FIN307 MATLAB

FINANICAL TOOLBOX CHAPTER 1 – Financial Data Analysis PART03

MATLAB: ACF and PACF

Plot ACF and PACF for

- MA(1), MA(2), AR(1), AR(2), ARMA(1,1)

Build in functions: **autocorr, parcorr**

Stationary Models and the Autocorrelation Function

Let $\{X_t\}$ be a time series with $E(X_t^2) < \infty$. The **mean function** of $\{X_t\}$ is

$$\mu_X(t) = E(X_t).$$

The **covariance function** of $\{X_t\}$ is

$$\gamma_X(r, s) = \mathrm{Cov}(X_r, X_s) = E[(X_r - \mu_X(r))(X_s - \mu_X(s))]$$

for all integers r and s.

$\{X_t\}$ is **(weakly) stationary** if

(i) $\mu_X(t)$ is independent of t,

and

(ii) $\gamma_X(t + h, t)$ is independent of t for each h.

Let $\{X_t\}$ be a stationary time series. The **autocovariance function** (ACVF) of $\{X_t\}$ at lag h is

$$\gamma_X(h) = \mathrm{Cov}(X_{t+h}, X_t).$$

The **autocorrelation function** (ACF) of $\{X_t\}$ at lag h is

$$\rho_X(h) \equiv \frac{\gamma_X(h)}{\gamma_X(0)} = \mathrm{Cor}(X_{t+h}, X_t).$$

Determine the stationarity for

a) Random walk: $S_t = X_1 + X_2 + \ldots + X_t \quad t=1,2,\ldots$

b) MA(1), AR(1), MA(2), AR(2)

Example The random walk $\{S_t, t = 0, 1, 2, \dots\}$ (starting at zero) $\{S_t, t = 0, 1, \dots\}$

$S_t = X_1 + X_2 + \cdots + X_t$, for $t = 1, 2, \dots,$ where $\{X_t\}$ is iid noise.

$S_t - S_{t-1} = X_t$

$$E(X_t) = 0$$

$$E(X_t^2) = \sigma^2$$

$$E[S_t] = E[X_1 + X_2 + ... + X_t]$$
$$= E[X_1] + E[X_2] + ... + E[X_t] = 0$$

$$\gamma_S(t+h, t) = \mathrm{Cov}(S_{t+h}, S_t)$$
$$= \mathrm{Cov}(S_t + X_{t+1} + \cdots + X_{t+h}, S_t)$$
$$= \mathrm{Cov}(S_t, S_t)$$
$$= Cov[X_1^2] + Cov[X_2^2] + ... + Cov[X_t^2] + \{\text{all other mixture } Cov[X_r X_s]\}$$
$$= t\sigma^2$$

$\gamma_S(t+h, t)$ depends on t. $\{S_t\}$ is *not* stationary.

Example

MA(1) process $X_t = Z_t + \theta Z_{t-1}, \quad t = 0, \pm 1, \pm 2, \ldots,$ $\{Z_t\} \sim WN(0, \sigma^2)$

$$E(X_t) = E(Z_t + \theta Z_{t-1}) = E(Z_t) + \theta\, E(Z_{t-1}) = 0.$$

(i) $\mu_X(t)$ is independent of t,

$$\operatorname{cov}(X_t, X_{t+\tau})$$

$$= \operatorname{cov}(Z_t + \theta Z_{t-1}, Z_{t+\tau} + \theta Z_{t-1+\tau})$$

$$\sigma_{XY} = E(XY) - \mu_X \mu_Y$$

$$= E[(Z_t + \theta Z_{t-1})(Z_{t+\tau} + \theta Z_{t-1+\tau})] - E(Z_t + \theta Z_{t-1})E(Z_{t+\tau} + \theta Z_{t-1+\tau})$$

$$= E(Z_t Z_{t+\tau}) + \theta\, E(Z_t Z_{t-1+\tau}) + \theta\, E(Z_{t-1} Z_{t+\tau}) + \theta^2\, E(Z_{t-1} Z_{t-1+\tau}).$$

$Cov[Z_r Z_s] = 0, r \neq s$ $\{Z_t\} \sim WN(0, \sigma^2)$

$$\mathrm{cov}(X_t, X_{t+\tau})$$

$$= \mathrm{E}(Z_t Z_{t+\tau}) + \theta\,\mathrm{E}(Z_t Z_{t-1+\tau}) + \theta\,\mathrm{E}(Z_{t-1} Z_{t+\tau}) + \theta^2\,\mathrm{E}(Z_{t-1} Z_{t-1+\tau}).$$

$$\mathrm{cov}(X_t, X_{t+\tau}) = \begin{cases} \mathrm{E}(Z_t^2) + \theta^2\,\mathrm{E}(Z_{t-1}^2) = (1+\theta^2)\sigma^2, & \text{if } \tau = 0, \\ \theta\,\mathrm{E}(Z_t^2) = \theta\sigma^2, & \text{if } \tau = \pm 1, \\ 0, & \text{if } |\tau| > 1. \end{cases}$$

