Variational equilibrium model for minimum cost traffic network problems

G. Mastroeni[*] M. Pappalardo[†]

September 23, 2025

Abstract: We propose a variational formulation of one of the most important models for traffic network-problems, namely, the minimal-cost problem. This models share the peculiarity to be formulated by means of a suitable variational inequality. The Lagrangean approach to the study of this variational inequality allows us to consider dual variables, associated to the constraints of the feasible set, which may receive interesting interpretations in terms of potentials associated to the arcs and the nodes of the network. This interpretation is an extension of the classic one existing in the literature. Some examples show that a solution of the variational formulation can substantially differ from the classical one.

Key words: Network flows, variational inequalities, equilibrium problems, trasportation problems.

AMS Subject classification: 90B10.

[*]Department of Mathematics, via Buonarroti 2, 56127 PISA, Italy. Email: mastroen@dm.unipi.it

[†]Department of Applied Mathematics, via Bonanno 25/b, 56126 PISA, Italy. Email: pappalardo@dma.unipi.it

1 Introduction

The high increase of the volume of traffic registered in the recent years asks for effective mathematical models for the analysis of the road circulation especially in the urban areas.

Traffic assignment problems, that have been widely studied in the context of transportation network analysis, are characterized by several aspects among which we mention the management of the road network (streets to be made one-way only, semaphorical waiting times, etc.), the knowledge of the traffic demand between origin-destination nodes and the definition of the equilibrium flows.

In this paper we propose a variational models for the formulation of the equilibrium in traffic problems that generalizes the classical minimal-cost network-flow problem.

In our analysis, we refer to a road traffic network, where the users aim to optimize the cost of transfer from a certain origin to a given destination; anyway, the considered models can be extended to other applications as economic or computer networks.

Historically, equilibrium flows were defined as the extremizers of a suitable functional (for example, the total utility of the users). The natural criticism to this definition is based on the fact that there are very few situations in which it is possible to define such a functional. In order to do this it is necessary to have a deep deterministic knowledge of the behaviour of the users. Unlike the optimization models, the variational ones do not require the existence of such a functional. Variational inequality models for the equilibrium traffic assignment have been considered by several authors: we refer to [4,7] and references therein, for a detailed description of the development of this topic in the literature.

We begin with a brief description of some mathematical results that we will use in the sequel.

Let $K := \{x \in \mathbb{R}^n : g(x) \leq 0,\ h(x) = 0\}$. In its simplest form, a variational inequality consists in finding $y \in K$ such that

$$\langle M(y), x - y \rangle \geq 0, \quad \forall x \in K, \qquad VI(M,K)$$

where $M : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $g : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and $h : \mathbb{R}^n \longrightarrow \mathbb{R}^p$.

Similarly to constrained extremum problem, we can associate to VI suitable Lagrangian-type optimality conditions in order to obtain primal-dual formulations of VI as stated by the following well-known result:

Proposition 1.1 *Suppose that the following conditions hold:*
i) g is a convex differentiable function and there exists $\tilde{x} \in \mathbb{R}^n$ such that $g(\tilde{x}) < 0$;
ii) h is affine.
Then $y \in \mathbb{R}^n$ is a solution of VI(M,K) if and only if there exist $\lambda^* \in \mathbb{R}^p$, $\mu^* \in \mathbb{R}^m$ such that (y, μ^*, λ^*) is a solution of the system

$$\begin{cases} M(x) \ + \ \mu \nabla g(x) \ - \lambda \nabla h(x) = \ 0 \\ \langle \lambda, g(x) \rangle \ = \ 0 \\ \mu \geq 0,\ g(x) \leq 0,\ h(x) = 0. \end{cases} \tag{1}$$

We will show that the multipliers $\lambda_i^*,\ i = 1, \ldots, p$ and $\mu_j^*,\ j = 1, \ldots, m$, may receive suitable interpretations when VI represents the equilibrium condition of a traffic network problem. In

Section 2 we will consider a generalized minimum cost flow problem, formulated by means of a variational inequality where the operator M represents the cost associated to the arcs of the network. In Section 3 we will show that the dual variables can be interpreted in terms of potentials associated to the arcs and the nodes of the network, a generalization of the classic results existing in the literature.

In Section 4 we analyse the equivalence between an arc-flow model with capacity constraints and a model without capacities, but with a suitable penalised operator.

