Newton-Euler Dynamics

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dedicated to

George Leitmann

teacher, mentor, colleague, and friend

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Preface

Dynamics is the oldest and best established field of mathematical physics. It is based on relatively few, relatively simple, basic concepts, all of which should be familiar to the readers of this book. Thus one would suspect that the subject is an easy one to understand and apply, and, indeed, this is true for the class of two-dimensional problems normally considered in an introductory engineering course in dynamics. In 2-D problems, the motion is easy to visualize and the required mathematical tools and manipulations are minimal.

A few decades ago, the dynamics of interest in engineering applications was in fact mostly concerned with 2-D motion of rigid bodies in simple machines. In many engineering situations of current interest, however, the dynamics problems that arise are very complicated and, in particular, involve three-dimensional motion. Noteworthy examples are the motion of robotic manipulators and the motion of aerospace vehicles. Many such problems are difficult to visualize and require lengthy mathematical analysis. This analysis must be based on a sound understanding of the basic concepts of dynamics.

Because the complexity of the fundamentals of dynamics is low, it is one subject engineering undergraduates can master. Undergraduate courses such as thermodynamics and fluid mechanics are only introductions to these subjects, but Newton-Euler dynamics can be completed at the undergraduate level. Indeed, students using this book will know already all the basic concepts. It is the purpose of this book to teach students how to solve *any* dynamics problem by the Newton-Euler method.

Newton presented his Three Laws for a hypothetical object called a particle. This is a body having mass but no extension. In the generation or two after Newton, many scientists extended these Laws to collections of particles such as rigid bodies and fluids. By far the most important contributor to dynamics of rigid bodies was L. Euler, and for this reason the title of the book is *Newton-Euler Dynamics.*

The term Newton's "Laws" is misleading. We know now that these "Laws" are *experimentally* based. Today we use the principles of dynamics because they give a sufficiently accurate mathematical *model* of physical phenomena. Further, the accuracy of these Laws is dependent on the choice of reference frame used to measure the motion of a body.

My approach to the subject is to present the principles as clearly, generally, and concisely as possible, and then to build on them in an orderly way. No shortcuts will be introduced to solve simple problems that will have to be "unlearned" and replaced for more difficult situations. I will not introduce concepts such as "inertial force", "centrifugal force", etc. These terms will be left where they belong, on the acceleration side of Newton's Second Law.

It will come as no surprise to readers of this book that the central equation of *kinetics* is Newton's Second Law, $\underline{F} = m\underline{a}$. The procedure for finding the resultant force $\underline{F}$ on an object (free-body diagrams and addition and resolution of forces) is developed in statics and therefore well-known. The procedure for determining the acceleration $\underline{a}$ is termed *kinematics* and is almost always the most difficult part of a dynamics analysis. Therefore, this book begins with a thorough study of kinematics.

Kinematics is presented as the motion (velocity and acceleration) of points and reference frames relative to another reference frame. Later on, this is applied to the motion of a rigid body by identifying the center of mass with a point and the body with a body-fixed reference frame. Newton's Second Law then gives the motion of this point and this frame with respect to an inertial frame.

I take great pains not to confuse "reference frames" and "coordinate systems".[1] In my presentation of kinematics, one of the fundamental results is Euler's Theorem.[2] Although not of direct use in dynamics (it is concerned with *finite* rotations), it has many implications in dynamics. For example, it leads in a natural way to the definition of angular velocity; it also provides a way of visualizing 3-D motion.

The background needed for the book is a physics course in mechanics, an engineering course in statics, a mathematics course in differential equations, and, possibly, a first engineering course in dynamics.

This book is meant to be used two ways. First, as a text for a second undergraduate course in dynamics, usually an elective in most mechanical engineering programs. Assuming the first course focuses on 2-D motion, the present book is focused on 3-D motion. It also can be used as a text for a first dynamics course. This is possible because the

[1] For example, one book refers to an "inertial coordinate system".

[2] This theorem is not even mentioned in some dynamics texts.

"review" of 2-D kinematics is in fact comprehensive. When used in this latter way, the book should be supplemented by more 2-D applications. In my own first course in dynamics, I cover the book and supplemental 2-D material in 50 fifty-minute lectures.

The origin of this book is a set of lecture notes developed and used over many years by Professor George Leitmann of the University of California at Berkeley. Starting with these notes, I developed my own set of lecture notes as a text in the dynamics class I teach at Santa Clara University. The approach and spirit of the book remain those of Professor Leitmann, but I bear full responsibility for any errors that may be found in it. I am thankful to Professor Leitmann not only for granting me permission to use freely material from his notes, but also for many stimulating discussions regarding dynamics.

Mark D. Ardema
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June, 2004

Chapter 1

Introduction and Basic Concepts

1.1 Fundamental Definitions and Assumptions

The subject of dynamics is conveniently divided into two topics.

Kinematics is the study of motion, that is, of the evolution of the position of geometric objects over time, without regard for the cause of the motion.

Kinetics is the study of the relation between forces and motion. The basis of kinetics is Newton's three laws of motion.

Since typically much of the complexity of dynamics problems is in the description of the motion, kinematics must be rigorously developed and thoroughly understood before the study of kinetics is undertaken.

There are two basic types of problems in dynamics. The first problem is: given the motion of a body, what are the forces acting on it (or more precisely, what systems of forces can produce the given motion)? For example, suppose that the path of the manipulator portion of a multi-link robot is specified so as to perform some useful function. Dynamic analysis would then be used to determine the forces required to produce this path.

The second type of problem in dynamics is the reverse of the first: given the forces, what is the motion. As an example, it might be desired to know the motion of an airplane for a set of specified forces acting on it.

Of course there are also situations of "mixed" type: some of the forces and some aspects of the motion may be known and the remaining information is to be determined.

As with any field of mathematical physics, Newtonian dynamics is based on a set of assumptions. These are four in number:

1. The mathematical description (model) of the world in which physical objects move is a *three-dimensional* (3-D) *Euclidean space.* In such a space, all the familiar results of elementary geometry and trigonometry are valid. The familiar Pythagorean Theorem for calculating distances and the parallelogram method of adding vectors are valid. We will need to use these results many times in the sequel. Planar (2-D) and rectilinear (1-D) motion are regarded as special cases of 3-D motion.

2. In this 3-D Euclidean space, there exist inertial frames of reference. A frame of reference is any set of three non-collinear directions used for defining positions in 3-D space. An *inertial (Newtonian, Galilean) frame* is one in which Newton's Laws of Motion are valid to a sufficient degree of accuracy.

3. The quantities *mass* and *time* are universal; that is, their measured values are the same for all observers (in all reference frames).

4. Physical objects are rigid bodies or collections of rigid bodies. A *rigid body* is commonly defined as an object that has mass and does not deform even when forces are applied; later on we will give an alternative definition that is more useful mathematically.

Assumptions (1)–(3) used to be regarded as physical "laws", but they are now to be regarded as *engineering approximations.* They are in fact highly accurate for almost all engineering problems. Assumption (4) is also an approximation since all known objects deform under the application of forces. These assumptions allow us to construct a *mathematical model*; that is, to formulate a mathematical description of a physical situation.

For most engineering applications, a reference frame fixed on the earth surface is a satisfactory inertial frame. However, if greater accuracy is desired, or if the motion is over "long" distances at high speeds (for example long distance aircraft or spacecraft flight), then an inertial frame fixed at the center of the earth and not rotating with it is required. This special case will be taken up in detail later. Similarly, it must be decided whether or not an object may be modeled as a rigid body; that is, if the deformations of the body may be neglected.

In any case, it must be decided that the four fundamental assumptions are sufficiently valid before dynamic analysis may be based on them.

1.2 Position, Velocity, and Acceleration of a Point

A fundamental concept of kinematics is that motion is *relative*: The description of the motion of a point depends on the observer; that is, the attributes of the motion, such as position, velocity and acceleration, are relative to the observer.

To define the attributes of the motion, we associate a *frame of reference* with the observer, namely three non-collinear vectors called the *basis vectors* of the observer. A commonly used triad of basis vectors is three mutually perpendicular, right-handed, unit vectors, say $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$, with origin O. The position of a point P relative to reference frame $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ is defined by the vector[1]

$$\underline{r} = \vec{OP} \tag{1.1}$$

as shown in Fig. 1-1.

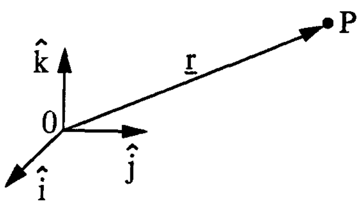


Fig. 1-1

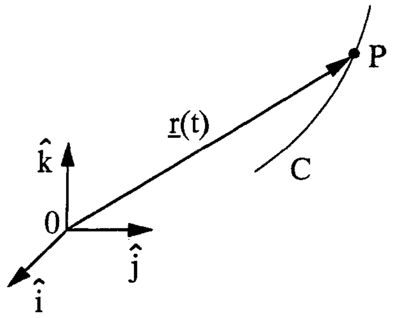


Fig. 1-2

Now consider the motion of point P as its position relative to $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ changes with time t. That is, the position vector of P is a function of time, $\underline{r}(t)$, so that P moves along a curve C (Fig. 1-2). The *velocity*, $\underline{v}(t)$, of P relative to $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ is the time rate of change of $\underline{r}(t)$ as measured in $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$. In other words, if $\underline{r}(t)$ and $\underline{r}(t + \Delta t)$ are the position vectors of P at times t and $t + \Delta t$, respectively, as shown in Fig. 1-3, then

$$\boxed{\underline{v}(t) = \frac{d\underline{r}(t)}{dt} = \lim_{\Delta t \to 0} \frac{\underline{r}(t + \Delta t) - \underline{r}(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta \underline{r}(t)}{\Delta t}} \tag{1.2}$$

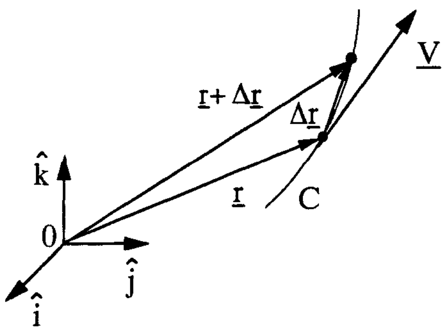


Fig. 1-3

Note that $\underline{v}(t)$ is tangent to curve C at position $\underline{r}(t)$.

Similarly, the *acceleration* $\underline{a}(t)$ of P relative to $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ is

$$\boxed{\underline{a}(t) = \frac{d\underline{v}(t)}{dt} = \lim_{\Delta t \to 0} \frac{\underline{v}(t + \Delta t) - \underline{v}(t)}{\Delta t}} \tag{1.3}$$

It is important to understand that the vectors $\underline{r}$, $\underline{v}$, and $\underline{a}$ depend on the choice of reference frame; in general, a point will have different position, velocity and acceleration vectors in different frames. For example, with respect to a frame with origin at the point, $\underline{r} = \underline{0}$, $\underline{v} = \underline{0}$, and $\underline{a} = \underline{0}$. As another example, to a passenger seated in an airplane, a flight attendant walking down the aisle toward him or her sees the attendant moving at say, 2 miles/hour. On the other hand, an observer on the ground sees the attendant moving in the opposite direction at say, 800 miles/hour.

As a more complicated example, consider the motion of the tip P of the nose on one of the horses on a merry-go-round. To a rider on the horse, the nose has a fixed position and therefore the velocity and acceleration of P are zero. To an observer standing on the rotating platform, the nose has a simple up-and-down motion so that the velocity and acceleration vectors of P are always vertical. To a person standing on the ground, the motion of the nose is obviously much more complicated. It will be one of our purposes in later chapters to determine motion in such complex situations.

The purposes of kinematics are to establish the motion of points relative to reference frames and of reference frames relative to each other. In the merry-go-round situation just discussed, for example, to determine the motion of the horse's nose from the point of view of the different observers, reference frames are associated with each observer (Fig. 1-4).

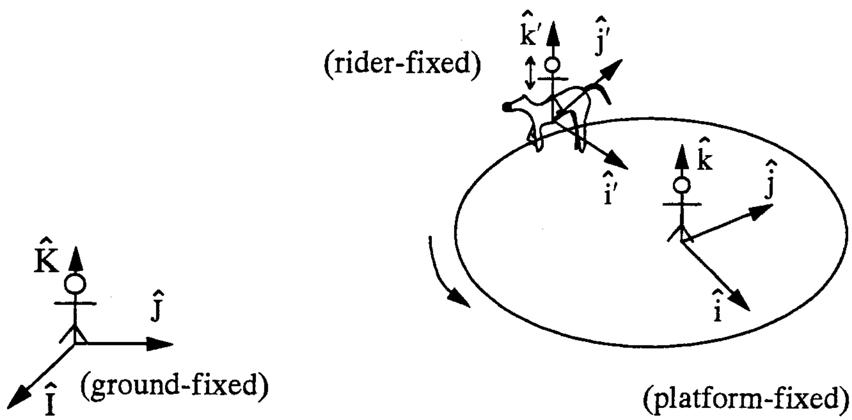


Fig. 1-4

Knowing the motion of these frames relative to each other then gives us the motion of point P in any of these frames, provided we know it in one, which of course we do; relative to the frame associated with the rider on the horse, the velocity and acceleration of the tip of the nose are zero.

There is one class of reference frame that is so natural and familiar to us that we frequently omit specifying it. These are frames attached to the earth surface. In such a frame we almost always measure the motion of our machines and their components. For example, when one says "I drove to Los Angeles and averaged 60 miles per hour", one means, more precisely, "I drove to Los Angeles and averaged 60 miles per hour relative to a frame of reference fixed in the earth". Relative to a frame fixed in our car, of course, we have traveled at zero speed and gone nowhere.

Notes

1 A brief review of vector algebra and calculus is found in Appendix A. Vectors will be underlined throughout this book, except that unit vectors are denoted by $(\hat{\ })$.

Chapter 2

Review of Planar Kinematics

2.1 Plane Motion of a Point; Rectangular Components of Velocity and Acceleration

In this chapter, the kinematics of motion in a plane will be developed. Consider a point P moving relative to reference frame $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ with origin O (Fig. 2-1). Without loss of generality, we take the plane of the motion to be the $(\hat{\imath}, \hat{\jmath})$-plane, that is the plane perpendicular to unit vector $\hat{k}$. For brevity, in the rest of this chapter, we shall refer to the frame as the $\{\hat{\imath}, \hat{\jmath}\}$ frame.

The position vector of the point may be resolved into rectangular components (see Appendix A):

$$\underline{r} = x\hat{\imath} + y\hat{\jmath} \tag{2.1}$$

Equation (2.1) expresses $\underline{r}$ as the sum of two vectors. We call (x, y) the *rectangular coordinates* of $\underline{r}$.

To obtain the velocity and acceleration of the point, Eqns. (1.2) and (1.3) are invoked:[1]

$$\boxed{\underline{v} = \frac{d\underline{r}}{dt} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath}} \tag{2.2}$$

$$\boxed{\underline{a} = \frac{d\underline{v}}{dt} = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath}} \tag{2.3}$$

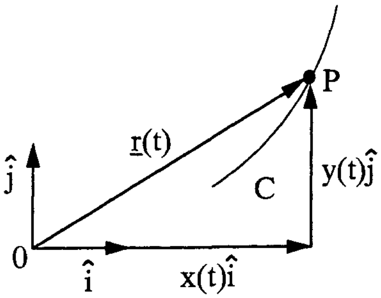


Fig. 2-1

where the fact that $\hat{\imath}$ and $\hat{\jmath}$ are constant vectors has been used (and thus $\dot{\hat{\imath}} = \dot{\hat{\jmath}} = \ddot{\hat{\imath}} = \ddot{\hat{\jmath}} = \underline{0}$). We call $(\dot{x}, \dot{y})$ the rectangular components of velocity, and $(\ddot{x}, \ddot{y})$ the rectangular components of acceleration.

It is often convenient to express $\underline{v}$ and $\underline{a}$ in components along directions other than $\hat{\imath}$ and $\hat{\jmath}$; two of the most common and useful will be considered later in this Chapter.

2.2 Example

Consider a nozzle spraying water with an exit velocity $\underline{v}_0$ at an angle θ with the horizontal, as shown on Fig. 2-2. It is desired to find: (i) the rectangular coordinates of a water particle P, leaving the nozzle at time $t = 0$, at some later time t, relative to the reference frame shown, (ii) the maximum height, h, of the water particle, (iii) the distance the water

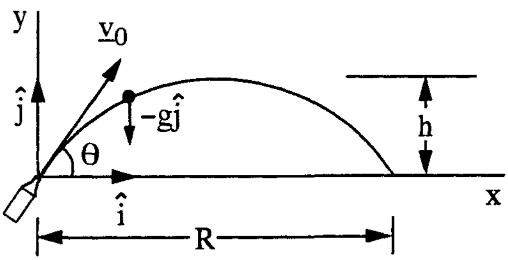


Fig. 2-2

particle has traveled, R, when it returns to its original height, and (iv) the shape of the path of the particle. There is a constant acceleration downward on the particle of magnitude g, the gravitational acceleration.

Expressing the acceleration in rectangular components, Eqn. (2.3),

$$-g\hat{\jmath} = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath}$$

so that

$$\ddot{x} = 0 , \qquad \ddot{y} = -g$$

Integrating twice, we get

$$\begin{aligned} \dot{x} &= A , & \dot{y} &= -gt + C \\ x &= At + B , & y &= -\frac{1}{2}gt^2 + Ct + D \end{aligned}$$

where A, B, C, and D are constants of integration.

The initial conditions

$$x(0) = 0 , \qquad \dot{x}(0) = \nu_0 \cos\theta , \qquad y(0) = 0 , \qquad \dot{y}(0) = \nu_0 \sin\theta$$

give the constants of integration as

$$A = \nu_0 \cos\theta , \qquad B = 0 , \qquad C = \nu_0 \sin\theta , \qquad D = 0$$

The coordinates of the water particle at any time t are therefore

$$x(t) = \nu_0 t \cos\theta , \qquad y(t) = \nu_0 t \sin\theta - \frac{1}{2}gt^2$$

The maximum height h occurs at time t_h when $dy/dt = 0$:

$$\frac{dy}{dt} = \nu_0 \sin\theta - gt_h = 0$$

which gives $t_h = \dfrac{\nu_0}{g}\sin\theta$ so that

$$\begin{aligned} h &= y(t_h) = (\nu_0 \sin\theta)\left(\frac{\nu_0}{g}\sin\theta\right) - \frac{1}{2}g\left(\frac{\nu_0}{g}\sin\theta\right)^2 \\ &= \frac{\nu_0^2}{2g}\sin^2\theta \end{aligned}$$

The time t_R at which the original height is reached is given by

$$y(t_R) = \nu_0 t_R \sin\theta - \frac{1}{2} g t_R^2 = 0$$

that is,

$$t_R = \frac{2\nu_0}{g} \sin\theta$$

which is twice t_h. Thus the range R is

$$R = x(t_R) = \frac{\nu_0^2}{g} \sin 2\theta$$

The path is determined by eliminating t:

$$y = (\nu_0 \sin\theta)\left(\frac{x}{\nu_0 \cos\theta}\right) - \frac{1}{2} g \left(\frac{x}{\nu_0 \cos\theta}\right)^2 = x\tan\theta - \frac{g x^2}{2\nu_0^2} \sec^2\theta$$

which is the equation of a parabola opening downward and passing through $(x, y) = (0, 0)$ and $(R, 0)$.

2.3 Tangential-Normal Components

Introduce two new unit vectors as follows: $\hat{e}_t$ is collinear with the velocity vector $\underset{\sim}{\nu}$ (and hence is tangent to C, the curve on which P is moving), and $\hat{e}_n$ is perpendicular to $\hat{e}_t$ and always pointing in the direction of curvature of C. Let θ be the angle between unit vectors $\hat{e}_t$ and $\hat{\imath}$; we get two different situations,depending on whether θ is increasing or decreasing (Fig. 2-3).

Now define the *speed* of point P as the magnitude of the velocity of P, that is, as $\nu = |\underset{\sim}{\nu}|$. Then

$$\boxed{\underset{\sim}{\nu} = \nu \hat{e}_t} \tag{2.4}$$

$$\underset{\sim}{a} = \dot{\nu}\hat{e}_t + \nu\dot{\hat{e}}_t \tag{2.5}$$

From Eqn. (2.4), $(\nu, 0)$ are the *tangential-normal components of* $\underset{\sim}{\nu}$.

In general, $\hat{e}_t$ changes direction over time and thus $\dot{\hat{e}}_t$ is not zero. What is needed is the resolution into tangential-normal components of $\dot{\hat{e}}_t$. To do this, consider the relationship between the two sets of unit

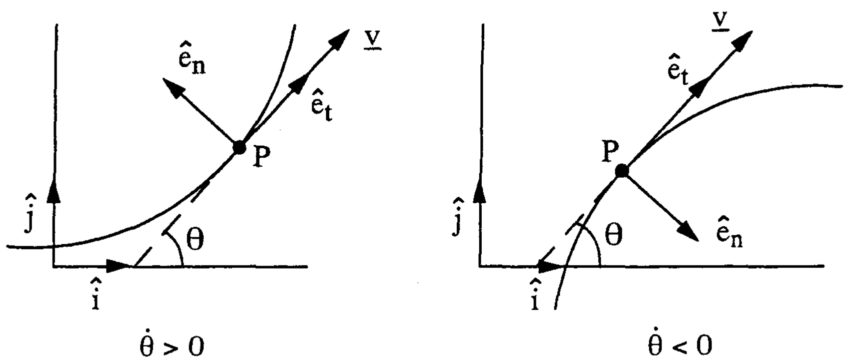


Fig. 2-3

vectors (Fig. 2-4). Writing $\hat{e}_t$ and $\hat{e}_n$ in components along $\hat{\imath}$ and $\hat{\jmath}$ for the case $\dot{\theta} > 0$ gives

$$\begin{aligned} \hat{e}_t &= \cos\theta\hat{\imath} + \sin\theta\hat{\jmath} \\ \hat{e}_n &= -\sin\theta\hat{\imath} + \cos\theta\hat{\jmath} \end{aligned} \tag{2.6}$$

This is an example of a *unit vector transformation.* These transformations always have the same form, namely cosine and sine terms appearing diagonally, and either one or three negative terms. We will need to derive and use unit vector transformations many times later in this book.

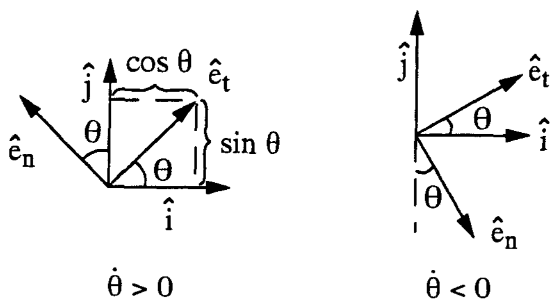


Fig. 2-4

Differentiating the first of Eqns. (2.6) and using the second, we have

$$\dot{\hat{e}}_t = -\dot{\theta}\sin\theta\hat{\imath} + \dot{\theta}\cos\theta\hat{\jmath} = \dot{\theta}\hat{e}_n \tag{2.7}$$

For $\dot{\theta} < 0$, the same analysis gives

$$\begin{aligned} \hat{e}_t &= \cos\theta\hat{\imath} + \sin\theta\hat{\jmath} \\ \hat{e}_n &= \sin\theta\hat{\imath} - \cos\theta\hat{\jmath} \end{aligned} \tag{2.8}$$

$$\dot{\hat{e}}_t = -\dot{\theta}\hat{e}_n \tag{2.9}$$

Thus for all values of $\dot{\theta}$,

$$\dot{\hat{e}}_t = |\dot{\theta}|\hat{e}_n \tag{2.10}$$

Substituting Eqn. (2.10) into Eqn. (2.5):

$$\boxed{\underline{a} = \dot{\nu}\hat{e}_t + \nu|\dot{\theta}|\hat{e}_n} \tag{2.11}$$

and we see that the *tangential-normal components of* $\underline{a}$ are $(\dot{\nu},\ \nu|\dot{\theta}|)$.

These components may be written in another way. Let

$$\rho = \frac{\nu}{|\dot{\theta}|} \tag{2.12}$$

Note that $\rho \geq 0$. ρ is called the *radius of curvature* of curve C at the point P. If the curve is given by $y = f(x)$, then a result from elementary calculus is the formula

$$\rho = \frac{\left[1 + (df/dx)^2\right]^{3/2}}{|d^2f/dx^2|} \tag{2.13}$$

Substituting Eqn. (2.12) into Eqn. (2.11) gives an alternative form for the tangential-normal components of $\underline{a}$:

$$\boxed{\underline{a} = \dot{\nu}\hat{e}_t + \frac{\nu^2}{\rho}\hat{e}_n} \tag{2.14}$$

Suppose now that the motion is on a circle of radius R (Fig. 2-5). Then $S = R\theta$ so that $\dot{S} = R\dot{\theta} = \nu$, and thus $\rho = R$ and the normal component of acceleration is ν^2/R.

Several observations may be made regarding Eqns. (2.11) and (2.14). First, note that $\hat{e}_n$ is undefined and $\rho = \infty$ for $\dot{\theta} = 0$; this occures if curve C is a straight line or P is at an inflection point (Fig. 2-6). Note also that there is always non-zero acceleration if the curve is bending,

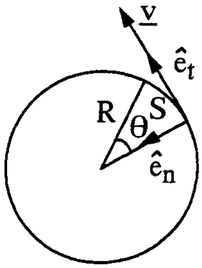


Fig. 2-5

and that the component of acceleration normal to the tangent of the curve is always in the direction of the curvature. It is obvious that the tangential-normal components depend on the curve C and cannot be specified unless C is known; for this reason, they are sometimes called *intrinsic* components. Finally, it is important to understand that $\underline{v}$ and $\underline{a}$ as given by Eqns. (2.4) and (2.11) or (2.14) are the velocity and acceleration of the point P with respect to reference frame $\{\hat{\imath}, \hat{\jmath}\}$, expressed in components along $\{\hat{e}_t, \hat{e}_n\}$.

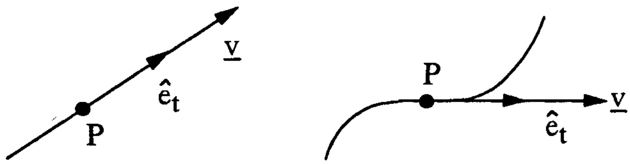


Fig. 2-6

2.4 Example

Suppose that for the problem of Section 2.2, $\nu_0 = 25$ ft/sec and $\theta = 35°$. We want the radius of curvature, ρ, of the stream both at the nozzle and at the point of maximum height.

Since ρ is asked for, tangential-normal components are a natural choice; thus from Eqn. (2.14):

$$\underline{a} = a_t \hat{e}_t + a_n \hat{e}_n$$

so that

$$-g\hat{j} = \dot{\nu}\hat{e}_t + \frac{\nu^2}{\rho}\hat{e}_n$$

From Fig. 2-7 the unit vector transformations at the nozzle are

$$\hat{i} = \hat{e}_t \cos\theta + \hat{e}_n \sin\theta$$
$$\hat{j} = \hat{e}_t \sin\theta - \hat{e}_n \cos\theta$$

so that

$$-g \sin\theta \hat{e}_t + g\cos\theta \hat{e}_n = \dot{\nu}\hat{e}_t + \frac{\nu_0^2}{\rho}\hat{e}_n$$

which implies

$$-g\sin\theta = \dot{\nu}_0$$
$$g\cos\theta = \frac{\nu_0^2}{\rho}$$

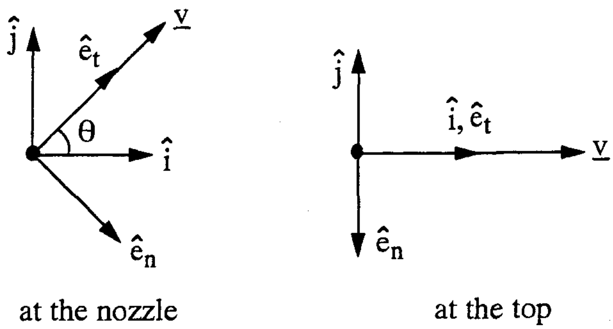


Fig. 2-7

Solving the second of these for ρ,

$$\rho = \frac{\nu_0^2}{g\cos\theta} = 23.7 \text{ ft}$$

From Fig. 2-7 the unit vector transformations at the top are

$$\hat{i} = \hat{e}_t$$
$$\hat{j} = -\hat{e}_n$$

so that

$$g\hat{e}_n = \dot{\nu}\hat{e}_t + \frac{\nu^2}{\rho}\hat{e}_n$$

which implies $\dot{\nu} = 0$ and $g = \nu^2/\rho$. Since the speed at the top is $\dot{x} = \nu_0 \cos\theta$ (see Section 2.2):

$$\rho = \frac{\nu_0^2 \cos^2\theta}{g} = 13.02 \text{ ft}$$

where $g = 32.2$ ft/sec^2 was used.

2.5 Example

A car is rounding a curve of radius 250 m. At a certain instant, the car increases its speed at a rate of 2.0 m/s^2 and the magnitude of the acceleration of the car is measured as 3.0 m/s^2. What is the speed of the car at that instant?

From Eqn. (2.14), the square of the magnitude of the acceleration is (see Fig. 2-8)

$$a^2 = a_t^2 + a_n^2 = \dot{v}^2 + \frac{\nu^4}{R^2}$$

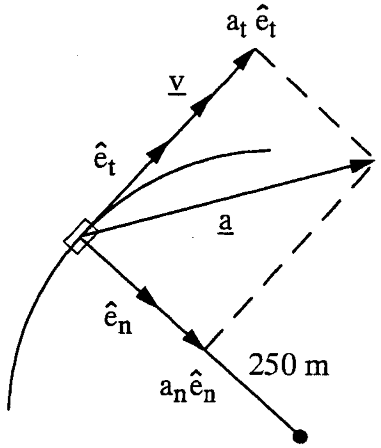


Fig. 2-8

Solving for ν yields

$$\begin{aligned}\nu &= \left[R^2(a^2 - \dot{\nu}^2)\right]^{\frac{1}{4}} = \left[250^2(3^2 - 2^2)\right]^{\frac{1}{4}} \\ &= 23.64 \text{ m/s}\end{aligned}$$

2.6 Radial-Transverse Components

Radial-transverse components are especially useful when motion is due to a force directed at a fixed point, such as may occure in spaceflight. Introduce unit vectors $\{\hat{e}_r, \hat{e}_\theta\}$ with $\hat{e}_r$ along the position vector and $\hat{e}_\theta$ perpendicular to it such that $\{\hat{e}_r, \hat{e}_\theta, \hat{k}\}$ form a right-hand system as shown in Fig. 2-9.[2] The pair of numbers $(r,\ \theta)$ are called the *polar coordinates* of point P. Now,

$$\begin{aligned}&\underline{r} = r\hat{e}_r \\ &\underline{\nu} = \dot{\underline{r}} = \dot{r}\hat{e}_r + r\dot{\hat{e}}_r\end{aligned} \tag{2.15}$$

The derivative $\dot{\hat{e}}_r$ could be determined by using unit vector transformations as before, but an alternative method, based directly on the

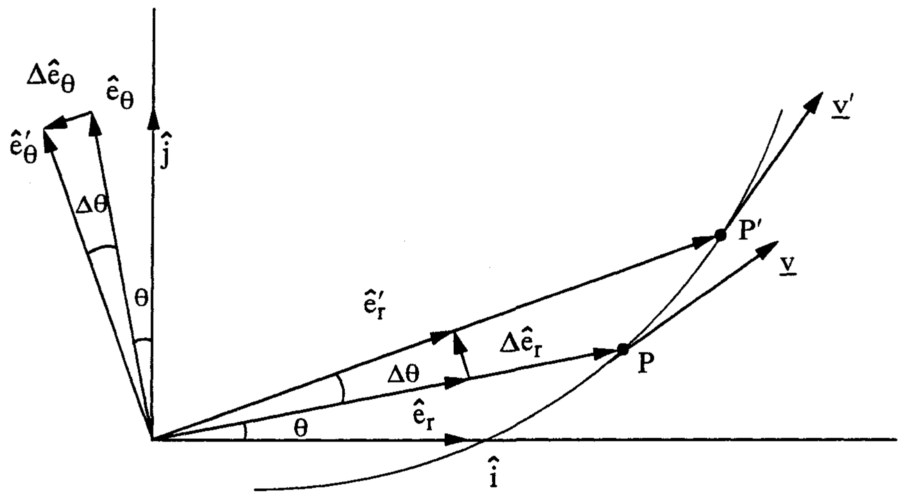


Fig. 2-9

definition of a derivative, will be used:

$$\begin{aligned}\dot{\hat{e}}_r &= \lim_{\Delta t \to 0} \frac{\Delta \hat{e}_r}{\Delta t} = \lim_{\Delta t \to 0} \frac{(1 \cdot \Delta\theta)\hat{e}_\theta}{\Delta t} = \dot{\theta}\hat{e}_\theta \\ \dot{\hat{e}}_\theta &= \lim_{\Delta t \to 0} \frac{\Delta \hat{e}_\theta}{\Delta t} = \lim_{\Delta t \to 0} \frac{(1 \cdot \Delta\theta)(-\hat{e}_r)}{\Delta t} = -\dot{\theta}\hat{e}_r\end{aligned} \tag{2.16}$$

Substituting Eqn. (2.16) in Eqn. (2.15), we get the velocity expressed in *radial-transverse components:*

$$\boxed{\underline{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta} \tag{2.17}$$

The components of acceleration are determined similarly

$$\begin{aligned}\underline{a} = \dot{\underline{v}} &= \ddot{r}\hat{e}_r + \dot{r}\dot{\hat{e}}_r + \dot{r}\dot{\theta}\hat{e}_\theta + r\ddot{\theta}\hat{e}_\theta + r\dot{\theta}\dot{\hat{e}}_\theta \\ &= \ddot{r}\hat{e}_r + \dot{r}\dot{\theta}\hat{e}_\theta + \dot{r}\dot{\theta}\hat{e}_\theta + r\ddot{\theta}\hat{e}_\theta - r\dot{\theta}^2\hat{e}_r\end{aligned}$$

$$\boxed{a = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{e}_r + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{e}_\theta} \tag{2.18}$$

2.7 Example

Fig. 2-10 shows a repair truck with a telescoping, rotating boom. At the instant shown, the length of the boom, ℓ, is 20 ft and is increasing at the constant rate of 6 in/sec, and θ is 30° and decreasing at the constant rate of 0.075 rad/sec. The truck is parked. We want to find the velocity and acceleration of point B at the end of the boom relative to the ground at this instant.

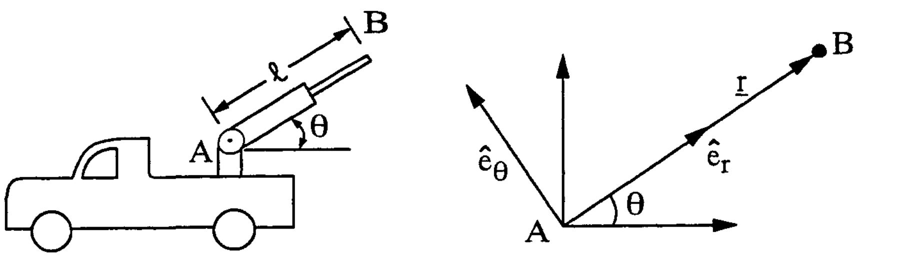


Fig. 2-10

Since we are given data in terms of r, θ, and their derivatives, radial-transverse components are selected. From the data,

$$r = 20\ , \qquad \dot{r} = 0.5\ , \qquad \ddot{r} = 0$$
$$\theta = 30\left(\frac{\pi}{180}\right)\ , \qquad \dot{\theta} = -0.075\ , \qquad \ddot{\theta} = 0$$

Thus from Eqns. (2.17) and (2.18),

$$\underline{\nu}_B = (0.5\hat{e}_r - 1.5\hat{e}_\theta)\ \text{ft/sec}$$
$$\underline{a}_B = (-0.1125\hat{e}_r - 0.075\hat{e}_\theta)\ \text{ft/sec}^2$$

The magnitudes of these vectors are

$$\nu_B = \sqrt{\nu_r^2 + \nu_\theta^2} = 1.581\ \text{ft/sec}$$
$$a_B = \sqrt{a_r^2 + a_\theta^2} = 0.135\ \text{ft/sec}^2$$

2.8 Angular Velocity

Consider two reference frames, moving with respect to each other such that the rectangular unit vectors $\hat{I}, \hat{J}$ of the first and $\hat{\imath}, \hat{\jmath}$, of the second remain in the same plane. Thus $\hat{k} = \hat{K}$ and we will refer to these two frames as the $\{\hat{I}, \hat{J}\}$ and $\{\hat{\imath}, \hat{\jmath}\}$ frames, respectively. The position of $\{\hat{\imath}, \hat{\jmath}\}$ relative to $\{\hat{I}, \hat{J}\}$, may be specified by the location of the origin plus the angular orientation θ (Fig. 2-11). It is convenient to define the *angular velocity* of $\{\hat{\imath}, \hat{\jmath}\}$ relative to $\{\hat{I}, \hat{J}\}$ as

$$\boxed{\underline{\omega} = \dot{\theta}\hat{k} = \dot{\theta}\hat{K}} \tag{2.19}$$

Any vector $\underline{Q}$ must satisfy certain properties, one of which is the commutative property:

$$\underline{Q}_1 + \underline{Q}_2 = \underline{Q}_2 + \underline{Q}_1$$

Consider two plane rotations of $\{\hat{\imath}, \hat{\jmath}\}$ relative to $\{\hat{I}, \hat{J}\}$; it is obvious that the final position of $\{\hat{\imath}, \hat{\jmath}\}$ is the same regardless of the order of the rotations; that is

$$\Delta\theta_1\hat{k} + \Delta\theta_2\hat{k} = \Delta\theta_2\hat{k} + \Delta\theta_1\hat{k}$$

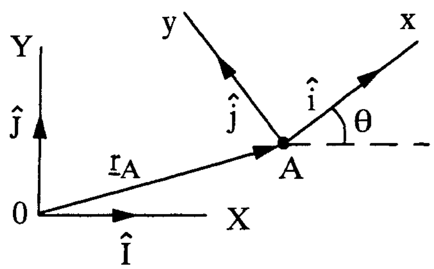


Fig. 2-11

Dividing by Δt and taking the limit $\Delta t \to 0$ gives

$$\underline{\omega}_1 + \underline{\omega}_2 = \underline{\omega}_2 + \underline{\omega}_1$$

showing that $\underline{\omega}$ satisfies the required property.[3] It is clear that $\underline{\omega}$ completely characterizes the rotation; its direction specifies the axis of rotation and its magnitude gives the rate of rotation.

2.9 Relative Motion of Reference Frames

Let

$$\frac{D(\,)}{Dt} = \text{time rate of change with respect to reference frame } \{\hat{I}, \hat{J}\}$$

$$\frac{d(\,)}{dt} = \text{time rate of change with respect to reference frame } \{\hat{\imath}, \hat{\jmath}\}$$

For any scalar Q (for example, mass), $DQ/Dt = dQ/dt = \dot{Q}$ but for a vector $\underline{Q}$, $D\underline{Q}/Dt \neq d\underline{Q}/dt$ in general. Consider a rotation of reference frame $\{\hat{\imath}, \hat{\jmath}\}$ with respect to $\{\hat{I}, \hat{J}\}$. From Fig. 2-12,[4] in the limit as $\Delta\theta \to 0$,

$$\Delta\hat{\imath} = \Delta\theta\hat{\jmath} = \Delta\theta\hat{k} \times \hat{\imath}$$

$$\frac{D\hat{\imath}}{Dt} = \lim_{\Delta t \to 0} \frac{\Delta\hat{\imath}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta\theta\hat{k}}{\Delta t} \times \hat{\imath} = \dot{\theta}\hat{k} \times \hat{\imath} = \underline{\omega} \times \hat{\imath} \tag{2.20}$$

Also,

$$\frac{D\hat{\jmath}}{Dt} = \underline{\omega} \times \hat{\jmath}\,; \qquad \frac{D\hat{k}}{Dt} = \underline{\omega} \times \hat{k} = \dot{\theta}\hat{k} \times \hat{k} = \underline{0} \tag{2.21}$$

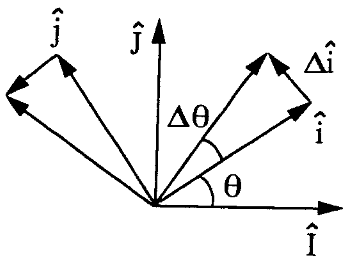


Fig. 2-12

Now if $\underline{Q}$ is any vector, we can write $\underline{Q} = Q_x\hat{\imath} + Q_y\hat{\jmath}$; differentiating with respect to time in the $\{\hat{I}, \hat{J}\}$ frame:

$$\begin{aligned}
\frac{D\underline{Q}}{Dt} &= \frac{D}{Dt}(Q_x\hat{\imath} + Q_y\hat{\jmath}) \\
&= \frac{DQ_x}{Dt}\hat{\imath} + Q_x\frac{D\hat{\imath}}{Dt} + \frac{DQ_y}{Dt}\hat{\jmath} + Q_y\frac{D\hat{\jmath}}{Dt} \\
&= \frac{dQ_x}{dt}\hat{\imath} + \frac{dQ_y}{dt}\hat{\jmath} + Q_x\underline{\omega} \times \hat{\imath} + Q_y\underline{\omega} \times \hat{\jmath} \\
&= \frac{d}{dt}(Q_x\hat{\imath} + Q_y\hat{\jmath}) + \underline{\omega} \times (Q_x\hat{\imath} + Q_y\hat{\jmath})
\end{aligned}$$

$$\boxed{\frac{D\underline{Q}}{Dt} = \frac{d\underline{Q}}{dt} + \underline{\omega} \times \underline{Q}} \tag{2.22}$$

This very important relation is called the *basic kinematic equation* (BKE). We will see later that it is the same for 3-D motion as it is for 2-D.

2.10 Relative Velocity and Acceleration

Let frame $\{\hat{\imath}, \hat{\jmath}\}$ have angular velocity $\underline{\omega}$ and its origin have position $\underline{r}_B$ relative to frame $\{\hat{I}, \hat{J}\}$, and let point A move with respect to both frames (Fig. 2-13). It is our goal to relate the velocity and acceleration of point A as measured in one frame to the velocity and acceleration of A as measured in the other. From Fig. 2-13,

$$\underline{r}_A = \underline{r}_B + \underline{r} \tag{2.23}$$

Let

$$\underline{v}_A = \frac{D\underline{r}_A}{Dt} \quad = \quad \text{velocity of } A \text{ with respect to frame } \{\hat{I}, \hat{J}\}$$

$$\underline{v}_B = \frac{D\underline{r}_B}{Dt} \quad = \quad \text{velocity of } B \text{ with respect to frame } \{\hat{I}, \hat{J}\}$$

$$\underline{v}_r = \frac{d\underline{r}}{dt} \quad = \quad \text{velocity of } A \text{ with respect to frame } \{\hat{\imath}, \hat{\jmath}\}$$

$$\underline{a}_A = \frac{D\underline{v}_A}{Dt} \quad = \quad \text{acceleration of } A \text{ with respect to frame } \{\hat{I}, \hat{J}\}$$

$$\underline{a}_B = \frac{D\underline{v}_B}{Dt} \quad = \quad \text{acceleration of } B \text{ with respect to frame } \{\hat{I}, \hat{J}\}$$

$$\underline{a}_r = \frac{d\underline{v}_r}{dt} \quad = \quad \text{acceleration of } A \text{ with respect to frame } \{\hat{\imath}, \hat{\jmath}\}$$

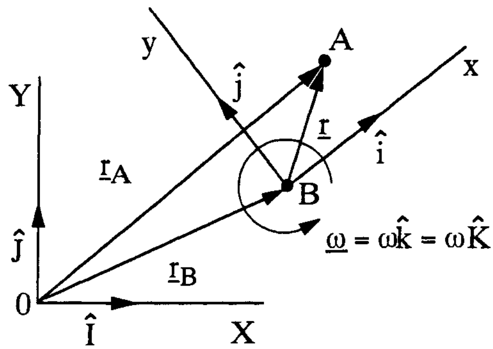


Fig. 2-13

Now let $\underline{Q} = \underline{r}_A$ in the BKE, Eqn. (2.22), and use Eqn. (2.23), hence

$$\frac{D\underline{r}_A}{Dt} = \frac{d\underline{r}_A}{dt} + \underline{\omega} \times \underline{r}_A$$

$$\begin{aligned} \underline{v}_A &= \frac{d\underline{r}_B}{dt} + \frac{d\underline{r}}{dt} + \underline{\omega} \times \underline{r}_B + \underline{\omega} \times \underline{r} \\ &= \underline{v}_r + \frac{d\underline{r}_B}{dt} + \underline{\omega} \times \underline{r}_B + \underline{\omega} \times \underline{r} \end{aligned}$$

$$\boxed{\underline{v} = \underline{v}_r + \underline{v}_B + \underline{\omega} \times \underline{r}} \qquad (2.24)$$

where $D\underline{r}_B/Dt = d\underline{r}_B/dt + \underline{\omega} \times \underline{r}_B$ by the BKE.

Next let $\underline{Q} = \underline{v}_A$ in the BKE. Then

$$\frac{D\underline{v}_A}{Dt} = \frac{d\underline{v}_A}{dt} + \underline{\omega} \times \underline{v}_A$$

$$\begin{aligned} \underline{a}_A &= \frac{d\underline{v}_r}{dt} + \frac{d\underline{v}_B}{dt} + \frac{d\underline{\omega}}{dt} \times \underline{r} + \underline{\omega} \times \frac{d\underline{r}}{dt} + \underline{\omega} \times \underline{v}_r + \underline{\omega} \times \underline{v}_B \\ &\quad + \underline{\omega} \times (\underline{\omega} \times \underline{r}) \\ &= \underline{a}_r + \frac{d\underline{\omega}}{dt} \times \underline{r} + 2\underline{\omega} \times \underline{v}_r + \underline{\omega} \times (\underline{\omega} \times \underline{r}) + \frac{d\underline{v}_B}{dt} + \underline{\omega} \times \underline{v}_B \end{aligned}$$

$$\boxed{\underline{a}_A = \underline{a}_r + \underline{\dot{\omega}} \times \underline{r} + 2\underline{\omega} \times \underline{v}_r + \underline{\omega} \times (\underline{\omega} \times \underline{r}) + \underline{a}_B} \tag{2.25}$$

where $D\underline{v}_B/Dt = d\underline{v}_B/dt + \underline{\omega} \times \underline{v}_B$ was used.

Eqns. (2.24) and (2.25) are called the *relative velocity and acceleration equations*; they relate the motion of a point as seen in one reference frame to the motion of the same point as seen in another frame. It is important to have a clear understanding of the meaning of all the vectors that enter into Eqns. (2.24) and (2.25).

A special case that frequently occurs is that point B is fixed in $\{\hat{I}, \hat{J}\}$ (coincides with O for example); then $\underline{v}_B = \underline{a}_B = \underline{0}$. Another special case is A fixed in $\{\hat{\imath}, \hat{\jmath}\}$, so that $\underline{v}_r = \underline{a}_r = \underline{0}$.

In general, we must distinguish the reference frame when taking the time derivative of a vector, and we have done so except for the vector $\underline{\omega}$. But this is acceptable because $\underline{\omega}$ has the same derivative in either reference frame. To see this, put $\underline{Q} = \underline{\omega}$ in the BKE:

$$\frac{D\underline{\omega}}{Dt} = \frac{d\underline{\omega}}{dt} + \underline{\omega} \times \underline{\omega} = \frac{d\underline{\omega}}{dt} = \underline{\dot{\omega}} \tag{2.26}$$

Clearly, this property is also true for any vector parallel to $\underline{\omega}$.

2.11 Example

We now solve the problem of Section 2.7 by using the relative velocity and acceleration equations, Eqns. (2.24) and (2.25), respectively. Since some of the data is given relative to the boom (the extension of its upper part) and the answers are required relative to the ground, a boom-fixed frame $\{\hat{\imath}, \hat{\jmath}\}$ and a ground-fixed frame $\{\hat{I}, \hat{J}\}$, both with origin at point A, are attached as shown on Fig. 2-14.[5]

First, the relative velocity Eqn. (2.24) is employed, with suitable change of labeling:

$$\underline{v}_B = \underline{v}_A + \underline{v}_r + \underline{\omega} \times \underline{r}$$

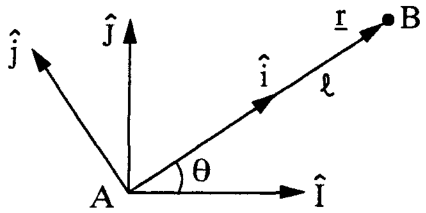


Fig. 2-14

In this equation, $\underline{v}_B$ is the velocity of B in $\{\hat{I}, \hat{J}\}$, that is with respect to the ground; $\underline{v}_A$ is the velocity of the origin of $\{\hat{\imath}, \hat{\jmath}\}$, point A, in $\{\hat{I}, \hat{J}\}$; $\underline{v}_r$ is the velocity of B in $\{\hat{\imath}, \hat{\jmath}\}$; $\underline{\omega}$ is the angular velocity of $\{\hat{\imath}, \hat{\jmath}\}$ with respect to $\{\hat{I}, \hat{J}\}$; and $\underline{r}$ is the position of B in $\{\hat{\imath}, \hat{\jmath}\}$. Thus,

$$\underline{v}_A = \underline{0}\ , \qquad \underline{v}_r = \dot{\ell}\hat{\imath}\ , \qquad \underline{\omega} = \dot{\theta}\hat{k}\ , \qquad \underline{r} = \ell\hat{\imath}$$

and hence

$$\begin{aligned} \underline{v}_B &= \underline{0} + \dot{\ell}\hat{\imath} + (\dot{\theta}\hat{k}) \times (\ell\hat{\imath}) = \dot{\ell}\hat{\imath} + \dot{\theta}\ell\hat{\jmath} \\ &= (0.5\hat{\imath} - 1.5\hat{\jmath})\ \text{ft/sec} \end{aligned}$$

Next, Eqn. (2.25) is used for the acceleration of point B in $\{\hat{I}, \hat{J}\}$

$$\underline{a}_B = \underline{a}_A + \underline{a}_r + 2\underline{\omega} \times \underline{v}_r + \underline{\omega} \times (\underline{\omega} \times \underline{r}) + \dot{\underline{\omega}} \times \underline{r}$$

The new quantities in this equation are

$$\underline{a}_A = \underline{0}\ , \qquad \underline{a}_r = \underline{0}\ , \qquad \dot{\underline{\omega}} = \underline{0}$$

Thus

$$\begin{aligned} \underline{a}_B &= 2(\dot{\theta}\hat{k}) \times (\dot{\ell}\hat{\imath}) + \dot{\theta}\hat{k} \times \left[(\dot{\theta}\hat{k}) \times (\ell\hat{\imath})\right] = -\dot{\theta}^2\ell\hat{\imath} + 2\dot{\theta}\dot{\ell}\hat{\jmath} \\ &= (-0.1125\hat{\imath} - 0.075\hat{\jmath})\ \text{ft/sec}^2 \end{aligned}$$

These results agree with those of Section 2.7.

2.12 Example

Two cars move as illustrated in Fig. 2-15. The position of car A is defined by a point A on the car, and the position of car B is defined by a point B on the car. Thus, when we speak of velocity and acceleration of a car, we mean the velocity and acceleration of the point defining its position in a given reference frame. Car A approaches the intersection at a speed of 90 km/hr in the direction shown and is at a distance of $x = 30$ m from the intersection; its speed increases at a rate of 2 m/s^2. At the same instant of time, car B travels on a curve of radius $R = 200$ m, passing through the intersection at a speed of 72 km/hr in the direction shown; its speed decreases at a rate of 4 m/s^2. It is desired to determine the velocity and acceleration of car A (point A) relative to car B (that is as seen by an occupant fixed in car B).

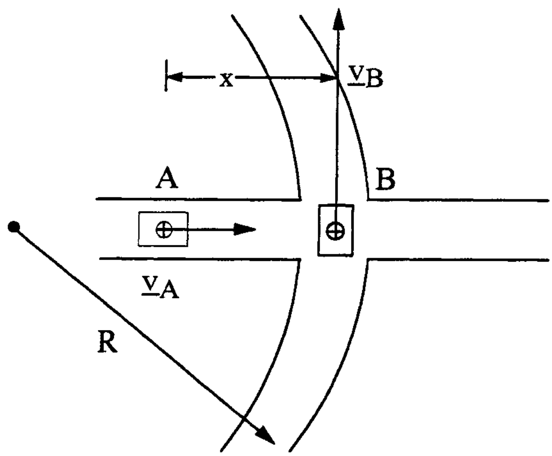


Fig. 2-15

To solve this problem, two reference frames are needed. The first, $\{\hat{I}, \hat{J}\}$, is fixed in the ground with origin at the instantaneous location of car B (point B), and $\hat{J}$ along car B's velocity relative to the ground. This frame is needed because we are given the problem's data relative to this frame (Fig. 2-16). The second frame, $\{\hat{\imath}, \hat{\jmath}\}$, is attached to car B with origin at point B; this frame is needed because the answers to the problem are requested relative to this frame. Note that at the instant of interest, both frames are in identical positions; this does not mean they

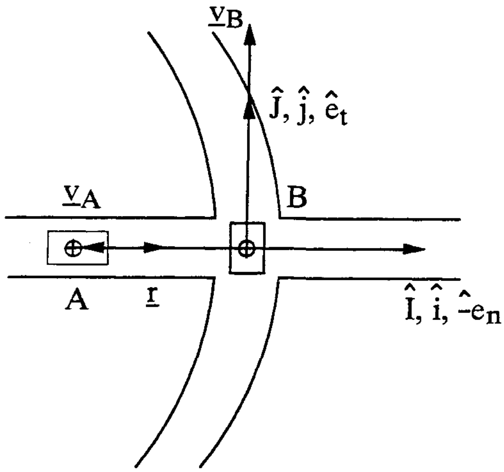


Fig. 2-16

are identical frames, however, because an instant later, frame $\{\hat{\imath}, \hat{\jmath}\}$ will have moved relative to the ground (and thus to frame $\{\hat{I}, \hat{J}\}$).

Referring to Fig. 2-16, we can write the following:

$$\begin{aligned}
\underline{\nu}_A &= 90\hat{I}\frac{Km}{h} = 25\hat{I}\frac{m}{s} = 25\hat{\imath}\frac{m}{s} \\
\underline{\nu}_B &= 72\hat{J}\frac{Km}{h} = 20\hat{J}\frac{m}{s} = 20\hat{\jmath}\frac{m}{s} \\
\underline{r} &= -30\hat{I}m = -30\hat{\imath}m \\
\underline{\omega} &= \frac{\nu_B}{R}\hat{K} = \frac{20}{200}\hat{K} = \frac{1}{10}\hat{K}\frac{rad}{s} = \frac{1}{10}\hat{k}\frac{rad}{s}
\end{aligned}$$

This last expression follows from the fact that the rate of rotation ω for motion at a speed ν_B on a circle of radius R is $\omega = \nu_B/R$, see Eqn. (2.12). Note that we can resolve vectors in either reference frame and get identical results; we choose the $\{\hat{\imath}, \hat{\jmath}\}$ frame. Solving Eqn. (2.24) for $\underline{\nu}_r$, the quantity of interest:

$$\begin{aligned}
\underline{\nu}_r &= \underline{\nu}_A - \underline{\nu}_B - \underline{\omega} \times \underline{r} = 25\hat{\imath} - 20\hat{\jmath} - \left(\frac{1}{10}\hat{k}\right) \times \left(-30\hat{\imath}\right) \\
&= \left(25\hat{\imath} - 17\hat{\jmath}\right)\frac{m}{s}
\end{aligned}$$

Now let's interpret the three terms in Eqn. (2.24) for this problem. The motion of car A relative to car B is composed of three separate motions: the translation of car A, the translation of B, and the turning of B. To see their effects, we "turn-off" two of these at a time. Parking car B at the intersection leaves the motion of car A only, which is clearly in the $+\hat{\imath}$ direction at A's speed relative to the occupants of B; this is the $\underline{\nu}_A$ term. Next park car A and let car B go straight through the intersection without turning; clearly, the occupants of B will see A going in the $-\hat{\jmath}$ direction with car B's speed (the $\underline{\nu}_B$ term). To visualize the $\underline{\omega} \times \underline{r}$ term, park car A and place B on a turntable at the intersection that rotates with angular speed ν_B/R. It should be clear that to car B, car A is moving in the $+\hat{\jmath}$ direction with speed $\nu_B x/R$.

An interesting question is: Does car A see car B moving with the negative of the velocity with which B sees A moving? To answer this question, we attach reference frame $\{\hat{\imath}, \hat{\jmath}\}$ to car A and $\{\hat{I}, \hat{J}\}$ to the ground at car A's instantaneous location. Now the relative velocity equation reads

$$\underline{\nu}_B = \underline{\nu}_r + \underline{\nu}_A + \underline{\omega} \times \underline{r}$$

so that

$$\underline{\nu}_r = \underline{\nu}_B - \underline{\nu}_A + \underline{\omega} \times \underline{r}$$

But now $\underline{\omega} = \underline{0}$ because $\{\hat{\imath}, \hat{\jmath}\}$ is not rotating with respect to $\{\hat{I}, \hat{J}\}$ and hence the answer to the question is no. This is easy to see because if both cars are parked and B is placed on a turntable, B will not move relative to A, but A will move relative to B.

Let's now return to the original problem and solve Eqn. (2.25) for the acceleration of car A relative to B:

$$\underline{a}_r = \underline{a}_A - \dot{\underline{\omega}} \times \underline{r} - 2\underline{\omega} \times \underline{\nu}_r - \underline{\omega} \times (\underline{\omega} \times \underline{r}) - \underline{a}_B$$

The new quantities in this equation are

$$\begin{aligned}
\underline{a}_A &= 2\hat{\imath}\frac{m}{s^2} \\
\underline{a}_B &= -\dot{\nu}_B\hat{\jmath} - \frac{\nu_B^2}{R}\hat{\imath} = (-2\hat{\imath} - 4\hat{\jmath})\frac{m}{s^2} \\
\dot{\underline{\omega}} &= \frac{\dot{\nu}_B}{R}\hat{k} = -0.02\hat{k}\frac{rad}{s^2}
\end{aligned}$$

The expression for $\underline{a}_B$ follows from Eqn. (2.14) with $\hat{e}_t = \hat{\jmath}$ and $\hat{e}_n = -\hat{\imath}$. Carrying out the computations:

$$\begin{aligned}\underline{a}_r &= 2\hat{\imath} - (-0.02\hat{k}) \times (-30\hat{\imath}) - 2\left(\frac{1}{10}\hat{k}\right) \times (25\hat{\imath} - 17\hat{\jmath}) \\ &\quad - \left(\frac{1}{10}\hat{k}\right) \times \left[\left(\frac{1}{10}\hat{k}\right) \times (-30\hat{\imath})\right] - (-2\hat{\imath} - 4\hat{\jmath}) \\ &= (0.3\hat{\imath} - 1.6\hat{\jmath})\frac{m}{s^2}\end{aligned}$$

Notes

1 A dot will frequently denote a time derivative, for example $\dot{x} = dx/dt$.

2 Note that this θ differs from that used earlier.

3 Establishing this property for 3-D motion will be much more difficult.

4 See also Appendix A.

5 Alternatively, a boom-fixed frame aligned with frame $\{\hat{I}, \hat{J}\}$ at the instant of interest could be used.

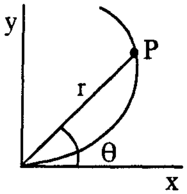
Problems

(Note: In the following problems, where appropriate, assume a flat earth, constant gravitational acceleration g in the vertical direction of 32.2 ft/s^2 or 9.81 m/s^2, and no air resistance.)

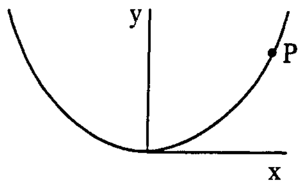
2/1 A point moves at a constant speed of 5 ft/s along a path given by $y = 10e^{-2x}$, where x and y are in ft. Find the acceleration of the point when $x = 2$ ft.

2/2 A point moves at constant speed ν along a curve defined by $r = A\theta$, where A is a constant. Find the normal and tangential components of acceleration.

2/3 A point travels along a parabola $y = kx^2$, k a constant, such that the horizontal component of velocity, $\dot{x}$, remains a constant. Determine the acceleration of the point as a function of position.

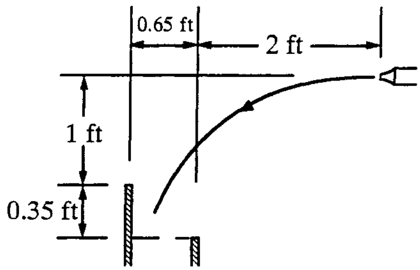


Problem 2/2



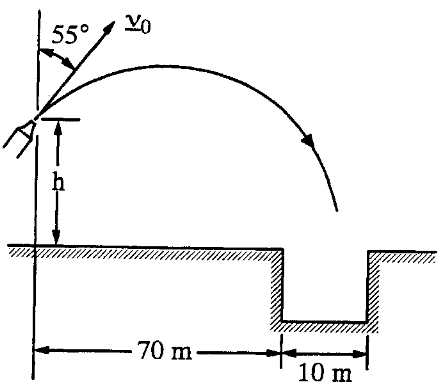
Problem 2/3

2/4 Grain is being discharged from a nozzle into a vertical shute with an initial horizontal velocity $\underline{v}_0$. Determine the range of values of v_0 for which the grain will enter the shute.



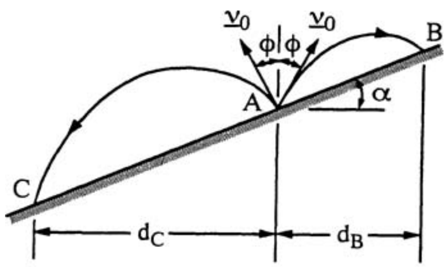
Problem 2/4

2/5 A nozzle is located at a height h above the ground and discharges water at a speed $v_0 = 25$ m/s at an angle of 55° with the vertical. Determine the range of values of h for which the water enters the trench in the ground.



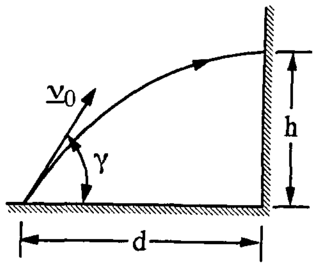
Problem 2/5

2/6 A rotating water sprinkler is positioned at point A on a lawn inclined at an angle $\alpha = 10°$ relative to the horizontal. The water is discharged with a speed of $\nu_0 = 8$ ft/s at an angle of $\phi = 40°$ to the vertical. Determine the horizontal distances d_C and d_B where the water lands.



