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Traffic Equilibrium and Variational Inequalities

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We consider the general traffic equilibrium network model where the travel cost on each link of the transportation network may depend on the flow on this as well as other links of the network. The model has been designed in order to handle situations where there is interaction between traffic on different links (e.g., two-way streets, intersections) or between different modes of transportation on the same link. For this model, we use the techniques of the theory of variational inequalities to establish existence of a traffic equilibrium pattern, to design an algorithm for the construction of this pattern and to derive estimates on the speed of convergence of the algorithm.

THE MODEL

In the well known traffic equilibrium problem, a fixed travel demand is prescribed for every origin-destination pair of nodes of the transportation network and one has to determine the user-optimized traffic pattern with the equilibrium property that, once established, no user may decrease his travel cost by making a unilateral decision to change his route.

The travel demand d_w associated with the typical origin-destination pair w will be distributed among the paths of the network which connect w. We let Fp denote the flow on the path p. Thus,

$$d_w = \textstyle\sum_{\text{all paths joining } w} F_p. \tag{1}$$

We group together the path flows F_p into a vector $\mathbf{F} \in R^N$ (N is the number of paths in the network) which determines the path flow pattern. A flow pattern $\mathbf{F}$ induces a link load pattern $\mathbf{f} \in R^n$ (n is the number of links in the network) through the equation

$$\mathbf{f} = A\mathbf{F}, \tag{2}$$

where A is the arc-chain incidence matrix of the network. Thus $\mathbf{F}$ lies in

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the convex set $\mathscr{K}$ of the nonnegative vectors in R^N that satisfy (1) and $\mathbf{f}$ must lie in the convex set $\kappa = A\mathscr{K}$ in R^n.

In a user-optimized network, the user's criterion for selecting a travel path is personal travel cost. We assume that each user on link a of the network has a travel cost c_a that depends, in an a priori specified fashion, on the load pattern $\mathbf{f}$. Thus, grouping together the link costs into a vector $\mathbf{c}$ in R^n, we have

$$\mathbf{c} = \hat{\mathbf{c}}(\mathbf{f}), \tag{3}$$

where $\hat{\mathbf{c}}$ is a continuously differentiable function from κ to R^n. The simplest case arises when the travel cost c_a on every link a depends solely upon the flow f_a on a. This is the *standard model* which has been studied extensively (e.g., [1, 2, 7, 9, 10]).

Although the standard model is satisfactory for many applications, it is incapable of handling situations where the travel cost on each link of the transportation network also depends upon the flow on other links of the network. Examples of such situations arise in two-way streets where the user's travel cost in each direction depends not only on the traffic volume in that direction but also on the traffic volume in the opposite direction. Another example arises in urban uncontrolled intersections where the travel cost on a link depends on the traffic on all intersecting links. An *extended model*, capable of handling situations where the travel cost c_a on every link a is allowed to depend also on the load on links other than a has been considered by various authors.[3, 11] Interestingly, it was shown in [4] that, by an appropriate transformation, this model can also handle networks where different modes of transportation interact on the same link. In such a multimodal traffic equilibrium model, each mode of transportation has an individual cost function and, at the same time, contributes to its own and other modes' cost functions in an individual way.

As mentioned above, a user-optimized flow pattern, once established, has the equilibrium property that no user has any incentive of making a unilateral decision to change his route. Mathematically, the above property is characterized by a set of *equilibrium conditions* various equivalent formulations of which have been described in the literature.[1, 2, 13] Here we will find it convenient to use the formulation proposed recently by SMITH[12]: A load pattern $\bar{\mathbf{f}} \in \kappa$ is user-optimized if and only if

$$\hat{\mathbf{c}}(\bar{\mathbf{f}}) \cdot (\mathbf{f} - \bar{\mathbf{f}}) \geq 0, \qquad \text{for all} \quad \mathbf{f} \in \kappa. \tag{4}$$

In the standard model, the equilibrium conditions (4) can be visualized as the Kuhn-Tucker conditions of a convex nonlinear programming problem (associated system-optimized network). Based on this observation, theorems on the existence, uniqueness and stability of the equilibrium pattern as well as algorithms for its computation have been devised (e.g. [2, 6, 7, 9]).

As observed in [3], a similar reduction to a convex minimization problem is also possible in the case of the extended model, provided that the travel cost functions have the property that the Jacobian matrix

$$[\partial \hat{\mathbf{c}}/\partial \mathbf{f}] \tag{5}$$

is symmetric. However, the above symmetry condition is very restrictive and it is unlikely that it will hold in most practical applications.

Smith[12] establishes a uniqueness result in the general, nonsymmetric, case under the assumption that the function $\hat{\mathbf{c}}$ satisfies a reasonable monotonicity condition. However, the problem of existence and numerical computation of equilibria remains open.

