4) SEQUENCES AND SERIES

Defn: A sequence is a function whose domain is the set of natural numbers and the range is the set of real numbers.

- A sequence is denoted by $(x_n)_{n=1}^{\infty}$ or (x_n)

- A sequence may also be defined recursively as follows

$X_1 = 1,\ x_2 = 2,\ x_{n+1} = x_n + 3x_{n-1}$

$X_{4}=$ $x_3 = x_2 + 3x_1 = 5$

$x_4 = x_3 + 3x_2 = 11$

$x_5 = x_4 + 3x_3 = 26$.

Defn: We say that a sequence (X_n) converges to a limit x and write the ~~limit~~ $\lim_{n\to\infty}(x_n) = x$ if given $\varepsilon > 0$, there exists $N(\varepsilon)$ such that $|X_n - x| < \varepsilon$ whenever $n > N(\varepsilon)$.

Remark: A sequence which converges to zero is called a null sequence.

Example 1

Show that the sequence $(x_n) = \left(\frac{n}{n+1}\right)$ converges to 1

Soln

Given $\varepsilon > 0$, we find $N(\varepsilon)$ such that $|x_n - x| < \varepsilon$ whenever $n > N(\varepsilon)$

$\left|\frac{n}{n+1} - 1\right| < \varepsilon \Rightarrow \frac{n-(n+1)}{n+1} < \varepsilon$

$\Rightarrow \left|\frac{-1}{n+1}\right| < \varepsilon$

$\frac{1}{n+1} < \varepsilon \Rightarrow n+1 > \frac{1}{\varepsilon}$

$\Rightarrow n > \frac{1}{\varepsilon} - 1 = N(\varepsilon)$

Example 2

- Show that $(x_n) = \left(\frac{1}{5^n}\right)$ converges to 0

Soln

Given $\varepsilon > 0$, we find $N(\varepsilon)$

$\left|\frac{1}{5^n} - 0\right| < \varepsilon \Rightarrow \frac{1}{5^n} < \varepsilon \Rightarrow$

$5^n > \frac{1}{\varepsilon}$

$\log 5^n > \log \frac{1}{\varepsilon} \Rightarrow$

$n \log 5 > \log \frac{1}{\varepsilon}$

$\Rightarrow n > \dfrac{\log \frac{1}{\varepsilon}}{\log 5} = N(\varepsilon)$

Example 3

Show that the sequence

$(x_n) = (\sqrt{n+1} - \sqrt{n})$

converges to 0

Given $\varepsilon > 0$, we find $N(\varepsilon)$

$\left| \sqrt{n+1} - \sqrt{n} - 0 \right| < \varepsilon$

$\sqrt{n+1} - \sqrt{n} = \dfrac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{(\sqrt{n+1} + \sqrt{n})}$

$= \dfrac{1}{\sqrt{n+1} + \sqrt{n}} < \dfrac{1}{\sqrt{n} + \sqrt{n}}$

$= \dfrac{1}{2\sqrt{n}}$

Now $|x_n - 0| < \left|\frac{1}{2\sqrt{n}} - 0\right| < \varepsilon$

$\Rightarrow \frac{1}{2\sqrt{n}} < \varepsilon \Rightarrow \sqrt{n} > \frac{1}{2\varepsilon}$

~~$2\sqrt{n} < \varepsilon 2\sqrt{n}$~~

$n > \frac{1}{4\varepsilon^2} = N(\varepsilon)$

Evaluate the following limits

i) $\lim\limits_{x \to 3} \dfrac{x^2 - 4}{x + 2}$

(ii) $\lim\limits_{n \to \infty} \left(\dfrac{n}{n+1}\right)$

iii) $\lim\limits_{x \to \infty} \left(\dfrac{4x^2 + 3x + 1}{5x^2 + 2x - 1}\right)$