Problem 2/6

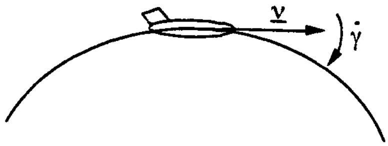
2/7 A ball is thrown with velocity $\underline{\nu}_0$ against a vertical wall a distance d away. Determine the maximum height h at which the ball can strike the wall and the corresponding angle γ, in terms of ν_0, d, and g.



Problem 2/7

2/8 In Problem 2/7, is the ball ascending or descending when it strikes the wall? What minimum speed ν_0 is needed to strike the wall at all?

2/9 A condition of "weightlessness" may be obtained by an airplane flying a curved path in the vertical plane as shown. If the plane's speed is $\nu = 800$ km/h, what must be the rate of rotation of the airplane $\dot{\gamma}$ to obtain this condition at the top of its loop?

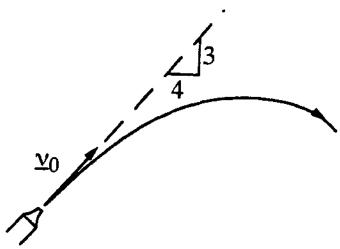


Problem 2/9

2/10 The speed of a car is increasing at a constant rate from 60 mi/h to 75 mi/h over a distance of 600 ft along a curve of 800 ft radius. What is the magnitude of the total acceleration of the car after it has traveled 400 ft along the turn?

2/11 Consider again the situation of Problem 2/5. Determine the radius of curvature of the stream both as it leaves the nozzle and at its maximum height.

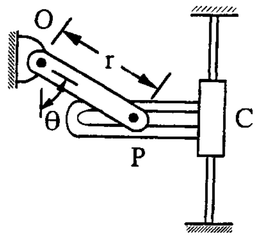
2/12 Consider again the situation of Section 2.2. It was observed that the radius of curvature of the stream of water as it left the nozzle was 35 ft. Find the speed ν_0 with which the water left the nozzle, and the radius of curvature of the stream when it reaches its maximum height.



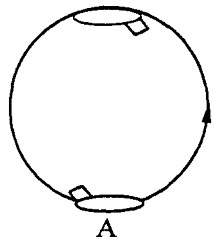
Problem 2/12

2/13 The velocity of a point at a certain instant is $\underline{\nu} = 3\hat{\imath} + 4\hat{\jmath}$ ft/s, and the radius of curvature of its path is 6.25 ft. The speed of the point is decreasing at the rate of 2 ft/s^2. Express the velocity and acceleration of the point in tangential-normal components.

2/14 Link OP rotates about O, and pin P slides in the slot attached to collar C. Determine the velocity and acceleration of collar C as a function of θ for the following cases:
(i) $\dot{\theta} = \omega$ and $\ddot{\theta} = 0$,
(ii) $\dot{\theta} = 0$ and $\ddot{\theta} = \alpha$.



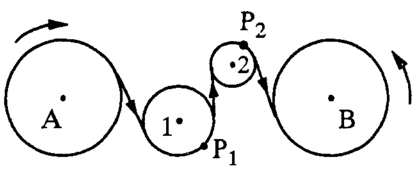
Problem 2/14



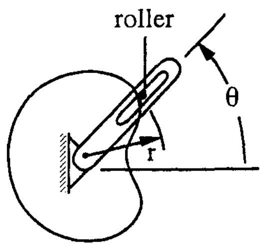
Problem 2/15

2/15 At the bottom A of a vertical inside loop, the magnitude of the total acceleration of the airplane is 3g. If the airspeed is 800 mph and is increasing at the rate of 20 mph per second, determine the radius of curvature of the path at A.

2/16 Tape is being transferred from drum A to drum B via two pulleys. The radius of pulley 1 is 1.0 in and that of pulley 2 is 0.5 in. At P_1, a point on pulley 1, the normal component of acceleration is 4 in/s^2 and at P_2, a point on pulley 2, the tangential component of acceleration is 3 in/s^2. At this instant, compute the speed of the tape, the magnitude of the total acceleration at P_1, and the magnitude of the total acceleration at P_2.



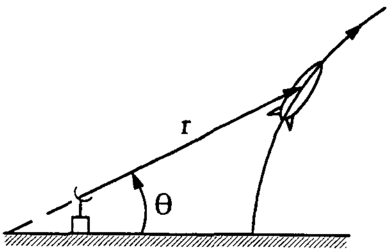
Problem 2/16



Problem 2/17

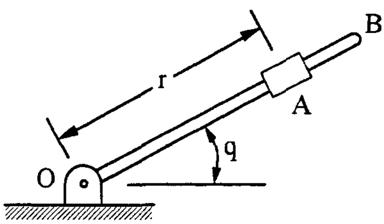
2/17 The shape of the stationary cam is that of a limacon, defined by $r = b - c\cos\theta$, $b > c$. Determine the magnitude of the total acceleration as a function of θ if the slotted arm rotates with a constant angular rate $\omega = \dot{\theta}$ in the counter clockwise direction.

2/18 A radar used to track rocket launches is capable of measuring r, $\dot{r}$, $\dot{\theta}$, and $\ddot{\theta}$. The radar is in the vertical plane of the rocket's flight path. At a certain time, the measurements of a rocket are $r = 35,000$ m, $\dot{r} = 1600$ m/s, $\dot{\theta} = 0$, and $\ddot{\theta} = -0.0072$ rad/s^2. What direction is the rocket heading relative to the radar at this time? What is the radius of curvature of its path?

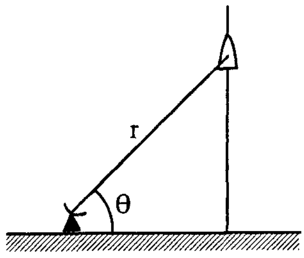


Problem 2/18

2/19 A collar A slides on a thin rod OB such that $r = 60t^2 - 20t^3$, with r in meters and t in seconds. The rod rotates according to $\theta = 2t^2$, with θ in radians. Determine the velocity and total acceleration of the collar when $t = 1s$, using radial-transverse components.



Problem 2/19



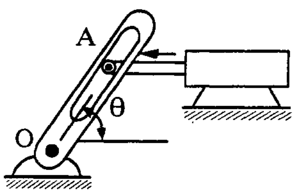
Problem 2/21

2/20 Consider the same situation as in Problem 2/19, but with $r = 1.25t^2 - 0.9t^3$ and $\theta = \frac{1}{2}\pi(4t - 3t^3)$. Answer the same questions.

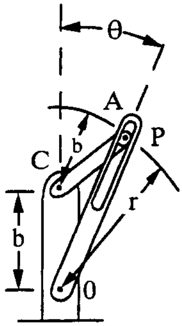
2/21 A vertically ascending rocket is tracked by radar as shown. When $\theta = 60°$, measurements give $r = 30,000$ ft, $\ddot{r} = 70$ ft/s^2, and $\dot{\theta} = 0.02$ rad/s. Determine the magnitudes of the velocity and the acceleration of the rocket at this instant.

2/22 The path of fluid particles in a certain centrifugal pump is closely approximated by $r = r_0 e^{n\theta}$ where r_0 and n are constants. If the pump turns at a constant rate $\dot{\theta} = \omega$, determine the expression for the magnitude of the acceleration of a fluid particle when $r = R$.

2/23 The pin A at the end of the piston of the hydraulic cylinder has a constant speed 3 m/s in the direction shown. For the instant when $\theta = 60°$, determine $\dot{r}, \ddot{r}, \dot{\theta}$ and $\ddot{\theta}$, where $r = \overline{OA}$.



Problem 2/23



Problem 2/24

2/24 Slotted arm OA oscillates about O and drives crank CP via the pin at P. For an interval of time, $\dot{\theta} = \omega =$ constant. During this time, determine the magnitude of the acceleration of P as a function of θ. Also, show that the magnitudes of the velocity and acceleration of P are constant during this time interval. Solve this problem using radial-transverse components.

2/25 Solve problem 2/24 using the relative velocity and acceleration equations.

(The following four problems all concern the truck with the telescoping, rotating boom of Sections 2.7 and 2.11. The data is the same except as indicated. It is desired to find the velocity and acceleration of point B at the end of the boom relative to the ground. Work all these problems using the relative velocity and acceleration equations.)

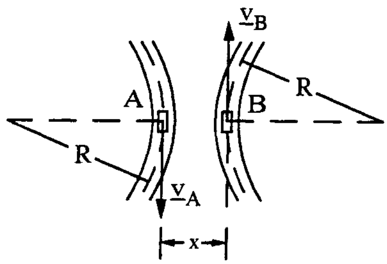
2/26 The rate of increase of the length of the boom is increasing at the rate of 0.5 in/s^2.

2/27 The rate of increase of θ is decreasing at the rate of 0.01 rad/s^2.

2/28 The truck is moving straight ahead at a constant speed of 3 ft/s.

2/29 The truck is moving straight ahead at a speed of 3 ft/s and is accelerating at a rate of 0.5 ft/s^2.

2/30 Two cars, labeled A and B, are traveling on curves with constant equal speeds of 72 km/hr. The curves both have radius $R = 100$m and their point of closest approach is $x = 30$m. Find the velocity of B relative to the occupants of A at the point of closest approach.

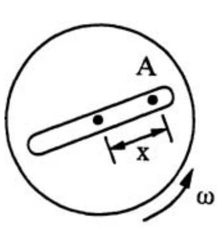


Problem 2/30

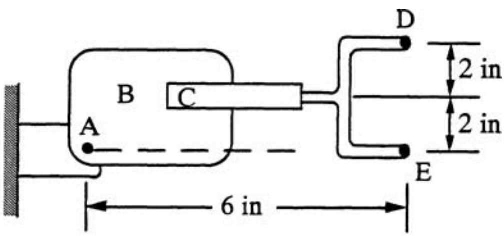
2/31 For the same conditions of Problem 2/30, find the acceleration of B relative to A.

2/32 For the same conditions of Problem 2/30, find the acceleration of B relative to A if A is speeding up at the rate of 3 m/s^2 and B is slowing down at the rate of 6 m/s^2.

2/33 At a certain instant, the disk is rotating with an angular speed of $\omega = 15$ rad/s and the speed is increasing at a rate of 20 rad/s^2. The slider moves in the slot in the disk at the constant rate $\dot{x} = 120$ in/s and at the same instant is at the center of the disk. Obtain the acceleration of the slider at this instant using both Eqn. (2.18) and Eqn. (2.25).



Problem 2/33

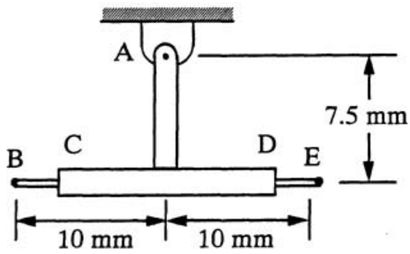


Problem 2/34

2/34 Shown is an automated welding device. Plate B rotates about point A, and the welding bracket with tips D and E moves in a cylinder C attached to B. At a certain instant, bracket DE is moving to the right with respect to plate B at a constant rate of 3 in/s and B is rotating counter clockwise about A at a constant rate of 1.6 rad/s. Determine the velocity and acceleration of tip E at that instant.

2/35 For the same situation as in Problem 2/34, determine the velocity and acceleration of tip D.

2/36 Bracket ACD is rotating clockwise about A at the constant rate of 2.4 rad/s. When in the position shown, rod BE is moving to the right relative to the bracket at the constant rate of 15 mm/s. Find the velocity and acceleration of point B.



Problem 2/36

2/37 Same as Problem 2/36, except that the rotation of the bracket is speeding up at a rate of 0.3 rad/s^2.

2/38 Same as Problem 2/36, except that the rod is slowing down at the rate of 2 mm/s^2.

2/39 Find the velocity and acceleration of point E for the situation in Problem 2/36.

Chapter 3

Coordinate Systems, Components, and Transformations

3.1 Rectangular Coordinates and Components

Consider again, as in Section 1.2, a point P moving on a curve C in 3-D Euclidean space (Fig. 1-2). At time t, the point has position vector $\underline{r}$ relative to a reference frame with unit vectors $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$. Recall that the velocity and acceleration vectors of the point P are defined by Eqns. (1.2) and (1.3), respectively.

The position vector may be resolved into components along $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$:

$$\underline{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} \tag{3.1}$$

This expresses $\underline{r}$ as the sum of three vectors (Fig. 3-1); (x, y, z) are called the *rectangular components*, or coordinates, of $\underline{r}$.

Differentiating Eqn. (3.1) gives the velocity and acceleration of point P in rectangular components:

$$\boxed{\underline{v} = \frac{d\underline{r}}{dt} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath} + \dot{z}\hat{k}} \tag{3.2}$$

$$\boxed{\underline{a} = \frac{d\underline{v}}{dt} = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath} + \ddot{z}\hat{k}} \tag{3.3}$$

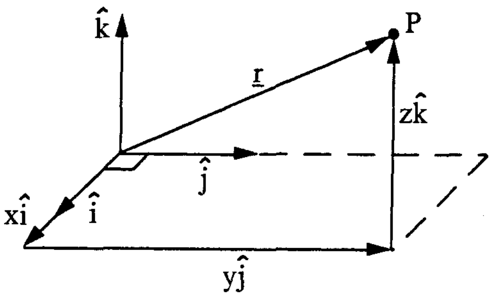


Fig. 3-1

As an example, suppose a point moves with constant speed ν along a curve given by $x = e^r$, $y = \sin r$, and $z = r^2$ where r is a known function of time. Then the position vector of the point at any time is

$$\underline{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} = e^r\hat{\imath} + \sin r\hat{\jmath} + r^2\hat{k}$$

and from Eq. (3.2) the velocity is

$$\underline{\nu} = \left(e^r\hat{\imath} + \cos r\hat{\jmath} + 2r\hat{k}\right)\frac{dr}{dt}$$

The speed of the point is therefore

$$\nu = \sqrt{e^{2r} + \cos^2 r + 4r^2}\,\frac{dr}{dt}$$

so that we may write

$$\underline{\nu} = \frac{\nu}{\sqrt{e^{2r} + \cos^2 r + 4r^2}}\left(e^r\hat{\imath} + \cos r\hat{\jmath} + 2r\hat{k}\right)$$

In a similar way, the acceleration may be found from Eqn. (3.3) and written in terms of r and ν.

Just as in 2-D motion, it is frequently convenient to use coordinates and components other than rectangular for describing 3-D motion. In the next few sections we will develop the most widely used 3-D coordinate systems and express the velocity and acceleration vectors in terms of these coordinates. We will also give a general method for deriving relations for other coordinate systems.

3.2 Intrinsic Components

Intrinsic components are a generalization to 3-D of the 2-D tangential-normal components discussed in Section 2.3.

Let a point move from position P to position P' in time Δt along curve C (Fig. 3-2). From Eqn. (1.2), we know that the velocity vector $\underline{\nu}$ is tangent to C at P. Now introduce a unit vector $\hat{e}_t$ collinear with $\underline{\nu}$ (and hence also tangent to C at P). Our goal is to define two more unit vectors, say $\hat{e}_n$ and $\hat{e}_b$, that form a right-hand orthogonal unit vector triad with $\hat{e}_t$.

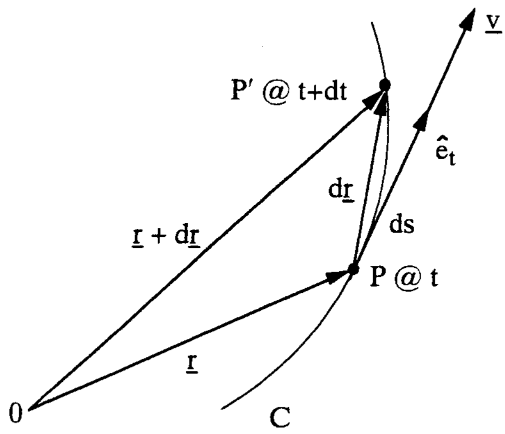


Fig. 3-2

If s is arc length measured along C, then as $dt \to 0$, $ds \to |d\underline{r}|$ so that

$$d\underline{r} = |d\underline{r}|\hat{e}_t = ds\ \hat{e}_t$$

and thus

$$\underline{\nu} = \frac{d\underline{r}}{dt} = \frac{ds}{dt}\hat{e}_t = \dot{s}\hat{e}_t \tag{3.4}$$

where $\dot{s} = \nu = |\underline{\nu}|$. Now define

$$\hat{e}_n = \frac{1}{f}\frac{d\hat{e}_t}{ds} \tag{3.5}$$

where

$$\boxed{f = \left|\frac{d\hat{e}_t}{ds}\right|} \tag{3.6}$$

is called the *flexure* (of curve C at P). We next show that $\hat{e}_n$ is a unit vector normal to $\hat{e}_t$. The magnitude of $\hat{e}_n$ is

$$|\hat{e}_n| = \frac{1}{f}\left|\frac{d\hat{e}_t}{ds}\right| = 1$$

so that $\hat{e}_n$ is a unit vector.

Consider the product $\hat{e}_t \cdot \hat{e}_t$:

$$\begin{aligned}
&\hat{e}_t \cdot \hat{e}_t = 1 \\
&\frac{d}{ds}\left(\hat{e}_t \cdot \hat{e}_t\right) = 0 \\
&2f\hat{e}_n \cdot \hat{e}_t = 0
\end{aligned}$$

Thus $\hat{e}_n$ is a unit vector normal to $\hat{e}_t$. The third unit vector may now be defined as

$$\hat{e}_b = \hat{e}_t \times \hat{e}_n \tag{3.7}$$

The vectors $\{\hat{e}_t, \hat{e}_n, \hat{e}_b\}$ are called the *Frenet Triad.* These vectors also satisfy the relations:

$$\begin{aligned}
\hat{e}_n &= \hat{e}_b \times \hat{e}_t \\
\hat{e}_t &= \hat{e}_n \times \hat{e}_b
\end{aligned} \tag{3.8}$$

To show these, we need to use the triple vector product relation, Eqn. A.16. Thus, for example,

$$\hat{e}_b \times \hat{e}_t = \left(\hat{e}_t \times \hat{e}_n\right) \times \hat{e}_t = \left(\hat{e}_t \cdot \hat{e}_t\right)\hat{e}_n - \left(\hat{e}_t \cdot \hat{e}_n\right)\hat{e}_t = \hat{e}_n$$

Next we develop a set of differential equations that give the evolution of the unit vectors $\{\hat{e}_t, \hat{e}_n, \hat{e}_b\}$ along the curve C. Consider the derivatives of $\hat{e}_t$, $\hat{e}_n$, and $\hat{e}_b$ with respect to path length and use Eqns. (3.5), (3.7), and (3.8):

$$\begin{aligned}
\hat{e}_t' &= \frac{d\hat{e}_t}{ds} = \frac{d}{ds}(\hat{e}_n \times \hat{e}_b) = \hat{e}_n' \times \hat{e}_b + \hat{e}_n \times \hat{e}_b' \\
\hat{e}_n' &= \hat{e}_b' \times \hat{e}_t + \hat{e}_b \times \hat{e}_t' = -\hat{e}_t \times \hat{e}_b' - f\hat{e}_n \times \hat{e}_b \\
\hat{e}_b' &= \hat{e}_t' \times \hat{e}_n + \hat{e}_t \times \hat{e}_n' = f\hat{e}_n \times \hat{e}_n + \hat{e}_t \times \hat{e}_n = \hat{e}_t \times \hat{e}_n
\end{aligned}$$

Now consider

$$
\begin{aligned}
0 &= f\hat{e}_n \times \hat{e}_n = \hat{e}_n \times \hat{e}_t' = \hat{e}_n \times (\hat{e}_n' \times \hat{e}_b + \hat{e}_n \times \hat{e}_b') \\
&= \hat{e}_n \times (\hat{e}_n' \times \hat{e}_b) + \hat{e}_n \times (\hat{e}_n \times \hat{e}_b') \\
&= (\hat{e}_n \cdot \hat{e}_b)\hat{e}_n' - (\hat{e}_n \cdot \hat{e}_n')\hat{e}_b + (\hat{e}_n \cdot \hat{e}_b')\hat{e}_n - (\hat{e}_n \cdot \hat{e}_n)\hat{e}_b' \\
&= (\hat{e}_n \cdot \hat{e}_b')\hat{e}_n - \hat{e}_b'
\end{aligned}
$$

so that

$$
\hat{e}_b' = (\hat{e}_n \cdot \hat{e}_b')\hat{e}_n = \tau\hat{e}_n \tag{3.9}
$$

where

$$
\boxed{\tau = \hat{e}_n \cdot \hat{e}_b'} \tag{3.10}
$$

is called the *torsion* (of curve C at P). Then

$$
\hat{e}_n' = -\hat{e}_t \times \hat{e}_b' - f\hat{e}_n \times \hat{e}_b = -\hat{e}_t \times \tau\hat{e}_n - f\hat{e}_t = -\tau\hat{e}_b - f\hat{e}_t
$$

To summarize, we have the *Frenet Formulas*

$$
\boxed{\begin{aligned}
\frac{d\hat{e}_t}{ds} &= f\hat{e}_n \\
\frac{d\hat{e}_b}{ds} &= \tau\hat{e}_n \\
\frac{d\hat{e}_n}{ds} &= -\tau\hat{e}_b - f\hat{e}_t
\end{aligned}} \tag{3.11}
$$

These are vector differential equations for the evolution of the unit vectors. For a given curve C, it is clear that $\hat{e}_t$, $\hat{e}_n$, $\hat{e}_b$, f, and τ are functions of position along the curve (Fig. 3-3) and for this reason they are called *intrinsic* quantities.

We are now in position to write velocity and acceleration in intrinsic components. The velocity is

$$
\boxed{\underset{\sim}{\nu} = \nu\hat{e}_t} \tag{3.12}
$$

and acceleration is found by differentiating Eqn. (3.12) and using Eqns. (3.11):

$$
\underset{\sim}{a} = \dot{\nu}\hat{e}_t + \nu\dot{\hat{e}}_t = \dot{\nu}\hat{e}_t + \nu\,\frac{d\hat{e}_t}{ds}\,\frac{ds}{dt}
$$

$$
\boxed{\underset{\sim}{a} = \dot{\nu}\hat{e}_t + \nu^2 f\hat{e}_n} \tag{3.13}
$$

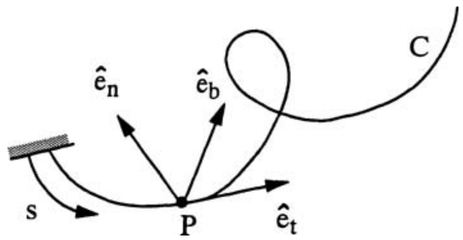


Fig. 3-3

Thus $\underline{a}$ lies in the $(\hat{e}_t, \hat{e}_n)$ plane; this is called the *osculating plane.*

As a special case, consider plane motion, that is motion restricted to the $(\hat{e}_t, \hat{e}_n)$ plane. Then

$$\hat{e}_b = \hat{e}_t \times \hat{e}_n = \underline{\text{constant}} = \hat{k}$$

and

$$\frac{d\hat{e}_b}{dt} = \frac{d\hat{e}_b}{ds}\frac{ds}{dt} = \nu\tau\hat{e}_n = \underline{0}$$

Thus $\tau = 0$ and torsion is zero for plane curves. Recall that for plane motion, acceleration expressed in tangential-normal components is (Eqn. (2.14)):

$$\underline{a}_{\text{plane}} = \dot{\nu}\hat{e}_t + \frac{\nu^2}{\rho}\hat{e}_n$$

Comparing this with Eqn. (3.13), we see that $f = 1/\rho$ and the flexure is the reciprocal of the curvature in the osculating plane. This gives an interpretation of f and τ; f is a measure of the amount of "bending" in the curve and τ is a measure of the "twisting" out-of-plane.

To gain a better understanding of flexure and torsion, consider a helical spring as shown in Fig. 3-4. First, suppose the coils of the spring are pushed flat so that they all lie in $\{\hat{\imath}, \hat{\jmath}\}$ plane as shown in Fig. 3-5. Then $\hat{e}_b (= \hat{k})$ is a constant vector, and from Eqn. (3.10) the torsion τ is zero. Also, from Eqn. (3.5), $d\hat{e}_t = f\hat{e}_n ds$ so that $\hat{e}_n d\theta = \hat{e}_n f R d\theta$ and consequently the flexure f is equal to the reciprocal of R, as just shown above.

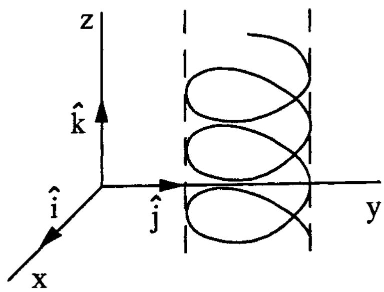


Fig. 3-4

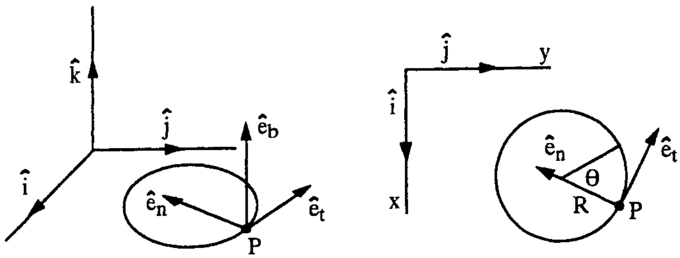


Fig. 3-5

As the spring is extended, flexure decreases and torsion increases. In the limit as the spring is extended to a vertical wire (Fig. 3-6), $\hat{e}_t (= \hat{k})$ is a constant and thus from Eqn. (3.6) $f = 0$. Also, since $\hat{e}_n$ and $\hat{e}_b$ are not defined in this limit, the torsion becomes infinite.

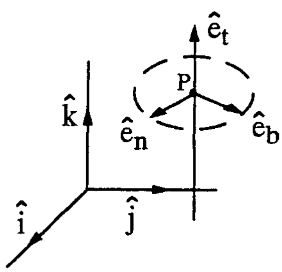


Fig. 3-6

3.3 Example

A flexible cable is in equilibrium in a horizontal plane under action of known external forces and tension (Fig. 3-7). We want the equations defining the shape of the cable.

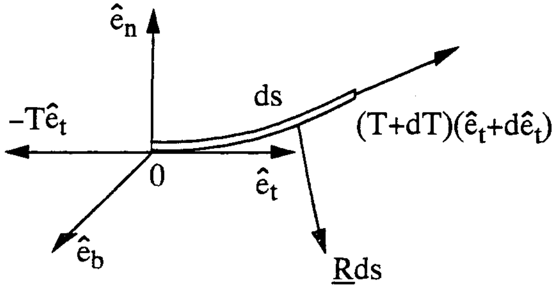


Fig. 3-7

Consider an infinitesimal element of the cable. Place unit vectors such that the cable is in the $(\hat{e}_n, \hat{e}_t)$ plane; one end is at 0 and the force at this end is along the $\hat{e}_t$ direction (tangent to the cable). Let $\underline{R}$ be the external known force per unit length (for example the weight of the cable) and let T be the tension in the left end. Resolve $\underline{R}$ into components along the unit vectors:

$$\underline{R} = R_1\hat{e}_t + R_2\hat{e}_n + R_3\hat{e}_b$$

For equilibrium, the sum of the forces must be zero; using Eqn. (3.5):

$$-T\hat{e}_t + (T+dT)(\hat{e}_t + f\,ds\hat{e}_n) + (R_1\hat{e}_t + R_2\hat{e}_n + R_3\hat{e}_b)\,ds = \underline{0}$$

The component equations of this vector equation are, ignoring the second order $dT\ ds$ term,

$$dT + R_1 ds = 0$$
$$Tf ds + R_2 ds = 0$$
$$R_3 ds = 0$$

which imply

$$\frac{dT}{ds} + R_1 = 0$$
$$Tf + R_2 = 0$$
$$R_3 = 0$$

The interpretation of these three equations is as follows: The first is a differential equation for the change of tension along s; the second gives the relation between tension and curvature; and the third gives the condition that is necessary to keep the cable in a plane.

3.4 General Approach to Coordinate Systems and Components

The choice of a coordinate system and the corresponding component representation of the position, velocity, and acceleration vectors is a decision of the analyst. The criterion in this selection should be convenience and minimization of algebraic manipulation. Indeed, an unwise choice of coordinate system in 3-D problems can increase the work required to obtain a solution several-fold.

Because of the importance of coordinate system selection, and because the best choice is problem dependent, in this section we will give a general method of coordinate selection and resolution of $\underline{r}$, $\underline{v}$, and $\underline{a}$. In the following two sections, we will apply this method to obtain the two most useful systems (other than rectangular) – cylindrical and spherical.[1] The geometry of the motion, the form of the problem data, and the desired form of the solution should all be taken into account when selecting a coordinate system.

The general procedure is as follows:

1. Choose coordinates $x_i = (x_1, x_2, x_3)$ and corresponding unit vectors directions $\hat{e}_i = (\hat{e}_1, \hat{e}_2, \hat{e}_3)$. The three coordinates x_i may be any combination of three lengths or angles that fully and unambiguously specify the position of a point in 3-D space relative to a given reference frame.

2. Form the functions $x(x_i)$, $y(x_i)$, $z(x_i)$ where (x, y, z) are the rectangular coordinates of the point.

3. Form the functions $\hat{\imath}(\hat{e}_i)$, $\hat{\jmath}(\hat{e}_i)$, $\hat{k}(\hat{e}_i)$ where $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ are the rectangular unit vectors.

4. Differentiate the functions in step (2) to get $\dot{x}(x_i)$, $\dot{y}(x_i)$, $\dot{z}(x_i)$ and $\ddot{x}(x_i)$, $\ddot{y}(x_i)$, $\ddot{z}(x_i)$.

5. Substitute the results of steps (3) and (4) into Eqns. (3.2) and (3.3) to get the components of $\underline{\nu}$ and $\underline{a}$ associated with the new coordinate system:

$$\underline{\nu} = \nu_1 \hat{e}_1 + \nu_2 \hat{e}_2 + \nu_3 \hat{e}_3 \tag{3.14}$$

$$\underline{a} = a_1 \hat{e}_1 + a_2 \hat{e}_2 + a_3 \hat{e}_3 \tag{3.15}$$

The unit vectors $\{\hat{e}_1,\ \hat{e}_2,\ \hat{e}_3\}$ must be linearly independent and usually are restricted to be right-hand orthogonal. If this is the case, the magnitudes of $\underline{\nu}$ and $\underline{a}$ are given by

$$\boxed{\nu = |\underline{\nu}| = \sqrt{\nu_1^2 + \nu_2^2 + \nu_3^2}} \tag{3.16}$$

$$\boxed{a = |\underline{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}} \tag{3.17}$$

and are the same in any coordinate system.

3.5 Cylindrical Coordinates and Components

Fig. 3-8 shows the position of a point P relative to a reference frame $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$. The cylindrical coordinates of P are defined as follows. Drop a line from P parallel to $\hat{k}$ until it intersects the $\{\hat{\imath}, \hat{\jmath}\}$ plane; call this point Q. Let $r = |\overrightarrow{OQ}|$ and let θ be the angle between $\hat{\imath}$ and $\overrightarrow{OQ}$. The three numbers (r, θ, z) are then the *cylindrical coordinates* of P; it is clear that these three numbers completely and unambiguously locate P relative to the $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ reference frame.

The natural choice of unit vectors associated with coordinates (r, θ, z) are $\{\hat{e}_r, \hat{e}_\theta, \hat{k}\}$ as shown on Fig. 3-8; this completes step (1) of the five-step procedure of the previous section. From the figure,

$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}} \tag{3.18}$$

which completes step (2). For step (3), we need the unit vector transformation from $\{\hat{e}_r, \hat{e}_\theta, \hat{k}\}$ to $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$. To this end, consider the view

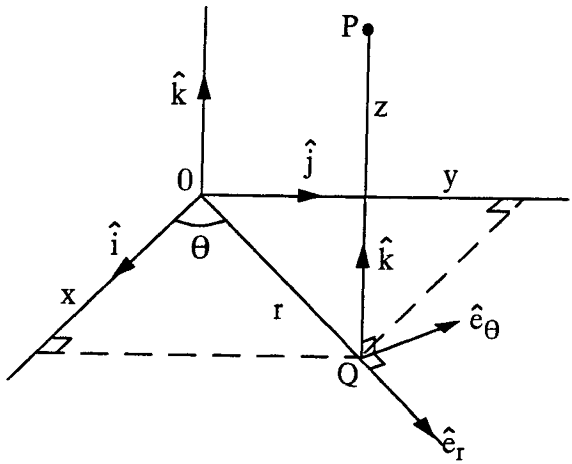


Fig. 3-8

down $\hat{k}$, as shown in Fig. 3-9; from this figure,

$$\begin{aligned} \hat{\imath} &= \cos\theta\hat{e}_r - \sin\theta\hat{e}_\theta \\ \hat{\jmath} &= \sin\theta\hat{e}_r + \cos\theta\hat{e}_\theta \\ \hat{k} &= \hat{k} \end{aligned} \tag{3.19}$$

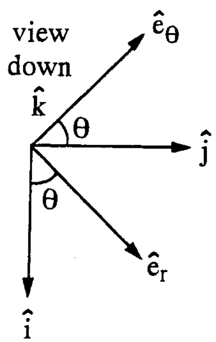


Fig. 3-9

For step (4), Eqns. (3.18) are differentiated:

$$\begin{aligned} \dot{x} &= \dot{r}\cos\theta - r\dot{\theta}\sin\theta \\ \dot{y} &= \dot{r}\sin\theta - r\dot{\theta}\cos\theta \\ \dot{z} &= \dot{z} \end{aligned} \tag{3.20}$$

$$\begin{aligned}
\ddot{x} &= \ddot{r}\cos\theta - 2\dot{r}\dot{\theta}\sin\theta - r\ddot{\theta}\sin\theta - r\dot{\theta}^2\cos\theta \\
\ddot{y} &= \ddot{r}\sin\theta + 2\dot{r}\dot{\theta}\cos\theta + r\ddot{\theta}\cos\theta - r\dot{\theta}^2\sin\theta \\
\ddot{z} &= \ddot{z}
\end{aligned} \tag{3.21}$$

Finally, Eqns. (3.19), (3.20) and (3.21) are substituted into Eqns. (3.2) and (3.3), and simplified:

$$\boxed{\underline{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta + \dot{z}\hat{k}} \tag{3.22}$$

$$\boxed{\underline{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{e}_\theta + \ddot{z}\hat{k}} \tag{3.23}$$

These equations give the velocity and acceleration vectors of point P relative to reference frame $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$, expressed in components along the cylindrical unit vectors $\{\hat{e}_r, \hat{e}_\theta, \hat{k}\}$. Comparing these equations with Eqns. (2.17) and (2.18) we see that the cylindrical components of $\underline{v}$ and $\underline{a}$ are the same as the radial-transverse components with an additional perpendicular dimension.

3.6 Example

Consider the screw-jack shown in Fig. 3-10. The jack starts from rest and increases its rotational speed linearly with time, $\omega = Kt$. The lead of the screw is L. It is desired to find the velocity and acceleration of point A, expressed in cylindrical coordinates, after the screw has turned one revolution from rest.

The cylindrical coordinates of A are (r, θ, z) as shown in Fig. 3-10. We first determine the time it takes to make one revolution (take $\theta = 0$ when $t = 0$):

$$\begin{aligned}
\dot{\theta} &= \omega = Kt \\
\theta &= \int Kt\,dt = \frac{1}{2}Kt^2 \\
\ddot{\theta} &= K
\end{aligned}$$

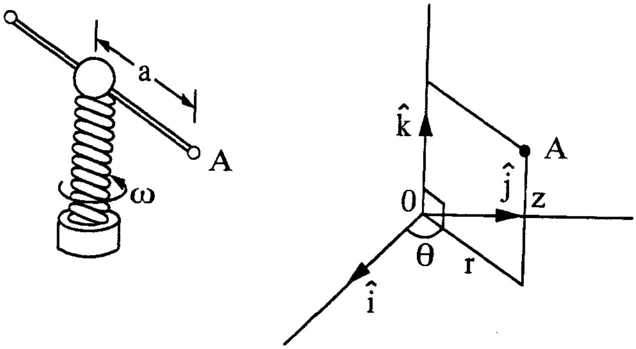


Fig. 3-10

After one revolution,

$$
\begin{aligned}
2\pi &= \frac{Kt^2}{2} \\
t &= 2\sqrt{\pi/K} \\
\dot{\theta} &= 2\sqrt{\pi K}
\end{aligned}
$$

Also,

$$r = a\,, \qquad \dot{r} = 0\,, \qquad \ddot{r} = 0$$

The lead of a screw is its advance per revolution, that is $z = L$ when $\theta = 2\pi$. Also, z is proportional to θ, say $z = c\theta$; evaluating after one revolution gives $c = L/2\pi$ and thus, after one revolution,

$$z = L\,, \qquad \dot{z} = L\sqrt{\frac{K}{\pi}}\,, \qquad \ddot{z} = \frac{LK}{2\pi}$$

We are now ready to compute the cylindrical coordinates, after one revolution.

The cylindrical components of velocity are (Eqn. (3.22)):

$$
\begin{aligned}
\nu_r &= \dot{r} = 0 \\
\nu_\theta &= r\dot{\theta} = 2a\sqrt{\pi K} \\
\nu_z &= \dot{z} = L\sqrt{K/\pi}
\end{aligned}
$$

so that

$$\underline{\nu} = 2a\sqrt{\pi K}\hat{e}_\theta + L\sqrt{K/\pi}\hat{k}$$

Similarly, from Eqn. (3.23), the components of acceleration are:

$$\begin{aligned} a_r &= \ddot{r} - r\dot{\theta}^2 = -4a\pi K \\ a_\theta &= r\ddot{\theta} + 2\dot{r}\dot{\theta} = aK \\ a_z &= \ddot{z} = LK/2\pi \end{aligned}$$

and therefore

$$\underline{a} = -4a\pi K\hat{e}_r + aK\hat{e}_\theta + \frac{LK}{2\pi}\hat{k}$$

The magnitudes of these vectors are found from Eqns. (3.16) and (3.17):

$$\nu = \sqrt{\nu_r^2 + \nu_\theta^2 + \nu_z^2} = \frac{K}{\pi}\sqrt{4a^2\pi^2 + L^2}$$

$$a = \sqrt{a_r^2 + a_\theta^2 + a_z^2} = aK\sqrt{1 + 16\pi^2 + \frac{L^2}{4\pi^2 a^2}}$$

3.7 Spherical Coordinates and Components

Consider again the position of a point P relative to $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$, Fig. 3-11. Again drop a line from P parallel to $\hat{k}$ until it intersects the $\{\hat{\imath}, \hat{\jmath}\}$ plane at Q. Now let $R = |\overrightarrow{OP}|$ and ϕ be the angle between $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ with r and θ defined as before. The *spherical coordinates* of P are (R, θ, ϕ).

The associated unit vectors are as follows: $\hat{e}_R$ is along $\overrightarrow{OP}$, $\hat{e}_\phi$ is perpendicular and in the POQ plane, and $\hat{e}_\theta$ is as before.

The coordinate transformation equations are

$$\boxed{\begin{aligned} x &= r\cos\theta = R\cos\phi\cos\theta \\ y &= r\sin\theta = R\cos\phi\sin\theta \\ z &= R\sin\phi \end{aligned}} \tag{3.24}$$

(The first two of these follow from $r = R\cos\phi$ and Eqns. (3.18).)

The unit vector transformation from $\{\hat{e}_R, \hat{e}_\theta, \hat{e}_\phi\}$ to $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ is best determined in two steps by introducing an intermediate unit vector $\hat{e}$

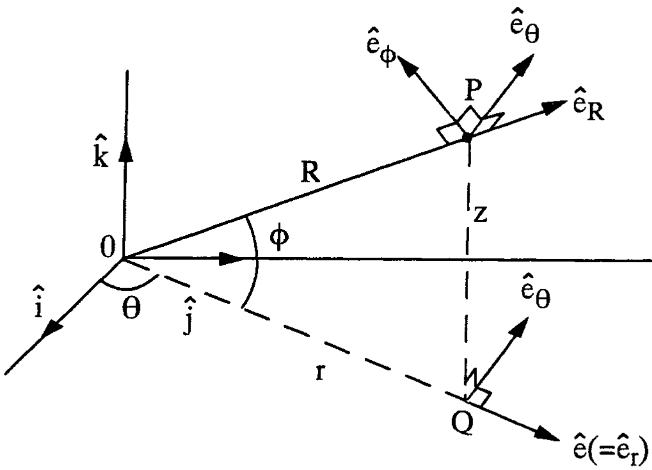


Fig. 3-11

(the same as unit vector $\hat{e}_r$ of cylindrical coordinates, Fig. 3-11). Then, referring to Fig. 3-12,

$$\hat{\imath} = \cos\theta\hat{e} - \sin\theta\hat{e}_\theta$$
$$\hat{\jmath} = \sin\theta\hat{e} + \cos\theta\hat{e}_\theta$$

and

$$\hat{e} = \cos\phi\hat{e}_R - \sin\phi\hat{e}_\phi$$
$$\hat{k} = \sin\phi\hat{e}_R + \cos\phi\hat{e}_\phi$$

so that

$$\begin{aligned}\hat{\imath} &= \cos\theta(\cos\phi\hat{e}_R - \sin\phi\hat{e}_\phi) - \sin\theta\hat{e}_\theta \\ \hat{\jmath} &= \sin\theta(\cos\phi\hat{e}_R - \sin\phi\hat{e}_\phi) + \cos\theta\hat{e}_\theta \\ \hat{k} &= \sin\phi\hat{e}_R + \cos\phi\hat{e}_\phi\end{aligned} \tag{3.25}$$

Differentiating Eqns. (3.24) and substituting the result along with Eqns. (3.25) into Eqns. (3.2) and (3.3) gives $\underline{v}$ and $\underline{a}$ resolved in spherical components:

$$\boxed{\underline{v} = \dot{R}\hat{e}_R + R\dot{\phi}\hat{e}_\phi + R\dot{\theta}\cos\phi\hat{e}_\theta} \tag{3.26}$$

$$\boxed{\begin{aligned}\underline{a} = &\left(\ddot{R} - R\dot{\phi}^2 - R\dot{\theta}^2\cos^2\phi\right)\hat{e}_R + \left[\frac{1}{R}\,\frac{d}{dt}\left(R^2\dot{\phi}\right) + R\dot{\theta}^2\right.\\ &\left.\sin\phi\cos\phi\right]\hat{e}_\phi + \left[\frac{\cos\phi}{R}\,\frac{d}{dt}\left(R^2\dot{\theta}\right) - 2R\dot{\phi}\dot{\theta}\sin\phi\right]\hat{e}_\theta\end{aligned}} \tag{3.27}$$

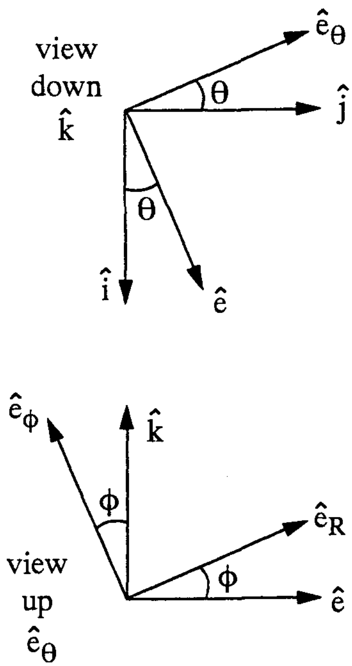


Fig. 3-12

3.8 Coordinate Transformations

It is often desirable to express a vector given in terms of one set of components in terms of another set of components. Any vector $\underline{Q}$ (and in particular $\underline{v}$ and $\underline{a}$) may be written in any set of components:

$$\underline{Q} = \underbrace{Q_x\hat{\imath} + Q_y\hat{\jmath} + Q_z\hat{k}}_{\text{(rectangular)}} \tag{3.28}$$

$$= \underbrace{Q_r\hat{e}_r + Q_\theta\hat{e}_\theta + Q_z\hat{k}}_{\text{(cylindrical)}} \tag{3.29}$$

$$= \underbrace{Q_R \hat{e}_R + Q_\theta \hat{e}_\theta + Q_\phi \hat{e}_\phi}_{\text{(spherical)}} \tag{3.30}$$

The question is, what is the relationship between these components?

As a first case, suppose we are given the rectangular components of $\underset{\sim}{Q}$, (Q_x, Q_y, Q_z), and we want to find the cylindrical components, (Q_r, Q_θ, Q_z). All we need are $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ in terms of $\hat{e}_r$, $\hat{e}_\theta$, $\hat{k}$ and these are just Eqns. (3.19). Substituting these into Eqn. (3.28):

$$\underset{\sim}{Q} = (Q_x \cos\theta + Q_y \sin\theta)\,\hat{e}_r + (-Q_x \sin\theta + Q_y \cos\theta)\,\hat{e}_\theta + Q_z \hat{k}$$

and thus

$$\begin{aligned} Q_r &= Q_x \cos\theta + Q_y \sin\theta \\ Q_\theta &= -Q_x \sin\theta + Q_y \cos\theta \\ Q_z &= Q_z \end{aligned} \tag{3.31}$$

These equations may be conveniently written in matrix notation:

$$\begin{pmatrix} Q_r \\ Q_\theta \\ Q_z \end{pmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} Q_x \\ Q_y \\ Q_z \end{pmatrix}$$

$$\boxed{(Q_{r\theta z}) = [T_\theta]\,(Q_{xyz})} \tag{3.32}$$

To continue further, we will need to know some basic rules of *matrix algebra*, which now will be briefly reviewed. Consider a linear system of algebraic equations:

$$\begin{aligned} y_1 &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = \sum_{i=1}^{3} a_{1i}x_i \\ y_2 &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = \sum_{i=1}^{3} a_{2i}x_i \\ y_3 &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = \sum_{i=1}^{3} a_{3i}x_i \end{aligned}$$

These may be written in the compact forms

$$\begin{aligned} y_j &= \sum_{i=1}^{3} a_{ji}x_i\,; \qquad j = 1, 2, 3 \\ (y) &= [A](x) \end{aligned}$$

where $[A]$ is a 3×3 matrix with elements

$$[A] = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

This explains the way to multiply a matrix times a vector

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Two matrices are multiplied together as follows:

$$[C] = [A][B] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} & a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} & a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33} \end{bmatrix}$$

$$c_{ij} = \sum_{k=1}^{3} a_{ik}b_{kj} \; ; \qquad i, j = 1, 2, 3$$

We will also need the *inverse* of a matrix; that is, given $[A]$, the matrix $[A]^{-1}$ such that

$$[A]^{-1}[A] = [I] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where $[I]$ is the *identity* (or *unit*) matrix and has the property such that if (x) is any vector then

$$[I](x) = (x)$$

Now consider the inverse of the case we considered earlier; that is: given (Q_r, Q_θ, Q_z), find (Q_x, Q_y, Q_z). This transformation is easily found by multiplying Eqn. (3.32) by $[T_\theta]^{-1}$:

$$[T_\theta]^{-1}(Q_{r\theta z}) = [T_\theta]^{-1}[T_\theta](Q_{xyz}) = [I](Q_{xyz}) = (Q_{xyz})$$

$$\boxed{(Q_{xyz}) = [T_\theta]^{-1}(Q_{r\theta z})} \tag{3.33}$$

where

$$[T_\theta] = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \; ; \quad [T_\theta]^{-1} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.34}$$

As another case, suppose (Q_x, Q_y, Q_z) are known and it is desired to find (Q_R, Q_θ, Q_ϕ). Substituting Eqn. (3.25) into Eqn. (3.28) gives:

$$\begin{aligned} Q_R &= Q_x \cos\theta\cos\phi + Q_y \sin\theta\cos\phi + Q_z \sin\phi \\ Q_\theta &= -Q_x \sin\theta + Q_y \cos\theta \\ Q_\phi &= -Q_x \cos\theta\sin\phi - Q_y \sin\theta\sin\phi + Q_z \cos\phi \end{aligned}$$

$$\begin{pmatrix} Q_R \\ Q_\theta \\ Q_\phi \end{pmatrix} = \begin{bmatrix} \cos\theta\cos\phi & \sin\theta\cos\phi & \sin\theta \\ -\sin\theta & \cos\theta & 0 \\ -\cos\theta\sin\phi & -\sin\theta\sin\phi & \cos\phi \end{bmatrix} \begin{pmatrix} Q_x \\ Q_y \\ Q_z \end{pmatrix}$$

$$\boxed{(Q_{R\theta\phi}) = [T_{\phi\theta}](Q_{xyz}) = [T_\phi][T_\theta](Q_{xyz})} \tag{3.35}$$

where

$$[T_{\phi\theta}] = [T_\phi][T_\theta] = \begin{bmatrix} \cos\theta\cos\phi & \sin\theta\cos\phi & \sin\phi \\ -\sin\theta & \cos\theta & 0 \\ -\cos\theta\sin\phi & -\sin\theta\sin\phi & \cos\phi \end{bmatrix} \tag{3.36}$$

$$[T_\phi] = [T_{\phi\theta}][T_\theta]^{-1} = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix} \tag{3.37}$$

As a final case, given (Q_r, Q_θ, Q_z) we find (Q_R, Q_θ, Q_ϕ). From Eqns. (3.32) and (3.35):

$$\boxed{(Q_{R\theta\phi}) = [T_\phi][T_\theta](Q_{xyz}) = [T_\phi](Q_{r\theta z})} \tag{3.38}$$

3.9 Examples

First, suppose it is desired to express the velocity and acceleration of point A in the example of Section 3.6 in rectangular components. This is accomplished by using Eqn. (3.33):

$$\begin{pmatrix} \nu_x \\ \nu_y \\ \nu_z \end{pmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 0 \\ 2a\sqrt{\pi K} \\ L\sqrt{K/\pi} \end{pmatrix}$$

$$\begin{aligned} \nu_x &= -2a\sqrt{\pi K}\sin\theta \\ \nu_y &= 2a\sqrt{\pi K}\cos\theta \\ \nu_z &= L\sqrt{K/\pi} \end{aligned}$$

$$\underline{\nu} = 2a\sqrt{\pi K}\left(-\sin\theta\hat{\imath} + \cos\theta\hat{\jmath} + \frac{L}{2a\pi}\hat{k}\right)$$

$$\begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} -4a\pi K \\ aK \\ LK/2\pi \end{pmatrix}$$

$$\begin{aligned} a_x &= -4a\pi K\cos\theta - aK\sin\theta \\ a_y &= -4a\pi K\sin\theta + aK\cos\theta \\ a_z &= LK/2\pi \\ \underline{a} &= aK\left[(-4\pi\cos\theta - \sin\theta)\hat{\imath} + (-4\pi\sin\theta + \cos\theta)\hat{\jmath} + \tfrac{L}{2\pi a}\hat{k}\right] \end{aligned}$$

As another example, suppose at a certain instant a point P has spherical coordinates (R, θ, ϕ) relative to a certain reference frame and the velocity of the point is parallel to the x-axis of the frame (Fig. 3-13). It is desired to find the spherical components of velocity. Since we know the rectangular components to be $(\nu, 0, 0)$, we could use Eqn. (3.35) directly. Instead, we will use Eqns. (3.32) and (3.38) sequentially, thereby also obtaining the cylindrical components as an intermediate step:

$$\begin{pmatrix} \nu_r \\ \nu_\theta \\ \nu_z \end{pmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} \nu \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \nu\cos\theta \\ -\nu\sin\theta \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \nu_R \\ \nu_\theta \\ \nu_\phi \end{pmatrix} = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix} \begin{pmatrix} \nu\cos\theta \\ -\nu\sin\theta \\ 0 \end{pmatrix} = \begin{pmatrix} \nu\cos\theta\cos\phi \\ -\nu\sin\theta \\ -\nu\cos\theta\sin\phi \end{pmatrix}$$

$$\underline{\nu} = \nu\cos\theta\cos\phi\hat{e}_R - \nu\sin\theta\hat{e}_\theta - \nu\cos\theta\sin\phi\hat{e}_\phi$$

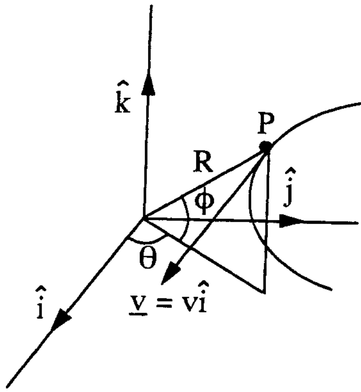


Fig. 3-13

Notes

1 The intrinsic components described in Section 3.2 are not often the best choice in 3-D problems.

Problems

3/1 A point moves from rest from position (2,1,0) with acceleration components $a_x = 2+t$, $a_y = t^2$, and $a_x = t^3 - 2$. Find the position vector as a function of t.

3/2 The position vector of a point is $\underline{r} = t^2\hat{\imath} + t^3\hat{\jmath} - t^4\hat{k}$. Find the velocity of the point when $t = 2$. What is the component of the velocity in the direction of the vector $6\hat{\imath} - 2\hat{\jmath} + 3\hat{k}$? What is the acceleration of the point at $t = 2$?

3/3 A point travels along a space curve at constant speed ν. The curve is given by $x = ar$, $y = br^2$, $z = cr^3$, where a, b and c

are constants. Find the velocity and acceleration of the point at position (a, b, c).

3/4 A point moves along a helix at constant speed ν. The equation of the helix in cylindrical coordinates is given by $r = \text{const} = a$; $z = p\theta/2\pi$, $p = \text{const}$. Calculate the cylindrical components of the velocity and acceleration.

3/5 A point moves along a helix defined by the equation $r = \text{const} = a$; $z = p\theta/2\pi$, $p = \text{const}$. If the point moves along the curve with acceleration of constant magnitude, find the position of the point as a function of time.

3/6 A point can move freely on the surface of a sphere of radius R. Express the velocity and acceleration of the point in terms of
(a) Spherical coordinates;
(b) Cylindrical coordinates;
(c) Cartesian coordinates.

3/7 A point moves with constant speed v along a space curve given by

$$x = \cos\theta \qquad y = \sin\theta \qquad z = \theta$$

where $\theta = \theta(t)$. Find the velocity and acceleration of the particle as functions of θ.

3/8 The rectangular components of a point are $x = r\cos\dfrac{v}{t}$, $y = r\sin\dfrac{v}{t}$, and $z = vt + \dfrac{1}{2}at^2$. Determine the magnitudes of the velocity and acceleration of the point as functions of time. The quantities r, v, and a are all constants.

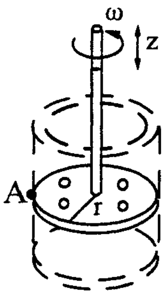
3/9 The velocity and acceleration of a point at a certain instant are $\underline{\nu} = 6\hat{\imath} - 3\hat{\jmath} + 2\hat{k}$ and $\underline{a} = 3\hat{\imath} - \hat{\jmath} - 5\hat{k}$. Determine the angle between $\underline{\nu}$ and $\underline{a}$, and $\dot{\nu}$.

3/10 The velocity and acceleration of a point are $\underline{\nu} = 6\hat{\imath} - 2\hat{\jmath} + \hat{k}$ ft/s and $\underline{a} = 2\hat{\imath} + 3\hat{\jmath} - 6\hat{k}$ ft/s^2 at a certain instant. Show that $\underline{\nu}$ and $\underline{a}$ are perpendicular at this instant and use this fact to determine $\dot{\nu}$. Also determine the flexure f of the path of the point at this instant.

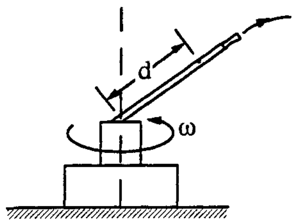
3/11 At a certain instant, a point P has velocity $\underset{\sim}{v} = 4\hat{\imath} + 3\hat{\jmath}$ m/s and its acceleration $\underset{\sim}{a}$ has a magnitude of 10 m/s^2 and makes an angle of 30° with $\underset{\sim}{v}$. Calculate the flexture f for this instant and determine the component of $\underset{\sim}{a}$ tangent to the point's path.

3/12 The screw-jack of Section 3.6 has a lead of 1.25in and turns at a constant rate of 4 rev/s. Calculate the magnitudes of the velocity and acceleration of point A.

3/13 A mixing chamber has a rotating, translating element of radius r as shown. The translating motion is periodic and given by $z = z_0 \sin 2\pi n t$, and the constant rotation speed is ω. Determine the maximum magnitude of acceleration experienced by a point A on the rim of the element.



Problem 3/13



Problem 3/14

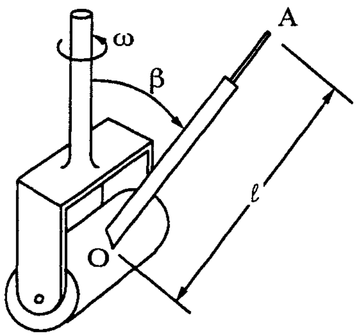
3/14 The rotating nozzle turns at a constant rate $\dot{\theta} = \omega$. Water flows through the nozzle at a constant speed v. Determine the magnitudes of the velocity and acceleration of a water particle at distance d along the nozzle.

3/15 Referring to Fig. 3-11, show that the time rates of change of the spherical coordinate unit vectors may be expressed as:

$$\dot{\hat{e}}_R = \dot{\phi}\hat{e}_\theta + \dot{\theta}\cos\phi\hat{e}_\theta$$
$$\dot{\hat{e}}_\theta = -\dot{\theta}\cos\phi\hat{e}_R + \dot{\theta}\sin\phi\hat{e}_\phi$$
$$\dot{\hat{e}}_\phi = -\dot{\phi}\hat{e}_R - \dot{\theta}\sin\phi\hat{e}_\theta$$

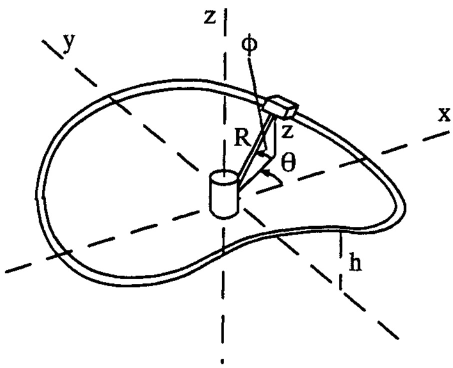
3/16 Using the results of Problem 3/15, obtain the acceleration components in spherical coordinates by differentiation of the position vector, $\underset{\sim}{r} = R\hat{e}_R$.

3/17 The telescoping, rotating boom is attached to a rotating shaft. The rates ω, $\dot{\beta}$, and $\dot{\ell}$ are all constant. Find the spherical acceleration components of the end A of the boom at the instant when $\ell = 1.2$ in, and $\beta = 45°$. The rates are $\omega = 2$ rad/s, $\dot{\beta} = 3/2$ rad/s, and $\dot{\ell} = 0.9$ in/s.



Problem 3/17

3/18 For the same situation as in Problem 3/17, ω is constant and equal to $\pi/3$ rad/s. Arm OA is being elevated at the constant rate $\dot{\beta} = -2\pi/3$. The end of the boom moves according to $\ell = 50 + 200t^2$, where ℓ is in inches and t in seconds. At time $t = 0$, $\beta = 90°$. Determine the spherical components of the acceleration and the magnitude of the acceleration of point A when $t = 1/2$ seconds.



Problem 3/19

3/19 The cars of an amusement park ride are attached to a shaft that rotates at a constant angular rate $\omega = \dot{\theta}$ about the vertical axis

z. The track upon which the cars ride rises and falls according to $z = (h/2)(1 - \cos 2\theta)$. Use the methods of Section 3.5 to determine the cylindrical components of the velocity of a car as it passes the position $\theta = \pi/4$ rad.

3/20 For the situation of Problem 3/19, determine the cylindrical components of acceleration.

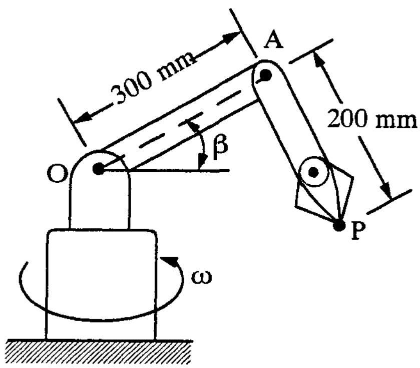
3/21 For the situation of Problem 3/19, use the methods of Section 3.7 to find the spherical components of velocity.

3/22 For the situation of Problem 3/19, find the spherical components of acceleration.

3/23 Use the results of Problem 3/19 and the transformation relations of Section 3.8 to find the spherical components of velocity.

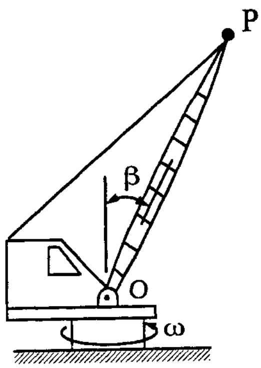
3/24 Use the results of Problem 3/22 and the transformation relations of Section 3.8 to find the cylindrical components of acceleration.

3/25 The robotic device shown is used to position a small part at point P. Calculate the acceleration of P and its magnitude at an instant when $\beta = 30°$, $\dot{\beta} = 10$ degree/s, and $\ddot{\beta} = 20$ degree/s^2. The base of the robot rotates at the constant rate of $\omega = 40$ degree/s and arms OA and AP remain perpendicular.



Problem 3/25

3/26 The revolving crane has a boom length of $\overline{OP} = 24$ m and is turning at a constant rate of 2 rev/m. Simultaneously, the boom is being lowered at the constant rate $\dot{\beta} = 0.10$ rad/s. Determine the magnitudes of the velocity and acceleration of the point P when the position $\beta = 30°$ is passed.



Problem 3/26

3/27 The position vector of a point has rectangular components $(x, y, z) = (5, 2, 3)$. What are (a) the cylindrical components and (b) the spherical components of the point's position? Obtain matrices $[T_\theta]$ and $[T_\phi]$ and use them in the solution.

Chapter 4

Relative Motion

4.1 Introductory Remarks

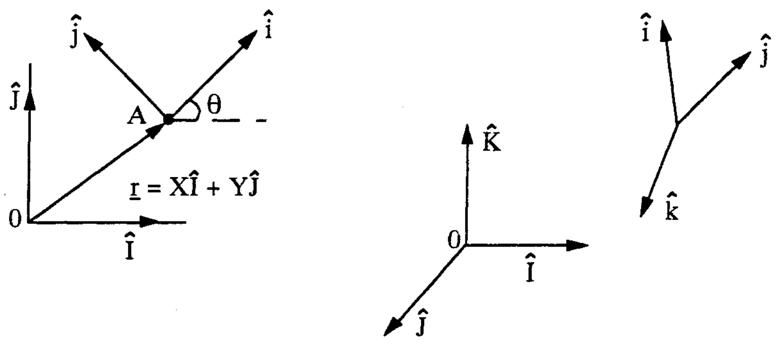
In this chapter it is our primary goal to relate the velocity and acceleration of a point as seen in one reference frame to the velocity and acceleration of the same point as seen in another frame, which is in general 3-D motion with respect to the first frame. That is, we want to derive equations similar to Eqns. (2.24) and (2.25) for 3-D motion.