It turns out that under the monotonicity assumption the equilibrium conditions (4) have the structure of a *variational inequality* whose theory was developed in order to study free boundary problems in partial differential equations. Using the theory of variational inequalities, as presented in [5], we are able to establish, in the next section, the existence of a unique equilibrium pattern, without imposing any symmetry conditions, and then, in the following section, to design an algorithm for the computation of this load pattern. The algorithm proceeds by iteration and every step requires the computation of an equilibrium pattern for a quadratic standard model. We also derive explicit estimates on the speed of convergence of the algorithm. Finally, for illustration purposes, we use the algorithm to estimate the user-optimized equilibrium pattern for a simple network with two-way streets.

In a subsequent publication we will apply similar techniques to the traffic equilibrium problem with elastic demands.

EXISTENCE AND UNIQUENESS OF THE USER-OPTIMIZED TRAFFIC EQUILIBRIUM

WE WILL ASSUME throughout that the assigned travel cost function $\hat{\mathbf{c}}:\kappa \to R^n$ is continuously differentiable and satisfies the strong monotonicity condition

$$(\hat{\mathbf{c}}(\mathbf{f}) - \hat{\mathbf{c}}(\bar{\mathbf{f}}))\cdot(\mathbf{f} - \bar{\mathbf{f}}) \geq \alpha\,|\mathbf{f} - \bar{\mathbf{f}}|^2, \qquad \text{for all} \quad \mathbf{f}, \bar{\mathbf{f}} \in \kappa, \tag{6}$$

where α is some positive constant.

A necessary and sufficient condition for (6) is that the (not necessarily symmetric) $n \times n$ Jacobian matrix $[\partial \hat{\mathbf{c}}/\partial \mathbf{f}]$ be positive definite at every $\mathbf{f} \in \kappa$.* This assumption is expected to hold in most situations where there is interaction between traffic on different links since, in that case, al-

* See Appendix I.

though $\hat{c}_a$ may depend on the entire load pattern $\mathbf{f}$, it is reasonable to expect that the main dependence is on f_a itself, so that $\partial\hat{c}_a/\partial f_a \gg \partial\hat{c}_a/\partial f_b \geq 0$, $b \neq a$. On the other hand, when the extended model arises from a multimodal network equilibrium problem, the interaction is not necessarily weak. Indeed, in that case $\partial\hat{c}_a/\partial f_a$ may be the marginal cost of users of a certain mode due to changes in traffic volume of the same mode while $\partial\hat{c}_a/\partial f_b$ may be the marginal cost of users of the same mode due to changes in traffic volume of a different mode and hence it should not generally be expected that $\partial\hat{c}_a/\partial f_a \gg \partial\hat{c}_a/\partial f_b$.[4] Thus in multimodal networks the monotonicity condition (6) is not always satisfied and, as a matter of fact, the possibility of multiple equilibria has been demonstrated in [8].

In the special case where the Jacobian matrix

$$[\partial\hat{\mathbf{c}}/\partial\mathbf{f}] \tag{7}$$

is symmetric, it was shown in [3] that any user-optimized load pattern is the minimum over κ of the strictly convex functional $S(\mathbf{f})$ defined by the line integral

$$S(\mathbf{f}) = \int_0^{\mathbf{f}} \hat{\mathbf{c}}(\mathbf{f})\cdot d\mathbf{f}. \tag{8}$$

Based on this observation, theorems on the existence, uniqueness and stability of the equilibrium pattern as well as algorithms for its computation have been devised for the symmetric case (see e.g. [3]). The simplest example arises when the users' travel cost functions are affine, i.e.,

$$\hat{\mathbf{c}}(\mathbf{f}) = G\mathbf{f} + \mathbf{h}, \tag{9}$$

where G is a symmetric positive definite $n \times n$ matrix and $\mathbf{h}$ is an assigned n-vector. In this case the functional (8) becomes quadratic,

$$S(\mathbf{f}) = \tfrac{1}{2}\,\mathbf{f}\cdot G\mathbf{f} + \mathbf{h}\cdot\mathbf{f}. \tag{10}$$

We now return to the general, nonsymmetric, case and we first state, for completeness, a uniqueness result due to Smith.[12]

THEOREM 1. *There is at most one user-optimized load pattern.*

Proof. Let $\mathbf{f}^1, \mathbf{f}^2 \in \kappa$ be two user-optimized load patterns. We write the equilibrium conditions (4), first for $\bar{\mathbf{f}} = \mathbf{f}^1$ and $\mathbf{f} = \mathbf{f}^2$ and then for $\bar{\mathbf{f}} = \mathbf{f}^2$ and $\mathbf{f} = \mathbf{f}^1$, thus obtaining

$$\begin{aligned} \hat{\mathbf{c}}(\mathbf{f}^1)\cdot(\mathbf{f}^2 - \mathbf{f}^1) &\geq 0, \\ \hat{\mathbf{c}}(\mathbf{f}^2)\cdot(\mathbf{f}^1 - \mathbf{f}^2) &\geq 0. \end{aligned} \tag{11}$$

Adding the above two inequalities and using the monotonicity condition (6) we obtain $|\mathbf{f}^1 - \mathbf{f}^2| = 0$ which implies that $\mathbf{f}^1 = \mathbf{f}^2$. The proof is complete.

