Duality relations for variational inequalities with applications to network flows

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DUALITY RELATIONS FOR VARIATIONAL INEQUALITIES WITH APPLICATIONS TO NETWORK FLOWS

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1. Introduction.

In recent works [1,4] a duality scheme for variational inequalities has been introduced by means of separation arguments, following the image space approach developed in [7] for constrained extremum problems. In this work we consider another duality scheme for variational inequalities, based on the classical duality theory for extremum problems (see [11] and [12]) which will turn out to be equivalent to the one introduced in [1].

Consider the following variational inequality:

$$\text{find } \overline{x} \in \mathbb{K} \text{ such that } \langle F(\overline{x}), x - \overline{x} \rangle \geq 0, \quad \forall x \in \mathbb{K} \tag{VI}$$

where $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and $\mathbb{K} \subseteq \mathbb{R}^n$.

It is known that variational inequalities are closely related to optimization problems; in particular, if $\mathbb{K}$ is convex and there exists f such that $F = \nabla f$ (in this case we say that (VI) has the integrability property), (VI) is the classical first–order necessary optimality condition for the problem

$$\min f(x) \text{ s.t. } x \in \mathbb{K}. \tag{P}$$

Anyway, it is possible to connect (VI) with a suitable constrained extremum problem, even when (VI) has not the integrability property. It is easy to show that $\overline{x}$ is a solution of (VI) if and only if it is a solution of the following problem:

$$\min \langle F(\overline{x}), x - \overline{x} \rangle \text{ s.t. } x \in \mathbb{K} \tag{$P_{\overline{x}}$}$$

whose first–order optimality condition, in the case in which (VI) has the integrability property, is the same of the one of problem (P) at the point $\overline{x}$. Therefore it will be natural to associate to (VI) a variational inequality which represents a first–order optimality condition of a dual of problem $(P_{\overline{x}})$: such variational inequality will be defined as *dual* of (VI). In Sections 3,4 we will show how the present scheme allows to recover some of the duality results obtained in [1,4] by means of separation arguments. In particular we will see that, if we consider Fenchel dual of problem $(P_{\overline{x}})$, we obtain (up to a simmetry) the dual variational inequality introduced by Mosco [10]. In Section 5 we provide an application to a generalized minimal–cost network–flow problem formulated as a variational inequality, giving an interpretation to the related dual variables in terms of potential associated to the nodes and the arcs of the network.

In the sequel we will study a variational inequality defined on $\mathbb{R}^n$ where $\langle \cdot, \cdot \rangle$ is the Euclidean scalar product and we will consider the following notations. If $Q \subseteq \mathbb{R}^n$ is a set, we put

$$\delta(x|Q) := \begin{cases} 0, & \text{if } x \in Q, \\ +\infty, & \text{otherwise.} \end{cases}$$

Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be convex and $g : \mathbb{R}^n \longrightarrow \mathbb{R}$ be concave, then

$$f^*(x^*) := \sup\{\langle x^*, x\rangle - f(x) \ : \ x \in \mathbb{R}^n\},$$
$$g_*(x^*) := \inf\{\langle x^*, x\rangle - g(x) \ : \ x \in \mathbb{R}^n\}$$

are the Fenchel conjugates, in convex and concave sense respectively, of the functions f and g and are defined on the dual space $(\mathbb{R}^n)^* \cong \mathbb{R}^n$.

$$\partial f(x_0) := \{x^* \in \mathbb{R}^n \ : \ f(x) \geq f(x_0) + \langle x^*, x - x_0\rangle, \quad \forall x \in \mathbb{R}^n\}.$$

is the subdifferential of the convex function f at $x_0 \in \mathbb{R}^n$. If $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ the (effective) domain of f is

$$dom f := \{x \in \mathbb{R}^n \ : \ f(x) < +\infty\}.$$

Let $C \subseteq \mathbb{R}^m$ be a closed convex cone, the positive polar of C is the set

$$C^* := \{x^* \in \mathbb{R}^m \ : \ \langle x^*, x\rangle \geq 0, \quad \forall x \in C\}.$$

A function $G : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is said C–concave iff

$$G(tx + (1-t)y) - tG(x) - (1-t)G(y) \in C, \quad \forall x, y \in \mathbb{R}^n, \ \forall t \in [0,1].$$

Let $x, y \in \mathbb{R}^m$; we denote $x \geq_C y$ iff $x - y \in C$.

2. Lagrangean and parametric representation of an extremum problem.

Let us recall some results concerning duality theory for constrained extremum problems, due to Rockafellar [11]. Consider the following problem

$$\min f(x) \text{ s.t. } x \in Q, \tag{P}$$

where $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ and $Q \subseteq \mathbb{R}^n$.

Definition 2.1. Let Λ be a vector space; the function $L : \mathbb{R}^n \times \Lambda \longrightarrow [-\infty, +\infty]$ is said a *Lagrangean representation of* P if the following conditions hold:
i) for each fixed $x \in Q$ the function $L(x, \cdot)$ is closed and concave,

ii) $$\sup\{L(x,\lambda) \ : \ \lambda \in \Lambda\} = \begin{cases} f(x), & \text{if } x \in Q, \\ +\infty, & \text{otherwise.} \end{cases}$$

Remark 2.1. It is immediate that (P) is equivalent to the following problem:

$$\inf_{x \in \mathbb{R}^n} \sup_{\lambda \in \Lambda} L(x, \lambda).$$

Definition 2.2. Given a Lagrangean representation L, the *dual problem of* (P) is

$$\sup_{\lambda \in \Lambda} \inf_{x \in \mathbb{R}^n} L(x, \lambda). \tag{D}$$

Definition 2.3. Given a Lagrangean representation L, the pair $(x_0, \lambda_0) \in \mathbb{R}^n \times \Lambda$ is a *saddle point of* L if

$$L(x_0, \lambda) \leq L(x_0, \lambda_0) \leq L(x, \lambda_0), \qquad \forall x \in \mathbb{R}^n, \ \forall \lambda \in \Lambda.$$

Theorem 2.1. [11] *A pair $(x_0, \lambda_0) \in \mathbb{R}^n \times \Lambda$ is a saddle point of L if and only if x_0 solves (P) and λ_0 solves (D).* □

In order to obtain a Lagrangean representation of the problem, we can previously consider a parametric representation of (P).