2 The arc-flow model

In this section we will consider a variational inequality model formulated in terms of the flows on the arcs. This kind of model is easier to handle in the applications, since the real data are very often related to the arcs of the network, instead of the paths: an example is given by the capacities, which, in the real applications, are, in general, given on the arcs.

We will propose a variational arc-flow model which is an extension of the classic minimal-cost network flow problem; in particular, we will obtain a generalization of the Bellman dual optimality conditions.

In this section we will consider the following further assumptions and notations:

- f_i is the flow on the arc $A_i := (r, s)$.
- $f := (f_1, \dots, f_n)^T$ is the vector of the flows on all arcs.
- We assume that each arc A_i is associated with an upper bound d_i on its capacity, $d := (d_1, \dots, d_n)$.
- $c_i(f)$ is the cost of the arc A_i, $\forall i = 1, \dots, n$ and $c(f) := (c_1(f), \dots, c_n(f))^T$; we assume that $c(f) \geq 0$.
- q_j is the balance at the node j, $j = 1, \dots, p$ and $q := (q_1, \dots, q_p)^T$.
- $\Gamma = (\gamma_{ij}) \in \mathbb{R}^p \times \mathbb{R}^n$ is the node-arc incidence matrix whose elements are

$$\gamma_{ij} = \begin{cases} -1, & \text{if } i \text{ is the initial node of the arc } A_j, \\ +1, & \text{if } i \text{ is the final node of the arc } A_j, \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

Definition 2.1 *The feasible set of the arc-flow model is defined by:*

$$K_f := \{f \in \mathbb{R}^n \ : \ \Gamma f = q, \ 0 \leq f \leq d\}.$$

The equilibrium condition of the arc-flow model can be expressed by the variational inequality:

$$\text{find } f^* \in K_f \ \text{ s.t. } \ \langle c(f^*), f - f^* \rangle \geq 0, \quad \forall f \in K_f. \tag{3}$$

The problem (3) can be interpreted as a generalized formulation of the minimal-cost network-flow problem; in fact, when the function $c(f)$ is independent of f, so that $c(f) := (c_{ij}, (i,j) \in A)$, then (3) collapses to the classic minimal-cost problem.

Example 2.1 We solve the variational inequality (3). By the gap function approach for variational inequalities, we obtain a solution of (3) by solving the following constrained extremum problem:

$$\min_{f \in K_f} g(f) := \sup_{x \in K_f} \{\langle c(f), f - x\rangle - \|x - f\|^2\}.$$

Let

$$\Gamma = \begin{pmatrix} -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix},$$

$$q = (-2, 0, 0, 0, 0, 2)^T, \qquad d = (2, 1, 1, 1, 1, 1, 2, 2)^T.$$

The cost function is defined by $c(f) := Cf$ where C is the diagonal matrix with components on the diagonal given by the vector $D := (5.5, 1, 2, 3, 4, 50, 3.5, 1.5)$.

The solution of (3) is given by the vector

$$f^* := \begin{pmatrix} 1.00000000000000 \\ 1.00000000000000 \\ 0.15753013039398 \\ 0.84246986960602 \\ 0.88494592553125 \\ 0.11505407446875 \\ 1.04247605592524 \\ 0.95752394407476 \end{pmatrix}$$

The potentials at the nodes are

$$\lambda^* := \begin{pmatrix} 5.81504424778761 \\ 0.31504424778761 \\ 3.53982300884955 \\ 0 \\ -2.21238938053097 \\ -3.64867256637168 \end{pmatrix}$$

Suppose, now, that the cost function is a constant $c(f) := D$, where $D := (5.5, 1, 2, 3, 4, 50, 3.5, 1.5)$. In this case the variational inequality (3) collapses to the classic minimal-cost flow problem. The optimal solution and the potentials at the nodes are given by the vectors:

$$f^* := (1, 1, 0, 1, 1, 0, 1, 1)^T,$$

$$\lambda^* := \begin{pmatrix} 10.79508381675025 \\ 5.29508381675034 \\ 8.68163105415608 \\ 3.79965839866407 \\ 1.79965839866444 \\ 0.29965839866425 \end{pmatrix}.$$

Observe that we have chosen the constant vector D equal to the diagonal of the matrix C in the previous example.