We now proceed to establish existence of the user-optimized load pattern by employing ideas from the theory of variational inequalities [5]. We fix a symmetric positive definite matrix G and a positive number ρ, to be selected below. For some fixed feasible load pattern $\bar{\mathbf{f}}$ we consider the user-optimized problem for the originally given mathematical network, with the same travel demands (hence the same feasible set κ) but with new travel cost functions given by

$$\tilde{\mathbf{c}}(\mathbf{f}) = G\mathbf{f} + \mathbf{h}, \tag{12}$$

where

$$\mathbf{h} = \rho\hat{\mathbf{c}}(\bar{\mathbf{f}}) - G\bar{\mathbf{f}}. \tag{13}$$

In other words, we have to determine $\hat{\mathbf{f}} \in \kappa$ satisfying the user-optimized equilibrium conditions (4) for the cost function $\tilde{\mathbf{c}}(\mathbf{f})$, viz.,

$$\hat{\mathbf{f}} \cdot G(\mathbf{f} - \hat{\mathbf{f}}) + \mathbf{h} \cdot (\mathbf{f} - \hat{\mathbf{f}}) \geq 0, \qquad \text{for all} \quad \mathbf{f} \in \kappa. \tag{14}$$

Since G is symmetric and positive definite, the above user-optimized problem has a unique solution $\hat{\mathbf{f}}$ which, as shown above, will be the unique minimum over κ of the strictly convex function (10). We let

$$T_\rho : \kappa \to \kappa \tag{15}$$

denote the map which carries $\bar{\mathbf{f}} \in \kappa$ into the minimum over κ of the function (10) with $\mathbf{h}$ given by (13). The following proposition indicates the relevance of the map T_ρ.

LEMMA 1. *Every fixed point of T_ρ is a user-optimized load pattern of the original nonsymmetrical problem.*

Proof. If for some $\bar{\mathbf{f}} \in \kappa$, $\hat{\mathbf{f}} = T_\rho \bar{\mathbf{f}} = \bar{\mathbf{f}}$, then, by vitrue of (13), (14) takes the form

$$\bar{\mathbf{f}} \cdot G(\mathbf{f} - \bar{\mathbf{f}}) + (\rho\hat{\mathbf{c}}(\bar{\mathbf{f}}) - G\bar{\mathbf{f}}) \cdot (\mathbf{f} - \bar{\mathbf{f}}) \geq 0, \qquad \text{for all} \quad \mathbf{f} \in \kappa \tag{16}$$

or,

$$\hat{\mathbf{c}}(\bar{\mathbf{f}}) \cdot (\mathbf{f} - \bar{\mathbf{f}}) \geq 0, \qquad \text{for all} \quad \mathbf{f} \in \kappa \tag{17}$$

which shows that $\bar{\mathbf{f}}$ is indeed a user optimized load pattern for the original problem. The proof is complete.

In view of the above lemma our goal is to show that the transformation T_ρ has a fixed point. To this end we have

LEMMA 2. *Let ν be the maximum over κ of the maximum eigenvalue of the positive definite symmetric matrix*

$$[\partial\hat{\mathbf{c}}/\partial\mathbf{f}]^T G^{-1} [\partial\hat{\mathbf{c}}/\partial\mathbf{f}]. \tag{18}$$

Then, for

$$0 < \rho < 2\alpha/\nu, \tag{19}$$

the map T_ρ is a contraction on κ, with respect to the norm

$$\|\mathbf{f}\| = (\mathbf{f}\cdot G\mathbf{f})^{1/2}, \tag{20}$$

and we have

$$\|T_\rho\mathbf{f}^1 - T_\rho\mathbf{f}^2\| \le \lambda\|\mathbf{f}^1 - \mathbf{f}^2\|, \qquad \text{for all} \quad \mathbf{f}^1, \mathbf{f}^2 \in \kappa, \tag{21}$$

with

$$\lambda = [1 - (1/\mu)\rho(2\alpha - \nu\rho)]^{1/2}, \tag{22}$$

where μ is the maximum eigenvalue of G.