Definition 2.4. Given a vector space Y, the function $F : \mathbb{R}^n \times Y \longrightarrow [-\infty, +\infty]$ is said a *parametric representation of* (P) if it fulfils the following conditions:
i) for each fixed $x \in \mathbb{R}^n$, the function $F(x, \cdot)$ is closed and convex,

ii) $$F(x, 0) = \begin{cases} f(x), & \text{if } x \in Q, \\ +\infty, & \text{otherwise.} \end{cases}$$

The following result clarifies the relations between the Lagrangean and the parametric representation of (P).

Theorem 2.2. [11] *Given the problem (P) we have:*

i) if F is a parametric representation of (P), then a Lagrangean representation can be obtained defining

$$L(x, \lambda) := \inf\{F(x, y) + \langle \lambda, y \rangle \ : \ y \in Y\}. \tag{2.1}$$

ii) Vice versa, if L is a Lagrangean representation of (P), then a parametric representation can be obtained defining

$$F(x,y) := \sup\{L(x,\lambda) - \langle \lambda, y\rangle \ : \ \lambda \in \Lambda\}. \qquad \square$$

Let us examine now two particular cases which will also be considered in the sequel.

1) Lagrangean Dual. Given the constrained extremum problem

$$\min f(x) \text{ s.t. } g(x) \leq 0$$

with $g : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, consider the following parametric representation

$$F_L(x,y) := \begin{cases} f(x), & \text{if } g(x) \leq y, \\ +\infty, & \text{otherwise.} \end{cases}$$

The corresponding Lagrangean representation is given by

$$L_L(x,\lambda) = \inf\{F(x,y) + \langle \lambda, y\rangle \ : \ y \in \mathbb{R}^n\} = \begin{cases} f(x) + \langle \lambda, g(x)\rangle, & \text{if } \lambda \geq 0, \\ -\infty, & \text{otherwise.} \end{cases}$$

2) Fenchel Dual. Given the problem

$$\min f(x) - g(x) \text{ s.t. } x \in \mathbb{R}^n$$

where f is convex and g is concave, the dual of Fenchel is defined by

$$\max g_*(x^*) - f^*(x^*) \text{ s.t. } x^* \in \mathbb{R}^n.$$

This dual problem can be obtained by means of the following parametric representation:

$$F_F(x,y) := f(x) - g(x+y).$$

The corresponding Lagrangean representation is given by

$$L_F(x,x^*) = \inf\{F(x,y) + \langle x^*, y\rangle \ : \ y \in \mathbb{R}^n\} = f(x) - \langle x^*, x\rangle + g_*(x^*)$$

and it fulfils the following relation

$$\sup_{x^* \in \mathbb{R}^n} \inf_{x \in \mathbb{R}^n} L_F(x,x^*) = \sup\{g_*(x^*) - f^*(x^*) \ : \ x^* \in \mathbb{R}^n\}.$$

3. A duality scheme for variational inequalities.

Consider the following variational inequality:

$$\text{find } \overline{x} \in \mathbb{K} \text{ such that } \langle F(\overline{x}), x - \overline{x}\rangle \geq 0, \ \forall x \in \mathbb{K} \qquad (VI)$$

with $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and $\mathbb{K} \subseteq \mathbb{R}^n$ a convex set. It is well–known that, if there exists $f \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$ such that $F = \nabla f$, (VI) represents the classical first–order optimality condition for the problem

$$\min f(x) \text{ s.t } x \in \mathbb{K}.$$

If $F \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$, a necessary and sufficient condition for the existence of f is that the Jacobian matrix $\nabla F(x)$ is symmetric for each $x \in \mathbb{R}^n$. In this case we say that (VI) has the *integrability property.*

Let us introduce, now, a first duality scheme that could be suitable in the case in which (VI) has the integrability property; let (D) a dual problem (in the sense of Definition 2.2) of the problem (P);

$$\begin{tikzcd}
(P) \arrow[rr] \arrow[d, "\begin{array}{l}\text{First Order}\\ \text{Optimality}\\ \text{Condition}\end{array}"] & & (D) \arrow[d, "\begin{array}{l}\text{First Order}\\ \text{Optimality}\\ \text{Condition}\end{array}"] \\
(VI) & & (DVI)
\end{tikzcd} \tag{3.1}$$

where (DVI) denotes the dual variational inequality associated to (VI).

Unfortunately, not always the problem (VI) has the integrability property and therefore the previous scheme can not be considered in general. To overcome this difficulty we will introduce another constrained extremum problem that is closely related to the problem (P) in the case in which (VI) has the integrability property:

$$\min\langle F(\overline{x}), x - \overline{x}\rangle \quad \text{s.t.} \quad x \in \mathbb{K}. \tag{$P_{\overline{x}}$}$$

The following result is of immediate proof but, since it will be very useful in the sequel, we state it in a proposition.

Proposition 3.1. *$\overline{x} \in \mathbb{K}$ is a solution for (VI) if and only if it is a global optimal solution of the problem $(P_{\overline{x}})$.* □

Remark 3.1. When (P) has the integrability property, the problem $(P_{\overline{x}})$ is equivalent to the linearization of (P) at the point $\overline{x}$.

