (8) *and let x^* be a solution of the* VI(X, F). *Suppose that there exists vectors $\pi^* \in \mathbb{R}^m$ and $\mu^* \in \mathbb{R}^p$ such that*

$$\begin{aligned} &F(x^*)+\sum_{i=1}^{m} \pi_i^* \nabla g_i(x^*)+\sum_{j=1}^{p} \mu_j^* \nabla h_j(x^*)=0, \\ &\pi_i^* \geqslant 0, \quad \pi_i^* g_i(x^*)=0, \qquad i=1,2,\ldots,m, \end{aligned} \tag{31}$$

and that

$$z^{\mathrm{T}}[\nabla F(x^*)+\sum_{i\in I_1} \pi_i^* \nabla^2 g_i(x^*)+\sum_{j=1}^{p} \mu_j^* \nabla^2 h_j(x^*)]z>0$$

for all $z \neq 0$ such that

$$\begin{aligned} &z^{\mathrm{T}}\nabla g_i(x^*)=0 \quad \forall i\in I_1=\{i:\, \pi_i^*>0\}, \\ &z^{\mathrm{T}}\nabla g_i(x^*)\leqslant 0 \quad \forall i\in I_2=\{i:\, g_i(x^*)=0,\, \pi_i^*=0\}, \\ &z^{T}\nabla h_j(x^*)=0 \quad \forall j=1,2,\ldots,p. \end{aligned}$$

Then x^* is a locally unique solution of the VI(X, F).

Note that expression (31) constitutes the set of first-order necessary conditions for x^* to be a solution to problem VI(X, F); see Proposition 2.2.

Both Proposition 3.3 and Theorem 3.11 can be specialized to the nonlinear complementarity problem. However, due to the special structure of NCP(F), a stronger result can be established:

Theorem 3.12 [141]. *Let $F:\mathbb{R}^n \to \mathbb{R}^n$ be once continuously differentiable and let x^* be a solution to the* NCP(F). *Define the index sets*

$$I_+=\{i:\, x_i^*>0\} \quad \text{and} \quad I_0=\{i:\, x_i^*=0,\, F_i(x^*)=0\} \tag{32}$$

and the vectors $x_{I_+}=(x_i:\, i\in I_+)$, $x_{I_0}=(x_i:\, i\in I_0)$, $F_{I_+}(x)=(F_i(x):\, i\in I_+)$, $F_{I_0}(x)=(F_i(x):\, i\in I_0)$. If there exists no vector $(x_{I_+}, x_{I_0})\neq 0$ satisfying

$$\begin{aligned} &\nabla_{I_+}F_{I_+}(x^*)x_{I_+}+\nabla_{I_0}F_{I_+}(x^*)x_{I_0}=0, \\ &\nabla_{I_+}F_{I_0}(x^*)x_{I_+}+\nabla_{I_0}F_{I_0}(x^*)x_{I_0}\geqslant 0, \\ &x_{I_0}\geqslant 0 \quad \text{and} \quad (x_{I_0})^{\mathrm{T}}[\nabla_{I_+}F_{I_0}(x^*)x_{I_+}+\nabla_{I_0}F_{I_0}(x^*)x_{I_0}]=0, \end{aligned} \tag{33}$$

where $\nabla_S F_T(x)$ denotes the partial differentiation of $F_T(x)$ with respect to the components of x_S, then x^* is a locally unique solution of the NCP(F).

The system (33) is an example of a *mixed homogeneous linear complementarity problem.* There are many conditions under which (33) will have only zero as the unique solution; the following corollary provides one such condition.

Corollary 3.4 [141, 157]. *Let F, x^*, I_+ and I_0 be given as in Theorem* 3.12. *If for each subset J of I_0 including the empty set, the matrix*

$$\begin{bmatrix} \nabla_{I_+} F_{I_+}(x^*) & \nabla_J F_{I_+}(x^*) \\ \nabla_{I_+} F_J(x^*) & \nabla_J F_J(x^*) \end{bmatrix}$$

is nonsingular, then x^* is a locally unique solution of the NCP(F). *In particular, if $\nabla F(x^*)$ is nondegenerate (i.e., if each principal submatrix of $\nabla F(x^*)$ is nonsingular), then the same conclusion holds.*

An important special case of this result is worthy of note; namely, when the set I_0 is empty:

Definition 3.11. A solution x^* of the NCP(F) is said to satisfy the *strict complementarity property* if $x^* + F(x^*) > 0$. If this property holds, then x^* is said to be a *nondegenerate solution.*

Corollary 3.5. *Let $F: \mathbb{R}^n \to \mathbb{R}^n$ be once continuously differentiable and let x^* be a nondegenerate solution of the* NCP(F). *Let $I_+ = \{i: x_i^* > 0\}$ be the index set of the positive x^* variables. If $\nabla_{I_+} F_{I_+}(x^*)$ is nonsingular, then x^* is a locally unique solution of the* NCP(F).

This corollary can be established directly by invoking the classical inverse function theorem [194, Chapter 5]. Under the nondegeneracy assumption, the nonlinear complementarity problem reduces to a system of nonlinear equations in a neighborhood of the solution; this observation will have other implications when the sensitivity of the solution is discussed in Section 5. For other specialized results for the nonlinear and linear complementarity problems, the interested reader is referred to [179, 182, 183, 132, 215].

4. Algorithms

The history of algorithms for solving the finite-dimensional variational inequality and nonlinear complementarity problems is relatively short (again, we restrict our attention to finite-dimensional problems; see [15] for a discussion of the infinite-dimensional problem). Cottle [32] developed the first algorithm for solving the nonlinear complementarity problem which extended the principal pivoting algorithm for the linear case; this algorithm is not applicable to the variational inequality problem in general. Between the time Cottle's algorithm was published and the appearance of the PIES algorithm [5], there was essentially no significant development of algorithms for solving variational inequalities. The fixed-point and homotopy algorithms, although applicable to the nonlinear complementarity problem, have not been employed in solving the general variational inequality problem. Motivated by the interest generated by the PIES algorithm and by the desire to unify the convergence theory, Pang and Chan [207] summarize the various iterative algorithms as well as the various approaches to prove convergence; Dafermos [42] also provides an interesting framework for the proof of convergence of these methods. In this section, we shall first give a brief review of the results in [207] and then proceed to discuss more recent developments. As in the previous section, space limits our ability to discuss every algorithm which has been developed for variational inequalities and nonlinear complementarity problems; the references at the end of the paper should provide the missing detail.

A general approach for solving the variational inequality VI(X, F) consists of creating a sequence $\{x^k\} \subseteq X$ such that each x^{k+1} solves problem VI(X, F^k):

$$F^k(x^{k+1})^{\mathrm{T}}(y - x^{k+1}) \geqslant 0 \quad \forall y \in X, \tag{34}$$

where $F^k(x)$ is some approximation to $F(x)$. The two basic choices for this approximation are that F^k is either a linear or nonlinear function. For the linear approximations:

$$F^k(x) = F(x^k) + A(x^k)(x - x^k) \tag{35}$$

where $A(x^k)$ is an $n \times n$ matrix, several methods exist which differ in the choice of $A(x^k)$:

$$\begin{aligned}
A(x^k) &= \nabla F(x^k) && \text{(Newton's method)}\\
&\approx \nabla F(x^k) && \text{(Quasi-Newton)}\\
&= D(x^k) && \text{(Linearized Jacobi)}\\
&= \begin{cases} L(x^k) + D(x^k)/\omega^* \\ U(x^k) + D(x^k)/\omega^* \end{cases} && \text{(Successive Overrelaxation (SOR))}\\
&= \tfrac{1}{2}[\nabla F(x^k) + \nabla F(x^k)^{\mathrm{T}}] && \text{(Symmetrized Newton)}\\
&= G && \text{(Projection)}
\end{aligned}$$

where

$D(x^k) \equiv$ the diagonal part of $\nabla F(x^k)$,

$L(x^k) \equiv$ the lower triangular part of $\nabla F(x^k)$,

$U(x^k) \equiv$ the upper triangular part of $\nabla F(x^k)$,

$\omega^* \equiv$ a scalar parameter $\in (0, 2)$,

$G \equiv$ a fixed, symmetric, positive definite matrix.

In what follows, the convergence theory for each of these methods will be discussed.

Let us begin with the most powerful of the abovementioned algorithms: Newton's method. In order to discuss the convergence of this method, we must discuss the notion of a *regular solution* which was introduced by Robinson [218] under the broader context of generalized equations:

Definition 4.1 [218]. Let x^* be a solution of the problem VI(X, F). Then x^* is called *regular* if there exists a neighborhood N of x^* and a scalar $\delta > 0$ such that for every vector y with $\|y\| < \delta$, there is a unique vector $x(y) \in N$ that solves the *perturbed linearized variational inequality problem* VI(X, F^y) where $F^y : \mathbb{R}^n \to \mathbb{R}^n$ is defined by

$$F^y(x) = F(x^*) + y + \nabla F(x^*)(x - x^*);$$

moreover, as a function of the perturbed vector y, the solution $x(y)$ is Lipschitz continuous; i.e., there exists a constant $L > 0$ such that whenever $\|y\| < \delta$ and $\|z\| < \delta$, one has

$$\|x(y) - x(z)\| \leqslant L\|y - z\|.$$

It is easy to see that when $X = \mathbb{R}^n$, the regularity of a solution x^* of the VI(X, F) is equivalent to the nonsingularity of the Jacobian matrix $\nabla F(x^*)$. More generally, if the set X is defined by differentiable inequalities of the form (8), the following result provides a set of sufficient conditions for regularity.

Proposition 4.1 [218]. *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be once continuously differentiable, $g : \mathbb{R}^n \to \mathbb{R}^m$ and $h : \mathbb{R}^n \to \mathbb{R}^p$ be twice continuously differentiable, X be defined by* (8), *and let x^* be a solution to problem* VI(X, F). *Suppose that there exist vectors $\pi^* \in \mathbb{R}^m$, $\mu^* \in \mathbb{R}^p$ such that the first-order conditions* (31) *hold. Suppose also that the following conditions hold:*

(a) *linear independence of the gradients of the binding constraints; i.e., the vectors*

$$(\nabla g_i(x^*) \colon i \in I_+ \cup I_0), \qquad (\nabla h_j(x^*) \colon j = 1, 2, \ldots, p),$$

where $I_+ = \{i \colon \pi_i^* > 0\}$, $I_0 = \{i \colon g_i(x^*) = 0,\ \pi_i^* = 0\}$ are linearly independent;

(b) *the strong second-order condition*

$$z^{\mathrm{T}}\left[\nabla F(x^*) + \sum_{i \in I_+} \pi_i^* \nabla^2 g_i(x^*) + \sum_{j=1}^{p} \mu_j^* \nabla^2 h_j(x^*)\right] z > 0$$

holds for all $z \neq 0$ such that

$$z^{\mathrm{T}} \nabla g_i(x^*) = 0 \quad \forall i \in I_+,$$

$$z^{\mathrm{T}} \nabla h_j(x^*) = 0, \quad j = 1, 2, \ldots, p.$$

Then (x^*, π^*, μ^*) is a regular solution of the GCP($\mathbb{R}^n \times \mathbb{R}^m_+ \times \mathbb{R}^p$, H) *formulation of the* VI(X, F), *where H is defined by equation* (9).

In the case of the nonlinear complementarity problem, the set of sufficient conditions in Proposition 4.1 can be weakened to the point where the resulting conditions become necessary as well:

Proposition 4.2 [218]. *Let $F:\mathbb{R}^n \to \mathbb{R}^n$ be once continuously differentiable and let x^* be a solution of the* NCP(F). *A necessary and sufficient condition for x^* to be regular is that the following two conditions hold with I_+, I_0 being defined in* (32):

(a) *the matrix $\nabla_{I_+}F_{I_+}(x^*)$ is nonsingular*;

(b) *the Schur complement $\nabla_{I_0}F_{I_0}(x^*) - \nabla_{I_+}F_{I_0}(x^*)[\nabla_{I_+}F_{I_+}(x^*)]^{-1}\nabla_{I_0}F_{I_+}(x^*)$ is a P-matrix.*

Note that the conditions in Propositions 4.1 and 4.2 are stronger than their respective conditions in Theorems 3.11 and 3.12. Thus, under the conditions of the above propositions, x^* will be a locally unique solution of the respective problems.

Given the notion of regularity, we may now state the basic convergence result for Newton's method when applied to the problem VI(X, F):

Theorem 4.1 [121]. *Let X be a nonempty, closed and convex subset of $\mathbb{R}^n$, $F:\mathbb{R}^n \to \mathbb{R}^n$ be once continuously differentiable, and x^* be a regular solution of* VI(X, F). *Then there exists a neighborhood N of x^* such that whenever the initial vector x^0 is chosen in N, the entire sequence $\{x^k\}$ generated by Newton's method is well-defined and converges to x^*. Furthermore, if $\nabla F(x^*)$ is Lipschitz continuous around x^*, then the convergence is quadratic; i.e., there exists a constant $c > 0$ such that for all k sufficiently large,*

$$\|x^{k+1} - x^*\| \leq c\|x^k - x^*\|^2. \tag{36}$$

The quadratic convergence of Newton's method is one of its most important features. However, as in the case of solving a system of nonlinear equations, there are several well-recognized drawbacks with this method: (i) the evaluation of the $n \times n$ Jacobian matrix $\nabla F(x^k)$ at each iteration, (ii) the solution of each variational inequality subproblem VI(X, F^k), and (iii) the local nature of the method. While many remedies exist for the first two drawbacks, the global convergence remains a rare property for most of the modified methods; see [207] for conditions ensuring such global convergence.

In order to overcome the first drawback of having to generate the full Jacobian matrix at each iteration, *Quasi-Newton* methods can be employed. In the classs of *secant methods*, the matrix $A(x^k)$ in the linear approximation scheme is updated from one iteration to the next by a simple small-rank matrix; see Josephy [122] and Subramanian [253]. Such secant methods significantly reduce the work to evaluate $\nabla F(x^k)$ but do not ease the computational effort involved in solving the resulting subproblems. Also, the secant methods have a rate of convergence which is at best superlinear.

The other major class of linear approximation methods involve the use of a symmetric matrix $A(x^k)$ at each iteration; the *linearized Jacobi, symmetrized Newton and projection algorithms* fall under this category. The motivation behind these so-called *symmetrized methods* is that each subproblem can be cast as an optimization problem, thereby easing the task of obtaining its solution (if X is polyhedral, each subproblem becomes a quadratic program). This is an especially attractive feature because numerically efficient and robust optimization software are becoming more readily available. However, the price one pays for this "ease of use" is that the convergence of the resulting methods typically requires much stronger properties on the mapping F, and that the quadratic convergence rate of the basic Newton method is lost.

The projection method takes $A(x^k)$ to be a fixed symmetric, positive definite matrix G for all k and has the advantage that it is globally convergent; however, it is only applicable to the problem $\mathrm{VI}(X, F)$ when F is strongly monotone. In essence, a projection method is just a simple fixed-point iteration of the mapping H defined by (12). This method originated in the infinite-dimensional variational inequality problems and was one of the first algorithms used on finite-dimensional problems [82, 4, 56, 39, 40, 24, 79]. The linearized Jacobi method, by taking $A(x^k)$ to be a diagonal matrix at each iteration, leads to an optimization problem with a separable quadratic objective function. The symmetrized Newton method, also called the contracting ellipsoid method in [86, 88], is a simple variant of the general linearization scheme in which the symmetric part of $\nabla F(x)$ is used to define the linear approximation.

Besides the symmetric methods, other linear approximation schemes can easily be envisioned. The most notable is the *linearized successive overrelaxation* (SOR) *method* described above. In particular, this method is potentially very useful in solving the nonlinear complementarity problem because in this case, each resulting linear complementarity subproblem is defined by a triangular matrix M and is thus easily solvable. The drawback of the SOR methods is that they require very restrictive assumptions on the mapping F in order for convergence to hold. For further information on this algorithm, the reader is referred to [6, 194, 197].

In Pang and Chan [207], very general convergence results for the class of linear approximation methods are presented. In the following theorem, we summarize these results for the symmetrized methods discussed above.

Theorem 4.2 [207]. *Let X be a nonempty, closed and convex subset of $\mathbb{R}^n$ and let $F : \mathbb{R}^n \to \mathbb{R}^n$ be given.*

(a) *Suppose that F is Lipschitz continuous and strongly monotone with constants β and γ respectively; i.e., for all vectors $x, y \in X$,*

$$\|F(x) - F(y)\|_2 \leqslant \beta \|x - y\|_2,$$
$$[F(x) - F(y)]^{\mathrm{T}}(x - y) \geqslant \gamma \|x - y\|_2^2.$$

Let G be a symmetric positive definite matrix with smallest and largest eigenvalues *given by ν^{-1}, η respectively. If $\nu^2\beta^2 < 2\gamma/\eta$, then for any initial vector x^0 the sequence $\{x^k\}$ generated by the projection algorithm with the matrix G is uniquely defined and converges to the unique solution of the* VI(X, F).