Proof. Consider $\bar{\mathbf{f}}^1, \bar{\mathbf{f}}^2 \in \kappa$. Set $\hat{\mathbf{f}}^1 = T_\rho\bar{\mathbf{f}}^1$, $\hat{\mathbf{f}}^2 = T_\rho\bar{\mathbf{f}}^2$. Then, (14) yields

$$\hat{\mathbf{f}}^1\cdot G(\mathbf{f} - \hat{\mathbf{f}}^1) + (\rho\hat{\mathbf{c}}(\bar{\mathbf{f}}^1) - G\bar{\mathbf{f}}^1)\cdot(\mathbf{f} - \hat{\mathbf{f}}^1) \ge 0, \qquad \text{for all} \quad \mathbf{f} \in \kappa, \tag{23}$$

$$\hat{\mathbf{f}}^2\cdot G(\mathbf{f} - \hat{\mathbf{f}}^2) + (\rho\hat{\mathbf{c}}(\bar{\mathbf{f}}^2) - G\bar{\mathbf{f}}^2)\cdot(\mathbf{f} - \hat{\mathbf{f}}^2) \ge 0, \qquad \text{for all} \quad \mathbf{f} \in \kappa. \tag{24}$$

Writing (23) with $\mathbf{f} = \hat{\mathbf{f}}^2$, (24) with $\mathbf{f} = \hat{\mathbf{f}}^1$ and adding up, we obtain

$$\|\hat{\mathbf{f}}^1 - \hat{\mathbf{f}}^2\|^2 \le \{\bar{\mathbf{f}}^1 - \bar{\mathbf{f}}^2 - \rho G^{-1}[\hat{\mathbf{c}}(\bar{\mathbf{f}}^1) - \hat{\mathbf{c}}(\bar{\mathbf{f}}^2)]\}\cdot G(\hat{\mathbf{f}}^1 - \hat{\mathbf{f}}^2). \tag{25}$$

Applying Schwarz's inequality, we have

$$\|\hat{\mathbf{f}}^1 - \hat{\mathbf{f}}^2\|^2 \le \|\bar{\mathbf{f}}^1 - \bar{\mathbf{f}}^2 - \rho G^{-1}[\hat{\mathbf{c}}(\bar{\mathbf{f}}^1) - \hat{\mathbf{c}}(\bar{\mathbf{f}}^2)]\|\,\|\hat{\mathbf{f}}^1 - \hat{\mathbf{f}}^2\| \tag{26}$$

whence, after dividing through by $\|\hat{\mathbf{f}}^1 - \hat{\mathbf{f}}^2\|$, squaring and expanding the right hand side, we obtain

$$\begin{aligned}\|\hat{\mathbf{f}}^1 - \hat{\mathbf{f}}^2\|^2 \le \|\bar{\mathbf{f}}^1 - \bar{\mathbf{f}}^2\|^2 &- 2\rho(\hat{\mathbf{c}}(\bar{\mathbf{f}}^1) - \hat{\mathbf{c}}(\bar{\mathbf{f}}^2)) \\ &\cdot(\bar{\mathbf{f}}^1 - \bar{\mathbf{f}}^2) + \rho^2(\hat{\mathbf{c}}(\bar{\mathbf{f}}^1) - \hat{\mathbf{c}}(\bar{\mathbf{f}}^2))\cdot G^{-1}(\hat{\mathbf{c}}(\bar{\mathbf{f}}^1) - \hat{\mathbf{c}}(\bar{\mathbf{f}}^2)).\end{aligned} \tag{27}$$

On account of (6) and since G is symmetric and positive definite, (27) implies that*

$$\begin{aligned}\|\hat{\mathbf{f}}^1 - \hat{\mathbf{f}}^2\|^2 \le \|\bar{\mathbf{f}}^1 - \bar{\mathbf{f}}^2\|^2 - 2\rho\alpha(\bar{\mathbf{f}}^1 - \bar{\mathbf{f}}^2)\cdot(\bar{\mathbf{f}}^1 - \bar{\mathbf{f}}^2) + \rho^2(\bar{\mathbf{f}}^1 - \bar{\mathbf{f}}^2) \\ \cdot[\partial\hat{\mathbf{c}}/\partial\mathbf{f}]^T G^{-1}[\partial\hat{\mathbf{c}}/\partial\mathbf{f}](\bar{\mathbf{f}}^1 - \bar{\mathbf{f}}^2)\end{aligned} \tag{28}$$

or, finally,

$$\|\hat{\mathbf{f}}^1 - \hat{\mathbf{f}}^2\|^2 \le \|\bar{\mathbf{f}}^1 - \bar{\mathbf{f}}^2\|^2 - \rho[2\alpha - \nu\rho](\bar{\mathbf{f}}^1 - \bar{\mathbf{f}}^2)\cdot(\bar{\mathbf{f}}^1 - \bar{\mathbf{f}}^2) \tag{29}$$

where ν is the maximum over κ of the maximum eigenvalue of the positive definite symmetric matrix (18).