Remark 2.1 *The arc-flow model can be easily related to the path-flow model. Some further notations are:*

- *m is the total number of the considered paths and $F := (F_1, \ldots, F_m)^T$ is the vector of the relative flows.*
- *We will suppose that, the nodes of the couple W_j are connected by the (oriented) paths, R_i, $i \in P_j \subseteq \{1, \ldots, m\}$, $\forall j = 1, \ldots, \ell$.*
- *ρ_j is the traffic demand for W_j, $j = 1, \ldots, \ell$, $\rho := (\rho_1, \ldots, \rho_\ell)^T$.*
- *$\Phi = (\phi_{ij}) \in \mathbb{R}^\ell \times \mathbb{R}^m$ is the couplets-paths incidence matrix whose elements are*

$$\phi_{ij} = \begin{cases} 1, & \text{if } W_i \text{ is connected by the path } R_j, \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

- *Let $\Delta = \{\delta_{is}\}$ be the Kronecker matrix, where*

$$\delta_{is} = \begin{cases} 1, & \text{if } A_i \in R_s \quad i = 1, \ldots n \\ 0, & \text{if } A_i \notin R_s \quad s = 1, \ldots m \end{cases}$$

  and assume that the cost $C_s(F)$ can be expressed as the sum of the costs on the arcs of R_s:

$$C_s(F) = \sum_{i=1}^{n} \delta_{is} c_i(f).$$

In order to obtain an arc-flow formulation of the path-flow model it is necessary to define the feasible set of the path-arc-flow model

$$K_{f,F} := \{f \in \mathbb{R}^n : f = \Delta F, \Phi F = \rho, \; F \geq 0\}.$$

Proposition 2.1 *$VI(c, K_f)$ is equivalent to $VI(c, K_{f,F})$.*

Proof. *By the Kronecker matrix the flows on the arcs can be expressed in terms of the flows on the paths $f_i = \sum_{s=1}^{m} \delta_{is} F_s$. Therefore $f = \Delta F$* and *$C(F) = \Delta^T c(f)$. The previous relations lead to the following equalities:*

$$\langle C(H), F - H \rangle = c^T(f^*) \Delta(F - H) = \langle c(f^*), f - f^* \rangle,$$

where we have put $f^* := \Delta H$. □

An advantage of the path-arc-flow model lies, for example, in the possibility of adding capacity constraints on the arcs in the feasible set $K_{f,F}$, even though the traffic demand is related to the couples O-D. We remark that in order to adopt the arc-flow model in the standard form (3), it is necessary that the traffic demand is related only to the nodes of the network.

3 Potentials and dual variables

We can apply Proposition 1.1 in order to obtain a primal-dual formulation of the variational inequality (3).

Proposition 3.1 *f^* is a solution of the variational inequality (3) if and only if there exists $(\lambda^*, \mu^*) \in \mathbb{R}^{p \times n}$ such that (f^*, λ^*, μ^*) is a solution of the system*

$$\begin{cases} c(f) \ + \ \lambda\Gamma \ + \ \mu \ \geq \ 0 \\ \langle c(f) + \lambda\Gamma + \mu, f \rangle \ = \ 0 \\ \langle f - d, \mu \rangle = 0 \\ 0 \leq f \leq d \ , \ \Gamma f = q, \ \mu \geq 0 \end{cases} \tag{5}$$

We remark that, in order to follow the notations used in the theory of potentials, with no loss of generality, we have changed in (5) the sign of the multiplier λ (w.r.t. the statement of Proposition 1.1). Now we analyse the system (5)

The case with no capacity constraints

Suppose, at first, that there are no capacity constraints on the arcs so that $d_{ij} = +\infty$, $\forall (i,j) \in A$. Then (5) becomes

$$\begin{cases} c(f) \ + \ \lambda\Gamma \ \geq \ 0 \\ \langle c(f) + \lambda\Gamma, f \rangle \ = \ 0 \\ \Gamma f = q, \ f \geq 0 \end{cases} \tag{6}$$

The question that now arises is to establish whether it is possible to find an equivalent formulation of (6) in terms of an equilibrium principle. It is easy to see that f^* fulfils (6) if and only if there exists $\lambda^* \in \mathbb{R}^p$ such that, $\forall (i,j) \in A$:

$$f^*_{ij} > 0 \Longrightarrow c_{ij}(f^*) = \lambda^*_i - \lambda^*_j \tag{7}$$

$$f^*_{ij} = 0 \Longrightarrow c_{ij}(f^*) \geq \lambda^*_i - \lambda^*_j. \tag{8}$$