(b) *Let F be once continuously differentiable. Suppose that x^* solves the* VI(X, F) *and that $\nabla F(x^*)$ has positive diagonals. Let $D(x^*)$ and $B(x^*)$ be the diagonal and off-diagonal parts of $\nabla F(x^*)$ respectively. If*

$$\|D(x^*)^{-1/2}B(x^*)D(x^*)^{-1/2}\|_2 < 1, \tag{37}$$

then there exists a neighborhood N of x^* such that whenever the initial vector x^0 is chosen in this neighborhood, the sequence $\{x^k\}$ generated by the linearized Jacobi method is uniquely defined and converges to x^*.

(c) *Let F be once continuously differentiable. Suppose that x^* solves the* VI(X, F) *and that $\nabla F(x^*)$ is positive definite. Let $A(x^*)$ and $C(x^*)$ denote the symmetric and skew-symmetric parts of $\nabla F(x^*)$ respectively. If*

$$\|C(x^*)\|_2 < \lambda_{\min}(A(x^*)) \tag{38}$$

where $\lambda_{\min}(A(x^*))$ denotes the least eigenvalue of $A(x^*)$, then the conclusion of part (b) *holds for the symmetrized Newton method.*

Moreover, the convergence of each of these methods is geometric; i.e., there exists a constant $r \in (0, 1)$ such that for a certain vector norm and for all k,

$$\|x^{k+1} - x^*\| \leq r\|x^k - x^*\|. \tag{39}$$

Note that the assumptions required in the above theorem all imply that the solution x^* is regular in the sense of Definition 4.1. Thus, the symmetrized methods not only need stronger conditions for convergence than those for the basic Newton algorithm, they also possess a slower rate of convergence. Nevertheless, in light of the advantages listed previously, the symmetrized methods can often prove to be a viable alternative to Newton's method.

In terms of the practical performance of the abovementioned algorithms, one can expect that the closer $\nabla F(x^*)$ is to being symmetric, the better the symmetrized Newton method will behave since (38) would hold identically if this Jacobian were symmetric; i.e., $\|C(x^*)\| = 0$. In the case of the linearized Jacobi method, the more "diagonally dominant" $\nabla F(x^*)$ is, the better the method can be expected to perform. Indeed, one has the following result.

Proposition 4.3 [207]. *Let A be a row diagonally dominant and strictly column diagonally dominant matrix with positive diagonal entries; i.e.,*

$$a_{ii} \geq \sum_{j \neq i} |a_{ij}| \quad \forall i,$$

$$a_{jj} > \sum_{i \neq j} |a_{ij}| \quad \forall j.$$

Let $A = D + B$ be the decomposition of A into its diagonal and off-diagonal parts. Then

$$\|D^{-1/2}BD^{-1/2}\|_2 < 1.$$

The same conclusion holds if A is column diagonally dominant and strictly row diagonally dominant with positive diagonal entries.

In the case of the projection algorithm, the convergence depends on the constants β and γ; the smaller β and the larger γ, the better the method will perform. These conclusions are drawn from the analysis of the magnitude of the constant r in (39). We refer to the dissertation by Hammond [86] and the papers by Hammond and Magnanti [87, 88] for further discussion on the convergence of the symmetrized methods.

In the category of nonlinear approximation schemes for solving VI(X, F), the most popular is the nonlinear Jacobi (also called relaxation or diagonalization) algorithm [5, 8, 73, 82]. When applied to the PIES model, this method actually reduces to the PIES algorithm. The basic idea of this algorithm is to extend the linearized Jacobi method by producing a separable nonlinear map at each iteration

$$F^k(x) = (\ldots, F_i^k(x_i), \ldots)^{\mathrm{T}} : \mathbb{R}^n \to \mathbb{R}^n$$

where

$$F_i^k(x_i) = F_i(x_1^k, x_2^k, \ldots, x_{i-1}^k, x_i, x_{i+1}^k, \ldots, x_n^k).$$

Due to the separability of F^k, each subproblem VI(X, F^k) can again be cast into the form of an optimization problem; it was precisely this feature which contributed to the success of the PIES model in applications. As shown in Pang and Chan [207], the local convergence of this method is analogous to the linearized Jacobi method in Theorem 4.2; i.e., some form of diagonal dominance is required.

As we see from Theorems 4.1 and 4.2, all the methods discussed above, with the exception of the projection method, are locally convergent. In order to enlarge the domain of convergence for these methods, the idea of backtracking by means of a linesearch step is useful. As is well-known in the solution of nonlinear equations [46], backtracking via the use of valid merit functions is essential in order to insure *global* convergence of iterative schemes such as Newton's method. The trouble with general variational inequalities and nonlinear complementarity problems is that valid merit functions which are relatively easy to compute are very difficult if not impossible to find. Marcotte and Dussalt [164, 166, 167] have shown that when Newton's method is applied to a variational inequality problem over a bounded polyhedral set with a monotone mapping F, it provides a descent direction for the gap function (Definition 2.5); i.e., the gap function is a valid merit function for the monotone variational inequality problem. Based on this result, Marcotte and Dussault propose a gap-decreasing method which, under the joint assumptions of strong monotonicity and strict complementarity [166, Definition 3], is shown to be globally convergent and locally equivalent to the basic Newton algorithm. However, the evaluation of the gap function involves the solution of a linear programming problem which, in general, precludes its use in a backtracking scheme. For the general variational inequality problem, Harker [99] provides a "spacer step" which is similar to the PARTANS idea in nonlinear programming. Although this idea is not theoretically shown to accelerate convergence, extensive numerical results indicate that this approach can yield substantial computational savings. Finally, Preckel [211] and Rutherford [231, 232] provide heuristic backtracking schemes for the nonlinear complementarity problem which, although not proven to provide a descent direction, are very effective. In summary, the theory of merit functions for variational inequalities and nonlinear complementarity problems is currently weak at best.

Another useful idea to improve upon the above iterative methods was proposed recently in Pang [202]: *inexact iterative methods*. The central idea is to define each iterate x^{k+1} as an inexact solution of the subproblem VI(X, F^k); in other words, at each iteration one would only solve the subproblem up to a certain degree of accuracy (by perhaps another iterative scheme). The use of an inexact method immediately raises the question of how one measures the accuracy of the approximate solution to a variational inequality problem. Presumably, one wants a practical measure which can be employed in the stopping rule when solving the subproblems. In the case of the nonlinear complementarity problem NCP(F), one such measure is defined by

$$\rho(x) = \|\min(x, F(x))\| \tag{40}$$

where $\|\cdot\|$ is a certain vector norm. The following result justifies the use of $\rho(x)$ as a measure of inexactness.

Proposition 4.4 [202]. *Let $F:\mathbb{R}^n \to \mathbb{R}^n$ be given.*

(a) *If x^* is a regular solution of the* NCP(F), *then there exists a scalar $c > 0$ and a neighborhood N of x^* such that for all $x \in N$,*

$$\|x - x^*\| \leq c\rho(x). \tag{41}$$

(b) *Suppose that F is a uniform P function. If x^* is the unique solution of the* NCP(F), *then there exists a constant $c > 0$ such that* (41) *holds for all vectors $x \in \mathbb{R}^n$.*

Part (a) of the above proposition states that under the sole assumption of regularity, the quantity $\rho(x)$ in (40) is a reasonable measure of the inexactness in the neighborhood of the solution in the sense that the smaller $\rho(x)$ is, the closer one can expect the vector x is to x^*. Part (b) shows that the same conclusion holds globally for all vectors $x \in \mathbb{R}^n$ if F satisfies a stronger property.

Using $\rho(x)$ as the measure of inaccuracy and the following rule to define x^{k+1}:

$$\|\min(x^{k+1}, F^k(x^{k+1}))\| \leq \nu_k \|\min(x^k, F(x^k))\|$$

where ν_k is a prescribed positive scalar, Pang [202] has established the convergence of a large family of Newton-type methods for solving the NCP(F) under basically the same conditions as for the exact methods. Moreover, in the case of the basic

Newton method, the inexact version is shown to possess the same quadratic rate of convergence provided that one chooses the scalars ν_k to satisfy

$$\nu_k \leqslant \delta \|\min(x^k, F(x^k))\|$$

for any positive constant δ. Preliminary computational results in [202] illustrate the potential advantages of the inexact methods, especially with a careful choice of the inexact rule which is employed.

For the more general variational inequality problem VI(X, F), the same inexact theory remains valid with the use of the following measure of inaccuracy:

$$\rho(x) = \|x - \text{proj}_{I,X}(x - F(x))\|.$$

The reader is referred to [202] for the details of the abovementioned methods, and to the paper by Pang [203] for a discussion of the related issue of error bounds for the linearly constrained variational inequality problem.

Another class of iterative methods which are applicable to the variational inequality problem VI(X, F) where X is a compact polyhedral set (which excludes the nonlinear complementarity problem) are the *simplicial decomposition methods.* These methods have been applied to the traffic assignment problem (see Section 7) and numerical results [145, 209] indicate that they can be effective computational techniques for large-scale problems. The main idea behind the simplicial decomposition methods is derived from that of *column generation* in linear programming. Since X is a compact polyhedron, it can be expressed as the convex hull of its extreme points. At the beginning of each iteration, a variational inequality subproblem VI(X^k, F^k) is solved (either exactly or approximately), where X^k denotes the convex hull of a subset of the extreme points of X. By using a merit function such as the gap function to guide the addition and deletion of extreme points, a new subset X^{k+1} is obtained and the process continues. Due to the fact that each set X^k is the convex hull of a finite number of explicitly known vectors, the solution of each subproblem VI(X^k, F^k) should be relatively easy provided that the number of extreme points is small. Indeed, the ability to control the number of extreme points is vital. Lawphongpanich and Hearn [145] show that if the gap function (16) is used as the merit function in the above scheme to control the set X^k and if F^k is taken to be the given mapping F for all iterations k, then the method will terminate in a finite number of *major iterations* if F is strongly monotone. Pang and Yu [209] propose using the linear approximation mappings discussed previously for each F^k and establish convergence results.

As mentioned above, it is vital that the number of extreme points in X^k not grow without control if the simplicial decomposition method is to be practically effective. Recognizing this fact, Lawphongpanich and Hearn [147] introduce the notion of *restricted simplicial decomposition* in which one places a limit on the number of extreme points kept in each set X^k. So far, this scheme is shown to work only for nonlinear programs; the case of the variational inequality problem has not been considered.

Another approach has been proposed by Fukushima [79] for solving the VI(X, F), and is similar in spirit to the simplicial decomposition algorithm in attempting to simplify the solution of the subproblems VI(X, F^k). While simplicial decomposition can be considered as providing *inner approximations* X^k to the set X, Fukushima proposes the use of *outer approximations.* Fukushima's analysis is limited to the projection method, it would be interesting to investigate how this outer approximation idea could be applied to other linear approximation algorithms (especially to the basic Newton method).

The next variant of the basic iterative schemes to be mentioned is the *proximal point algorithm.* This algorithm was originally developed for finding the zero of a maximal monotone operator; see Moreau [185] and Rockafellar [228, 229]. When specialized to the VI(X, F), the algorithm works as follows. Given x^k, let x^{k+1} be an approximate solution to the variational inequality problem VI(X, $F+\varepsilon_k I$) where ε_k is an arbitrary positive scalar and I is the identity map. Note that if F is monotone, then the mapping $F^k = F + \varepsilon_k I$ is strongly monotone. Thus, each subproblem VI(X, F^k) has a unique solution. Rockafellar [228] has shown that if each x^{k+1} is chosen according to an appropriate inexact rule, then the sequence $\{x^k\}$ is bounded if and only if the problem VI(X, F) has a solution. Moreover, if the sequence $\{x^k\}$ is bounded, then it converges to a certain solution of VI(X, F). Since each F^k is strongly monotone, any of the abovementioned algorithms can be used to solve the subproblem VI(X, F^k) for x^{k+1}. Related work includes the papers [221, 246, 247, 248, 255]; the papers by Spingarn contain some interesting ideas that have not been fully explored.

There are several other approaches for solving the variational inequlity and nonlinear complementarity problems. For the variational inequality problem, Lüthi [153] provides an ellipsoid algorithm, Nguyen and Dupis [193] and Marcotte [163] develop cutting-plane based methods, and Smith [241, 242, 243] provides an algorithm which is based upon a Lyapunov stability argument. For the complementarity problem, Stone [252] reports on the direct use of a nonlinear optimization system (MINOS) to solve the mathematical programming formulation (15) of the NCP(F), Habetler and Kostreva [84] and Kostreva [136] provide some direct pivot methods for the same problem, various approaches have been taken which are based on the fixed-point and/or nonlinear equation formulation of this problem [67, 68, 261, 254, 131], and McLinden [174, 175] describes a conceptual algorithm for solving generalized complementarity problems with maximal monotone mappings. Recently, Kojima et al. [134] described a continuation method for the nonlinear complementarity problem with a uniform P-function that is closely related to McLinden's work. This method is extended to the case of a monotone complementarity problem in a subsequent paper [135] by the same authors. Finally, Mangasarian and McLinden [158] provide bounds on the L_1 norm of components of the solution vector for the nonlinear complementarity problem; these bounds may prove useful in the design of algorithms for this class of problems by iteratively bounding the region in which the solution must lie.

To close this section, let us mention three recent reports [105, 205, 206] which were completed while the present paper was undergoing its review for publication. In two of these reports [105], [205], the authors propose solving the variational inequality and nonlinear complementarity problems as a system of "B-differentiable" equations by an extension of the classical Newton method and its damped-version for continuously differentiable equations; in [206], the idea of continuation is suggested as a means to globalize the Newton method described in this section. Both of these suggestions are promising but require further study.

5. Sensitivity analysis

As in the case of a mathematical program, sensitivity analysis is an important part in the solution of variational inequalities and nonlinear complementarity problems. In the context of equilibrium modeling, the need for sensitivity analysis is even more acute due to the nature of the modeling effort. When one computes equilibria, the results are used to predict the behavior of the underlying system; that is, the modeling effort is descriptive or "positive" rather than normative. In such instances, one must carefully explore the sensitivity and stability of the equilibria to changes in model parameters such as demand elasticities, conjectural variations, etc. Ideally, one would like to compute all equilibria when the parameters vary over their range of values. However, such a task is computationally very complex (if at all possible) and thus, one must oftentimes resort to analyzing the effects of parameters changes on a single equilibrium point.

In nonlinear programming, sensitivity analysis is a well-studied subject. The book by Fiacco [64] describes the general theory and provides various computational procedures (see also Banks et al. [16]). Over the past several years, there has been an increasing number of extensions of these results to variational inequalities and nonlinear complementarity problems. In this section, we give a summary of the principal sensitivity results for the latter two problems.

Consider the family of *parameterized variational inequality problems*:

$$\{\mathrm{VI}(X_\lambda, F(\cdot, \lambda)) \colon \lambda \in \Lambda\}, \tag{42}$$

where Λ is a subset of $\mathbb{R}^l$ (the space of parameters), $F: \mathbb{R}^{n+l} \to \mathbb{R}^n$ and for each $\lambda \in \Lambda$, X_λ is a nonempty, closed and convex subset of $\mathbb{R}^n$. For each $\lambda \in \Lambda$, let $S(\lambda)$ denote the (possibly) empty solution set of $\mathrm{VI}(X_\lambda, F(\cdot, \lambda))$. The central problem of sensitivity analysis deals with the investigation of the behavior of the family $\{S(\lambda)\}$ as the parameter vector λ varies over the set Λ. There are two general types of analysis that one might want to perform: local and global. In local sensitivity analysis, a specific value λ^* and a solution $x^* \in S(\lambda^*)$ are given, and one is interested in analyzing the change of x^* as λ varies in a small neighborhood of λ^*. In this case, the set Λ is taken to be such a neighborhood. Some typical questions to be answered are: will $S(\lambda)$ remain nonempty for all $\lambda \in \Lambda$? If the problem $\mathrm{VI}(X_{\lambda^*}, F(\cdot, \lambda^*))$ has

x^* as a locally unique solution, will $\mathrm{VI}(X_\lambda, F(\cdot, \lambda))$ also have locally unique solutions for $\lambda \in \Lambda$? If so, what can one say about the continuity and/or differentiability of the solution function at the point λ^*? How can one calculate the derivatives if they exist? In global sensitivity analysis, the set Λ is not restricted to be the neighborhood of a certain base point λ^*. In this case, the continuity of $S(\lambda)$ as a point-to-set map is of particular interest.

A useful apparatus for obtaining the local sensitivity results is the notion of a regular solution (see Definition 4.1). With regularity, we state a general result for the family (42) where X_λ is independent of λ; i.e., for the case where the parameters solely affect the function F (this includes the nonlinear complementarity problem as a special case). the proof of the result follows from the representation of the variational inequality problem as a generalized equation for which an implicit-function theorem has been established by Robinson [217].

Theorem 5.1 [218, 221]. *Let X be a nonempty, closed and convex subset of $\mathbb{R}^n$ and let $F:\mathbb{R}^{n+l} \to \mathbb{R}^n$ be continuously differentiable. Suppose that x^* is a regular solution of the problem* $\mathrm{VI}(X, F(\cdot, \lambda^*))$. *Then there exist neighborhoods N of λ^* and W of x^* such that for each $\lambda \in N$, there is a unique vector $x(\lambda)$ in W which solves the perturbed* $\mathrm{VI}(X, F(\cdot, \lambda))$; *moreover, $x(\lambda)$ is Lipschitz continuous as a function of $\lambda \in N$. Finally, $x(\lambda)$ is B-differentiable at λ^* if X is polyhedral.*

Note that the above theorem states that under the assumption of a regular solution, each perturbed problem $\mathrm{VI}(X, F(\cdot, \lambda))$ has a locally unique solution provided that λ is within a neighborhood of λ^*.