(ii) $\gamma_X(t+h, t)$ is independent of t for each h. → MA(1) is a weakly stationary process

autocorrelation function $\gamma_X(\tau) = \mathrm{cov}(X_t, X_{t+\tau})$ for any t.

$$\rho_X(\tau) = \begin{cases} 1, & \text{if } \tau = 0, \\ \frac{\theta}{1+\theta^2} & \text{if } \tau = \pm 1, \\ 0, & \text{if } |\tau| > 1. \end{cases}$$

MA(1) process

AR(1)

Example

$$X_t = \phi X_{t-1} + Z_t, \quad t = 0, \pm 1, \ldots, \qquad \{Z_t\} \sim \mathrm{WN}(0, \sigma^2), |\phi| < 1$$

$EX_t = 0.$

(i) $\mu_X(t)$ is independent of t,

covariance function

$$\gamma_X(0) = \mathrm{Cov}(X_t, X_t)$$

$$= \mathrm{Cov}(\phi X_{t-1} + Z_t, \phi X_{t-1} + Z_t)$$

$$= \phi^2 \gamma_X(0) + \sigma^2$$

$$\gamma_X(0) = \sigma^2 / \left(1 - \phi^2\right).$$

$$\gamma(h) = \gamma(-h)$$

covariance function $h > 0$

$$\gamma_X(h) = \mathrm{Cov}(X_t, X_{t-h})$$

$$= \mathrm{Cov}(\phi X_{t-1}, X_{t-h}) + \mathrm{Cov}(Z_t, X_{t-h})$$ Z_t is uncorrelated with X_s for each $s < t$.

$$= \phi\gamma_X(h-1) + 0$$ Z_t is uncorrelated with X_{t-1}

$$\vdots$$

$$= \phi^h \gamma_X(0).$$ (ii) $\gamma_X(t+h, t)$ is independent of t for each h. stationary

autocorrelation function

$$\rho_X(h) = \frac{\gamma_X(h)}{\gamma_X(0)} = \phi^{|h|}, \quad h = 0, \pm 1, \ldots .$$

Partial Autocorrelation Function (PACF)

- In general, a partial correlation is a conditional correlation.
- It is the correlation between two variables under the assumption that we know and take into account the values of some other set of variables.
- For instance, consider a regression context in which y = response variable and x_1, x_2, and x_3 are predictor variables. The partial correlation between y and x_3 is the correlation between the variables determined taking into account how both y and x_3 are related to x_1 and x_2.

- Note that this is also how the parameters of a regression model are interpreted. Think about the difference between interpreting the regression models:

$$y = \beta_0 + \beta_1 x^2 \text{ and } y = \beta_0 + \beta_1 x + \beta_2 x^2$$

In the first model, β_1 can be interpreted as the linear dependency between x^2 and y. In the second model, β_2 would be interpreted as the linear dependency between x^2 and y WITH the dependency between x and y already accounted for.

Consider

$$
\begin{aligned}
r_t &= \phi_{0,1} + \phi_{1,1} r_{t-1} + e_{1t}, \\
r_t &= \phi_{0,2} + \phi_{1,2} r_{t-1} + \phi_{2,2} r_{t-2} + e_{2t}, \\
r_t &= \phi_{0,3} + \phi_{1,3} r_{t-1} + \phi_{2,3} r_{t-2} + \phi_{3,3} r_{t-3} + e_{3t}, \\
r_t &= \phi_{0,4} + \phi_{1,4} r_{t-1} + \phi_{2,4} r_{t-2} + \phi_{3,4} r_{t-3} + \phi_{4,4} r_{t-4} + e_{4t}, \\
\vdots &\quad \vdots
\end{aligned}
$$

Determine:

- The 1^{st} order partial autocorrelation
- The 2^{nd} order (lag) partial autocorrelation
- The 3^{rd} order (lag) partial autocorrelation

AutoRegressive and Moving Average

Simulate

a) MA(1), AR(1), MA(2), AR(2)

b) Compare their ACF and PACF

Conditional Mean Model	ACF	PACF
AR(p)	Tails off gradually	Cuts off after p lags
MA(q)	Cuts off after q lags	Tails off gradually
ARMA(p,q)	Tails off gradually	Tails off gradually

Plot the sample autocorrelation function (ACF) and partial autocorrelation function (PACF).

```
figure
subplot(2,1,1)
autocorr(Y)
subplot(2,1,2)
parcorr(Y)
```