It is obvious at once that this will not be so easy in 3-D, as it was in 2-D. First, it was easy to visualize 2-D motion because all rotation is about a fixed direction in space. How can general 3-D motion usefully be visualized?

Second, we will need to specify the position of one reference frame relative to the position of another reference frame. Once again this is easy in 2-D; for example, the coordinates of the origin and one angle obviously suffices (Fig. 4-1). It is not obvious how to do this in 3-D.

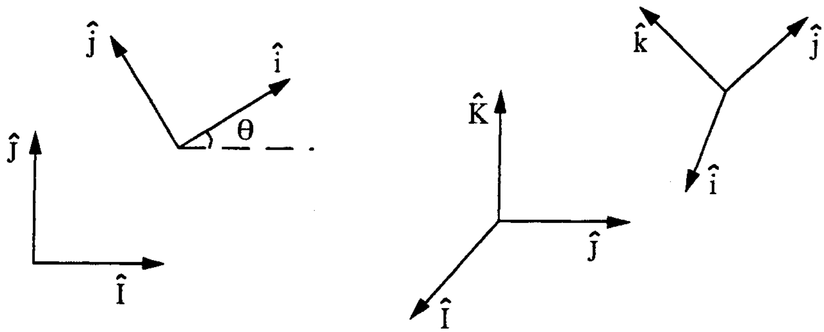
Third, and most important for our purposes, we need a definition of the angular velocity of one frame relative to another that is a useful and consistent extension of the concept in 2-D motion, as defined by Eqn. (2.19) (see Fig. 4-2).

Euler's Theorem leads to answers to all these questions. Although the theorem itself is usually not of direct use in dynamics, it has many corollaries and implications that are vital for the development of 3-D



2-D: (X,Y,θ) 3-D: ?

Fig. 4-1



2-D: $\underline{\omega} = \dot{\theta}\hat{k}$ 3-D: $\underline{\omega} = \underline{?}$

Fig. 4-2

kinematics. In the following section the theorem will be stated and the proof outlined.

4.2 Euler's Theorem

Euler's Theorem states that "Any displacement of one reference frame (or rigid body, since we will later identify rigid bodies with body-fixed frames) with respect to another frame can be brought about by a single rotation about some line." Before discussing the implications of this theorem, the proof will be summarized.

To begin the proof, consider two reference frames $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ and $\{\hat{I}, \hat{J}, \hat{K}\}$ initially lined up with $\hat{\imath}$ along $\hat{I}$, etc. Now displace $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ relative to $\{\hat{I}, \hat{J}, \hat{K}\}$ keeping the origin fixed (Fig. 4-3). Denote the angle between $\hat{\jmath}$ and $\hat{K}$ by $(\hat{\jmath}, \hat{K})$, etc; there are nine such angles relating the new positions of the axes to the old positions. The cosines of these angles are called the *direction cosines* of the displacement.

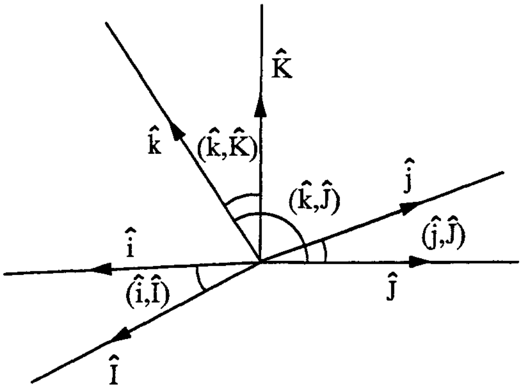


Fig. 4-3

Denote the direction cosines by α_{ij}; that is

$$\begin{array}{lll}
\cos(\hat{\imath}, \hat{I}) = \hat{\imath} \cdot \hat{I} = \alpha_{11}\,, & \cos(\hat{\imath}, \hat{J}) = \hat{\imath} \cdot \hat{J} = \alpha_{12}\,, & \cos(\hat{\imath}, \hat{K}) = \hat{\imath} \cdot \hat{K} = \alpha_{13} \\
\cos(\hat{\jmath}, \hat{I}) = \hat{\jmath} \cdot \hat{I} = \alpha_{21}\,, & \cos(\hat{\jmath}, \hat{J}) = \hat{\jmath} \cdot \hat{J} = \alpha_{22}\,, & \cos(\hat{\jmath}, \hat{K}) = \hat{\jmath} \cdot \hat{K} = \alpha_{23} \\
\cos(\hat{k}, \hat{I}) = \hat{k} \cdot \hat{I} = \alpha_{31}\,, & \cos(\hat{k}, \hat{J}) = \hat{k} \cdot \hat{J} = \alpha_{32}\,, & \cos(\hat{k}, \hat{K}) = \hat{k} \cdot \hat{K} = \alpha_{33}
\end{array} \qquad (4.1)$$

This defines a 3×3 matrix $[A] = [\alpha_{ij}]$, which will hence forth be denoted A. Some properties of A (not proved here) are:

(i) $\quad AA^{\top} = A^{\top}A = I$

$\quad (A^{\top} = \text{transpose of } A,\ I = \text{identity matrix}) \qquad (4.2)$

(ii) $\quad |A| = 1 \qquad (|A| = \text{determinant of } A) \qquad (4.3)$

(iii) $$\begin{pmatrix} \hat{\imath} \\ \hat{\jmath} \\ \hat{k} \end{pmatrix} = A \begin{pmatrix} \hat{I} \\ \hat{J} \\ \hat{K} \end{pmatrix} \qquad (4.4)$$

Now let P be a point fixed in $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ (i.e. it rotates with $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$). After the displacement, P has moved to a new position P' relative to $\{\hat{I}, \hat{J}, \hat{K}\}$, as shown in Fig. 4-4. Before the rotation, $\underline{r}$, the position vector of P, has the same components in either frame:

$$\underline{r} = x\hat{I} + y\hat{J} + z\hat{K} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} \tag{4.5}$$

After the displacement, the components will generally be different:

$$\underline{r}' = x'\hat{I} + y'\hat{J} + z'\hat{K} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} \tag{4.6}$$

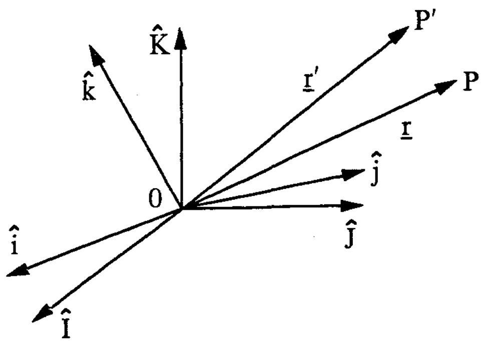


Fig. 4-4

Next we show that $\underline{r} = A\underline{r}'$ for any P and thus that A characterizes the rotation; it is called the *rotation matrix.* Take the dot product of the second form of $\underline{r}'$ in Eqn. (4.6) with $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ sequentially and then dot the first form of $\underline{r}'$ with $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ sequentially and equate:

$$x = x'\hat{I}\cdot\hat{\imath} + y'\hat{J}\cdot\hat{\imath} + z'\hat{K}\cdot\hat{\imath}$$

$$y = x'\hat{I}\cdot\hat{\jmath} + y'\hat{J}\cdot\hat{\jmath} + z'\hat{K}\cdot\hat{\jmath}$$

$$z = x'\hat{I}\cdot\hat{k} + y'\hat{J}\cdot\hat{k} + z'\hat{K}\cdot\hat{k}$$

Thus

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = [a_{ij}] \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

$$\underline{r} = A\underline{r}' \tag{4.7}$$

To prove Euler's Theorem, we need to show that there is some point P, and hence a line $\underline{r} = \overline{OP}$, fixed in $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$, whose position in $\{\hat{I}, \hat{J}, \hat{K}\}$ is unchanged during the displacement; that is, $\underline{r} = \underline{r}'$ for some P. Thus, for this P,

$$A\underline{r}' = \underline{r}' \tag{4.8}$$

In other words, A leaves $\underline{r}'$ unchanged.

Eqn. (4.8) is an example of an eigenvalue problem. We will encounter such problems again later. In general, $\underline{X}$ is an *eigenvector* of a matrix Q with *eigenvalue* λ if

$$\boxed{Q\underline{X} = \lambda\underline{X}}$$

In other words, $\underline{X}$ is a vector such that Q changes (possibly) its magnitude but not its direction.

Our problem is to find an eigenvector of the rotation matrix A with eigenvalue $\lambda = 1$. Suppose there exists an $\underline{r}'$ such that A leaves it unchanged. Then

$$A\underline{r}' = \underline{r}' = I\underline{r}'$$

where I is the identity matrix (see Section 3.8), and

$$(A - I)\underline{r}' = \underline{0}$$

There will be a non-trivial solution[1] of this homogeneous equation if

$$|A - I| = 0$$

Using Eqns. (4.2) and (4.3) and some properties of determinants:

$$A - I = A - AA^{\top} = A(I - A^{\top})$$

$$|A - I| = |A|\,|I - A^{\top}| = |I - A^{\top}| = -|A^{\top} - I| = -|A - I|$$

But this implies that $|A - I| = 0$ and, therefore, an eigenvector exists and there is an $\underline{r}'$ such that $\underline{r}' = A\underline{r}'$ and a corresponding P such that $\overline{OP} = \overline{OP}'$. This completes the proof of Euler's Theorem.

One interesting implication of the theorem is that a sequence of rotations of a rigid body relative to a reference frame may be replaced by a single rotation. To demonstrate this in a simple example, hold your right arm out to the side with the palm of your hand facing forward (your hand will be a rigid body in this demonstration). Now do three rotations of your arm: first rotate forward 90°, then up 90°, and finally 90° back to your right side. Is your hand in the same position in which it started out? If not, what single rotation (what axis and what angle) will get it back to its original position?

It was stated earlier that Euler's Theorem will tell us how many independent parameters are required to specify the position of one reference frame relative to another. To see this, consider frame $\{\hat{\imath},\hat{\jmath},\hat{k}\}$ to be displaced relative to frame $\{\hat{I},\hat{J},\hat{K}\}$, Fig. 4-5. Let the line $\overline{AP}$ be such that a simple rotation of $\{\hat{\imath},\hat{\jmath},\hat{k}\}$ about $\overline{AP}$ brings $\{\hat{\imath},\hat{\jmath},\hat{k}\}$ lined up with $\{\hat{I},\hat{J},\hat{K}\}$; the existence of such a line is guaranteed by the theorem.

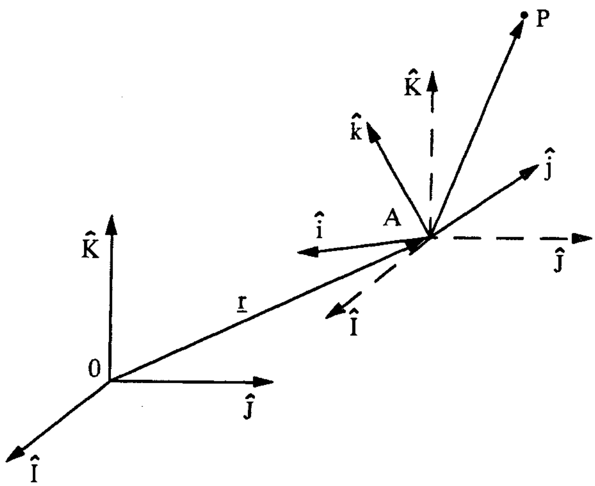


Fig. 4-5

Let α and β be the angles defining this line (Fig. 4-6), and let θ be the rotation angle required for alignment of the two frames. The location of $\{\hat{\imath},\hat{\jmath},\hat{k}\}$ relative to $\{\hat{I},\hat{J},\hat{K}\}$ is then fully specified by the *six* quantities $(X,Y,Z,\alpha,\beta,\theta)$, where X,Y,Z are the coordinates of A, the origin of $\{\hat{\imath},\hat{\jmath},\hat{k}\}$, in $\{\hat{I},\hat{J},\hat{K}\}$.

Many sets of three angles have been devised for the purpose of specifying the orientation of one frame relative to another (the three defined

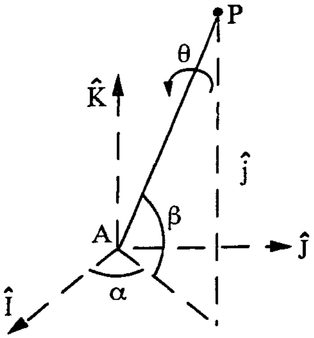


Fig. 4-6

above are not usually the best choice). Later we shall use a set of three angles called Euler's angles to specify the orientation of one reference frame relative to another.

4.3 Finite Rotations

Consider the rotation of frame $\{\hat{\imath}, \hat{\jmath}\}$ relative to $\{\hat{I}, \hat{J}\}$ in 2-D motion (Fig. 4-7). It is clear the rotation has both magnitude (θ) and direction ($\hat{k}$) and can be characterized by $\theta\hat{k}$. For two successive rotations, it is also clear that

$$\theta_1\hat{k} + \theta_2\hat{k} = \theta_2\hat{k} + \theta_1\hat{k}$$

and thus we can replace these two rotations by a single one, $(\theta_1 + \theta_2)\hat{k}$.

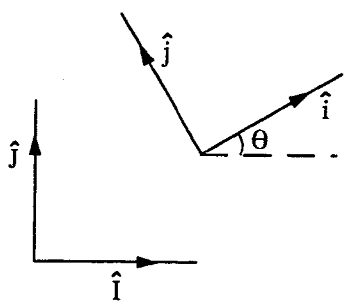


Fig. 4-7

Is this same property true in 3-D motion? That is, does $\theta\hat{e}$, the rotation by θ about some line $\hat{e}$ (Fig. 4-8) satisfy the commutative property of vectors? The following counter example shows that this is not generally true.

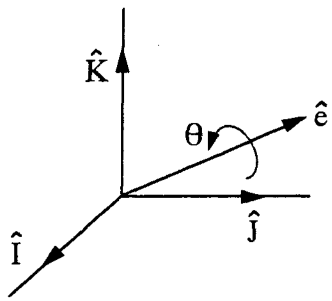


Fig. 4-8

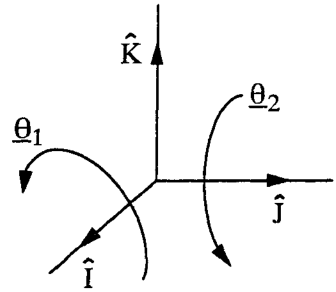


Fig. 4-9

Consider a frame $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ initially lined up with frame $\{\hat{I}, \hat{J}, \hat{K}\}$. It then undergoes two rotations relative to $\{\hat{I}, \hat{J}, \hat{K}\}$; $\theta_1\hat{e}_1$ is a counterclockwise rotation of 90° about $\hat{I}$ and $\theta_2\hat{e}_2$ is a counterclockwise 90° rotation about $\hat{J}$ (Fig. 4-9). First suppose the order of rotations is $\theta_1\hat{e}_1$, and then $\theta_2\hat{e}_2$ (Fig. 4-10); and then suppose the order is $\theta_2\hat{e}_2$ and then $\theta_1\hat{e}_1$ (Fig. 4-11). The figures show that $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ ends up in a different position in each case, and thus

$$\theta_1\hat{e}_1 + \theta_2\hat{e}_2 \neq \theta_2\hat{e}_2 + \theta_1\hat{e}_1$$

This result has important implications for analysis of finite rotations of a rigid body.

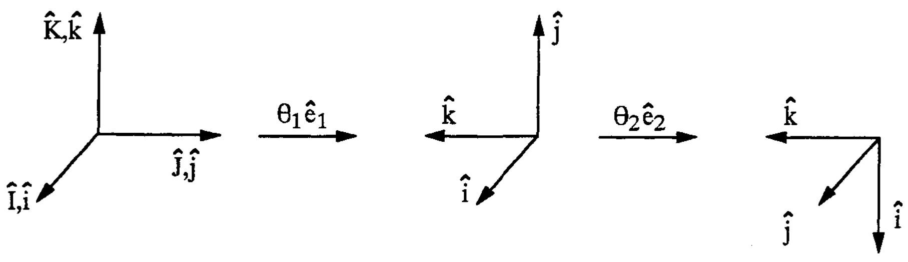


Fig. 4-10

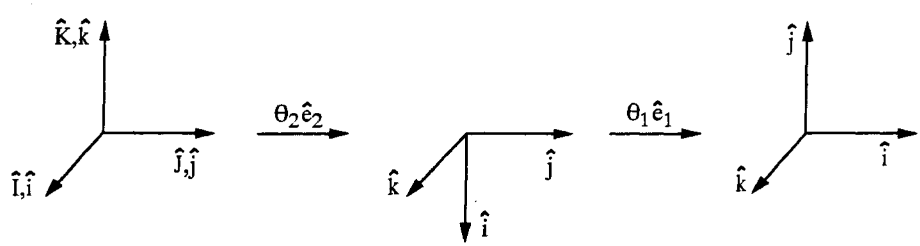


Fig. 4-11

4.4 Infinitesimal Rotations and Angular Velocity and Acceleration

Euler's Theorem gives us a convenient way of looking at the motion of a rigid body; namely, we can visualize the motion as a *sequence of infinitesimal rotations* about a constantly changing line in space, the existence of such a line following from Euler's theorem. If we denote the angular change by $\Delta\theta$ and the line by unit vector $\hat{e}$, then each infinitesimal rotation is characterized by $\Delta\theta\hat{e}$. It is natural now to define the angular velocity of one reference frame relative to another

$$\boxed{\underline{\omega} = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t}\hat{e} = \dot{\theta}\hat{e}} \tag{4.9}$$

Before committing to this definition, however, we need to check to see that infinitesimal rotations $\Delta\theta\hat{e}$ commute.

Let a frame $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ be given an infinitesimal displacement $\Delta\theta\hat{e}$ relative to another frame $\{\hat{I}, \hat{J}, \hat{K}\}$ (Fig. 4-12). Any point, say A, travels to a new point, say A', along a circular arc (Fig. 4-13) in the displacement. Consider the plane passing through A and A' perpendicular to $\hat{e}$; let P be the intercept of this plane and the $\hat{e}$ axis. Also, let $\underline{r}$ and $\underline{r}'$ be the position vectors of A and A', respectively, and let $\hat{a}$ be the unit vector $\overrightarrow{AP}/|\overline{AP}|$. For infinitesimal rotations, the magnitude of $\Delta\underline{r}$ is $|\overline{AP}|\Delta\theta = |\underline{r}|\sin\phi\Delta\theta$ and the direction of $\Delta\underline{r}$ is $\hat{e}\times\hat{a}$, where $\sin\phi = |\overline{AP}|/|\underline{r}|$ was used. Thus

$$\Delta\underline{r} = (|\underline{r}|\sin\phi\Delta\theta)\,(\hat{e}\times\hat{a})$$

and

$$\hat{e}\times\underline{r} = |\hat{e}|\;|\underline{r}|\sin\phi\,(\hat{e}\times\hat{a})$$

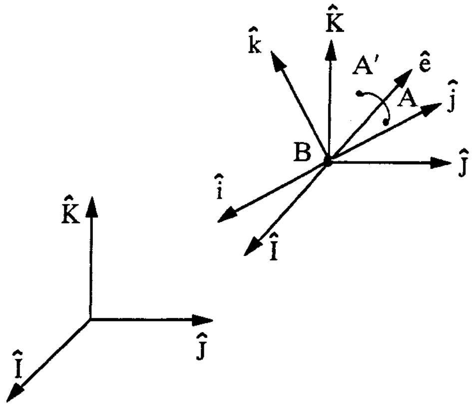


Fig. 4-12

so that

$$\Delta \underline{r} = \Delta\theta \hat{e} \times \underline{r} = \Delta \underline{\theta} \times \underline{r}$$

where $\Delta\underline{\theta} = \Delta\theta\hat{e}$.

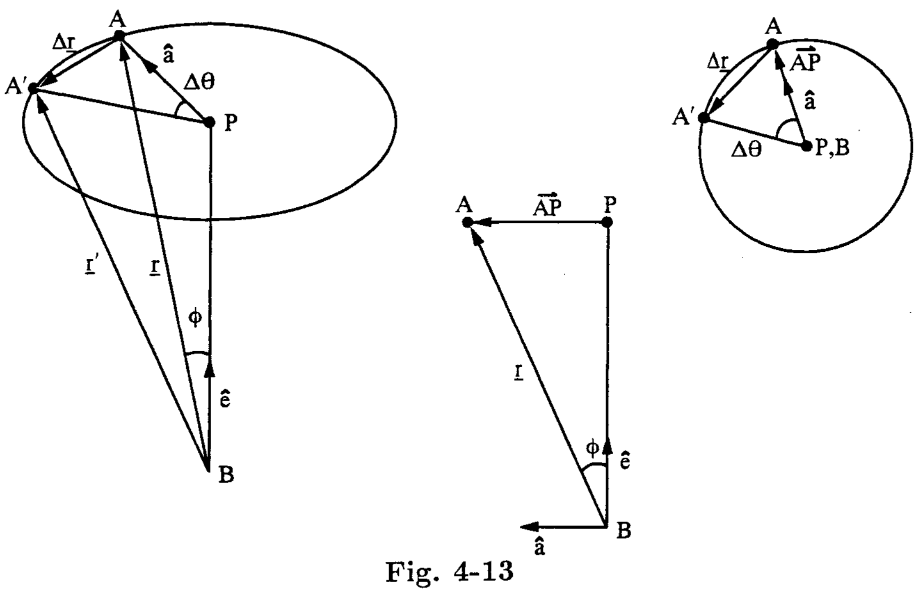


Fig. 4-13

Next consider two sequential displacements, $A \to A' \to A''$ with $\underline{r} \to \underline{r}' \to \underline{r}''$ about first $\hat{e}_1$ and then $\hat{e}_2$. Then

$$\Delta\underline{r}_1 = \Delta\underline{\theta}_1 \times \underline{r}\,; \qquad \Delta\underline{r}_2 = \Delta\underline{\theta}_2 \times \underline{r}'$$

Since $\underline{r}' = \underline{r} + \Delta\underline{r}_1$, to first order:

$$\Delta\underline{r}_2 = \Delta\underline{\theta}_2 \times (\underline{r} + \Delta\underline{r}_1) = \Delta\underline{\theta}_2 \times (\underline{r} + \Delta\underline{\theta}_1 \times \underline{r}) = \Delta\underline{\theta}_2 \times \underline{r}$$

The total displacement is then

$$\begin{aligned} \Delta\underline{r}_1 + \Delta\underline{r}_2 &= \Delta\underline{\theta}_1 \times \underline{r} + \Delta\underline{\theta}_2 \times \underline{r} = (\Delta\underline{\theta}_1 + \Delta\underline{\theta}_2) \times \underline{r} \\ &= (\Delta\underline{\theta}_2 + \Delta\underline{\theta}_1) \times \underline{r} = \Delta\underline{r}_2 + \Delta\underline{r}_1 \end{aligned}$$

This proves that for infinitesimal rotations $\Delta\underline{\theta} = \Delta\theta\hat{e}$ is a vector with the commutative property. This allows us to define *angular velocity* as in Eqn. (4.9).

Now let $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ have a general displacement (translation and rotation) with respect to $\{\hat{I}, \hat{J}, \hat{K}\}$ and let A and $\underline{r}$ be fixed in $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ (Fig. 4-14). Then $\underline{r}_A = \underline{r}_B + \underline{r}$ so that $\Delta\underline{r}_A = \Delta\underline{r}_B + \Delta\underline{r}$ and if the displacement is infinitesimal,

$$\Delta\underline{r}_A = \Delta\underline{r}_B + \Delta\underline{\theta} \times \underline{r} \tag{4.10}$$

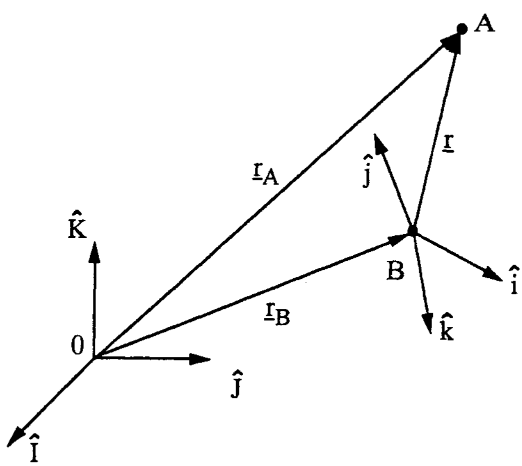


Fig. 4-14

Divide by Δt and pass to the limit

$$\lim_{\Delta t \to 0} \frac{\Delta \underline{r}_A}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta \underline{r}_B}{\Delta t} + \lim_{\Delta t \to 0} \frac{\Delta \underline{\theta} \times \underline{r}}{\Delta t}$$

Using Eqn. (4.9), this becomes

$$\frac{D\underline{r}_A}{Dt} = \frac{D\underline{r}_B}{Dt} + \underline{\omega} \times \underline{r}$$

or

$$\boxed{\underline{v}_A = \underline{v}_B + \underline{\omega} \times \underline{r}} \tag{4.11}$$

where $D(\ )/Dt$ is the time derivative relative to $\{\hat{I}, \hat{J}, \hat{K}\}$. A special case that frequently arises is point B fixed at 0 (Fig. 4-15); then

$$\underline{r}_B = \underline{0}\,, \qquad \underline{v}_B = \underline{0}\,, \qquad \underline{v}_A = \underline{\omega} \times \underline{r}_A$$

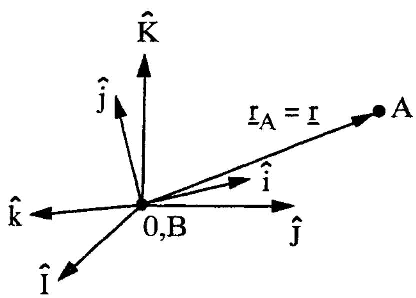


Fig. 4-15

The *angular acceleration* of $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ with respect to $\{\hat{I}, \hat{J}, \hat{K}\}$ will be needed later; it is defined by

$$\boxed{\underline{\alpha} = \frac{D\underline{\omega}}{Dt}} \tag{4.12}$$

4.5 Example

A right circular cone of dimensions h and r, as shown on Fig. 4-16, rolls without slipping on a plane. The center of the face of the cone, point

B, travels around the axis Z, which is perpendicular to the plane, at a constant speed ν. It is desired to find: (i) the angular velocity of the cone, (ii) the velocity of point P on the face of the cone, and (iii) the angular acceleration of the cone.

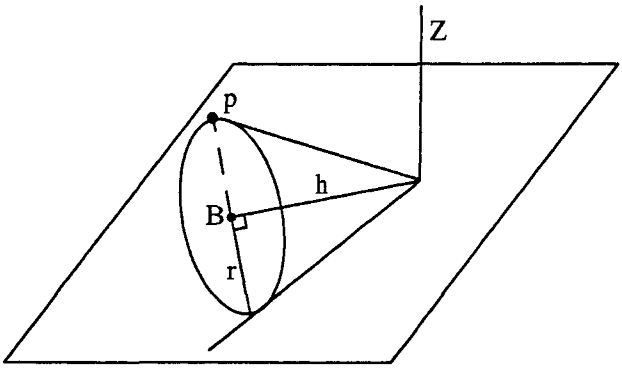


Fig. 4-16

It will be shown in the next Chapter that all body-fixed reference frames have the same angular velocity relative to a non-body-fixed frame; this allows us to define the "angular velocity of a rigid body" as the angular velocity of any body-fixed frame. Thus, we need to introduce a body-fixed frame and a frame fixed in the plane such that when Eqn. (4.11) is applied the only unknown will be $\underline{\omega}$.

First, by the definition of rolling-without-slipping, all of the body points instantaneously in contact with the plane have zero velocity relative to the plane at that instant. Thus this line of points lies on the axis about which the body is instantaneously rotating, the rotation axis of Euler's Theorem. Since angular velocity is defined as lying on this line, we have at once

$$\underline{\omega} = \omega \hat{I}$$

where $\{\hat{I}, \hat{J}, \hat{K}\}$ is a frame fixed in the plane as shown on Fig. 4-17. It remains only to find ω, the magnitude of $\underline{\omega}$.

Introduce a body-fixed frame with origin at B, the center of the base of the cone, as shown on Fig. 4-17. Eqn. (4.11) gives

$$\underline{\nu}_A = \underline{\nu}_B + \underline{\omega} \times \underline{r}_A$$

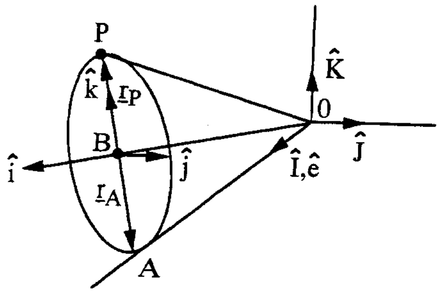


Fig. 4-17

with

$$\underline{v}_A = \underline{0}\,, \qquad \underline{v}_B = -v\hat{J}\,, \qquad \underline{\omega} = \omega\hat{I}\,, \qquad \underline{r}_A = -r\hat{k}$$

The required unit vector transformation is (Fig. 4-18):

$$\hat{k} = \cos\alpha\hat{K} - \sin\alpha\hat{I}$$

Substituting and solving for v,

$$\underline{0} = -v\hat{J} + (\omega\hat{I}) \times (-r)(\cos\alpha\hat{K} - \sin\alpha\hat{I})$$

$$v = r\omega\cos\alpha = r\omega\frac{h}{\sqrt{h^2 + r^2}}$$

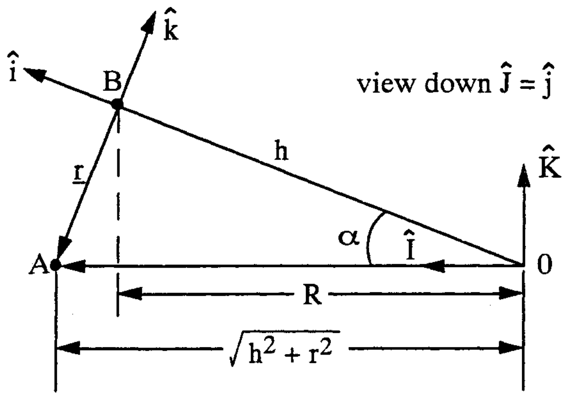


Fig. 4-18

where Fig. 4-18 was used. Thus

$$\omega = \nu\sqrt{\frac{1}{r^2} + \frac{1}{h^2}}$$

and

$$\underline{\omega} = \nu\sqrt{\frac{1}{r^2} + \frac{1}{h^2}}\,\hat{I}$$

The velocity of point P may now be determined from Eqn. (4.11):

$$\begin{aligned}\underline{\nu}_P &= \underline{\nu}_B + \underline{\omega} \times \underline{r}_P = -\nu\hat{J} + (\omega\hat{I}) \times (r\hat{k}) = (-\nu - \omega r\cos\alpha)\hat{J} \\ &= -2\nu\hat{J}\end{aligned}$$

Finally, the angular acceleration of the cone, $\underline{\alpha}$, is computed using its definition, Eqn. (4.12). Now we must be careful because as time evolves, $\underline{\omega}$ will always point along the line of contact and thus revolves around the Z axis. To account for this we write

$$\underline{\omega} = \nu\sqrt{\frac{1}{r^2} + \frac{1}{h^2}}\hat{e}$$

where $\hat{e}$ is always along the line of contact. Applying Eqn. (4.12),

$$\underline{\alpha} = \frac{D\underline{\omega}}{Dt} = \nu\sqrt{\frac{1}{r^2} + \frac{1}{h^2}}\,\frac{D\hat{e}}{Dt}$$

where D/Dt is the time derivative with respect to the plane-fixed frame. Considering an infinitesmal rotation of the cone, Fig. 4-19 gives

$$\Delta\hat{e} = \Delta\theta(-\hat{J})$$

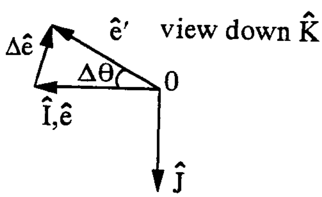


Fig. 4-19

so that

$$\frac{D\hat{e}}{Dt} = \lim_{\Delta t \to 0} \frac{\Delta \hat{e}}{\Delta t} = -\dot{\theta} \hat{J}$$

Since B is moving around a circle of radius R, with $R = h^2/\sqrt{r^2 + h^2}$ (see Fig. 4-18), $\nu = R\dot{\theta}$, so that

$$\underline{\alpha} = \nu \sqrt{\frac{1}{r^2} + \frac{1}{h^2}} \left(-\frac{\nu}{R}\right) \hat{J} = -\frac{\nu^2}{h^2} \left(\frac{r}{h} + \frac{h}{r}\right) \hat{J}$$

4.6 Basic Kinematic Equation

Next, we derive an equation giving the time rate of change of a vector fixed in one frame relative to that in another. Let $\underline{\eta}$ be a vector fixed in $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ (Fig. 4-20) and let $D(\ )/Dt$ be the time derivative in another frame $\{\hat{I}, \hat{J}, \hat{K}\}$. From the figure,

$$\underline{\eta} = \underline{r}_P - \underline{r}_Q = \underline{r}_p - \underline{r}_q$$

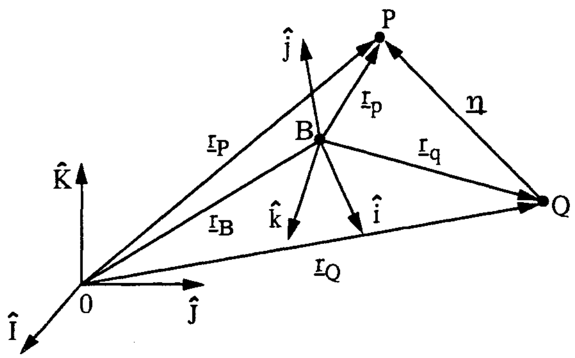


Fig. 4-20

In an infinitesimal displacement, from Eqn. (4.10),

$$\Delta \underline{r}_P = \Delta \underline{r}_B + \Delta \underline{\theta} \times \underline{r}_p$$

$$\Delta \underline{r}_Q = \Delta \underline{r}_B + \Delta \underline{\theta} \times \underline{r}_q$$

so that

$$\Delta \underline{r}_P - \Delta \underline{r}_Q = \Delta \underline{\theta} \times \underline{r}_p - \Delta \underline{\theta} \times \underline{r}_q$$
$$\Delta \left(\underline{r}_P - \underline{r}_Q \right) = \Delta \underline{\theta} \times \left(\underline{r}_p - \underline{r}_q \right)$$

Dividing by Δt and letting $\Delta t \to 0$

$$\lim_{\Delta t \to 0} \frac{\Delta \underline{\eta}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta \underline{\theta}}{\Delta t} \times \underline{\eta}$$

$$\frac{D\underline{\eta}}{Dt} = \underline{\omega} \times \underline{\eta} \qquad (\underline{\eta} \text{ fixed in } \{\hat{\imath}, \hat{\jmath}, \hat{k}\}) \tag{4.13}$$

In particular, since $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ are of course fixed in $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$,

$$\frac{D\hat{\imath}}{Dt} = \underline{\omega} \times \hat{\imath}\,, \qquad \frac{D\hat{\jmath}}{Dt} = \underline{\omega} \times \hat{\jmath}\,, \qquad \frac{D\hat{k}}{Dt} = \underline{\omega} \times \hat{k}\,, \tag{4.14}$$

Now let $\underline{Q}$ be any vector. We wish to derive an expression relating the time derivative of $\underline{Q}$ in one frame, say $d\underline{Q}/dt$ in $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$, to the derivative as seen in another frame (Fig. 4-21), say $D\underline{Q}/Dt$ in $\{\hat{I}, \hat{J}, \hat{K}\}$. Vector $\underline{Q}$ may be written in terms of components in either frame:

$$\underline{Q} = Q_X \hat{I} + Q_Y \hat{J} + Q_Z \hat{K} = Q_x \hat{\imath} + Q_y \hat{\jmath} + Q_z \hat{k}$$

so that[2]

$$\frac{D\underline{Q}}{Dt} = \dot{Q}_X \hat{I} + \dot{Q}_Y \hat{J} + \dot{Q}_Z \hat{K}$$
$$\frac{d\underline{Q}}{dt} = \dot{Q}_x \hat{\imath} + \dot{Q}_y \hat{\jmath} + \dot{Q}_z \hat{k}$$

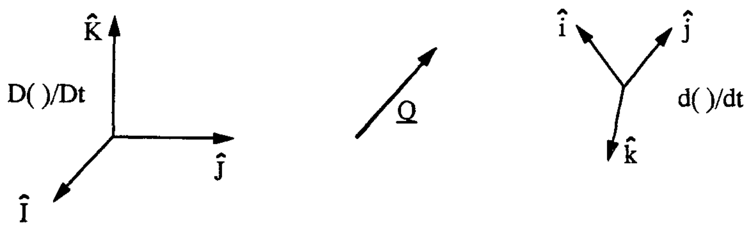


Fig. 4-21

Now consider:

$$\begin{aligned}\frac{D\underline{Q}}{Dt} &= \frac{D}{Dt}(Q_x\hat{\imath} + Q_y\hat{\jmath} + Q_z\hat{k}) \\ &= \dot{Q}_x\hat{\imath} + \dot{Q}_y\hat{\jmath} + \dot{Q}_z\hat{k} + Q_x\frac{D\hat{\imath}}{Dt} + Q_y\frac{D\hat{\jmath}}{Dt} + Q_z\frac{D\hat{k}}{Dt} \\ &= \frac{d\underline{Q}}{dt} + Q_x\underline{\omega}\times\hat{\imath} + Q_y\underline{\omega}\times\hat{\jmath} + Q_z\underline{\omega}\times\hat{k} \\ &= \frac{d\underline{Q}}{dt} + \underline{\omega}\times(Q_x\hat{\imath} + Q_y\hat{\jmath} + Q_z\hat{k})\end{aligned}$$

$$\boxed{\frac{D\underline{Q}}{Dt} = \frac{d\underline{Q}}{dt} + \underline{\omega}\times\underline{Q}} \tag{4.15}$$

As before, this is called the *basic kinematic equation (BKE)*. Comparison with Eqn. (2.22) shows that this is the same equation as for 2-D.

4.7 Some Properties of Angular Velocity

Let

$\underline{\omega}$ = angular velocity of frame $\{\hat{\imath},\hat{\jmath},\hat{k}\}$ with respect to $\{\hat{I},\hat{J},\hat{K}\}$
$\underline{\Omega}$ = angular velocity of frame $\{\hat{I},\hat{J},\hat{K}\}$ with respect to $\{\hat{\imath},\hat{\jmath},\hat{k}\}$

By the BKE, Eqn. (4.15), for any vector $\underline{Q}$,

$$\frac{D\underline{Q}}{Dt} = \frac{d\underline{Q}}{dt} + \underline{\omega}\times\underline{Q}$$

$$\frac{d\underline{Q}}{dt} = \frac{D\underline{Q}}{Dt} + \underline{\Omega}\times\underline{Q} = \frac{d\underline{Q}}{dt} + \underline{\omega}\times\underline{Q} + \underline{\Omega}\times\underline{Q}$$

Thus

$$(\underline{\omega}+\underline{\Omega})\times\underline{Q} = \underline{0}$$

Since this must hold for any vector $\underline{Q}$, this implies that $\underline{\omega} = -\underline{\Omega}$; that is, the angular velocity of one reference frame as viewed from another is the negative of the angular velocity of the second relative to the first frame. This is easy to visualize in 2-D motion, but not so easy in 3-D.

It is also easy to show by the BKE that:

$$\frac{D\underline{\omega}}{Dt} = \frac{d\underline{\omega}}{dt} = \dot{\underline{\omega}}$$

$$\frac{D\underline{\Omega}}{Dt} = \frac{d\underline{\Omega}}{dt} = \dot{\underline{\Omega}}$$

4.8 Relative Velocity and Acceleration Equations

Let point A be in general 3-D motion relative to two reference frames; the two frames are also in general motion relative to each other (Fig. 4-22). From this figure:

$$\underline{r}_A = \underline{r}_B + \underline{r} \tag{4.16}$$

Let

$$\underline{\nu}_A = \frac{D\underline{r}_A}{Dt} = \text{velocity of } A \text{ w.r.t. } \{\hat{I}, \hat{J}, \hat{K}\}$$

$$\underline{\nu}_B = \frac{D\underline{r}_B}{Dt} = \text{velocity of } B \text{ w.r.t. } \{\hat{I}, \hat{J}, \hat{K}\}$$

$$\underline{\nu}_r = \frac{d\underline{r}}{dt} = \text{velocity of } A \text{ w.r.t. } \{\hat{\imath}, \hat{\jmath}, \hat{k}\}$$

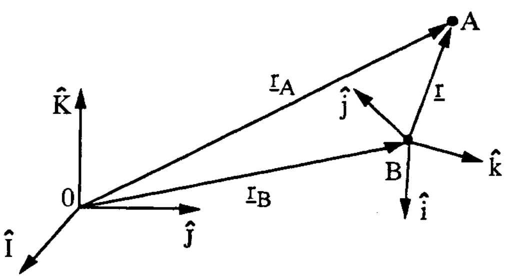


Fig. 4-22

Differentiate Eqn. (4.16) and use the BKE, Eqn. (4.15):

$$\frac{D\underline{r}_A}{Dt} = \frac{D\underline{r}_B}{Dt} + \frac{D\underline{r}}{Dt} = \frac{D\underline{r}_B}{Dt} + \frac{d\underline{r}}{dt} + \underline{\omega} \times \underline{r}$$

$$\boxed{\underline{\nu}_A = \underline{\nu}_B + \underline{\nu}_r + \underline{\omega} \times \underline{r}} \tag{4.17}$$

This is the *relative velocity equation.* Two common special cases are

$$\begin{array}{ll} \text{(i)} & \underline{r} \text{ fixed in} \{\hat{\imath}, \hat{\jmath}, \hat{k}\} \Rightarrow \underline{v}_A = \underline{v}_B + \underline{\omega} \times \underline{r} \\ \text{(ii)} & B \text{ fixed in} \{\hat{I}, \hat{J}, \hat{K}\} \Rightarrow \underline{v}_A = \underline{v}_r + \underline{\omega} \times \underline{r} \end{array}$$

Turning to acceleration, let

$$\underline{a}_A = \frac{D\underline{v}_A}{Dt} = \text{acceleration of } A \text{ w.r.t. } \{\hat{I}, \hat{J}, \hat{K}\}$$
$$\underline{a}_B = \frac{D\underline{v}_B}{Dt} = \text{acceleration of } B \text{ w.r.t. } \{\hat{I}, \hat{J}, \hat{K}\}$$
$$\underline{a}_r = \frac{d\underline{v}_r}{dt} = \text{acceleration of } A \text{ w.r.t. } \{\hat{\imath}, \hat{\jmath}, \hat{k}\}$$

Differentiate Eqn. (4.17) and use the BKE:

$$\begin{aligned} \frac{D\underline{v}_A}{Dt} &= \frac{D\underline{v}_B}{Dt} + \frac{D\underline{v}_r}{Dt} + \frac{D\underline{\omega}}{Dt} \times \underline{r} + \underline{\omega} \times \frac{D\underline{r}}{Dt} \\ &= \underline{a}_B + \frac{d\underline{v}_r}{dt} + \underline{\omega} \times \underline{v}_r + \dot{\underline{\omega}} \times \underline{r} + \underline{\omega} \times \left(\frac{d\underline{r}}{dt} + \underline{\omega} \times \underline{r} \right) \end{aligned}$$

$$\boxed{\underline{a}_A = \underline{a}_B + \underline{a}_r + 2\underline{\omega} \times \underline{v}_r + \dot{\underline{\omega}} \times \underline{r} + \underline{\omega} \times (\underline{\omega} \times \underline{r})} \tag{4.18}$$

This is the *relative acceleration equation.* Two of the terms in this equation have been given names:

(i) $2\underline{\omega} \times \underline{v}_r$ is the *Coriolis acceleration.*
(ii) $\underline{\omega} \times (\underline{\omega} \times \underline{r})$ is the *centripetal acceleration.*

The relative velocity and acceleration equations thus turn out to be exactly the same in 3-D as in 2-D. Indeed, it was our goal to define $\underline{\omega}$ in just the right way for this to be true.

4.9 Composition Relations for Angular Velocities and Accelerations

In general, the angular velocity $\underline{\omega}$ can be very difficult to visualize and compute for two reference frames moving with respect to each other in 3-D. However, if two frames move such that a line of one remains fixed with respect to the other, say a line in the direction $\hat{k}$ for example, then finding $\underline{\omega}$ is easy (Fig. 4-23):

$$\underline{\omega} = \dot{\theta}\hat{K} = \dot{\theta}\hat{k}$$

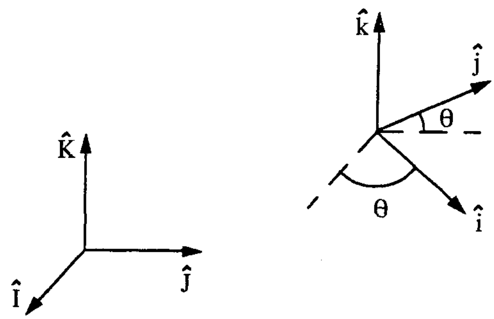


Fig. 4-23

To deal with more complex situations we introduce as many *intermediate frames* as are needed to make the relative motion between two successive frames of a simple type. To this end, consider a sequence of n reference frames in arbitrary motion with respect to each other (Fig. 4-24). Let

$$\begin{aligned} \underline{\omega}_{i/j} \quad = \quad & \text{angular velocity of frame } \{\hat{\imath}_i, \hat{\jmath}_i, \hat{k}_i\} \\ & \text{with respect to frame } \{\hat{\imath}_j, \hat{\jmath}_j, \hat{k}_j\} \end{aligned} \qquad (4.19)$$

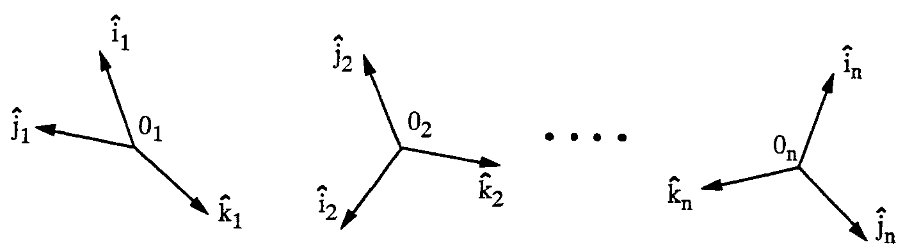


Fig. 4-24

We will show that

$$\boxed{\underline{\omega}_{n/1} = \underline{\omega}_{2/1} + \underline{\omega}_{3/2} + \ldots + \underline{\omega}_{n/n-1} = \sum_{i=2}^{n} \underline{\omega}_{i/i-1}} \tag{4.20}$$

Denote

$$\boxed{\left(\frac{d(\;)}{dt}\right)_i = \text{time derivative w.r.t. } \{\hat{\imath}_i, \hat{\jmath}_i, \hat{k}_i\}} \tag{4.21}$$

From the BKE, for any vector $\underline{Q}$,

$$\left(\frac{d\underline{Q}}{dt}\right)_1 = \left(\frac{d\underline{Q}}{dt}\right)_2 + \underline{\omega}_{2/1} \times \underline{Q}$$

$$\left(\frac{d\underline{Q}}{dt}\right)_2 = \left(\frac{d\underline{Q}}{dt}\right)_3 + \underline{\omega}_{3/2} \times \underline{Q}$$

$$\vdots$$

$$\left(\frac{d\underline{Q}}{dt}\right)_{n-1} = \left(\frac{d\underline{Q}}{dt}\right)_n + \underline{\omega}_{n/n-1} \times \underline{Q}$$

$$\left(\frac{d\underline{Q}}{dt}\right)_1 = \left(\frac{d\underline{Q}}{dt}\right)_n + \underline{\omega}_{n/1} \times \underline{Q}$$

By repetitive substitution

$$\begin{aligned}\left(\frac{d\underline{Q}}{dt}\right)_n + \underline{\omega}_{n/1} \times \underline{Q} \quad &= \quad \underline{\omega}_{2/1} \times \underline{Q} + \underline{\omega}_{3/2} \times \underline{Q} + \underline{\omega}_{4/3} \times \underline{Q} \\ &\quad + \ldots + \underline{\omega}_{n/n-1} \times \underline{Q} + \left(\frac{d\underline{Q}}{dt}\right)_n\end{aligned}$$

$$\underline{0} = \left[\underline{\omega}_{n/1} - \left(\underline{\omega}_{2/1} + \underline{\omega}_{3/2} + \ldots + \underline{\omega}_{n/n-1}\right)\right] \times \underline{Q}$$

Since $\underline{Q}$ is arbitrary, this implies

$$\underline{\omega}_{n/1} = \underline{\omega}_{2/1} + \underline{\omega}_{3/2} + \cdots + \underline{\omega}_{n/n-1} \tag{4.22}$$

which was to be proved.

Considering next angular accelerations, let

$$\underline{\alpha}_{i/j} = \text{angular acceleration of } \{\hat{\imath}_i, \hat{\jmath}_i, \hat{k}_i\} \text{ w.r.t. } \{\hat{\imath}_j, \hat{\jmath}_j, \hat{k}_j\} \tag{4.23}$$

This is defined by

$$\boxed{\underline{\alpha}_{i/j} = \left(\frac{d\underline{\omega}_{i/j}}{dt}\right)_j} \tag{4.24}$$

By the BKE

$$\underline{\alpha}_{i/j} = \left(\frac{d\underline{\omega}_{i/j}}{dt}\right)_j = \left(\frac{d\underline{\omega}_{i/j}}{dt}\right)_i + \underline{\omega}_{i/j} \times \underline{\omega}_{i/j} = \left(\frac{d\underline{\omega}_{i/j}}{dt}\right)_i$$

This shows that $\alpha_{i/j}$ may be computed by differentiating $\underline{\omega}_{i/j}$ in either the i^{th} frame or in the j^{th} frame (but not in general in any other frame).

We now develop a relationship between the angular accelerations of three frames in relative motion. Start with the relation among angular velocities, $n = 3$ in Eqn. (4.22):

$$\underline{\omega}_{3/1} = \underline{\omega}_{2/1} + \underline{\omega}_{3/2}$$

Differentiate this in frame 1 and use the BKE:

$$\left(\frac{d\underline{\omega}_{3/1}}{dt}\right)_1 = \left(\frac{d\underline{\omega}_{2/1}}{dt}\right)_1 + \left(\frac{d\underline{\omega}_{3/2}}{dt}\right)_1$$

$$\underline{\alpha}_{3/1} = \underline{\alpha}_{2/1} + \left(\frac{d\underline{\omega}_{3/2}}{dt}\right)_2 + \underline{\omega}_{2/1} \times \underline{\omega}_{3/2}$$

$$\boxed{\underline{\alpha}_{3/1} = \underline{\alpha}_{2/1} + \underline{\alpha}_{3/2} + \underline{\omega}_{2/1} \times \underline{\omega}_{3/2}} \tag{4.25}$$

The relation for higher numbers of reference frames is obtained similarly, that is by differentiation of the appropriate relative velocity equation. The analysis is very tedious, and the result is very complicated for the general case.

For the case of four reference frames, the result is

$$\boxed{\begin{aligned}\underline{\alpha}_{4/1} = {} & \underline{\alpha}_{4/3} + \underline{\alpha}_{3/2} + \underline{\alpha}_{2/1} + \underline{\omega}_{3/2} \times \underline{\omega}_{4/3} + \underline{\omega}_{2/1} \times \underline{\omega}_{4/3} \\ & + \underline{\omega}_{2/1} \times \underline{\omega}_{3/2}\end{aligned}} \tag{4.26}$$

Note that there can be angular acceleration even when angular velocities are constant. This cannot happen in 2-D motion, for which all angular velocity cross-products are zero.

4.10 Summary of Relative Motion

To avoid confusion when dealing with multiple (more than two) reference frames, the key equations will now be written in a more general form. The BKE relating the time derivative of any vector $\underline{Q}$ in two frames is:

$$\boxed{\left(\frac{d\underline{Q}}{dt}\right)_j = \left(\frac{d\underline{Q}}{dt}\right)_i + \underline{\omega}_{i/j} \times \underline{Q}} \tag{4.27}$$

were the definitions (4.19) and (4.21) are to be used.

The equations relating the velocity and acceleration of a point A as seen in two different frames are

$$\boxed{\underline{v}_{A/j} = \underline{v}_{A/i} + \underline{v}_{O_i/j} + \underline{\omega}_{i/j} \times \underline{r}_{A/i}} \tag{4.28}$$

$$\boxed{\underline{a}_{A/j} = \underline{a}_{A/i} + \underline{a}_{O_i/j} + 2\underline{\omega}_{i/j} \times \underline{v}_{A/i} + \underline{\alpha}_{i/j} \times \underline{r}_{A/i} + \underline{\omega}_{i/j} \times \left(\underline{\omega}_{i/j} \times \underline{r}_{A/i}\right)} \tag{4.29}$$

where $\underline{v}_{A/i}$ and $\underline{a}_{A/i}$ are the velocity and acceleration of point A relative to frame i, respectively; $\underline{v}_{O_i/j}$ and $\underline{a}_{O_i/j}$ are the velocity and acceleration of the origin O_i of the i^{th} frame relative to frame j; $\underline{r}_{A/i}$ is the position vector of A in frame i; and $\underline{\alpha}_{i/j}$ is as defined in Eqn. (4.24).

To summarize, the general procedure for solving kinematics problems is as follows:

1. Introduce a sufficient number of reference frames to make the rotation of successive reference frames sufficiently simple.

2. Introduce convenient coordinate systems in each frame as required.

3. Find all angular velocities and accelerations.

4. Choose a frame in which to resolve vectors into convenient components and make all required unit vector transformations.

5. Apply the relative velocity and acceleration equations.

This procedure will now be illustrated with an example.

4.11 Example

A thin disk AB rotates about a shaft OB which itself rotates about the fixed axis Z (Fig. 4-25). At the instant shown, the disk's rotational speed is p and is increasing at rate $\dot{p}$; the rotational speed about the Z axis is ω and is increasing at rate $\dot{\omega}$; and points O, B, and A are all in the (Z, Y) plane as shown. The shaft OB is rigidly attached to the vertical shaft so that angle ϕ with the (X, Y) plane is fixed. We want to find $\underline{v}_A$ and $\underline{a}_A$, the velocity and acceleration of point A, respectively, relative to axes X, Y, Z fixed in the ground.

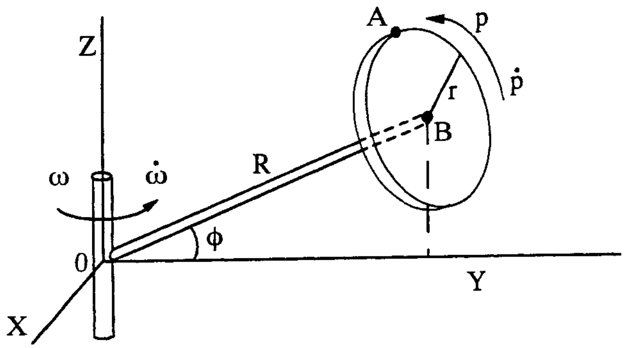


Fig. 4-25

To isolate the two angular motions, introduce three reference frames (Fig. 4-26):

1. fixed in ground;

2. fixed in shaft OB and lined up with frame (1) at instant shown; and

3. fixed in disk with origin at B and y-axis through A.

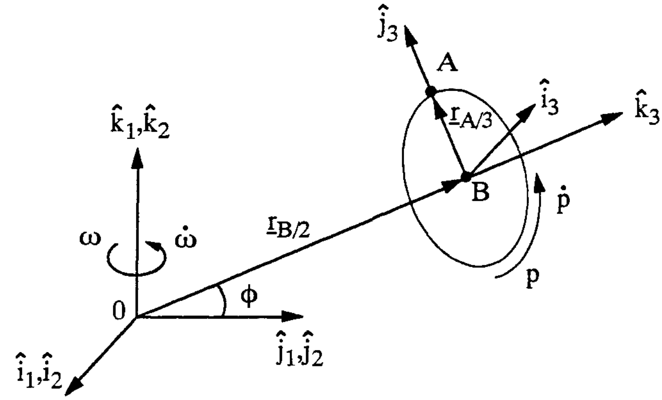


Fig. 4-26

We first find $\underline{v}_A$. The relative velocity equation, Eqn. (4.28), is used to relate the velocity of A as seen in frame (3) to that as seen in frame (1):

$$\underline{v}_A = \underline{v}_{A/1} = \underline{v}_{B/1} + \underline{v}_{A/3} + \underline{\omega}_{3/1} \times \underline{r}_{A/3}$$

location of A in (3)
ang. vel. of (3) w.r.t. (1)
velocity of A w.r.t. (3)
velocity of B w.r.t. (1)
velocity of A w.r.t. (1)

From the figure

$$\underline{r}_{A/3} = r\hat{j}_3 \ , \ \ \underline{r}_{B/2} = R\hat{k}_3 \ , \ \ \underline{\omega}_{2/1} = \omega\hat{k}_1 \ , \ \ \underline{\omega}_{3/2} = p\hat{k}_3 \ ,$$
$$\underline{v}_{A/3} = \underline{0} \ \ [A \text{ fixed in } (3)]$$

Using Eqn. (4.20):

$$\underline{\omega}_{3/1} = \sum_{i=2}^{3} \underline{\omega}_{i/i-1} = \underline{\omega}_{2/1} + \underline{\omega}_{3/2} = \omega\hat{k}_1 + p\hat{k}_3$$

All that remains is to find $\underline{v}_{B/1}$, the velocity of B relative to frame (1). We do this three ways, in order of increasing generality. First, as a 2-D problem in tangential-normal components. Note that B moves in a plane parallel to the $\{\hat{i}_1, \hat{j}_1\}$ plane on a circle with radius $R\cos\phi$ and center a distance $R\sin\phi$ above O on the z_1 axis (Fig. 4-27). Thus

$$\underline{v}_{B/1} = (\omega R\cos\phi)(\hat{e}_t) = \omega R\cos\phi(-\hat{i}_1)$$

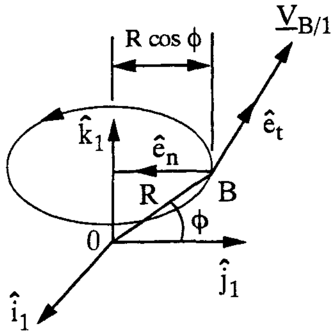


Fig. 4-27

Second, as a 3-D problem in spherical coordinates. Applying Eqn. (3.26):

$$\underline{v}_{B/1} = \dot{R}\hat{e}_R + R\dot{\phi}\hat{e}_\phi + R\dot{\theta}\cos\phi\hat{e}_\theta = R\omega\cos\phi(-\hat{i}_1)$$

Finally, as a problem in relative motion between frames (1) and (2).

Applying Eqn. (4.28) for this case (Fig. 4-28):

$$\begin{aligned}\underline{v}_{B/1} &= \underline{v}_{0/1} + \underline{v}_{B/2} + \underline{\omega}_{2/1} \times \underline{r}_{B/2} = \underline{0} + \underline{0} + \omega\hat{k}_2 \times R\hat{k}_3 \\ &= \omega R(-\cos\phi\hat{i}_1)\end{aligned}$$

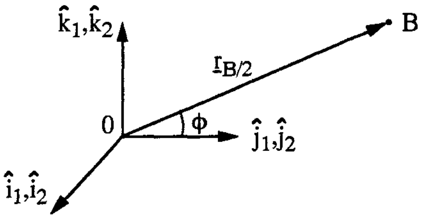


Fig. 4-28

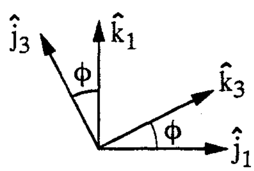


Fig. 4-29

It is now necessary to choose a reference frame and coordinate system in which to resolve vectors; we choose rectangular coordinates in frame (1). The required unit vector transformations are

$$\hat{\imath}_2 = \hat{\imath}_1 \ , \qquad \hat{\jmath}_2 = \hat{\jmath}_1 \ , \qquad \hat{k}_2 = \hat{k}_1$$

and (see Fig. 4-29):

$$\begin{aligned}
\hat{k}_3 &= \cos\phi\hat{\jmath}_1 + \sin\phi\hat{k}_1 \\
\hat{\jmath}_3 &= -\sin\phi\hat{\jmath}_1 + \cos\phi\hat{k}_1 \\
\hat{\imath}_3 &= -\hat{\imath}_1
\end{aligned}$$

Now substitute

$$\begin{aligned}
\underline{\nu}_A &= -(R\omega\cos\phi)\hat{\imath}_1 + \underline{0} + \left[\omega\hat{k}_1 + p(\cos\phi\hat{\jmath}_1 + \sin\phi\hat{k}_1)\right] \\
&\quad \times r(-\sin\phi\hat{\jmath}_1 + \cos\phi\hat{k}_1) \\
&= -(R\omega\cos\phi)\hat{\imath}_1 + (\omega\hat{k}_1 + p\cos\phi\hat{\jmath}_1 + p\sin\phi\hat{k}_1) \\
&\quad \times r(-\sin\phi\hat{\jmath}_1 + \cos\phi\hat{k}_1) \\
&= -(R\omega\cos\phi)\hat{\imath}_1 + r(\omega\sin\phi\hat{\imath}_1 + p\cos^2\phi\hat{\imath}_1 + p\sin^2\phi\hat{\imath}_1) \\
&= (r\omega\sin\phi + rp - R\omega\cos\phi)\hat{\imath}_1
\end{aligned}$$

Thus the velocity of A is in the $\hat{\imath}_1$ direction.

As a check to see if this makes sense, consider two special cases. First, $\phi = 0$ (Fig. 4-30):

$$\underline{\nu}_{A_{\phi=0}} = (rp - R\omega)\hat{\imath}_1$$

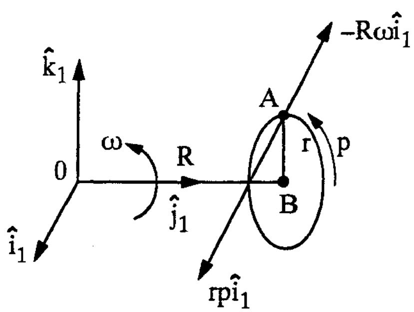


Fig. 4-30

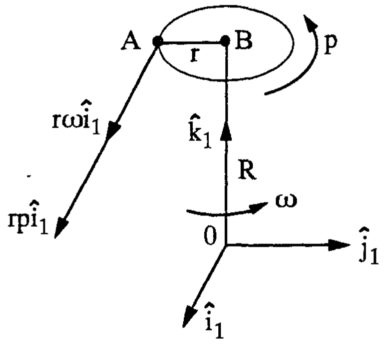


Fig. 4-31

This agrees with inspection of the figure. Second, take $\phi = 90°$ (Fig. 4-31):

$$\underline{v}_{A_{\phi}=90} = (r\omega + rp)\hat{\imath}_1$$

This also agrees with inspection.