The notion of B-differentiability used in Theorem 5.1 first appeared in Robinson [221, 222] where various properties were derived. Further properties of a B-differentiable function are obtained in [205, 223, 237]. In particular, Shapiro [237] established that in finite-dimensional Euclidean space, a locally Lipschitzian function is B-differentiable at a point if and only if it is directionally differentiable there; in this case, the B-derivative coincides with the directional derivative. In what follows, we apply Theorem 5.1 to the family (42) where each X_λ is defined by

$$X_\lambda = \{x \in \mathbb{R}^n \colon g(x, \lambda) \leqslant 0, \quad h(x, \lambda) = 0\}, \tag{43}$$

where $g:\mathbb{R}^{n+l} \to \mathbb{R}^m$, $h:\mathbb{R}^{n+l} \to \mathbb{R}^p$ are continuously differentiable. Under a suitable constraint qualification, each member problem in the family (42) can be formulated as a mixed nonlinear complementarity problem:

$$\begin{aligned} &F(x, \lambda) + \sum_{i=1}^{m} \pi_i \nabla_x g_i(x, \lambda) + \sum_{j=1}^{p} \mu_j \nabla_x h_j(x, \lambda) = 0, \\ &\pi_i \geqslant 0, \quad \pi_i g_i(x, \lambda) = 0, \qquad i = 1, 2, \ldots, m, \\ &h_j(x, h) = 0, \quad j = 1, 2, \ldots, p. \end{aligned} \tag{44}$$

For each $\lambda \in \Lambda$, define the Lagrangian function $L: \mathbb{R}^{n+m+p} \to \mathbb{R}^n$ by

$$L(x, \pi, \mu, \lambda) = F(x, \lambda) + \sum_{i=1}^{m} \pi_i \nabla_x g_i(x, \lambda) + \sum_{j=1}^{p} \mu_j \nabla_x h_j(x, \lambda). \tag{45}$$

Now suppose that $z^* = (x^*, \pi^*, \mu^*)$ is a regular solution of (44) with $\lambda = \lambda^*$. Let I_+, I_0 be the two index sets defined by

$$I_+ = \{i: \pi_i^* > 0\}, \qquad I_0 = \{i: g_i(x^*) = 0, \pi_i^* = 0\} \tag{46}$$

(cf. Proposition 4.1). As a mixed nonlinear complementarity problem, the regularity of z^* can be characterized by the following two conditions which are slight generalizations of those in Proposition 4.2 for a pure NCP(F):

(a) the following matrix is nonsingular,

$$M(z^*, \lambda^*) = \begin{bmatrix} \nabla_x L(z^*, \lambda^*) & (\nabla_x g_{I_+}(x^*, \lambda^*))^{\mathrm{T}} & (\nabla_x h(x^*, \lambda^*))^{\mathrm{T}} \\ -\nabla_x g_{I_+}(x^*, \lambda^*) & 0 & 0 \\ -\nabla_x h(x^*, \lambda^*) & 0 & 0 \end{bmatrix}; \tag{47}$$

(b) the following matrix is a P-matrix,

$$[\nabla_x g_{I_0}(x^*, \lambda^*) \;\; 0 \;\; 0] M(z^*, \lambda^*)^{-1} \begin{bmatrix} \nabla_x g_{I_0}(x^*, \lambda^*)^{\mathrm{T}} \\ 0 \\ 0 \end{bmatrix}.$$

Specializing Theorem 5.1 to the problem (44), one can state the following result which summarizes the principal local sensitivity properties for the family (42) with each X_λ of the form (43).

Theorem 5.2 [218, 221, 141, 142]. *Let $F: \mathbb{R}^{n+l} \to \mathbb{R}^n$ be once continuously differentiable and let $g: \mathbb{R}^{n+l} \to \mathbb{R}^m$, $h: \mathbb{R}^{n+l} \to \mathbb{R}^p$ be twice continuously differentiable. Suppose that $z^* = (x^*, \pi^*, \mu^*)$ is a regular solution of* (44) *with $\lambda = \lambda^*$. Let I_+ and I_0 be as defined in* (46). *Then there exists a neighborhood N of λ^* such that*

(a) *for each $\lambda \in N$, the system* (44) *has a locally unique solution $(x(\lambda), \pi(\lambda), \mu(\lambda))$ with*

$$(x(\lambda^*), \pi(\lambda^*), \mu(\lambda^*)) = (x^*, \pi^*, \mu^*);$$

(b) *as a function of $\lambda \in N$, $(x(\lambda), \pi(\lambda), \mu(\lambda))$ is Lipschitz continuous;*

(c) *the solution function $(x(\lambda), \pi(\lambda), \mu(\lambda))$ is directionally differentiable at λ^* along any direction. The directional derivative along a direction d is given by the vector $(\bar{x}, \bar{\pi}, \bar{\mu})$ which is the unique solution to the following system:*

$$\begin{aligned} &\nabla_x L(z^*, \lambda^*)\bar{x} + \nabla_x g_{I_+}(z^*, \lambda^*)^{\mathrm{T}} \bar{\pi}_{I_+} + \nabla_x g_{I_0}(z^*, \lambda^*)^{\mathrm{T}} \bar{\pi}_{I_0} \\ &\quad + \nabla_x h(z^*, \lambda^*)^{\mathrm{T}} \bar{\mu} + \nabla_\lambda L(z^*, \lambda^*) d = 0, \\ &\nabla_x g_{I_+}(z^*, \lambda^*)\bar{x} + \nabla_\lambda g_{I_+}(z^*, \lambda^*) d = 0, \\ &\nabla_x g_{I_0}(z^*, \lambda^*)\bar{x} + \nabla_\lambda g_{I_0}(z^*, \lambda^*) d \leqslant 0, \qquad \bar{\pi}_{I_0} \geqslant 0, \\ &[\nabla_x g_{I_0}(z^*, \lambda^*)\bar{x} + \nabla_\lambda g_{I_0}(z^*, \lambda^*) d]^{\mathrm{T}} \bar{\pi}_{I_0} = 0, \\ &\nabla_x h(z^*, \lambda^*)\bar{x} + \nabla_\lambda h(z^*, \lambda^*) d = 0, \\ &\bar{\pi}_i = 0 \quad \forall i \in \{1, 2, \ldots, m\} \backslash \{I_+ \cup I_0\}; \end{aligned} \tag{48}$$

(d) *as a function of the direction vector d, the solution* $(\bar{x}, \bar{\pi}, \bar{\mu})$ *of the system* (48) *is Lipschitz continuous.*

Some remarks concerning the above theorem are in order. First, according to the characterization of regularity mentioned above, it is easy to deduce the fact that the system (48) does indeed have a unique solution $(\bar{x}, \bar{\pi}, \bar{\mu})$ for each vector d. Moreover, $(\bar{x}, \bar{\pi}, \bar{\mu})$ can be computed by solving a mixed linear complementarity problem. Second, part (a) of Theorem 5.2 does not assert that $x(\lambda)$ solves the perturbed $\mathrm{VI}(X_\lambda, F(\cdot, \lambda))$. In order for this condition to hold, we must invoke some convexity property of the set X_λ which is not assumed in the theorem (cf. part (b) of Proposition 2.2). Third, although Theorem 5.2 is derived as a special case of Theorem 5.1, the former result may be considered as treating a broader family of parameterized problems (42) than the latter result due to the fact that the set X_λ is permitted to vary according to λ in Theorem 5.2; no such changes are permitted in Theorem 5.1.

In general, B-differentiability is weaker than Fréchet differentiability. Theorem 5.2 does not assert the Fréchet differentiability of the solution function $(x(\lambda), \pi(\lambda), \mu(\lambda))$ at λ^*. Extending a result by Pang [204] for the nonlinear complementarity problem, we state the following necessary and sufficient conditions for such a Fréchet differentiability property to hold.

Theorem 5.3 [204, 206]. *Let* F, g, h, $z^* = (x^*, \pi^*, \mu^*)$, I_+, I_0 *and* d *be given as in Theorem* 5.2 *and the matrix* $M(z^*, \lambda^*)$ *be given by* (47). *A necessary and sufficient condition for the solution function* $(x(\lambda), \pi(\lambda), \mu(\lambda))$ *to be Fréchet differentiable at* λ^* *is that either the index set* I_0 *is empty, or the matrix*

$$\nabla_\lambda g_{I_0}(x^*, \lambda^*) - [\nabla_x g_{I_0}(x^*, \lambda^*) \;\; 0 \;\; 0] M(z^*, \lambda^*)^{-1} \begin{bmatrix} \nabla_\lambda L(z^*, \lambda^*) \\ \nabla_\lambda g_{I_+}(x^*, \lambda^*) \\ \nabla_\lambda h(x^*, \lambda^*) \end{bmatrix}$$

is equal to zero. If $(x(\lambda), \pi(\lambda), \mu(\lambda))$ *is indeed Fréchet differentiable at* λ^*, *then*

$$\nabla \begin{bmatrix} x(\lambda^*) \\ \pi_{I_+}(\lambda^*) \\ \mu(\lambda^*) \end{bmatrix} = M(z^*, \lambda^*)^{-1} \begin{bmatrix} \nabla_\lambda L(z^*, \lambda^*) \\ \nabla_\lambda g_{I_+}(x^*, \lambda^*) \\ \nabla_\lambda h(x^*, \lambda^*) \end{bmatrix}$$

and $\nabla \pi_i(\lambda^*) = 0 \;\forall i \notin I_+$.

Note that the case where $I_0 = \emptyset$ in Theorem 5.3 corresponds to the strict complementarity property holding; i.e., $\pi^* + g(x^*, \pi^*) > 0$ (cf. Definition 3.11).

An important feature of the results in Theorems 5.2 and 5.3 is that they are derived under the assumption of a regular solution to the system (44). A consequence of such a regularity condition is the uniqueness of the *multiplier vectors* $\pi(\lambda)$, $\mu(\lambda)$.

In two recent reports [212, 213], Qui and Magnanti have generalized Theorem 5.2 by allowing non-unique multipliers; related papers include Kyparisis [144] and Dafermos [43]. The reader is referred to the references for details. The extent to which Theorem 5.3 remains valid in the absence of the regularity assumption has yet to be examined.

We next turn our attention to some global sensitivity result for the family (42). Results of this type differ from the local results in two major aspects. First, they all require stronger assumptions. Fortunately, this weakness is compensated by the fact that no differentiability assumption is required. Of course, without the latter assumption, no differentiability conclusion can be drawn.

The following result makes use of the notion of strong monotonicity (cf. Definition 3.1(d)). This result applies to the family (42) where the set X_λ is independent of λ.

Theorem 5.4 [44, 45]. *Let X be a nonempty, closed and convex subset of $\mathbb{R}^n$, and for each $\lambda \in \Lambda$, let $F(\cdot, \lambda): \mathbb{R}^n \to \mathbb{R}^n$ be continuous and strongly monotone with respect to X with modulus α_λ. Let $x(\lambda)$ be the unique solution of the* VI(X, $F(\cdot, \lambda)$). *Then for any $\lambda, \lambda' \in \Lambda$,*

$$\|x(\lambda')-x(\lambda)\|_2 \leq \|F(x(\lambda'), \lambda')-F(x(\lambda'), \lambda)\|_2/\alpha_\lambda. \tag{49}$$

The expression (49) can be used to establish the continuity of the solution function $x(\lambda)$ under various additional assumptions. For example, if for each $x \in X$, the function $F(x, \cdot)$ is Lipschitz continuous in λ and if the Lipschitz constant is independent of x, then it follows from (49) that $x(\lambda)$ is continuous. Similarly, if for each $x \in X$, the function $F(x, \cdot)$ is continuous in λ and if the modulus α_λ of strong monotonicity is independent of λ, then the same conclusion holds.

In the case of the nonlinear complementarity problem, the strong monotonicity assumption in Theorem 5.4 can be replaced with the uniform P-property (cf. Definition 3.7(b)) and a similar conclusion can be established.

Unlike Theorem 5.1, Theorem 5.4 cannot be specialized to the problem (44) due to the fact that the following mapping can never be strongly monotone (or be a uniform P-function) in (x, π, μ) jointly:

$$H\begin{pmatrix} x \\ \pi \\ \mu \\ \lambda \end{pmatrix} = \begin{pmatrix} L(x, \pi, \mu, \lambda) \\ -g(x, \lambda) \\ h(x, \lambda) \end{pmatrix}.$$

Instead of deriving a refinement of Theorem 5.4 for the problem (44), we state a general result asserting the continuity of the solution map $S(\lambda)$. This result is applicable to the general family (42) where each set X_λ is not assumed to be of any particular structure. For this purpose, we consider each set X_λ as the image of the vector $\lambda \in \Lambda$ under the point-to-set map $X: \Lambda \to \mathbb{R}^n$. The first two parts of the following result are quoted from Harker and Pang [104], and the last two parts can be easily proven.

Theorem 5.5 [104]. *Let Λ be a nonempty and closed subset of $\mathbb{R}^l$, $F:\mathbb{R}^{n+l}\to\mathbb{R}^n$ be continuous, and $X:\Lambda\to\mathbb{R}^n$ be a nonempty-valued, continuous (i.e., both upper and lower semi-continuous) point-to-set map. Define $S:\Lambda\to\mathbb{R}^n$ to be the solution map of the family of parameterized variational inequalities* (42). *Then*

(a) *S is a closed point-to-set map*;

(b) *if X is convex-valued and compact-valued, then S is nonempty-valued, compact-valued and upper semi-continuous*;

(c) *if X is convex-valued and if for each $\lambda\in\Lambda$, $F(\cdot,\lambda):\mathbb{R}^n\to\mathbb{R}^n$ is coercive, then the conclusion of* (b) *holds*;

(d) *if in addition to the assumptions in* (b) *or* (c), *S is single-valued at a particular value λ^*, then S is continuous at λ^*.*

The final approach to be mentioned in this paper which is used to analyze the sensitivity, or more specifically, the *stability* of the solution vector to changes in problem parameters involves the use of either *degree theory* or *Lyapunov stability analysis.* In general, very little research has been done in this area. Ha [83] presents the application of degree theory in the study of the stability of solutions to the nonlinear complementarity problem, and M. J. Smith [244] illustrates on the traffic assignment problem how Lyapunov stability analysis can be used to check the stability or instability of equilibrium points. While sensitivity and stability analysis are related, stability analysis is more concerned with the dynamics of reaching equilibria while sensitivity analysis simply deals with the question of whether or not the equilibrium set is continuous and differentiable. Given the increasing importance of nonlinear dynamics and in particular, *chaos theory* [115], a great deal of future research should be dedicated to the analysis of the stability of variational inequalities.

6. Generalizations

There are several generalizations of the problems VI(X, F) and GCP(X, F) in which the fixed set X and the mapping F are replaced by point-to-set mappings [15]. In this section, we define these *generalized* variational inequality and complementarity problems and summarize their properties; applications of these problems will be discussed in the next section.

To begin, let us define the most general of the extensions to the basic variational inequality problem:

Definition 6.1. Let X and F be point-to-set mappings (called *multi-functions*) from $\mathbb{R}^n$ into subsets of $\mathbb{R}^n$. The *generalized quasivariational inequality problem*, denoted by GQVI(X, F), is to find vectors $x^*\in X(x^*)$ and $y^*\in F(x^*)$ such that

$$(x-x^*)^{\mathrm{T}}y^*\geqslant 0 \quad \forall x\in X(x^*).$$

If the map X is constant (i.e., if $X(x) = X \ \forall x$), then the GQVI(X, F) is called the *generalized variational inequality problem* and is denoted by GVI(X, F). If the map F is point-to-point and if X is point-to-set, the GQVI(X, F) is called the *quasivariational inequality problem* and is denoted by QVI(X, F).

Just as in the case of the standard VI(X, F), the GVI(X, F) is closely related to the mathematical program (4). Indeed, if f is a convex and real-valued function and if X is a closed, convex set, then a vector x^* is a global minimum solution of (4) if and only if x^* solves the GVI(X, ∂f) where ∂f denotes the *subdifferential mapping* of f; i.e., ∂f is the set of subgradient vectors of f at x. We refer the reader to Rockafellar [225] for further discussion on the connection between the GVI and a convex mathematical program, and to [224, 226] for conditions under which an arbitrary multi-function F is the subdifferential mapping of a convex, real-valued function. In general, a vector x^* solves the GVI(X, F) if and only if x^* is a *zero* of the operator $F(\cdot) + N_X(\cdot)$ where for each vector x, $N_X(x)$ is the normal cone defined in (2); i.e., if and only if $0 \in F(x^*) + N_X(x^*)$. Extensive studies on the GVI can be found in Fang's Ph.D. dissertation [56] and in the paper [61]. An application of the GVI to the general network equilibrium problem can be found in [62].