By means of the problem $(P_{\overline{x}})$, we are able to consider the following general duality scheme for variational inequalities:

$$
\begin{tikzcd}
(VI) \arrow[dd, leftrightarrow] & (DVI) \\
 & \text{Saddle Point Condition for } L \arrow[u] \\
(P_{\overline{x}}) \arrow[r, leftrightarrow] & \text{Lagrangean Representation } L \text{ of } P_{\overline{x}} \arrow[u]
\end{tikzcd}
\tag{3.2}
$$

In this scheme instead of considering directly a dual of problem $(P_{\overline{x}})$, we have considered the saddle point condition of a suitable Lagrangean representation L; in fact it is known (Theorem 2.1) that a saddle point of a Lagrangean representation is a vector containing both the solutions of the primal problem $(P_{\overline{x}})$ and of its dual (dependent on the chosen representation). The dual variational inequality (DVI) must be defined in order to guarantee that $\overline{x}^*$ is one of the solutions of (DVI) if and only if $(\overline{x}, \overline{x}^*)$ is a saddle point of the Lagrangean representation on a suitable product space.

As already mentioned, the previous scheme allows to recover the dual variational inequality introduced by Mosco [10]. Consider the following generalized variational inequality:

$$\text{find } \overline{x} \in \mathbb{R}^n \text{ such that } \langle F(\overline{x}), x - \overline{x} \rangle \geq f(\overline{x}) - f(x),\ \forall x \in \mathbb{R}^n \tag{GVI}$$

where $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and $f : \mathbb{R}^n \longrightarrow [-\infty, +\infty]$ is convex and lower semicontinuous.

Remark 3.2. The problem (GVI) collapses into the problem (VI) if $f(\cdot) = \delta(\cdot|\mathbb{K})$.

In the hypothesis that F has an inverse, in [10] it has been introduced the following dual variational inequality:

$$\text{find } \overline{x}^* \in \mathbb{R}^n \text{ such that } \langle -F^{-1}(\overline{x}^*), x^* - \overline{x}^* \rangle \geq f^*(\overline{x}^*) - f^*(x^*),\ \forall x^* \in \mathbb{R}^n \tag{$DGVI$}$$

and the following result has been proven.

Theorem 3.1. *If F has an inverse and f is a proper convex and lower semicontinuous function, then $\overline{x}$ is a solution for (GVI) if and only if $\overline{x}^* = -F(\overline{x})$ is a solution of $(DGVI)$.* □

Following the scheme (3.1), consider the following constrained extremum problem:

$$\min \langle F(\overline{x}), x - \overline{x} \rangle - f(\overline{x}) + f(x) \text{ s.t. } x \in \mathbb{R}^n. \tag{$P_{\overline{x}}$}$$

Remark 3.3. It is immediate that $\overline{x}$ is a solution of (GVI) if and only if $\overline{x}$ is a global optimal solution of $(P_{\overline{x}})$.

Let us consider the Lagrangean representation L_F of $(P_{\overline{x}})$, which allows to obtain the Fenchel dual problem of $(P_{\overline{x}})$: consider the objective function as difference of the two functions $\langle F(\overline{x}), x-\overline{x}\rangle - f(\overline{x})$ and $-f(x)$, then

$$L_F(\overline{x}; x, x^*) = \langle F(\overline{x}), x-\overline{x}\rangle - f(\overline{x}) - \langle x^*, x\rangle - f^*(-x^*).$$

The following result, proven in [4], holds.

Theorem 3.2. *If F possesses an inverse and f is a proper convex and lower semicontinuous function, then*

i) $\overline{x}$ is a solution for (GVI) if and only if $(\overline{x}, F(\overline{x})) \in \mathbb{R}^n \times \mathbb{R}^n$ is a saddle point for $L_F(\overline{x}; x, x^*)$.

ii) $\overline{x}^* = -F(\overline{x})$ is a solution for $(DGVI)$ if and only if $(\overline{x}, F(\overline{x})) \in \mathbb{R}^n \times \mathbb{R}^n$ is a saddle point for $L_F(\overline{x}; x, x^*)$. □

Remark 3.4. Theorem 3.1 follows immediately from Theorem 3.2; moreover Theorem 3.1. can be proven, more generally, in a Hausdorff locally convex space [10].

Mosco dual variational inequality has been obtained by means of Fenchel duality. Similarly, if we consider the classical Lagrangean duality applied to the problem $(P_{\overline{x}})$, we can recover the results obtained in [4]. Consider the variational inequality

$$\text{find } \overline{x} \in \mathbb{K} \text{ such that } \langle F(\overline{x}), x-\overline{x}\rangle \geq 0, \quad \forall x \in \mathbb{K} \tag{VI}$$

where $\mathbb{K} = \{x \in \mathbb{R}^n \; : \; g(x) \geq_C 0\}$ with $g : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and $C \subseteq \mathbb{R}^m$ a closed convex cone.

We recall that $\overline{x}$ is a solution for (VI) if and only if $\overline{x}$ is a global optimal solution of the problem

$$\min \langle F(\overline{x}), x-\overline{x}\rangle \text{ s.t. } x \in \mathbb{K}. \tag{$Q_{\overline{x}}$}$$

Following the scheme (3.1) consider the classical Lagrangean function associated to $(Q_{\overline{x}})$:

$$L_L(\overline{x}; x, \lambda) = \langle F(\overline{x}), x-\overline{x}\rangle - \langle \lambda, g(x)\rangle.$$

In order to characterize the saddle point condition for L_L, it is useful to introduce an equivalent formulation of the problem (VI) by means of the following system:

$$\begin{cases} F(x) - \lambda \nabla g(x) = 0 \\ \langle \lambda, g(x)\rangle = 0 \\ \lambda \in C^*, \; g(x) \geq_C 0 \end{cases} \tag{S}$$

The following results have been proven in [4].