We observe that the dual variables corresponding to the constraints on flow conservation can be interpreted in terms of potentials at the nodes of the network. Actually, from (7) we deduce that

$$f^*_{ij} > 0 \Longrightarrow \lambda^*_i - \lambda^*_j \geq 0, \tag{9}$$

that is, a necessary condition for the arc (i,j) to have a positive flow is that the difference of potential between the nodes i and j is positive. Vice versa, from (9) we deduce that

$$\lambda^*_i - \lambda^*_j < 0 \Longrightarrow f^*_{ij} = 0, \tag{10}$$

that is, the negativity of the difference of potentials between nodes i and j is a sufficient condition in order to have $f^*_{ij} = 0$.

The case of capacity constraints

A straightforward extension of the relations (7) and (8) can be obtained in the presence of capacity constraints applying directly Proposition 3.1. We can state the following equilibrium principle, which is of immediate proof.

Theorem 3.1 *f^* is a solution of the variational inequality (3) if and only if there exist $\lambda^* \in \mathbb{R}^p$ and $\mu^* \in \mathbb{R}^n_+$ such that, $\forall (i,j) \in A$:*

$$0 < f^*_{ij} < d_{ij} \Longrightarrow c_{ij}(f^*) = \lambda^*_i - \lambda^*_j, \quad \mu^*_{ij} = 0, \tag{11}$$

$$f^*_{ij} = 0 \Longrightarrow c_{ij}(f^*) \geq \lambda^*_i - \lambda^*_j, \quad \mu^*_{ij} = 0, \tag{12}$$

$$f^*_{ij} = d_{ij} \Longrightarrow c_{ij}(f^*) = \lambda^*_i - \lambda^*_j - \mu^*_{ij}. \tag{13}$$

We observe that the relations (9) and (10) are still valid and that (11) – (13) collapse to (7) and (8) when $d_{ij} = +\infty$, $\forall (i,j) \in A$.

Example 3.1 Consider the problem introduced in the Example 2.1. Note that the existence of a positive flow between the nodes i and j implies that $\lambda_i > \lambda_j$, according to (9).

We also remark that the multipliers μ_{ij} that appear in (13) can be interpreted as an additional cost to be added to $c_{ij}(f^*)$ in order to achieve the equivalence with the difference of potentials $\lambda^*_i - \lambda^*_j$. This aspect of the analysis will be developed in the next section.

4 About the equivalence between a model with capacities and one without capacities

In this section we will deepen the analysis of the model with capacity constraints. When the feasible set contains capacity constraints on the arcs a further research direction is given by the reformulation of the model with capacities by means of a model without capacities, but with a different operator. Denote by $K_f(g)$ the feasible set in the presence of the general capacity constraints $g(f) \leq 0$:

$$K_f(g) := \{f \in \mathbb{R}^n \ : \ \Gamma f = q, \ f \geq 0 \ g(f) \leq 0\}.$$

For the sake of simplicity, in this section we shall denote by K_f the uncapacitated feasible set (obtained by dropping the constraint $g(f) \leq 0$).

Under suitable assumptions, it is possible to show that f^* is a solution of the variational inequality $VI(c, K_f)$ if and only if it is a solution of the following:

$$\langle c(f^*) - \Psi'_f(\alpha; f^*, f^*), f - f^* \rangle \geq 0, \quad \forall f \in K_f, \qquad VI_\alpha(c, K_f)$$

where $\Psi : \mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is a suitable function depending on the parameter $\alpha \in \mathbb{R}^k$ and Ψ'_f denotes the gradient of Ψ, w.r.t. its third component.
Let us begin with a first result.