The quasivariational inequality problem was introduced by Bensoussan and Lions [20, 21, 22] in their study of impulse control theory; thus, the original setting of the problem was in an infinite-dimensional metric space. As an attempt to combine this infinite-dimensional work and the finite-dimensional generalized variational inequality problem, Chan and Pang [28] defined the generalized quasivariational inequality problem as given in Definition 6.1.

Separately, the generalized complementarity problem GCP(X, F) was extended by Saigal [233] to the case where F is a point-to-set mapping. The extended problem was studied rather extensively by McLinden [174, 175, 176] which also deals extensively with the GVI (see also the work by Spingarn [246]). Motivated by the development in the generalized variational inequality setting, Chan and Pang [28] further extended Saigal's problem to the following:

Definition 6.2. Let Y and F be point-to-set mappings from $\mathbb{R}^n$ into subsets of $\mathbb{R}^n$, with Y being (convex) cone-valued. Let m be a point-to-point mapping from $\mathbb{R}^n$ into itself. The *generalized implicit complementarity problem*, denoted by GICP(m, Y, F), is to find vectors $x^* \in m(x^*) + Y(x^*)$ and $y^* \in \mathbb{R}^n$ such that

$$y^* \in F(x^*) \cap Y(x^*)^* \quad \text{and} \quad (x^* - m(x^*))^{\mathrm{T}} y^* = 0.$$

If the map Y is a constant and if m is identically equal to zero, we denote the problem GICP(m, Y, F) as SGCP(Y, F); this latter problem is Saigal's extension of the GCP. If $Y(x)$ is equal to the nonnegative orthant $\mathbb{R}^n_+$ for all x and if F is point-to-point, then the problem GICP(m, Y, F) becomes the *implicit complementarity problem* which is denoted by ICP(m, F).

As one expects, there exists a close relationship between the GQVI and the GICP; this relationship is now made explicit:

Proposition 6.1 [28]. *The solutions sets of the* GICP(m, Y, F) *and the* GQVI(X, F) *where* $X(x) = m(x) + Y(x)$ *are equal.*

Similar to the variational inequality problem VI, the most fundamental existence result for the GQVI(X, F) requires some type of compactness assumption on the multi-function X and a continuity assumption on F. The proof of the following result makes use of the well-known Kakutani fixed point theorem.

Theorem 6.1 [28]. *Let X and F be point-to-set mappings from $\mathbb{R}^n$ into nonempty, convex subsets of $\mathbb{R}^n$. Suppose that there exists a nonempty, compact and convex set C such that*

(i) *$X(x)$ is contained in C for all $x \in C$;*

(ii) *F is compact-valued and upper semi-continuous on C;*

(iii) *X is continuous on C.*

Then there exists a solution to the GQVI(X, F).

Condition (i) in the above theorem is the key compactness assumption required on the multifunction X. By replacing this assumption with a coercivity condition on the multi-function F, one can state the following existence result for the GQVI.

Theorem 6.2 [28]. *Let X and F be point-to-set mappings from $\mathbb{R}^n$ into nonempty, convex subsets of $\mathbb{R}^n$. Suppose that*

(i) *there exists a vector $x^0 \in \bigcap_{x \in \mathbb{R}^n} X(x)$ satisfying*

$$\lim_{x \in X(x), \|x\| \to \infty} \left[\inf_{y \in F(x)} (x - x^0)^T y / \|x\| \right] = \infty;$$

(ii) *F is compact-valued and upper semi-continuous on $\mathbb{R}^n$;*

(iii) *there exists a $\rho_0 > 0$ such that $X(x) \cap B_\rho$ is a continuous mapping for all $\rho \geq \rho_0$.*

Then there exists a solution to the GQVI(X, F).

Specializing this theorem to the GICP, we obtain the following existence result.

Corollary 6.1 [28]. *Let $\tilde{Y}$ be a nonempty, closed, convex and solid cone in $\mathbb{R}^n$. Let m be a continuous point-to-point mapping from $\mathbb{R}^n$ into itself, and let F be a point-to-set mapping from $\mathbb{R}^n$ into nonempty convex subsets of $\mathbb{R}^n$. Let $X(x) = m(x) + \tilde{Y}$. Suppose that*

(i) *there exists a vector u such that for all vectors $x \in \mathbb{R}^n$ one has $u - m(x) \in \tilde{Y}$;*

(ii) *the mapping F is strongly copositive with respect to X at the point u; i.e., there exists a constant $\alpha > 0$ and a vector $v \in F(u)$ such that for all $x \in X(x)$ and for all $y \in F(x)$,*

$$(y - v)^T (x - u) \geq \alpha \|x - u\|_2^2;$$

(iii) *F is compact-valued and upper semi-continuous on $\mathbb{R}^n$.*

Then there exists a solution to the GICP(m, Y, F) *where* $Y(x) = \tilde{Y}$ *for all* x.

Notice that if $\tilde{Y}$ is the nonnegative orthant, then condition (i) in the above corollary simply states that the mapping m is bounded above by the vector u.

There are several other existence results for the GQVI and the GICP; these results can be found in [56, 28]. Uniqueness results for the GQVI(X, F) are non-existent. Indeed, any such result would require very strong assumptions on the multi-functions X and F. As far as algorithms are concerned, very few exist for the GQVI. A simple projection algorithm is described in [28] for the problem GQVI(X, F) where F is point-to-point and $X(x) = m(x) + \tilde{X}$, and several iterative methods for the ICP are discussed by Pang [197, 199]. Finally, few, if any, sensitivity results are available for both the GQVI and the GICP (see, however, [176]). In summary, these two problems provide a fertile area for future research.

We close this section by briefly discussing the problem of a mathematical program which contains variational inequality/nonlinear complementarity constraints. This problem is defined as follows:

$$\text{minimize } f(x, y) \quad \text{subject to } x \in X \text{ and } y \in Y(x) \tag{50}$$

where $f: \mathbb{R}^{n+m} \to \mathbb{R}$, X is a nonempty and closed subset of $\mathbb{R}^n$, and for each $x \in X$, $Y(x)$ is the solution set of a variational inequality

$$Y(x) = \{y \in U(x) \colon (z - y)^{\mathrm{T}} F(x, y) \geq 0 \ \forall z \in U(x)\}$$

where $U: X \to \mathbb{R}^m$ is a multi-function and $F: \mathbb{R}^{n+m} \to \mathbb{R}^m$ is a point-to-point mapping. By letting S denote the feasible region of the mathematical program (50); i.e.,

$$S = \{(x, y) \in \mathbb{R}^{n+m} \colon x \in X, y \in Y(x)\},$$

Theorem 5.5 gives sufficient conditions for S to be a closed set. Thus, the following existence result for an optimal solution to (50) can be proven by employing a coercivity assumption on the objective function f.

Theorem 6.3 [104]. *Let X be a nonempty and closed subset of $\mathbb{R}^n$. Let $f: \mathbb{R}^{n+m} \to \mathbb{R}$, $F: \mathbb{R}^{n+m} \to \mathbb{R}^m$ be continuous, and $U: X \to \mathbb{R}^m$ be a nonempty-valued and continuous point-to-set map. Suppose that program* (50) *is feasible and that*

$$\lim_{(x,y) \in S, \|(x,y)\| \to \infty} f(x, y) = \infty.$$

Then there exists an optimal solution to (50).

In general, problem (50) is very difficult to solve due to the nonconvexity of the feasible region S. Many heuristics have been proposed [3, 103, 165], but no well-tested and globally convergent algorthms exist. Thus, the problem of a mathematical program with variational inequality/nonlinear complementarity constraints also provides an area for future research.

7. Applications

In this section, we survey four major applications in which the theory and algorithms of variational inequalities and nonlinear complementarity problems have made a substantial impact. This impact consists of two parts: theoretical and algorithmic insights. Conversely, these applications have generated new questions in variational inequality/nonlinear complementarity theory, many of which have yet to be answered. In terms of theoretical insight, the use of variational inequalities often aids in better understanding the conditions under which an equilibrium exists and is unique, and in the comparative statics (sensitivity analysis) of the equilibria. Algorithmic insight represents the understanding which one can obtain through the numerical solution of a model. While it is not always necessary to numerically solve a model in order to reach certain conclusions about the properties of the equilibrium, such numerical solution is often the only way to obtain any insights into the qualitative properties of large-scale models [173].

After formulating the various applications, we will present the qualitative properties of these models with the tools of Section 3, discuss specialized algorithms for these applications, and then describe *real* applications of the models; i.e., the use of these models and associated algorithms in problems arising from real engineering, public policy and strategic planning issues. As in previous sections, space limits our ability to provide an exhaustive survey of applications; however, the Reference Section does contain such a comprehensive list. One purpose for this section is to encourage the creative use of the variational inequality/nonlinear complementarity framework in new and innovative application areas.

The first application area to be surveyed is in fact the most general: the *Nash equilibrium of an n-person noncooperative game* [192]:

Definition 7.1. Given a set N of n players in a noncooperative game where each player $i \in N$ is represented by a strategy vector $x_i \in X_i \subseteq \mathbb{R}^{m_i}$ (m_i being a positive integer), and a utility function $u_i : X \to \mathbb{R}$ where $X \equiv \prod_{i \in N} X_i$ and $u = (u_1, u_2, \ldots, u_n)^{\mathrm{T}}$, a *Nash equilibrium* $x^* \in X$ *of the game* $\mathrm{NE}(X, u)$ is defined as a point at which no player can unilaterally increase his utility:

$$u_i(x_i^*, x_{N\setminus\{i\}}^*) \geqslant u_i(x_i, x_{N\setminus\{i\}}^*) \quad \forall x_i \in X_i$$

where $x_{N\setminus\{i\}} = (x_j : j \in N, j \neq i)$.

The fact that this problem could be formulated as a variational inequality $\mathrm{VI}(X, F)$ dates back to the early papers by Lions and Stampacchia [152] and Bensoussan [19]. The following result can be easily proven by summing the first-order optimality conditions arising from each player's utility maximization problem; the fact that X is the full Cartesian product of the individual strategy sets X_i makes the summation of the first-order conditions sufficient as well.

Proposition 7.1 [80]. *Let X_i be nonempty, closed and convex subset of $\mathbb{R}^{m_i}$, $m_i \geq 1$ and $u_i : X \to \mathbb{R}$ be once continuously differentiable and pseudo-concave with respect to x_i for all $i \in N$. Then $x^* \in X$ is a solution to the Nash equilibrium problem* NE(X, u) *if and only if $x^* \in X$ is a solution to the following variational inequality* VI(X, F):

$$\sum_{i \in N} F_i(x^*)^{\mathrm{T}}(x_i - x_i^*) \geq 0 \quad \forall x \in X \tag{51}$$

where

$$F(x) = (F_i(x) : i \in N), \quad F_i : X \to \mathbb{R}^{m_i} \text{ such that } F_i(x) = -\nabla_{x_i} u_i(x). \tag{52}$$

Thus, the Nash equilibrium problem can very naturally be formulated as a variational inequality problem. The existence and uniqueness of a Nash equilibrium can be proven by using any one of the results listed in Section 3. For example, Nash's [192] famous existence result for noncooperative games follows directly from Theorem 3.1:

Proposition 7.2 [192]. *Let X_i be a nonempty, compact and convex subset of $\mathbb{R}^{m_i}$, $m_i \geq 1$, and $u_i : X \to \mathbb{R}$ be once continuously differentiable and pseudo-concave with respect to x_i for all $i \in N$. Then a solution $x^* \in X$ to the Nash equilibrium problem* NE(X, u) *exists.*

Other applications of the results of Section 3 to the Nash equilibrium problem can be found in Border [25] and Aubin [12].

The concept of a Nash equilibrium can be extended to include additional joint constraints on players' actions which cut across all players simultaneously. Such constraints make the strategy sets X_i of each player dependent on the full strategy vector x and hence, the regular Nash equilibrium concept is not applicable. As discussed in Harker [102], the introduction of such joint constraints is controversial from the viewpoint of economics, but is a very useful mathematical device. Thus, the Nash equilibrium concept can be extended in order to form the notion of *generalized Nash equilibrium* or *social equilibrium* in Ichiishi [116]. The generalized Nash equilibrium problem GNE(X, K, u) is formally defined as follows:

Definition 7.2. Given a set N of n players in a noncooperative game where each player $i \in N$ is represented by a strategy vector $x_i \in X_i \subseteq \mathbb{R}^{m_i}$ (m_i being a positive integer), a point-to-set mapping $K_i : X \to X_i$, and a utility function $u_i : X \to \mathbb{R}$ where $X \equiv \prod_{i \in N} X_i$, $K = \prod_{i \in N} K_i : X \to X$, and $u = (u_1, u_2, \ldots, u_n)^{\mathrm{T}}$, a *generalized Nash equilibrium $x^* \in X$ of the game* GNE(X, K, u) is defined as a point at which no player can unilaterally increase his utility *given the constraints imposed on him by the other players*:

$$u_i(x_i^*, x_{N\setminus\{i\}}^*) \geq u_i(x_i, x_{N\setminus\{i\}}^*) \quad \forall x_i \in K_i(x^*)$$

where $x_{N\setminus\{i\}} = (x_j : j \in N, j \neq i)$.

Thus, the multi-function K represents the constraints which are imposed across the players of the game.

Bensoussan [19] was the first to recognize that the game GNE(X, K, u) can be case into the form of a quasivariational inequality (Definition 6.1); this formulation was recently extended in Baiocchi and Capelo [15]. For the finite-dimensional problem, Harker [102] has recently established the following result.

Proposition 7.3 [102]. *Let X_i be nonempty, closed and convex subset of $\mathbb{R}^{m_i}$, $m_i \geq 1$, $K_i : X \to X_i$ be a nonempty and closed map, and $u_i : \mathrm{gr}\, K_i \to \mathbb{R}$ be once continuously differentiable and pseudo-concave with respect to x_i for all $i \in N$, where $\mathrm{gr}\, K_i$ denotes the graph of the map K_i. Then $x^* \in K(x^*)$ is a solution to the generalized Nash equilibrium problem* GNE(X, K, u) *if and only if $x^* \in K(x^*)$ is a solution to the following quasivariational inequality* QVI(K, F):

$$\sum_{i \in N} F_i(x^*)^{\mathrm{T}}(x_i - x_i^*) \geq 0 \quad \forall x \in K(x^*) \tag{53}$$

where F is defined in (52).

The existence of a solution to the generalized Nash game follows from the results of Section 6. For example, the general existence result in Ichiishi [116] can be obtained as a special case of Theorems 6.1 and 6.2:

Proposition 7.4. *Let X_i be a nonempty, convex and compact subset of $\mathbb{R}^{m_i}$, $K_i : X \to X_i$ be an upper and lower semicontinuous map in which $K_i(x)$ is nonempty, closed and convex for all $x \in X$, u_i is continuous on $\mathrm{gr}\, K_i$, and $u_i(x_i, x_{N\setminus\{i\}})$ is quasi-concave in $x_i \in K_i(x)$ for all $x \in X$ and for all $i \in N$. Then a solution $x^* \in K(x^*)$ to the generalized Nash equilibrium problem* GNE(X, K, u) *exists.*

In Harker [102], specializations of this result to a quasivariational inequality defined over a *sectional correspondence* (which includes the case where K is polyhedral) are explored.

The need for the numerical solution of the Nash equilibrium problem has always been and continues to be a major impetus for the development of algorithms for complementarity problems and variational inequalities. For example, the well-known Lemke algorithm [149, 150] for linear complementarity problems was developed in order to solve bi-matrix games. Nash's existence result was published in 1950, and it was not until 1964 that Lemke and Howson created the first algorithm to compute Nash equilibria in the special case of bi-matrix games. The Lemke–Howson discovery opened up a new area of research devoted to the computation of game-theoretic equilibria. In recent years, several algorithms based on those presented in Section 4 have been specialized to the Nash equilibrium problem [91, 162, 169, 171]. For the generalized Nash equilibrium problem, Harker [102] represents the only algorithmic work to date.

An extension of the simple Nash equilibrium problem is the so-called Stackelberg or *leader-follower* problem in which a market leader L attempts to maximize his utility subject to the equilibrium conditions of the set of followers N. This problem is a special case of the general problem of optimization subject to variational inequality constraints (50). The purpose of this model is to explicitly incorporate the reactions of the competitors in order to arrive at a "better" decision than would be achieved if one naively assumed that competitors would not react to the leader's strategy.

Applications of the Nash equilibrium and associated Stackelberg equilibrium concepts abound; see the Reference section for a listing of a sample of this long list. Mathiesen and Lont [171] provide an application to the prediction of prices and quantities in the world steel market, and Choi et al. [30] provides an application of the Stackelberg model in a marketing situation in which firms compete over price and product attributes. In Haurie et al. [110], a novel use of variational inequalities is made in solving a stochastic, dynamic model of the European natural gas market. Finally, Harker [101] presents a large-scale application of variational inequalities and quasivariational inequalities in the analysis of the privatization of mass transit services in the United States.

In summary, the wealth of real-world applications attests to the usefulness and computational efficiency of the variational inequality approach in modeling noncooperative games. In fact, the remainder of the applications to be discussed in this section can formally be considered as special cases of either the Nash or generalized Nash equilibrium concept [116]. Thus, the Nash equilibrium problem has been and will continue to be an active area of theoretical, computational and applied research in the variational inequality field.