Lemma 3.1. *Suppose that g is a differentiable C–concave function and that there exists $x_0 \in \mathbb{R}^n$ such that $g(x_0) \in \operatorname{int} C$. Then $\overline{x} \in \mathbb{K}$ is a solution of (VI) if and only if there exists $\overline{\lambda} \in \mathbb{R}^m$ such that $(\overline{x}, \overline{\lambda})$ is a solution of (S).* □

Lemma 3.2. *Suppose that g is a differentiable C–concave function and that there exists $x_0 \in \mathbb{R}^n$ such that $g(x_0) \in \operatorname{int} C$. Then $(\overline{x}, \overline{\lambda})$ is a saddle point of $L_L(\overline{x}; x, \lambda)$ on $\mathbb{R}^n \times C^*$ if and only if $(\overline{x}, \overline{\lambda})$ is a solution of (S).* □

Theorem 3.1. *Suppose that g is a differentiable C–concave function and that there exists $x_0 \in \mathbb{R}^n$ such that $g(x_0) \in \operatorname{int} C$. Then $\overline{x} \in Q$ is a solution of (VI) if and only if there exists $\overline{\lambda} \in \mathbb{R}^m$ such that $(\overline{x}, \overline{\lambda})$ is a saddle point of $L_L(\overline{x}; x, \lambda)$ on $\mathbb{R}^n \times C^*$* □

Remark 3.5. We observe that if $F = \nabla f$, then the system (S) collapses in the classical Kuhn–Tucker condition for the extremum problem

$$\min f(x) \text{ s.t. } x \in \mathbb{K}, \tag{P}$$

and therefore the schemes (3.1) and (3.2) coincide.

The previous results state equivalent formulations of the variational inequality in the primal–dual product space $\mathbb{R}^n \times \mathbb{R}^m$. If, in the system (S) or in the saddle point condition for the function L_L, it is possible to eliminate the dependance on the primal variable x, we obtain a dual problem associated to the variational inequality (VI).

In particular this is possible when the operator F possesses an inverse and the constraints g are linear, that is, $g(x) = Ax - b$ with $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ and $C = \mathbb{R}^m_+$. In this case (S) becomes:

$$\begin{cases} x = F^{-1}(A^T\lambda) \\ \langle \lambda, AF^{-1}(A^T\lambda) - b \rangle = 0 \\ \lambda \geq 0, \quad AF^{-1}(A^T\lambda) - b \geq 0 \end{cases} \tag{3.3}$$

The system (3.3) represents a dual complementarity problem associated to the variational inequality (VI). When the feasible region $\mathbb{K}$ of the variational inequality is a closed convex cone, it is also possible to define a dual complementarity problem associated to (VI). In this case the system (S) becomes:

$$\begin{cases} F(x) - \lambda = 0 \\ \langle \lambda, x \rangle = 0 \\ \lambda \in \mathbb{K}^*, \quad x \in \mathbb{K}. \end{cases}$$

Therefore the dual complementarity problem associated to (VI) is the following:

$$\text{find } \lambda \in \mathbb{K}^* \text{ such that } \langle \lambda, F^{-1}(\lambda) \rangle = 0, \; F^{-1}(\lambda) \in \mathbb{K}.$$

Remark 3.6. It is known that, when the feasible region is a closed convex cone $\mathbb{K}$, (VI) is equivalent to the complementarity problem

$$\text{find } x \in \mathbb{K} \text{ such that } \langle x, F(x) \rangle = 0, \; F(x) \in \mathbb{K}^*.$$

In this case it is, therefore, natural that the dual problem associated to (VI) is a complementarity problem.

A further example is given by Mosco dual variational inequality which can be obtained eliminating from the saddle point condition for the function $L_F(\overline{x}; x, x^*)$ the dependance on the variable x.

4. Connections with separation.

In this section we show that the introduced scheme is closely related to the one considered in [1,4] using separation arguments in a parametric image space associated to the variational inequality. We recall the duality scheme introduced in [1,4]. Given the generalized variational inequality (GVI) defined in the previous section, for each $y \in dom f$, let

$$\Phi(y; x) := \langle F(y), y - x\rangle + f(y) - f(x).$$

$\overline{x} \in dom f$ is a solution for (GVI) if and only if the following system, in the unknown x, is impossible:

$$\begin{cases} \Phi(\overline{x}; x) > 0 \\ x \in \mathbb{R}^n \end{cases}$$

Let V be a real vector space and $\phi : \mathbb{R}^n \times \mathbb{R}^n \times V \longrightarrow [-\infty, +\infty]$ be a function that fulfils the following properties:

$$0 \in \overline{V} := \{v \in V \ : \ \phi(y; x, v) = \Phi(y; x), \quad \forall (y, x) \in dom\phi(\cdot; \cdot, v)\} \tag{4.1}$$

$$dom\phi(\cdot; \cdot, 0) = dom\Phi(y; x) \tag{4.2}$$

Let V_0 a subset of V such that $0 \in V_0 \subseteq \overline{V}$; it is easy to see that $\overline{x} \in dom f$ is a solution of (GVI) if and only if the following system, in the unknowns x and v, is impossible:

$$\begin{cases} \phi(\overline{x}; x, v) > 0 \\ v \in V_0, \ x \in \mathbb{R}^n. \end{cases} \tag{$PS_{\overline{x}}$}$$

Consider the following subsets of the space $\mathbb{R} \times V$:

$$\mathcal{K}(y) := \{(u, v) \in \mathbb{R} \times V \ ,: \ u = \phi(y; x, v), \ x \in \mathbb{R}^n\},$$
$$\mathcal{H} := \{(u, v) \in \mathbb{R} \times V \ : \ u > 0, \ v \in V_0\};$$

the optimality condition for (GVI), in the parametric image space, is stated in the next proposition.