We select $\rho < 2\alpha/\nu$. Since $\|\bar{\mathbf{f}}^1 - \bar{\mathbf{f}}^2\|^2 \le \mu(\bar{\mathbf{f}}^1 - \bar{\mathbf{f}}^2)\cdot(\bar{\mathbf{f}}^1 - \bar{\mathbf{f}}^2)$, where μ is the maximum eigenvalue of G, (29) yields

$$\|\hat{\mathbf{f}}^1 - \hat{\mathbf{f}}^2\|^2 \le \{1 - (1/\mu)\rho(2\alpha - \nu\rho)\}\,\|\bar{\mathbf{f}}^1 - \bar{\mathbf{f}}^2\|^2. \tag{30}$$

* See Appendix II.

(30) shows that T_ρ is a contraction relative to the norm (20) with a constant λ given by (22) and this completes the proof of the lemma.

We can now easily establish the existence of a user-optimized load pattern, for the original extended model, using Lemmas 1 and 2 and Banach's fixed point theorem.

THEOREM 2. *There is a unique user-optimized load pattern* $\bar{\mathbf{f}}$ *satisfying the equilibrium conditions* (4). *Furthermore, if* $\mathbf{f}^0$ *is any feasible load pattern in* κ *and* ρ *satisfies* (19),

$$T_\rho^m \mathbf{f}^0 \to \bar{\mathbf{f}},\ m \to \infty, \tag{31}$$

and we have the estimate

$$\| T_\rho^m \mathbf{f}^0 - \bar{\mathbf{f}} \| \le \lambda^m/(1-\lambda) \| T_\rho \mathbf{f}^0 - \mathbf{f}^0 \|, \tag{32}$$

where λ *is the constant defined by* (22).

Proof. Let $k > m \ge 1$. Then, using the contraction property (21), we obtain

$$\begin{aligned} \| T_\rho^k \mathbf{f}^0 - T_\rho^m \mathbf{f}^0 \| &\le \textstyle\sum_{j=m}^{k-1} \| T_\rho^{j+1} \mathbf{f}^0 - T_\rho^j \mathbf{f}^0 \| \\ &\le \textstyle\sum_{j=m}^{k-1} \lambda^j \| T_\rho \mathbf{f}^0 - \mathbf{f}^0 \| \\ &\le \lambda^m/(1-\lambda) \| T_\rho \mathbf{f}^0 - \mathbf{f}^0 \|. \end{aligned} \tag{33}$$

Since $0 < \lambda < 1$, it follows that $\{T_\rho^k \mathbf{f}^0\}$ is a Cauchy sequence and must therefore converge to, say, $\bar{\mathbf{f}} \in \kappa$. Being a contraction, T_ρ is continuous and so

$$T_\rho \bar{\mathbf{f}} = T_\rho \lim_{k\to\infty} T_\rho^k \mathbf{f}^0 = \lim_{k\to\infty} T_\rho^{k+1} \mathbf{f}^0 = \bar{\mathbf{f}}. \tag{34}$$

This shows that $\bar{\mathbf{f}}$ is indeed a fixed point of T_ρ and therefore is the unique user-optimized load pattern. Finally, letting $k \to \infty$ in (33) we arrive at (32). The proof is complete.

In view of (32), if one desires to improve the speed of convergence of the sequence $\{T_\rho^k \mathbf{f}^0\}$, one should try to select ρ in such a way that the constant λ in (22) becomes as small as possible. This is accomplished by selecting

$$\rho = \alpha/\nu \tag{35}$$

which yields

$$\lambda = (1 - \alpha^2/\mu\nu)^{1/2}. \tag{36}$$

Since α and ν are eigenvalues of large matrices, in the applications it will usually be difficult to obtain close estimates for them. We should emphasize however that if we select ρ in the interval $(0, 2\alpha/\nu)$ where α is any lower bound over κ of the minimum eigenvalue of the symmetric part of the Jacobian $[\partial\hat{\mathbf{c}}/\partial\mathbf{f}]$ (see Appendix I), ν is any upper bound over κ of the maximum eigenvalue of the positive definite symmetric matrix (18), there is guarantee that the algorithm converges though not with "optimal" speed. Even when no estimate whatsoever for α and ν is available, we can still determine the range of ρ for which we have convergence by trial and error.

THE ALGORITHM

IN THIS SECTION we will develop an algorithm that determines the unique user-optimized load pattern which satisfies the equilibrium conditions (4) under the monotonicity condition (6). Using the results in [4] the same algorithm can be used for solving the multimodal network equilibrium problem under an analogous monotonicity condition. The algorithm is designed to follow the procedure suggested by the proof of the existence Theorem 2.

We begin with a symmetric, positive definite matrix G and we assume that we can determine the user-optimized load pattern when the travel cost functions have the special form

$$\tilde{\mathbf{c}}(\mathbf{f}) = G\mathbf{f} + \mathbf{h}. \tag{37}$$

Indeed, as it has been mentioned in the previous section, in this case the network equilibrium problem can be reduced to a convex quadratic programming problem. Two types of matrices G appear promising at the present time. First, when G is diagonal, the resulting quadratic programming (system-optimizing) problem corresponds to a standard network equilibrium problem which can be solved by any one of the existing special algorithms.[2, 6, 7, 9] Another possibility is to select a G "as close as possible" to the Jacobian matrix $[\partial\hat{\mathbf{c}}/\partial\mathbf{f}]$ in which case each step of the algorithm would require longer computations but, in return, convergence will be faster and fewer steps will be required to attain the same degree of accuracy.