The next application to be discussed in this section is the *traffic assignment* or *network equilibrium* model which is used to predict the steady-state volume of traffic on urban transportation networks. As discussed in the Introduction, this problem has been a major impetus for the development of algorithms for large-scale variational inequalities such as the simplicial decomposition algorithm described in Section 4, and the extension of sensitivity results to incorporate non-locally unique solutions [212]. The basis of the model is the concept of *user* or *Wardrop equilibrium* [263]. This concept is a behavioral principle which states that drivers compete noncooperatively for the network resources in order to minimize their travel costs. Thus, the traffic assignment model can be considered to be a special case of the Nash equilibrium problem [108]. Following Friesz [75] (see also [71, 72, 154] for reviews of this problem), let us define the notation necessary to formulate the traffic assignment problem:

$G(V, A) \equiv$ the network in which A represents the set of arcs and V the node set,
$W \equiv$ the set of origin-destination (O-D) pairs $w = (i, j) \in W$,
$P_w \equiv$ the set of paths connecting O-D pair $w \in W$,
$P = \bigcup_{w \in W} P_w$
$f_a \equiv$ the flow on arc $a \in A$, $f = (f_a \colon a \in A)$,

$h_p \equiv$ the flow on path $p \in P$, $h = (h_p\colon p \in P)$,

$\delta_{ap} \equiv \begin{cases} 1 & \text{if path } p \in P \text{ traverses arc } a \in A, \\ 0 & \text{otherwise,} \end{cases}$

$\Delta \equiv [\delta_{ap}]$, the arc-path incidence matrix,

$c_a(f) \equiv$ the average transportation cost function for arc $a \in A$,
$c(f) = (c_a(f)\colon a \in A)$,

$C_p(h) \equiv$ the average transportation cost function for path $p \in P$,
$C(h) = (C_p(h)\colon p \in P)$,

$u_w \equiv$ the minimum transportation cost between O–D pair $w \in W$, $u = (u_w\colon w \in W)$,
$= \min_{p \in P_w}\{C_p(h)\}$,

$T_w(u) \equiv$ the demand for transportation between O–D pair $w \in W$,
$T(u) = (T_w(u)\colon w \in W)$,

where

$$f = \Delta h, \qquad C(h) = \Delta^{\mathrm{T}} c(f).$$

The user equilibrium principle states that a driver will choose the minimum-cost path between every O–D pair and through this process, those utilized paths will have equal costs; paths with costs higher than the minimum will have no flow:

Definition 7.3. A flow-cost pattern (f^*, u^*) is a *user equilibrium* UE(c, T) if it satisfies the following conditions;

$$h_p[C_p(h) - u_w] = 0, \quad C_p(h) - u_w \geqslant 0, \quad h_p \geqslant 0, \qquad \forall w \in W,\ p \in P_w, \tag{54}$$

$$\sum_{p \in P_w} h_p - T_w(u) = 0, \quad u_w \geqslant 0, \qquad \forall w \in W. \tag{55}$$

Note that without the nonnegativity requirement $(u_w \geqslant 0)$, (54)–(55) is a mixed nonlinear complementarity problem (see Section 3).

By assuming that the arc costs, and hence the path costs, are strictly positive and that demands are nonnegative, the user equilibrium problem can be formulated as a pure nonlinear complementarity problem NCP(F):

Proposition 7.5 [1]. *Let $c_a(f)$ be strictly positive for all $a \in A$ and let $T_w(u) \geqslant 0$. Then (f^*, u^*) is a user equilibrium* UE(c, T) *if and only if (f^*, u^*) solves the following* NCP(F):

$$\sum_{p \in P} F_p(h, u) h_p + \sum_{w \in W} F_w(h, u) u_w = 0, \tag{56}$$

$$F_p(h, u) \geqslant 0, \qquad h_p \geqslant 0,$$

$$F_w(h, u) \geqslant 0, \qquad u_w \geqslant 0,$$

where

$$f = \Delta h, \qquad C(h) = \Delta^{\mathrm{T}} c(f),$$

$$F_p(h, u) = C_p(h) - u_w, \qquad F_w(h, u) = \sum_{p \in P_w} h_p - T_w(u).$$

Thus, the nonlinear complementarity problem is a "natural" formulation for the user equilibrium principle in that very weak conditions are necessary in order to obtain this result. However, this formulation requires the a priori enumeration of all paths in the network. If $T(u)$ is an invertible function of u with inverse

$$u\Phi(T), \tag{57}$$

then the user equilibrium principle can be formulated as a variational inequality over a polyhedral set X in which the paths variables can be treated endogeneously:

Proposition 7.6 [241, 39]. *Let $T(u)$ be an invertible function. Then (f^*, T^*) is a user equilibrium* UE(c, T) *if and only if (f^*, T^*) solves the following* VI(X, F):

$$c(f^*)^{\mathrm{T}}(f-f^*)-\Phi(T^*)^{\mathrm{T}}(T-T^*)\geqslant 0 \quad \forall (f, T)\in X \tag{58}$$

where

$$X=\left\{(f, T): f=\Delta h,\ \sum_{p\in P_w} h_p=T_w \quad \forall w\in W,\ h\geqslant 0,\ T\geqslant 0\right\}.$$

The set X defined above is a polyhedral cone. In the special case of fixed demand $(T(u)=T)$, the inverse function Φ drops out of the formulation, and the set X becomes a compact polyhedron. In this case, any linear program over X can be solved as a shortest path problem. This fact explains the efficiency of the simplicial decomposition algorithm when applied to the fixed-demand traffic assignment problem.

The existence and uniqueness of a user equilibrium can be established by employing several of the results listed in Section 3. For example, Harker [96] has used Smith's [245] specialization of Theorem 3.3 to prove the following:

Proposition 7.7 [96]. *Let $c_a(f)$ be a strictly positive and continuous function which is bounded away from zero for all $a\in A$, and let $T_w(u)$ be a strictly positive and continuous function which is bounded above for all $w\in W$. Then a user equilibrium (f^*, u^*) exists for the problem* NCP(F) *defined in Proposition* 7.5.

Uniqueness follows from the assumption of strict monotonicity of $c(f)$ and $-\Phi(T)$ in the variational inequality formulation of the problem (58). Note that this result implies that (f, u) is unique; path flows h are rarely unique in the traffic assignment problem.

Many specialized algorithms have been developed for the traffic assignment model since LeBlanc et al.'s [148] adaptation of the Frank–Wolfe algorithm to solve the optimization formulation of this problem [18]. The early work by Dafermos [41] and Florian and Spiess [74] explore the use of projection and the diagonalization algorithms for the traffic assignment problem, and the simplicial decomposition algorithm described in Section 4 have been widely applied to this situation

[145, 147, 209]; Nagurney [189] provides a computational comparison of some of these algorithms. The sensitivity analysis results of Section 5 have also been adapted in order to analyze the traffic assignment problem [45, 212].

An important extension of the traffic assignment is the so-called *equilibrium network design* or *signal optimization* problem in which one wishes to optimize some measure of system performance subject to the fact that drivers will behave in a user-equilibrium manner. Thus, the equilibrium network design problem is a special case of the Stackelberg problem and therefore, of a mathematical program with equilibrium constraints described in problem (50). The importance of this problem in the traffic literature is illustrated by the phenomenon of *Braess' paradox* [186, 251]. In this situation, the addition of a new arc in the network (i.e., a network investment) causes the network to become more congested, not less! Thus, the paradox points to the need to carefully analyze the addition or deletion of capacity to a network when the users of the network do not behave in a fully prescribed, system-optimal manner. Several algorithmic approaches have been suggested for the network design problem [3, 103, 165], but the computation of a global optimal solution to this problem remains to be solved.

To date, no real applications of the traffic assignment problem have been made in the case where the costs are asymmetric. The major cause of this paucity of applications is due to the inability to date to estimate such asymmetric cost functions; future research must and will be dedicated to this estimation problem.

An extension to the traffic assignment model can be obtained by replacing the O–D demand function $T(u)$ with a more basic system of commodity supply and demand functions at each node of the network and a behavioral principle of the movement of goods between these regions. That is, one can replace the transportation demand functions with a model which represents the *derived* nature of transportation demand. This *spatial price equilibrium model* was originally suggested by Samuelson in 1952 [234] and has been widely studied and applied since that date; see the reviews [75, 93, 97] for details on this history. In order to illustrate this model, let us define

$\pi_i \equiv$ the price of a homogeneous commodity at node $i \in V$, $\pi = (\pi_i : i \in V)$,
$D_i(\pi) \equiv$ the demand for the commodity at node $i \in V$, $D(\pi) = (D_i(\pi) : i \in V)$,
$S_i(\pi) \equiv$ the supply of the commodity at node $i \in V$, $S(\pi) = (S_i(\pi) : i \in V)$.

The spatial price equilibrium model states that if goods are shipped between two regions, then competition will bid the net profits down to zero; and if net profit is negative, then no shipments will occur between the regions. Stating this mathematically, one has:

Definition 7.4. A flow-price pattern (f^*, π^*) is a *spatial price equilibrium* SPE(c, S, D) if it satisfies the following conditions:

$$S_i(\pi) - D_i(\pi) + \sum_{w=(k,i)\in W} \sum_{p\in P_w} h_p - \sum_{w=(i,j)\in W} \sum_{p\in P_w} h_p = 0, \quad \pi_i \geqslant 0, \qquad \forall i \in V, \tag{59}$$

$$[\pi_i + C_p(h) - \pi_j]h_p = 0, \quad \pi_i + C_p(h) - \pi_j \geqslant 0, \quad h_p \geqslant 0,$$
$$\forall w = (i, j) \in W, \ p \in P_w. \tag{60}$$

By assuming that the arc costs are strictly positive and that the supply and demand functions at each node satisfy

$$\pi_i = 0 \Rightarrow D_i(\pi) \geqslant S_i(\pi), \tag{61}$$

the spatial price equilibrium problem can be formulated as a pure nonlinear complementarity problem NCP(F):

Proposition 7.8 [76]. *Let $c_a(f)$ be strictly positive for all $a \in A$ and let condition* (61) *hold for all $i \in V$. Then (f^*, π^*) is a spatial price equilibrium* SPE(c, S, D) *if and only if (f^*, π^*) solves the following* NCP(F):

$$\sum_{i \in V} F_i(h, \pi)\pi_i + \sum_{w \in W} \sum_{p \in P_w} F_p(h, \pi)h_p = 0, \tag{62}$$
$$F_i(h, \pi) \geqslant 0, \qquad \pi_i \geqslant 0,$$
$$F_p(h, \pi) \geqslant 0, \qquad h_p \geqslant 0,$$

where

$$f = \Delta f, \qquad C(h) = \Delta^{\mathrm{T}} c(f),$$
$$F_i(h, \pi) = S_i(\pi) - D_i(\pi) + \sum_{w=(k,i) \in W} \sum_{p \in P_w} h_p - \sum_{w=(i,j) \in W} \sum_{p \in P_w} h_p,$$
$$F_p(h, \pi) = \pi_i + C_p(h) - \pi_j \quad \textit{for } p \in P_{(i,j)}.$$

As in the case of the traffic assignment problem, if one assumes that $S(\pi)$ and $D(\pi)$ are invertible functions with inverses

$$\Psi(S) \geqslant 0, \qquad \Theta(D) \geqslant 0, \tag{63}$$

then the spatial price equilibrium principle can be formulated as a variational inequality over a polyhedral set X in which the paths can be treated endogeneously:

Proposition 7.9 [73, 78]. *Let $S(\pi)$, $D(\pi)$ be invertible functions with nonnegative inverses. Then (f^*, S^*, D^*) is a spatial price equilibrium* SPE(c, S, D) *if and only if (f^*, S^*, D^*) solves the following* VI(X, F):

$$\Psi(S^*)^{\mathrm{T}}(S - S^*) + c(f^*)^{\mathrm{T}}(f - f^*) - \Theta(D^*)^{\mathrm{T}}(D - D^*) \geqslant 0 \quad \forall (f, S, D) \in X \tag{64}$$

where

$$X = \Bigg\{(f, S, D) \colon S_i - D_i + \sum_{w=(k,i) \in W} \sum_{p \in P_w} h_p - \sum_{w=(i,j) \in W} \sum_{p \in P_w} h_p = 0 \ \forall i \in V,$$
$$f = \Delta h, \ S \geqslant 0, \ D \geqslant 0, \ h \geqslant 0\Bigg\}.$$

As in the case of the traffic assignment problem, the existence of a solution to the spatial price equilibrium problem can be established with several of the results in Section 3. Smith [245] provides a very general result using his specialization of Theorem 3.3 to the nonlinear complementarity formulation (62):

Proposition 7.10 [245]. *Let $c_a(f)$ be a strictly positive and continuous function which is bounded away from zero for all $a \in A$, and $S(\pi)$, $D(\pi)$ be continuous functions which satisfy condition* (61) *and the following condition: there exists a price vector $\bar{\pi}$ such that for all $i \in V$ and all price vectors π,*

$$\pi_i \geqslant \bar{\pi}_i \;\Rightarrow\; S_i(\pi) \geqslant D_i(\pi). \tag{65}$$

Then a spatial price equilibrium (f^*, π^*) exists for the problem NCP(F) *defined in Proposition* 7.8.

Specialized algorithms for the spatial price equilibrium problem abound, especially in cases where the tree structure of the O–D flows can be fully exploited [73, 120, 200]. Friesz et al. [78] and Nagurney [190] provide a computational comparison and review of the various methods for computing spatial equilibria. Sensitivity analysis results have also been specialized to the spatial price equilibrium problem [29, 44, 257].

Applications of the spatial price problem are numerous; Chapter 2 of Harker [97] provides a detailed description of the major applications. In fact, the PIES model [5] described in the Introduction can be considered as a sophisticated formulation of the spatial price equilibrium concept. In these applications, the use of the variational inequality formulation is important due to the asymmetries which typically arise in systems of demand and supply functions. In summary, the spatial price equilibrium model has been and will continue to be a useful tool in practical applications.

In recent years, there has emerged a literature which attempts to bring together several of the abovementioned models. Harker and Friesz [77, 97] build a model for predicting intercity freight flows which ties together the notions of Nash, user and spatial price equilibria. Harker [95] and Haurie and Marcotte [108] bring together the notion of Nash and spatial price equilibrium; this work is extended in Harker [100] to cooperative game theory. The Haurie–Marcotte paper also establishes the fact that the spatial price model can be considered as a special case of the Nash equilibrium problem. Finally, Harker [98] brings together the traffic assignment and Nash models to show another interesting paradox which can arise due to network externalities. The book [93] provides other examples of this fusion between the various models described above.

Last, but not least, we come to the *general or Walrasian equilibrium model.* The purpose of this model is to predict economic activity in a closed economy; i.e., to compute the equilibrium activities and prices in an economy when all interactions between the commodities comprising this economy have been incorporated. This well-known problem has become the basis for much of modern mathematical economics and has proven very useful in analyzing tax policy, international trade, issues in energy economics, etc. The early fixed-point methods for solving equilibrium problems [259, 236] were strongly motivated by the need to compute solutions of the general equilibrium model. Ichiishi [116] has established the linkage between this problem and the generalized Nash (or social equilibrium) problem; however, this problem has a sufficiently rich theoretical and applied history to be of immense interest on its own.

Manne [159] provides an excellent review of the various mathematical formulations of the general equilibrium problem. Following the formulation used by Mathiesen [168, 169] and Rutherford [231, 232], let us define the following notation:

$M \equiv$ the set of m activities in the economy,
$N \equiv$ the set of n goods in the economy,
$y_i \equiv$ the level of the ith activity, $y = (y_i : i \in M)$,
$\pi_j \equiv$ the price of the jth good, $\pi = (\pi_j : j \in N)$,
$b_j \equiv$ the initial endowment of the jth good, $b = (b_j : j \in N)$,
$c_i \equiv$ the constant component of the unit cost of operating the ith activity, $c = (c_i : i \in M)$,
$d_j(\pi) \equiv$ the demand function for the jth good, $d(\pi) = (d_j(\pi) : j \in N)$,
$A(\pi) = [a_{ij}(\pi)] \equiv$ the technological input–output matrix for the economy,
$a_{ij} > 0 \Rightarrow$ output, $a_{ij} < 0 \Rightarrow$ input.

Using this notation, the general equilibrium problem is defined as follows:

Definition 7.5. A price-activity pattern (π^*, y^*) is a *general equilibrium* GE(b, c, d, A) if this point satisfies the following conditions:

(i) $c - A(\pi)\pi \geqslant 0$, nonpositive profits for all activities;
(ii) $b + A(\pi)^{\mathrm{T}}y - d(\pi) \geqslant 0$, nonpositive excess demands for all goods;
(iii) $\pi \geqslant 0$, $y \geqslant 0$, nonnegativity of prices and activities;
(iv) $[c - A(\pi)\pi]^{\mathrm{T}}y = 0$, activities with negative profit are not performed;
(v) $[b + A(\pi)^{\mathrm{T}}y - d(\pi)]^{\mathrm{T}}\pi = 0$, Walras Law-excess supply only in the case of free goods.