Proposition 4.1. *Let V a real vector space and ϕ fulfilling (4.1) and (4.2); then $\overline{x} \in dom f$ is a solution of (GVI) if and only if*

$$\mathcal{K}(\overline{x}) \cap \mathcal{H} = \emptyset. \tag{4.3}$$

The condition (4.3) is stated by proving that $\mathcal{K}(\overline{x})$ and $\mathcal{H}$ lie in two disjoint level sets of suitable separation function. Consider the class of separation functions, depending on a parameter $(\theta, \lambda) \in \mathbb{R}_+ \times \Lambda$, $w(\theta, \lambda; \cdot, \cdot) : \mathbb{R} \times V \longrightarrow \mathbb{R}$, that fulfils the following conditions, for each $\lambda \in \Lambda$ and for each $\theta \geq 0$:

$$\begin{aligned} &w(\theta, \lambda; u, v) = \theta u + \gamma(\lambda, v), \\ &lev_{\geq 0}\gamma(\lambda; \cdot) \supseteq V_0, \text{ where } \gamma : \Lambda \times V \longrightarrow \mathbb{R}. \end{aligned} \tag{4.4}$$

Remark 4.1. The second condition in (4.4) is equivalent to

$$\mathcal{H} \subseteq lev_{\geq 0} w(\theta, \lambda; \cdot, \cdot), \quad \forall (\theta, \lambda) \in \mathbb{R}_+ \times \Lambda.$$

Remark 4.2. If there exists $\theta > 0$ and $\lambda \in \Lambda$ such that

$$\sup_{(u,v) \in \mathcal{K}(\overline{x})} w(\theta, \lambda; u, v) \leq 0, \tag{4.5}$$

or equivalently

$$\inf_{(u,v) \in \mathcal{K}(\overline{x})} -w(\theta, \lambda; u, v) \geq 0$$

then the condition (4.3) holds.

Definition 4.1. The function $M(y; \cdot, \cdot, \cdot) : \mathbb{R} \times \Lambda \times \mathbb{R}^n \longrightarrow \mathbb{R}$

$$M(y; \theta, \lambda, x) := \inf_{v \in V} \{-\theta\phi(y; x, v) - \gamma(\lambda; v)\} \tag{4.6}$$

is called the *generalized Lagrangean* associated to the system $(PS_{\overline{x}})$ where ϕ and γ fulfil conditions (4.1) (4.2), and (4.4) respectively.

The problem of finding a separation function that fulfils (4.5) can be formulated by means of a minimax problem that is defined as dual of the system $(PS_{\overline{x}})$.

Definition 4.2. [1] The *dual problem* $(PD_{\overline{x}})$ associated to the system $(PS_{\overline{x}})$ is defined by

$$\sup_{(\theta,\lambda) \in \mathbb{R}_+ \times \Lambda} \inf_{x \in \mathbb{R}^n} M(\overline{x}; \theta, \lambda, x) \tag{$PD_{\overline{x}}$}$$

where M is the generalized Lagrangean.

It is immediate to prove the following result.

Proposition 4.2. *Let F be a parametric representation of the problem $(P_{\overline{x}})$; let*

$$\gamma(\lambda; v) = -\langle \lambda, v \rangle, \qquad \phi(\overline{x}; x, v) = -F(x, v), \qquad \theta = 1, \qquad V = Y, \qquad V_0 = \{0\};$$

then the function M coincides with the Lagrangean representation of $(P_{\overline{x}})$ given by (2.1). □

Remark 4.3. Viceversa if a function ϕ that fulfils (4.1) and (4.2) is given, then it is easy to see that $-\phi$ is a parametric representation of the problem $(P_{\overline{x}})$, provided that$\phi(\overline{x}; x, \cdot)$ is closed and convex for each $x \in \mathbb{R}^n$.y

5. Application to network flow problems

In this section we are dealing with a generalized minimal-cost network-flow problem, formulated as a variational inequality. The main purpose here consists in giving an interpretation in terms of potentials of the dual variables introduced in Section 3.

Consider a network represented as a directed graph $G(N, A)$, where $N = \{N_1, \dots, N_m\}$ is the set of nodes and $A = \{A_1, \dots, A_n\}$ is the set of arcs. We assume that each arc $(N_i, N_j) \in A$ is associated with a finite upper bound d_{ij} on its capacity and with a cost c_{ij} of shipping one unit of the flow through the arc. On N we define the *surplus* function $q : N \longrightarrow \mathbb{R}$ with the following meaning: $q(N_j)$ represents the flow generated $(q(N_j) < 0)$ or absorbed $(q(N_j) > 0)$ in the node N_j, while $q(N_j) = 0$ means that in N_j there is no flow neither generated nor absorbed, namely N_j is a transit node.

Let C be the set of circuits that are the simple and closed paths of the digraph G. Without any loss of generality, we can assume that, if $(N_i, N_j) \in A$, then $(N_j, N_i) \notin A$.

Definition 5.1. A function $\pi : A \longrightarrow \mathbb{R}$ is called *difference of potentials* iff for each circuit $c = (N_{i_1}, \dots, N_{i_r}, N_{i_1}) \in C$ we have

$$\sum_{s=1}^{r} \delta_s(c)\pi(N_{i_s}, N_{i_{s+1}}) = 0 \quad \text{where} \quad \delta_s(c) = \begin{cases} +1, & \text{if } (N_{i_s}, N_{i_{s+1}}) \in A, \\ -1, & \text{otherwise.} \end{cases} \tag{5.1}$$

If there exists a function $p : N \longrightarrow \mathbb{R}$ such that $\forall (N_i, N_j) \in A$ we have $\pi(N_i, N_j) = p(N_j) - p(N_i)$, then the function $\pi : A \longrightarrow \mathbb{R}$ is a difference of potentials. In this case, the function p is said *potential function.*

In the sequel we will see how the relationship between the concept of duality and that of potential can be extended also to a generalized formulation of a minimal-cost flow problem.