Next consider the calculation of the acceleration of point A. For this calculation, we choose to write all vectors in rectangular components in frame (3). Angular acceleration $\underline{\alpha}_{3/1}$ will needed later; using Eqns. (4.24) and (4.27), and noting that $\hat{k}_2$ is a vector fixed in frame (1) and $\hat{k}_3$ is fixed in (2):

$$\underline{\alpha}_{2/1} = \left(\frac{d\underline{\omega}_{2/1}}{dt}\right)_1 = \left(\frac{d(\omega\hat{k}_2)}{dt}\right)_1 = \dot{\omega}\hat{k}_2$$

$$\underline{\alpha}_{3/2} = \left(\frac{d\underline{\omega}_{3/2}}{dt}\right)_2 = \left(\frac{d(p\hat{k}_3)}{dt}\right)_2 = \dot{p}\hat{k}_3$$

Now apply Eqn. (4.25):

$$\begin{aligned} \underline{\alpha}_{3/1} &= \underline{\alpha}_{2/1} + \underline{\alpha}_{3/2} + \underline{\omega}_{2/1} \times \underline{\omega}_{3/2} \\ &= \dot{\omega}(\cos\phi\hat{\jmath}_3 + \sin\phi\hat{k}_3) + \dot{p}\hat{k}_3 + \omega p(\cos\phi\hat{\jmath}_3 + \sin\phi\hat{k}_3) \times \hat{k}_3 \\ &= (\omega p\cos\phi)\hat{\imath}_3 + (\dot{\omega}\cos\phi)\hat{\jmath}_3 + (\dot{\omega}\sin\phi + \dot{p})\hat{k}_3 \end{aligned}$$

where the unit vector transformation $\hat{k}_2 = \cos\phi\hat{\jmath}_3 + \sin\phi\hat{k}_3$ was used.

The relative acceleration equation relating acceleration in frame (1) to that in frame (3) is

$$\begin{aligned}\underline{a}_A = \underline{a}_{A/1} &= \underline{a}_{B/1} + \underline{a}_{A/3} + \underline{\omega}_{3/1} \times (\underline{\omega}_{3/1} \times \underline{r}_{A/3}) \\ &+ 2\underline{\omega}_{3/1} \times \underline{v}_{A/3} + \underline{\alpha}_{3/1} \times \underline{r}_{A/3}\end{aligned}$$

Computation of terms:

$$\underline{a}_{A/3} = \underline{0}$$

$$\begin{aligned}\underline{a}_{B/1} &= \left(\frac{d\underline{v}_{B/1}}{dt}\right)_1 = \left[\frac{d(\underline{\omega}_{2/1} \times R\hat{k}_3)}{dt}\right]_1 \\ &= \left(\frac{d\underline{\omega}_{2/1}}{dt}\right)_1 \times R\hat{k}_3 + \underline{\omega}_{2/1} \times R\left(\frac{d\hat{k}_3}{dt}\right)_1 \\ &= \underline{\alpha}_{2/1} \times R\hat{k}_3 + \underline{\omega}_{2/1} \times R\left[\left(\frac{d\hat{k}_3}{dt}\right)_2 + \underline{\omega}_{2/1} \times \hat{k}_3\right] \\ &= \dot{\omega}\hat{k}_2 \times R\hat{k}_3 + \omega\hat{k}_2 \times R(\omega\hat{k}_2 \times \hat{k}_3) \\ &= \dot{\omega}R\left[(\cos\phi\hat{j}_3 + \sin\phi\hat{k}_3) \times \hat{k}_3\right] + \omega^2 R(\cos\phi\hat{j}_3 + \sin\phi\hat{k}_3) \\ &\quad \times \left[(\cos\phi\hat{j}_3 + \sin\phi\hat{k}_3) \times \hat{k}_3\right] \\ &= \dot{\omega}R\cos\phi\hat{i}_3 - \omega^2 R\cos^2\phi\hat{k}_3 + \omega^2 R\sin\phi\cos\phi\hat{j}_3\end{aligned}$$

$$\begin{aligned}\underline{\omega}_{3/1} \times (\underline{\omega}_{3/1} \times \underline{r}_{A/3}) &= (\underline{\omega}_{3/2} + \underline{\omega}_{2/1}) \times \left[(\underline{\omega}_{3/2} + \underline{\omega}_{2/1}) \times r\hat{j}_3\right] \\ &= (p\hat{k}_3 + \omega\hat{k}_2) \times \left[(p\hat{k}_3 + \omega\hat{k}_2) \times r\hat{j}_3\right] \\ &= \left[p\hat{k}_3 + \omega(\cos\phi\hat{j}_3 + \sin\phi\hat{k}_3)\right] \\ &\quad \times \left\{\left[p\hat{k}_3 + \omega(\cos\phi\hat{j}_3 + \sin\phi\hat{k}_3)\right] \times r\hat{j}_3\right\} \\ &= -r(p + \omega\sin\phi)^2\hat{j}_3 + \omega r\cos\phi(p + \omega\sin\phi)\hat{k}_3\end{aligned}$$

$$2\underline{\omega}_{3/1} \times \underline{v}_{A/3} = \underline{0}$$

$$\begin{aligned}\underline{\alpha}_{3/1} \times \underline{r}_{A/3} &= \left(\frac{d\underline{\omega}_{3/1}}{dt}\right)_1 \times \underline{r}_{A/3} \\ &= \left[\omega p\cos\phi\hat{i}_3 + \dot{\omega}\cos\phi\hat{j}_3 + (\dot{\omega}\sin\phi + \dot{p})\hat{k}_3\right] \times (r\hat{j}_3) \\ &= \omega pr\cos\phi\hat{k}_3 - (\dot{\omega}\sin\phi + \dot{p})r\hat{i}_3\end{aligned}$$

Note that to get $\underline{a}_{B/1}$ we could have used any of the three methods used to get $\underline{v}_{B/1}$; instead, a fourth method has been used, the definition of $\underline{a}_{B/1}$.

Collecting terms and simplifying:

$$\begin{aligned}\underline{a}_A &= \dot{\omega} r \left(\frac{R}{r}\cos\phi - \sin\phi - \frac{\dot{p}}{\dot{\omega}}\right)\hat{\imath}_3 \\ &+\omega^2 r\left[\frac{R}{r}\sin\phi\cos\phi - \left(\frac{p}{\omega}+\sin\phi\right)^2\right]\hat{\jmath}_3 \\ &+\omega^2 r\cos\phi\left(-\frac{R}{r}\cos\phi + 2\frac{p}{\omega}+\sin\phi\right)\hat{k}_3\end{aligned}$$

This is the acceleration of point A, a point fixed on the disk, relative to a reference frame fixed in the ground, expressed in rectangular components in a frame fixed in the disk.

Notes

1 There will always be the trivial solution $\underline{r}' = \underline{0}$ which of course is true because the origin remains fixed.

2 Recall that $D(\ )/Dt = d(\ )/dt = (\dot{\ })$ for a scalar.

Problems

In all the following problems, assume all motions are relative to the ground, unless stated otherwise.

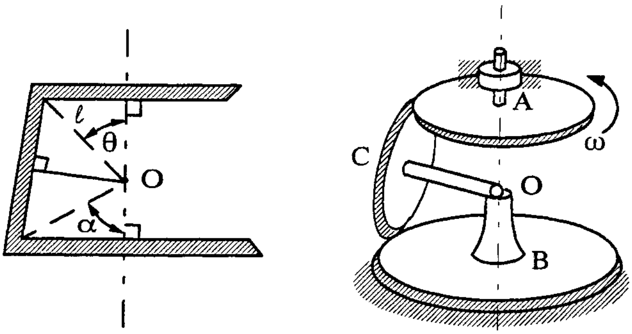
4/1 Solve Problem 3/12 using the relative motion equations.

4/2 Solve Problem 3/14 using the relative motion equations.

4/3 Solve Problem 3/25 using the relative motion equations.

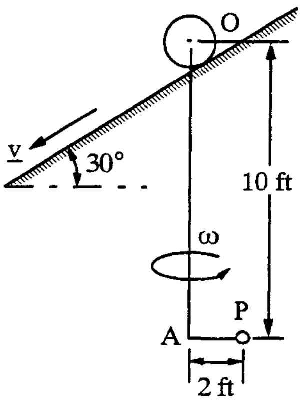
4/4 Solve Problem 3/26 using the relative motion equations.

4/5 Gear B is fixed and gear A turns about axis AB with angular speed ω. Find the angular velocity of gear C which is attached at point O with a ball joint.



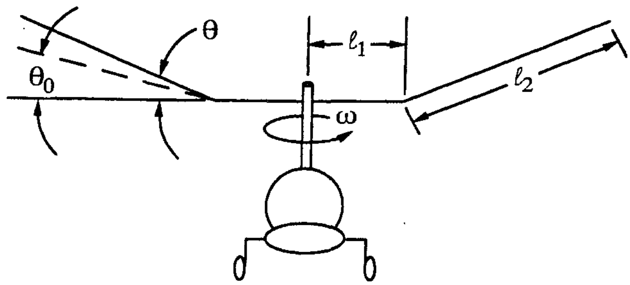
Problem 4/5

4/6 A roller moves down an inclined plane at constant speed $\underline{v} = 10$ ft/sec. The rod OAP moves with the roller such that OA is constrained to remain vertical, and rotates about OA at angular speed $\omega = 4$ radians/sec. What is the velocity of point P at the instant AP is in the plane of $\underline{v}$ and OA?



Problem 4/6

4/7 The helicopter blade shown rotates about the vertical axis at constant angular speed ω. The blade oscillates about a neutral position θ_0 such that θ is given by $\theta = \theta_0 + \theta_A \sin mt$, where θ_A is the amplitude of the oscillation. Find the velocity and acceleration of the tip of the blade for any value of θ.



Problem 4/7

4/8 Two points A and B move such that the distance between them remains 4 in. At a certain instant,

$$\underline{v}_A = (3\hat{\imath} - 2\hat{\jmath} + 4\hat{k})\text{in/s}$$

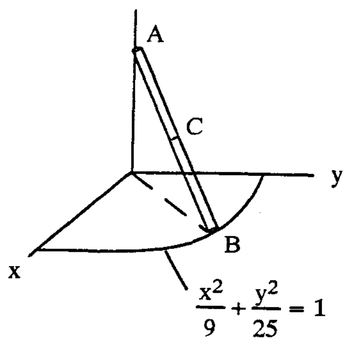
$$\underline{v}_B = (3\hat{\imath} + 6\hat{\jmath} - 2\hat{k})\text{in/s}$$

$$\underline{a}_A = (2\hat{\imath} + 5\hat{k})\text{in/s}$$

$$\dot{\omega} = (\hat{\imath} + 3\hat{\jmath})\text{rad/s}^2$$

relative to a frame $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ where $\underline{\omega}$ is the angular velocity of a frame with origin at A with unit vector $\hat{\imath}$ along $\overrightarrow{AB}$. Find the acceleration of point B at this instant.

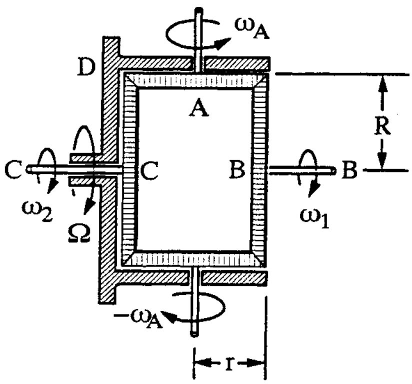
4/9 A rigid rod is constrained such that one end moves along the z-axis while the other end moves along the curve $(x^2/9) + (y^2/25) = 1$



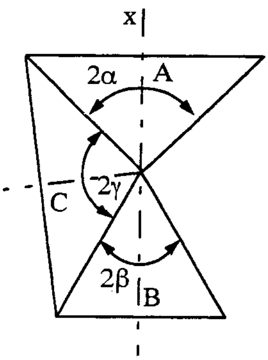
Problem 4/9

What is the velocity of the center of the rod when the end on the z-axis is at $z = 3$ in., has a velocity of $\dot{z} = 2$ in/sec, and $x > 0$, $y > 0$? The length of the rod is 5 in.

4/10 The differential of a car is designed such that the rear wheels can rotate about their axes BB and CC at different angular speeds ω_1 and ω_2, respectively. The two bevel gears attached to the wheel axes have radius R, and the two mating gears have radius r and rotate about their axes at angular speeds ω_A and $-\omega_A$, respectively. If the drive shaft rotates about the wheel axes at angular speed Ω, show that $2\Omega = \omega_1 + \omega_2$ and $2\omega_A = \dfrac{R}{r}(\omega_2 - \omega_1)$.



Problem 4/10



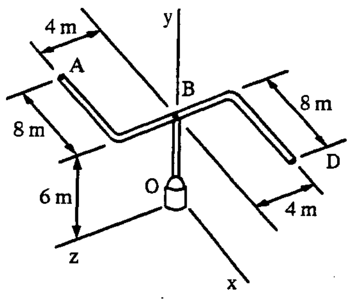
Problem 4/11

4/11 Cone A and cone B both turn about the x-axis with angular speed ω, but in opposite directions. Cone C rolls between A and B without slipping. Find the angular velocity of cone C.

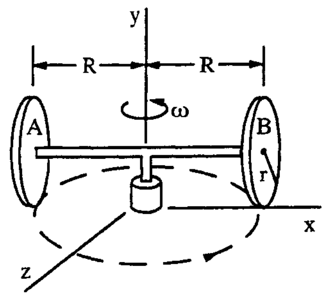
4/12 Rod ABD is rigidly connected to rod OB and rotates about the ball joint at O with angular velocity $\underline{\omega} = \omega_x\hat{\imath} + \omega_y\hat{\jmath} + \omega_z\hat{k}$. If $\underline{\nu}_A = (3\hat{\imath} + 14\hat{\jmath} + \nu_{A_z}\hat{k})$ and $\omega_x = 1.5$ rad/s at a certain instant, determine the angular velocity $\underline{\omega}$ of the assembly and the velocity of point D.

4/13 Same as Problem 4/12, except that $\omega_x = -1.5$ rad/s.

4/14 Two disks of radius r are mounted on an axle of length $2R$ and roll without slipping on the horizontal (x, y) plane at a constant angular rate ω. Find the angular velocity of disk A.

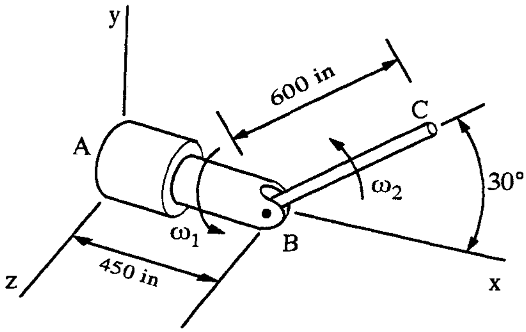


Problem 4/12



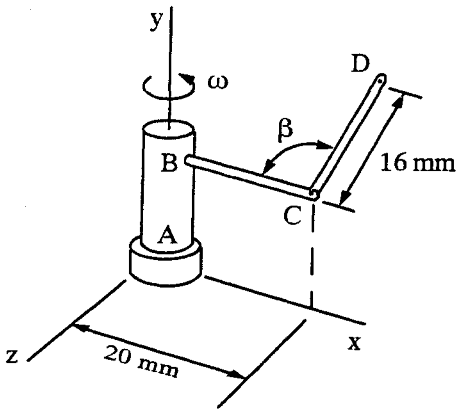
Problem 4/14

4/15 The shaft AB rotates with a constant angular speed of $\omega_1 = 3$ rad/s, and rod BC rotates with respect to the shaft with angular speed ω_2 as shown. At a certain instant, rod BC is in the vertical (x, y) plane, makes an angle of 30° with the horizontal (x, y) plane, and is rotating at a rate of 4 rad/s with the rotation rate increasing at the rate of 5 rad/s^2. Determine the velocity and acceleration of point C.



Problem 4/15

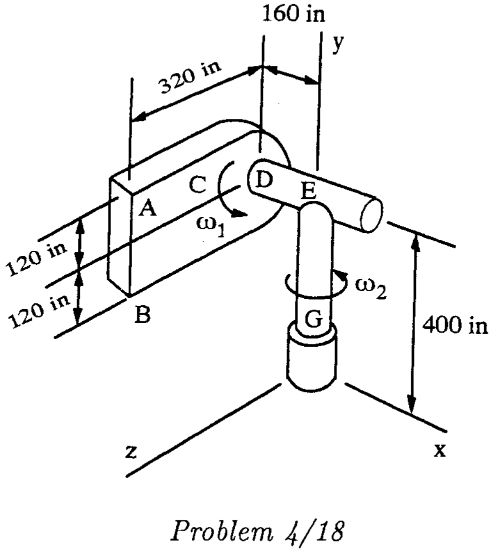
4/16 Shaft AB and rod BC rotate with constant rate $\omega = 0.6$ rad/s about a vertical axis as shown. Rod CD is hinged to rod BC such that it moves in a vertical plane about point C relative to rod BC. At the instant when $\beta = 120°$ and $\dot{\beta} = 0.45$ rad/s, compute (i) the angular acceleration of rod CD, (ii) the velocity of D, and (iii) the acceleration of D.



Problem 4/16

4/17 Same as Problem 4/16, except that $\beta = 60°$.

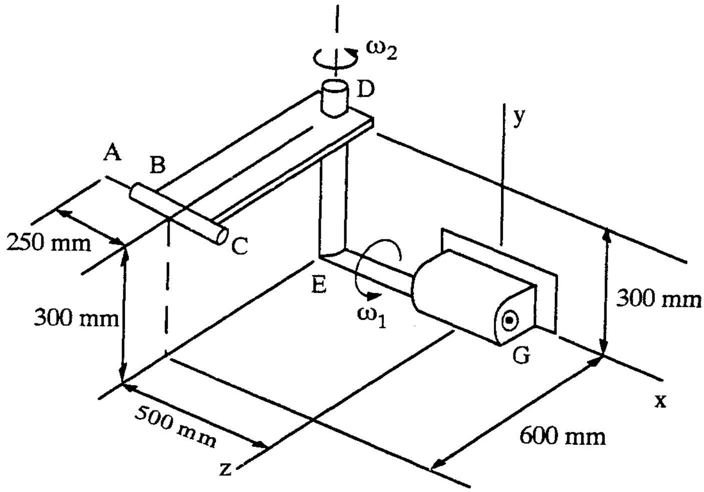
4/18 Bracket ABC rotates about component DEG at a constant angular rate of $\omega_1 = 2.4$ rad/s, which in turn rotates about the vertical axis at a constant rate of $\omega_2 = 2$ rad/s. Find the velocity and acceleration of point A.



Problem 4/18

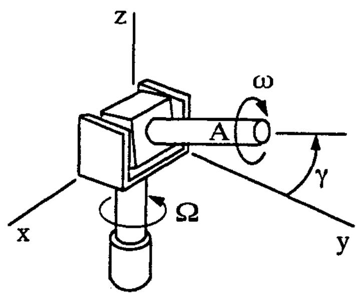
4/19 Same as Problem 4/18, except find the velocity and acceleration of point B.

4/20 A thin rod with tip A slides in cylinder BC. Bracket BCD rotates about rod DE with constant rate ω_2, and the welded rods DEG rotate about the x-axis with constant rate ω_1. At a certain instant, the rod with tip A is moving out of cylinder BC at a speed of 180 mm/s, $\omega_1 = 1.2$ rad/s, and $\omega_2 = 1.6$ rad/s. Find the velocity and acceleration of tip A at that instant.



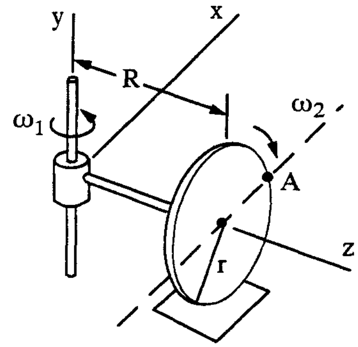
Problem 4/20

4/21 The universal joint shown revolves at the constant rate $\Omega = 4$ rad/s about the z-axis, and the shaft A rotates relative to the joint at the constant rate $\omega = 3$ rad/s. If γ is decreasing at the constant rate of $\pi/4$ rad/s, find the angular velocity of shaft A when $\gamma = 30°$.



Problem 4/21

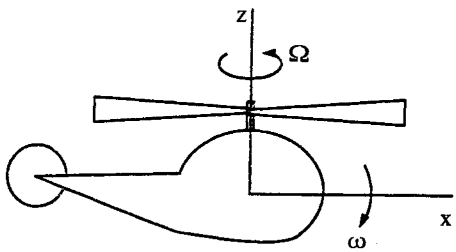
4/22 The disk of radius r rolls without slipping on a horizontal surface and makes one complete turn of radius R about the vertical y-axis in time T. Determine the angular velocity of the disk.



Problem 4/22

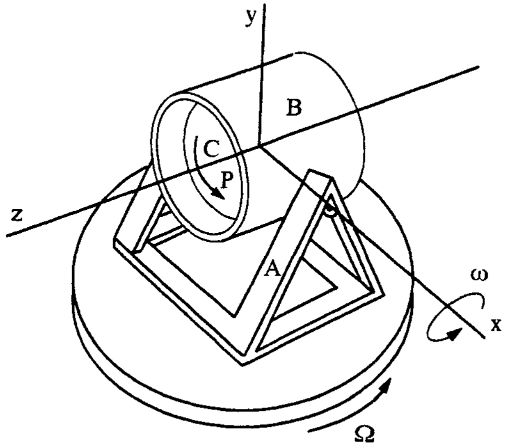
4/23 For the situation of Problem 4/22, determine the velocity of point A on the disk.

4/24 The helicopter is nosing over at the constant angular rate ω and its rotor blades are revolving at the constant angular rate Ω. Find the angular acceleration of the rotor blades.



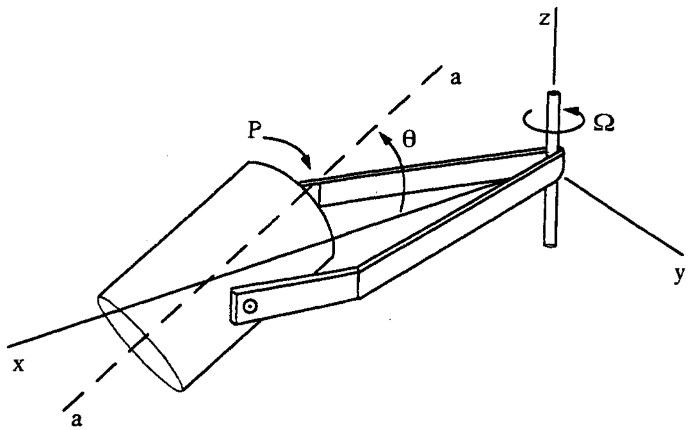
Problem 4/24

4/25 Shown is a spacecraft simulator. The astronauts ride in a drum C which rotates within drum B at the constant angular rate of P. Drum B rotates with respect to frame A at the constant rate of ω, and the entire assembly rotates about the vertical at the constant rate Ω. For $P = 0.9$ rad/s, $\omega = 0.15$ rad/s, and $\Omega = 0.2$ rad/s, determine the angular velocity and acceleration of drum C when the axis of the drum is horizontal.



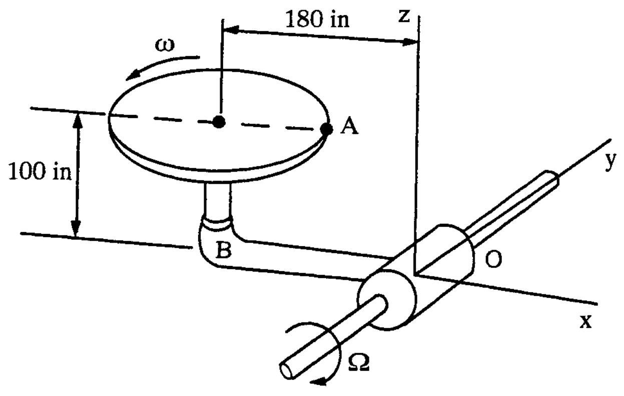
Problem 4/25

4/26 A flight simulator consist of a drum that rotates with constant rate P and tilts with constant rate $\dot{\theta}$ relative to a frame. The frame is fixed to the vertical shaft, which rotates at the constant rate of Ω. Find the angular acceleration of the drum in terms of P, $\dot{\theta}$, ω, and θ, where θ is the angle the drum makes with the horizontal.



Problem 4/26

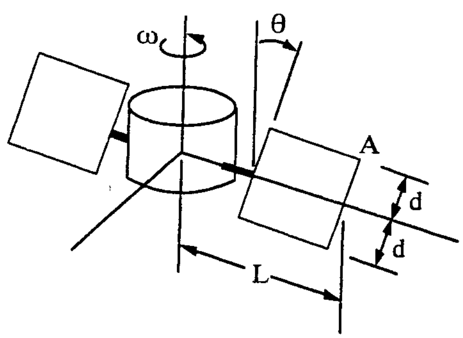
4/27 The circular disk, of radius 100 in, rotates with a constant angular speed of $\omega = 240$ rev/min with respect to bracket BO, and bracket



Problem 4/27

BO rotates about the x-axis with constant angular speed $\Omega = 30$ rev/min. Determine the velocity and acceleration of point A on the disk at the instant shown.

4/28 Two solar panels are being rotated relative to a spacecraft at the rate $\dot{\theta} = 0.25$ rad/s, while the spacecraft is spinning at the constant rate $\omega = 0.5$ rad/s in space. Find the angular velocity and acceleration of the panels, and the velocity and acceleration of point A. The dimensions are $L = 8$ ft, $d = 2$ ft, and $\theta = 30°$.

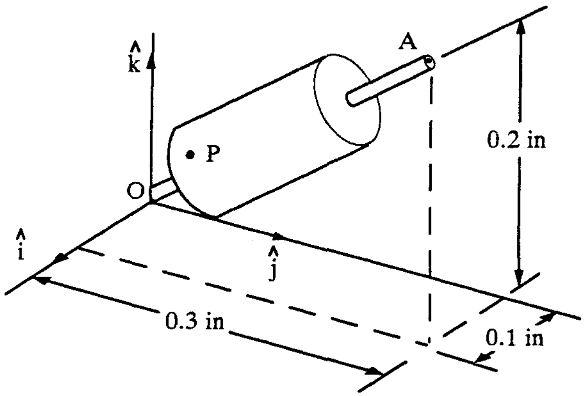


Problem 4/28

4/29 For Problem 4/21, find the angular acceleration of shaft A.

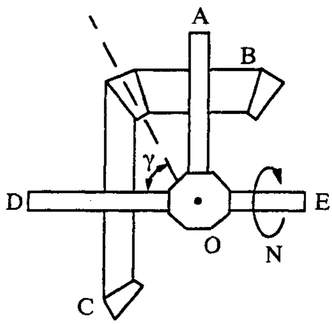
4/30 For Problem 4/22, determine the angular acceleration of the disk and the acceleration of point A on the disk.

4/31 The cylinder rotates about fixed axis OA with an angular speed of 1200 rev/m. At a certain instant, point P on the cylinder has a velocity of $\underline{v} = 3.86\hat{\imath} - 4.20\hat{k}$ m/s. Find the magnitude of the velocity of P and the radius r from axis OA to point P.



Problem 4/31

4/32 Shaft DE rotates about a fixed axis and shaft OA is rigidly attached to DE. Gear B is free to rotate about shaft OA and gear C is fixed. Gears B and C mesh at an angle $\gamma = \tan^{-1} 3/2$ as shown. Shafts DE and OA rotate at a speed of $N = 60$ rev/min the direction shown. Find the angular velocity and the angular acceleration of gear B.

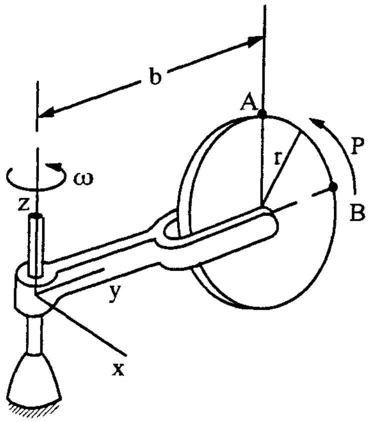


Problem 4/32

4/33 If gear C of Problem 4/32 is now given a constant rotational speed of 20 rev/m about axis DE in the same direction as N, and N

retains its constant rotation of 60 rev/m, calculate the angular velocity and the angular acceleration of gear B.

4/34 The circular disk of radius r rotates with a constant angular speed P in a bracket. Simultaneously, the bracket rotates with constant angular speed ω about a vertical shaft. Determine the angular velocity $\underset{\sim}{\omega}$ and the angular acceleration $\underset{\sim}{\alpha}$ of the disk.



Problem 4/34

4/35 For the situation in Problem 4/34, determine the velocity and acceleration of point A on the disk.

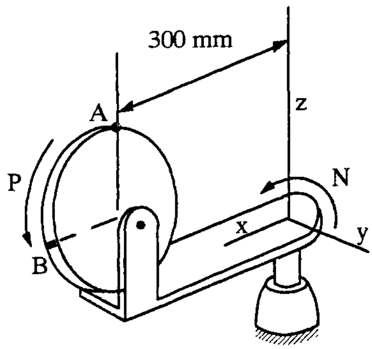
4/36 For the situation in Problem 4/34, determine the velocity and acceleration of point B on the disk.

4/37 Repeat Problem 4/34 if P and ω are increasing at the rates $\dot{P}$ and $\dot{\omega}$, respectively.

4/38 Repeat Problem 4/35 if P and ω are increasing at the rates of $\dot{P}$ and $\dot{\omega}$.

4/39 Repeat Problem 4/36 if P and ω are increasing at the rates of $\dot{P}$ and $\dot{\omega}$.

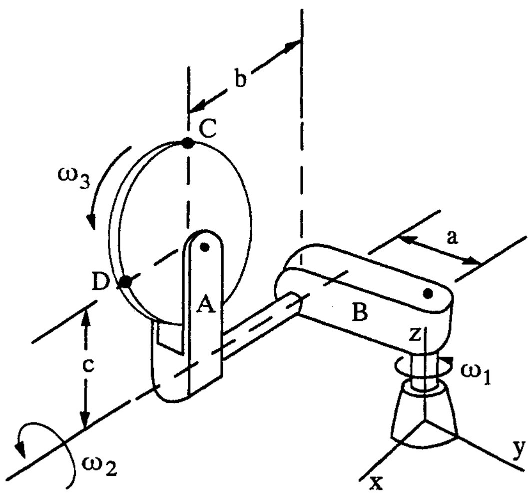
4/40 A circular disk of radius $r = 100$ mm rotates at a constant angular speed $P = 10\pi$ rad/s in a bracket as shown. At the same time, the bracket rotates about a vertical shaft at the constant angular rate of $N = 4\pi$ rad/s. Determine: (i) the angular velocity of the disk, (ii) the angular acceleration of the disk, (iii) the velocity of point A on the disk, and (iv) the acceleration of point A.



Problem 4/40

4/41 For the conditions of Problem 4/40, replace parts (iii) and (iv) by: (iii) the velocity of point B and (iv) the acceleration of point B.

4/42 The circular disk of radius r has constant angular velocity ω_3 relative to bracket A, bracket A has constant angular velocity ω_2 relative to bracket B, and bracket B rotates about the vertical with constant angular velocity ω_1. Determine: (i) the angular velocity of the disk, (ii) the angular acceleration of the disk, (iii) the velocity of point C on the disk, and (iv) the acceleration of point C.



Problem 4/42

4/43 For the conditions of Problem 4/42, replace parts (iii) and (iv) by: (iii) the velocity of point D and (iv) the acceleration of point D.

Chapter 5

Foundations of Kinetics

5.1 Newton's Laws of Motion

Newton's three laws of motion relate the forces on a particle to the particle's motion. A *particle* is a conceptual object that has mass but no volume; it is sometimes called a "point mass." Subsequent to Newton, several physicists, most notably L. Euler, generalized Newton's laws to certain types of collections of particles, including rigid bodies.

Consider the motion of a particle of mass m relative to an inertial frame of reference (Fig. 5-1). Suppose that at some instant of time the particle is at position $\underline{r}$ and is subjected to forces $\underline{F}_1, \underline{F}_2, \ldots, \underline{F}_n$ with vector sum (resultant) $\underline{F} = \sum_i \underline{F}_i$. Then Newton's Laws of Motion are the following.

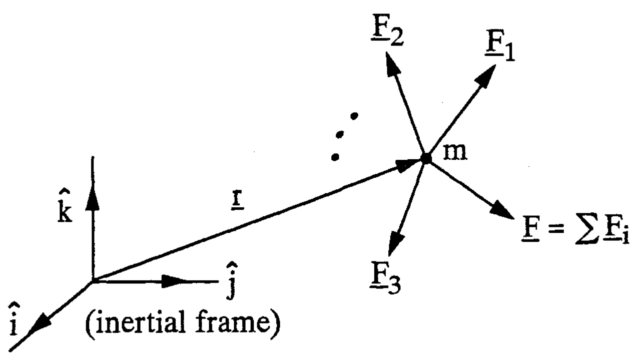


Fig. 5-1

Newton's First Law. If the resultant force on a particle is zero during an interval of time ($\underline{F}(t) = \underline{0}$), then the particle's velocity relative to an inertial frame of reference is constant during that interval of time ($\underline{v}(t) = \underline{const}$).

Newton's Second Law. The acceleration of a particle in an inertial frame at an instant of time is proportional to the resultant force acting on it at that time. The constant of proportionality is called the mass and is a property of the particular particle; thus, at instant t,

$$\boxed{\underline{F}(t) = m\underline{a}(t)} \tag{5.1}$$

The acceleration vector, $\underline{a}(t)$, has been defined previously in Eqn. (1.3).

Newton's Third Law. Given any two particles P_1 and P_2 with masses m_1 and m_2, the force exerted by P_1 on P_2, say $\underline{F}_{12}$, is equal and opposite to that exerted by P_2 on P_1, $\underline{F}_{21}$, and these forces act on a line adjoining the two particles; that is, $\underline{F}_{21} = -\underline{F}_{12}$ (Fig. 5-2).

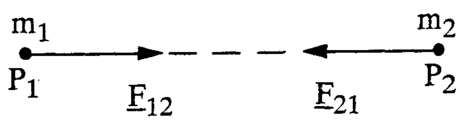


Fig. 5-2

This last law applies to all forces, including contact forces, elastic forces, and field forces such as gravity. Note that it is not enough that the forces be equal and opposite; they must also act on the line adjoining the particles. This is sometimes called the strong form of the third law, and is required for classical dynamics.[1]

Some remarks on these laws are in order. First, it is clear that the First Law is not an independent statement but is a consequence of the Second Law; thus the "First Law" should be called a corollary to the Second.

Second, of the many ways to interpret the Second Law we adopt the following. Suppose a reference frame is defined and a number of forces are applied to a mass particle sequentially, and the resulting accelerations are measured (Fig. 5-3). Then, if (i) the directions $\underline{a}_1$ and $\underline{F}_1$, $\underline{a}_2$ and

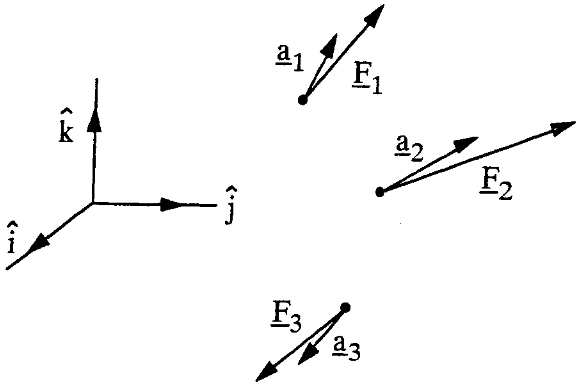


Fig. 5-3

$\underline{F}_2$, etc. are all the same, to an acceptable degree of accuracy, and (ii) the ratios $|\underline{F}_1|/|\underline{a}_1|$, $|\underline{F}_2|/|\underline{a}_2|$, etc. are all the same, to an acceptable degree of accuracy, then the chosen frame is an inertial frame of reference; that is, Newton's Laws may be applied in that frame.

Third, it should be clear that not all reference frames are inertial. For example, when we round a corner in a car, we are thrown sideways, in spite of the fact that there is no sideways force acting on us. This apparent inconsistency is due to our frame of reference (which is fixed in the car) not being inertial.

Fourth, in engineering a reference frame fixed in the earth usually suffices as an inertial frame, but a more accurate one is one with origin at the earth's center and not rotating with the earth. The difference between the two will be investigated in the next Chapter.

Finally, Newton's Second Law may be also written in terms of the particle's linear momentum, defined as

$$\boxed{\underline{L}(t) = m\underline{v}(t)} \tag{5.2}$$

Since the particle's mass is constant, Eqn. (5.1) may be written as

$$\boxed{\underline{F}(t) = \frac{d\underline{L}(t)}{dt}} \tag{5.3}$$

5.2 Center of Mass

Consider a collection of mass particles (Fig. 5-4). Let $\underline{r}_i$ be the position vector of one of the particles (which has mass m_i) relative to an inertial frame. Define the *center of mass* of the collection of mass particles, labelled point G, as the point with position vector

$$\boxed{\underline{\bar{r}} = \frac{\sum\limits_i m_i \underline{r}_i}{m}} \tag{5.4}$$

where the sum is over all the particles and $m = \sum\limits_i m_i$ is the total mass.

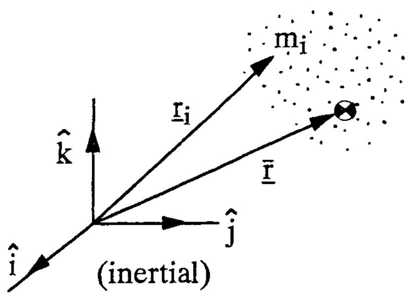


Fig. 5-4

Let

$$\underline{F}^e_{ij} = j^{\text{th}} \text{ external force on particle } i$$
$$\underline{F}^i_{ij} = j^{\text{th}} \text{ internal force on particle } i$$

Then Newton's Second Law, Eqn. (5.1), applied to particle i gives

$$\sum_j \underline{F}^e_{ij} + \sum_j \underline{F}^i_{ij} = m_i \underline{a}_i \tag{5.5}$$

where the vector sum is over all external and internal forces acting on particle i. The internal forces are due to the other particles in the system. Now add Eqns. (5.5) for all the particles to get

$$\sum_i \sum_j \underline{F}^e_{ij} + \sum_i \sum_j \underline{F}^i_{ij} = \sum_i m_i \underline{a}_i \tag{5.6}$$

The first term in this equation is the sum of all the external forces acting on all the particles, which we will call $\underline{F}^e$. The second term is zero by Newton's Third Law. Consider the last term and use Eqn. (5.4):

$$\begin{aligned}\sum_i m_i \underline{a}_i &= \sum_i m_i \frac{d^2\underline{r}_i}{dt^2} = \sum_i \frac{d^2}{dt^2}(m_i\underline{r}_i) = \frac{d^2}{dt^2}\left(\sum_i m_i\underline{r}_i\right) \\ &= \frac{d^2}{dt^2}(m\underline{\overline{r}}) = m\frac{d^2\underline{\overline{r}}}{dt^2} = m\underline{\overline{a}}\end{aligned}$$

Thus Eqn. (5.6) becomes

$$\boxed{\underline{F}^e = m\underline{\overline{a}}} \tag{5.7}$$

This perhaps unexpected result says that the motion of the center of mass of a collection of particles is the same as that of a particle with the total mass of all the particles, when subjected to the same resultant external force.

In terms of linear momentum, Eqn. (5.7) is

$$\boxed{\underline{F}^e = \frac{d\underline{\overline{L}}}{dt}} \tag{5.8}$$

where $\underline{\overline{L}} = m\underline{\overline{v}}$.

Now suppose there are n collections of mass particles, Fig. 5-5. Then the center of mass of the collection is

$$\begin{aligned}\underline{\overline{r}} &= \frac{\sum_i m_i\underline{r}_i}{m} = \frac{\left(\sum_i m_i\underline{r}_i\right)_1 + \left(\sum_i m_i\underline{r}_i\right)_2 + \cdots + \left(\sum_i m_i\underline{r}_i\right)_n}{m_1 + m_2 + \cdots + m_n} \\ &= \frac{m_1\underline{\overline{r}}_1 + m_2\underline{\overline{r}}_2 + \cdots + m_n\underline{\overline{r}}_n}{m_1 + m_2 + \cdots + m_n} = \frac{\sum_{i=1}^{n} m_i\underline{\overline{r}}_i}{\sum_{i=1}^{n} m_i}\end{aligned} \tag{5.9}$$

where m_i is the mass of the i^{th} collection and $\underline{\overline{r}}_i$ is the position vector of its center of mass. This relation is of use in finding the center of mass of complex bodies.

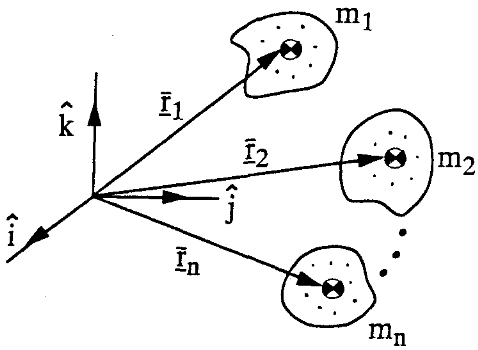


Fig. 5-5

5.3 Example

A projectile of mass 20 lb is fired from point O with speed $\nu = 300$ ft/s in the vertical (x, y) plane at the inclination shown on Fig. 5-6. When it reaches the top of its trajectory at P, it explodes into three fragments

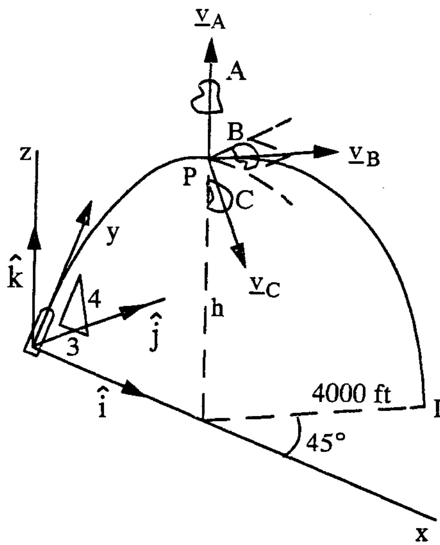


Fig. 5-6

A, B, and C. Immediately after the explosion, fragment A is observed to rise vertically with a speed of 179 ft/s; fragment B is seen to have a horizontal velocity. Fragment B eventually lands at point D. The masses of the three fragments A, B, and C are measured as 5,9, and 6 lb, respectively. It is desired to determine the velocity of fragment C just after the explosion. Aerodynamic drag on the projectile and the fragments may be neglected.

The velocity of the projectile at launch is $\underline{\nu} = \left(\frac{3}{5}\right)(300)\hat{\imath} + \left(\frac{4}{5}\right)(300)\hat{k}$. Using the results of the example of Section 2.2, we obtain

$$t_P = \frac{\nu_z}{g} = \left(\frac{4}{5}\right)(300)/(32.2) = 7.45 \text{ s}$$

$$h = z_P = \frac{\nu_z^2}{2g} = \frac{\left(\frac{4}{5}\right)^2 (300)^2}{(2)(32.2)} = 894 \text{ ft}$$

Because B has no vertical component of velocity of P, the time to reach the ground is the same as the time of the projectile to reach P, namely 7.45s. Further, its horizontal component of velocity remains constant so that $\nu_B = s/t = 4000/7.45 = 537$ ft/s.

Since all forces due to the explosion are internal to the system of particles, Eqn. (5.8) gives $\underline{0} = d\underline{L}/dt$. Thus the linear momentum is the same before and after the explosion:

$$m\underline{\nu} = m_A\underline{\nu}_A + m_B\underline{\nu}_B + m_C\underline{\nu}_C$$

$$(20)\left(\frac{3}{5}\right)(300)\hat{\imath} = (5)(179)\hat{k} + 9(537)(\cos 45°\hat{\imath} + \sin 45°\hat{\jmath}) + 6\underline{\nu}_c$$

$$\underline{\nu}_C = (30\hat{\imath} - 570\hat{\jmath} - 149\hat{k})\text{ft/s}$$

Hence the speed of fragment C just after the explosion is

$$\nu_C = \sqrt{(30)^2 + (570)^2 + (149)^2} = 590 \text{ ft/s}$$

The center of mass of the system has continued on the same path that the projectile had before the explosion.

5.4 Rigid Bodies

A collection of mass particles is called *rigid* (constitutes a *rigid body*) if there exists a reference frame in which each particle's position remains constant over time. Such a reference frame is called *body-fixed.* This definition is very useful, because it allows us to identify the motion of rigid bodies with that of reference frames.

Fig. 5-7 shows a rigid body, an inertial frame, and a body-fixed frame. Vector $\underline{d}_i$ is the position of mass particle i in the body-fixed frame. According to Eqn. (5.4) the position vector of the center of mass relative to the body-fixed frame is

$$\boxed{\underline{\bar{d}} = \frac{\sum_i m_i \underline{d}_i}{m}} \tag{5.10}$$

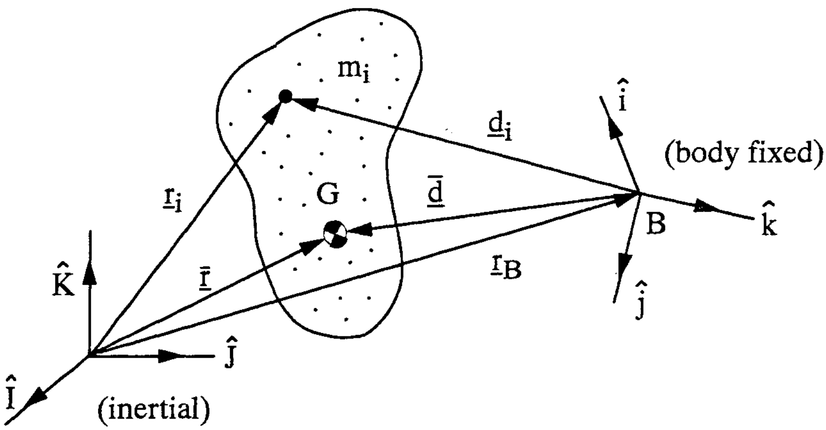


Fig. 5-7

and also, analogous to Eqn. (5.9), for a collection of rigid bodies,

$$\boxed{\underline{\bar{d}} = \frac{\sum_{i=1}^{n} m_i \underline{\bar{d}}_i}{\sum_{i=1}^{n} m_i}} \tag{5.11}$$

Resolve $\underline{d}_i$ and $\underline{\bar{d}}$ in the body-fixed frame:

$$\underline{d}_i = x_i \hat{\imath} + y_i \hat{\jmath} + z_i \hat{k}$$
$$\underline{\bar{d}} = \bar{d}_x \hat{\imath} + \bar{d}_y \hat{\jmath} + \bar{d}_z \hat{k}$$

By the definition of a rigid body, $\underline{d}_i$, x_i, y_i, z_i, $\overline{\underline{d}}$, $\overline{d}_x$, $\overline{d}_y$, and $\overline{d}_z$ are all constants. From the figure,

$$\underline{r}_i = \underline{r}_B + \underline{d}_i$$
$$\overline{\underline{r}} = \underline{r}_B + \overline{\underline{d}}$$

Neither the point G nor the point B need be "in" the body. For a torus, or "donut shaped" body, for example, G will be "outside" the body. Point B, the origin of the body-fixed frame, may be any convenient body-fixed point.

For the purposes of calculation, it is usually desirable to model a rigid body as a *continuum of mass*, that is, we let the number of mass particles approach infinity while their masses approach zero in such a way that the body's mass density is accurately modeled. Thus, for example, Eqns. (5.4) and (5.10) become

$$\boxed{\overline{\underline{r}} = \frac{1}{m}\int_{\text{body}} \underline{r}\, dm} \tag{5.12}$$

$$\boxed{\overline{\underline{d}} = \frac{1}{m}\int_{\text{body}} \underline{d}\, dm} \tag{5.13}$$

where $m = \displaystyle\int_{\text{body}} dm.$

5.5 Example

A rigid body is formed from three thin[2] homogenous rods as shown on Fig. 5-8. It is desired to find the center of mass of the body.

First, we establish that the center of mass of each individual rod is at its midpoint. From Eqn. (5.13) and Fig. 5-9:

$$\overline{\underline{d}} = \frac{1}{m}\int_{\text{body}} \underline{d}\, dm = \frac{1}{m}\int_0^\ell (x\hat{\imath})(\rho dx) = \frac{\rho\hat{\imath}}{m}\int_0^\ell x dx = \frac{\rho\ell^2}{2m}\hat{\imath} = \frac{\ell}{2}\hat{\imath}$$

where $\rho = m/\ell$ is the mass per unit length.

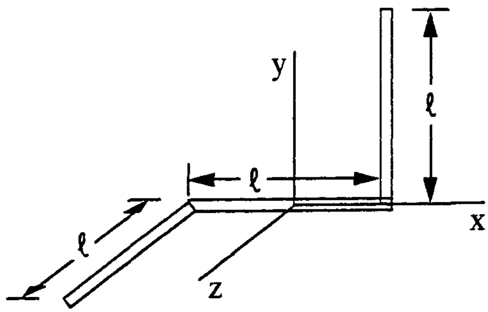


Fig. 5-8

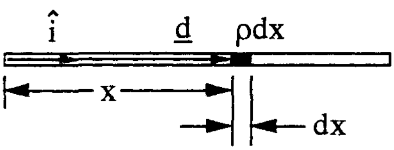


Fig. 5-9

Now, labeling the three rods as shown on Fig. 5-10,

$$\underline{d}_1 = \frac{\ell}{2}\hat{k} - \frac{\ell}{2}\hat{i}\,, \qquad \underline{d}_2 = \underline{0}\,, \qquad \underline{d}_3 = \frac{\ell}{2}\hat{i} + \frac{\ell}{2}\hat{j}$$

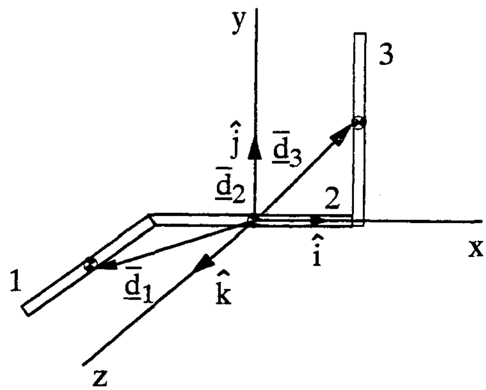


Fig. 5-10

Using Eqn. (5.11):

$$\underline{\overline{d}} = \frac{\sum_{i=1}^{3} m_i \underline{\overline{d}}_i}{\sum_{i=1}^{3} m_i} = \frac{1}{3}(\underline{\overline{d}}_1 + \underline{\overline{d}}_2 + \underline{\overline{d}}_3) = \frac{\ell}{6}(\hat{\jmath} + \hat{k})$$

so that $\overline{d}_x = 0$, $\overline{d}_y = \ell/6$, $\overline{d}_z = \ell/6$ are the rectangular coordinates of the center of mass relative to the axes shown.

5.6 Example

A machine part is made of sheet stock of homogeneous material of thickness t as shown on Fig. 5-11. The density of the material is ρ. We want to determine the center of mass of the part relative to the axes shown.

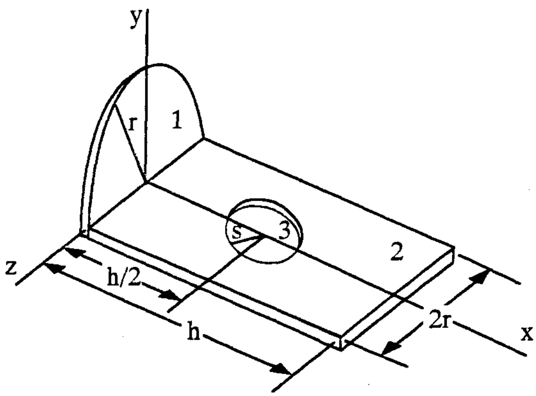


Fig. 5-11

First, we label the components of the part as shown. Considering component 1, the semi-circle, first, the volume, mass, and center of mass are:

$$v_1 = \frac{1}{2}\pi r^2 t\,, \qquad m_1 = \frac{1}{2}\pi\rho r^2 t\,, \qquad \underline{\overline{d}}_1 = \frac{4r}{3\pi}\hat{\jmath}$$

For component 2,

$$v_2 = 2rht\,, \qquad m_2 = 2\rho rht\,, \qquad \underline{\overline{d}}_2 = \frac{h}{2}\hat{\imath}$$

and for component 3,

$$v_3 = \pi s^2 t \,, \qquad m_3 = \pi \rho s^2 t \,, \qquad \underline{\bar{d}}_3 = \frac{h}{2}\hat{\imath}$$

Now applying Eqn. (5.11):

$$\underline{\bar{d}} = \frac{m_1\underline{\bar{d}}_1 + m_2\underline{\bar{d}}_2 - m_3\underline{\bar{d}}_3}{m_1 + m_2 - m_3} = \frac{\left(rh^2 - \frac{\pi s^2 h}{2}\right)\hat{\imath} + \frac{2}{3}r^3\hat{\jmath}}{\frac{1}{2}\pi r^2 + 2rh - \pi s^2}$$

Note that the contribution of the hole must be subtracted, and that the result does not depend on t or ρ.

5.7 Rigid Body Motion

Some of the above results, and others as well, are summarized in the following theorem.

Theorem.

1. The mass center of a rigid body moves under the sum of all external forces acting on the body according to Newton's second law for the motion of a particle.
2. The angular velocities of all body-fixed frames, relative to any other frame, are the same.
3. The motion of a rigid body (that is, the position of all of its particles as a function of time) is fully specified by the motion of any of its body-fixed frames.

We have just proved part (1) of this theorem. Parts (2) and (3) will be proved in the next two sections.

The theorem has many important and far-reaching consequences in dynamics, which we summarize:

1. If we only interested in the translational motion of a rigid body (for example, if the dimensions of the body are in some sense "small" or if the rotational motion is not of interest), then we need consider only the motion of the mass center, which is governed by Eqn.

(5.7). As an example, the dimensions of a transport airplane may seem to be "large," but if the purpose of analysis is the airplane performance over a long flight, considering only the motion of the mass center is usually sufficient.

As another example, if the path of a spacecraft is all that is of importance, considering only the motion of the center of mass is sufficient. If, however, the orientation of the spacecraft is important,[3] then the rotational dynamics must be included (Fig. 5-12).

2. Because the angular velocity of all body-fixed frames is the same, without ambiguity we may define the *angular velocity of a rigid body* as the angular velocity of any body-fixed frame with respect to an inertial frame.

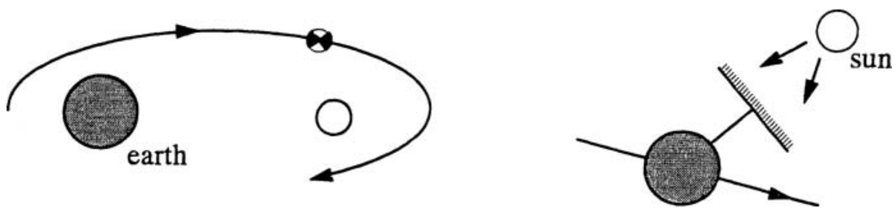


Fig. 5-12

3. If we develop kinematics as the description of the motion of points and reference frames, then we will have the kinematics of rigid bodies as a special case. This is because the translational motion of a rigid body is that of a point, the center of mass, and the rotational motion is that of a body-fixed frame. This general view of kinematics will have advantages later on.

5.8 Proof That the Motion of a Rigid Body Is Specified By the Motion of Any Body-Fixed Frame

Refer again to Fig. 5-7 which shows a rigid body, a body-fixed frame, and an inertial frame. From the figure

$$\underline{r}_i = \underline{r}_B + \underline{d}_i \tag{5.14}$$

Resolve these vectors into components as follows:

$$\begin{aligned} \underline{r}_i &= X_i\hat{I} + Y_i\hat{J} + Z_i\hat{K} \\ \underline{d}_i &= x_i\hat{\imath} + y_i\hat{\jmath} + z_i\hat{k} \\ \underline{r}_B &= X_B\hat{I} + Y_B\hat{J} + Z_B\hat{K} \end{aligned} \tag{5.15}$$

The coordinates of mass particle i in the body-fixed frame, x_i, y_i, and z_i, are known constants. Now take the dot product of the first of these equations with $\hat{I}, \hat{J}$, and $\hat{K}$ sequentially and use Eqn. (5.14) and the last two of Eqns. (5.15):

$$\begin{aligned} X_i &= \hat{I}\cdot\underline{r}_i = X_B + x_i\hat{\imath}\cdot\hat{I} + y_i\hat{\jmath}\cdot\hat{I} + z_i\hat{k}\cdot\hat{I} \\ Y_i &= \hat{J}\cdot\underline{r}_i = Y_B + x_i\hat{\imath}\cdot\hat{J} + y_i\hat{\jmath}\cdot\hat{J} + z_i\hat{k}\cdot\hat{J} \\ Z_i &= \hat{K}\cdot\underline{r}_i = Z_B + x_i\hat{\imath}\cdot\hat{K} + y_i\hat{\jmath}\cdot\hat{K} + z_i\hat{k}\cdot\hat{K} \end{aligned} \tag{5.16}$$

In matrix form, these equations are

$$\begin{pmatrix} X_i \\ Y_i \\ Z_i \end{pmatrix} = \begin{pmatrix} X_B \\ Y_B \\ Z_B \end{pmatrix} + [A]\begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix} \tag{5.17}$$

where $[A]$ is the rotion matrix, that is, the matrix of the direction cosines giving the orientation of $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ with respect to $\{\hat{I}, \hat{J}, \hat{K}\}$ (see Section 4.2). But only three of these nine direction cosines are independent and therefore the location of mass particle m_i relative to the inertial frame is fully specified by the six quantities X_B, Y_B, Z_B and the three independent direction cosines.

5.9 Proof That All Body-Fixed Frames Have the Same Angular Velocity

Fig. 5-13 shows a rigid body, two body-fixed frames, $\{\hat{\imath}_1, \hat{\jmath}_1, \hat{k}_1\}$ and $\{\hat{\imath}_2, \hat{\jmath}_2, \hat{k}_2\}$, and a non-body-fixed frame $\{\hat{I}, \hat{J}, \hat{K}\}$. Point B is any arbitrary body-fixed point.

In an infinitesimal displacement of the rigid body, B, O_1, and O_2 undergo displacements $\Delta\underline{r}$, $\Delta\underline{r}_{O_1}$, $\Delta\underline{r}_{O_2}$ relative to $\{\hat{I}, \hat{J}, \hat{K}\}$. (There are of course no displacements relative to $\{\hat{\imath}_1, \hat{\jmath}_1, \hat{k}_1\}$ and $\{\hat{\imath}_2, \hat{\jmath}_2, \hat{k}_2\}$).

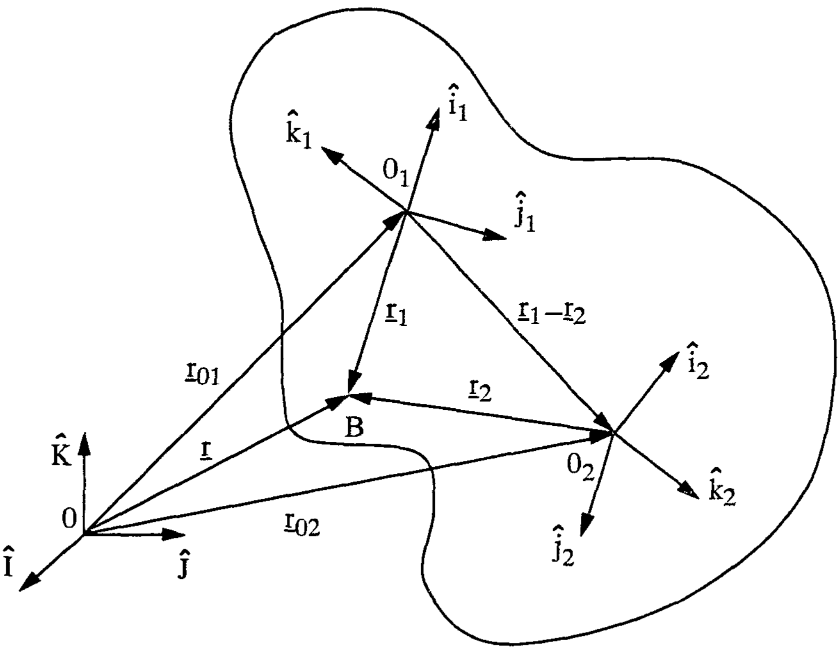


Fig. 5-13

Let $\Delta\underline{\theta}_1$ and $\Delta\underline{\theta}_2$ be the corresponding equivalent rotations of frames (1) and (2).

Since the displacement is infinitesimal (see Section 4.4)

$$\begin{aligned}
\Delta\underline{r} &= \Delta\underline{r}_{O_1} + \Delta\underline{\theta}_1 \times \underline{r}_1 \\
\Delta\underline{r} &= \Delta\underline{r}_{O_2} + \Delta\underline{\theta}_2 \times \underline{r}_2 \\
\Delta\underline{r}_{O_2} &= \Delta\underline{r}_{O_1} + \Delta\underline{\theta}_1 \times (\underline{r}_1 - \underline{r}_2)
\end{aligned} \tag{5.18}$$

Put the last of these in the second and use the first:

$$\begin{aligned}
\Delta\underline{r} &= \Delta\underline{r}_{O_1} + \Delta\underline{\theta}_1 \times (\underline{r}_1 - \underline{r}_2) + \Delta\underline{\theta}_2 \times \underline{r}_2 \\
\Delta\underline{r} &= \Delta\underline{r}_{O_1} + \Delta\underline{r} - \Delta\underline{r}_{O_1} - \Delta\underline{\theta}_1 \times \underline{r}_2 + \Delta\underline{\theta}_2 \times \underline{r}_2 \\
\underline{0} &= (\Delta\underline{\theta}_2 - \Delta\underline{\theta}_1) \times \underline{r}_2
\end{aligned} \tag{5.19}$$

Since B was arbitrary (and hence $\underline{r}_2$ as well) we must have

$$\Delta\underline{\theta}_2 = \Delta\underline{\theta}_1$$

Dividing by Δt and taking the limit $\Delta t \to 0$ gives

$$\underline{\omega}_2 = \underline{\omega}_1$$

which was to be proved.