Note that the conditions (i)–(v) directly translate into a nonlinear complementarity problem:

Proposition 7.11. *A price-activity pattern* (π^*, y^*) *is a general equilibrium* GE(b, c, d, A) *if and only if* (π^*, y^*) *solves the following* NCP(F):

$$F^y(\pi, y)^{\mathrm{T}}y + F^\pi(\pi, y)^{\mathrm{T}}\pi = 0, \tag{66}$$

$$F^y(\pi, y) \geqslant 0, \quad y \geqslant 0,$$

$$F^\pi(\pi, y) \geqslant 0, \quad \pi \geqslant 0,$$

where

$$F^y(\pi, y) = c - A(\pi)\pi,$$
$$F^\pi(\pi, y) = b + A(\pi)^{\mathrm{T}} y - d(\pi).$$

The above formulation involves prices and activities jointly. In the case where $A(\pi)$ is a constant function (i.e., $A(\pi) = A$, a constant matrix) which represents a Leontief production technology, one can provide a reduced formulation of the general equilibrium problem as a variational inequality defined over a polyhedral set X which is defined solely in terms of the price vector π:

Proposition 7.12. *A price vector* π^* *solves the* VI(X, F) *where*

$$X = \{\pi\colon c - A\pi \geqslant 0,\ \ \pi \geqslant 0\},$$
$$F(\pi) = b - d(\pi),$$

if and only if there exists a vector y^* *such that* (π^*, y^*) *solves* GE(b, c, d, A); y^* *is the vector of multipliers for the constraints in* X.

The above variational inequality formulation has not been used in applications to date; its use is an open and interesting area for future research.

In terms of the existence of a solution to the general equilibrium problem, the mathematical economics literature contains an enormous number of results [12, 25, 94]. One simple result follows directly from Theorem 3.1 and the variational inequality formulation in Proposition 7.12. Since the demand functions in a general equilibrium model are homogeneous of degree zero, one can without loss of generality normalize prices to be on the unit simplex. Thus, defining

$$\tilde{X} = X \cap \left\{\pi\colon \sum_{j\in N} \pi_j = 1\right\},$$

one has the following result.

Proposition 7.13. *Let* $d(\pi)$ *be a continuous function which is positively homogeneous of degree zero*; *i.e.*, $d(\alpha\pi) = d(\pi)\ \forall \alpha \in \mathbb{R}_{+}$. *Then there exists a solution* π^* *to the variational inequality formulation* VI$(\tilde{X}, F)$ *of the general equilibrium problem defined in Proposition* 7.12.

Concerning algorithms for the general equilibrium problem, Scarf's simplicial labeling method [236] is the major technique used in applications; see, for example, the works by Shoven [238] and Whalley [265]. In recent years, however, Newton's method for the nonlinear complementarity formulation has proven to be very effective. The papers by Eaves [55], Mathiesen [168, 169, 170], Manne and Preckel [160, 210, 211], and Rutherford [231, 232], where Newton's method is called the approach of *successive linear complementarity problems* (SLCP), have established the practical efficiency of this method in solving general equilibrium problems. However, as shown in detail by Mathiesen [170], this algorithm is not proven to be convergent due to the income effects inherent in such a model. Thus, the algorithm often works well, but its convergence properties are not fully understood. These same papers plus the Reference list also describe numerous applications of this problem.

In summary, the general equilibrium problem provides the seeds for further development of the algorithms described in Section 4, and remains as one of the most useful equilibrium models in practice.

8. Summary and future research needs

This paper has reviewed the state-of-the-art in the theory, computation and application of finite-dimensional variational inequalities and nonlinear complementarity problems. The study of these two problems is a young, fruitful and growing field of intellectual endeavor. The advances in algorithmic power have enabled researchers involved in applications to attack larger and more interesting problems arising in economics, public policy, engineering, and strategic planning. However, a great deal of work remains to be done; in what follows, we shall briefly state some open and interesting research topics in this area.

Two major research topics exist in the area of the theory of variational inequalities and nonlinear complementarity problems. The first topic involves the further development of the theory of parametric variational inequalities; i.e., the theory of sensitivity analysis. As discussed in Section 5, the regularity of the solution point is necessary for any meaningful sensitivity results to be achieved. Can this requirement be weakened? Qiu and Magnanti [213] have started the investigation of removing the local uniqueness requirements; much more work is needed. Finally, the theory of the *stability* of solutions to variational inequalities is currently weak as best.

An issue related to the abovementioned sensitivity analysis topic is that of *stochastic variational inequalities.* In almost all real applications of these techniques, the data definining the model is very "noisy" in the sense of having a fairly wide distribution of values [195]. The current variational inequality/nonlinear complementarity techniques ignore the stochastic nature of the data. What is needed in this area is the ability to compute the probability distribution of the equilibria, or at least the means, variances, medians, etc. From a practical viewpoint, this extension is vital if the model outputs are to be useful; one simply needs more than a point prediction in most applications.

In terms of algorithmic developments, the modification of the various techniques to be *globally convergent* and the development of *parallel* algorithms [208] are a top priority. Researchers involved in applications will continually push back the frontier of problem size and complexity; we must provide the algorithms which can solve these ever more complex problems. As briefly mentioned at the closing of Section 4, the ideas of continuation should provide a useful technique for the globalization of the locally convergent iterative methods [206]. Another promising approach for the development of globally convergent iterative schemes is that which is outlined in the two recent papers [205, 105]. Also, the ability to compute more than one (all?) equilibria would be extremely useful for the same reasons that the stochastic methods are needed—point predictions are simply not sufficient. Finally, as mentioned in Section 4, the state-of-the art in the computation of solutions to mathematical programs with variational inequality/nonlinear complementarity constraints is weak at best; future research must be devoted to this important problem area.

The current state of knowledge concerning quasi-variational inequalities/implicit complementarity problems in all of the abovementioned areas—theory, computation, and sensitivity analysis—is very weak. As illustrated in Harker [102], these problems can be very useful in practice. Much more research is needed in all three areas in order to provide researchers with the analytic and algorithmic tools to effectively employ these problems in practice.

Finally, this field has been and will continue to foster new and innovative applications. For example, the Haurie et al. [110] paper illustrates how variational inequalities can be used for analyzing stochastic, dynamic equilibrium problems. The reemergence of interest in dynamic modeling, as evidenced by the growth of *chaos theory* [115], attests to the fact that the modeling and computation of equilibria will continue to be an important intellectual pursuit. Given the success to date of variational inequalities and nonlinear complementarity problems, these techniques should be well suited to aid in the applications of the future.

Note added in proof. Since the completion of this survey, many developments have occurred in the field; the papers in this and its accompanying volume are a sample of these new results.

Acknowledgments

To survey an entire field is never easy and requires a great deal of help from many colleagues. Thus, the authors would like to thank the following individuals for their helpful comments on an earlier version of this paper: B. Curtis Eaves, Shu-Cherng Fang, Phil Jones, Mike Kostreva, Steve Robinson and Tom Rutherford.

References

[1] H.Z. Aashtiani and T.L. Magnanti, "Equilibria on a congested transportation network," *SIAM Journal on Algebraic and Discrete Methods* 2 (1981) 213–226.

[2] H.Z. Aashtiani and T.L. Magnanti, "A linearization and decomposition algorithm for computing urban traffic equilibria," *Proceedings of the* 1982 *IEEE International Large Scale Systems Symposium* (1982) 8–19.

[3] M. Abdulaal and L.J. LeBlanc, "Continuous equilibrium network design models," *Transportation Research* 13B (1979) 19–32.

[4] M. Aganagic, "Variational inequalities and generalized complementarity problems," Technical Report SOL 78-11, Systems Optimization Laboratory, Department of Operations Research, Stanford University (Stanford, CA 1978).

[5] B.H. Ahn, *Computation of Market Equilibria for Policy Analysis: The Project Independence Evaluation Study (PIES) Approach* (Garland, NY, 1979).

[6] B.H. Ahn, "A Gauss-Seidel iteration method for nonlinear variational inequality problems over rectangles," *Operations Research Letters* 1 (1982) 117–120.

[7] B.H. Ahn, "A parametric network method for computing nonlinear spatial equilibria," Research report, Department of Management Science, Korea Advanced Institute of Science and Technology (Seoul, Korea, 1984).

[8] B.H. Ahn and W.W. Hogan, "On convergence of the PIES algorithm for computing equilibria," *Operations Research* 30 (1982) 281–300.

[9] E. Allgower and K. Georg, "Simplicial and continuation methods for approximating fixed points and solutions to systems of equations," *SIAM Review* 22 (1980) 28–85.

[10] R. Asmuth, "Traffic network equilibrium," Technical Report SOL 78-2, Systems Optimization Laboratory, Department of Operations Research, Stanford University (Stanford, CA, 1978).

[11] R. Asmuth, B.C. Eaves and E.L. Peterson, "Computing economic equilibria on affine networks with Lemke's algorithm," *Mathematics of Operations Research* 4 (1979) 207–214.

[12] J.P. Aubin, *Mathematical Methods of Game and Economic Theory* (North-Holland, Amsterdam, 1979).

[13] M. Avriel, *Nonlinear Programming: Analysis and Methods* (Prentice-Hall, Englewood Cliffs, NJ, 1976).

[14] S.A. Awoniyi and M.J. Todd, "An efficient simplicial algorithm for computing a zero of a convex union of smooth functions," *Mathematical Programming* 25 (1983) 83–108.

[15] C. Baiocchi and A. Capelo, *Variational and Quasivariational Inequalities: Application to Free-Boundary Problems* (Wiley, New York, 1984).

[16] B. Banks, J. Guddat, D. Klatte, B. Kummer and K. Tammer, *Nonlinear Parametric Optimization* (Birkhauser, Basel, 1983).

[17] V. Barbu, *Optimal Control of Variational Inequalities* (Pitman Advanced Publishing Program, Boston, 1984).

[18] M.J. Beckman, C.B. McGuire, and C.B. Winston, *Studies in the Economics of Transportation* (Yale University Press, New Haven, CT, 1956).

[19] A. Bensoussan, "Points de Nash dans le cas de fonctionnelles quadratiques et jeux differentials linéaires a N personnes," *SIAM Journal on Control* 12 (1974) 460–499.

[20] A. Bensoussan, M. Goursat and J.L. Lions, "Contrôle impulsionnel et inéquations quasivariationnelles stationnaires," *Comptes Rendus Academie Sciences Paris* 276 (1973) 1279–1284.

[21] A. Bensoussan and J.L. Lions, "Nouvelle formulation de problèmes de contrôle impulsionnel et applications," *Comptes Rendus Academie Sciences Paris* 276 (1973) 1189–1192.

[22] A. Bensoussan and J.L. Lions, "Nouvelles méthodes en contrôle impulsionnel," *Applied Mathematics and Optimization* 1 (1974) 289–312.

[23] C. Berge, *Topological Spaces* (Oliver and Boyd, Edinburgh, Scotland, 1963).

[24] D.P. Bertsekas and E.M. Gafni, "Projection methods for variational inequalities with application to the traffic assignment problem," *Mathematical Programming Study* 17 (1982) 139–159.

[25] K.C. Border, *Fixed Point Theorems with Applications to Economics and Game Theory* (Cambridge University Press, Cambridge, 1985).

[26] F.E. Browder, "Existence and approximation of solutions of nonlinear variational inequalities," *Proceeding of the National Academy of Sciences, U.S.A.* 56 (1966) 1080–1086.

[27] M. Carey, "Integrability and mathematical programming models: a survey and parametric approach," *Econometrica* 45 (1977) 1957–1976.

[28] D. Chan and J.S. Pang, "The generalized quasi-variational inequality problem," *Mathematics of Operations Research* 7 (1982) 211–222.

[29] G.S. Chao and T.L. Friesz, "Spatial price equilibrium sensitivity analysis," *Transportation Research* 18B (1984) 423–440.
[30] S.C. Choi, W.S. DeSarbo and P.T. Harker, "Product positioning under price competition," *Management Science* 36 (1990) 265–284.
[31] R.W. Cottle, *Nonlinear Programs with Positively Bounded Jacobians.* Ph.D. dissertation, Department of Mathematics, University of California (Berkeley, CA, 1964).
[32] R.W. Cottle, "Nonlinear programs with positively bounded Jacobians," *SIAM Journal on Applied Mathematics* 14 (1966) 147–158.
[33] R.W. Cottle, "Complementarity and variational problems," *Symposia Mathematica* XIX (1976) 177–208.
[34] R.W. Cottle and G.B. Dantzig, "Complementary pivot theory of mathematical programming," *Linear Algebra and Its Applications* 1 (1968) 103–125.
[35] R.W. Cottle, F. Giannessi and J.L. Lions, eds., *Variational Inequalities and Complementarity Problems: Theory and Applications* (Wiley, New York, 1980).
[36] R.W. Cottle, G.J. Habetler and C.E. Lemke, "Quadratic forms semi-definite over convex cones," in: H.W. Kuhn, ed., *Proceedings of the Princeton Symposium on Mathematical Programming* (Princeton University Press, Princeton, NJ, 1970) 551–565.
[37] R.W. Cottle, J.S. Pang and V. Venkateswaran, "Sufficient matrices and the linear complementarity problem," *Linear Algebra and its Applications* 114/115 (1989) 231–249.
[38] R.W. Cottle and A.F. Veinott, Jr., "Polyhedral sets having a least element," *Mathematical Programming* 3 (1972) 238–249.
[39] S. Dafermos, "Traffic equilibria and variational inequalities," *Transportation Science* 14 (1980) 42–54.
[40] S. Dafermos, "The general multimodal network equilibrium problem with elastic demand," *Networks* 12 (1982) 57–72.
[41] S. Dafermos, "Relaxation algorithms for the general asymmetric traffic equilibrium problem," *Transportation Science* 16 (1982) 231–240.
[42] S. Dafermos, "An iterative scheme for variational inequalities," *Mathematical Programming* 26 (1983) 40–47.
[43] S. Dafermos, "Sensitivity analysis in variational inequalities," *Mathematics of Operations Research* 13 (1988) 421–434.
[44] S. Dafermos and A. Nagurney, "Sensitivity analysis for the general spatial economic equilibrium problem," *Operations Research* 32 (1984) 1069–1086.
[45] S. Dafermos and A. Nagurney, "Sensitivity analysis for the asymmetric network equilibrium problem," *Mathematical Programming* 28 (1984) 174–184.
[46] J.E. Dennis Jr. and R.B. Schnabel, *Numerical Methods for Unconstrained Optimization and Nonlinear Equations* (Prentice-Hall, Englewood Cliffs, NJ, 1983).
[47] I.C. Dolcetta and U. Mosco, "Implicit complementarity problems and quasi-variational inequalities," in: R.W. Cottle, F. Giannessi and J.L. Lions, eds., *Variational Inequalities and Complementarity Problems: Theory and Applications* (Wiley, New York, 1980) 75–87.
[48] B.C. Eaves, "On the basic theorem of complementarity," *Mathematical Programming* 1 (1971) 68–75.
[49] B.C. Eaves, "The linear complementarity problem," *Management Science* 17 (1971) 612–634.
[50] B.C. Eaves, "Homotopies for computation of fixed points," *Mathematical Programming* 3 (1972) 1–22.
[51] B.C. Eaves, "A short course in solving equations with PL homotopies," in: R.W. Cottle and C.E. Lemke eds., *Nonlinear Programming: SIAM-AMS Proceedings 9* (American Mathematical Society, Providence, RI, 1976) pp. 73–143.
[52] B.C. Eaves, "Computing stationary points," *Mathematical Programming Study* 7 (1978) 1–14.
[53] B.C. Eaves, "Computing stationary points, again," in: O.L. Mangasarian, R.R. Meyer and S.M. Robinson, eds., *Nonlinear Programming 3* (Academic Press, New York, 1978) pp. 391–405.
[54] B.C. Eaves, "Where solving for stationary points by LCPs is mixing Newton iterates," in: B.C. Eaves, F.J. Gould, H.O. Peitgen and M.J. Todd, eds., *Homotopy Methods and Global Convergence* (Plenum Press, New York, 1983) pp. 63–78.
[55] B.C. Eaves, "Thoughts on computing market equilibrium with SLCP," Technical Report, Department of Operations Research, Stanford University (Stanford, CA, 1986).

[56] S.C. Fang, *Generalized Variational Inequality, Complementarity and Fixed Point Problems: Theory and Application.* Ph.D. dissertation, Northwestern University (Evanston, IL, 1979).

[57] S.C. Fang, "An iterative method for generalized complementarity problems," *IEEE Transactions on Automatic Control* AC-25 (1980) 1225–1227.

[58] S.C. Fang, "Traffic equilibria on multiclass user transportation networks analyzed via variational inequalities," *Tamkang Journal of Mathematics* 13 (1982) 1–9.

[59] S.C. Fang, "Fixed point models for the equilibrium problems on transportation networks," *Tamkang Journal of Mathematics* 13 (1982) 181–191.

[60] S.C. Fang, "A linearization method for generalized complementarity problems," *IEEE Transactions on Automatic Control* AC 29 (1984) 930–933.

[61] S.C. Fang and E.L. Peterson, "Generalized variational inequalities," *Journal of Optimization Theory and Application* 38 (1982) 363–383.

[62] S.C. Fang and E.L. Peterson, "General network equilibrium analysis," *International Journal of Systems Sciences* 14 (1983) 1249–1257.