Let $f \in \mathbb{R}^n$ be the flow-vector on arcs. We will say that a flow f is feasible if and only if it satisfies the following conditions:

$$0 \leq f \leq d \tag{5.2}$$

and

$$\sum_{r \in E_j} f_{rj} - \sum_{s \in U_j} f_{js} = q_j, \quad \forall j = 1, \dots, k, \tag{5.3}$$

where

$$E_j = \{i : (N_i, N_j) \in A\} \text{ and } U_j = \{i : (N_j, N_i) \in A\}.$$

The conditions (5.2) represent the capacity constraints while (5.3) are the conditions of flow conservation in the network. We shall denote by K the feasible flow set, that is

$$K := \{f \in \mathbb{R}^n \ : \ \Gamma f = q, \ 0 \leq f \leq d\}, \tag{5.4}$$

where $\Gamma = (\gamma_{ij}) \in \mathbb{R}^m \times \mathbb{R}^n$ is the node-arc incidence matrix whose elements are

$$\gamma_{ij} = \begin{cases} -1, & \text{if } N_i \text{ is the initial node of the arc } A_j, \\ +1, & \text{if } N_i \text{ is the final node of the arc } A_j, \\ 0, & \text{otherwise.} \end{cases} \tag{5.5}$$

Consider the following problem:

$$\text{find } f^* \in K \ \text{ s.t. } \ \langle c(f^*), f^* \rangle \leq \langle c(f^*), f \rangle, \quad \forall f \in K. \tag{5.6}$$

The problem (5.6) is a generalized formulation of a minimal-cost network-flow problem; in fact, when the function $c(f)$ is independent of f, so that $c(f) = (c_{ij})$ is a positive vector, then (5.6) collapses to the classical minimal-cost problem [5].

We observe that f^* is a solution of (5.6) if and only if f^* is a global minimum of the following problem:

$$\min \langle c(f^*), f - f^* \rangle \text{ s.t. } f \in K \tag{5.7}$$

Following the general duality scheme (3.2) for variational inequalities, we define a dual problem for (5.6). To this aim, we assume f^* as a solution of (5.6) or equivalently of the following extremum problem:

$$\min \langle c(f^*), f \rangle \text{ s.t. } f \in K. \tag{5.8}$$

Since f^* is fixed, (5.8) is a linear program to which we can apply the corresponding well known duality theory. Our purpose here consists in stating conditions on f^* which determine a solution of (5.6). Consider the dual problem of (5.8):

$$\max\left[\langle \lambda, q \rangle + \langle \mu, d \rangle\right] \ \text{ s.t. } \lambda\Gamma + \mu \leq c(f^*), \ \mu \leq 0. \tag{5.9}$$

From the classical duality theory for Linear Programming we have: the necessary and sufficient condition in order that both (5.8) and (5.9) have optimal solutions is that both of them have feasible solutions. In that case their extrema coincide.

Applying the Complementarity Slackness Theorem to problems (5.8) and (5.9), and with the assumption that f^* is a solution of (5.8), we obtain the following relations which have been already introduced in [2].

$$\langle \mu, d - f^* \rangle = 0, \quad \langle c(f^*) - \lambda\Gamma - \mu, f^* \rangle = 0. \tag{5.10}$$

As previously mentioned, in this *dualization* approach we assumed to have a solution of problem (5.6). In [2] the following problem is formulated by means of the theory of gap functions and its solution provide simultaneously a solution of the variational inequality (5.6) and the associated dual variables.

$$\min[\langle c(f), f\rangle + \langle \lambda, q\rangle + \langle \mu, d\rangle] \text{ s.t. } f \in K,\ c(f) + \mu + \lambda \geq 0,\ \mu \geq 0. \tag{5.11}$$

The resolution of this problem may be difficult since in general the involved functions are not convex. Nevertheless, in some cases the resolution of (5.11) is easy: for instance, if $c(f) = Cf$ with C an $n \times n$ matrix, then the domain of (5.11) is convex; morever, if C is a positive definite, then (5.11) is a quadratic problem.

Now we will study the relationships between a solution of (5.11) and a solution of the variational inequality (5.6). These relationships can be obviously useful when the resolution of (5.11) is an easy task. Consider the following sets

$$\begin{aligned} \tilde{H}_f &= \{(\tilde{\lambda}, \tilde{\mu}) \in \mathbb{R}^n \times \mathbb{R}^m_- \ :\ c(f) - \tilde{\lambda} - \tilde{\mu} \geq 0\} \\ H_f &= \{(\lambda, \mu) \in \mathbb{R}^n \times \mathbb{R}^m_+ \ :\ c(f) + \lambda + \mu \geq 0\} \end{aligned}$$

Remark 5.1. If the variational inequality (5.6) has at least one solution, say f^*, then the set $\tilde{H}_{f^*}$ (and hence H_{f^*}) is not empty. This fact follows from the necessary condition for optimality of problem (5.8).

The following result clarifies the relations (5.10) and gives equivalence between the primal/dual problems and problem (5.11).

Proposition 5.1. *Suppose that both problems (5.8) and (5.9) have feasible solutions. Then (f^*, λ^*, μ^*) is an optimal solution of (5.11) if and only if it fulfils the following conditions:*
(i) $(\tilde{\lambda}^* = -\lambda^*, \tilde{\mu}^* = -\mu^*)$ is an optimal solution of (5.9);
(ii) f^* is an optimal solution of (5.8).