MA and AR selection

MATLAB: ACF and PACF

Plot ACF and PACF for

- MA(1), MA(2), AR(1), AR(2), ARMA(1,1)

Build in functions: **autocorr, parcorr**

Financial Data Analysis

Three properties:

✓Stationarity

✓Causality

✓Invertibility

Invertibility of MA Processes

MA(1) $\quad X_t = Z_t + \theta Z_{t-1}, \quad t = 0, \pm 1, \pm 2, \ldots, \qquad \{Z_t\} \sim WN(0, \sigma^2)$

$$\begin{aligned}
Z_t &= X_t - \theta Z_{t-1} \\
&= X_t - \theta(X_{t-1} - \theta Z_{t-2}) \\
&= X_t - \theta X_{t-1} + \theta^2 Z_{t-2} \\
&= X_t - \theta X_{t-1} + \theta^2 (X_{t-2} - \theta Z_{t-3}) \\
&= X_t - \theta X_{t-1} + \theta^2 X_{t-2} - \theta^3 Z_{t-3} \\
&= \ldots \\
&= X_t - \theta X_{t-1} + \theta^2 X_{t-2} - \theta^3 X_{t-3} + \theta^4 X_{t-4} + \ldots + (-\theta)^n Z_{t-n}
\end{aligned}$$

$$Z_t = X_t - \theta X_{t-1} + \theta^2 X_{t-2} - \theta^3 X_{t-3} + \theta^4 X_{t-4} + \ldots + (-\theta)^n Z_{t-n}$$

$$(-\theta)^n Z_{t-n} = Z_t - \sum_{j=0}^{n-1} (-\theta)^j X_{t-j}$$

if $|\theta| < 1$ $$\mathrm{E}\left(Z_t - \sum_{j=0}^{n-1} (-\theta)^j X_{t-j}\right)^2 = \mathrm{E}\left(\theta^{2n} Z_{t-n}^2\right) \underset{n\to\infty}{\longrightarrow} 0$$

the sum is convergent in the mean square sense.

$$Z_t = \sum_{j=0}^{\infty} (-\theta)^j X_{t-j}$$ inverted MA(1) to an infinite AR. **invertible process**.

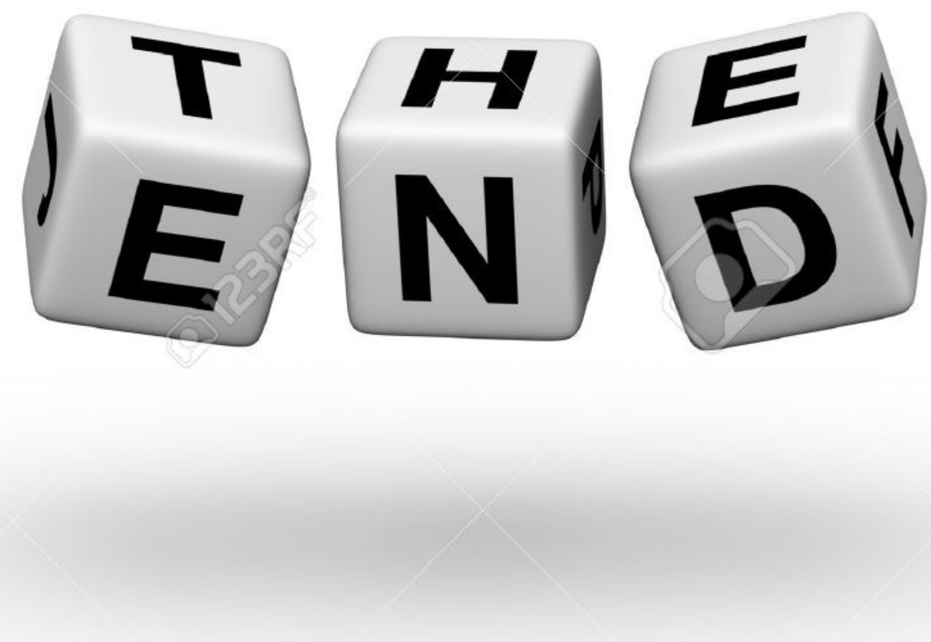
AR(1) Model and Causality

$$X_t = \phi X_{t-1} + Z_t, \quad t = 0, \pm 1, \ldots, \{Z_t\} \sim \mathrm{WN}(0, \sigma^2), |\phi| < 1$$

$$\begin{aligned} X_t &= \phi X_{t-1} + Z_t \\ &= \phi^2 X_{t-2} + Z_t + \phi Z_{t-1} \\ &= \ldots \\ &= \phi^N X_{t-N} + \sum_{j=0}^{N-1} \phi^j Z_{t-j} \end{aligned}$$

$$N \to \infty$$

$$X_t = \sum_{j=0}^{\infty} \phi^j Z_{t-j}$$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