[63] S.C. Fang and E.L. Peterson, "An economic equilibrium model on a multicommodity network," *International Journal of Systems Sciences* 16 (1985) 479–490.

[64] A.V. Fiacco, *Introduction to Sensitivity and Stability Analysis in Nonlinear Programming* (Academic Press, New York, 1983).

[65] A.V. Fiacco and J. Kyparisis, "Sensitivity analysis in nonlinear programming under second order assumptions," in: A. Bagchi and H. Th. Jongen, eds., *Systems and Optimization* (Springer, Berlin, 1985) pp. 74–97.

[66] M. Fiedler and V. Ptak, "On matrices with nonpositive off-diagonal elements and positive principal minors," *Czechoslovak Mathematics Journal* 12 (1962), 382–400.

[67] M.L. Fisher and F.J. Gould, "A simplicial algorithm for the nonlinear complementarity problem," *Mathematical Programming* 6 (1974) 281–300.

[68] M.L. Fisher and J.W. Tolle, "The nonlinear complementarity problem: existence and determination of solutions," *SIAM Journal of Control and Optimization* 15 (1977), 612–623.

[69] C.S. Fisk and D.E. Boyce, "Alternative variational inequality formulations of the network equilibrium—travel choice problem," *Transportation Science* 17 (1983) 454–463.

[70] C.S. Fisk and S. Nguyen, "Solution algorithms for network equilibrium models with asymmetric user costs," *Transportation Science* 16 (1982) 316–381.

[71] M. Florian, ed., *Traffic Equilibrium Methods* (Springer, Berlin, 1976).

[72] M. Florian, "Nonlinear cost network models in transportation analysis," *Mathematical Programming Study* 26 (1986) 167–196.

[73] M. Florian and M. Los, "A new look at static spatial price equilibrium models," *Regional Science and Urban Economics* 12 (1982) 579–597.

[74] M. Florian and H. Spiess, "The convergence of diagonalization algorithms for asymmetric network equilibrium problems," *Transportation Research* 16B (1982) 447–483.

[75] T.L. Friesz, "Network equilibrium, design and aggregation," *Transportation Research* 19A (1985) 413–427.

[76] T.L. Friesz, R.L. Tobin, T.E. Smith and P.T. Harker, "A nonlinear complementary formulation and solution procedure for the general derived demand network equilibrium problem," *Journal of Regional Science* 23 (1983) 337–359.

[77] T.L. Friesz and P.T. Harker, "Freight network equilibrium: a review of the state of the art," in: A. Daughety, ed., *Analytical Studies in Transportation Economics* (Cambridge University Press, Cambridge, 1985) 161–206.

[78] T.L. Friesz, P.T. Harker and R.L. Tobin, "Alternative algorithms for the general network spatial price equilibrium problem," *Journal of Regional Science* 24 (1984) 473–507.

[79] M. Fukushima, "A relaxed projection method for variational inequalities," *Mathematical Programming* 35 (1986) 58–70.

[80] D. Gabay and H. Moulin, "On the uniqueness and stability of Nash-equilibria in noncooperative games," in: A. Bensoussan, P. Kleindorfer and C.S. Tapiero, eds., *Applied Stochastic Control in Econometrics and Management Science* (North-Holland, Amsterdam, 1980) pp. 271–292.

[81] C.B. Garcia and W.I. Zangwill, *Pathways to Solutions, Fixed Points and Equilibria* (Prentice-Hall, Englewood Cliffs, NJ, 1981).

[82] R. Glowinski, J.L. Lions and R. Trémolières, *Analyses Numérique des Inéquations Variationalles: Methodes Mathematiques de l'Informatique* (Dunod, Paris, 1976).

[83] C.D. Ha, "Application of degree theory in stability of the complementarity problem," *Mathematics of Operations Research* 12 (1987) 368–376.

[84] G.J. Habetler and M.M. Kostreva, "On a direct algorithm for nonlinear complementarity problems," *SIAM Journal of Control and Optimization* 16 (1978) 504–511.

[85] G.J. Habetler and A.L. Price, "Existence theory for generalized nonlinear complementarity problems," *Journal of Optimization Theory and Applications* 7 (1971) 223–239.

[86] J.H. Hammond, *Solving Asymmetric Variational Inequality Problems and Systems of Equation with Generalized Nonlinear Programming Algorithms.* Ph.D. dissertation, Department of Mathematics, M.I.T. (Cambridge, MA, 1984).

[87] J.H. Hammond and T.L. Magnanti, "Generalized descent methods for asymmetric systems of equations," *Mathematics of Operations Research* 12 (1987) 678–699.

[88] J.H. Hammond and T.L. Magnanti, "A contracting ellipsoid method for variational inequality problems," Working Paper OR 160-87, Operations Research Center, M.I.T. (Cambridge, MA, 1987).

[89] T.H. Hansen, *On the Approximation of a Competitive Equilibrium.* Ph.D. dissertation, Department of Economics, Yale University (New Haven, CT, 1968).

[90] T.H. Hansen and H. Scarf, "On the approximation of Nash equilibrium points in an N-person noncooperative game," *SIAM Journal of Applied Mathematics* 26 (1974) 622–637.

[91] P.T. Harker, "A variational inequality approach for the determination of oligopolistic market equilibrium," *Mathematical Programming* 30 (1984) 105–111.

[92] P.T. Harker, "A generalized spatial price equilibrium model," *Papers of the Regional Science Association* 54 (1984) 25–42.

[93] P.T. Harker, ed., *Spatial Price Equilibrium: Advances in Theory, Computation and Application. Lecture Notes in Economics and Mathematical Systems, Vol 249* (Springer, Berlin, 1985).

[94] P.T. Harker, "Existence of competitive equilibria via Smith's nonlinear complementarity result," *Economics Letters* 19 (1985) 1–4.

[95] P.T. Harker, "Alternative models of spatial competition," *Operations Research* 34 (1986) 410–425.

[96] P.T. Harker, "A note on the existence of traffic equilibria," *Applied Mathematics and Computation* 18 (1986) 277–283.

[97] P.T. Harker, *Predicting Intercity Freight Flows* (VNU Science Press, Utrecht, The Netherlands, 1987).

[98] P.T. Harker, "Multiple equilibria behaviors on networks," *Transportation Science* 22 (1988), 39–46.

[99] P.T. Harker, "Accelerating the convergence of the diagonalization and projection algorithms for finite-dimensional variational inequalities," *Mathematical Programming* 41 (1988) 29–59.

[100] P.T. Harker, "The core of a spatial price equilibrium game," *Journal of Regional Science* 27 (1987) 369–389.

[101] P.T. Harker, "Privatization of urban mass transportation: application of computable equilibrium models for network competition," *Transportation Science* 22 (1988) 96–111.

[102] P.T. Harker, "Generalized Nash games and quasivariational inequalities," to appear in: *European Journal of Operational Research.*

[103] P.T. Harker and S.C. Choi, "A penalty function approach for mathematical programs with variational inequality constraints," Working paper 87-09-08, Department of Decision Sciences, University of Pennsylvania (Philadelphia, PA, 1987).

[104] P.T. Harker and J.S. Pang, "Existence of optimal solutions to mathematical programs with equilibrium constraints," *Operations Research Letters* 7 (1988) 61–64.

[105] P.T. Harker and J.S. Pang, "A damped-Newton method for the linear complementarity problem," in: E.L. Allgower and K. Georg, eds., *Computational Solution of Nonlinear Systems of Equations. AMS Lectures on Applied Mathematics* 26 (1990) 265–284.

[106] P.T. Harker and J.S. Pang, *Equilibrium Modeling With Variational Inequalities: Theory, Computation and Application,* in preparation.

[107] P. Hartman and G. Stampacchia, "On some nonlinear elliptic differential functional equations," *Acta Mathematica* 115 (1966) 153–188.

[108] A. Haurie and P. Marcotte, "On the relationship between Nash-Cournot and Wardrop equilibria," *Networks* 15 (1985) 295–308.

[109] A. Haurie and P. Marcotte, "A game-theoretic approach to network equilibrium," *Mathematical Programming Study* 26 (1986) 252–255.

[110] A. Haurie, G. Zaccour, J. Legrand and Y. Smeers, "A stochastic dynamic Nash-Cournot model for the European gas market," Working Paper G-87-24, École des Hautes Études Commeriales, Université de Montréal (Montréal, Que., 1987).

[111] D.W. Hearn, "The gap function of a convex program," *Operations Research Letters* 1 (1982) 67–71.

[112] D.W. Hearn, S. Lawphongpanich and S. Nguyen, "Convex programming formulation of the asymmetric traffic assignment problem," *Transportation Research* 18B (1984) 357–365.

[113] D.W. Hearn, S. Lawphongpanich and J.A. Ventura, "Restricted simplicial decomposition: computation and extensions," *Mathematical Programming Study* 31 (1987) 99–118.

[114] W. Hildenbrand and A.P. Kirman, *Introduction to Equilibrium Analysis* (North-Holland, Amsterdam, 1976).

[115] A.V. Holden, ed., *Chaos* (Princeton University Press, Princeton, NJ, 1986).

[116] T. Ichiishi, *Game Theory for Economic Analysis* (Academic Press, New York, 1983).

[117] C.M. Ip, *The Distorted Stationary Point Problem.* Ph.D. dissertation, School of Operations Research and Industrial Engineering, Cornell University (Ithaca, NY, 1986).

[118] K. Jittorntrum, "Solution point differentiability without strict complementarity in nonlinear programming," *Mathematical Programming Study* 21 (1984) 127–138.

[119] P.C. Jones, G. Morrison, J.C. Swarts and E. Theise, "Nonlinear spatial price equilibrium algorithms: a computational comparison," *Microcomputers in Civil Engineering* 3 (1988) 265–271.

[120] P.C. Jones, R. Saigal and M. Schneider, "Computing nonlinear network equilibria," *Mathematical Programming* 31 (1985) 57–66.

[121] N.H. Josephy, "Newton's method for generalized equations," Technical Report No. 1965, Mathematics Research Center, University of Wisconsin (Madison, WI, 1979).

[122] N.H. Josephy, "Quasi-Newton methods for generalized equations," Technical Report No. 1966, Mathematics Research Center, University of Wisconsin (Madison, WI, 1979).

[123] N.H. Josephy, "A Newton method for the PIES energy model," Technical Summary Report No. 1977, Mathematics Research Center, University of Wisconsin (Madison, WI, 1979).

[124] S. Karamardian, "The nonlinear complementarity problem with applications, parts I and II," *Journal of Optimization Theory and Applications* 4 (1969) 87–98 and 167–81.

[125] S. Karamardian, "Generalized complementarity problem," *Journal of Optimization Theory and Applications* 8 (1971) 161–167.

[126] S. Karamardian, "The complementarity problem," *Mathematical Programming* 2 (1972) 107–129.

[127] S. Karamardian, "Complementarity problems over cones with monotone and pseudomonotone maps," *Journal of Optimization Theory and Applications* 18 (1976) 445–454.

[128] S. Karamardian, "An existence theorem for the complementarity problem," *Journal of Optimization Theory and Applications* 18 (1976) 445–454.

[129] W. Karush, *Minima of Functions of Several Variables with Inequalities as Side Conditions.* M.S. thesis, Department of Mathematics, University of Chicago (Chicago, IL, 1939).

[130] D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and Their Application* (Academic Press, New York, 1980).

[131] M. Kojima, "Computational methods for solving the nonlinear complementarity problem," *Keio Engineering Reports* 27 (1974) 1–41.

[132] M. Kojima, "A unification of the existence theorems of the nonlinear complementarity problem," *Mathematical Programming* 9 (1975) 257–277.

[133] M. Kojima, "Strongly stable stationary solutions in nonlinear programming," in: S.M. Robinson, ed., *Analysis and Computation of Fixed Points* (Academic Press, New York, 1980) pp. 93–138.

[134] M. Kojima, S. Mizuno, and T. Noma, "A new continuation method for complementarity problems with uniform P-functions," *Mathematical Programming* 43 (1989) 107–114.

[135] M. Kojima, S. Mizuno, and T. Noma, "Limiting behavior of trajectories generated by a continuation method for monotone complementarity problems," Research Report No. B-199, Department of Information Sciences, Tokyo Institute of Technology (Tokyo, Japan, 1988).

[136] M.M. Kostreva, "Block pivot methods for solving the complementarity problem," *Linear Algebra and Its Application* 21 (1978) 207–215.

[137] M.M. Kostreva, "Elasto-hydrodynamic lubrication: a nonlinear complementarity problem," *International Journal for Numerical Methods in Fluids* 4 (1984) 377–397.

[138] H. Kremers and D. Talman, "Solving the nonlinear complementarity problem with lower and upper bounds," FEW330, Department of Econometrics, Tilburg University (Tilburg, The Netherlands, 1988).

[139] H.W. Kuhn, "Simplicial approximation of fixed points," *Proceedings of the National Academy of Sciences U.S.A.* 61 (1968) 1238–1242.

[140] H.W. Kuhn and A.W. Tucker, "Nonlinear programming," in: J. Neyman, ed., *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability,* (University of California Press, Berkeley, CA, 1951) pp. 481–492.

[141] J. Kyparisis, "Uniqueness and differentiability of solutions of parametric nonlinear complementarity problems," *Mathematical Programming* 36 (1986) 105–113.

[142] J. Kyparisis, "Sensitivity analysis framework for variational inequalities," *Mathematical Programming* 38 (1987) 203–213.

[143] J. Kyparisis, "Perturbed solutions of variational inequality problems over polyhedral sets," *Journal of Optimization Theory and Applications* 57 (1988) 295–305.

[144] J. Kyparisis, "Sensitivity analysis for nonlinear programs and variational inequalities with nonunique multipliers," Working paper, Department of Decision Sciences and Information Systems, Florida International University (Miami, FL, 1987).

[145] S. Lawphongpanich and D.W. Hearn, "Simplicial decomposition of asymmetric traffic assignment problem," *Transportation Research* 18B (1984) 123–133.

[146] S. Lawphongpanich and D.W. Hearn, "Bender's decomposition for variational inequalities," *Mathematical Programming (Series B)* 48 (1990) 231–247, this issue.

[147] S. Lawphongpanich and D.W. Hearn, "Restricted simplicial decomposition with application to the traffic assignment problem," *Ricera Operativa* 38 (1986) 97–120.

[148] L.J. LeBlanc, E.K. Morlok and W.P. Pierskalla, "An efficient approach to solving the road network equilibrium traffic assignment problem," *Transportation Research* 9 (1974) 309–318.

[149] C.E. Lemke, "Bimatrix equilibrium points and mathematical programming," *Management Science* 11 (1965) 681–689.

[150] C.E. Lemke and J.T. Howson, "Equilibrium points of bimatrix games," *SIAM Review* 12 (1964) 45–78.

[151] Y.Y. Lin and J.S. Pang, "Iterative methods for large convex quadratic programs: a survey," *SIAM Journal on Control and Optimization* 25 (1987) 383–411.

[152] J.L. Lions and G. Stampacchia, "Variational inequalities," *Communications on Pure and Applied Mathematics* 20 (1967) 493–519.

[153] H.J. Lüthi, "On the solution of variational inequality by the ellipsoid method," *Mathematics of Operations Research* 10 (1985) 515–522.

[154] T.L. Magnanti, "Models and algorithms for predicting urban traffic equilibria," in: M. Florian, ed., *Transportation Planning Models* (North-Holland, Amsterdam, 1984) pp. 153–185.

[155] O. Mancino and G. Stampacchia, "Convex programming and variational inequalities," *Journal of Optimization Theory and Application* 9 (1972) 3–23.

[156] O.L. Mangasarian, "Equivalence of the complementarity problem to a system of nonlinear equations," *SIAM Journal on Applied Mathematics* 31 (1976) 89–92.

[157] O.L. Mangasarian, "Locally unique solutions of quadratic programs, linear and nonlinear complementarity problems," *Mathematical Programming* 19 (1980) 200–212.

[158] O.L. Mangasarian and L. McLinden, "Simple bounds for solutions of monotone complementarily problems and convex programs," *Mathematical Programming* 32 (1985) 32–40.

[159] A.S. Manne, "On the formulation and solution of economic equilibrium models," *Mathematical Programming Study* 23 (1985) 1–22.

[160] A.S. Manne and P.V. Preckel, "A three-region intertemporal model of energy, international trade and capital flows," *Mathematical Programming Study* 23 (1985) 56–74.

[161] P. Marcotte, "Network optimization with continuous control parameters," *Transportation Science* 17 (1983) 181–197.

[162] P. Marcotte, "Quelques notes et résultats nouveaux sur les problème d'equilibre d'un oligopole," *R.A.I.R.O. Recherche Opérationnelle* 18 (1984) 147–171.

[163] P. Marcotte, "A new algorithm for solving variational inequalities with application to the traffic assignment problem," *Mathematical Programming* 33 (1985) 339–351.

[164] P. Marcotte, "Gap-decreasing algorithms for monotone variational inequalities," paper presented at the ORSA/TIMS Meeting, Miami Beach, October 1986.

[165] P. Marcotte, "Network design with congestion effects: a case of bi-level programming," *Mathematical Programming* 34 (1986) 142–162.