Proof ("only if"). Let us show (i). The feasibility of $(\tilde{\lambda}^*, \tilde{\mu}^*)$ for (5.9) follows from the feasibility of (f^*, λ^*, μ^*) for (5.11). Furthermore, for each (f, λ, μ) feasible for (5.11) we have

$$\langle c(f^*), f^*\rangle + \langle \lambda^*, q\rangle + \langle \mu^*, d\rangle \leq \langle c(f), f\rangle + \langle \lambda, q\rangle + \langle \mu, d\rangle.$$

This occurs in particular for each point (f^*, λ, μ) feasible for (5.11), that is, for each $(\lambda, \mu) \in H_{f^*}$ we have

$$\begin{aligned} \langle c(f^*), f^*\rangle + \langle \lambda^*, q\rangle + \langle \mu^*, d\rangle &\leq \langle c(f^*), f^*\rangle + \langle \lambda, q\rangle + \langle \mu, d\rangle \\ \langle \lambda^*, q\rangle + \langle \mu^*, d\rangle &\leq \langle \lambda, q\rangle + \langle \mu, d\rangle \ \forall (\lambda, \mu) \in H_{f^*} \\ \langle -\lambda^*, q\rangle + \langle -\mu^*, d\rangle &\geq \langle -\lambda, q\rangle + \langle -\mu, d\rangle \ \forall (\lambda, \mu) \in H_{f^*} \\ \langle \tilde{\lambda}^*, q\rangle + \langle \tilde{\mu}^*, d\rangle &\geq \langle \tilde{\lambda}, q\rangle + \langle \tilde{\mu}, d\rangle \ \forall (\tilde{\lambda}, \tilde{\mu}) \in \tilde{H}_{f^*}. \end{aligned}$$

Hence $(\tilde{\lambda}^*, \tilde{\mu}^*)$ is an optimal solution of (5.9). Now, let us show (ii). The feasibility of f^* for (5.8) easily follows from the feasibility of (f^*, λ^*, μ^*) for (5.11). Furthermore, we have:

$$c(f^*) + \lambda^* \Gamma + \mu^* \geq 0$$

that implies
$$\langle c(f^*), f\rangle + \langle \lambda^*, \Gamma f\rangle + \langle \mu^*, f\rangle \geq 0, \quad \forall f \in K.$$
or equivalently
$$\langle c(f^*), f\rangle + \langle \lambda^*, q\rangle + \langle \mu^*, d\rangle + \langle \mu^*, f - d\rangle \geq 0.$$
Since $f - d \leq 0$ and $\mu^* \geq 0$ we have $\langle c(f^*), f\rangle + \langle \lambda^*, q\rangle + \langle \mu^*, d\rangle \geq 0$ that is

$$\langle c(f^*), f\rangle \geq \langle -\lambda^*, q\rangle + \langle -\mu^*, d\rangle = \langle \tilde{\lambda}^*, q\rangle + \langle \tilde{\mu}^*, d\rangle = \langle c(f^*), f^*\rangle.$$

Therefore f^* is an optimal solution of (5.8).The last equation follows from the Strong Duality Theorem and the fact that $(\tilde{\lambda}^*, \tilde{\mu}^*)$ is a solution of (5.9) for (i).

("if") At every point (f, λ, μ) feasible for (5.11), the objective function is non–negative; in fact

$$\langle c(f), f\rangle + \langle \lambda, q\rangle + \langle \mu, d\rangle = \langle c(f) + \lambda\Gamma + \mu, f\rangle + \langle \mu, d - f\rangle \geq 0;$$

furthermore, it can be written as

$$\langle c(f), f\rangle - [\langle -\lambda, q\rangle + \langle -\mu, d\rangle].$$

Now, if f^* is a optimal solution of (5.8) and if $(-\lambda^*, -\mu^*)$ is an optimal solution of (5.9), then from the Strong Duality Theorem, we have

$$\langle c(f^*), f^*\rangle - [\langle -\lambda^*, q\rangle + \langle -\mu^*, d\rangle] = 0$$

hence (f^*, λ^*, μ^*) is a optimal solution of (5.11). □

Therefore, a solution of (5.11) will yield optimum solutions for the primal (5.8) and dual (5.9) problems simultaneously.

Corollary 5.1. [2] *Necessary and sufficient condition in order that $f^* \in K$ is a solution of the variational inequality (5.6) (and, therefore, also for the extremum problem (5.8)) is that there exist $\lambda^* \in \mathbb{R}^n$ and $\mu^* \in \mathbb{R}^m_+$ such that (f^*, λ^*, μ^*) is an optimal solution of (5.11).* □

Now we observe how the results of the duality theory are connected to the theory of *gap functions* associated to the variational inequality. To this end, let us recall the definition of *gap function.*

Definition 5.2. A function $g : \mathbb{R}^n \longrightarrow \mathbb{R}$ is said *gap function* associated to the variational inequality iff

(i) $g(f) \geq 0, \ \forall \ f \in K$

(ii) $g(\bar{f}) = 0$ iff $\bar{f}$ is a solution of (VI).

Proposition 5.2. *The function*

$$\phi(f) := \langle c(f), f\rangle - \max_{(\tilde{\lambda},\tilde{\mu})\in\tilde{H}_f} [\langle\tilde{\lambda}, q\rangle + \langle\tilde{\mu}, d\rangle]$$

is a gap function for the variational inequality (5.6).

Proof.

$$\phi(f) = \langle c(f), f\rangle - \max_{(\tilde{\lambda},\tilde{\mu})\in\tilde{H}_f} [\langle\tilde{\lambda}, q\rangle + \langle\tilde{\mu}, d\rangle] = \langle c(f), f\rangle + \min_{(\tilde{\lambda},\tilde{\mu})\in\tilde{H}_f} [\langle -\tilde{\lambda}, q\rangle + \langle -\tilde{\mu}, d\rangle]$$
$$= \min_{(\tilde{\lambda},\tilde{\mu})\in\tilde{H}_f} [\langle c(f), f\rangle + \langle -\tilde{\lambda}, q\rangle + \langle -\tilde{\mu}, d\rangle] = \min_{(\lambda,\mu)\in H_f} [\langle c(f), f\rangle + \langle \lambda, q\rangle + \langle \mu, d\rangle] \cdot$$

Now we have to show that ϕ satisfies properties *(i)* and *(ii)* of Definition 5.2.

(i) $\forall f \in K$ and $\forall(\lambda,\mu) \in H_f$ we have

$$\langle c(f), f\rangle + \langle\lambda, q\rangle + \langle\mu, d\rangle = \langle c(f) + \lambda\Gamma + \mu, f\rangle + \langle\mu, d - f\rangle \geq 0$$

Hence we have $\phi(f) \geq 0$, $\forall f \in K$.