Once G has been selected we pick the positive parameter ρ as in (35).

We are now ready to describe the algorithm. We start with an arbitrary feasible link load pattern $\mathbf{f}^{(0)}$.

Step i ($i = 1, 2, 3, \cdots$). We compute

$$\mathbf{h}^{i-1} = \rho\hat{\mathbf{c}}(\mathbf{f}^{i-1}) - G\mathbf{f}^{i-1}. \tag{38}$$

We then determine the unique user-optimized load pattern $\mathbf{f}^i$, corresponding to travel cost functions of the special form

$$\tilde{\mathbf{c}}^{i-1}(\mathbf{f}) = G\mathbf{f} + \mathbf{h}^{i-1}. \tag{39}$$

Note that in the terminology of the previous section,

$$\mathbf{f}^i = T_\rho\mathbf{f}^{i-1} = T_\rho{}^i\mathbf{f}^0. \tag{40}$$

Thus, the sequence $\mathbf{f}^0, \mathbf{f}^1, \mathbf{f}^2, \cdots$ of feasible load patterns converges, by

Theorem 2, to the unique user-optimized load pattern $\bar{\mathbf{f}}$. By (32) we may estimate the distance of $\mathbf{f}^k$ from $\bar{\mathbf{f}}$ through

$$\|\mathbf{f}^k - \bar{\mathbf{f}}\| \le \lambda^k/(1-\lambda)\,\|\mathbf{f}^1 - \mathbf{f}^0\|, \tag{41}$$

where

$$\lambda = [1 - \alpha^2/\mu\nu]^{1/2}. \tag{42}$$

This estimate allows to determine a priori the number of iterations required in order to obtain the user-optimized equilibrium, within any degree of accuracy.

We should note that estimate (41) is conservative and faster convergence is to be expected. To this effect, we note that one may easily derive the estimate

$$\|\mathbf{f}^k - \bar{\mathbf{f}}\| \le \lambda^{k-i}/(1-\lambda)\,\|\mathbf{f}^{i+1} - \mathbf{f}^i\|, \tag{43}$$

for any $i = 0, 1, 2, \cdots$, so that, if by any chance $\|\mathbf{f}^{i+1} - \mathbf{f}^i\|$ turns out to be small for some i, then we may use (43) to obtain a sharper estimate of the required number of iterations.

As mentioned above, it seems reasonable to select a G which approximates as closely as possible $[\partial\hat{\mathbf{c}}/\partial\mathbf{f}]$. Since the Jacobian matrix varies from point to point, it is plausible to assume that if at each step of the algorithm one uses a different G which approximates the Jacobian matrix $[\partial\hat{\mathbf{c}}/\partial\mathbf{f}]$ at this point (e.g., by selecting G to be the symmetric part of $[\partial\hat{\mathbf{c}}/\partial\mathbf{f}]$ at that point), faster (quadratic) convergence may be obtained. This variant of the algorithm will be studied in a future article.

EXAMPLE

IN ORDER to illustrate the application of the algorithm, we calculate the user-optimized traffic pattern for the simple network shown in Figure 1. Nodes x and y are connected by two two-way streets and by one one-way street with direction from x to y. Thus, the network consists of five links a_1, a_2, a_3, b_1, b_2, where a_1, a_2, a_3 are directed from x to y, and b_1, b_2 are the returns of a_1, a_2, respectively.

The travel demands are

$$d_{(x,y)} = 210, \qquad d_{(y,x)} = 120. \tag{44}$$

Further, the personal travel cost functions are given by

$$\begin{aligned}
\hat{c}_{a_1} &= 10f_{a_1} + 5f_{b_1} + 1000\\
\hat{c}_{a_2} &= 15f_{a_2} + 5f_{b_2} + 950\\
\hat{c}_{a_3} &= 20f_{a_3} + 3000\\
\hat{c}_{b_1} &= 20f_{b_1} + 2f_{a_1} + 1000\\
\hat{c}_{b_2} &= 25f_{b_2} + f_{a_2} + 1300.
\end{aligned} \tag{45}$$

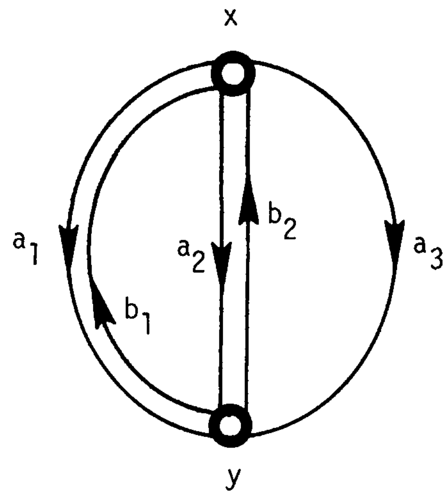

Figure 1

The maximum α for which condition (6) is satisfied is the minimum eigenvalue of the symmetric part of the matrix $[\partial\hat{\mathbf{c}}/\partial\mathbf{f}]$*. A computation yields $\alpha = 8.9$.