5.10 Gravitation

Newton's *Universal Law of Gravition* says that for two particles of masses m_1 and m_2 a distance r apart, each attracts the other with force (see Fig. 5-14)

$$\boxed{\underline{F}_1 = \frac{Km_1m_2}{r^2}\hat{e}_r = -\underline{F}_2} \tag{5.20}$$

where $\hat{e}_r$ is a unit vector along the line connecting the two particles and K is the universal gravitional constant. Experimental data gives the value of K as 6.673×10^{-11} m^3/(Kg·sec^2). The magnitude of the gravitional force is

$$F = |\underline{F}_1| = |\underline{F}_2| = \frac{Km_1m_2}{r^2} \tag{5.21}$$

In the usual case, one of the masses will be the earth. Let $m_1 = M$ be the mass of the earth and $m_2 = m$ be the mass of a particle; then the gravitional force acting on the particle is

$$\underline{F} = -\frac{KM}{r^2}m\hat{e}_r \tag{5.22}$$

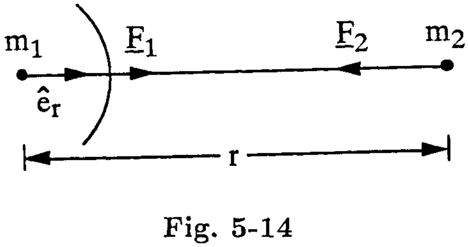


Fig. 5-14

This equation is a consequence of assuming the earth to have radial mass symmetry and the fact that a radially symmetric body acts gravitationally as if it were a particle at it's center.[4] Since the earth is not exactly radially symmetric, Eqn. (5.22) is an approximation. In fact, the earth is neither a perfect sphere (it is slightly flattened at the poles and bulges at the equator) nor homogeneous (it has local variations in density).

In engineering, we are usually concerned with objects on, or very near, the surface of the earth. In this case we may take $r = R$, the radius of the earth. The symbol $\underline{g}$ is used for the gravitational force per unit mass:

$$\underline{g} = \frac{1}{m}\underline{F} = -\frac{KM}{R^2}\hat{e}_r \tag{5.23}$$

The magnitude of this vector is

$$\boxed{g = \frac{KM}{R^2}} \tag{5.24}$$

According to Eqn. (5.23), $\underline{g}$ is a vector always pointing towards the center of the earth (Fig. 5-15).

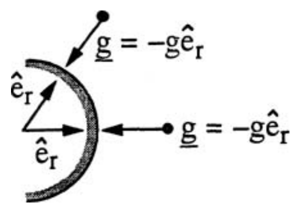


Fig. 5-15

Because the earth does not have perfect radial mass symmetry, g is not a constant but is slightly different at different places on the earth's surface. Experimental data shows that the value of g varies between 9.815 and 9.833 m/sec^2 (32.20 and 32.26 ft/sec^2) over the earth's surface. Standard values of 9.81 m/sec^2 and 32.2 ft/sec^2 are usually used for engineering calculations.

From Eqn. (5.23) the gravitational force due to the earth acting on a particle of mass m on or near the surface of the earth is

$$\underline{F} = m\underline{g} = -mg\hat{e}_r \tag{5.25}$$

The magnitude of this force is

$$\boxed{W = mg} \tag{5.26}$$

This force is commonly called the *weight* of the particle. It turns out, however, that the contact force from the earth[5] acting on a particle resting on the earth, which is the common definition of "weight", is influenced by the earth's rotation, as will be demonstrated in the next Chapter.

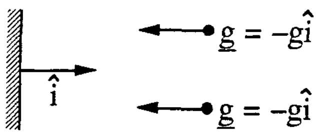


Fig. 5-16

In many engineering applications the motions take place over a distance much shorter than the radius of the earth. In this case, the "flat earth" assumption is made (Fig. 5-16); now the gravitational force has approximately a constant direction:

$$\boxed{\underline{F} = -mg\hat{i}} \tag{5.27}$$

5.11 Degrees of Freedom and Holonomic Constraints

It has been shown that the location of a point (equivalently, a particle) relative to a reference frame is specified by two parameters (say its rectangular coordinates x and y) in 2-D motion and by three parameters (say x, y, and z) in 3-D motion. We say that the particle has two or three *degress of freedom*, respectively.

Similarly, the location of one reference frame (equivalently, a rigid body) is specified relative to another frame by three parameters in 2-D motion (say x, y and θ as shown in Fig. 5-17); thus a rigid body in 2-D motion has threee degrees of freedom. In Section 4.2 it was established that a rigid body in 3-D motion has six degrees of freedom. Table 5-1 summarizes the degrees of freedom of particles and rigid bodies in unconstrained 2-D and 3-D motion.

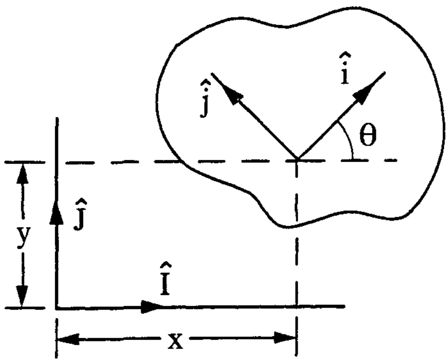


Fig. 5-17

Degrees of Freedom	2-D Motion	3-D Motion
Particle	2	3
Rigid Body	3	6

Table 5-1

In many cases the motion is constrained in some way. Consider a particle in 3-D motion. A constraint of the type

$$f(x, y, z) = 0 \tag{5.28}$$

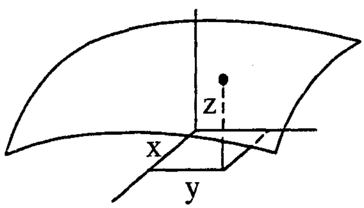


Fig. 5-18

on the motion of the particle is called a *holonomic* constraint. This constraint defines a surface in space upon which the particle is constrained to move (Fig. 5-18). Since Eqn. (5.28) can be solved for one of the coordinates, say $z = \phi(x, y)$, it is clear that now it takes only two parameters to specify the particles location so that the degrees of freedom (DOF) are now two. In general, each independent holonomic constraint reduces the DOF of the motion of a particle or a rigid body by one. This is illustrated in Fig. 5-19 for a particle in 3-D motion, where L is the number of independent holonomic constraints.

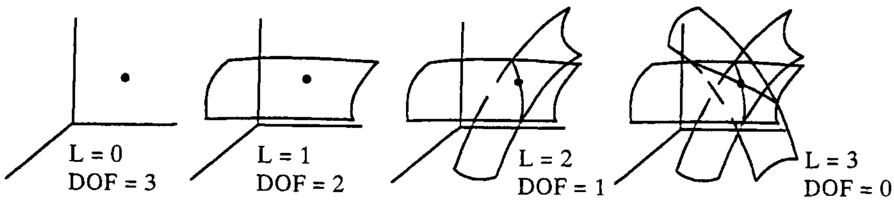


Fig. 5-19

One way to view the 2-D motion of a particle is that it is the case of 3-D motion with one holonomic constraint, $z = 0$ (Fig. 5-20).

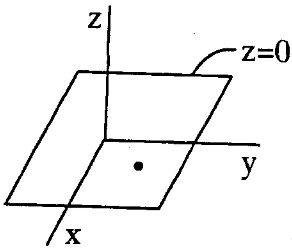


Fig. 5-20

Notes

1 This requirement is not always met, for example, by particles possessing properties in addition to mass, such as electric charge.

2 That is, it may be idealized that all the mass lies on the rod axis.

3 For example it may be required to keep its solar panels pointing at the sun.

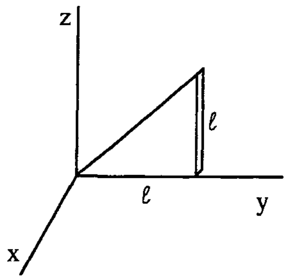
4 The proof of this is due to Newton.

5 As measured, for example, by a spring-loaded scale.

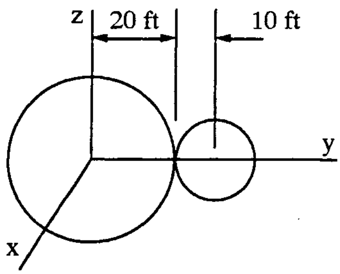
Problems

5/1 Find the center of mass of the homogeneous thin triangular plate of mass M relative to the axes shown using Eqn. (5.13).

5/2 A satellite may be idealized as made up of two radially symmetric spherical bodies as shown. The radius of the larger body is 20 ft and its mass is 60 slugs. The radius of the smaller body is 10 ft and its mass is 20 slugs. Find the center of mass relative to the axes shown, using Eqn. (5.9).

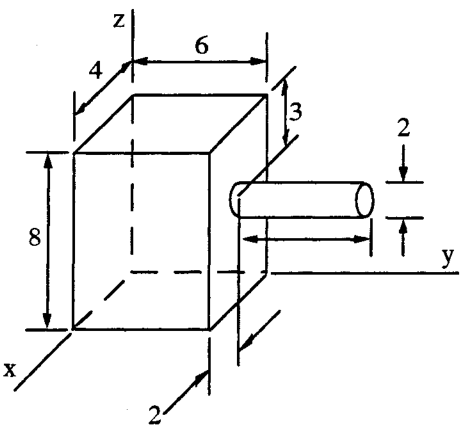


Problem 5/1



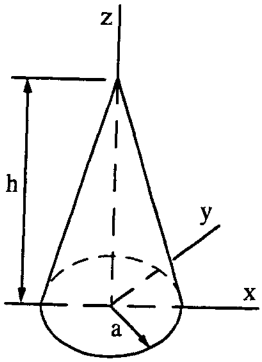
Problem 5/2

5/3 The assembly shown consists of two homogeneous objects each of the same density. Find the center of mass relative to the axes shown. All dimensions are in inches.

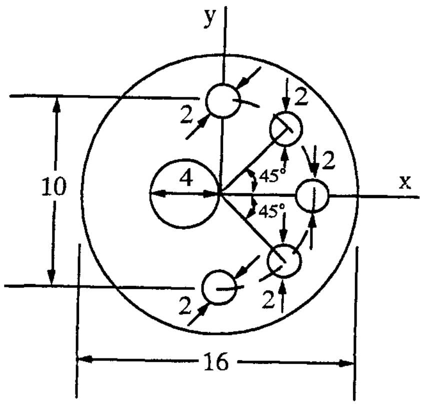


Problem 5/3

5/4 The total mass of the homogeneous right circular cone is M. Find the center of mass relative to the axes shown, using Eqn. (5.13).



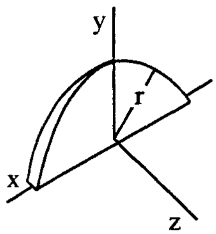
Problem 5/4



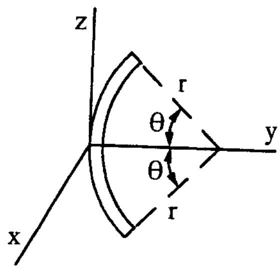
Problem 5/5

5/5 Find the center of mass relative to the axes shown of the thin uniform plate. The plate has six circular holes.

5/6 Find the center of mass of the thin, uniform, semicircular disk relative to the axes shown, using Eqn. (5.13). The mass of the plate is M.



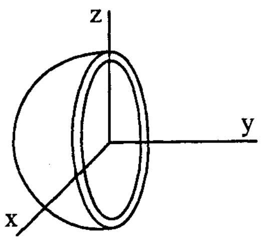
Problem 5/6



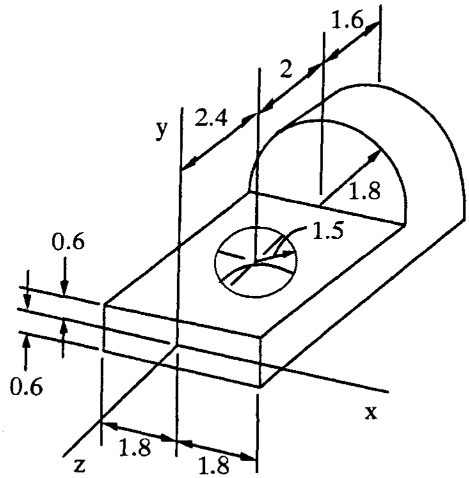
Problem 5/7

5/7 A thin uniform wire is bent into the shape of a circular arc lying in the (x, y) plane as shown. The mass per unit length of the wire is M. Determine the center of mass relative to the axes shown.

5/8 Determine the center of mass relative to the axes shown of the thin, uniform, hemispherical shell of mass M and radius R.



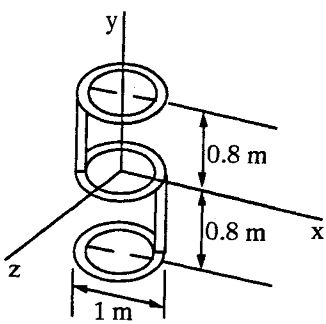
Problem 5/8



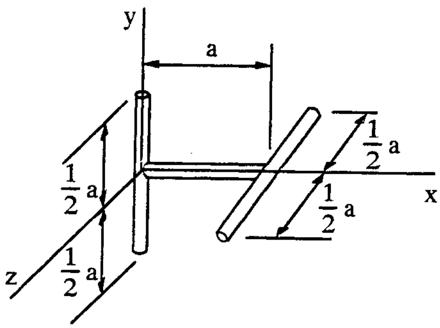
Problem 5/9

5/9 The machined element is made of a homogeneous material of mass density ρ. All dimensions are in inches. Find the center of mass relative to the axes shown.

5/10 Homogeneous thin wire is used to form the figure shown. Find the center of mass relative to the axes shown.



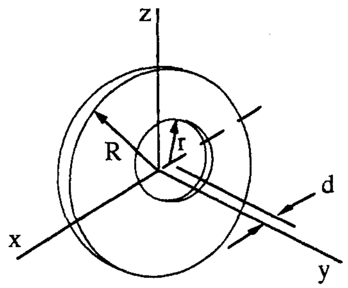
Problem 5/10



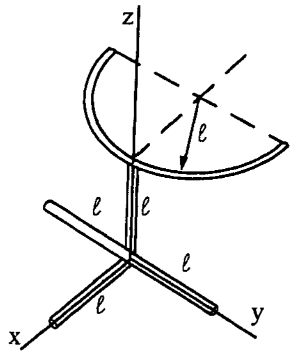
Problem 5/11

5/11 Three thin uniform rods, each of mass m and length a, are joined as shown. Find the center of mass of the assembly relative to the axes shown.

5/12 The figure shows a thin homogeneous disk with an off-center hole. Find the center of mass relative to the axes shown. The mass per unit area of the material is ρ.



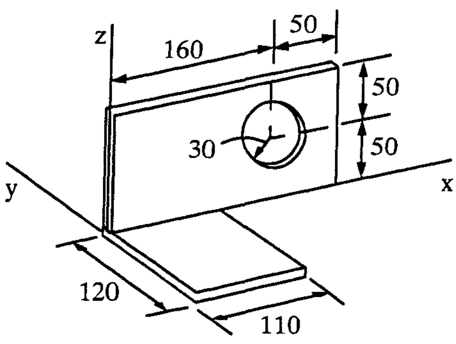
Problem 5/12



Problem 5/13

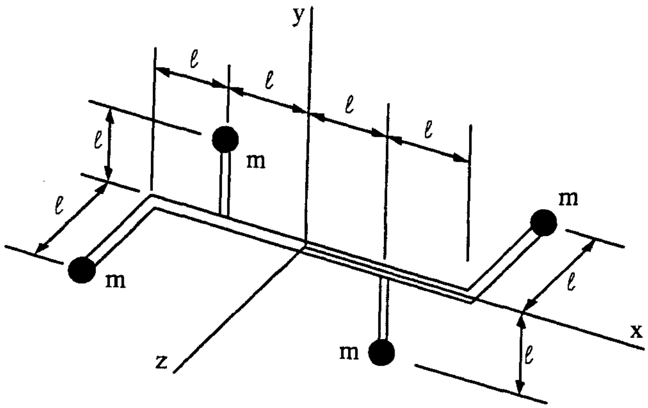
5/13 Thin homogeneous rods are welded together as shown. The mass per unit length of the rods is ρ. Determine the center of mass of the assembly relative to the axes shown.

5/14 The machine part is made of homogeneous thin plate of mass per unit area 13.45 Kg. All dimensions are in mm. Determine the center of mass relative to the axes shown.



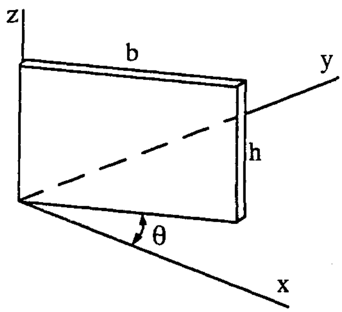
Problem 5/14

5/15 Four mass particles are connected by rods of negligible weight as shown. Determine the center of mass relative to the axes shown.



Problem 5/15

5/16 Find the center of mass relative to the axes shown of the thin uniform plate of mass m.



Problem 5/16

5/17 A person weight 150 lb at sea level. How much does she weigh at an altitude of 500 mi above sea level? Take the radius of the earth to be 3960 mi.

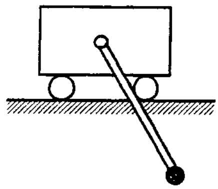
5/18 Suppose a person weighs 150 lb at the surface of the earth. How much would he or she weigh at the surface of the moon? The mass of the earth is 81 times that of the moon, and the radius of the earth is 11/3 the radius of the moon. Neglect the gravitational attraction of the earth on the person at the moon.

5/19 Two particles in 3-D motion are constrained to lie on a surface $f(x, y, z) = 0$. How many DOF does this system of particles have?

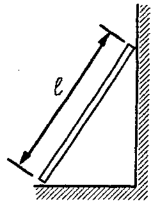
5/20 How many DOF does a simple planar pendulum have?

5/21 A spherical pendulum consists of a particle which moves under the gravitational force on a frictionless spherical surface. How many DOF does a spherical pendulum have?

5/22 A simple pendulum is attached to a cart that rolls on a track. How many DOF does this system have?



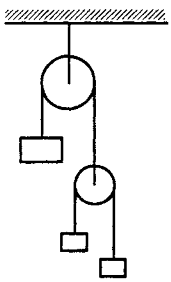
Problem 5/22



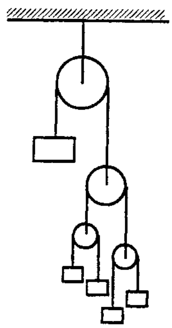
Problem 5/23

5/23 A ladder of length ℓ leans against a wall. How many DOF does it have?

5/24 The figure shows three masses attached to two ropes that pass over two pulleys. State the number of DOF.



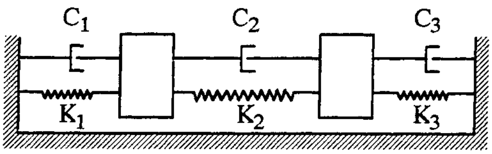
Problem 5/24



Problem 5/25

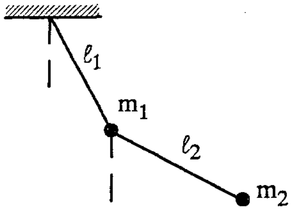
5/25 The figure shows five masses attached to four ropes that pass over four pulleys. How many DOF are there?

5/26 Two masses are connected by springs and dampers as shown. State the number of DOF of this system. Does the removal of the springs and dampers affect the number of DOF?

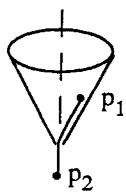


Problem 5/26

5/27 The figure shows a simple double plane pendulum. State the number of DOF of the system.



Problem 5/27



Problem 5/28

5/28 Two heavy particles are connected by a massless, inextensible string as shown. p_1 is constrained to move on the surface of an inverted circular cone and p_2 passes through a hole in the apex of the cone. How many DOF are there?

Chapter 6

Kinetics Of The Mass Center Of A Rigid Body

6.1 Equations of Motion, Two Dimensions

In Chapter 5 it was shown that the motion of the mass center of a rigid body is subject to the same equation as is the motion of a mass particle; that is,

$$\boxed{\underline{F}^e = m\overline{\underline{a}}} \tag{6.1}$$

where $\underline{F}^e$ is the resultant of the external forces acting on the rigid body, m is the body's mass, and $\overline{\underline{a}}$ is the acceleration of the mass center in an inertial reference frame.

In order to obtain scalar equations of motion we now resolve the acceleration of the center of mass in rectangular components (see Eqn. (2.3)):

$$\overline{\underline{a}} = \ddot{\overline{x}}\hat{\imath} + \ddot{\overline{y}}\hat{\jmath} \tag{6.2}$$

Also resolve the resultant external force acting on the body into rectangular components:

$$\underline{F}^e = F_x\hat{\imath} + F_y\hat{\jmath} \tag{6.3}$$

Then from Eqn. (6.1) the equations of motion of the mass center in 2-D motion in rectangular components are

$$\boxed{F_x = m\ddot{x}\ , \qquad F_y = m\ddot{y}} \tag{6.4}$$

As shown in Chapter 2, for 2-D motion it is frequently convenient to resolve $\underline{a}$ in tangential-normal components; in this case the equations of motion are (see Eqn. (2.14)):

$$\boxed{F_t = m\dot{\nu}\ , \qquad F_n = m\frac{\nu^2}{\rho} = m\nu|\dot{\theta}|} \tag{6.5}$$

where $\underline{F}^e = F_t\hat{e}_t + F_n\hat{e}_n$ In radial-transverse components (see Eqn. (2.18)), the equations are

$$\boxed{F_r = m(\ddot{r} - r\dot{\theta}^2)\ , \qquad F_\theta = m(2\dot{r}\dot{\theta} + r\ddot{\theta})} \tag{6.6}$$

where $\underline{F}^e = F_r\hat{e}_r + F_\theta\hat{e}_\theta$. The kinematic quantities in Eqns. (6.5) and (6.6) must refer to the motion of the mass center relative to an inertial frame.

6.2 Example

A small mass P of mass 2 kg slides in a smooth[1] guide in the vertical plane, as shown on Fig. 6-1. If P has constant horizontal velocity of 4.8 m/s at the top of the guide, where the radius of curvature is $\rho = 0.5$ m, what is the force $\underline{R}$ exerted by the guide on P at that point? The guide is fixed in an inertial frame.

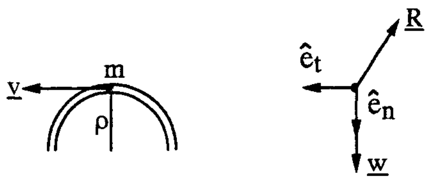


Fig. 6-1

Let $\underline{R} = R_t\hat{e}_t + R_n\hat{e}_n$. Applying Eqns. (6.5),

$$F_t = m\dot{v} \Longrightarrow R_t = 0$$

$$F_n = m\frac{v^2}{\rho} \Longrightarrow R_n + W = \frac{mv^2}{\rho}$$

$$R_n = m\left(\frac{v^2}{\rho} - g\right) = 72.5 \text{ N}$$

so that $\underline{R} = 72.5\hat{e}_n$ N.

6.3 Aircraft Equations of Motion in a Vertical Plane

Consider the motion of an aircraft in a vertical plane. The curvature of the surface and rotation of the earth are neglected. Because we are interested only in the translational motion of the aircraft, only the motion of the center of mass needs to be considered. There are three forces acting on the aircraft (Fig. 6-2):

$\underline{W}$, the aircraft weight

$\underline{T}$, the engine thrust

$\underline{A}$, the total aerodynamic force

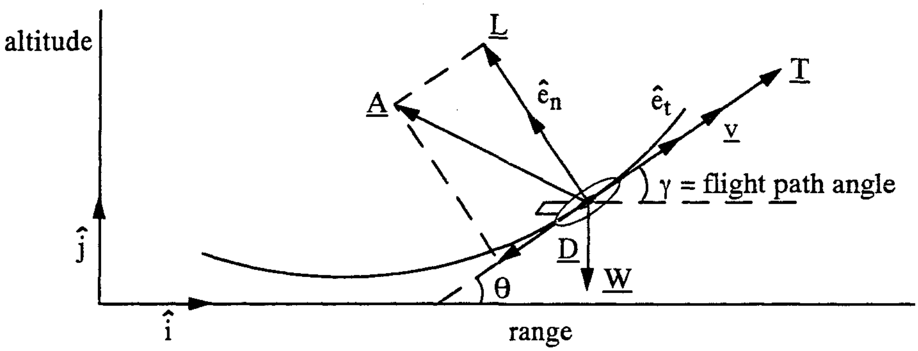


Fig. 6-2

The aerodynamic force is traditionally resolved into components along the velocity vector (called the drag, $\underline{D}$) and perpendicular to it (the lift,

$\underline{L}$). Consequently the resultant external force on the aircraft is

$$\underline{F}^e = \underline{W} + \underline{T} + \underline{D} + \underline{L}$$

Resolving this into tangential – normal components:

$$\underline{F}^e = -W \sin\gamma \hat{e}_t - W \cos\gamma \hat{e}_n + T\hat{e}_n + L\hat{e}_n - D\hat{e}_t$$

where γ, called the flight path angle, is the angle between $\underline{\nu}$ and the horizontal unit vector $\hat{i}$, and thus it is the same as θ on Fig. 2-3. Note that it has been assumed that the thrust vector lies along the velocity vector, which may not be a good assumption for some aircraft.

We now apply Newton's Second Law for the mass center of the aircraft, written in tangential-normal components. From Eqns. (6.5),

$$(T - D - W \sin\gamma)\hat{e}_t + (L - W\cos\gamma)\hat{e}_n = m(\dot{\nu}\hat{e}_t + \nu\dot{\gamma}\hat{e}_n)$$

The two scalar component equations of this vector equations are[2]

$$\begin{aligned} m\dot{\nu} &= T - D - W\sin\gamma \\ m\nu\dot{\gamma} &= L - W\cos\gamma \end{aligned}$$

Also, the components of $\underline{\nu}$ may be written as

$$\begin{aligned} \dot{x} &= \nu\cos\gamma \\ \dot{y} &= \nu\sin\gamma \end{aligned}$$

The collection of these equations may be written as a system of first order differential equations with the derivative terms isolated on the left-hand side:

$$\begin{aligned} \dot{\nu} &= \frac{1}{m}(T - D - W\sin\gamma) \\ \dot{\gamma} &= \frac{1}{m\nu}(L - W\cos\gamma) \\ \dot{x} &= \nu\cos\gamma \\ \dot{y} &= \nu\sin\gamma \end{aligned}$$

These equations are said to be in *state variable form* (ν, γ, x, y are the *state* variables). All equations of motion resulting from Newton's laws may be put into this form, a convenient one for further analysis and computational solution.

The equations may be used to solve either of the two basic problems of dynamics. First, given independent control forces T and L (D depends on L), what is the motion of the aircraft (that is, what are ν, γ, x and y as functions of time)? It is clear that this type of analysis involves integration. Second, given a specific motion, what are the control forces required to produce that motion; this analysis would require differentiation.

6.4 Equations of Motion, Three Dimensions

Proceeding in the same way as in Section 6.1, we obtain the equations of motion for 3-D motion in rectangular components as

$$\boxed{F_x = m\ddot{\bar{x}}\,, \qquad F_y = m\ddot{\bar{y}}\,, \qquad F_z = m\ddot{\bar{z}}} \tag{6.7}$$

In cylindrical components, Eqn. (3.23), the equations are

$$\boxed{\begin{aligned} F_r &= m(\ddot{r} - r\dot{\theta}^2) \\ F_\theta &= m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) \\ F_z &= m\ddot{z} \end{aligned}} \tag{6.8}$$

and in spherical components, Eqn. (3.27), they are

$$\boxed{\begin{aligned} F_R &= m(\ddot{R} - R\dot{\phi}^2 - R\dot{\theta}^2\cos^2\phi) \\ F_\phi &= m\left[\frac{1}{R}\frac{d}{dt}(R^2\dot{\phi}) + R\dot{\theta}^2\sin\phi\cos\phi\right] \\ F_\theta &= m\left[\frac{\cos\phi}{R}\frac{d}{dt}(R^2\dot{\theta}) - 2R\dot{\phi}\dot{\theta}\sin\phi\right] \end{aligned}} \tag{6.9}$$

As before, all kinematic quantities in these equations must refer to the motion of the center of mass in an inertial frame.

6.5 Example

Consider again the screw-jack of Section 3.6. Suppose $a = 0.1$ m, $K = 10$ rad/s^2, $L = 0.5$ m, and that the ball A has mass $m = 0.2$ kg. Then

$$F_r = -4a\pi Km = -2.51$$
$$F_\theta = aKm = 0.20$$
$$F_z = \frac{LKm}{2\pi} = 0.16$$

so that

$$\underline{F} = (-2.51\hat{e}_r + 0.20\hat{e}_\theta + 0.16\hat{k})\ \mathrm{N}$$

and $|\underline{F}| = 2.52$ N. This is the force exerted on ball A by its supporting arm.

6.6 Motion in Inertial and Non-Inertial Frames

First we will demonstrate that if one reference frame, say $\{\hat{\imath}_1, \hat{\jmath}_1, \hat{k}_1\}$, is inertial, then a second frame, say $\{\hat{\imath}_2, \hat{\jmath}_2, \hat{k}_2\}$, will also be inertial if the second frame translates with constant velocity with no rotation relative to the first frame. Let the center of mass of a rigid body of mass m have accelerations $\overline{\underline{a}}_1$ and $\overline{\underline{a}}_2$ relative to frame (1) and frame (2), respectively. Because frame (2) has constant velocity and no rotation relative to frame (1), Eqn. (4.18) gives $\overline{\underline{a}}_1 = \overline{\underline{a}}_2$. Since frame (1) is inertial, $\underline{F}^e = m\overline{\underline{a}}_1$, and consequently $\underline{F}^e = m\overline{\underline{a}}_2$ so that frame (2) is inertial as well.

Now let frame (1) be inertial and let frame (2) be in general motion relative to frame (1) with angular velocity $\underline{\omega}$ (Fig. 6-3). Since frame (1) is inertial,

$$\underline{F}^e = m\overline{\underline{a}}_1$$

Substituting Eqn. (4.18) gives

$$\boxed{\underline{F}^e = m\left[\underline{a}_{O_2} + \overline{\underline{a}}_2 + \underline{\omega} \times (\underline{\omega} \times \overline{\underline{r}}_2) + 2\underline{\omega} \times \overline{\underline{v}}_2 + \dot{\underline{\omega}} \times \overline{\underline{r}}_2\right]} \qquad (6.10)$$

This clearly shows that in general frame (2) will not be inertial, that is that $\underline{F}^e \neq m\overline{\underline{a}}_2$.

If one were to insist that Newton's Second Law be valid in frame (2), then Eqn. (6.10) must be written as

$$\underline{F}^e - m\left[\underline{a}_{O_2} + \underline{\omega} \times (\underline{\omega} \times \overline{\underline{r}}_2) + 2\underline{\omega} \times \overline{\underline{v}}_2 + \dot{\underline{\omega}} \times \overline{\underline{r}}_2\right] = m\overline{\underline{a}}_2$$

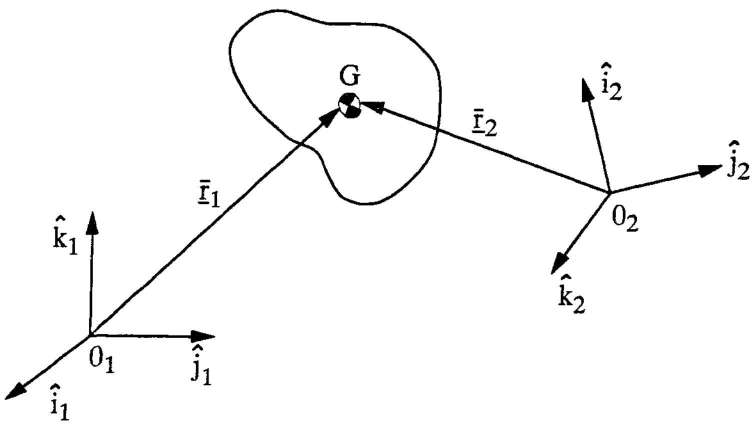


Fig. 6-3

so that many of the acceleration terms must be regarded as "fictitious forces." This is the origin of labels such as "centrifugal force" for the term $-m\underline{\omega} \times (\underline{\omega} \times \overline{\underline{r}}_2)$. This viewpoint, however, is unnecessarily complicated and potentially misleading. It is much better to use Eqn. (6.10) and regard these terms as they properly are, that is as acceleration terms.

6.7 Example – Rotating Cylindrical Space Station

Experience has shown that long-duration space travel causes severe physiological problems for space travelers, due to the lack of a gravitational force, some of which are irreversible. One solution to this problem is to spin the space station to create an "artificial gravity."

Consider a space station in the shape of a cylindrical shell that is rotating about its axis of symmetry with angular velocity $\underline{\omega}$ relative to an inertial frame (Fig. 6-4). Introduce an inertial frame $\{\hat{I}, \hat{J}\}$ and a

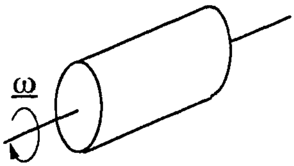


Fig. 6-4

station-fixed frame $\{\hat{\imath}, \hat{\jmath}\}$ as shown in Fig. 6-5. Consider a point A on the surface of the cylinder with position vector $\underline{r}_A = r\hat{\imath}$. Then, from Eqns. (4.17) and (4.18), the velocity and acceleration of A relative to $\{\hat{I}, \hat{J}\}$ are

$$\underline{v}_A = (-\omega\hat{k}) \times (r\hat{\imath}) = -\omega r\hat{\jmath}$$
$$\underline{a}_A = (-\omega\hat{k}) \times \left[(-\omega\hat{k}) \times (r\hat{\imath})\right] = -\omega^2 r\hat{\imath}$$

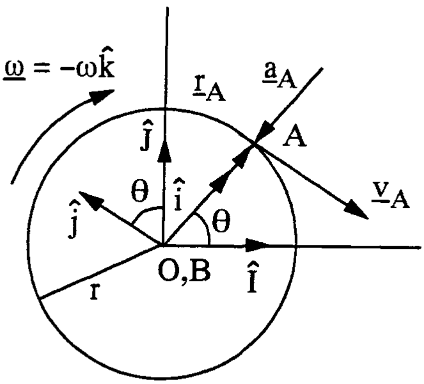


Fig. 6-5

For a person on the rim at A, neglecting the distance between the person's center of mass and the rim,

$$\underline{F}^e = m\underline{a}_A$$

To experience the same contact force from the wall as he or she would from the surface of the earth (Fig. 6-6)

$$-mg\hat{\imath} = -m\omega^2 r\hat{\imath}$$

where g is the gravitational acceleration at the earth surface. The speed at which the craft must be rotated to simulate the earth's gravity is then $\omega = \sqrt{g/r}$, and $\nu_A = \sqrt{gr}$.

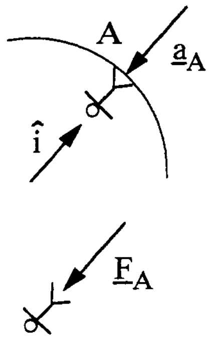


Fig. 6-6

As long as the person remains standing on the inside of the cylinder, it will still seem to him or her to be no different than standing on the surface of the earth (except that his or her head will feel somewhat lighter than his or her feet!). When he or she begins to move inside the station, however, the Coriolis term $2\underline{\omega} \times \underline{\nu}_r$ comes into play, and motion relative to the station will be quite different than motion relative to the surface of the earth.

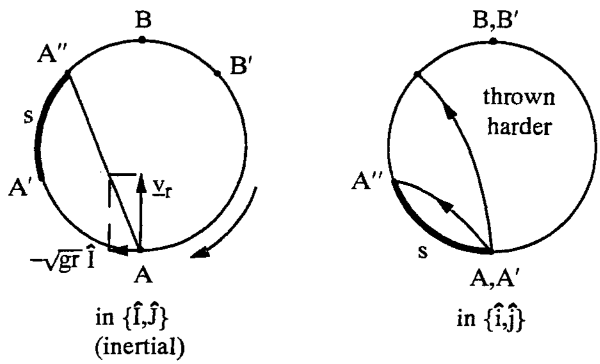


Fig. 6-7

To illustrate this, suppose the person at A wants to throw an object to another person at B, directly opposite A (Fig. 6-7). If he or she throws the object directly at B with velocity $-\nu_r \hat{J}$, then it will have constant velocity $-\sqrt{gr}\hat{I} + \nu_r \hat{J}$ in the inertial frame (there are no forces on the object after it leaves his or her hand) and consequently travels in a straight line relative to $\{\hat{I}, \hat{J}\}$. But while the object is in flight, B has moved to B' in $\{\hat{I}, \hat{J}\}$, and hence the object appears to the occupants of the station to have a curved path and hits "behind" B at A''. It is clear that the person at A must "lead" the person at B by throwing the object "ahead" of him or her (Fig. 6-8).

One way to do this would be to throw the object tangentially along the wall with (relative) speed $\sqrt{gr}$. The object would then have zero inertial velocity in the inertial frame, and consequently zero velocity thereafter, and would appear to the space travelers to move along the cylinder wall until it reaches the person at B; this is the path labeled 1 in Fig. 6-8.

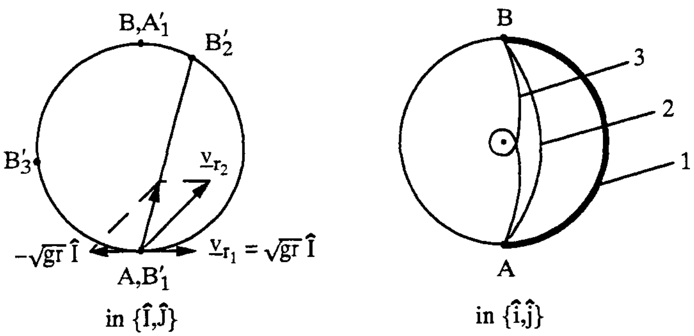


Fig. 6-8

Of course, many paths are possible from A to B, some of which even have "loops." These paths could be determined from Eqns. (4.17) and (4.18) and turn out to be *limacons*.

6.8 Inertial Frames of Reference

In most dynamic analyses, a reference frame fixed relative to the surface of the earth is adopted as an inertial frame. Experiments show, however, that a frame "fixed relative to the stars," and not rotating with the earth, is a far more accurate inertial frame; that is, in such a frame Newton's laws are satisfied to a far greater accuracy.

It is our main purpose in this and the following sections to determine the error associated with assuming that a frame fixed in the earth is inertial, and to identify situations in which this assumption should not be used.

The location of a point on the surface of the earth is specified by longitude and latitude, both angles (Fig. 6-9). Longitude is the angle between a plane passing through the point and the polar axis and a reference plane passing through the polar axis.[3] Latitude, denoted λ, is the angle between a plane parallel to the equatorial plane passing through the point and the equatorial plane itself. It will be assumed that the earth is a perfect sphere with spherical mass symmetry.

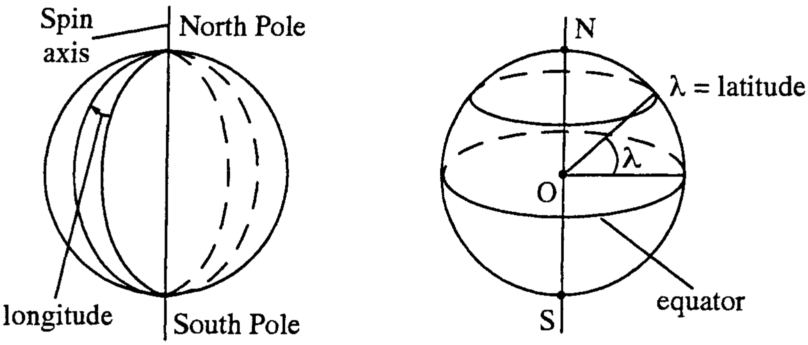


Fig. 6-9

Now consider a point B fixed on the surface of the earth (Fig. 6-10). A reference frame $\{\hat{I}, \hat{J}, \hat{K}\}$ with origin at the center of the earth and "fixed with respect to the stars", with $\hat{J}$ along the polar axis, will be taken as inertial; thus $\hat{I}$ and $\hat{K}$ are in the equitorial plane. Another frame, $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$, is fixed to the surface of the earth with $\hat{k}$ vertical, $\hat{\jmath}$ pointing North, and $\hat{\imath}$ pointing East; the origin is at point B which has latitude λ. The angular velocity of $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ with respect to $\{\hat{I}, \hat{J}, \hat{K}\}$ is $\underline{\omega_e} = \omega_e \hat{J}$, where ω_e is the rotation rate of the earth (one revolution per day, or 0.727×10^{-4} rad/sec). Although the rotation rate of the earth varies slightly over time, the change is extremely small, and we will take

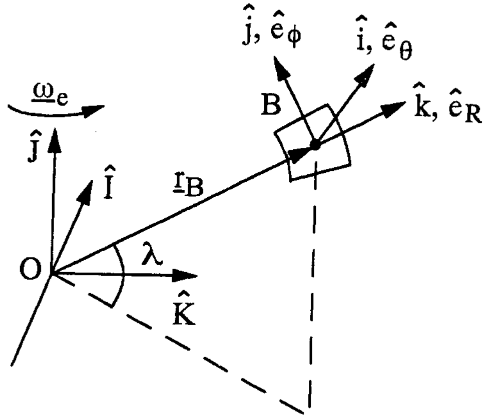


Fig. 6-10

$\dot{\underline{\omega}}_e = \underline{0}$.[4] From the figure, $\underline{r}_B = R\hat{k}$ where R is the radius of the earth (an average value of 6378 Km will be used).

The contact force acting on an object at rest at point B on the surface of the earth is shown in Fig. 6-11. Since there are only two forces acting, the contact force, $\underline{N}$, and the gravitational force, Eqn. (6.1) gives

$$\underline{N} - mg\hat{k} = m\underline{a}_B \tag{6.11}$$

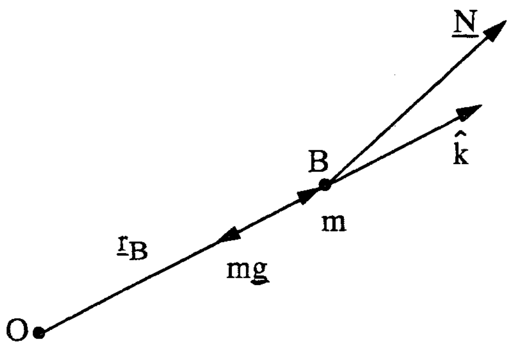


Fig. 6-11

Comparing Figs. 3-11 and 6-10, we note that $\{\hat{\imath}, \hat{\jmath}, \hat{k}\} = \{\hat{e}_\theta, \hat{e}_\phi, \hat{e}_R\}$, the latter being the unit vectors of spherical coordinates, with $\lambda = \phi$ and

$|\underline{r}_B| = R$. Consequently, applying Eqn. (3.27),

$$\begin{aligned} a_R &= -R\omega_e^2 \cos^2\lambda \\ a_\theta &= 0 \\ a_\phi &= R\omega_e^2 \sin\lambda\cos\lambda \end{aligned}$$

so that

$$\underline{a}_B = -R\omega_e^2\cos^2\lambda\hat{k} + R\omega_e^2\sin\lambda\cos\lambda\hat{\jmath} \tag{6.12}$$

(This also could have been determined by noting that B is traveling around the polar axis at rate ω_e on a circle of radius $R\cos\lambda$.) Substituting Eqn. (6.12) into (6.11) and solving for the contact force,

$$\boxed{\underline{N} = (mR\omega_e^2\sin\lambda\cos\lambda)\hat{\jmath} + (mg - mR\omega_e^2\cos^2\lambda)\hat{k}} \tag{6.13}$$

Note that if the earth's rotation is neglected, this reduces to the familiar formula

$$\underline{N} = mg\hat{k} \tag{6.14}$$

Comparing Eqns. (6.13) and (6.14), the extra terms in (6.13) are of order ω_e^2, a very small number. The first term is a "side" component of force, pushing North in the Northern Hemisphere and South in the Southern; it is zero at both poles and along the equator. The second term is the vertical component of the force. The earth rotation term in this component has the effect of reducing the apparent weight of the object; note that this term is zero at both poles. The maximum percentage error, which occurs at the equator, in using Eqn. (6.14) instead of (6.13) is $(R\omega_e^2/g)100 = 0.35\%$.[5]

Equation (6.13) causes ambiguity in what we mean by the term "weight". If weight is defined by $W = mg$, then W will not be equal to the contact force. Alternatively, we could define weight as the magnitude of the contact force:

$$W = |\underline{N}| = m\sqrt{(R\omega_e^2\sin\lambda\cos\lambda)^2 + (g - R\omega_e^2\cos^2\lambda)^2}$$

Another choice would be to define it as the vertical component of the contact force

$$W = m(g - R\omega_e^2\cos^2\lambda)$$

This later choice is perhaps best because it is this force which is measured by a spring-loaded scale. Finally, some textbooks resolve this problem by defining a new gravitational acceleration, say g', by

$$g' = g - R\omega_e^2 \cos^2 \lambda$$

With this definition, $W = mg'$ and is equal to the contact force, but now the gravitational acceleration, g', varies with latitude. Including the earth rotation gives values of g' from 9.780 to 9.833 m/sec^2 (32.09 to 32.26 ft/sec^2).

An even more accurate inertial frame is one that is fixed at the sun and whose axes have fixed directions relative to the "fixed stars". The mean distance from the sun to the earth, R_s, is about 1.5×10^8 Km, and the angular speed of the earth about the sun, ω_s, is one revolution per year, or 1.20×10^{-7} rad/sec. Thus the percentage error in using an earth fixed instead of a sun fixed frame is $(R_s\omega_s^2/g)100 = 0.02\%$. Even a sun-fixed frame is not perfectly inertial because the stars, of course, are not fixed in position. Although they are moving rapidly relative to each other, the stars are so far away that their apparent relative motion is extremely small, and a sun-fixed frame is a very accurate inertial frame.

Suppose that we are able to speed up the rotation of the earth. Eventually a rotation speed, say ω_e', would be reached such that the vertical component of Eqn. (6.13) becomes zero; that is $g = R{\omega_e'}^2$ at the equator. An object on the equator would than be "weight-less". The speed of the object would be $\nu = R\omega_e' = \sqrt{gR} = 7.9$ Km/sec. This is in fact approximately the speed of a satellite in a low circular earth orbit.

6.9 Motion Near the Surface of the Earth

Now let a point A move close to point B (Fig. 6-12). By close, we mean that we may take

$$|\underline{r}| \ll R = |\underline{r}_B| \approx |\underline{r}_A|$$

From Eqn. (4.18) the acceleration of point A in the inertial frame, $\underline{a}_A$, is related to its acceleration in the surface-fixed frame, $\underline{a}_r$, by

$$\underline{a}_A = \underline{a}_B + \underline{a}_r + \underline{\omega}_e \times (\underline{\omega}_e \times \underline{r}) + 2\underline{\omega}_e \times \underline{\nu}_r + \dot{\underline{\omega}}_e \times \underline{r}$$

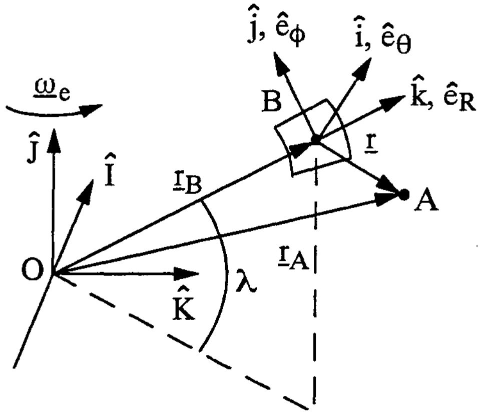


Fig. 6-12

Since ω_e is small, ω_e^2 will be very small, and we will neglect terms with ω_e^2 compared to terms with ω_e. Clearly, the centripetal term depends on ω_e^2, and the Coriolis term on ω_e. It has been shown in Eqn. (6.12) that $\underline{a}_B$ depends on ω_e^2, and we are neglecting $\dot{\underline{\omega}}_e \times \underline{r}$. Thus, approximately, $\underline{a}_A = \underline{a}_r + 2\underline{\omega}_e \times \underline{\nu}_r$, and Eqn. (6.10) gives the following result for the translational motion of a rigid body whose mass center is at A:

$$\boxed{m\underline{a}_r = \underline{F}^e - 2m\underline{\omega}_e \times \underline{\nu}_r} \tag{6.15}$$

The term $2m\underline{\omega}_e \times \underline{\nu}_r$, sometimes called the "Coriolis effect," is usually the largest term neglected when Newton's Laws are written in a frame attached to the earth surface. This term will be important either when the body is moving at a high rate of speed relative to the earth surface, or when all forces acting on the object are very small.

The Coriolis effect is often given as the reason for many phenomena, such as

1. Bathtub whirlpools turning in one direction in the Northern hemisphere and in the opposite direction in the Southern. This is true in the absence of other, larger effects.

2. Pig's tails curling in one direction in the Northern hemisphere and in the other direction in the Southern. A survey showed this to be not true!

3. Many weather patterns. This is true. It is a contributing factor in the large-scale circulation of air masses and in small-scale disturbances such as hurricanes.

6.10 Projectile Motion

It was observed in the previous section that the Coriolis effect will be important when $|\underline{v}_r|$ is large, that is when an object moves at high speed relative to the surface of the earth. Examples are high-speed airplanes, rockets, and projectiles. We will consider in this section the motion of a projectile launched with initial velocity $\underline{v}_{r_0}$ and subject to only the constant gravitational force.

Since A stays close to B by assumption, the force acting on the projectile is $\underline{F}^e = -mg\hat{k}$, approximately. Thus, the equation of motion (6.15) reduces to:

$$\underline{a}_r = -g\hat{k} - 2\underline{\omega}_e \times \underline{v}_r \tag{6.16}$$

Suppose the projectile is launched with initial velocity $\underline{v}_{r_0}$ at a point B on the surface of the earth at latitude λ, with azimuth angle θ and elevation angle ϕ (Fig. 6-13). Spherical coordinates are an obvious choice for this problem; writing $\underline{v}_{r_0} = v_0\hat{e}_R$, Eqn. (3.35) gives

$$\begin{pmatrix} v_{0x} \\ v_{0y} \\ v_{0z} \end{pmatrix} = [\mathrm{T}_{\phi\theta}]^{-1} \begin{pmatrix} v_0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} v_0 \cos\theta\cos\phi \\ v_0 \sin\theta\cos\phi \\ v_0 \sin\phi \end{pmatrix}$$

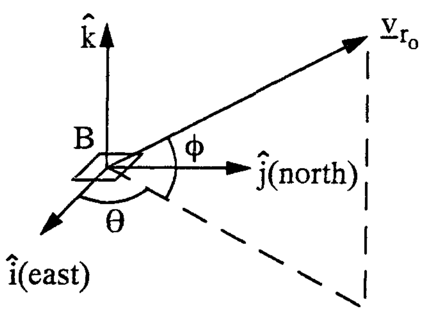


Fig. 6-13

so that

$$\underline{\nu}_{r_0} = \nu_0 \cos\theta \cos\phi \hat{\imath} + \nu_0 \sin\theta \cos\phi \hat{\jmath} + \nu_0 \sin\phi \hat{k} \tag{6.17}$$

Writing $\underline{\omega}_e$ in components along $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$:

$$\underline{\omega}_e = \omega_e \hat{J} = \omega_e \cos\lambda \hat{\jmath} + \omega_e \sin\lambda \hat{k} \tag{6.18}$$

Thus,

$$\begin{aligned}\underline{\omega}_e \times \underline{\nu}_{r_0} = \omega_e \nu_0 \Big[&(\cos\lambda \sin\phi - \sin\lambda \sin\theta \cos\phi)\hat{\imath} \\ &+ \sin\lambda \cos\theta \cos\phi \hat{\jmath} - \cos\lambda \cos\theta \cos\phi \hat{k}\Big]\end{aligned} \tag{6.19}$$

We will now integrate Eqn. (6.16). Setting $\underline{\nu}_r = d\underline{r}/dt$ and $\underline{a}_r = d^2\underline{r}/dt^2$, this equation becomes

$$\frac{d^2\underline{r}}{dt^2} + 2\underline{\omega}_e \times \frac{d\underline{r}}{dt} = \underline{g} \tag{6.20}$$

Integrating once gives

$$\frac{d\underline{r}}{dt} + 2\underline{\omega}_e \times \underline{r} = \underline{g}t + \underline{A} \tag{6.21}$$

where $\underline{A}$ is a vector constant of integration. Substituting Eqn. (6.21) in (6.20) and neglecting terms with ω_e^2, we get

$$\frac{d^2\underline{r}}{dt^2} + 2\underline{\omega}_e \times \underline{g}t + 2\underline{\omega}_e \times \underline{A} = \underline{g}$$

Integrating twice gives

$$\frac{d\underline{r}}{dt} + \underline{\omega}_e \times \underline{g}t^2 + 2\underline{\omega}_e \times \underline{A}t = \underline{g}t + \underline{B} \tag{6.22}$$

and

$$\underline{r} + \frac{1}{3}\underline{\omega}_e \times \underline{g}t^3 + \underline{\omega}_e \times \underline{A}t^2 - \frac{1}{2}\underline{g}t^2 = \underline{B}t + \underline{C} \tag{6.23}$$

where $\underline{B}$ and $\underline{C}$ are two more vector constants.

The constants, $\underline{A}$, $\underline{B}$, and $\underline{C}$ are found from applying initial conditions to Eqns. (6.21), (6.22), and (6.23):

$$\begin{aligned}\underline{\nu}_{r_0} &= \underline{A} \\ \underline{\nu}_{r_0} &= \underline{B} \\ \underline{0} &= \underline{C}\end{aligned}$$

Thus, Eqn. (6.23) becomes

$$\underline{r} = -\frac{1}{3}\underline{\omega}_e \times \underline{g}t^3 - \underline{\omega}_e \times \underline{\nu}_{r_0}t^2 + \frac{1}{2}\underline{g}t^2 + \underline{\nu}_{r_0}t \tag{6.24}$$

To write this in components, we write $\underline{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ and use Eqns. (6.17), (6.18), and (6.19); the result is

$$\begin{aligned} x &= \frac{1}{3}g\omega_e(\cos\lambda)t^3 - \nu_0\omega_e(\sin\phi\cos\lambda - \sin\theta\cos\phi\sin\lambda)t^2 \\ &\qquad +\nu_0(\cos\theta\cos\phi)t \\ y &= -\nu_0\omega_e(\cos\theta\cos\phi\sin\lambda)t^2 + \nu_0(\sin\theta\cos\phi)t \\ z &= \nu_0\omega_e(\cos\theta\cos\phi\cos\lambda)t^2 - \frac{1}{2}gt^2 + \nu_0(\sin\phi)t \end{aligned} \tag{6.25}$$

Note that from the last of these equations, the time for the projectile to return to the earth surface ($z = 0$) is

$$t_f = \frac{2\nu_0\sin\phi}{g - 2\nu_0\omega_e\cos\theta\cos\phi\cos\lambda} \tag{6.26}$$

The range would then be determined from substitution into the first two of Eqns. (6.25):

$$R = \sqrt{x^2(t_f) + y^2(t_f)} \tag{6.27}$$

As a simple example, suppose the projectile is launched into the equatorial plane. Then $\theta = 0$ and $\lambda = 0$, and Eqns. (6.25) become:

$$\begin{aligned} x &= \frac{1}{3}g\omega_e t^3 - \nu_0\omega_e\sin\phi t^2 + \nu_0\cos\phi t \\ y &= 0 \\ z &= \nu_0\omega_e\cos\phi t^2 - \frac{1}{2}gt^2 + \nu_0\sin\phi t \end{aligned} \tag{6.28}$$

Thus, the projectile stays in the equatorial plane. Eqn. (6.26) reduces to

$$t_f = \frac{2\nu_0\sin\phi}{g - 2\nu_0\omega_e\cos\phi} \tag{6.29}$$

Substitution into the equation for x gives the range as

$$R = x(t_f) = \frac{\nu_0^2\sin 2\phi}{g} - \frac{4\omega_e\nu_0^3\sin\phi}{3g^2}\left(1 - 4\cos^2\phi\right) \tag{6.30}$$

The percentage error associated with neglecting the earth's rotation is then

$$\text{error} = \frac{200\omega_e\nu_0\left(1 - 4\cos^2\phi\right)}{3g\cos\phi} \tag{6.31}$$

6.11 Example – Large Scale Weather Patterns

Suppose a low pressure area develops in the earth's atmosphere at a latitude λ. Then the air molecules will tend to move from the relatively higher pressure surrounding regions towards the low pressure area (Fig. 6-14). A typical air molecule may be viewed as a "projectile launched" in the local horizontal plane with speed ν, with azimuth say θ, and elevation $\phi = 0$. In this case Eqns. (6.17) and (6.19) become

$$\underline{\nu}_r = \nu\cos\theta\hat{\imath} + \nu\sin\theta\hat{\jmath}$$

$$\underline{\omega}_e \times \underline{\nu}_r = \omega_e\nu(-\sin\lambda\sin\theta\hat{\imath} + \sin\lambda\cos\theta\hat{\jmath} - \cos\lambda\cos\theta\hat{k})$$

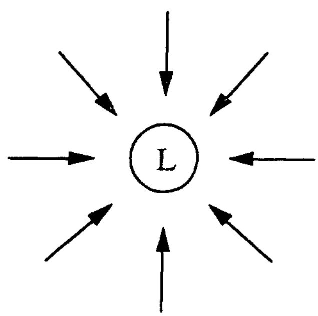


Fig. 6-14

The equation of motion (6.15) gives

$$m\underline{a}_r = -mg\hat{k} + \underline{B} - 2m\omega_e\nu(-\sin\lambda\sin\theta\hat{\imath} + \sin\lambda\cos\theta\hat{\jmath} - \cos\lambda\cos\theta\hat{k}) \qquad (6.32)$$

The buoyant force, $\underline{B}$, is assumed to be just sufficient to hold the air molecule in the local horizontal ($\{\hat{\imath}, \hat{\jmath}\}$) plane; thus

$$\underline{B} = mg\hat{k} - 2m\omega_e\nu\cos\lambda\cos\theta\hat{k}$$

and Eqn. (6.32) becomes

$$\underline{a}_r = 2\omega_e\nu\sin\lambda(\sin\theta\hat{\imath} - \cos\theta\hat{\jmath}) \qquad (6.33)$$

Now we transform to radial-transverse components (Fig. 6-15); the unit vector transformations are (see Section 2.6)

$$\hat{\imath} = \cos\theta\hat{e}_r - \sin\theta\hat{e}_\theta$$

$$\hat{\jmath} = \sin\theta\hat{e}_r + \cos\theta\hat{e}_\theta$$

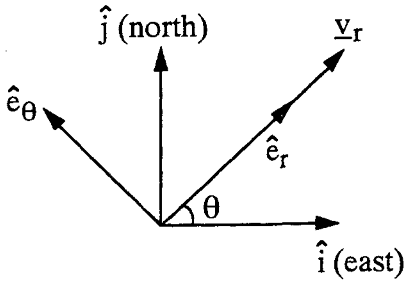


Fig. 6-15

Substituting these into Eqn. (6.33) gives

$$\underline{a}_r = -2\omega_e \nu \sin \lambda \hat{e}_\theta \tag{6.34}$$

First consider the Northern Hemisphere, where $\lambda > 0$. From Eqn. (6.34) the acceleration of the air molecule will be in the negative $\hat{e}_\theta$ direction; that is, the molecule's path will "bend to the right" (Fig. 6-16). Since all the molecules bend to the right, they must end up by

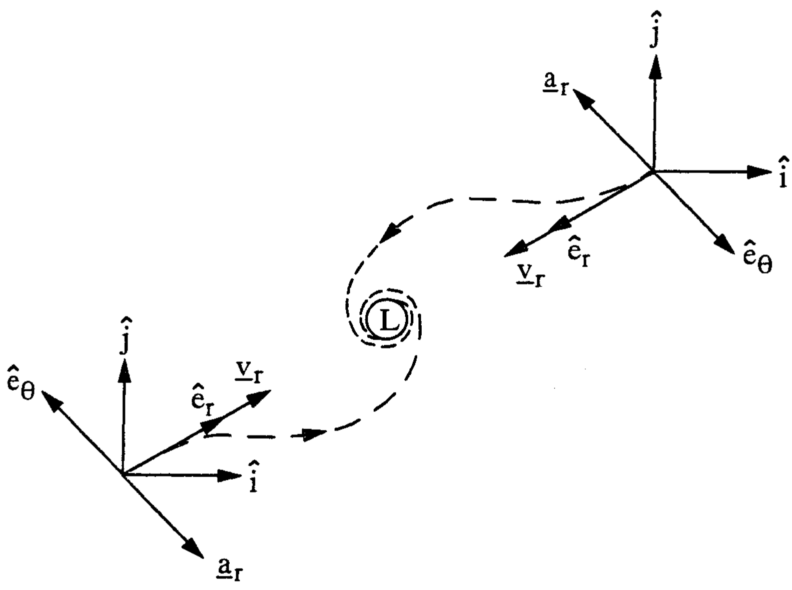


Fig. 6-16

spiraling into the low pressure region in a counter-clockwise direction, as viewed from above. The paths of the molecules are called streamlines in fluid mechanics, and a fundamental principle of fluid mechanics is that streamlines can never cross.[6] In the Southern Hemisphere, $\lambda < 0$, it is easy to see that the air molecules would circulate in a clock-wise direction.

The large scale circulation of the atmosphere caused by pressure differences and the Coriolis effect are large contributors to weather patterns. These circulations are easily visible on satellite pictures of the earth. One manifestation of this pattern is the characteristic of storms moving eastward onto the West Coast of the U.S. beginning with a southernly wind.

6.12 Aircraft Equations of Motion for 3-D Flight

As a final example of the motion of a rigid body's mass center, we will derive the equations of motion of an aircraft in 3-D flight. A reference frame fixed in the earth surface will be considered inertial,[7] a good approximation for low speed flight, and the gravitational force will be taken constant in magnitude and direction.

Symmetric flight of a symmetric airplane is assumed. A symmetric airplane has a plane of symmetry such that the right-hand side of the airplane is a mirror-image of the left-hand side. In symmetric flight, the aircraft is maneuvered in such a way that all aerodynamic forces (lift and drag) are kept in the plane of symmetry (this is sometimes called coordinated turning). Basically, an airplane turns by rolling to create a horizontal component of the lift vector.