(ii) ("if") $\phi(\bar{f}) = 0$ implies

$$\langle c(\bar{f}), \bar{f}\rangle = \max_{(\tilde{\lambda},\tilde{\mu})\in\tilde{H}_{\bar{f}}} [\langle\tilde{\lambda}, q\rangle + \langle\tilde{\mu}, d\rangle] = \langle\tilde{\lambda}^*, q\rangle + \langle\tilde{\mu}^*, d\rangle$$

with some $(\tilde{\lambda}^*, \tilde{\mu}^*) \in \tilde{H}_{\bar{f}}$; therefore $c(\bar{f}) - \tilde{\lambda}^*\Gamma - \tilde{\mu}^* \geq 0$. Now $\forall f \in K$, $\langle c(\bar{f}), f\rangle - \langle\tilde{\lambda}^*, \Gamma f\rangle - \langle\tilde{\mu}^*, f\rangle \geq 0$ implies

$$\langle c(\bar{f}), f\rangle\rangle \geq \langle\tilde{\lambda}^*, q\rangle + \langle\tilde{\mu}^*, d\rangle + \langle\tilde{\mu}^*, f - d\rangle \geq \langle\tilde{\lambda}^*, q\rangle + \langle\tilde{\mu}^*, d\rangle = \langle c(\bar{f}), \bar{f}\rangle.$$

Hence $\bar{f}$ is a solution of the variational inequality (5.6). ("only if") It follows from the fact that the variational inequality (5.6) is equivalent to the problem (5.8) and from the Strong Duality Theorem. □

¿From Proposition 5.1 and Corollary 5.1 we have the following result which can be obtained also directly from the duality theory for linear programming. Moreover, when $c(f)$ is indipendent of f, we can observe that the variational inequality (5.6) collapses to the classical minimal-cost flow problem, and that problem (5.11) corrisponds to its stationarity condition.

Corollary 5.2. $(\bar{f}, \bar{\lambda}, \bar{\mu})$ *is an optimal solution of (5.11) with zero optimal value iff* $\bar{f} \in K$, $(\bar{\lambda}, \bar{\mu}) \in H_{\bar{f}}$ *anf the following equalities hold:*

$$\langle\bar{\mu}, d - \bar{f}\rangle = 0, \ \langle c(\bar{f}) + \bar{\lambda}\Gamma + \bar{\mu}, \bar{f}\rangle = 0. \qquad \square$$

Remark 5.2. From Proposition 5.2 we observe that the problem (5.11) is the minimization of the *gap function* $\phi(f)$ associated to the variational inequality (5.6).

Finally we will give an interpretation of the dual variables, associated to the variational inequality, in terms of potentials associated to arcs and to nodes of the network. Let us take up again the relations (5.10). A feasible flow f is a optimal solution of the problem (5.6) if and only if there exists $(\lambda, \mu) \in H_f$ such that (f, λ, μ) satisfies the following relations

$$\langle \mu, d - f \rangle = 0, \ \langle c(f) + \lambda \Gamma + \mu, f \rangle = 0 \tag{5.12}$$

from which the following implications can be deduced:

$$\begin{cases} f_{ij} > 0 \Longrightarrow \lambda_i - \lambda_j - \mu_{ij} = c_{ij}(f) \\ \mu_{ij} > 0 \Longrightarrow f_{ij} = d_{ij} \end{cases} \tag{5,13}$$

Consider the case of positive costs c_i. The relations (5.13) can be written in the following way:

$$\begin{cases} f_{ij} > 0 \Longrightarrow \lambda_i - \lambda_j \geq 0 \\ f_{ij} > 0, \ c_{ij}(f) = 0 \Longrightarrow \lambda_i - \lambda_j = \mu_{ij} \\ \lambda_i < \lambda_j \Longrightarrow f_{ij} = 0 \\ \lambda_i = \lambda_j \Longrightarrow \mu_{ij} = 0 \\ \lambda_i > \lambda_j \Longrightarrow \begin{cases} f_{ij} = d_{ij} & \text{if } c_{ij}(f) = 0 \\ 0 \leq f_{ij} \leq d_{ij} & \text{if } c_{ij}(f) > 0 \end{cases} \end{cases} \tag{5.14}$$

Now, if we define the function $\pi : A \longrightarrow \mathbb{R}$ in the following way

$$\pi(N_i, N_j) = \lambda_i - \lambda_j \quad \forall (N_i, N_j) \in A,$$

and if we introduce the function $p(N_i) = \lambda_i$, then we see that $\pi(N_i, N_j)$ is difference of potentials and that $p(N_i)$ is potential function. Recall that the digraph G has been supposed asymmetric, that is $(N_i, N_j) \in A \Longrightarrow (N_j, N_i) \notin A$.

From this point of view, the conditions (5.14) can be interpreted in the following way: there exists a positive flow between the nodes N_i and N_j when there is a positive difference of potentials; furthermore, if the cost of the arc (N_i, N_j) becomes zero, then the flow f_{ij} is satured, that is it becomes equal to the capacity of the arc, and the pressure takes its maximum, that is it equals the difference of potentials between the nodes. Hence, the dual variables relative to the constraints on flow conservation are interpreted as potentials at the nodes.

In [2] it is noted that, if (f, λ, μ) satisfies (5.10), then $(f, \lambda, \mu + \Delta)$ fulfils (5.10) if $c(f)$ is replaced with $c(f) - \Delta$. This can have the following meaning: if the costs recieve a variation of Δ in such a way that $\langle \Delta, d - f \rangle = 0$, then we can have a new equilibrium, where the potentials at the nodes remain unchanged, and where the potentials at the arcs change in the opposite direction, with respect to the costs. Hence an increase (decrease) in the cost of an arc can be seen as a way of decreasing (increasing) the pressure (difference of potential) in the arc.

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