Proposition 4.1 *Assume that $\Psi : \mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ fulfils the following assumptions, $\forall \alpha \in \mathbb{R}^k$:*

1. *$\Psi(\alpha; f, f) = 0, \quad \forall f \in K_f$;*

2. *$\Psi(\alpha; f, \cdot)$ is a differentiable concave function on K_f , $\quad \forall f \in K_f$.*

Then f^* is a solution of $VI_\alpha(c, K_f)$ iff it is a solution of the variational inequality

$$\langle c(f^*), f - f^*\rangle - \Psi(\alpha; f^*, f) \geq 0, \quad \forall f \in K_f, \tag{14}$$

Proof. Let f^* be a solution of (14). Taking into account assumption *1*, this is equivalent to the fact that f^* is an optimal solution of the problem

$$\min_{f\in K_f} [\langle c(f^*), f - f^*\rangle - \Psi(\alpha; f^*, f)]. \tag{15}$$

By assumption *2*, we have that (15) is a convex problem, so that $VI_\alpha(c, K_f)$ is a necessary and sufficient optimality condition for (15), which completes the proof. $\square$

We now have:

Theorem 4.1 *Suppose that $\Psi(\alpha; f^*, f) := \sum_{i=1}^{v} \alpha_i(g_i(f^*) - g_i(f))$, where $g : \mathbb{R}^n \longrightarrow \mathbb{R}^v$ is convex, $\alpha_i \geq 0$, $i = 1, \ldots, v$ and $\exists f \in \mathbb{R}^n$ such that $g(f) < 0$. Then f^* is a solution of $VI_\alpha(c, K_f(g))$ iff there exists $\alpha \in \mathbb{R}^v_+$ such that f^* is a solution of $VI_\alpha(c, K_f)$ with $\sum_{i=1}^{v} \alpha_i g_i(H) = 0$.*

Proof. First of all, we observe that Ψ fulfils the assumptions *1,2* of the Proposition 4.1. Assume that f^* is a solution of $VI_\alpha(c, K_f)$ for a suitable $\alpha \in \mathbb{R}^v$ with $\sum_{i=1}^{v} \alpha_i g_i(f^*) = 0$.

By Proposition 4.1 we have that

$$\langle c(f^*), f - f^*\rangle \;\geq\; \Psi(\alpha; f^*, f), \quad \forall f \in K_f(g),$$

taking into account that $K_f(g) \subseteq K_f$. Since, in our hypotheses,

$$\Psi(\alpha; f^*, f) := -\sum_{i=1}^{v} \alpha_i g_i(f),$$

we obtain that $\Psi(\alpha; f^*, f) \geq 0$, $\forall f \in K_f(g)$, so that f^* is a solution of $VI(c, K_f(g))$.

Let f^* be a solution of $VI(c, K_f(g))$.
By Proposition 1.1, it is known that f^* is a solution of $VI(c, K_f(g))$ iff there exists $(\mu^*, \lambda^*, s^*) \in \mathbb{R}^{v\times p\times n}$ such that $(f^*, \mu^*, \lambda^*, s^*)$ is a solution of the system

$$\begin{cases} c(f) \;-\; \lambda\Gamma \;+\; \mu\nabla g(f) \;- s \;=\; 0 \\ \langle \mu, g(f)\rangle \;=\; \langle s, f\rangle = 0 \\ g(f) \leq 0 \;,\; \Gamma f = q, \;\; f \geq 0 \\ \mu \geq 0, \;\; s \geq 0 \end{cases} \tag{16}$$

Consider now $VI_\alpha(c, K_f)$. If we put $\alpha := \mu^*$, then $VI_{\mu^*}(c, K_f)$ becomes

$$\langle c(f^*) + \mu^*\nabla g(f^*), f - f^*\rangle \geq 0, \quad \forall f \in K_f.$$

Still by Proposition 1.1, we have that f^* is a solution of $VI_{\mu^*}(c, K_f)$ iff there exists $(\lambda^0, s^0) \in \mathbb{R}^{p\times n}$ such that (f^*, λ^0, s^0) is a solution of the system

$$\begin{cases} c(f) \;+\; \mu^*\nabla g(f) - \;\lambda\Gamma \;\; - s \;=\; 0 \\ \langle s, f\rangle = 0 \\ \Gamma f = q, \;\; f \geq 0, \;\; s \geq 0 \end{cases} \tag{17}$$

Therefore, if $(f^*, \mu^*, \lambda^*, s^*)$ is a solution of (16) then (f^*, λ^*, s^*) is also a solution of (17) and f^* solves $VI_{\mu^*}(c, K_f)$.

$\square$

Remark 4.1 By the proof of Theorem 4.1, it follows that the parameter α, which ensures the equivalence between the capacitated VI and the uncapacitated VI, can be chosen as the multiplier μ^*, associated to the constraint $g(f) \leq 0$.

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