[166] P. Marcotte and J.P. Dussault, "A modified Newton method for solving variational inequalities," *Proceeding of the 24th IEEE Conference on Decision and Control*, pp. 1433–1436.

[167] P. Marcotte and J.P. Dussault, "A note on a globally convergent Newton method for solving monotone variational inequalities," *Operations Research Letters* 6 (1987) 35–42.

[168] L. Mathiesen, "Computation of economic equilibria by a sequence of linear complementarity problems," *Mathematical Programming Study* 23 (1985) 144–162.

[169] L. Mathiesen, "Computational experience in solving equilibrium models by a sequence of linear complementarity problems," *Operations Research* 33 (1985) 1225–1250.

[170] L. Mathiesen, "An algorithm based on a sequence of linear complementarity problems applied to a Walrasian equilibrium model: an example," *Mathematical Programming* 37 (1987) 1–18.

[171] L. Mathiesen and A. Lont, "Modeling market equilibria: an application to the world steel market," Working Paper MU04, Center for Applied Research, Norwegian School of Economics and Business Administration (Bergen, Norway, 1983).

[172] L. Mathiesen and E. Steigum, Jr., "Computation of unemployment equilibria in a two-country multi-period model with neutral money," Working Paper, Center for Applied Research, Norwegian School of Economics and Business Administration (Bergen, Norway, 1985).

[173] L. McKenzie, "Why compute economic equilibria?," in: *Computing Equilibria: How and Why* (North-Holland, Amsterdam, 1976).

[174] L. McLinden, "The complementarity problem for maximal monotone multifunctions," in: R.W. Cottle, F. Giannessi and J.L. Lions, eds., *Variational Inequalities and Complementarity Problems* (Academic Press, New York, 1980) pp. 251–270.

[175] L. McLinden, "An analogue of Moreau's proximation theorem, with application to the nonlinear complementarity problem," *Pacific Journal of Mathematics* 88 (1980) 101–161.

[176] L. McLinden, "Stable monotone variational inequalities," *Mathematical Programming (Series B)* 48 (1990) 303–338, this issue.

[177] N. Megiddo, "A monotone complementarity problem with feasible solutions but no complementary solutions," *Mathematical Programming* 12 (1977) 131–132.

[178] N. Megiddo, "On the parametric nonlinear complementarity problem," *Mathematical Programming Study* 7 (1978) 142–159.

[179] N. Megiddo and M. Kojima, "On the existence and uniqueness of solutions in nonlinear complementarity theory," *Mathematical Programming* 12 (1977) 110–130.

[180] G.J. Minty, "Monotone (non-linear) operators in Hilbert space," *Duke Mathematics Journal* 29 (1962) 341–346.

[181] J.J. Moré, "The application of variational inequalities to complementarity problems and existence theorems," Technical Report 71-90, Department of Computer Sciences, Cornell University (Ithaca, NY, 1971).

[182] J.J. Moré, "Classes of functions and feasibility conditions in nonlinear complementarity problems," *Mathematical Programming* 6 (1974) 327–338.

[183] J.J. Moré, "Coercivity conditions in nonlinear complementarity problems," *SIAM Review* 17 (1974) 1–16.

[184] J.J. Moré and W.C. Rheinboldt, "On P- and S-functions and related classes of n-dimensional nonlinear mappings," *Linear Algebra and Its Applications* 6 (1973) 45–68.

[185] J.J. Moreau, "Proximitè et dualitè dans un espace Hilberiten," *Bulletin of the Society of Mathematics of France* 93 (1965) 273–299.

[186] J.D. Murchland, "Braess' paradox of traffic flow," *Transportation Research* 4 (1970) 391–394.

[187] K.G. Murty, *Linear Complementarity, Linear and Nonlinear Programming* (Helderman, Berlin, 1988).

[188] A. Nagurney, "Comparative tests of multimodal traffic equilibrium methods," *Transportation Research* 18B (1984) 469–485.

[189] A. Nagurney, "Computational comparisons of algorithms for general asymmetric traffic equilibrium problems with fixed and elastic demand," *Transportation Research* 20B (1986) 78–84.

[190] A. Nagurney, "Computational comparisons of spatial price equilibrium methods," *Journal of Regional Science* 27 (1987) 55–76.

[191] A. Nagurney, "Competitive equilibrium problems, variational inequalities and regional science," *Journal of Regional Science* 27 (1987) 503–517.

[192] J.F. Nash, "Equilibrium points in n-person games," *Proceedings of the National Academy of Sciences* 36 (1950) 48–49.

[193] S. Nguyen and C. Dupuis, "An efficient method for computing traffic equilibria in networks with asymmetric transportation costs," *Transportation Science* 18 (1984) 185-202.

[194] J.M. Ortega and W.C. Rheinboldt, *Iterative Solution of Nonlinear Equations in Several Variables* (Academic Press, New York, 1970).

[195] A.R. Pagan and J.H. Shannon, "Sensitivity analysis for linearized computable general equilibrium models," in: J. Piggott and J. Whalley, eds., *New Developments in Applied General Equilibrium Analysis* (Cambridge University Press, Cambridge, 1985) pp. 104-118.

[196] J.S. Pang, *Least-Element Complementarity Theory.* Ph.D. dissertation, Department of Operations Research, Stanford University (Stanford, CA, 1976).

[197] J.S. Pang, "The implicit complementarity problem", in: O.L. Managasarian, R.R. Meyer and S.M. Robinson, eds., *Nonlinear Programming* 4 (Academic Press, New York, 1981) 487-518.

[198] J.S. Pang, "A column generation technique for the computation of stationary points," *Mathematics of Operations Research* 6 (1981) 213-244.

[199] J.S. Pang, "On the convergence of a basic iterative method for the implicit complementarity problem," *Journal of Optimization Theory and Application* 37 (1982) 149-162.

[200] J.S. Pang, "Solution of the general multicommodity spatial equilibrium problem by variational and complementarity methods," *Journal of Regional Science* 24 (1984) 403-414.

[201] J.S. Pang, "Variational inequality problems over product sets: applications and iterative methods," *Mathematical Programming* 31 (1985) 206-219.

[202] J.S. Pang, "Inexact Newton methods for the nonlinear complementarity problem," *Mathematical Programming* 36 (1986) 54-71.

[203] J.S. Pang, "A posteriori error bounds for linearly constrained variational inequality problems," *Mathematics of Operations Research* 12 (1987) 474-484.

[204] J.S. Pang, "Two characterization theorems in complementarity theory," *Operations Research Letters* 7 (1988) 27-31.

[205] J.S. Pang, "Newton's method for B-differentiable equations," to appear in: *Mathematics of Operations Research.*

[206] J.S. Pang, "Solution differentiability and continuation of Newton's method for variational inequality problems over polyhedral sets," to appear in: *Journal of Optimization Theory and Applications.*

[207] J.S. Pang and D. Chan, "Iterative methods for variational and complementarity problems," *Mathematical Programming* 24 (1982) 284-313.

[208] J.S. Pang and J.M. Yang, "Parallel Newton methods for the nonlinear complementarity problem," *Mathematical Programming (Series B)* 42 (1988) 407-420.

[209] J.S. Pang and C.S. Yu, "Linearized simplicial decomposition methods for computing traffic equilibria on networks," *Networks* 14 (1984) 427-438.

[210] P.V. Preckel, "Alternative algorithms for computing economic equilibria," *Mathematical Programming Study* 23 (1985) 163-172.

[211] P.V. Preckel, "A modified Newton method for the nonlinear complementarity problem and its implementation," paper presented at the ORSA/TIMS Meeting, Miami Beach, FL, October 1986.

[212] Y. Qiu and T.L. Magnanti, "Sensitivity analysis for variational inequalities defined on polyhedral sets," *Mathematics of Operations Research* 14 (1989) 410-432.

[213] Y. Qiu and T.L. Magnanti, "Sensitivity analysis for variational inequalities," Working Paper OR 163-87, Operations Research Center, M.I.T. (Cambridge, MA, 1987).

[214] A. Reinoza, *A Degree For Generalized Equations.* Ph.D. dissertation, Department of Industrial Engineering, University of Wisconsin (Madison, WI, 1979).

[215] A. Reinoza, "The strong positivity conditions," *Mathematics of Operations Research* 10 (1985) 54-62.

[216] W.C. Rheinboldt, *Numerical Analysis of Parameterized Nonlinear Equations* (Wiley, New York, 1986).

[217] S.M. Robinson, "Generalized equations and their solutions, part I: basic theory," *Mathematical Programming Study* 10 (1979) 128-141.

[218] S.M. Robinson, "Strongly regular generalized equations," *Mathematics of Operations Research* 5 (1980) 43-62.

[219] S.M. Robinson, "Generalized equations and their solutions, part II: applications to nonlinear programming," *Mathematical Programming Study* 19 (1982) 200-221.

[220] S.M. Robinson, "Generalized equations," in: A. Bachem, M. Grötschel and B. Korte, eds., *Mathematical Programming: The State of the Art* (Springer, Berlin, 1982) pp. 346–367.

[221] S.M. Robinson, "Implicit B-differentiability in generalized equations," Technical Summary Report No. 2854, Mathematics Research Center, University of Wisconsin (Madison, WI, 1985).

[222] S.M. Robinson, "Local structure of feasible sets in nonlinear programming, part III: stability and sensitivity," *Mathematical Programming Study* 30 (1987) 45–66.

[223] S.M. Robinson, "An implicit-function theorem for a class of nonsmooth functions," to appear in: *Mathematics of Operations Research.*

[224] R.T. Rockafellar, "Characterization of the subdifferentials of convex functions," *Pacific Journal of Mathematics* 17 (1966) 497–510.

[225] R.T. Rockafellar, "Convex functions, monotone operators, and variational inequalities," *Theory and Applications of Monotone Operators: Proceedings of the NATO Advanced Study Institute, Venice, Italy* (Edizioni Oderisi, Gubbio, Italy, 1968) pp. 35–65.

[226] R.T. Rockafellar, "On the maximal monotonicity of subdifferential mappings," *Pacific Journal of Mathematics* 33 (1970) 209–216.

[227] R.T. Rockafellar, *Convex Analysis* (Princeton University Press, Princeton, NJ, 1970).

[228] R.T. Rockafellar, "Augmented Lagrangian and application of the proximal point algorithm in convex programming," *Mathematics of Operations Research* 1 (1976) 97–116.

[229] R.T. Rockafellar, "Monotone operators and the proximal point algorithm," *SIAM Journal on Control and Optimization* 14 (1976) 877–898.

[230] R.T. Rockafellar, "Lagrange multipliers and variational inequalities," in: R.W. Cottle, F. Giannessi, and J.L.Lions, eds., *Variational Inequalities and Complementarity Problems: Theory and Applications* (Wiley, New York, 1980) pp. 303–322.

[231] T.F. Rutherford, *Applied General Equilibrium Modeling.* Ph.D. dissertation Department of Operations Research, Stanford University (Stanford, CA, 1986).

[232] T.F. Rutherford, "Implementation issues and computational performance solving applied general equilibrium models with SLCP," Discussion Paper 837, Cowles Foundation for Research in Economics, Yale University (New Haven, CT, 1987).

[233] R. Saigal, "Extension of the generalized complementarity problem," *Mathematics of Operations Research* 1 (1976) 260–266.

[234] P.A. Samuelson, "Spatial price equilibrium and linear programming," *American Economic Review* 42 (1952) 283–303.

[235] H.E. Scarf, "The approximation of fixed points of a continuous mapping," *SIAM Journal on Applied Mathematics* 15 (1967) 1328–1342.

[236] H.E. Scarf and T. Hansen, *Computation of Economic Equilibria* (Yale University Press, New Haven, CT, 1973).

[237] A. Shapiro, "On concepts of directional differentiability," Research Report 73/88(18), Department of Mathematics and Applied Mathematics, University of South Africa (Pretoria, South Africa, 1988).

[238] J.B. Shoven, "Applying fixed points algorithms to the analysis of tax policies," in: S. Karmardian and C.B. Garcia, eds., *Fixed Points: Algorithms and Applications* (Academic Press, New York, 1977) pp. 403–434.

[239] J.B. Shoven, "The application of fixed point methods to economics," in: B.C. Eaves, F.J. Gould, H.O. Peitgen, and M.J. Todd, eds., *Homotopy Methods and Global Convergence* (Plenum Press, New York, 1983) pp. 249–262.

[240] S. Smale, "A convergent process of price adjustment and global Newton methods," *Journal of Mathematical Economics* 3 (1976) 107–120.

[241] M.J. Smith, "The existence, uniqueness and stability of traffic equilibria," *Transportation Research* 13B (1979) 295–304.

[242] M.J. Smith, "The existence and calculation of traffic equilibria," *Transportation Research* 17B (1983) 291–303.

[243] M.J. Smith, "A descent algorithm for solving monotone variational inequality and monotone complementarity problems," *Journal of Optimization Theory and Application* 44 (1984) 485–496.

[244] M.J. Smith, "The stability of a dynamic model of traffic assignment- an application of a method of Lyapunov," *Transportation Science* 18 (1984) 245–252.

[245] T.E. Smith, "A solution condition for complementarity problems: with an apilication to spatial price equilibrium," *Applied Mathematics and Computation* 15 (1984) 61–69.

[246] J.E. Spingarn, "Partial inverse of a monotone operator," *Applied Mathematics and Optimization* 10 (1983) 247–265.

[247] J.E. Spingarn, "Applications of the method of partial inverses to convex programming: decomposition," *Mathematical Programming* 32 (1985) 199–223.

[248] J.E. Spingarn, "On computation of spatial economic equilibria," Discussion Paper 8731, Center for Operations Research and Econometrics, Université Catholique de Louvain (Louvain-la-Neuve, Belgium, 1987).

[249] G. Stampacchia, "Variational inequalities," in *Theory and Applications of Monotone Operators, Proceedings of the NATO Advanced Study Institute, Venice, Italy* (Edizioni Oderisi, Gubbio, Italy, 1968) pp. 102–192.

[250] R. Steinberg and R.E. Stone, "The prevalence of paradoxes in transportation equilibrium problems," Working paper, AT&T Bell Laboratories (Holmdel, NJ, 1987).

[251] R. Steinberg and W.I. Zangwill, "The prevalence of Braess' paradox," *Transportation Science* 17 (1983) 301–319.

[252] J.C. Stone, "Sequential optimization and complementarity techniques for computing economic equilibria," *Mathematical Programming Study* 23 (1985) 173–191.

[253] P.K. Subramanian, "Gauss-Newton methods for the nonlinear complementarity problem," Technical Summary Report No. 2845, Mathematics Research Center, University of Wisconsin (Madison, WI, 1985).

[254] P.K. Subramanian, "Fixed-point methods for the complementarity problem," Technical Summary Report No. 2857, Mathematics Research Center, University of Wisconsin (Madison, WI, 1985).

[255] P.K. Subramanian, "A note on least two norm solutions of monotone complementarity problems," *Applied Mathematics Letters* 1 (1988) 395–397.

[256] A. Tamir, "Minimality and complementarity properties associated with Z-functions and M-functions," *Mathematical Programming* 7 (1974) 17–31.

[257] R.L. Tobin, "General spatial price equilibria: sensitivity analysis for variational inequality and nonlinear complementarity formulations," in: P.T. Harker, ed., *Spatial Price Equilibrium: Advances in Theory, Computation and Application, Lecture Notes in Economics and Mathematical Systems, Vol. 249* (Springer, Berlin, 1985) pp. 158–195.

[258] R.L. Tobin, "Sensitivity analysis for variational inequalities," *Journal of Optimization Theory and Applications* 48 (1986) 191–204.

[259] M.J. Todd, *The Computation of Fixed Points and Applications* (Springer, Berlin, 1976).

[260] M.J. Todd, "A note on computing equilibria in economics with activity models of production", *Journal of Mathematical Economics* 6 (1979) 135–144.

[261] G. Van der Laan and A.J.J. Talman, "Simplicial approximation of solutions to the nonlinear complementarity problem with lower and upper bounds," *Mathematical Programming* 38 (1987) 1–15.

[262] J.A. Ventura and D.W. Hearn, "Restricted simplicial decomposition for convex constrained problems," Research Report No. 86-15, Department of Industrial and Systems Engineering, University of Florida (Gainesville, FL, 1986).

[263] J.G. Wardrop, "Some theoretical aspects of road traffic research," *Proceedings of the Institute of Civil Engineers, Part II* (1952) 325–378.

[264] L.T. Watson, "Solving the nonlinear complementarity problem by a homotopy method," *SIAM Journal on Control and Optimization* 17 (1979) 36–46.

[265] J. Whalley, "Fiscal harmonization in the EEC: some preliminary findings of fixed point calculations," in: S. Karamardian and C.B. Garcia, eds., *Fixed Points: Algorithms and Applications* (Academic Press, New York, 1977) pp. 435–472.

[266] Y. Yamamoto, "A path following algorithm for stationary point problems," *Journal of the Operations Research Society of Japan* 30 (1987) 181–198.

[267] Y. Yamamoto, "Fixed point algorithms for stationary point problems," in: M. Zri and K. Tanabe, eds., *Mathematical Programming: Recent Developments and Applications* (KTK Scientific Publishers, Tokyo, 1989) pp. 283–308.