We select

$$G = \begin{bmatrix} 10 & 0 & 0 & 0 & 0 \\ 0 & 15 & 0 & 0 & 0 \\ 0 & 0 & 20 & 0 & 0 \\ 0 & 0 & 0 & 20 & 0 \\ 0 & 0 & 0 & 0 & 25 \end{bmatrix} \tag{46}$$

in equation (12).

After a computation we find that the maximum eigenvalue of the matrix (18) is $\nu = 29.8$. Then, using (35), we get $\rho = 0.30$.

To construct T_ρ we first note that equation (13) here takes the form

$$\begin{aligned} h_{a_1} &= -7\bar{f}_{a_1} + 1.5\bar{f}_{b_1} + 300 \\ h_{a_2} &= -10.5\bar{f}_{a_2} + 1.5\bar{f}_{b_2} + 285 \\ h_{a_3} &= -14\bar{f}_{a_3} + 900 \\ h_{b_1} &= -14\bar{f}_{b_1} + 0.6\bar{f}_{a_1} + 300 \\ h_{b_2} &= -17.5\bar{f}_{b_2} + 0.3\bar{f}_{a_2} + 390. \end{aligned} \tag{47}$$

Next we note that the standard user-optimized problem, for a two-node, n-link network with quadratic cost functions has been explicitly solved in

* See Appendix I.

[2]. The solution to our problem is given by

$$\begin{aligned}
f_{a_1} &= 96.9 - 0.0538h_{a_1} + 0.0308h_{a_2} + 0.0230h_{a_3} \\
f_{a_2} &= 64.6 + 0.0308h_{a_1} - 0.0462h_{a_2} + 0.0154h_{a_3} \\
f_{a_3} &= 48.5 + 0.0230h_{a_1} + 0.0154h_{a_2} - 0.0384h_{a_3} \\
f_{b_1} &= 66.7 + 0.0222(h_{b_2} - h_{b_1}) \\
f_{b_2} &= 53.3 - 0.0222(h_{b_2} - h_{b_1}),
\end{aligned} \tag{48}$$

whenever the above formula yields $f_{a_3} > 0$, or by

$$\begin{aligned}
f_{a_1} &= 126 + 0.04(h_{a_2} - h_{a_1}) \\
f_{a_2} &= 84 - 0.04(h_{a_2} - h_{a_1}) \\
f_{a_3} &= 0 \\
f_{b_1} &= 66.7 + 0.0222(h_{b_2} - h_{b_1}) \\
f_{b_2} &= 53.3 - 0.0222(h_{b_2} - h_{b_1}),
\end{aligned} \tag{49}$$

when (48) yields $f_{a_3} \leq 0$.

To apply the algorithm we start with initial feasible pattern $f^0_{a_1} = 70$, $f^0_{a_2} = 70$, $f^0_{a_3} = 70$, $f^0_{b_1} = 60$, $f^0_{b_2} = 60$. Using these **f**'s we compute the **h**'s through (47), then new **f**'s through (48) or (49), then new **h**'s through (47) and so on. The result of the computation is given in Table I.

Using (36) we obtain $\lambda = 0.95$. Applying estimate (43) with $i = 9$ and $k = 10$ we deduce that $\mathbf{f}^{10}$ is correct to the order of 2 units. Actually, the exact solution to the problem is $f^*_{a_1} = 120.0$, $f^*_{a_2} = 90.0$, $f^*_{a_3} = 0$, $f^*_{b_1} = 70.0$, $f^*_{b_2} = 50.0$ so that the accuracy is much better than that (of the order of 0.01). Thus, even estimate (43) is very conservative. The original estimate (32) is even more conservative because, with $\lambda = 0.95$, it would require 148 iterations to guarantee accuracy of the order of 1%.

APPENDIX I

WE SHOW that the strong monotonicity condition (6) is satisfied if and only if the Jacobian matrix $[\partial\hat{\mathbf{c}}/\partial\mathbf{f}]$ is positive definite and that the maximum α for which (6) holds is the minimum α_1 over κ of the maximum eigenvalue of the symmetric part of the above Jacobian.