Five reference frames will be needed as follows (Fig. 6-17):

1. Inertial, fixed on earth surface.
2. Local horizontal, attached to aircraft, origin at center of mass G, lined up with frame (1).
3. Intermediate, with $\hat{\imath}_3$ along projection of velocity vector $\underline{v}$ in the horizontal plane and $\hat{k}_3$ vertical.
4. Intermediate, $\hat{\imath}_4$ along $\underline{v}$ and $\hat{\jmath}_4$ along $\hat{\jmath}_3$.
5. Wind axes, $\hat{\imath}_5$ along $\underline{v}$, $\hat{k}_5$ along lift vector $\underline{L}$, $\hat{\jmath}_5$ perpendicular to the plane of symmetry.

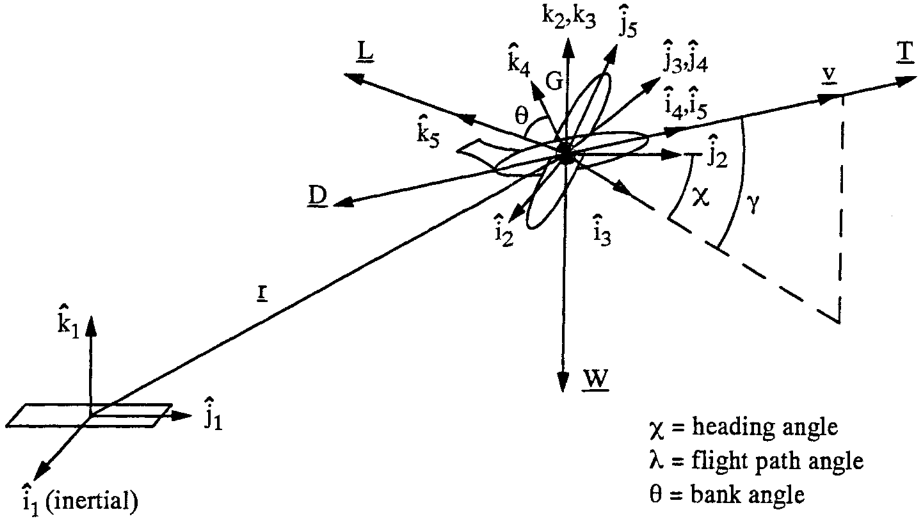


Fig. 6-17

Frame (1) is inertial and frame (5) is the one in which the aerodynamic forces are specified. The other three frames are needed to transition between these two.

The forces acting on the airplane are

$\underline{D}$ = drag, along $\underline{\nu}$ in opposite direction
$\underline{L}$ = lift, perpendicular to $\underline{\nu}$ and in the aircraft plane of symmetry
$\underline{T}$ = thrust, along $\underline{\nu}$
$\underline{W}$ = weight, vertical downward.

Referring to Fig. 6-17, the forces, and the velocity, are written as follows:

$$\begin{aligned}
\underline{W} &= -W\hat{k}_1 = -W\hat{k}_2 = -W\hat{k}_3 \\
\underline{T} &= T\hat{\imath}_4 = T\hat{\imath}_5 \\
\underline{D} &= -D\hat{\imath}_4 = -D\hat{\imath}_5 \\
\underline{L} &= L\hat{k}_5 \\
\underline{\nu} &= \nu\hat{\imath}_4 = \nu\hat{\imath}_5
\end{aligned}$$

We choose to resolve all vectors in frame (4); $\underline{a}$ is then determined as:

$$\underline{a} = \left(\frac{d\underline{\nu}}{dt}\right)_1 = \dot{\nu}\hat{\imath}_4 + \nu\left(\frac{d\hat{\imath}_4}{dt}\right)_1 = \dot{\nu}\hat{\imath}_4 + \nu\underline{\omega}_{4/1} \times \hat{\imath}_4$$

$$
\begin{aligned}
\underline{\omega}_{4/1} &= \underline{\omega}_{4/3} + \underline{\omega}_{3/2} + \underline{\omega}_{2/1} \\
\underline{\omega}_{4/3} &= -\dot{\gamma}\hat{j}_4 \,, \qquad \underline{\omega}_{3/2} = -\dot{\chi}\hat{k}_3 \,, \qquad \underline{\omega}_{2/1} = \underline{0} \\
\underline{\bar{a}} &= \dot{\nu}\hat{i}_4 + \nu\left(-\dot{\gamma}\hat{j}_4 - \dot{\chi}\hat{k}_3\right) \times \hat{i}_4
\end{aligned}
$$

The required unit vector transformations are (Fig. 6-18):

$$
\begin{aligned}
\hat{k}_3 &= \cos\gamma\hat{k}_4 + \sin\gamma\hat{i}_4 \\
\hat{k}_5 &= \cos\theta\hat{k}_4 - \sin\theta\hat{j}_4
\end{aligned}
$$

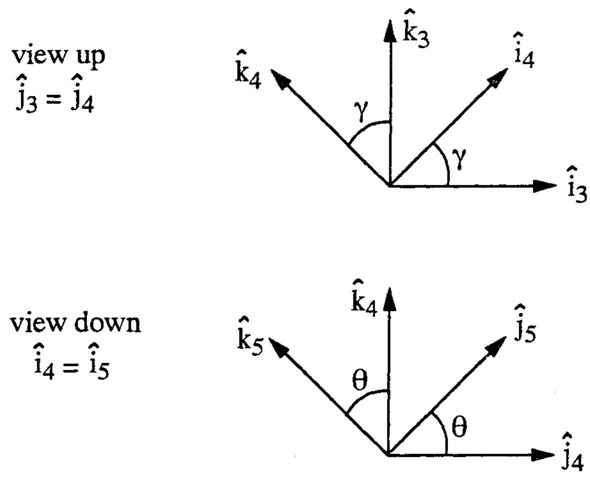


Fig. 6-18

Consequently, the resolutions of the acceleration and force vectors in frame (4) are:

$$
\begin{aligned}
\underline{\bar{a}} &= \dot{\nu}\hat{i}_4 + \nu\dot{\gamma}\hat{k}_4 - \nu\dot{\chi}\cos\gamma\hat{j}_4 \\
\underline{F}^e &= -W(\cos\gamma\hat{k}_4 + \sin\gamma\hat{i}_4) + T\hat{i}_4 - D\hat{i}_4 + L(\cos\theta\hat{k}_4 - \sin\theta\hat{j}_4)
\end{aligned}
$$

Since frame (1) is inertial, the motion of the aircraft mass center is governed by:

$$
\begin{aligned}
&\underline{F}^e = m\underline{\bar{a}} \\
&(T - D - W\sin\gamma)\hat{i}_4 - L\sin\theta\hat{j}_4 + (L\cos\theta - W\cos\gamma)\hat{k}_4 \\
&\qquad = m(\dot{\nu}\hat{i}_4 - \nu\dot{\chi}\cos\gamma\hat{j}_4 + \nu\dot{\gamma}\hat{k}_4)
\end{aligned}
$$

The components of this vector equation are:

$$T - D - W\sin\gamma = m\dot{\nu}$$
$$-L\sin\theta = -m\nu\dot{\chi}\cos\gamma$$
$$L\cos\theta - W\cos\gamma = m\nu\dot{\gamma}$$

Rearranging to put these equations in state variable form:

$$\begin{aligned}\dot{\nu} &= \frac{1}{m}(T - D - W\sin\gamma)\\ \dot{\chi} &= \frac{L\sin\theta}{m\nu\cos\gamma}\\ \dot{\gamma} &= \frac{L\cos\theta - W\cos\gamma}{m\nu}\end{aligned}$$

Noting that χ and γ are the spherical coordinates of the velocity vector,

$$\begin{aligned}\dot{x} &= \nu\cos\gamma\sin\chi\\ \dot{y} &= \nu\cos\gamma\cos\chi\\ \dot{h} &= \nu\sin\gamma\end{aligned}$$

This system of six differential equations has many applications in aircraft flight dynamics. For $\theta = 0$ they reduce to the plane motion equations derived in Section 6.3.

Notes

1 i.e. friction may be ignored.

2 It is clear that the component equations would contain more terms if written in components other than tangential-normal, and thus these are the preferable components for this problem.

3 The reference plane is usually taken as the plane passing through Greenwich, England.

4 Over the last few years the largest fluctuation in the earth's rotation rate has been measured as 5×10^{-18} rad/sec^2.

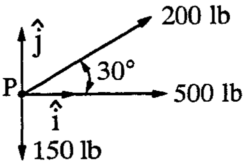
5 It should be noted that the earth is not a perfect sphere with spherical mass symmetry, and this also causes variations in earth contact forces.

6 Otherwise, two mass particles would be in the same place at the same time.

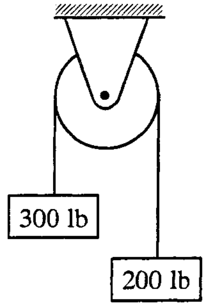
7 That is, we will neglect the Coriolis effect and all other effects of the earth's rotation. The equation of motion will therefore be Eqn. (6.1).

Problems

6/1 Find the resultant of the coplanar forces acting on the particle P as shown.



Problem 6/1



Problem 6/5

6/2 A force $\underline{F} = 5\hat{\imath}+6\hat{\jmath}-3\hat{k}$ lb acts on a particle of mass 2 slugs. Express the acceleration vector in $\{\hat{\imath},\hat{\jmath},\hat{k}\}$ components where $\{\hat{\imath},\hat{\jmath},\hat{k}\}$ is an inertial frame.

6/3 The position vector of a particle of mass 5 slugs w.r.t. an inertial frame $\{\hat{\imath},\hat{\jmath},\hat{k}\}$ is

$$\underline{r} = (1-2t^2)\hat{\imath} + 2t^3\hat{\jmath} + (5t^2-t)\hat{k} \text{ ft}$$

Find the force acting on the particle at time $t = 25$s.

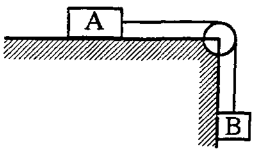
6/4 If the velocity of a particle of mass 5 kg is

$$\underline{v} = 6\cos 2t\hat{\imath} + 2t^2\hat{\jmath} - 6\sin 2t\hat{k}$$

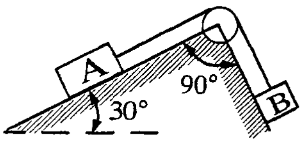
where $\{\hat{\imath},\hat{\jmath},\hat{k}\}$ is an inertial frame, find the force acting on the particle at $t = 2s$.

6/5 Two objects weighing 300 and 200 lb are attached to opposite ends of a light inextensible string passing over a light frictionless pulley. Find the acceleration of each object and the tension in the string.

6/6 Blocks A and B weighing 40 and 50 lb, respectively, are connected by a rope passing over a pulley. Find the acceleration of block B. Neglect the mass of the rope and pulley and all friction.



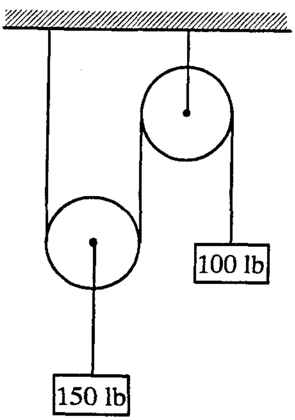
Problem 6/6



Problem 6/7

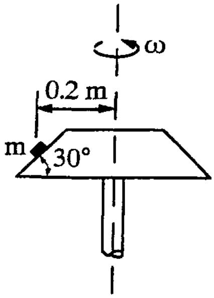
6/7 Block A and B weighing 35 and 70 lb are connected by a rope passing over a pulley as shown. Find the acceleration of block A. Assume that the rope is flexible, inextensible, and weightless and that the pulley is frictionless and of negligible mass. The surfaces are frictionless.

6/8 The 150 and 100 lb weights are attached to a rope passing over pulleys as shown. Assume that the pulleys are massless and frictionless and that the rope is weightless, flexible, and inextensible. Find the accelerations of each weight and the tension in the rope.

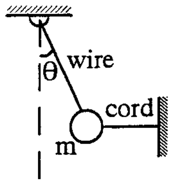


Problem 6/8

6/9 A small object of mass m is placed on a rotating conical surface as shown. If the coefficient of static friction is 0.8, calculate the steady angular speed ω of the cone about the vertical axis for which the object will not slip.



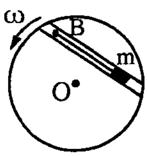
Problem 6/9



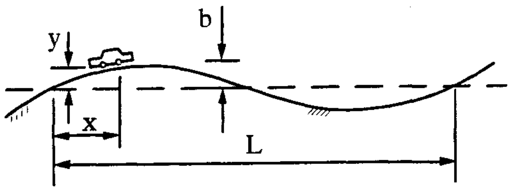
Problem 6/10

6/10 A simple pendulum consisting of a small mass and supporting wire is initially constrained by a cord as shown. Determine the ratio k of the tension T in the wire immediately after the chord is cut to that in the wire before the cord is cut.

6/11 A slider of mass m is free to slide smoothly in a 45° slot in the disk, which rotates with a constant angular speed ω in a horizontal plane about its center O. The slider is held in position by an elastic cord attached to point B on the disk. Determine the tension T in the cord.



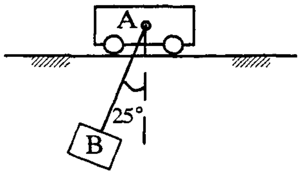
Problem 6/11



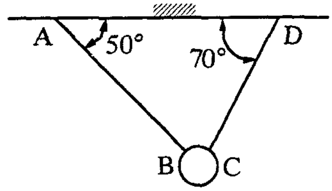
Problem 6/12

6/12 A stretch of highway has a series of humps and dips that may be modeled by the formula $y = b\sin(2\pi x/L)$. What is the maximum speed at which a car of mass m can go over a hump and still maintain contact with the road? If the car continues with this speed, what is the vertical force N produced by the road on the car's wheels at the bottom of a dip?

6/13 Block B, of mass 15 kg, is suspended from a 2.5 m cord attached to cart A, of mass 20 kg. Determine (i) the acceleration of the cart, and (ii) the tension in the cord, immediately after the system is released from rest in the position shown.



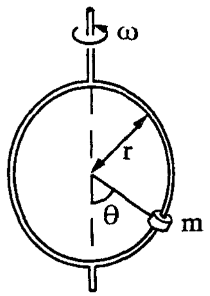
Problem 6/13



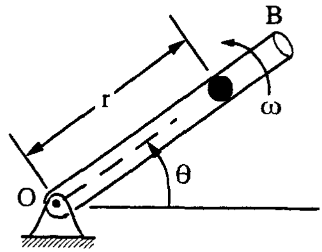
Problem 6/14

6/14 A small sphere of weight W is supported by two wires as shown. Suppose wire AB is cut; determine the tension in wire CD both before and after AB has been cut.

6/15 A small collar of mass $m = 250$ g slides smoothly on a circular hoop of radius $r = 500$ mm. The hoop rotates about the vertical AB axis at a steady angular rate of $\omega = 7.5$ rad/s. Find the three values of θ for which the collar will not slide on the hoop.



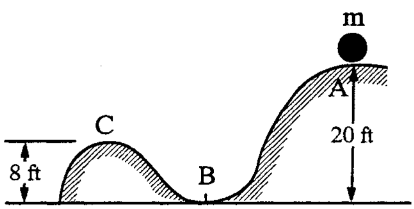
Problem 6/15



Problem 6/16

6/16 A particle slides in a frictionless tube OB, which rotates with constant velocity ω about a horizontal axis through O. If the particle is initially at rest relative to the tube and $r = a$, $\theta = 0$ when $t = 0$, find the reaction force between the particle and the tube as a function of time.

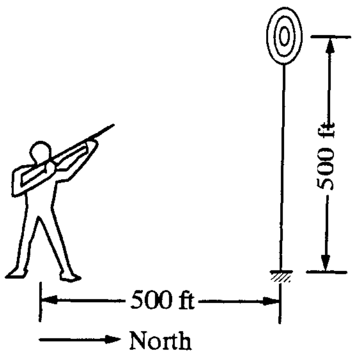
6/17 A small mass of mass m is released from rest at point A on a frictionless surface. If the radius of curvature at point B is 10 ft, what is the force exerted by the surface on the particle when it reaches B? What is the minimum value of the radius of curvature at point C so that the particle will not leave the surface?



Problem 6/17

In the following six problems, assume the earth is a homogeneous sphere rotating at a constant rate.

6/18 A man aims a gun and fires a bullet with a initial speed of 1000 ft/sec at a target which is mounted on top of a tower 500 ft high and located 500 ft north of the man. The man is located at a latitude $\lambda = 45°$. Determine the deflection of the bullet at the target due to gravity and the rotation of the earth.



Problem 6/18

6/19 A particle is dropped from a vertical distance h (measured radially) above the earth's surface. Find the particle's deflection at impact from the vertical due to the earth's rotation ω_e as a function of latitude λ.

6/20 A car of mass M moves southward with a velocity v and an acceleration a relative to the earth. Find the total lateral frictional force exerted on the tires as a result of the earth's rotation as a function of latitude λ.

6/21 At latitude 30°N, a bullet is fired toward a target located 3,000 ft north of the firing point. The initial speed of the bullet is 1,000 ft/sec. Find the horizontal deviation due to the earth's rotation neglecting the earth's curvature.

6/22 A projectile is fired vertically upward (i.e., in the direction of the local plumb line). Acknowledging the earth's rotation, will it fall back to its original position? Explain your answer.

6/23 A locomotive of mass M travels northwest on a horizontal track with a constant speed ν relative to the earth. Find the total force exerted on the track by the locomotive as a function of the latitude λ.

Chapter 7

Angular Momentum and Inertia Matrix

7.1 Definition of Angular Momentum

Consider a rigid body composed of a collection of mass particles (Fig. 7-1) and introduce an inertial frame $\{\hat{I}, \hat{J}, \hat{K}\}$ with origin at 0 and a body-fixed frame $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ with origin at B. The position of mass particle i, with mass m_i, resolved into rectangular components in the body-fixed frame, is given by

$$\underline{d}_i = x_i \hat{\imath} + y_i \hat{\jmath} + z_i \hat{k} \tag{7.1}$$

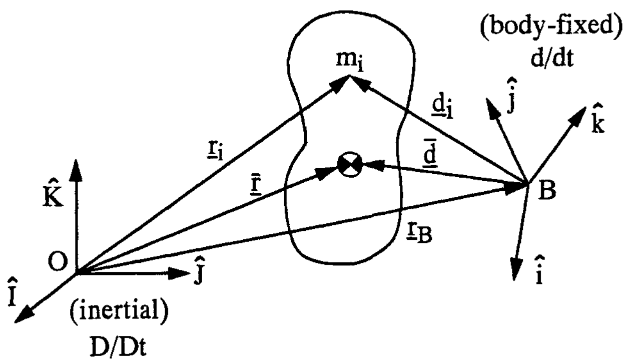


Fig. 7-1

Vector $\underline{d}_i$ and scalars x_i, y_i, and z_i are all constants with respect to the body-fixed frame. Let $\underline{\omega}$ be the angular velocity of the rigid body (that is, the angular velocity of frame $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ with respect to frame $\{\hat{I}, \hat{J}, \hat{K}\}$), and resolve $\underline{\omega}$ in the same way:

$$\underline{\omega} = \omega_x \hat{\imath} + \omega_y \hat{\jmath} + \omega_z \hat{k} \tag{7.2}$$

Define the *linear momentum* of mass particle i relative to point B by

$$\underline{L}_i = m_i \frac{D\underline{r}_i}{Dt} \tag{7.3}$$

where D/Dt is the time derivative in the inertial frame. Summing over the mass particles of the body gives the *body's linear momentum* as

$$\boxed{\underline{L} = \sum_i m_i \frac{D\underline{r}_i}{Dt}} \tag{7.4}$$

In Section 5.2, we showed that the motion of the mass center is governed by Eqn. (5.8), repeated here:

$$\underline{F}^e = \frac{D\overline{\underline{L}}}{Dt} \tag{7.5}$$

where $\overline{\underline{L}} = m\overline{\underline{v}} = m\frac{D\overline{\underline{r}}}{Dt}$ and $\overline{\underline{r}}$ is given by Eqn. (5.4). In order to determine the motion of other body-fixed points, we need to consider the body's angular momentum.

Analogous to Eqn. (7.4), define the body's *angular momentum* (more descriptively called *moment of momentum*) with respect to point B by

$$\boxed{\underline{H}_B = \sum_i \underline{d}_i \times \left(m_i \frac{D\underline{d}_i}{Dt} \right)} \tag{7.6}$$

This is important because in the next chapter the time rate-of-change of $\underline{H}_B$ will be related to the sum of the external force moments acting on the body just as Eqn. (7.5) relates the time rate-of-change of $\overline{\underline{L}}$ to the sum of the external forces. Note that $\underline{r}_i$ is used in Eqn. (7.4) but that $\underline{d}_i$ is used in Eqn. (7.6).

Since $\underline{d}_i$ is fixed in $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$, the BKE, Eqn. (4.15), gives

$$\frac{D\underline{d}_i}{Dt} = \frac{d\underline{d}_i}{dt} + \underline{\omega} \times \underline{d}_i = \underline{\omega} \times \underline{d}_i$$

Substituting this into Eqn. (7.6) and using the vector triple product relation, Eqn. (A.16),

$$\begin{aligned}\underline{H}_B &= \sum_i m_i \underline{d}_i \times (\underline{\omega} \times \underline{d}_i) = \sum_i m_i \left[(\underline{d}_i \cdot \underline{d}_i)\underline{\omega} - (\underline{d}_i \cdot \underline{\omega})\underline{d}_i\right] \\ &= \sum_i m_i \left[d_i^2 \underline{\omega} - (\underline{\omega} \cdot \underline{d}_i)\underline{d}_i\right] \end{aligned} \tag{7.7}$$

The components of this equation are

$$\begin{aligned} H_{B_x} &= \underline{H}_B \cdot \hat{i} = \sum_i m_i \left[\left(x_i^2 + y_i^2 + z_i^2\right)\omega_x - \left(\omega_x x_i + \omega_y y_i + \omega_z z_i\right) x_i\right] \\ &= \sum_i m_i \left(y_i^2 + z_i^2\right)\omega_x - \sum_i m_i y_i x_i \omega_y - \sum_i m_i z_i x_i \omega_z \\ H_{B_y} &= \underline{H}_B \cdot \hat{j} = \sum_i m_i \left(x_i^2 + z_i^2\right)\omega_y - \sum_i m_i y_i x_i \omega_x - \sum_i m_i y_i z_i \omega_z \\ H_{B_z} &= \underline{H}_B \cdot \hat{k} = \sum_i m_i \left(x_i^2 + y_i^2\right)\omega_z - \sum_i m_i z_i x_i \omega_x - \sum_i m_i z_i y_i \omega_y \end{aligned} \tag{7.8}$$

7.2 Moments and Products of Inertia

Define

$$\begin{aligned} &I_{xx} = \sum_i m_i \left(y_i^2 + z_i^2\right) \quad I_{yy} = \sum_i m_i \left(x_i^2 + z_i^2\right) \quad I_{zz} = \sum_i m_i \left(x_i^2 + y_i^2\right) \\ &I_{xy} = \sum_i m_i x_i y_i \quad I_{yz} = \sum_i m_i y_i z_i \quad I_{zx} = \sum_i m_i z_i x_i \end{aligned} \tag{7.9}$$

The I_{ii}, $i = 1, 2, 3$, are called *moments of inertia*; note that $I_{ii} \geq 0$. The I_{ij}, $i \neq j$ are called *products of inertia*; note that $I_{ij} = I_{ji}$.

For the purposes of computation, it is usually best to consider the rigid body as a continuum of mass so that Eqns. (7.9) become[1]

$$\boxed{\begin{aligned} I_{xx} &= \int_m (y^2 + z^2)dm \geq 0 \\ I_{yy} &= \int_m (x^2 + z^2)dm \geq 0 \\ I_{zz} &= \int_m (x^2 + y^2)dm \geq 0 \end{aligned}} \tag{7.10}$$

$$\boxed{\begin{aligned} I_{xy} &= \int_m xydm = I_{yx} \\ I_{yz} &= \int_m yzdm = I_{zy} \\ I_{zx} &= \int_m zxdm = I_{xz} \end{aligned}} \qquad \text{(7.10 continued)}$$

From Eqns. (7.8) and (7.9), the components of the relative angular momentum now may be written as

$$\begin{aligned} H_{B_x} &= I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z \\ H_{B_y} &= I_{yy}\omega_y - I_{yx}\omega_x - I_{yz}\omega_z \\ H_{B_z} &= I_{zz}\omega_z - I_{zx}\omega_x - I_{zy}\omega_y \end{aligned} \qquad (7.11)$$

In matrix notation this is

$$\begin{pmatrix} H_{B_x} \\ H_{B_y} \\ H_{B_z} \end{pmatrix} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

$$\boxed{\underline{H}_B = [I]\underline{\omega}} \qquad (7.12)$$

where $[I]$ is the inertia matrix (or tensor):

$$\boxed{[I] = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix}} \qquad (7.13)$$

These quantities characterize the mass distribution of a rigid-body.

From their definitions, Eqns. (7.9) or (7.10), the moments and products of inertia have an additive property. If a rigid body is composed of several, say n, component rigid bodies, rigidly connected, then its moments and products of inertia are:

$$\boxed{\begin{aligned} I_{xx} &= \sum_{j=1}^{n} I_{xx_j} \\ I_{yy} &= \sum_{j=1}^{n} I_{yy_j} \\ I_{zz} &= \sum_{j=1}^{n} I_{zz_j} \end{aligned}} \qquad (7.14)$$

$$\boxed{\begin{aligned} I_{xy} &= \sum_{j=1}^{n} I_{xy_j} \\ I_{yz} &= \sum_{j=1}^{n} I_{yz_j} \\ I_{zx} &= \sum_{j=1}^{n} I_{zx_j} \end{aligned}} \qquad \text{(7.14 continued)}$$

These are useful because many complex bodies may be thought of as composed of several simple ones.

Moments of inertia of homogeneous bodies with common shapes may be found in the Table of Appendix B.

7.3 Examples

First consider a thin rod of mass m and length ℓ with uniform mass density per unit length (Fig. 7-2). We want the moments and products of inertia about axes through the center of mass and about axes through an end of the rod. Let $\rho = m/\ell$ be the mass per unit length; then $dm = \rho d\bar{x} = \rho dx$. Note that for each mass particle of the rod:

$$\bar{y} = \bar{z} = y = z = 0$$

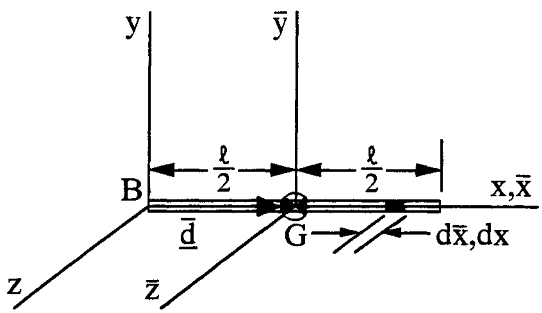


Fig. 7-2

Applying Eqs. (7.10) to the centroidal axes:

$$\begin{aligned}
\overline{I}_{xx} &= \int_m (\overline{y}^2 + \overline{z}^2)dm = 0 \\
\overline{I}_{yy} &= \int_m (\overline{x}^2 + \overline{z}^2)dm = \int_{-\ell/2}^{\ell/2} \overline{x}^2 \rho d\overline{x} \\
&= \rho \left[\frac{\overline{x}^3}{3}\right]_{-\ell/2}^{\ell/2} = \frac{\rho}{3}\left(\frac{\ell^3}{8} + \frac{\ell^3}{8}\right) = \frac{\rho\ell^3}{12} = \frac{m\ell^2}{12} \\
\overline{I}_{zz} &= \int_m (\overline{x}^2 + \overline{y}^2)dm = \int_{-\ell/2}^{\ell/2} \overline{x}^2 \rho d\overline{x} = \frac{m\ell^2}{12} \\
\overline{I}_{xy} &= \int_m \overline{x}\overline{y}dm = 0 \\
\overline{I}_{yz} &= \overline{I}_{zx} = 0
\end{aligned}$$

For the axes through the end of the rod:

$$\begin{aligned}
I_{xx} &= \int_m (y^2 + z^2)dm = 0 \\
I_{yy} &= \int_m (x^2 + z^2)dm = \int_0^\ell x^2 \rho dy = \rho \left[\frac{x^3}{3}\right]_0^\ell = \frac{\rho\ell^3}{3} = \frac{m\ell^2}{3} \\
I_{zz} &= \int_m (x^2 + y^2)dm = \int_0^\ell x^2 \rho dy = \frac{m\ell^2}{3} \\
I_{xy} &= I_{yz} = I_{zx} = 0
\end{aligned}$$

Next consider the rigid body consisting of three thin rods of length ℓ and mass m as shown in Fig. 5-8, repeated here as Fig. 7-3. We want

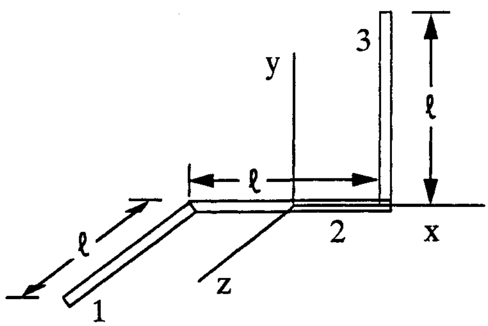


Fig. 7-3

the inertia matrix relative to the axes shown. Labeling the rods as shown in Fig. 7-3 and using the additive property, Eqns. (7.14),

$$\begin{aligned}
I_{xx} &= I_{xx_1} + I_{xx_2} + I_{xx_3} \\
&= \left[\int_m (y^2+z^2)dm\right]_1 + \left[\int_m (y^2+z^2)dm\right]_2 + \left[\int_m (y^2+z^2)dm\right]_3 \\
&= \int_0^\ell z^2\rho dz + 0 + \int_0^\ell y^2 \rho dy = \frac{1}{3}m\ell^2 + \frac{1}{3}m\ell^2 = \frac{2}{3}m\ell^2
\end{aligned}$$

$$\begin{aligned}
I_{yy} &= \left[\int_m (x^2+z^2)dm\right]_1 + \left[\int_m (x^2+z^2)dm\right]_2 + \left[\int_m (x^2+z^2)dm\right]_3 \\
&= \int_0^\ell \left[\left(-\frac{\ell}{2}\right)^2 + z^2\right]\rho dz + \int_{-\ell/2}^{\ell/2} x^2 \rho dx + \int_0^\ell \left(\frac{\ell}{2}\right)^2 \rho dy \\
&= \frac{1}{4}m\ell^2 + \frac{1}{3}m\ell^2 + \frac{1}{12}m\ell^2 + \frac{1}{4}m\ell^2 = \frac{11}{12}m\ell^2
\end{aligned}$$

$$\begin{aligned}
I_{zz} &= \left(\frac{\ell}{2}\right)^2 m + \frac{1}{12}m\ell^2 + \int_0^\ell \left[\left(\frac{\ell}{2}\right)^2 + y^2\right]\rho dy \\
&= \frac{1}{4}m\ell^2 + \frac{1}{12}m\ell^2 + \frac{1}{4}m\ell^2 + \frac{1}{3}m\ell^2 = \frac{11}{12}m\ell^2
\end{aligned}$$

$$\begin{aligned}
I_{xy} &= I_{xy_1} + I_{xy_2} + I_{xy_3} = \left[\int_m xydm\right]_1 + \left[\int_m xydm\right]_2 + \left[\int_m xydm\right]_3 \\
&= \int_0^\ell \left(-\frac{\ell}{2}\right)(0)\rho dz + \int_{-\ell/2}^{\ell/2} (x)(0)\rho dx + \int_0^\ell \left(\frac{\ell}{2}\right) y\rho dy \\
&= \left(\frac{\ell}{2}\right)\rho\left(\frac{\ell^2}{2}\right) = \frac{1}{4}m\ell^2
\end{aligned}$$

$$I_{yz} = \int (0)(z)\rho dz + \int (0)(0)\rho dx + \int (y)(0)\rho dy = 0$$

$$I_{zx} = \int_0^\ell (z)\left(-\frac{\ell}{2}\right)\rho dz + 0 + 0 = -\frac{1}{4}m\ell^2$$

Thus the inertia matrix, Eqn. (7.13), is

$$[I] = \frac{m\ell^2}{12}\begin{bmatrix} 8 & -3 & 3 \\ -3 & 11 & 0 \\ 3 & 0 & 11 \end{bmatrix}$$

7.4 Principal Axes and Principal Moments of Inertia

In this section we shall show that it is always possible to find, for any body-fixed point, a body-fixed frame with origin at that point in which the products of inertia, I_{ij}, $i \neq j$, vanish. The axes of such a frame are called *principal axes of inertia* (PAI), and the moments of inertia in such a frame are called *principal moments of inertia* (PMI). We will prove this statement constructively, that is by giving a procedure for finding the PAI and PMI.

Let $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ be any body fixed axes with moments and products of inertia I_{xx}, I_{yy}, I_{zz}, I_{xy}, I_{yz}, I_{zx}, and let $\{\hat{\imath}', \hat{\jmath}', \hat{k}'\}$ be principal axes with the same origin (assuming for the moment that such exist) with principal moments of inertia I_1, I_2, I_3 (Fig. 7-4).

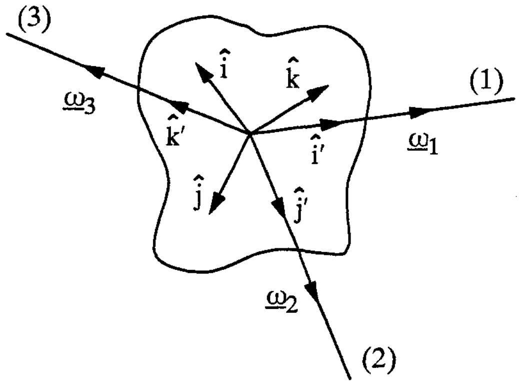


Fig. 7-4

Consider a special motion of the body, one with $\underline{\omega}$ parallel to $\hat{\imath}'$; then $\underline{\omega} = \omega\hat{\imath}'$ and from Eqn. (7.12):

$$\begin{aligned}
\underline{H}_B &= [I]\underline{\omega} \\
\begin{pmatrix} H'_{B_x} \\ H'_{B_y} \\ H'_{B_z} \end{pmatrix} &= \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \begin{pmatrix} \omega \\ 0 \\ 0 \end{pmatrix} \\
H'_{B_x} &= I_1\omega \\
H'_{B_y} &= 0
\end{aligned}$$

$$\begin{aligned} H'_{B_z} &= 0 \\ \underline{H}_B &= I_1 \omega \hat{i}' = I_1 \underline{\omega} \end{aligned} \tag{7.15}$$

More generally, if I is any one of the PMI and $\underline{\omega}$ is parallel to the corresponding PAI, then

$$\underline{H}_B = I\underline{\omega} \tag{7.16}$$

But $\underline{H}_{B_r}$, expressed in components along $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$, is

$$\underline{H}_B = [I]\underline{\omega} \tag{7.17}$$

where $\underline{\omega}$ in components along $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ is

$$\underline{\omega} = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

Equating Eqns. (7.16) and (7.17),

$$[I]\underline{\omega} = I\underline{\omega}$$

or

$$([I] - I[1])\underline{\omega} = \underline{0} \tag{7.18}$$

where [1] is the identity matrix.[2] We see that this is an *eigenvalue problem*, first encountered in Section 4.2.

Writing out Eqns. (7.18) in scalar form:

$$\begin{aligned} &(I_{xx} - I)\omega_x - I_{xy}\omega_y - I_{xz}\omega_z = 0 \\ &-I_{yx}\omega_x + (I_{yy} - I)\omega_y - I_{yz}\omega_z = 0 \\ &-I_{zx}\omega_x - I_{zy}\omega_y + (I_{zz} - I)\omega_z = 0 \end{aligned} \tag{7.19}$$

These are three homogeneous algebraic equations in the three unknowns ω_x, ω_y, ω_z. There will be a nontrivial solution if and only if the determinant of the coefficients vanishes:

$$\begin{vmatrix} I_{xx} - I & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} - I & -I_y \\ -I_{zx} & -I_{zy} & I_{zz} - I \end{vmatrix} = 0 \tag{7.20}$$

This is a cubic equation in I, with roots I_1, I_2, I_3, the PMI. To find the PAI, the PMI are substituted one at a time into Eqns. (7.19) and these equations are solved for $\underline{\omega}_1$, $\underline{\omega}_2$, $\underline{\omega}_3$ in turn; these define the directions of the PAI.

7.5 Example

We find the PMI and the corresponding PAI for the three-bar assembly considered earlier, in Sections 5.5 and 7.3 (Fig. 7-3). In Section 7.3 the inertia matrix was found to be:

$$[I] = \frac{m\ell^2}{12}\begin{bmatrix} 8 & -3 & 3 \\ -3 & 11 & 0 \\ 3 & 0 & 11 \end{bmatrix}$$

The PMI are the eigenvalues of matrix $[I]$, obtained from Eqn. (7.20):

$$\begin{vmatrix} 8-I & -3 & 3 \\ -3 & 11-I & 0 \\ 3 & 0 & 11-I \end{vmatrix} = 0$$

$$(8-I)(11-I)(11-I) - (3)(11-I)(3) - (11-I)(-3)(-3) = 0$$
$$(11-I)[(8-I)(11-I) - 9 - 9] = 0$$

$$I_1 = 11, \quad I^2 - 19I + 70 = 0$$
$$I_{2,3} = \frac{19 \pm \sqrt{19^2 - (4)(70)}}{2} = 5, 14$$

The PAI are the eigenvectors of $[I]$, obtained from Eqn. (7.19):

$$(8-I)\omega_x - 3\omega_y + 3\omega_z = 0$$
$$-3\omega_x + (11-I)\omega_y = 0$$
$$3\omega_x + (11-I)\omega_z = 0$$

For $I_1 = 11$:

$$\left.\begin{array}{l} -3\omega_{x_1} - 3\omega_{y_1} + 3\omega_{z_1} = 0 \\ -3\omega_{x_1} = 0 \\ 3\omega_{x_1} = 0 \end{array}\right\} \Rightarrow \begin{array}{l} \omega_{x_1} = 0 \\ \omega_{y_1} = \omega_{z_1} \end{array}$$

For $I_2 = 5$:

$$\left.\begin{array}{l} 3\omega_{x_2} - 3\omega_{y_2} + 3\omega_{z_2} = 0 \\ -3\omega_{x_2} + 6\omega_{y_2} = 0 \\ 3\omega_{x_2} + 6\omega_{z_2} = 0 \end{array}\right\} \Rightarrow \begin{array}{l} \omega_{y_2} = \frac{1}{2}\omega_{x_2} \\ \omega_{z_2} = -\frac{1}{2}\omega_{x_2} \end{array}$$

For $I_3 = 14$:

$$\left.\begin{array}{l} -6\omega_{x3} - 3\omega_{y3} + 3\omega_{z3} = 0 \\ -3\omega_{x3} - 3\omega_{y3} = 0 \\ 3\omega_{x3} - 3\omega_{z3} = 0 \end{array}\right\} \Rightarrow \begin{array}{l} \omega_{y3} = -\omega_{x3} \\ \omega_{z3} = \omega_{x3} \end{array}$$

The three PAI of the assembly, labeled (1), (2), and (3), are shown in Fig. 7-5. The PAI are always mutually perpendicular.

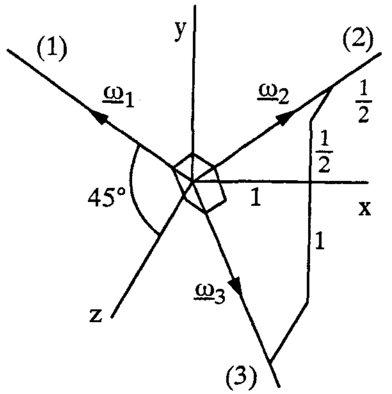


Fig. 7-5

7.6 Rotational Mass Symmetry

In practice, it is frequently possible to find the PAI by inspection. A common example is a body with rotational mass symmetry. In such a body, there is an axis, say $\hat{k}$, such that the mass properties depend only on (r, z), and not on θ, where (r, θ, z) are cylindrical coordinates (Fig. 7-6).

Then, for a mass at any point on one side of $\hat{k}$ there is an equal mass at a point equidistant on the other. Consider such a mass pair as shown in Fig. 7-7. For this mass pair:

$$m_i = m_j$$
$$x_i = -x_j$$
$$y_i = -y_j$$
$$z_i = z_j$$

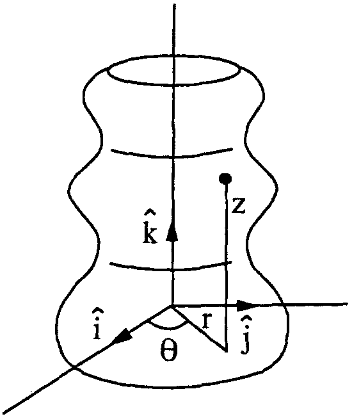


Fig. 7-6

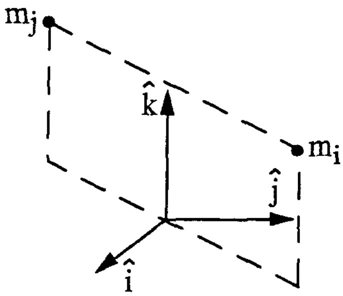


Fig. 7-7

Thus

$$
\begin{aligned}
m_i x_i z_i &= -m_j x_j z_j \\
m_i y_i z_i &= -m_j y_j z_j
\end{aligned}
$$

and

$$
\begin{aligned}
I_{xz} &= \sum_k m_k x_k y_k = 0 \\
I_{yz} &= \sum_k m_k y_k z_k = 0
\end{aligned}
$$

Consider another pair as shown in Figure 7-8. For this pair

$$
\begin{aligned}
m_i &= m_j \\
x_i &= -x_j \\
y_i &= y_j \\
z_i &= z_j \\
m_i x_i y_i &= -m_j x_j y_i
\end{aligned}
$$

and therefore

$$I_{xy} = \sum_k m_k x_k y_k = 0$$

This shows that for a body with rotational mass symmetry, the axis of symmetry, and any two orthogonal axes orthogonal to it, are PAI's. The inertia matrix relative to these axes is of the form

$$\boxed{[I] = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}, \quad \text{with } I_1 = I_2} \tag{7.21}$$

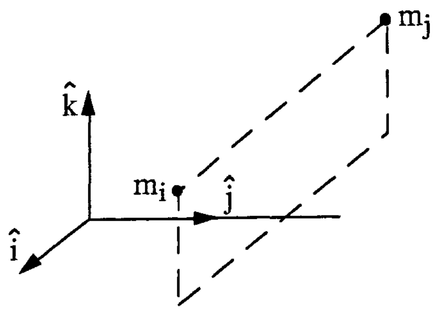


Fig. 7-8

7.7 Relation Between Angular Momenta

It is sometimes desirable to obtain the angular momentum of a rigid body relative to some point B if the angular momentum relative to the center of mass is known, or vice-versa. To this end consider Fig. 7-9, which shows a rigid body and three reference frames – one inertial, one body-fixed, and one arbitrary. From the figure

$$\underline{d}_i = \overline{\underline{d}} + \underline{\rho}_i \tag{7.22}$$

Using definition (7.6) and Eqn. (7.22):

$$\begin{aligned} \underline{H}_B &= \sum_i \underline{d}_i \times m_i \frac{D\underline{d}_i}{Dt} = \sum_i (\overline{\underline{d}} + \underline{\rho}_i) \times m_i \frac{D(\overline{\underline{d}} + \underline{\rho}_i)}{Dt} \\ &= \sum_i \overline{\underline{d}} \times m_i \frac{D\overline{\underline{d}}}{Dt} + \sum_i \underline{\rho}_i \times m_i \frac{D\overline{\underline{d}}}{Dt} + \sum_i \overline{\underline{d}} \times m_i \frac{D\underline{\rho}_i}{Dt} + \sum_i \underline{\rho}_i \times m_i \frac{D\underline{\rho}_i}{Dt} \end{aligned}$$

$$
\begin{aligned}
&= \underline{\bar{d}} \times m\frac{D\underline{\bar{d}}}{Dt} + \underbrace{\left(\sum_i m_i \underline{\rho}_i\right)}_{=\underline{0}} \times \frac{D\underline{\bar{d}}}{Dt} + \underline{\bar{d}} \times \underbrace{\frac{D\left(\sum_i m_i \underline{\rho}_i\right)}{Dt}}_{=\underline{0}} + \underline{\bar{H}} \\
&= \underline{\bar{d}} \times m\frac{D\underline{\bar{d}}}{Dt} + \underline{\bar{H}} \qquad (7.23)
\end{aligned}
$$

which is the desired relation.

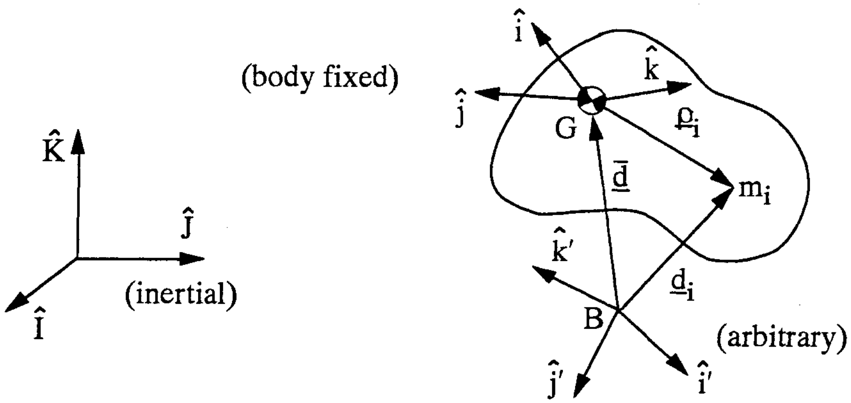


Fig. 7-9

7.8 Parallel Axis Theorem

Consider again the situation shown in Fig. 7-9 but now make the restriction that $\{\hat{\imath}', \hat{\jmath}', \hat{k}'\}$ is body fixed and parallel to $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$.

Let $\underline{\omega}$ be the angular velocity of the body. Since $\{\hat{\imath}', \hat{\jmath}', \hat{k}'\}$ is body fixed, the BKE gives

$$\frac{D\underline{\bar{d}}}{Dt} = \frac{d\underline{\bar{d}}}{dt} + \underline{\omega} \times \underline{\bar{d}} = \underline{\omega} \times \underline{\bar{d}}$$

Thus Eqn. (7.23) becomes

$$\underline{H}_B = \underline{\bar{H}} + \underline{\bar{d}} \times m(\underline{\omega} \times \underline{\bar{d}})$$

and using Eqns. (A.16) and (7.12)

$$[I]\underline{\omega} = [\bar{I}]\underline{\omega} + m\left[\bar{d}^2 \underline{\omega} - (\underline{\omega} \cdot \underline{\bar{d}})\underline{\bar{d}}\right] \qquad (7.24)$$

Writing the last term in components, we have

$$\begin{aligned}\overline{d}^2\underline{\omega}-(\underline{\omega}\cdot\underline{\overline{d}})\underline{\overline{d}} &= \left[(\overline{d}_y^2+\overline{d}_z^2)\omega_x-\overline{d}_x\overline{d}_y\omega_y-\overline{d}_x\overline{d}_z\omega_z\right]\hat{\imath}\\ &\quad+\left[-\overline{d}_y\overline{d}_x\omega_x+(\overline{d}_x^2+\overline{d}_z^2)\omega_y-\overline{d}_y\overline{d}_z\omega_z\right]\hat{\jmath}\\ &\quad\quad+\left[-\overline{d}_z\overline{d}_x\omega_x-\overline{d}_z\overline{d}_y\omega_y+(\overline{d}_x^2+\overline{d}_y^2)\omega_z\right]\hat{k}\end{aligned}$$

Substituting this back into Eqn. (7.24) and writing in components gives

$$\begin{bmatrix} I_{x'x'} & -I_{x'y'} & -I_{x'z'}\\ -I_{y'x'} & I_{y'y'} & -I_{y'z'}\\ -I_{z'x'} & -I_{z'y'} & I_{z'z'}\end{bmatrix}\begin{pmatrix}\omega'_x\\ \omega'_y\\ \omega'_z\end{pmatrix} \tag{7.25}$$

$$= \begin{bmatrix} \overline{I}_{xx}+m(\overline{d}_y^2+\overline{d}_z^2) & -\overline{I}_{xy}-m\overline{d}_x\overline{d}_y & -\overline{I}_{xz}-m\overline{d}_x\overline{d}_z\\ -\overline{I}_{yx}-m\overline{d}_y\overline{d}_x & \overline{I}_{yy}+m(\overline{d}_x^2+\overline{d}_z^2) & -\overline{I}_{yz}-m\overline{d}_y\overline{d}_z\\ -\overline{I}_{zx}-m\overline{d}_z\overline{d}_x & -\overline{I}_{zy}-m\overline{d}_z\overline{d}_y & \overline{I}_{zz}+m(\overline{d}_x^2+\overline{d}_y^2)\end{bmatrix}\begin{pmatrix}\omega_x\\ \omega_y\\ \omega_z\end{pmatrix}$$

But because the axes were parallel,

$$\begin{pmatrix}\omega_{x'}\\ \omega_{y'}\\ \omega_{z'}\end{pmatrix}=\begin{pmatrix}\omega_x\\ \omega_y\\ \omega_z\end{pmatrix}$$

and since $\underline{\omega}$ was arbitrary, Eqn. (7.25) implies

$$\boxed{\begin{aligned} I_{x'x'} &= \overline{I}_{xx}+m(\overline{d}_y^2+\overline{d}_z^2)\\ I_{y'y'} &= \overline{I}_{yy}+m(\overline{d}_x^2+\overline{d}_z^2)\\ I_{z'z'} &= \overline{I}_{zz}+m(\overline{d}_x^2+\overline{d}_y^2)\\ I_{x'y'} &= \overline{I}_{xy}+m\overline{d}_x\overline{d}_y\\ I_{x'z'} &= \overline{I}_{xz}+m\overline{d}_x\overline{d}_z\\ I_{y'z'} &= \overline{I}_{yz}+m\overline{d}_y\overline{d}_z\end{aligned}} \tag{7.26}$$

The result just derived, Eqns. (7.26), is called the Parallel Axis Theorem (PAT). It relates the I_{ij} relative to axes through the center of mass (G) to the I_{ij} relative to any set of axes parallel to those through G.

This result has many practical applications. For example, suppose we know the inertia matrix relative to one set of axes, say $[I]_1$, and we want it relative to another set of axes, say $[I]_2$, parallel to the first set (Fig. 7-10). This requires two applications of the PAT as follows:

$$\begin{aligned}
\overline{I}_{xx} &= I_{x_1x_1} - m(\overline{d}_{y_1}^2 + \overline{d}_{z_1}^2) \\
\overline{I}_{xy} &= I_{x_1y_1} - m\overline{d}_{x_1}\overline{d}_{y_1} \\
I_{x_2x_2} &= \overline{I}_{xx} + m(\overline{d}_{y_2}^2 + \overline{d}_{z_2}^2) \\
I_{x_2y_2} &= \overline{I}_{xy} + m\overline{d}_{x_2}\overline{d}_{y_2}
\end{aligned}$$

so that

$$\begin{aligned}
I_{x_2x_2} &= I_{x_1x_1} + m(\overline{d}_{y_2}^2 - \overline{d}_{y_1}^2 + \overline{d}_{z_2}^2 - \overline{d}_{z_1}^2) \\
I_{x_2y_2} &= I_{x_1y_1} + m(\overline{d}_{x_2}\overline{d}_{y_2} - \overline{d}_{x_1}\overline{d}_{y_1})
\end{aligned}$$

and similarly for the other moments and products of inertia.

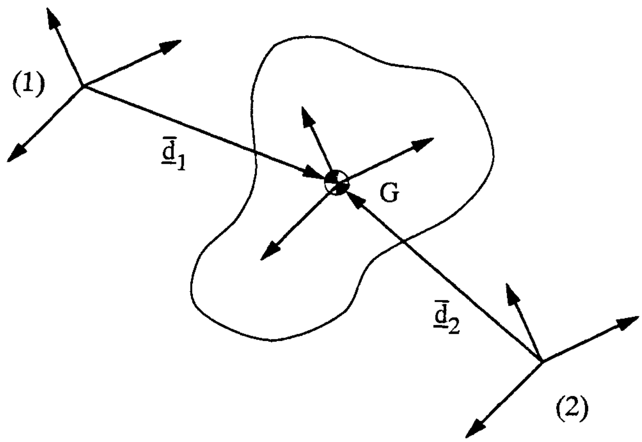


Fig. 7-10

It should be noted from Eqns. (7.26) that the moments of inertia are minimum with respect to axes through the center of mass, relative to all other parallel axes. For example for the x-axis (Fig. 7-11):

$$\overline{I}_{xx} \leq I_{x_1x_1}$$

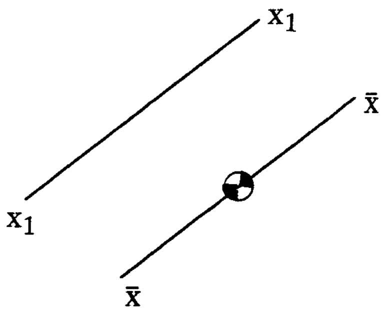


Fig. 7-11

7.9 Radius of Gyration

The inertia properties are sometimes characterized by *radii of gyration.* The radius of gyration about axis i is defined as

$$\boxed{k_i = \sqrt{\frac{I_{ii}}{m}}} \tag{7.27}$$

For example, for the thin rod of the example in Section 7.3,

$$k_y = \sqrt{\frac{I_{yy}}{m}} = \frac{\ell}{2\sqrt{3}}$$

and for the three-rod assembly,

$$k_x = \sqrt{\frac{I_{xx}}{m}} = \sqrt{\frac{2}{3}}\ell$$

The definition of k_i, Eqn. (7.27), shows that it is the perpendicular distance from axis i at which a particle of mass m would need to be placed to give the same moment of inertia as the actual body (Fig. 7-12).

7.10 Examples

Suppose we know the moments and products of inertia of a slender rod with respect to its center of mass (G), and want them with respect to an end of the rod (Fig. 7-2). In Section 7.3 we found this by direct integration; now we use the PAT. We have $\underline{\bar{d}} = \ell/2\hat{\jmath}$ and therefore

$$\begin{aligned} \bar{d}_x &= 0 \\ \bar{d}_y &= \frac{\ell}{2} \\ \bar{d}_z &= 0 \end{aligned}$$

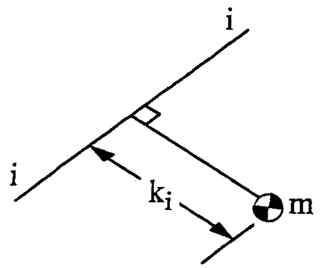


Fig. 7-12

Equations (7.26) give

$$\begin{aligned} I_{xx} &= \bar{I}_{xx} + m(\bar{d}_y^2 + \bar{d}_z^2) = \frac{1}{12}m\ell^2 + m\left(\frac{\ell}{2}\right)^2 = \frac{1}{3}m\ell^2 \\ I_{yy} &= \bar{I}_{yy} + m(\bar{d}_x^2 + \bar{d}_z^2) = 0 \\ I_{zz} &= \bar{I}_{zz} + m(\bar{d}_y^2 + \bar{d}_x^2) = \frac{1}{3}m\ell^2 \\ I_{xy} &= I_{yz} = I_{zx} = 0 \end{aligned}$$

which agrees with Section 7.3.

Consider again the three bar assembly last encountered in Section 7.3. We use the PAT to get the inertia matrix with respect to the axes shown in Fig. 5-10, repeated here as Fig. 7-13. From the figure

$$\begin{aligned} \underline{\bar{d}}_1 &= -\frac{\ell}{2}\hat{\imath} + \frac{\ell}{2}\hat{k} \\ \underline{\bar{d}}_2 &= \underline{0} \\ \underline{\bar{d}}_3 &= \frac{\ell}{2}\hat{\imath} + \frac{\ell}{2}\hat{\jmath} \end{aligned}$$

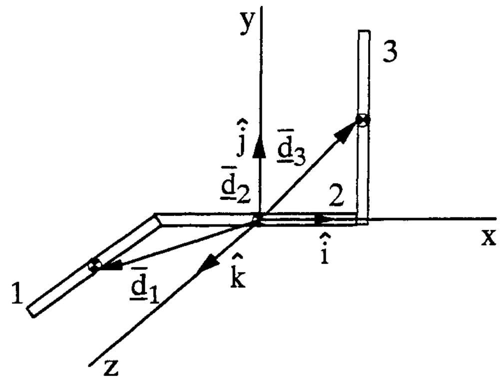


Fig. 7-13

so that

$$\begin{aligned} &\overline{d}_{x_1} = -\frac{\ell}{2}\,, && \overline{d}_{y_1} = 0\,, && \overline{d}_{z_1} = \frac{\ell}{2} \\ &\overline{d}_{x_2} = 0\,, && \overline{d}_{y_2} = 0\,, && \overline{d}_{z_2} = 0 \\ &\overline{d}_{x_3} = \frac{\ell}{2}\,, && \overline{d}_{y_3} = \frac{\ell}{2}\,, && \overline{d}_{z_3} = 0\,. \end{aligned}$$

Using Eqns. (7.14) and (7.26):

$$\begin{aligned} I_{xx} &= I_{xx_1} + I_{xx_2} + I_{xx_3} = \overline{I}_{xx_1} + m(\overline{d}^2_{y_1} + \overline{d}^2_{z_1}) + \overline{I}_{xx_2} \\ &\quad + m(\overline{d}^2_{y_2} + \overline{d}^2_{z_2}) + \overline{I}_{xx_3} + m(\overline{d}^2_{y_3} + \overline{d}^2_{z_3}) \\ &= \frac{1}{12}m\ell^2 + m\left(0 + \left(\frac{\ell}{2}\right)^2\right) + 0 + m(0+0) + \frac{1}{12}m\ell^2 \\ &\quad + m\left(\left(\frac{\ell}{2}\right)^2 + 0\right) = \frac{2}{3}m\ell^2 \\ I_{yy} &= \overline{I}_{yy_1} + m(\overline{d}^2_{x_1} + \overline{d}^2_{z_1}) + \overline{I}_{yy_2} + m(\overline{d}^2_{x_2} + \overline{d}^2_{z_2}) + \overline{I}_{yy_3} \\ &\quad + m(\overline{d}^2_{x_3} + \overline{d}^2_{z_3}) = \frac{1}{12}m\ell^2 + m\left(\left(\frac{\ell}{2}\right)^2 + \left(\frac{\ell}{2}\right)^2\right) \\ &\quad + \frac{1}{12}m\ell^2 + m(0+0) + 0 + m\left(\left(\frac{\ell}{2}\right)^2 + 0\right) = \frac{11}{12}m\ell^2 \\ I_{zz} &= 0 + \frac{1}{4}m\ell^2 + \frac{1}{12}m\ell^2 + 0 + \frac{1}{12}m\ell^2 + \frac{1}{2}m\ell^2 = \frac{11}{12}m\ell^2 \end{aligned}$$

$$\begin{aligned}
I_{xy} &= I_{xy_1} + I_{xy_2} + I_{xy_3} = \overline{I}_{xy_1} + m\overline{d}_{x_1}\overline{d}_{y_1} + \overline{I}_{xy_2} \\
&\quad + m\overline{d}_{x_2}\overline{d}_{y_2} + \overline{I}_{xy_3} + m\overline{d}_{x_3}\overline{d}_{y_3} \\
&= 0 + m\left(-\frac{\ell}{2}\right)(0) + 0 + m(0)(0) + 0 + m\left(\frac{\ell}{2}\right)\left(\frac{\ell}{2}\right) = \frac{1}{4}m\ell^2 \\
I_{yz} &= m\overline{d}_{y_1}\overline{d}_{z_1} + m\overline{d}_{y_2}\overline{d}_{z_2} + m\overline{d}_{y_3}\overline{d}_{z_3} = 0 \\
I_{zx} &= m\overline{d}_{z_1}\overline{d}_{x_1} + m\overline{d}_{z_2}\overline{d}_{x_2} + m\overline{d}_{z_3}\overline{d}_{x_3} = -\frac{1}{4}m\ell^2
\end{aligned}$$

We can also use the PAT to find the inertia matrix relative to axes through the center of mass of the assembly. The location of the center of mass, G, was found in Section 5.5 to be

$$\overline{\underline{d}} = \frac{\ell}{6}(\hat{\jmath} + \hat{k}) \quad \Rightarrow \quad \overline{d}_x = 0\ , \quad \overline{d}_y = \frac{\ell}{6}\ , \quad \overline{d}_z = \frac{\ell}{6}$$

This vector locates G of the assembly relative to the original axes; it could also have been determined by inspection. Now,

$$\begin{aligned}
\overline{I}_{xx} &= I_{xx} - m(\overline{d}_y^2 + \overline{d}_z^2) = \frac{2}{3}m\ell^2 - m\left(\left(\frac{\ell}{6}\right)^2 + \left(\frac{\ell}{6}\right)^2\right) = \frac{11}{18}m\ell^2 \\
\overline{I}_{yy} &= I_{yy} - m(\overline{d}_x^2 + \overline{d}_y^2) = \frac{11}{12}m\ell^2 - m\left(0 + \left(\frac{\ell}{6}\right)^2\right) = \frac{31}{36}m\ell^2 \\
\overline{I}_{zz} &= I_{zz} - m(\overline{d}_x^2 + \overline{d}_y^2) = \frac{11}{12}m\ell^2 - m\left(0 + \left(\frac{\ell}{6}\right)^2\right) = \frac{8}{9}m\ell^2 \\
\overline{I}_{xy} &= I_{xy} - m\overline{d}_x\overline{d}_y = \frac{1}{4}m\ell^2 - 0 = \frac{1}{4}m\ell^2 \\
\overline{I}_{yz} &= I_{yz} - m\overline{d}_y\overline{d}_z = 0 - m\left(\frac{\ell}{6}\right)\left(\frac{\ell}{6}\right) = \frac{1}{36}m\ell^2 \\
\overline{I}_{zx} &= I_{zx} - m\overline{d}_z\overline{d}_x = -\frac{1}{4}m\ell^2 - m\left(\frac{\ell}{6}\right)(0) = -\frac{1}{4}m\ell^2
\end{aligned}$$

As a final example, we determine the moments and products of inertia of the machine part depicted on Fig. 5-11, repeated here as Fig. 7-14. We now assume the thickness t is small, in the sense that it may be neglected relative to the other dimensions. For component 1,

$$\begin{aligned}
I_{xx_1} &= \frac{1}{2}m_1 r^2 = \frac{1}{4}\pi\rho r^4 t \\
I_{yy_1} &= I_{zz_1} = \frac{1}{4}m_1 r^2 = \frac{1}{8}\pi\rho r^4 t
\end{aligned}$$

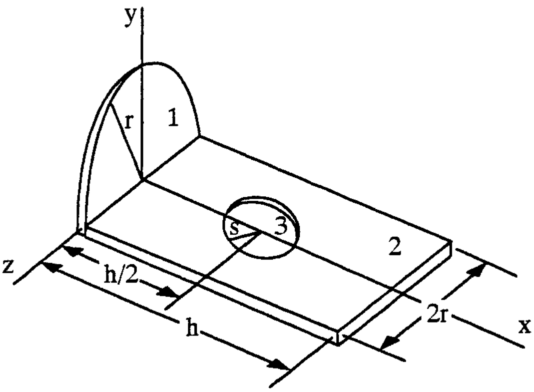


Fig. 7-14

For component 2,

$$
\begin{aligned}
I_{xx_2} &= \frac{1}{12}m_2(2r)^2 = \frac{2}{3}\rho r^3 ht \\
I_{yy_2} &= \frac{1}{12}m_2(4r^2 + h^2) + \frac{1}{4}m_2h^2 = \frac{2}{3}\rho(r^2 + h^2)rht \\
I_{zz_2} &= \frac{1}{12}m_2h^2 + \frac{1}{4}m_2h^2 = \frac{2}{3}\rho rh^3 t
\end{aligned}
$$

and for component 3,

$$
\begin{aligned}
I_{xx_3} &= \frac{1}{4}m_3s^2 = \frac{1}{4}\pi\rho s^4 t \\
I_{yy_3} &= \frac{1}{2}m_3s^2 + m_3\left(\frac{h}{2}\right)^2 = \left(\frac{1}{2}s^2 + \frac{1}{4}h^2\right)\pi\rho s^2 t \\
I_{zz_3} &= \frac{1}{4}m_3s^2 + m_3\left(\frac{h}{2}\right)^2 = \left(\frac{1}{4}s^2 + \frac{1}{4}h^2\right)\pi\rho s^2 t
\end{aligned}
$$

Thus, for example,

$$
I_{xx} = I_{xx_1} + I_{xx_2} - I_{xx_3} = \frac{1}{4}\rho r^4 t + \frac{2}{3}\rho r^3 ht - \frac{1}{4}\pi\rho s^4 t
$$

In obtaining these moments of inertia, the Table of Appendix B as well as the PAT have been used. The products of inertia are all zero because all three components have rotational mass symmetry about the specified axes.