iv) $\lim\limits_{x \to \infty} \dfrac{3^x + 5^x}{2^x + 7^x}$

v) $\lim\limits_{n \to \infty} \dfrac{(2n+1)! + (3n-4)!}{(5n+2)! + (7n+1)!}$

i) $\dfrac{9-4}{3+2} \cdot \dfrac{\cancel{5}}{\cancel{5}} = 1$

$\dfrac{(x-2)\cancel{(x+2)}}{\cancel{(x+2)}}$

$3 - 2 = 1$

ii) $\lim\limits_{n \to \infty} \left(\dfrac{n}{n+1}\right) = \lim\limits_{n \to \infty} \left(\dfrac{\frac{n}{n}}{\frac{n+1}{n}}\right)$

$= \dfrac{1}{1+0} = 1$

iii) $\lim_{x \to \infty} \dfrac{\left(\frac{4x^2}{x^2} + \frac{3x}{x^2} + \frac{1}{x^2}\right)}{\left(\frac{5x^2}{x^2} + \frac{2x}{x^2} + \frac{1}{x^2}\right)}$

$$\frac{4+0+0}{5+0-0} = \frac{4}{5}$$

iv) $\lim_{x \to \infty} \dfrac{3^x + 5^x}{2^x + 7^x}$

$\lim_{x \to \infty} \dfrac{\left(\frac{3^x}{7^x} + \frac{5^x}{7^x}\right)}{\left(\frac{2^x}{7^x} + \frac{7^x}{7^x}\right)}$

$$\frac{0+0}{0+1} = 0$$

v) $\lim_{n \to \infty} \dfrac{(2n+1)! + (3n-4)!}{(5n+2)! + (7n+1)!}$

$$\frac{\frac{(2n+1)!}{(7n+1)!} + \frac{(3n-4)!}{(7n+1)!}}{\frac{(5n+2)!}{(7n+1)!} + \frac{(7n+1)!}{(7n+1)!}}$$

$$= \frac{0+0}{0+1} = 0$$

Bounded sequences

- A sequence (x_n) is said to be bounded if there exists a real number M such that $|x_n| \leq M$

Examples

1. $\left(\frac{n}{n+1}\right)_{n=1}^{\infty}$ is bounded.

The lower bound is $\frac{1}{2}$ and the upper bound is 1. Indeed $\left|\frac{n}{n+1}\right| < 1$

2. $\left((-1)^n \cdot \frac{1}{n}\right)$ is bounded.

The lower bound $= -1$ and the upper bound is $\frac{1}{2}$

3. The sequence $\frac{n^2+1}{n}$ $\left(n + \frac{1}{n}\right)$ is bounded below. The lower bound is 2 and the upper bound is ∞. It is unbounded from above

Theorem

If the limit of a sequence exists, then it is unique.

Proof (by contradiction)

Suppose on the contrary that a sequence (X_n) converges to two distinct limits

L_1 and L_2 where $L_1 \neq L_2$

$L_1 - L_2 \neq 0$ $|L_1 - L_2| > 0$

Take $\varepsilon = \frac{1}{2}|L_1 - L_2| > 0$

- Given that $X_n \to L_1$

for $\frac{\varepsilon}{2} > 0$, $\exists N_1(\varepsilon)$ such that $|X_n - L_1| < \frac{\varepsilon}{2}$

whenever $n > N_1(\varepsilon)$.

Given that the sequence $(x_n) \to L_2$, for $\frac{\varepsilon}{2} > 0$, $\exists N_2(\varepsilon)$

such that $|x_n - L_2| < \frac{\varepsilon}{2}$

whenever $n > N_2(\varepsilon)$

Now, $|L_1 - L_2| = |L_1 - x_n + x_n - L_2|$

$\leq |L_1 - x_n| + |x_n - L_2|$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$

$= \varepsilon$

$\varepsilon = \frac{1}{2}|L_1 - L_2|$

$1 < \frac{1}{2}$. This is absurd

so the sequence (x_n) cannot converge to two distinct limits.