Indeed, note that

$$[\hat{\mathbf{c}}(\mathbf{f}) - \hat{\mathbf{c}}(\bar{\mathbf{f}})]\cdot(\mathbf{f} - \bar{\mathbf{f}}) = \int_0^1 (\mathbf{f} - \bar{\mathbf{f}})\cdot\frac{\partial\hat{\mathbf{c}}}{\partial\mathbf{f}}\bigg|_{\bar{\mathbf{f}}+\lambda(\mathbf{f}-\bar{\mathbf{f}})} (\mathbf{f}-\bar{\mathbf{f}})\,d\lambda$$

$$\geq \alpha_1 |\mathbf{f} - \bar{\mathbf{f}}|^2.$$

TABLE I

Iteration		a_1	a_2	a_3	b_1	b_2
0	**f**	70.0	70.0	70.0	60.0	60.0
	h	−100.0	−360.0	−80.0	−498.0	−639.0
1	**f**	89.4	76.9	43.7	63.6	56.4
	h	−230.4	−437.9	288.2	−536.8	−573.9
2	**f**	102.4	82.2	25.4	65.9	54.1
	h	−318.0	−496.9	544.4	−561.2	−532.1
3	**f**	111.2	86.2	12.6	67.3	52.7
	h	−377.4	−541.0	723.6	−575.5	−506.4
4	**f**	117.2	89.1	3.7	68.2	51.8
	h	−418.1	−572.9	848.2	−584.5	−489.8
5	**f**	119.8	90.2	0.0	68.8	51.2
	h	−435.4	−585.3	900.0	−591.3	−478.9
6	**f**	120.0	90.0	0.0	69.2	50.8
	h	−436.2	−583.8	900.0	−596.8	−472.0
7	**f**	120.1	89.9	0.0	69.5	50.5
	h	−436.4	−583.3	900.0	−601.0	−466.7
8	**f**	120.1	89.9	0.0	69.7	50.3
	h	−436.1	−583.6	900.0	−603.7	−463.2
9	**f**	120.1	89.9	0.0	69.9	50.1
	h	−435.8	−583.9	900.0	−606.5	−459.7
10	**f**	120.0	90.0	0.0	70.0	50.0

On the other hand, if $\bar{\mathbf{f}}$ is the point where the symmetric part of $[\partial\hat{\mathbf{c}}/\partial\mathbf{f}]$ has eigenvalue α_1, with corresponding eigenvector, say, $\mathbf{r}$, $|\mathbf{r}| = 1$, we select $\mathbf{f} = \bar{\mathbf{f}} + \mu\mathbf{r}$ and note that

$$\frac{[\hat{\mathbf{c}}(\mathbf{f}) - \hat{\mathbf{c}}(\bar{\mathbf{f}})]\cdot(\mathbf{f} - \bar{\mathbf{f}})}{|\mathbf{f} - \bar{\mathbf{f}}|^2} = \mathbf{r}\cdot\frac{\partial\hat{\mathbf{c}}}{\partial\mathbf{f}}\bigg|_{\bar{\mathbf{f}}}\mathbf{r} + 0(\mu) = \alpha_1 + 0(\mu)$$

which shows that α_1 is optimal.

APPENDIX II

TO SEE how one can pass from inequality (27) to inequality (28) define

$$g(\lambda) = [\hat{\mathbf{c}}(\bar{\mathbf{f}}^2 + \lambda(\bar{\mathbf{f}}^1 - \bar{\mathbf{f}}^2)) - \hat{\mathbf{c}}(\bar{\mathbf{f}}^2)]\cdot G^{-1}[\hat{\mathbf{c}}(\bar{\mathbf{f}}^2 + \lambda(\bar{\mathbf{f}}^1 - \bar{\mathbf{f}}^2)) - \hat{\mathbf{c}}(\bar{\mathbf{f}}^2)]$$

and note that $g(1) = [\hat{\mathbf{c}}(\bar{\mathbf{f}}^1) - \hat{\mathbf{c}}(\bar{\mathbf{f}}^2)]\cdot G^{-1}[\hat{\mathbf{c}}(\bar{\mathbf{f}}^1) - \hat{\mathbf{c}}(\bar{\mathbf{f}}^2)]$ but also, since $g(0) = 0$,

$$g(1) = \frac{dg}{d\lambda}\bigg|_{\lambda=\bar{\lambda}} = (\bar{\mathbf{f}}^1 - \bar{\mathbf{f}}^2)\cdot\left[\frac{\partial \hat{\mathbf{c}}}{\partial \mathbf{f}}\right]^T \mathrm{G}^{-1}\left[\frac{\partial \hat{\mathbf{c}}}{\partial \mathbf{f}}\right](\bar{\mathbf{f}}^1 - \bar{\mathbf{f}}^2)$$

where the Jacobian matrix $[\partial\hat{\mathbf{c}}/\partial\mathbf{f}]$ is evaluated at the point $\bar{\mathbf{f}}^2 + \bar{\lambda}(\bar{\mathbf{f}}^1 - \bar{\mathbf{f}}^2)$.

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