These examples show that determination of the moments and products of inertia of a complex body by Eqns. (7.10) (generally a difficult and tedious task) usually can be avoided by: (i) "breaking-up" the body into geometrically simple components, (ii) identifying and using symmetry properties, (iii) using values from tables, (iv) applying the parallel axis theorem, and (v) using the composition relations, Eqns. (7.14).

The determination of the mass, the location of the center of mass, and the moments and products of inertia of a rigid body is sometimes called *mass properties engineering.*

Notes

1 For a rigid body and a body-fixed frame, as we are considering, the moments and products of inertia are constant over time. There are more general situations, for example a rocket rapidly losing mass, in which these quantities change over time.

2 See Section 3.8.

Problems

7/1 Find the inertia matrix (i.e. all the moments and products of inertia) for the body described in Problem 5/1 relative to the axes shown. Do not use the table in Appendix B.

7/2 Find the inertia matrix for the body of Problem 5/2. You may use Appendix B.

7/3 Find the inertia matrix for the body of Problem 5/3. You may use Appendix B.

7/4 Find the inertia matrix for the body of Problem 5/4. Do not use Appendix B.

7/5 Find the inertia matrix for the body of Problem 5/5. You may use Appendix B.

7/6 Find the inertia matrix for the body of Problem 5/6. Do not use Appendix B.

7/7 Find the inertia matrix for the body of Problem 5/7. Do not use Appendix B.

7/8 Find the inertia matrix for the body of Problem 5/8. Do not use Appendix B.

7/9 Find the inertia matrix for the body of Problem 5/9. You may use Appendix B.

7/10 Find the inertia matrix for the body of Problem 5/10. The density of the material is 1.8 kg/m. You may use Appendix B.

7/11 Find the inertia matrix for the body of Problem 5/11. You may use Appendix B.

7/12 Find the inertia matrix for the body of Problem 5/12. You may use Appendix B.

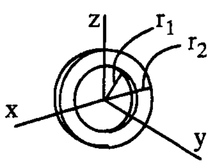
7/13 Find the inertia matrix for the body of Problem 5/13. You may use Appendix B.

7/14 Find the inertia matrix for the body of Problem 5/14. You may use Appendix B.

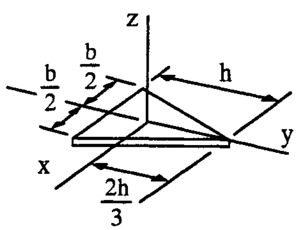
7/15 Find the inertia matrix for the body of Problem 5/15. You may use Appendix B.

7/16 Find the inertia matrix for the body of Problem 5/16. Do not use Appendix B.

7/17 Determine the moments of inertia of a thin homogeneous ring of mass, m, w.r.t. the axes shown.



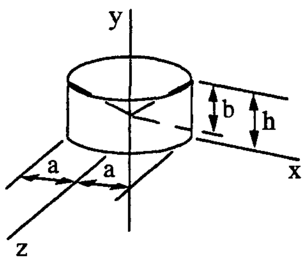
Problem 7/17



Problem 7/18

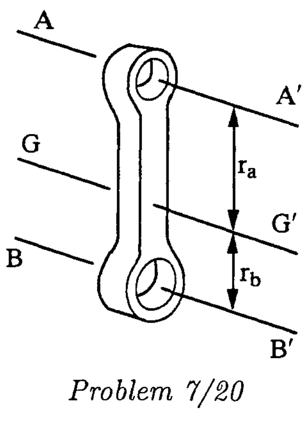
7/18 A thin plate of mass m is cut from homogeneous material in the shape of an isosceles triangle of base b and height h. Determine the moments of inertia relative to the centroidal axes shown.

7/19 A machine part is obtained by machining a conical cavity into a homogeneous circular cylinder. For $b = \frac{1}{2}h$, determine the moment of inertia and the radius of gyration of the part w.r.t. the centroidal y axis.



Problem 7/19

7/20 The figure shows a 2 kg connecting rod. It is known that the moments of inertia relative to axes passing through the centerlines of the bearings are $I_{AA'} = 78\ \mathrm{gm}^2$ and $I_{BB'} = 41\ \mathrm{gm}^2$, and that $r_a + r_b = 290$ mm. Determine the location of the centroidal axis GG' and the radius of gyration w.r.t. axis GG'.



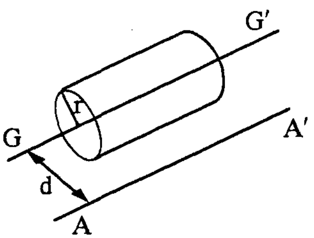
Problem 7/20

7/21 Figure 7/20 shows a 3.25 kg connecting rod. It is known that the moment of inertia relative to axis AA' is $I_{AA'} = 205\ \mathrm{gm}^2$ and that

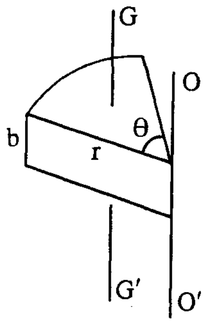
$r_a = 225$ mm and $r_b = 140$ mm. Find the moment of inertia of the rod relative to axis BB'.

7/22 Consider an arbitrary rigid body and three mutually perpendicular body-fixed axes x, y, z. Prove that the moment of inertia w.r.t. any one of the three axes cannot be larger than the sum of the moments of inertia w.r.t. the other two axes; that is, for example, that $I_{xx} \leq I_{yy} + I_{zz}$. Further prove that if the body has an axis of rotational mass symmetry and x is that axis, then $I_{yy} \geq \frac{1}{2} I_{xx}$ where y is any axis perpendicular to x.

7/23 The moment of inertia of a homogeneous cylinder of radius r about an axis parallel to the axis of symmetry of the cylinder is sometimes approximated by multiplying the mass of the cylinder times the square of the distance d between the two axes. Does this approximation improve or worsen as the ratio d/r increases? What percentage error results if (i) $d = 10r$, (ii) $d = 2r$?



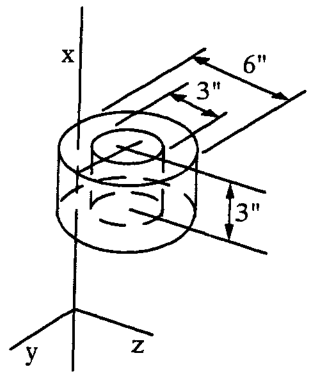
Problem 7/23



Problem 7/24

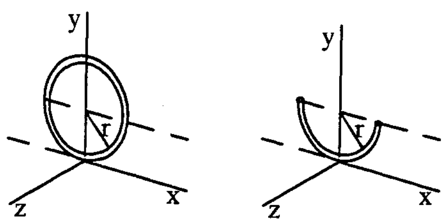
7/24 The homogeneous cylindrical wedge has mass m, radius r, and thickness b. Using Appendix B, write the moment of inertia about axis OO'. Then find the moment of inertia about axis GG' passing through the wedge's center of mass and parallel to OO'.

7/25 Determine the moment of inertia of the homogeneous steel cylinder with a concentric hole about the x axis, an axis parallel to the axis of symmetry and passing through the outer surface of the cylinder. The mass density of the steel is $\rho = 15.19$ lb sec^2/ft^4.



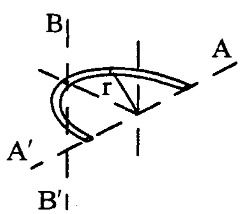
Problem 7/25

7/26 Find the moment of inertia about the tangent axis x for both a circular and a semi-circular thin homogeneous ring. The mass of each is m.



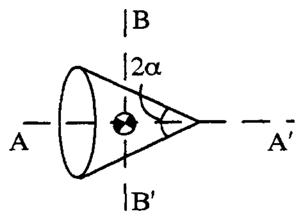
Problem 7/26

7/27 Find the moment of inertia of the thin homogeneous half-ring of mass m both about its diametrical axis AA' and about axis BB' perpendicular to the plane of the ring and passing through the midpoint of the ring.

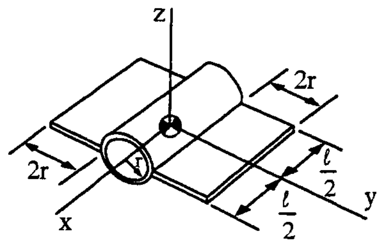


Problem 7/27

7/28 For a homogeneous cone, find the value of the half-cone angle α for which the moment of inertia $I_{AA'}$ w.r.t. the axis of symmetry equals the moment of inertia $I_{BB'}$ perpendicular to the axis of symmetry passing through the cone's center of mass.



Problem 7/28

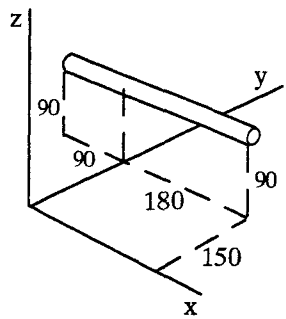


Problem 7/29

7/29 The figure shows a spacecraft that may be idealized as a thin cylindrical shell and two thin flat panels, all made of the same homogeneous material and having the same thickness. For stability when spun around the x-axis, the moment of inertia I_{xx} must be less than moment of inertia I_{yy}. Determine the value of ℓ relative to r that must be exceeded to achieve this.

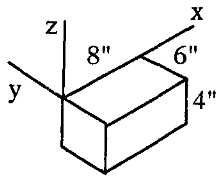
7/30 Consider the assembly of rods shown on Fig. 5/21. Determine the moment of inertia relative to axis x'.

7/31 Find the products of inertia relative to the axes shown of the uniform slender rod of mass 0.60 kg. All dimensions are in mm.



Problem 7/31

7/32 The figure shows a homogeneous rectangular block of 50 lb. Determine the products of inertia of the block relative to the axes shown.



Problem 7/32

Chapter 8

Angular Momentum Equations

8.1 Angular Momentum Equation

Recall that by Euler's Theorem (Section 4.2), the location of all points of a rigid body are determined by six independent variables. Equation (7.5) provides three of these, the coordinates of the mass center of the rigid body. The other three are determined by considering the effects of the *moments* of the forces acting on the body.

Figure 8-1 shows a rigid body, considered as a collection of mass particles, moving relative to an inertial reference frame. Point B is any point and G is the center of mass. From the figure:

$$\underline{d}_i = \overline{\underline{d}} + \underline{\rho}_i \tag{8.1}$$

$$\underline{d}_i = \underline{r}_i - \underline{r}_B \tag{8.2}$$

so that

$$\frac{D^2\underline{d}_i}{Dt^2} = \frac{D^2\underline{r}_i}{Dt^2} - \frac{D^2\underline{r}_B}{Dt^2} = \underline{a}_i - \underline{a}_B \tag{8.3}$$

where D/Dt is the time derivative in the inertial frame.

As usual, let:

$$\underline{F}^e_{ij} = j^{\underline{\text{th}}} \text{ external force acting on } m_i$$

$$\underline{F}^i_{ij} = j^{\underline{\text{th}}} \text{ internal force acting on } m_i$$

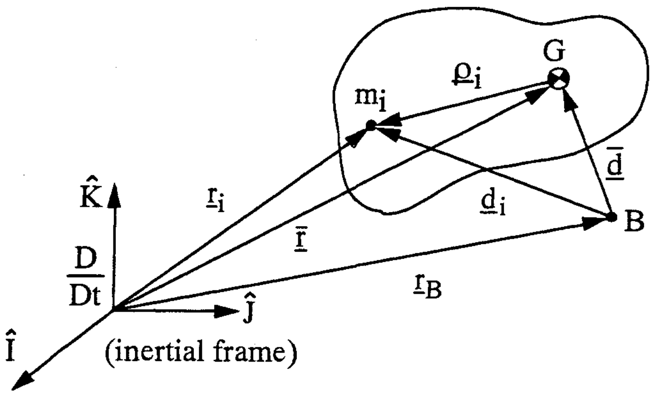


Fig. 8-1

The moments of these forces relative to point B are[1]

$$\underline{M}^e_{B_{ij}} = \underline{d}_i \times \underline{F}^e_{ij} \tag{8.4}$$

$$\underline{M}^i_{B_{ij}} = \underline{d}_i \times \underline{F}^i_{ij} \tag{8.5}$$

Summing over all forces and masses gives the total moment of the forces about B;

$$\underline{M}_B = \sum_i \left(\underline{d}_i \times \sum_j \underline{F}^e_{ij} \right) + \sum_i \left(\underline{d}_i \times \sum_j \underline{F}^i_{ij} \right) \tag{8.6}$$

By Newton's Third Law, the internal forces occur in equal and opposite pairs of forces which act on the line adjoining them. Figure 8-2 shows that the net moment due to such a pair is zero, so that

$$\sum_i \left(\underline{d}_i \times \sum_j \underline{F}^i_{ij} \right) = \underline{0}$$

Consequently Eqn. (8.6) becomes

$$\underline{M}_B = \sum_i \left(\underline{d}_i \times \sum_j \underline{F}^e_{ij} \right) \tag{8.7}$$

and, using Eq. (5.5),

$$\underline{M}_B = \sum_i \underline{d}_i \times m_i \underline{a}_i \tag{8.8}$$

Now recall Eqn. (7.6), the definition of angular momentum about B,

$$\underline{H}_B = \sum_i \underline{d}_i \times m_i \frac{D\underline{d}_i}{Dt}$$

Using Eqns. (8.1), (8.2), (8.3), and (8.8), we have

$$\begin{aligned}
\frac{D\underline{H}_B}{Dt} &= \sum_i \frac{D\underline{d}_i}{Dt} \times m_i \frac{D\underline{d}_i}{Dt} + \sum_i \underline{d}_i \times m_i \frac{D^2\underline{d}_i}{Dt^2} \\
&= \sum_i \underline{d}_i \times m_i \underline{a}_i - \sum_i \underline{d}_i \times m_i \underline{a}_B \\
&= \underline{M}_B - \sum_i \overline{\underline{d}} \times m_i \underline{a}_B - \sum_i \underline{\rho}_i m_i \times \underline{a}_B = \underline{M}_B - \overline{\underline{d}} \times m \underline{a}_B
\end{aligned}$$

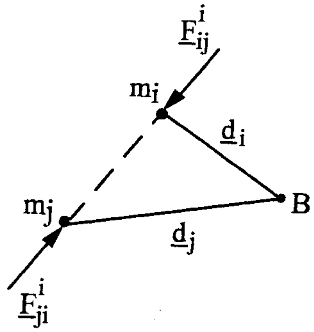


Fig. 8-2

Rearranging,

$$\boxed{\underline{M}_B = \frac{D\underline{H}_B}{Dt} + \overline{\underline{d}} \times m \underline{a}_B} \tag{8.9}$$

Thus far, B is any point, but it is usually possible, and desirable, to pick B such that the last term of Eqn. (8.9) is zero. This will happen when:

1. $\overline{\underline{d}} = \underline{0}$. This is the case $B = G$ and thus

$$\overline{\underline{M}} = \frac{D\overline{\underline{H}}}{Dt} \tag{8.10}$$

2. $\underline{a}_B = \underline{0}$. This is the case of B moving with constant velocity in the inertial frame:

$$\underline{M}_B = \frac{D\underline{H}_B}{Dt} \tag{8.11}$$

As a subcase, this equation applies if B is fixed in the inertial frame.

3. $\overline{\underline{d}}$ parallel to $\underline{a}_B$.

8.2 Euler's Equations

Now introduce a body-fixed frame $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ with origin at B and whose axes are principal axes of inertia. Further, point B is either fixed in the inertial frame or is at the center of mass, so that either Eqn. (8.10) or (8.11) applies. Let the angular velocity of $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ with respect to $\{\hat{I}, \hat{J}, \hat{K}\}$ be $\underline{\omega}$ and denote the time derivative in the body-fixed frame by d/dt. Then by the basic kinematic equation (Eqn. (4.15)),

$$\underline{M}_B = \frac{D\underline{H}_B}{Dt} = \frac{d\underline{H}_B}{dt} + \underline{\omega} \times \underline{H}_B \tag{8.12}$$

From Eqn. (7.12),

$$\underline{H}_B = [I]\underline{\omega}$$

$$\begin{pmatrix} H_{B_x} \\ H_{B_y} \\ H_{B_z} \end{pmatrix} = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \begin{pmatrix} I_{xx}\omega_x \\ I_{yy}\omega_y \\ I_{zz}\omega_z \end{pmatrix}$$

$$\underline{H}_B = I_{xx}\omega_x\hat{\imath} + I_{yy}\omega_y\hat{\jmath} + I_{zz}\omega_z\hat{k}$$

so that

$$\frac{d\underline{H}_B}{dt} = I_{xx}\dot{\omega}_x\hat{\imath} + I_{yy}\dot{\omega}_y\hat{\jmath} + I_{zz}\dot{\omega}_z\hat{k}$$

$$\underline{\omega} \times \underline{H}_B = \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \omega_x & \omega_y & \omega_z \\ I_{xx}\omega_x & I_{yy}\omega_y & I_{zz}\omega_z \end{bmatrix}$$

Thus the three component equations of Eqn. (8.12) are

$$\boxed{\begin{aligned} M_{B_x} &= I_{xx}\dot{\omega}_x - (I_{yy} - I_{zz})\omega_y\omega_z \\ M_{B_y} &= I_{yy}\dot{\omega}_y - (I_{zz} - I_{xx})\omega_z\omega_x \\ M_{B_z} &= I_{zz}\dot{\omega}_z - (I_{xx} - I_{yy})\omega_x\omega_y \end{aligned}} \tag{8.13}$$

where

$$\underline{M}_B = M_{B_x}\hat{\imath} + M_{B_y}\hat{\jmath} + M_{B_z}\hat{k} \tag{8.14}$$

Equations (8.13) are called *Euler's Equations.* It must be kept in mind that they apply only for body-fixed principal axes with origin either fixed or at the center of mass.

8.3 Summary of Rigid Body Motion

In practice, it is often better to use Eqns. (8.10) or (8.11) rather than Eqns. (8.13). The latter have more restrictive assumptions and don't usually save very much computation. Thus we will adopt the following as the complete equations defining rigid body motion:

$$\boxed{\begin{aligned} &\underline{F}^e = \frac{D}{Dt}\overline{\underline{L}} \\ &\overline{\underline{M}} = \frac{D}{Dt}\overline{\underline{H}}, \quad \text{or,} \quad \underline{M}_B = \frac{D}{Dt}\underline{H}_B \end{aligned}} \tag{8.15}$$

where, to recapitulate,

$\underline{F}^e$ = sum of external forces

$\overline{\underline{L}} = m\overline{\underline{v}}$ = linear momentum

$\overline{\underline{M}}$ = sum of moments of external forces about the center of mass

$\underline{M}_B$ = sum of moments of external forces about B, a point moving with constant velocity in an inertial frame

$\overline{\underline{H}} = [\overline{I}]\underline{\omega}$ = angular momentum about the center of mass

$\underline{H}_B = [I]_B\underline{\omega}$ = angular momentum about B

In this and the following two chapters, we shall apply Eqns. (8.15) to several types of problems.

In general, use of Eqns. (8.15) may be quite complicated, especially the calculation of the angular momentum $\underline{H}$. It is frequently possible, however, to choose body-fixed axes that simplify this calculation. Two obvious choices for orientation for body-fixed axes are as follows:

1. Along principal axes of inertia. In this case,

$$\underline{\omega} = \omega_x \hat{\imath} + \omega_y \hat{\jmath} + \omega_z \hat{k}$$

$$\begin{pmatrix} H_z \\ H_y \\ H_z \end{pmatrix} = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

$$\boxed{\underline{H} = I_{xx}\omega_x \hat{\imath} + I_{yy}\omega_y \hat{\jmath} + I_{zz}\omega_z \hat{k}} \tag{8.16}$$

(where $\underline{H}$ is $\overline{\underline{H}}$ or $\underline{H}_B$).

2. One axis along a body-fixed axis that stays fixed in an inertial frame, say $\hat{k}$, if such an axis exists, so that

$$\underline{\omega} = \omega \hat{k}$$

$$\begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix}$$

$$\boxed{\underline{H} = -I_{xz}\omega \hat{\imath} - I_{yz}\omega \hat{\jmath} + I_{zz}\omega \hat{k}} \tag{8.17}$$

Note that only three of the moments and products of inertia need be computed in this case. This is the case of fixed axis rotation, treated in the next section and covered in detail in the following chapter.

8.4 Examples

Suppose the three bar assembly depicted on Fig. 7-3 is rotated about the x-axis, which is fixed in an inertial frame. Then the second of the second of Eqns. (8.15) applies with point B taken as the origin of the axes shown on Fig. 7-3. At a certain instant suppose the rotation rate is ω and is increasing at the rate $\dot{\omega}$. First we compute $\underline{H}_B$ from Eqn. (7.12). In our case, $\underline{\omega} = \omega\hat{\imath}$; using the results of Section 7.3, we obtain

$$\underline{H}_B = [I]\underline{\omega}$$

$$\begin{pmatrix} H_{B_x} \\ H_{B_y} \\ H_{B_z} \end{pmatrix} = \frac{m\ell^2}{12}\begin{bmatrix} 8 & -3 & 3 \\ -3 & 11 & 0 \\ 3 & 0 & 11 \end{bmatrix}\begin{pmatrix} \omega \\ 0 \\ 0 \end{pmatrix} = \frac{m\ell^2\omega}{12}\begin{pmatrix} 8 \\ -3 \\ 3 \end{pmatrix}$$

$$\underline{H}_B = \frac{m\ell^2\omega}{12}(8\hat{\imath} - 3\hat{\jmath} + 3\hat{k})$$

Let $D(\ )/Dt$ and $d(\ )/dt$ denote the time derivatives w.r.t. an inertial and a body-fixed frame, respectively. Then,

$$\underline{M}_B = \frac{D\underline{H}_B}{Dt} = \frac{d\underline{H}_B}{dt} + \underline{\omega} \times \underline{H}_B$$

$$\begin{aligned} M_{B_x}\hat{\imath} + M_{B_y}\hat{\jmath} + M_{B_z}\hat{k} &= \frac{m\ell^2\dot{\omega}}{12}(8\hat{\imath} - 3\hat{\jmath} + 3\hat{k}) + (\omega\hat{\imath}) \\ &\quad \times \left(\frac{m\ell^2\omega}{12}\right)(8\hat{\imath} - 3\hat{\jmath} + 3\hat{k}) \end{aligned}$$

$$M_{B_x} = \frac{2}{3}m\ell^2\dot{\omega}\ ,\ M_{B_y} = -\frac{1}{4}m\ell^2(\dot{\omega} + \omega^2)\ ,\ M_{B_z} = \frac{1}{4}m\ell^2(\dot{\omega} - \omega^2)$$

These moment components, for example, might be exerted by bearings in which a body-fixed shaft along the x-axis rotates. Note that, because the axes are not PAI, Euler's equations do not apply.

Now suppose the three bar assembly is rotated about PAI (2), an axis fixed in an inertial frame, as shown on Fig. 7-5. A similar calculation gives

$$
\begin{aligned}
\begin{pmatrix} H_{B_1} \\ H_{B_2} \\ H_{B_3} \end{pmatrix} &= \frac{m\ell^2}{12}\begin{bmatrix} 11 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 14 \end{bmatrix}\begin{pmatrix} 0 \\ \omega \\ 0 \end{pmatrix} \\
\underline{H}_B &= \frac{5}{12}m\ell^2\omega\hat{\jmath}
\end{aligned}
$$

Thus

$$
\underline{M}_B = \frac{d\underline{H}_B}{dt} + \underline{\omega} \times \underline{H}_B = \frac{5}{12}m\ell^2\dot{\omega}\hat{\jmath}
$$

so that

$$
M_{B_x} = 0\,, \quad M_{B_y} = \frac{5}{12}m\ell^2\dot{\omega}\,, \quad M_{B_z} = 0
$$

In this case the moment has only a component along the axis of rotation; this moment is zero if the rotation rate is constant.

Finally, suppose the assembly is rotated about an axis parallel to the z-axis passing through the center of mass of the assembly. Using the results of Section 7.10, the calculation of $\overline{\underline{M}}$ is as follows:

$$
\begin{aligned}
\begin{pmatrix} \overline{H}_x \\ \overline{H}_y \\ \overline{H}_z \end{pmatrix} &= \frac{1}{36}m\ell^2\omega\begin{bmatrix} 22 & -9 & 9 \\ -9 & 31 & 1 \\ 9 & 10 & 32 \end{bmatrix}\begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} \\
\overline{\underline{H}} &= \frac{m\ell^2\omega}{36}(9\hat{\imath} + \hat{\jmath} + 32\hat{k}) \\
\overline{\underline{M}} &= \frac{d\overline{\underline{H}}}{dt} + \underline{\omega} \times \overline{\underline{H}} = \frac{m\ell^2}{36}\left[(9\dot{\omega} - \omega^2)\hat{\imath} + (\dot{\omega} + 9\omega^2)\hat{\jmath} + 32\dot{\omega}\hat{k}\right]
\end{aligned}
$$

8.5 Special Case of Planar Motion

An important special case is that of 2-D motion. In this case, each particle of the rigid body travels in parallel planes (Fig. 8-3). Letting $\hat{k}$ be a unit vector perpendicular to these planes, $\underline{\omega} = \omega\hat{k}$ and Eqn. (8.17) applies. Using the second of Eqns. (8.15),

$$
\begin{aligned}
\underline{M} &= \frac{D\underline{H}}{Dt} = \frac{d\underline{H}}{dt} + \underline{\omega} \times \underline{H} \\
&= \left(-I_{xz}\dot{\omega} + I_{yz}\omega^2\right)\hat{\imath} - \left(I_{yz}\dot{\omega} + I_{xz}\omega^2\right)\hat{\jmath} + I_{zz}\dot{\omega}\hat{k}
\end{aligned}
$$

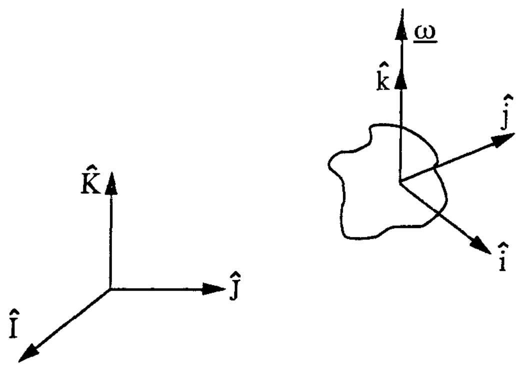


Fig. 8-3

so that

$$\boxed{\begin{aligned} M_x &= -I_{xz}\dot{\omega} + I_{yz}\omega^2 \\ M_y &= -I_{yz}\dot{\omega} - I_{xz}\omega^2 \\ M_z &= I_{zz}\dot{\omega} \end{aligned}} \tag{8.18}$$

where moments are taken about either the center of mass or a point moving with constant velocity.

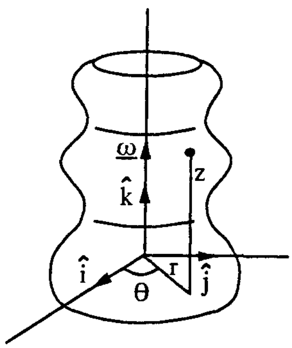


Fig. 8-4

As a subcase, suppose the body has rotational mass symmetry (Fig. 8-4) about the $\hat{k}$ axis (see Section 7.6). Then $I_{xz} = I_{yz} = 0$ and Eqn. (8.16) reduces to the familiar formula

$$\boxed{M_z = I_{zz}\dot{\omega}} \tag{8.19}$$

As another subcase, suppose, instead, that the body is thin; that is, all of its particles may be assumed to lie in a single plane (Fig. 8-5). In this case $I_{xz} = I_{yz} = 0$ also, and Eqn. (8.19) results.

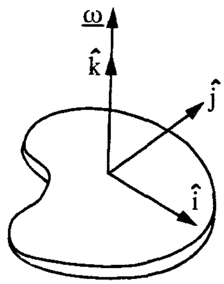


Fig. 8-5

8.6 Example

A sphere is projected along a horizontal surface with initial velocity $\underline{\nu}_0$ and no rotation. If the coefficient of kinetic friction is μ, we wish to determine the time it takes for the sphere to start rolling without sliding and $\underline{\nu}$ and $\underline{\omega}$ at that instant (Fig. 8-6).

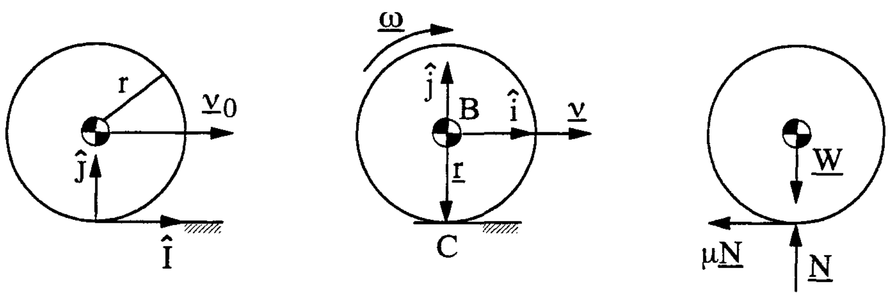


Fig. 8-6

The relevant equations are Eqns. (6.4) and (8.19), repeated here:

$$F_x = m\ddot{\bar{x}}\,, \quad F_y = m\ddot{\bar{y}}\,, \quad M_z = I_{zz}\dot{\omega}$$

These give, dropping the overbar on x,

$$-\mu N = m\ddot{x}$$
$$N - w = 0$$
$$-\mu N r = -\frac{2}{5} m r^2 \dot{\omega}$$

where $\overline{I} = \frac{2}{5} m r^2$ for a sphere. Using $w = mg$, the first two of these give

$$\begin{aligned} \ddot{x} &= -\mu g \\ \dot{x} &= -\mu g t + \dot{x}_0 \\ x &= -\frac{1}{2}\mu g t^2 + \dot{x}_0 + x_0 \end{aligned}$$

If we take $x = 0$ when $t = 0$,

$$\begin{aligned} x &= -\frac{1}{2}\mu g t^2 + \nu_0 t \\ \dot{x} &= -\mu g t + \nu_0 \end{aligned}$$

From the third equation,

$$\begin{aligned} \dot{\omega} &= \frac{5}{2}\frac{\mu g}{r} \\ \omega &= \frac{5}{2}\frac{\mu g}{r} t \end{aligned}$$

The relative velocity equation applied to point C is

$$\underline{\nu}_C = \underline{\nu}_B + \underline{\nu}_r + \underline{\omega} \times \underline{r}$$

where

$$\underline{\nu}_B = \dot{x}\hat{\imath}\,, \quad \underline{\nu}_r = \underline{0}\,, \quad \underline{\omega} = -\frac{5}{2}\frac{\mu g}{r} t \hat{k}$$

The sphere stops sliding when $\underline{\nu}_C = \underline{0}$; at this instant:

$$\begin{aligned} \underline{0} &= \dot{x}\hat{\imath} + \underline{0} + \left(-\frac{5}{2}\frac{\mu g}{r} t\right) \hat{k} \times (-r\hat{\jmath}) \\ t &= \frac{2\nu_0}{7\mu g} \\ \underline{\nu} &= \left[-\mu g \left(\frac{2\nu_0}{7\mu g}\right) + \nu_0\right] \hat{\imath} = \frac{5}{7}\nu_0 \hat{\imath} \\ \underline{\omega} &= -\frac{5}{2}\frac{\mu g}{r}\left(\frac{2\nu_0}{7\mu g}\right) \hat{k} = -\frac{5}{7}\frac{\nu_0}{r}\hat{k} \end{aligned}$$

Note that these results do not depend on the mass of the sphere.

8.7 Equivalent Force Systems

Next it will be established that any external force acting on a body at some point may be replaced with the same force acting at another point plus a moment of the force about the second point, and the motion will be the same.

On Fig. (8-7), one of the external forces on a rigid body, $\underline{F}_j^e$, acts at a point A. We will show that moving this force to another point C and adding a moment

$$\underline{M} = \vec{CA} \times \underline{F}_j^e \tag{8.20}$$

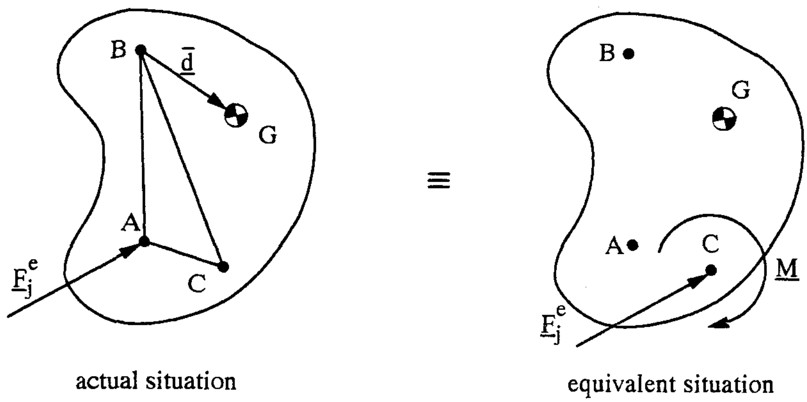


Fig. 8-7

is dynamically equivalent to the actual situation. Let B be any point; then, from Eqn. (8.9), the motion is governed by

$$\begin{aligned} \sum_j \underline{F}_j^e &= m\bar{\underline{a}} \\ \underline{M}_{B_a} &= \frac{D\underline{H}_B}{Dt} + \bar{\underline{d}} \times m\underline{a}_B \end{aligned} \tag{8.21}$$

where $\underline{M}_{B_a}$ is the sum of the moments of the external forces about point B in the actual situation. Clearly, moving $\underline{F}_j^e$ from A to C does not affect $\sum_j \underline{F}_j^e$, and thus $\bar{\underline{a}}$ is the same, but what about $\underline{M}_B$?

The contribution of $\underline{F}_j^e$ to $\underline{M}_{B_a}$ is

$$\underline{M}_{B_{aj}} = \overrightarrow{BA} \times \underline{F}_j^e$$

But $\overrightarrow{BA} = \overrightarrow{BC} + \overrightarrow{CA}$ so that

$$\underline{M}_{B_{aj}} = \overrightarrow{BC} \times \underline{F}_j^e + \overrightarrow{CA} \times \underline{F}_j^e = \underline{M}_{B_{ej}} + \underline{M}$$

where Eqn. (8.20) was used and $\underline{M}_{B_{ej}}$ is the moment of $\underline{F}_j^e$ about B in the equivalent situation. Thus $\underline{M}_B$ is the same in both the actual and equivalent situations, which was to be proved.

Now suppose B is selected as G, the center of mass (Fig. 8-8). Then Eqns. (8.21) reduce to

$$\sum_j \underline{F}_j^e = m\underline{\bar{a}}$$

$$\underline{\overline{M}} = \frac{D\underline{\overline{H}}}{Dt}$$

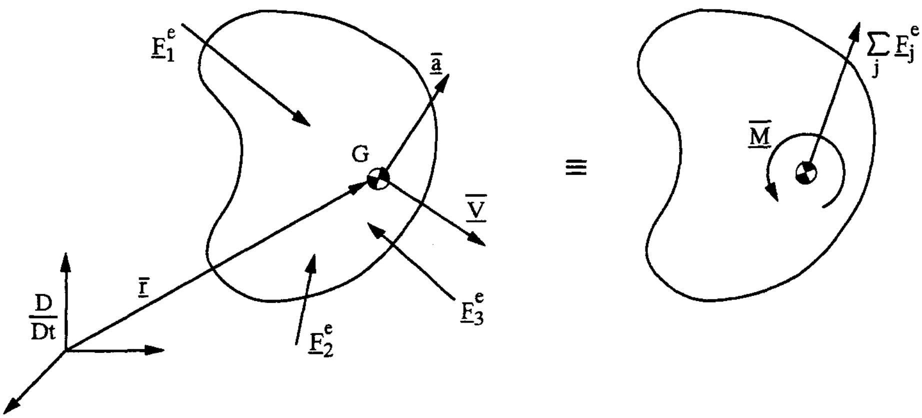


Fig. 8-8

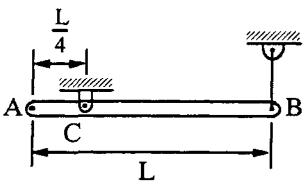
where $\underline{\overline{M}}$, as before, is the sum of the moments of all the external forces about G.

Notes

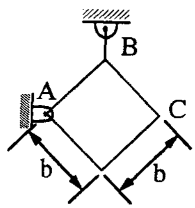
1 Up to now, the "point at which a vector acts" has been irrelevent. For moments, however, the point at which a force acts is important.

Problems

8/1 A thin uniform beam of length L and weight W is supported as shown. If the cable suddenly breaks at B, determine the acceleration of end B and the reaction at the pin support at C.

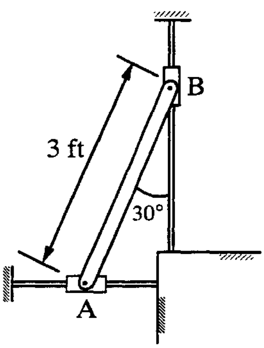


Problem 8/1



Problem 8/2

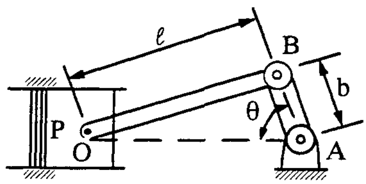
8/2 A thin uniform square plate of weight W is supported as shown. If the cable at B suddenly breaks, determine (i) the angular acceleration of the plate, (ii) the acceleration of corner C, and (iii) the reaction at A.



Problem 8/3

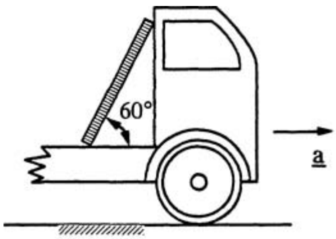
8/3 The ends of an 8 lb uniform thin rod AB are attached to collars of negligible weight that slide without friction along fixed rods. If rod AB is released from rest in the position shown, determine, immediately after release, (i) the angular acceleration of the rod, (ii) the reaction force at A, and (iii) the reaction force at B.

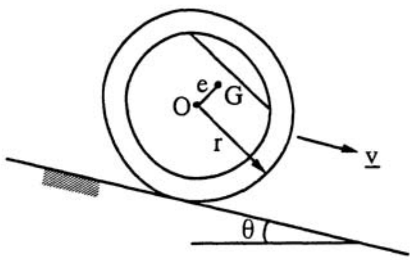
8/4 For the engine components shown, $\ell = 8$ in, $b = 3$ in, piston P weighs 4 lb, and connecting rod BO may be assumed to be a 5 lb uniform slender rod. Determine the forces exerted on the connecting rod at B and O when crank AB is rotating with a constant angular speed of 2000 rpm for the position $\theta = 0$. Neglect the static weight of the rod.



Problem 8/4

8/5 A uniform thin bar is propped up on a truckbed as shown. The coefficient of static friction between the bar and truck surfaces is 0.40. Determine the maximum horizontal acceleration a that the truck may have without causing the bar to slip.

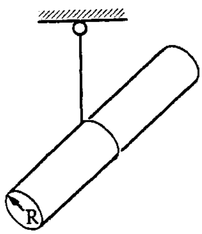




Problem 8/5 *Problem 8/6*

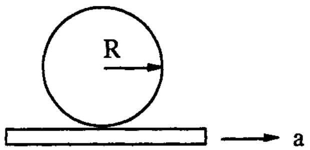
8/6 An unbalanced wheel, whose mass center G is displaced from the geometric center O by distance e, is rolling down an incline of angle θ. What is the least speed v for which the wheel loses momentary contact with the incline? What is the position of G when this occurs?

8/7 A homogeneous circular cylinder of radius R and weight W has a thread of negligible weight wrapped around its center. One end of the thread is fixed and the cylinder is allowed to fall. Find the acceleration of the center of the cylinder.

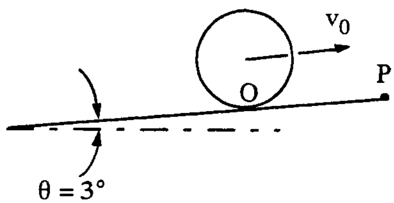


Problem 8/7

8/8 A homogeneous circular cylinder rests upon a flat slab which is given a constant acceleration of magnitude $a = 2g$. Find the angular acceleration of the cylinder when the coefficient of sliding friction f between the cylinder and the slab equals (a) 0.5, and (b) 0.75.



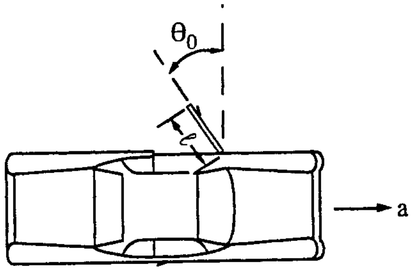
Problem 8/8



Problem 8/9

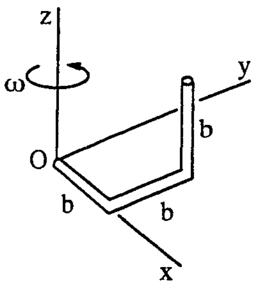
8/9 A uniform disk of mass $m = 1$ slug and radius $a = 1$ ft rolls without slipping in a vertical plane up a slope with inclination angle $\theta = 3°$. At point O, the center of the disk has a speed $v_0 = 6$ ft/sec. Determine the distance from O to P, where P is the point at which the disk comes temporarily to rest.

8/10 The door of a car is left open when the car starts to move from rest with a constant acceleration a. Assuming that the door can be considered to be a uniform rectangular plate of mass M, determine the angular motion of the door. In particular, with what angular speed does the door shut ($\theta = 90°$)? Neglect friction in the hinges and air resistance.

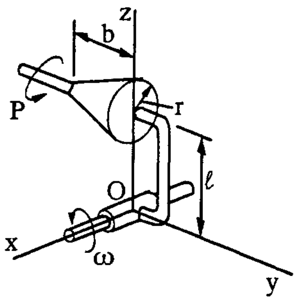


Problem 8/10

8/11 The assembly of three uniform thin rods revolves about the z axis with a constant angular speed ω. Find the angular momentum of the assembly about point O, $\underline{H}_0$.



Problem 8/11

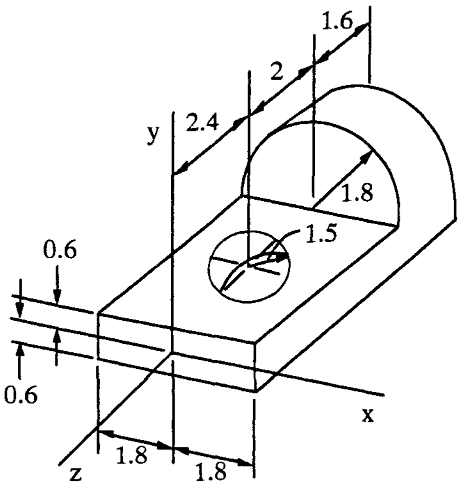


Problem 8/12

8/12 The homogeneous solid right-circular cone of mass m, length b, and radius r spins at a steady angular rate p about its axis of symmetry. Simultaneously, the bracket it is attached to revolves at the constant rate ω about the x axis. Determine the angular momentum $\underline{H}_0$ of the cone relative to point O.

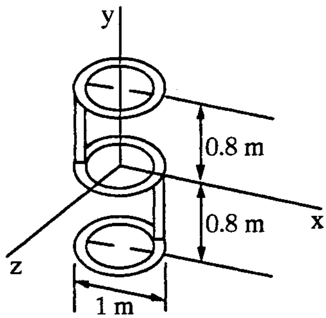
In the next seven problems, the object shown is rotated about the axis indicated. The axis is fixed in an inertial frame; the rotation rate is ω and is increasing at the rate $\dot{\omega} = \alpha$. It is desired to find the angular momentum relative to the origin of the axes shown and the moment that must be exerted to maintain the motion.

8/13 About the y axis.

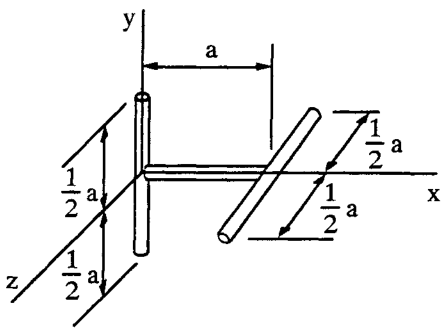


Problem 8/13

8/14 About the y axis. Use the moments and products of inertia computed in Problem 7/10.



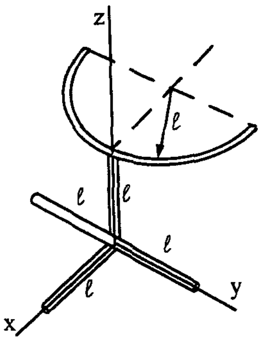
Problem 8/14



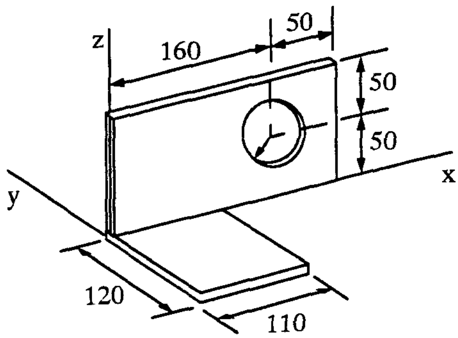
Problem 8/15

8/15 About the x axis.

8/16 About the z axis.



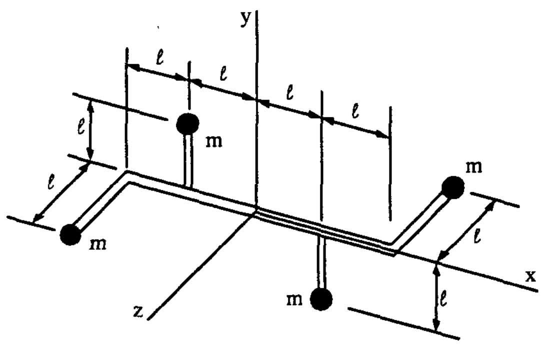
Problem 8/16



Problem 8/17

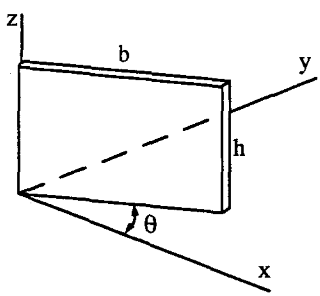
8/17 About the x axis.

8/18 About the x axis.



Problem 8/18

8/19 About the x axis.



Problem 8/19

Chapter 9

Fixed-Axis Rotation

9.1 Introductory Remarks

Using Eqns. (8.15) to derive the equations of motion of a complex rigid body in general 3-D motion can be very complicated and lengthy. In many engineering applications, however, the body has symmetry properties or the motion is constrained in some way or both. Exploiting these features makes the analysis considerably easier. In this and the following chapter, we consider two special, commonly occurring, kinds of motion – fixed-axes rotation and motion with one point fixed.

In fixed-axis rotation, a rigid body moves such that a line of body-fixed points remains stationary in an inertial frame of reference. There are many important engineering examples of this type of motion, for example wheels, crankshafts, compressors, turbines, pumps, and computer disk-drives. Motion of this type already has been considered in Section 8.4. We begin here by analyzing two simple problems.

9.2 Off-Center Disk

Figure 9-1 shows a thin homogeneous circular disk mounted on a thin rigid shaft[1] rotating in bearings which are fixed in an inertial frame of reference. The disk is mounted so that the plane of the disk is 90° to the shaft and its center G is offset a distance d from the shaft axis at B.

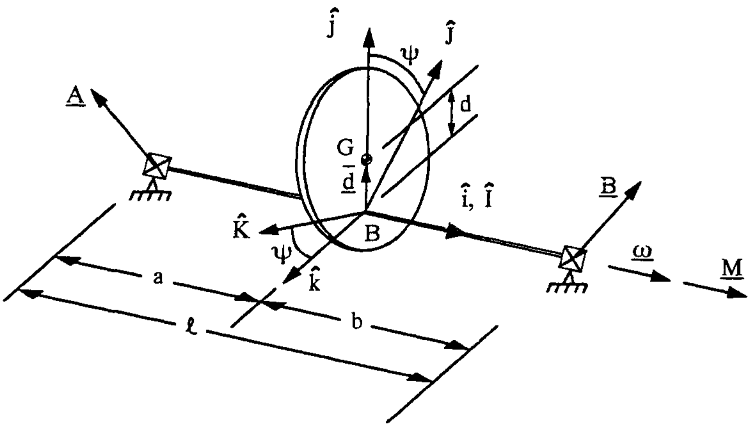


Fig. 9-1

The shaft rotates at a constant speed ω and there may be a moment $\underline{M}$.

Introduce the following reference frames:

1. $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ is body-fixed with origin B, $\hat{\imath}$ along shaft axis, and $\hat{\jmath}$ on a line connecting the shaft axis and G, the center of mass.

2. $\{\hat{I}, \hat{J}, \hat{K}\}$ is inertial with origin B and $\hat{I}$ along shaft axis, and such that $\hat{J}$ makes an angle ψ with $\hat{\jmath}$ at the instant shown.

From the figure, the unit vector transformations are

$$
\begin{aligned}
\hat{\imath} &= \hat{I} \\
\hat{\jmath} &= \cos\psi \hat{J} + \sin\psi \hat{K} \\
\hat{k} &= -\sin\psi \hat{J} + \cos\psi \hat{K}
\end{aligned}
$$

Also,

$$
\begin{aligned}
\underline{\omega} &= \omega\hat{\imath} = \omega\hat{I} \\
\underline{\bar{d}} &= d\hat{\jmath} \Rightarrow \bar{d}_x = 0\ , \quad \bar{d}_y = d\ , \quad \bar{d}_z = 0 \\
\underline{M} &= M\hat{I}
\end{aligned}
$$

The Parallel Axis Theorem, Eqns. (7.26), is used to get the moments and products of inertia relative to $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$:

$$
I_{xx} = \bar{I}_{xx} + m(\bar{d}_y^2 + \bar{d}_x^2) = \frac{1}{2}mr^2 + md^2 = 2I + md^2
$$

$$
\begin{aligned}
I_{yy} &= \frac{1}{4}mr^2 = I \\
I_{zz} &= I + md^2 \\
I_{xy} &= I_{yz} = I_{zx} = 0
\end{aligned}
$$

This verifies that the $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ are principal axes of inertia. Next we compute the angular momentum about point B and its time derivative; Eqn. (7.12) gives:

$$
\underline{H}_B = [I]\underline{\omega}
$$

$$
\begin{pmatrix} H_{B_x} \\ H_{B_y} \\ H_{B_z} \end{pmatrix} = \begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix} \begin{pmatrix} \omega \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} I_{xx}\omega \\ 0 \\ 0 \end{pmatrix}
$$

$$
\underline{H}_B = \omega I_{xx}\hat{\imath} = \omega I_{xx}\hat{I}
$$

Since ω, I_{xx}, and $\hat{I}$ are all constants,

$$
\frac{D\underline{H}_B}{Dt} = \underline{0}
$$

Now resolve the bearing forces in the $\{\hat{I}, \hat{J}, \hat{K}\}$ frame:

$$
\underline{A} = A_X\hat{I} + A_Y\hat{J} + A_Z\hat{K}\,, \qquad \underline{B} = B_X\hat{I} + B_Y\hat{J} + B_Z\hat{K}
$$

Using the first of Eqns. (8.15), we have[2]

$$
\begin{aligned}
&\underline{F}^e = m\underline{\bar{a}} = -md\omega^2\hat{\jmath} \\
&A_X\hat{I} + A_Y\hat{J} + A_Z\hat{K} + B_X\hat{I} + B_Y\hat{J} + B_Z\hat{K} \\
&\qquad = -md\omega^2(\cos\psi\hat{J} + \sin\psi\hat{K})
\end{aligned}
$$

The component equations are:

$$
\begin{aligned}
A_X + B_X &= 0 \\
A_Y + B_Y &= -md\omega^2\cos\psi \\
A_Z + B_Z &= -md\omega^2\sin\psi
\end{aligned}
\tag{9.1}
$$

Next consider the moment equation. Since B is fixed in the inertial frame, the second of Eqns. (8.15) gives

$$
\underline{M}_B = \frac{D\underline{H}_B}{Dt}
$$

$$-a\hat{I} \times \underline{A} + b\hat{I} \times \underline{B} + M\hat{I} = \frac{D\underline{H}_B}{Dt} = \underline{0}$$
$$-aA_Y\hat{K} + aA_Z\hat{J} + bB_Y\hat{K} - bB_Z\hat{J} + M\hat{I} = \underline{0}$$

The components of this are

$$\begin{aligned} &M = 0 \\ &aA_Z - bB_Z = 0 \\ &-aA_Y + bB_Y = 0 \end{aligned} \tag{9.2}$$

The components A_X, A_Y, A_Z, B_X, B_Y, B_Z, M are the unknowns of the problem. Solving Eqns. (9.1) and (9.2) for these components results in

$$\begin{aligned} A_X &= -B_X \\ A_Y &= -\frac{md\omega^2 b\cos\omega t}{l} \\ B_Y &= -\frac{md\omega^2 a\cos\omega t}{l} \\ A_Z &= -\frac{md\omega^2 b\sin\omega t}{l} \\ B_Z &= -\frac{md\omega^2 a\sin\omega t}{l} \\ M &= 0 \end{aligned} \tag{9.3}$$

Thus there are oscillatory forces in the bearings, sometimes called "dynamic loads". The forces in Eqn. (9.3) are the forces exerted by the bearings on the shaft; the forces exerted by the shaft on the bearings are, of course, equal and opposite. Note that no moment is required to keep the shaft spinning at constant speed.[3] Note also that values of A_X and B_X are not obtained; since bearings generally exert negligible axial force, we may take $A_X = B_X = 0$.

The bearing forces given in Eqns. (9.3) are plotted on Fig. 9-2. The magnitudes of the forces in each bearing are constant:

$$|\underline{A}| = \sqrt{A_Y^2 + A_Z^2} = \frac{md\omega^2 b}{l}$$

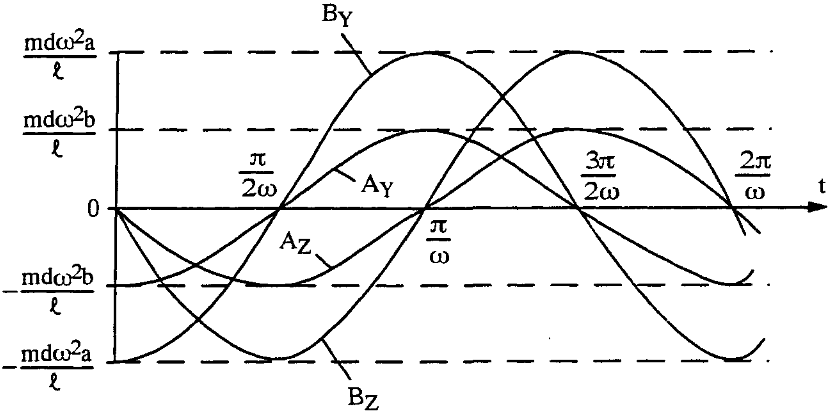


Fig. 9-2

$$|\underline{B}| = \sqrt{B_Y^2 + B_Z^2} = \frac{md\omega^2 a}{l}$$

Note that these forces are proportional to the mass of the disk and to the rotation speed squared. Finally, we remark that these forces rotate in-phase with the shaft as shown on Fig. 9-3.

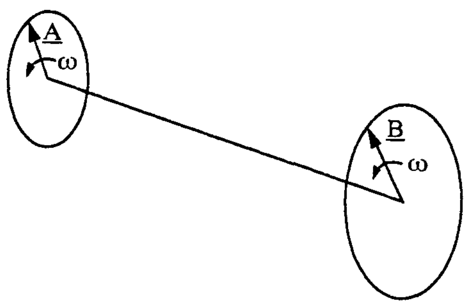


Fig. 9-3

9.3 Bent Disk

Now suppose a thin homogeneous disk is mounted on a rotating shaft such that the center of the disk is on the shaft axis but the disk is bent from being perpendicular to the shaft by angle β (Fig. 9-4). As before, the shaft rotates at a constant speed and a moment may be applied, and, as before, it is desired to find the forces $\underline{A}$ and $\underline{B}$ in the bearings and the moment $\underline{M}$.

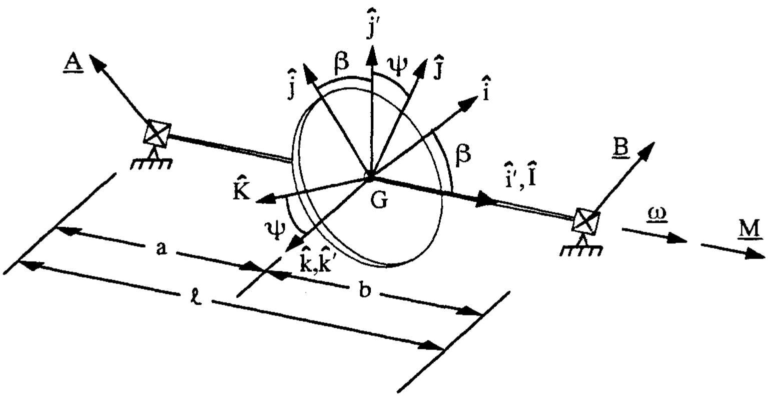


Fig. 9-4

In this case it is convenient to introduce three reference frames:

1. $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ is body-fixed along the principal axes of inertia, with $\hat{\jmath}$ and $\hat{k}$ in the plane of the disk.
2. $\{\hat{\imath}', \hat{\jmath}', \hat{k}'\}$ is body-fixed, with $\hat{\imath}$ along the axis of rotation and $\hat{k}'$ in the plane of the disk.
3. $\{\hat{I}, \hat{J}, \hat{K}\}$ is inertial, with $\hat{I}$ along the axis of rotation such that $\hat{J}$ makes an angle ψ with $\hat{\jmath}'$ at the instant shown.

For convenience, we choose to form $D\underline{\overline{H}}/Dt$ in the $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ axes and then transform to $\{\hat{I}, \hat{J}, \hat{K}\}$. The required unit vector transformations are (Fig. 9-5):

$$\hat{\imath}' = \cos\beta\hat{\imath} - \sin\beta\hat{\jmath}$$

$$\hat{\imath} = \cos\beta\hat{\imath}' + \sin\beta\hat{\jmath}'$$
$$\hat{\jmath} = -\sin\beta\hat{\imath}' + \cos\beta\hat{\jmath}'$$

$$\hat{\jmath}' = \cos\psi\hat{J} + \sin\psi\hat{K}$$
$$\hat{k}' = -\sin\psi\hat{J} + \cos\psi\hat{K}$$

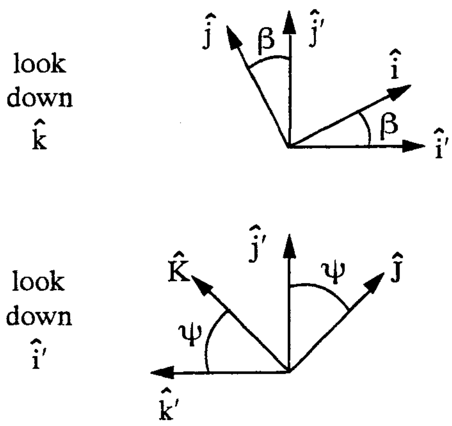


Fig. 9-5

In the $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ axes,

$$\begin{aligned} I_{yy} &= I_{zz} = \frac{1}{4}mr^2 = I \\ I_{xx} &= \frac{1}{2}mr^2 = 2I \\ I_{xy} &= I_{yz} = I_{zx} = 0 \end{aligned}$$

Also,

$$\underline{\omega} = \omega \hat{I} = \omega \hat{\imath}' = \omega(\cos\beta \hat{\imath} - \sin\beta \hat{\jmath})$$

The angular momentum is

$$\underline{\overline{H}} = [\overline{I}]\underline{\omega}$$

$$\begin{pmatrix} \overline{H}_x \\ \overline{H}_y \\ \overline{H}_z \end{pmatrix} = \begin{bmatrix} 2I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{pmatrix} \omega\cos\beta \\ -\omega\sin\beta \\ 0 \end{pmatrix} = \begin{pmatrix} 2I\omega\cos\beta \\ -I\omega\sin\beta \\ 0 \end{pmatrix}$$

$$\underline{\overline{H}} = I\omega(2\cos\beta\hat{\imath} - \sin\beta\hat{\jmath})$$

Taking the time derivative in an inertial (ground-fixed) frame results in

$$\begin{aligned} \frac{D\underline{\overline{H}}}{Dt} &= \frac{d\underline{\overline{H}}}{dt} + \underline{\omega} \times \underline{\overline{H}} \\ &= \underline{0} + \omega(\cos\beta\hat{\imath} - \sin\beta\hat{\jmath}) \times I\omega(2\cos\beta\hat{\imath} - \sin\beta\hat{\jmath}) = \frac{1}{2}I\omega^2 \sin 2\beta\hat{k} \end{aligned}$$

Now transform this first to the $\{\hat{\imath}', \hat{\jmath}', \hat{k}'\}$ axes, and then to the ground fixed $\{\hat{I}, \hat{J}, \hat{K}\}$ axes:

$$\frac{D\overline{\underline{H}}}{Dt} = \frac{1}{2} I\omega^2 \sin 2\beta \hat{k}' = \frac{1}{2} I\omega^2 \sin 2\beta(-\sin\psi \hat{J} + \cos\psi \hat{K})$$

Next resolve the forces in the inertial frame

$$\underline{A} = A_X \hat{I} + A_Y \hat{J} + A_Z \hat{K}\ , \qquad \underline{B} = B_X \hat{I} + B_Y \hat{J} + B_Z \hat{K}$$

The force equation gives

$$\underline{F}^e = m\underline{\overline{a}} = \underline{0}$$

where we have again neglected the weight of the disk. The components are

$$A_X + B_X = 0\ , \qquad A_Y + B_Y = 0\ , \qquad A_Z + B_Z = 0 \tag{9.4}$$

The moment equation gives

$$\begin{aligned}
&\underline{\overline{M}} = \frac{D\overline{\underline{H}}}{Dt} \\
&-a\hat{I} \times \underline{A} + b\hat{I} \times \underline{B} + M\hat{I} = \frac{D\overline{\underline{H}}}{Dt} \\
&-a\hat{I} \times (A_X \hat{I} + A_Y \hat{J} + A_Z \hat{K}) + b\hat{I} \times (B_X \hat{I} + B_Y \hat{J} + B_Z \hat{K}) + M\hat{I} \\
&\qquad = \frac{D\overline{\underline{H}}}{Dt} \\
&-aA_Y \hat{K} + aA_Z \hat{J} + bB_Y \hat{K} - bB_Z \hat{J} + M\hat{I} = \frac{D\overline{\underline{H}}}{Dt}
\end{aligned}$$

whose components are

$$\begin{aligned}
M &= 0 \\
aA_Z - bB_Z &= -\frac{1}{2} I\omega^2 \sin\psi \sin 2\beta \\
-aA_Y + bB_Y &= \frac{1}{2} I\omega^2 \cos\psi \sin 2\beta
\end{aligned} \tag{9.5}$$

Solving Eqns. (9.4) and (9.5) for the force and moment components results in:

$$\begin{aligned} A_X &= -B_X \\ A_Y &= -B_Y = -\frac{1}{2l} I\omega^2 \sin 2\beta \cos \omega t \\ A_Z &= -B_Z = -\frac{1}{2l} I\omega^2 \sin 2\beta \sin \omega t \\ M &= 0 \end{aligned} \tag{9.6}$$

where ψ has been replaced by ωt.

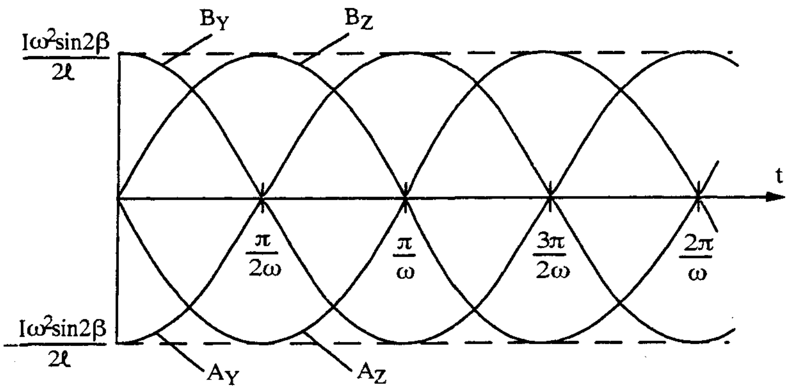


Fig. 9-6

Thus in this case there are also oscillatory dynamic loads in the bearings that are proportional to the square of the rotation rate; these force components are plotted on Fig. 9-6. As before, the bearing forces rotate with the shaft, but now these forces are out-of phase (Fig. 9-7); their magnitudes are the same:

$$|\underline{A}| = |\underline{B}| = \frac{I\omega^2 \sin 2\beta}{2l}$$

Finally, we note that for $\beta = 0$ and $\beta = \pi/2$ (Fig. 9-8) these forces vanish:

$$A_Y = B_Y = A_Z = B_Z = 0$$

Thus there are no dynamic loads when the disk is rotated about one of its principal axes of inertia.

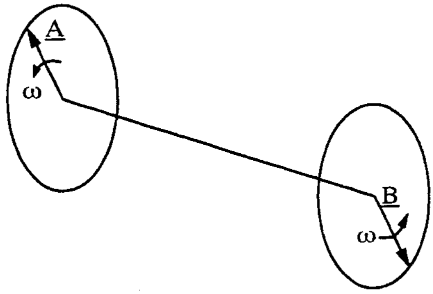


Fig. 9-7

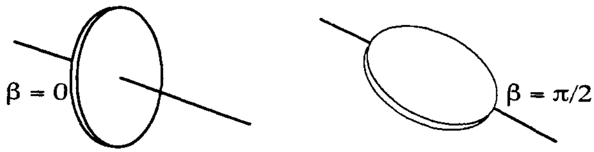


Fig. 9-8

9.4 Static and Dynamic Balancing

We have just seen that when a rigid body is rotated about a fixed axis, oscillatory forces in the shaft's supports may result, and that there are two sources of such forces. If a rigid body is rotated about an axis that does not pass through the body's center of mass then oscillatory forces will occur and we say that the body is *statically unbalanced.* On the other hand, if the body is rotated about an axis that is not a principal axis of inertia we say that the body is *dynamically unbalanced.* Static unbalance can be detected by static experiments, whereas dynamic unbalance cannot.

Oscillatory forces are destructive to bearings and consequently usually only small amounts of unbalance can be tolerated, particularly at high rotation speeds, and thus rotating masses sometimes must be *balanced.* The object of balancing is to reduce oscillatory dynamic loads by adding or redistributing mass such that the center of mass lies on the axis of rotation (*static balancing*) and a principal axis of inertia is lined up with the axis of rotation (*dynamic balancing*).

A common example of balancing is the need to balance the tire-and-wheel assemblies of our cars. This is done not by a detailed mass properties analysis of the assembly, which would be impractical, but by spinning the assembly and adding mass until the oscillatory forces are sufficiently small. Thus the effect is treated and not the cause. Also, the drive-trains of most cars are balanced before they leave the factory.

Another familiar example of the need for balancing occurs in the sporting goods industry. Many sports use round balls which must be adequately balanced. Golf balls, for example, are carefully checked for balance before being sold.

As another example, consider a bowling ball. Assuming a perfect sphere, it is statically balanced if the center of mass of the ball is at the geometric center. When the ball is placed in any attitude on a flat horizontal surface, it won't move if statically balanced. Will it role true (contact point traces out a straight line)? No, not if it is dynamically unbalanced. To be dynamically balanced, it must have rotational mass symmetry about every axis through the center (recall that the products of inertia vanish about an axis of rotational mass symmetry).[4]

Consider an otherwise massless bowling ball with two heavy lumps of mass at opposite ends of a diameter of the ball (Fig. 9-9). Is the ball statically balanced? Yes; when placed on a flat horizontal surface it will not roll. If rolled about an axis containing the mass lumps, it will roll true, but with very little resistance to acceleration. If rolled such that the mass lumps turn end over end, it will roll true with great resistance. If rolled at any other attitude, it will not roll true.

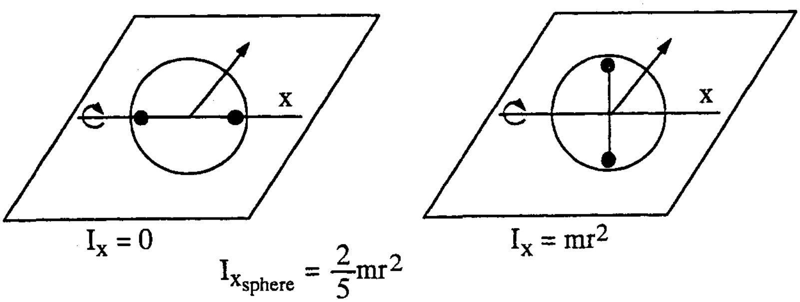


Fig. 9-9

9.5 General Case

We now consider the fixed axis rotation of an arbitrary rigid body that is neither statically nor dynamically balanced. The deadweight will not be neglected and the rotation rate will be allowed to vary.

Let $\{\hat{I}, \hat{J}, \hat{K}\}$ be inertial with $\hat{J}$ vertical and $\hat{K}$ along the shaft axis (Fig. 9-10). Let $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ be body fixed with origin at B, a point on the shaft, and $\hat{k}$ be along the shaft with ψ the angle between $\hat{\imath}$ and the local vertical. The shaft is rotating with angular velocity $\underline{\omega}$ and is subject to external moment $\underline{M}$. The other external forces are $m\underline{g}$, acting perpendicular to the shaft, and the bearing reactions $\underline{A}$ and $\underline{B}$. We want to find $\underline{A}$ and $\underline{B}$.

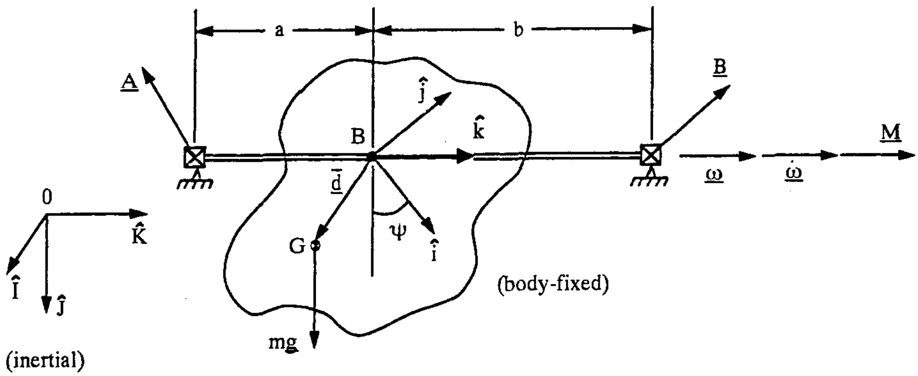


Fig. 9-10

The equations of motion as usual are Eqns. (8.15)

$$\begin{aligned} \underline{F}^e &= m\underline{\bar{a}} \\ \underline{M}_B &= \frac{D\underline{H}_B}{Dt} \quad \text{(because } B \text{ fixed in } \{\hat{I}, \hat{J}, \hat{K}\}) \end{aligned}$$

These give

$$\underline{A} + \underline{B} + mg\hat{J} = m\underline{\bar{a}}$$

$$M\hat{k} + \underline{\bar{d}} \times mg\hat{J} - a\hat{k} \times \underline{A} + b\hat{k} \times \underline{B} = \frac{D\underline{H}_B}{Dt}$$

We have

$$\underline{\bar{d}} = \bar{x}\hat{\imath} + \bar{y}\hat{\jmath} + \bar{z}\hat{k}$$

$$\begin{aligned}\underline{\omega} &= \omega\hat{k} = \dot{\psi}\hat{k} \\ \frac{D\underline{\omega}}{Dt} &= \dot{\omega}\hat{k} = \ddot{\psi}\hat{k} \quad \text{(because } \hat{k} \text{ is fixed in } \{\hat{I}, \hat{J}, \hat{K}\}\text{)}\end{aligned}$$

where $\underline{\omega}$ = angular velocity of $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ with respect to $\{\hat{I}, \hat{J}, \hat{K}\}$, that is, the angular velocity of the rigid body.

The relative acceleration equation, Eqn. (4.18), is used to relate the acceleration of G as seen in $\{\hat{I}, \hat{J}, \hat{K}\}$ to the acceleration of G as seen in $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$:

$$\overline{\underline{a}} = \underline{a}_B + \underline{a}_r + 2\underline{\omega} \times \underline{v}_r + \underline{\omega} \times (\underline{\omega} \times \overline{\underline{d}}) + \frac{D\underline{\omega}}{Dt} \times \overline{\underline{d}}$$

Noting that $\underline{a}_B = \underline{a}_r = \underline{v}_r = \underline{0}$, this becomes

$$\begin{aligned}\overline{\underline{a}} &= \omega\hat{k} \times \left[\omega\hat{k} \times (\overline{x}\hat{\imath} + \overline{y}\hat{\jmath} + \overline{z}\hat{k})\right] + \dot{\omega}\hat{k} \times (\overline{x}\hat{\imath} + \overline{y}\hat{\jmath} + \overline{z}\hat{k}) \\ &= -(\dot{\omega}\overline{y} + \omega^2\overline{x})\hat{\imath} + (\dot{\omega}\overline{x} - \omega^2\overline{y})\hat{\jmath}\end{aligned}$$

Also

$$\hat{J} = \cos\psi\hat{\imath} - \sin\psi\hat{\jmath}$$

Now resolve $\underline{A}$ and $\underline{B}$ in components in the body-fixed frame[5]

$$\begin{aligned}\underline{A} &= A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k} \\ \underline{B} &= B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}\end{aligned}$$

Then the components of the force equation become

$$\begin{aligned}&A_x + B_x = -mg\cos\psi - m\dot{\omega}\overline{y} - m\omega^2\overline{x} \\ &A_y + B_y = mg\sin\psi + m\dot{\omega}\overline{x} - m\omega^2\overline{y} \\ &A_z + B_z = 0\end{aligned} \tag{9.7}$$

For the moment equations, we need $\underline{H}_B$ and $D\underline{H}_B/Dt$. These are determined as follows

$$\underline{H}_B = [I]\underline{\omega} = [I]\omega\hat{k}$$

$$\begin{pmatrix} H_{B_x} \\ H_{B_y} \\ H_{B_z} \end{pmatrix} = \begin{bmatrix} \cdot & \cdot & -I_{xz} \\ \cdot & \cdot & -I_{yz} \\ \cdot & \cdot & I_{zz} \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} \quad (\{\hat{\imath}, \hat{\jmath}, \hat{k}\} \text{ not principal axes})$$

$$H_{B_x} = -I_{xz}\omega\ , \qquad H_{B_y} = -I_{yz}\omega\ , \qquad H_{B_z} = I_{zz}\omega$$

$$\underline{H}_B = -I_{xz}\omega\hat{\imath} - I_{yz}\omega\hat{\jmath} + I_{zz}\omega\hat{k}$$

$$\begin{aligned}\frac{D\underline{H}_B}{Dt} &= \frac{d\underline{H}_B}{dt} + \underline{\omega} \times \underline{H}_B = \dot{\omega}(-I_{xz}\hat{\imath} - I_{yz}\hat{\jmath} + I_{zz}\hat{k}) \\ &\quad + \omega\hat{k} \times \omega(-I_{xz}\hat{\imath} - I_{yz}\hat{\jmath} + I_{zz}\hat{k}) \\ &= (-\dot{\omega}I_{xz} + \omega^2 I_{yz})\hat{\imath} - (\dot{\omega}I_{yz} + \omega^2 I_{xz})\hat{\jmath} + \dot{\omega}I_{zz}\hat{k}\end{aligned}$$

Also

$$\begin{aligned}\overline{\underline{d}} \times mg\hat{J} &= mg[\overline{z}\sin\psi\hat{\imath} + \overline{z}\cos\psi\hat{\jmath} - (\overline{x}\sin\psi + \overline{y}\cos\psi)\hat{k}] \\ -a\hat{k} \times \underline{A} &= aA_y\hat{\imath} - aA_x\hat{\jmath} \\ b\hat{k} \times \underline{B} &= -bB_y\hat{\imath} + bB_x\hat{\jmath}\end{aligned}$$

The moment equations are then

$$\begin{aligned}&aA_y - bB_y = -mg\overline{z}\sin\psi - \dot{\omega}I_{xz} + \omega^2 I_{yz} \\ &-aA_x + bB_x = -mg\overline{z}\cos\psi - \dot{\omega}I_{yz} - \omega^2 I_{xz} \\ &M = mg(\overline{x}\sin\psi + \overline{y}\cos\psi) + \dot{\omega}I_{zz}\end{aligned} \tag{9.8}$$

Equations (9.7) and (9.8) are now solved simultaneously for the components of $\underline{A}$, $\underline{B}$, and $\underline{M}$. The result is:

$$\begin{aligned}A_x &= mg\frac{\overline{z}-b}{l}\cos\psi - \frac{\overline{x}}{l}mb\omega^2 + \frac{\omega^2}{l}I_{xz} - \frac{\overline{y}}{l}mb\dot{\omega} + \frac{\dot{\omega}}{l}I_{yz} \\ A_y &= -mg\frac{\overline{z}-b}{l}\sin\psi - \frac{\overline{y}}{l}mb\omega^2 + \frac{\omega^2}{l}I_{yz} - \frac{\overline{x}}{l}mb\dot{\omega} - \frac{\dot{\omega}}{l}I_{xz} \\ B_x &= -mg\frac{\overline{z}+a}{l}\cos\psi - \frac{\overline{x}}{l}ma\omega^2 - \frac{\omega^2}{l}I_{xz} - \frac{\overline{y}}{l}ma\dot{\omega} - \frac{\dot{\omega}}{l}I_{yz} \\ B_y &= \underbrace{mg\frac{\overline{z}+a}{l}\sin\psi}_{1} - \underbrace{\frac{\overline{y}}{l}ma\omega^2}_{2} - \underbrace{\frac{\omega^2}{l}I_{yz}}_{3} + \underbrace{\frac{\overline{x}}{l}ma\dot{\omega}}_{4} + \underbrace{\frac{\dot{\omega}}{l}I_{xz}}_{5}\end{aligned} \tag{9.9}$$

$$\begin{aligned}&A_z + B_z = 0 \\ &M = mg\overline{x}\sin\psi + mg\overline{y}\cos\psi + \dot{\omega}I_{zz}\end{aligned} \tag{9.10}$$

Let's now discuss the terms in these equations. First note that the first terms on the right-hand-sides of Eqns. (9.9) are due to the body weight; in the inertial frame these would be constant forces in the vertical direction. This is simply the deadweight that the bearings would have to support even if the shaft were not rotating. All the other terms on the right-hand-sides of Eqns. (9.9) would be oscillatory (dynamic) forces in the inertial frame.

The object of static balancing would be to alter or add mass so that $\overline{x} = \overline{y} = 0$. This gets rid of terms 2 and 4 in Eqns. (9.9). The object of dynamic balancing would be to alter or add mass so that $I_{xz} = I_{yz} = 0$. This gets rid of terms 3 and 5.

Finally, note from Eqn. (9.10) that if the body is statically unbalanced, then a moment is required to maintain the shaft rotation at a constant speed (unless deadweight is ignored). On the other hand, if the body is statically balanced, there will be a moment only if there is angular acceleration, and conversely.

Notes

1 We may consider the shaft to be part of the rigid body.

2 Notice that we are neglecting the weight of the disk in this equation. This is because such "deadweight" forces are usually negligible in this type of problem.

3 This would not be true if deadweight were included.

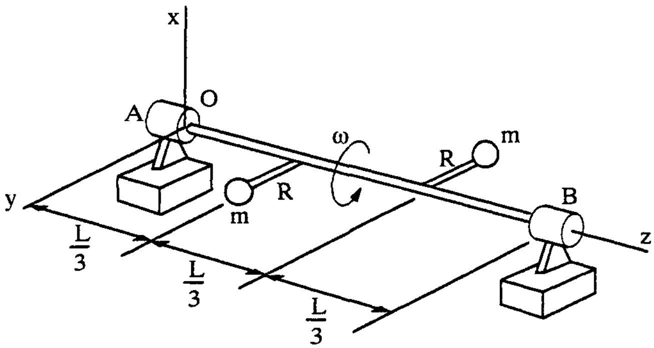
4 Actually, bowling balls are deliberately unbalanced to give them "feel" or "English".

5 In the examples of Sections 9.2 and 9.3 we resolved these vectors in the inertial frame.

Problems

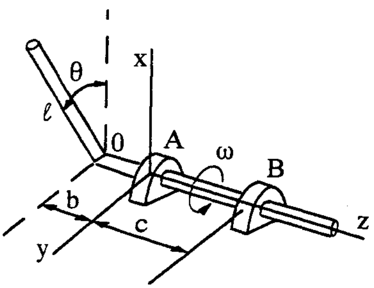
The following eight problems concern a rigid body or collection of bodies rotating with constant angular speed about an axis fixed in an inertial frame. For each problem, determine the following: (i) the angular momentum relative to the point O, (ii) the time rate of change of the angular momentum in an inertial frame, and (iii) the dynamic reaction forces and moments in the bearings or other external supports of the system. Also, state whether or not the system is statiscally or dynamically balanced or both. Ignore the dead weight. In all cases, start with Eqns. (8.15).

9/1 The figure shows two small masses connected to a shaft by rods of negligible mass.

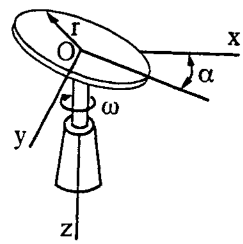


Problem 9/1

9/2 The figure shows a uniform slender bar of mass m and length ℓ connected to a slender shaft at one end.



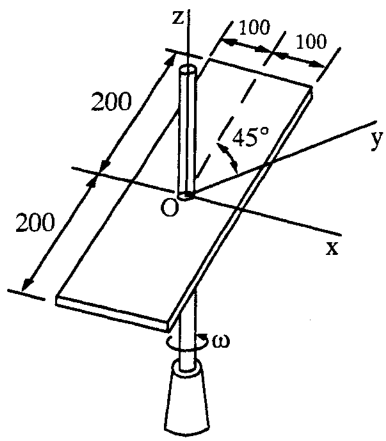
Problem 9/2



Problem 9/3

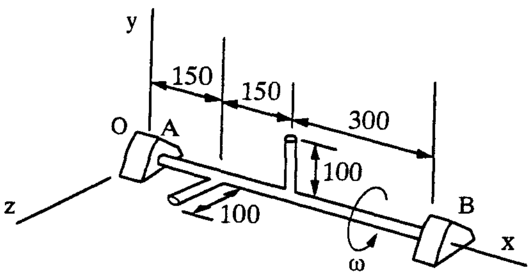
9/3 The figure shows a thin homogeneous circular disk of mass m and radius r mounted at its center on a shaft at angle α.

9/4 The figure shows a thin homogeneous rectangular plate with a mass of 3 kg attached to a shaft at an angle of 45°. The shaft rotates at an angular speed of $\omega = 20\pi$ rad/sec. All dimensions are in mm.



Problem 9/4

9/5 The figure shows two uniform thin rods, each of mass 300 g, attached to a slender shaft that rotates at a rate of 3 rad/sec. All dimensions are in mm.

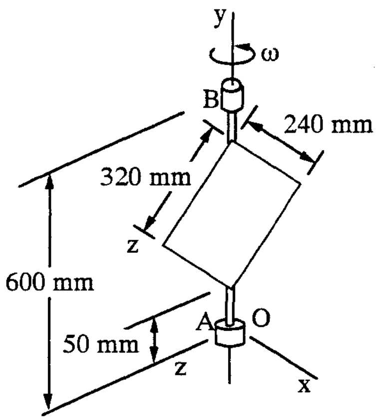


Problem 9/5

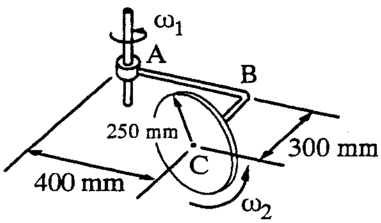
9/6 The figure shows a thin, homogeneous rectangular plate of mass 6 kg attached to a shaft rotating at angular speed $\omega = 5$ rad/sec.

9/7 The figure shows a thin uniform disk of mass 5 kg rotating at the constant rate $\omega_2 = 8$ rad/sec w.r.t. member ABC, which in turn rotates about a shaft at a constant rate $\omega_1 = 3$ rad/sec.

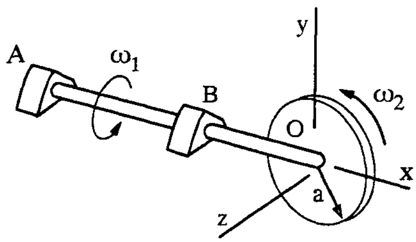
9/8 The figure shows a thin uniform disk of mass m and radius a rotating with angular speed ω_2 in a fork. The fork is attached to a shaft that rotates with angular speed ω_1.



Problem 9/6

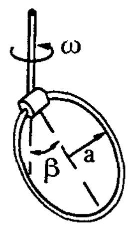


Problem 9/7



Problem 9/8

9/9 A thin uniform ring of radius a is attached by a collar at A to a vertical shaft that rotates with constant angular speed ω. Determine the constant angle β that the plane of the ring makes with the vertical. Also determine the maximum value of ω for which the ring remains vertical.



Problem 9/9

Chapter 10

Motion of a Rigid Body With One Point Fixed; Gyroscopic Motion

10.1 Instantaneous Axis of Zero Velocity

We now consider a more general class of rigid body motion, that is, motion in which one point remains fixed in an inertial frame. In this type of motion, attention is naturally focused on the angular orientation and motion of the body. We begin by showing that in this case there is at every instant a body-fixed line of points that is at rest.

Let point B of a rigid body have zero velocity at a given time with respect to frame $\{\hat{I}, \hat{J}, \hat{K}\}$ and let $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ be a body-fixed frame with origin B (Fig. 10-1). Denote the body's angular velocity (angular velocity of $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ with respect to $\{\hat{I}, \hat{J}, \hat{K}\}$) by $\underline{\omega}$. Suppose P is a point of the body along the line L as defined by $\underline{\omega}$. Then by Eqn. (4.17)

$$\underline{\nu}_P = \underline{\nu}_B + \underline{\nu}_r + \underline{\omega} \times \underline{r} = \underline{0}$$

where $\underline{\nu}_B = \underline{0}$ by assumption, $\underline{\nu}_r = \underline{0}$ because P is body-fixed, and $\underline{\omega} \times \underline{r} = \underline{0}$ because $\underline{\omega}$ and $\underline{r}$ are collinear. Therefore, all points along the line L are instantaneously at rest in $\{\hat{I}, \hat{J}, \hat{K}\}$ and L is an axis of instantaneous zero velocity.

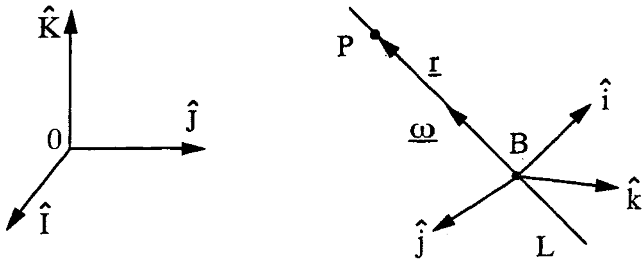


Fig. 10-1

10.2 Euler's Angles

It was concluded in Section 4.2 that three angles suffice to describe the orientation of a rigid body relative to a frame of reference. Although there are many sets of angles that have been devised to do this, the oldest, and still most common set, is Euler's Angles.

Let point B of a rigid body be fixed for all time in a reference frame $\{\hat{I}, \hat{J}, \hat{K}\}$. Without loss of generality, take the origin of this frame to be B (Fig. 10-2). Now consider the following three reference frames,

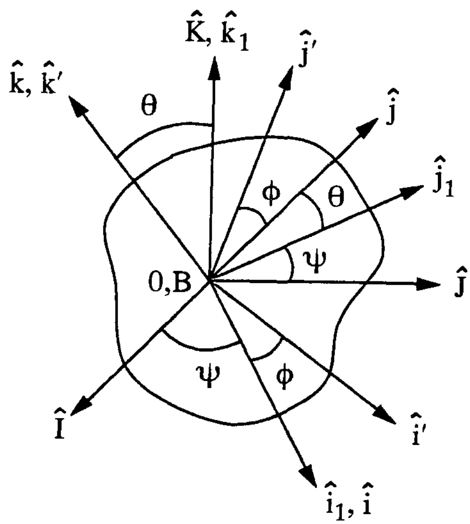


Fig. 10-2

obtained by successive rotations relative to $\{\hat{I}, \hat{J}, \hat{K}\}$:

1. $\{\hat{\imath}_1, \hat{\jmath}_1, \hat{k}_1\}$ is obtained by rotating through an angle ψ about $\hat{K}$, so that $\hat{k}_1 = \hat{K}$.

2. $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ is obtained by rotating through an angle θ about $\hat{\imath}_1$, so that $\hat{\imath} = \hat{\imath}_1$.

3. $\{\hat{\imath}', \hat{\jmath}', \hat{k}'\}$ is obtained by rotating through an angle ϕ about $\hat{k}$, so that $\hat{k}' = \hat{k}$.

The angles (ψ, θ, ϕ) are called *Euler's Angles.* The point of this exercise is that by proper selection of ψ, θ, and ϕ, we can get $\{\hat{\imath}', \hat{\jmath}', \hat{k}'\}$ to line up with any chosen body-fixed frame, and thus these three angles give the orientation of the body.

Let's look at this another way. Figure 10-3 shows a rigid body with body-fixed frame $\{\hat{\imath}', \hat{\jmath}', \hat{k}'\}$ moving with one point, B, fixed relative to frame $\{\hat{I}, \hat{J}, \hat{K}\}$.

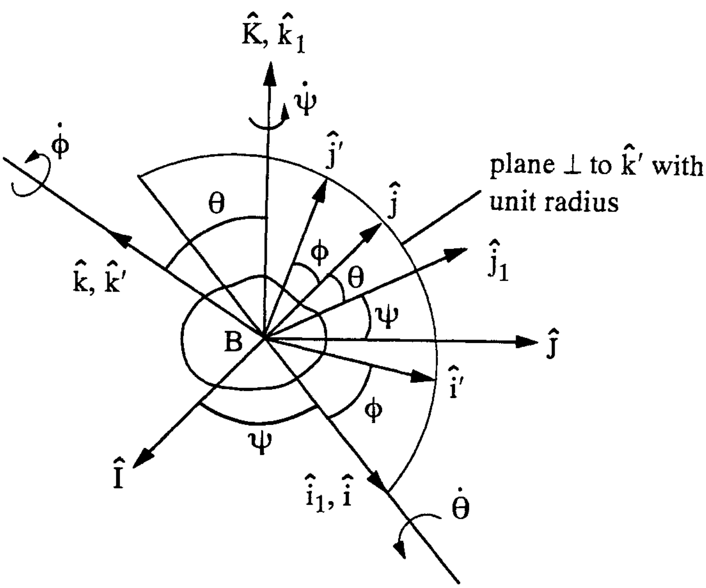


Fig. 10-3

Imagine a disk of unit radius passing through point B and perpendicular to body-fixed axis $\hat{k}'$. The intersection of this disk with the $\{\hat{I}, \hat{J}\}$

plane defines unit vectors $\hat{\imath}_1$ and $\hat{\imath}$; thus these unit vectors are always perpendicular to $\hat{k}'$ and in the $\{\hat{I}, \hat{J}\}$ plane. Note that $\hat{\imath}, \hat{\jmath}, \hat{\imath}'$, and $\hat{\jmath}'$ are all in the unit disk.

Next we determine the angular velocity of the rigid body. Let

$$\underline{\Omega} \quad = \quad \text{angular velocity of } \{\hat{\imath}, \hat{\jmath}, \hat{k}\} \text{ with respect to } \{\hat{I}, \hat{J}, \hat{K}\}$$
$$\underline{\omega} \quad = \quad \text{angular velocity of } \{\hat{\imath}', \hat{\jmath}', \hat{k}'\} \text{ with respect to } \{\hat{I}, \hat{J}, \hat{K}\}$$

so that $\underline{\omega}$ is the angular velocity of the rigid body. Consider infinitesimal rotations $\Delta\underline{\psi}$, $\Delta\underline{\theta}$ and $\Delta\underline{\phi}$. Since infinitesimal, they can be added:[1]

$$\underline{\Omega} = \lim_{\Delta t \to 0} \frac{1}{\Delta t}(\Delta\psi\hat{K} + \Delta\theta\hat{\imath}) = \dot{\psi}\hat{K} + \dot{\theta}\hat{\imath} \tag{10.1}$$

From Fig. 10-4,

$$\hat{K} = \cos\theta\hat{k} + \sin\theta\hat{\jmath} \tag{10.2}$$

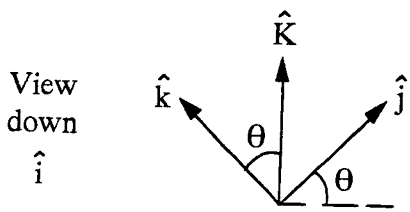


Fig. 10-4

so that

$$\boxed{\underline{\Omega} = \dot{\theta}\hat{\imath} + \dot{\psi}\sin\theta\hat{\jmath} + \dot{\psi}\cos\theta\hat{k}} \tag{10.3}$$

Also,

$$\underline{\omega} = \underline{\Omega} + \lim_{\Delta t \to 0} \frac{\Delta\phi\hat{k}}{\Delta t} = \underline{\Omega} + \dot{\phi}\hat{k}$$

$$\boxed{\underline{\omega} = \dot{\theta}\hat{\imath} + \dot{\psi}\sin\theta\hat{\jmath} + (\dot{\phi} + \dot{\psi}\cos\theta)\hat{k}} \tag{10.4}$$

These expressions also may be derived from Eqn. (4.20):

$$\underline{\omega} = \underline{\omega}_{4/1} = \sum_{i=2}^{4} \underline{\omega}_{i/i-1}$$

10.3 Transformations

It is often necessary to go back and forth between the various reference frames, that is, to express the components of a vector in one frame if they are given in another. As before,[2] it suffices to get the relations between the various unit vectors, which we now do.

From Fig. 10-5,

$$\begin{aligned}\hat{\imath}_1 &= \cos\psi\hat{I} + \sin\psi\hat{J}\\ \hat{\jmath}_1 &= -\sin\psi\hat{I} + \cos\psi\hat{J}\\ \hat{k}_1 &= \hat{K}\end{aligned}$$

so that

$$\begin{pmatrix}\hat{\imath}_1\\ \hat{\jmath}_1\\ \hat{k}_1\end{pmatrix} = \begin{pmatrix}\cos\psi & \sin\psi & 0\\ -\sin\psi & \cos\psi & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\hat{I}\\ \hat{J}\\ \hat{K}\end{pmatrix} = [T_\psi]\begin{pmatrix}\hat{I}\\ \hat{J}\\ \hat{K}\end{pmatrix} \tag{10.5}$$

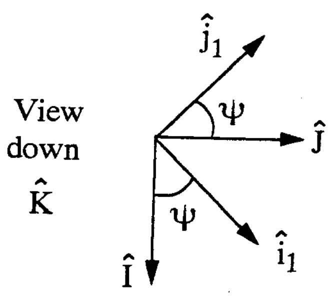


Fig. 10-5

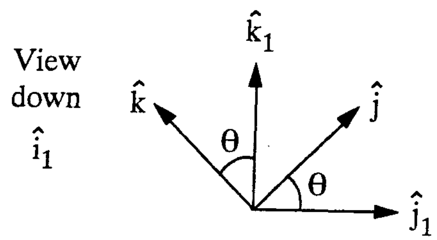


Fig. 10-6

From Fig. 10-6,

$$\begin{aligned}\hat{\imath} &= \hat{\imath}_1\\ \hat{\jmath} &= \cos\theta\hat{\jmath}_1 + \sin\theta\hat{k}_1\\ \hat{k} &= -\sin\theta\hat{\jmath}_1 + \cos\theta\hat{k}_1\end{aligned}$$

so that

$$\begin{pmatrix}\hat{\imath}\\ \hat{\jmath}\\ \hat{k}\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ 0 & \cos\theta & \sin\theta\\ 0 & -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\hat{\imath}_1\\ \hat{\jmath}_1\\ \hat{k}_1\end{pmatrix} = [T_\theta]\begin{pmatrix}\hat{\imath}_1\\ \hat{\jmath}_1\\ \hat{k}_1\end{pmatrix} \tag{10.6}$$

And finally from Fig. 10-7,

$$\begin{aligned} \hat{\imath}' &= \cos\phi\hat{\imath} + \sin\phi\hat{\jmath} \\ \hat{\jmath}' &= -\sin\phi\hat{\imath} + \cos\phi\hat{\jmath} \\ \hat{k}' &= \hat{k} \end{aligned}$$

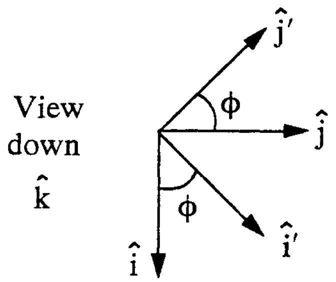


Fig. 10-7

so that

$$\begin{pmatrix} \hat{\imath}' \\ \hat{\jmath}' \\ \hat{k}' \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{\imath} \\ \hat{\jmath} \\ \hat{k} \end{pmatrix} = [T_\theta] \begin{pmatrix} \hat{\imath} \\ \hat{\jmath} \\ \hat{k} \end{pmatrix} \qquad (10.7)$$

Now consider any vector $\underset{\sim}{Q}$. Resolving into components:

$$\begin{aligned} \underset{\sim}{Q} &= Q_X\hat{I} + Q_Y\hat{J} + Q_Z\hat{K} = Q'_x\hat{\imath}' + Q'_y\hat{\jmath}' + Q'_z\hat{k}' \\ &= Q_x\hat{\imath} + Q_y\hat{\jmath} + Q_z\hat{k} = Q_{x_1}\hat{\imath}_1 Q_{y_1}\hat{\jmath}_1 Q_{z_1}\hat{k}_1 \end{aligned}$$

Then

$$\begin{aligned} \underset{\sim}{Q} &= (Q'_xQ'_yQ'_z)\begin{pmatrix} \hat{\imath}' \\ \hat{\jmath}' \\ \hat{k}' \end{pmatrix} = (Q'_xQ'_yQ'_x)[T_\phi]\begin{pmatrix} \hat{\imath} \\ \hat{\jmath} \\ \hat{k} \end{pmatrix} \\ &= (Q'_xQ'_yQ'_x)[T_\phi][T_\theta]\begin{pmatrix} \hat{\imath}_1 \\ \hat{\jmath}_1 \\ \hat{k}_1 \end{pmatrix} \\ &= (Q'_xQ'_yQ'_z)[T_\phi][T_\theta][T_\psi]\begin{pmatrix} \hat{I} \\ \hat{J} \\ \hat{K} \end{pmatrix} = (Q_XQ_YQ_Z)\begin{pmatrix} \hat{I} \\ \hat{J} \\ \hat{K} \end{pmatrix} \end{aligned}$$

Therefore

$$(Q_X Q_Y Q_Z) = (Q'_x Q'_y Q'_z)[T_\phi][T_\theta][T_\psi] \tag{10.8}$$

Taking the transpose of this equation:[3]

$$\begin{pmatrix} Q_X \\ Q_Y \\ Q_Z \end{pmatrix} = [T_\psi]^T[T_\theta]^T[T_\phi]^T \begin{pmatrix} Q'_x \\ Q'_y \\ Q'_z \end{pmatrix} = [T_\psi]^{-1}[T_\theta]^{-1}[T_\phi]^{-1} \begin{pmatrix} Q'_x \\ Q'_y \\ Q'_z \end{pmatrix} \tag{10.9}$$

Now multiply by $[T_\psi]$, then $[T_\theta]$, and then $[T_\phi]$:

$$\begin{pmatrix} Q'_x \\ Q'_y \\ Q'_z \end{pmatrix} = [T_\phi][T_\theta][T_\psi] \begin{pmatrix} Q_X \\ Q_Y \\ Q_Z \end{pmatrix} \tag{10.10}$$

Other transformations are found similarly.

10.4 Example – Thin Spherical Pendulum

Consider a uniform thin bar suspended at one end by a socket joint and moving in 3-D as shown on Fig. 10-8. We want to find $\underline{H}_B$, the angular

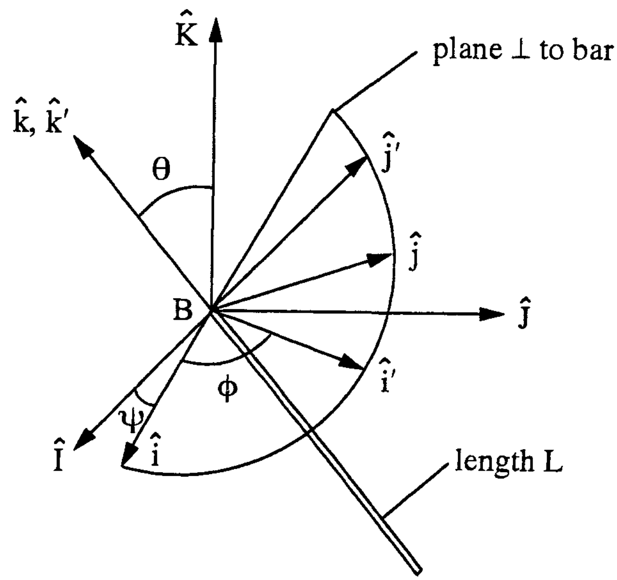


Fig. 10-8

momentum of the body about B, the point of suspension. Relative to body-fixed axes, the moments and products of inertia are

$$\begin{aligned} I_{x'x'} &= I_{y'y'} = \frac{1}{3}mL^2 \\ I_{z'z'} &= I_{x'y'} = I_{y'z'} = I_{z'x'} = 0 \end{aligned} \tag{10.11}$$

Comparing Eqns. (10.4) and (10.11) we see that we have $[I]$ in one frame and $\underline{\omega}$ in another, and thus we must transform to a common frame.

First, we choose the body-fixed frame $\{\hat{\imath}', \hat{\jmath}', \hat{k}'\}$ and transform $\underline{\omega}$; from Eqn. (10.7),

$$\begin{aligned} \begin{pmatrix} \omega'_x \\ \omega'_y \\ \omega'_z \end{pmatrix} &= [T_\phi] \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\psi}\sin\theta \\ \dot{\phi} + \dot{\psi}\cos\theta \end{pmatrix} \\ &= \begin{pmatrix} \dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi \\ -\dot{\theta}\sin\phi + \dot{\psi}\sin\theta\cos\phi \\ \dot{\phi} + \dot{\psi}\cos\theta \end{pmatrix} \end{aligned} \tag{10.12}$$

From Eqn. (7.12):

$$\begin{pmatrix} H'_{B_x} \\ H'_{B_y} \\ H'_{B_z} \end{pmatrix} = \begin{bmatrix} \frac{1}{3}mL^2 & 0 & 0 \\ 0 & \frac{1}{3}mL^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \omega'_x \\ \omega'_y \\ \omega'_z \end{pmatrix} \tag{10.13}$$

Combining Eqns. (10.12) and (10.13) gives

$$\begin{aligned} H'_{B_x} &= \frac{1}{3}mL^2(\dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi) \\ H'_{B_y} &= \frac{1}{3}mL^2(-\dot{\theta}\sin\phi + \dot{\psi}\sin\theta\cos\phi) \\ H'_{B_z} &= 0 \end{aligned}$$

or in vector form

$$\begin{aligned} \underline{H}_B &= \frac{1}{3}mL^2(\dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi)\hat{\imath}' \\ &\quad + \frac{1}{3}mL^2(-\dot{\theta}\sin\phi + \dot{\psi}\sin\theta\cos\phi)\hat{\jmath}' \end{aligned} \tag{10.14}$$

This is $\underline{H}_B$ in terms of Euler's angles and their rates in body-fixed components. Note that $\dot{\phi}$ does not appear; this is because all the mass lies on axis $\hat{k}'$. This is the same reason there is no $\hat{k}'$ component of $\underline{H}_B$.

Second, we choose to resolve $\underline{H}_B$ in components along $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$. We have

$$\begin{aligned} I_{xx} &= I_{yy} = \frac{1}{3}mL^2 \\ I_{zz} &= I_{xy} = I_{yz} = I_{zx} = 0 \end{aligned} \tag{10.15}$$

Note that because of the body's symmetry, the moments and products of inertia are constant, *even though the* $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ *are not body-fixed.* Now, from Eqns. (7.12), (10.4), and (10.15),

$$\begin{pmatrix} H_{B_x} \\ H_{B_y} \\ H_{B_z} \end{pmatrix} = \begin{bmatrix} \frac{1}{3}mL^2 & 0 & 0 \\ 0 & \frac{1}{3}mL^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\psi}\sin\theta \\ \dot{\phi} + \dot{\psi}\cos\theta \end{pmatrix}$$

which gives

$$\underline{H}_B = \frac{1}{3}mL^2\dot{\theta}\hat{\imath} + \frac{1}{3}mL^2\dot{\psi}\sin\theta\hat{\jmath} \tag{10.16}$$

Comparison of Eqns. (10.14) and (10.16) shows why it is preferable to write $\underline{H}_B$ in components along $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ rather than $\{\hat{\imath}', \hat{\jmath}', \hat{k}'\}$.

10.5 Gyroscopic Motion

Gyroscopic motion is the motion of a radially symmetric rigid body such that one point on the axis of symmetry remains fixed[4] relative to an inertial frame. Examples of such motion are tops, spinning rotors, and gyroscopes.

Figure 10-9 shows a body with radial mass symmetry moving with point B fixed. Choose an inertial frame $\{\hat{I}, \hat{J}, \hat{K}\}$ with origin at B, and a body-fixed frame $\{\hat{\imath}', \hat{\jmath}', \hat{k}'\}$ with $\hat{k}'$ the axis of symmetry. Frame $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ is as described before, and (ψ, θ, ϕ) are Euler's Angles. In gyroscopic motion, the line defined by $\hat{\imath}$ is called the *line of nodes* and the rates of the Euler Angles are given names:

$$\begin{aligned} \dot{\phi} &= \textit{spin} \\ \dot{\theta} &= \textit{nutation} \\ \dot{\psi} &= \textit{precession} \end{aligned}$$

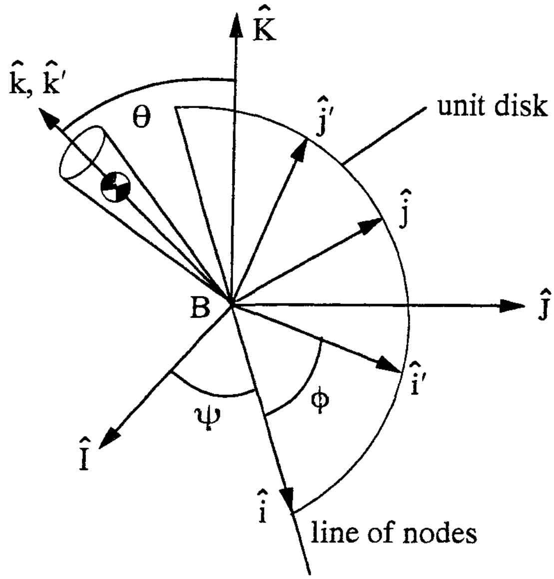


Fig. 10-9

As we saw in the previous section, because of the symmetry, the moments and products of inertia are constant in $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$, even though this frame is not body-fixed, and $\underline{H}_B$ is simplest when expressed in this frame. Thus we choose to resolve all vectors in this frame. Because of the symmetry,

$$\begin{aligned} I_{xx} &= I_{yy} = I_0 \\ I_{zz} &= I \\ I_{xy} &= I_{yz} = I_{zx} = 0 \end{aligned} \tag{10.17}$$

Our goal is now to derive the equations governing gyroscopic motion. From Eqns. (10.3) and (10.4) the angular velocities $\underline{\omega}$ and $\underline{\Omega}$ are related by:

$$\underline{\omega} = \underline{\Omega} + \dot{\phi}\hat{k} \tag{10.18}$$

and the components are

$$\begin{aligned} \omega_x &= \Omega_x = \dot{\theta} \\ \omega_y &= \Omega_y = \dot{\psi}\sin\theta \\ \omega_z &= \Omega_z + \dot{\phi} = \dot{\phi} + \dot{\psi}\cos\theta \end{aligned} \tag{10.19}$$

The angular momentum relative to B and its time derivative are determined as follows:

$$
\begin{aligned}
\underline{H}_B &= [I]\underline{\omega} \\
\begin{pmatrix} H_{B_x} \\ H_{B_y} \\ H_{B_z} \end{pmatrix} &= \begin{bmatrix} I_0 & 0 & 0 \\ 0 & I_0 & 0 \\ 0 & 0 & I \end{bmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \\
\underline{H}_B &= I_0\omega_x\hat{\imath} + I_0\omega_y\hat{\jmath} + I\omega_z\hat{k} = I_0\Omega_x\hat{\imath} + I_0\Omega_y\hat{\jmath} + I(\Omega_z + \dot{\phi})\hat{k}
\end{aligned}
$$

$$\frac{D\underline{H}_B}{Dt} = \frac{d\underline{H}_B}{dt} + \underline{\Omega} \times \underline{H}_B \tag{10.20}$$

where

$$\frac{d\underline{H}_B}{dt} = I_0\dot{\Omega}_x\hat{\imath} + I_0\dot{\Omega}_y\hat{\jmath} + I(\dot{\Omega}_z + \ddot{\phi})\hat{k} \tag{10.21}$$

$$
\begin{aligned}
\underline{\Omega} \times \underline{H}_B &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \Omega_x & \Omega_y & \Omega_z \\ I_0\Omega_x & I_0\Omega_y & I(\Omega_z + \dot{\phi}) \end{vmatrix} = [(I - I_0)\Omega_y\Omega_z \\
&\quad + I\dot{\phi}\Omega_y]\hat{\imath} + [(I_0 - I)\Omega_z\Omega_x - I\dot{\phi}\Omega_x]\hat{\jmath}
\end{aligned} \tag{10.22}
$$

Note that $\underline{\omega}$ must be used in Eqn. (7.12) but that $\underline{\Omega}$ must be used in Eqn. (10.20).

Since B is a point fixed in the inertial frame, the second of Eqns. (8.15) applies:

$$\underline{M}_B = \frac{D\underline{H}_B}{Dt}$$

From Eqns. (10.20), (10.21), and (10.22) the component equations are:

$$
\begin{aligned}
M_{B_x} &= I_0\dot{\Omega}_x - (I_0 - I)\Omega_y\Omega_z + I\dot{\phi}\Omega_y \\
M_{B_y} &= I_0\dot{\Omega}_y - (I - I_0)\Omega_z\Omega_x - I\dot{\phi}\Omega_x \\
M_{B_z} &= I(\dot{\Omega}_z + \ddot{\phi})
\end{aligned} \tag{10.23}
$$

Using Eqns. (10.19), these components in terms of Euler's angles are:

$$
\begin{aligned}
M_{B_x} &= I_0\ddot{\theta} - (I_0 - I)\dot{\psi}(\sin\theta)\dot{\psi}\cos\theta + I(\omega_z - \dot{\psi}\cos\theta)\dot{\psi}\sin\theta \\
M_{B_y} &= I_0\frac{d}{dt}(\dot{\psi}\sin\theta) - (I - I_0)\dot{\psi}(\cos\theta)\dot{\theta} - I(\omega_z - \dot{\psi}\cos\theta)\dot{\theta} \\
M_{B_z} &= I\dot{\omega}_z
\end{aligned} \tag{10.24}
$$

These may be written as:

$$\boxed{\begin{aligned} M_{B_x} &= I_0(\ddot{\theta} - \dot{\psi}^2 \sin\theta\cos\theta) + \omega_z I \dot{\psi}\sin\theta \\ M_{B_y} &= \frac{I_0}{\sin\theta}\frac{d}{dt}(\dot{\psi}\sin^2\theta) - I\dot{\theta}\omega_z \\ M_{B_z} &= I\dot{\omega}_z \end{aligned}} \tag{10.25}$$

where from Eqns. (10.19):

$$\boxed{\omega_z = \dot{\phi} + \dot{\psi}\cos\theta} \tag{10.26}$$

From Eqns. (8.15), we see that the same equations would have resulted if the center of mass had been chosen instead of point B as the point about which to take the moments. In this case we would have obtained:

$$\boxed{\begin{aligned} \overline{M}_x &= \overline{I}_0(\ddot{\theta} - \dot{\psi}^2 \sin\theta\cos\theta) + \omega_z \overline{I} \dot{\psi}\sin\theta \\ \overline{M}_y &= \frac{\overline{I}_0}{\sin\theta}\frac{d}{dt}(\dot{\psi}\sin^2\theta) - \overline{I}\dot{\theta}\omega_z \\ \overline{M}_z &= \overline{I}\dot{\omega}_z \end{aligned}} \tag{10.27}$$

Equations (10.25) (or (10.27)) are three coupled, second order, nonlinear, differential equations. Generally, they must be solved numerically. They can be used to solve the two types of problems:

(1) Given $\underline{M}$, find $\phi(t)$, $\theta(t)$, and $\psi(t)$; this requires integration.
(2) Given $\phi(t)$, $\theta(t)$ and $\psi(t)$, find $\underline{M}$; this requires differentiation.

The solutions to these equations can exhibit a wide variety of behavior. For example, consider the motion of a top and imagine the axis of the top tracing a curve on a unit sphere centered at the top's support point. Then further analysis and numerical integration reveal that the motions shown in Fig. 10-10 are possible. The motion in an actual situation depends on the parameters I and I_0, and on the initial conditions $\phi(0)$, $\dot{\phi}(0)$, $\theta(0)$, $\dot{\theta}(0)$, $\psi(0)$, $\dot{\psi}(0)$.

We will now discuss some important special cases of gyroscopic motion, for which analytical solutions are possible.

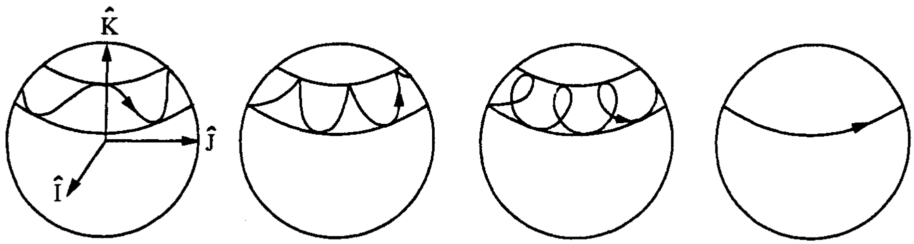


Fig. 10-10

10.6 Steady Precession

The fourth case shown in Fig. 10-10 is an example of *steady precession.* In general, steady precession is defined as motion for which:

$$\theta = \text{const.}, \qquad \dot{\psi} = \text{const.}, \qquad \dot{\phi} = \text{const.} \tag{10.28}$$

Equations (10.25) become in this case:

$$\begin{aligned} M_{B_x} &= \dot{\psi}\sin\theta(I\omega_z - I_0\dot{\psi}\cos\theta) \\ M_{B_y} &= 0 \\ M_{B_z} &= 0 \end{aligned} \tag{10.29}$$

Thus the external moment can have only an x-component for steady precession to be possible.

Consider, for example, a top (Fig. 10-11). Since gravity is the only force exerting a moment about B,

$$\underline{M}_B = \underline{\bar{r}} \times m\underline{g} = \bar{r}\hat{k} \times mg(-\sin\theta\hat{j} - \cos\theta\hat{k}) = \bar{r}mg\sin\theta\hat{\imath} \tag{10.30}$$

Since $\underline{M}_B$ has only an x-component, a top is capable of steady precession, in which case the motion is given by, from Eqns. (10.29) and (10.30),

$$\bar{r}mg = \dot{\psi}(I\omega_z - I_0\dot{\psi}\cos\theta) \tag{10.31}$$

We now return to the general case of steady precession and use Eqn. (10.26) in (10.29):

$$\boxed{M_{B_x} = M = \dot{\psi}^2\sin\theta\cos\theta(I - I_0) + \dot{\psi}\dot{\phi}\sin\theta I} \tag{10.32}$$

We see from this that if $M \neq 0$ then $\dot{\psi} \neq 0$; that is, if there is a moment, there must be precession. Provided that $\sin\theta\cos\theta(I - I_0) \neq 0$, Eqn. (10.32) will be a quadratic equation in $\dot{\psi}$; the solution is

$$\boxed{\dot{\psi} = \frac{I\dot{\phi}}{2(I_0 - I)\cos\theta}\left(1 \pm \sqrt{1 - \frac{4M(I_0 - I)\cos\theta}{I^2\dot{\phi}^2\sin\theta}}\right)} \qquad (10.33)$$

The roots will be real and distinct if

$$1 - \frac{4M(I_0 - I)\cos\theta}{I^2\dot{\phi}^2\sin\theta} > 0$$

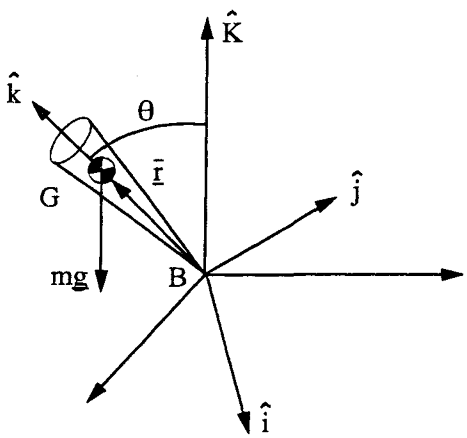


Fig. 10-11

Since for gyroscopes and tops $\dot{\phi}$ is typically very large, we use the binomial theorem to expand the roots given in Eqn. (10.33) in powers of $1/\dot{\phi} \ll 1$; the leading terms are

$$\boxed{\dot{\psi}_1 = \frac{M}{I\dot{\phi}\sin\theta}, \qquad \dot{\psi}_2 = \frac{I\dot{\phi}}{(I_0 - I)\cos\theta}} \qquad (10.34)$$

These are the two approximate values at which steady precession can occur; the second is usually much faster than the first, due to large $\dot{\phi}$. The one that occurs depends on the initial conditions at the start of the motion; the first one almost always occurs in practice.

If gravity is the only external force producing a moment and if $\dot{\psi}_1$ occurs, Eqns. (10.30) and (10.34) give

$$\dot{\psi}_1 = \frac{\bar{r}mg}{I\dot{\phi}}$$

This gives the precession of a top in terms of its geometry and its spin for steady precession. Note that as the spin decreases, the precession increases.

10.7 Example

A thin uniform rod of length L and mass m is attached with a hinge to a shaft that rotates with a constant angular speed ω, as shown on Fig. 10-12. We want to find the angle θ the rod makes with the vertical. It is instructive to work the problem two ways.

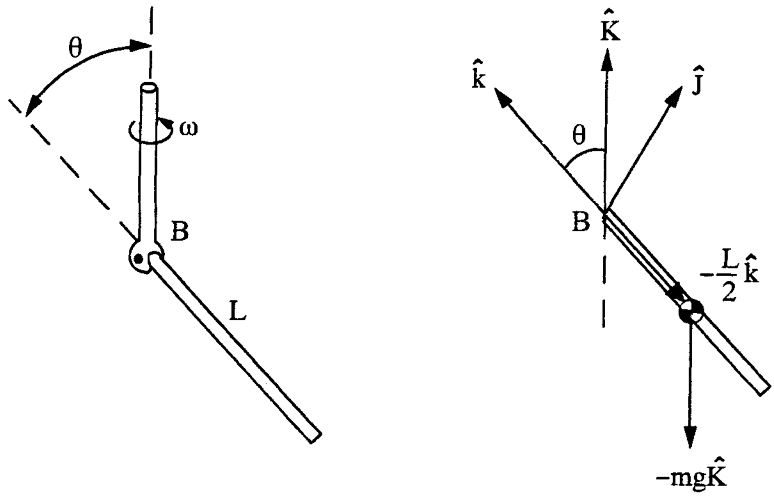


Fig. 10-12

First, we apply Eqn. (10.31). In the present case, $\bar{r} = -L/2$, $\dot{\psi} = \omega$, $I = 0$, and $I_0 = \dfrac{1}{3}mL^2$. Thus,

$$\frac{1}{2}g = \frac{1}{3}\omega^2 L\cos\theta$$

so that

$$\theta = \cos^{-1}\left(\frac{3g}{2L\omega^2}\right)$$

For a more complete analysis, we begin with the second of Eqns. (8.15):

$$\underline{M}_B = \frac{D\underline{H}_B}{Dt} = \frac{d\underline{H}_B}{dt} + \underline{\Omega} \times \underline{H}_B$$

The angular momentum about B is given by Eqn. (10.16); in the present terms, this becomes $\underline{H}_B = \frac{1}{3}mL^2\omega\sin\theta\hat{\jmath}$. Consequently, using Eqn. (10.3) and referring again to Fig. 10-12, we arrive at

$$\begin{aligned}
\left(-\frac{L}{2}\hat{k}\right) \times \left(-mg\hat{K}\right) &= \left(\omega\sin\theta\hat{\jmath} + \omega\cos\theta\hat{k}\right) \times \left(\frac{1}{3}mL^2\omega\sin\theta\hat{\jmath}\right) \\
\frac{1}{2}mgL\hat{k} \times \left(\cos\theta\hat{k} + \sin\theta\hat{\jmath}\right) &= \frac{1}{3}mL^2\omega^2\sin\theta\cos\theta(\hat{k}\times\hat{\jmath}) \\
\frac{1}{2}g\sin\theta &= \frac{1}{3}L\omega^2\sin\theta\cos\theta
\end{aligned}$$

The two roots of this equation are

$$\theta = 0\ , \qquad \text{and,} \qquad \theta = \cos^{-1}\left(\frac{3g}{2L\omega^2}\right)$$

For slow rotation speeds, specifically for $\omega < \sqrt{\frac{3g}{2L}}$, only the first root is possible and the rod remains vertical. For $\omega > \sqrt{\frac{3g}{2L}}$, both roots are possible, but only the second one is stable and therefore the rod describes a cone of angle $\theta = \cos^{-1}\left(\frac{3g}{2L\omega^2}\right)$ about the vertical.

Thus a drill, no matter how precisely aligned in the chuck, will want to assume a non zero angle with the vertical above a certain angular speed, and thus begin to "chatter".

10.8 Steady Precession with Zero Moment

Now suppose, in addition to steady precession, that there is no moment. In this case, from Eqn. (10.32),

$$M = \dot{\psi}[\dot{\psi}\sin\theta\cos\theta(I - I_0) + \dot{\phi}\sin\theta I] = 0$$

If $\dot{\psi} \neq 0$, solution for $\dot{\psi}$ in terms of the other variables is:

$$\boxed{\dot{\psi} = \frac{I\dot{\phi}}{(I_0 - I)\cos\theta}} \tag{10.35}$$

If the center of mass is chosen to take moments about, this equation becomes

$$\boxed{\dot{\psi} = \frac{\overline{I}\dot{\phi}}{(\overline{I}_0 - \overline{I})\cos\theta}} \tag{10.36}$$

From this we see that

1. If $\overline{I} < \overline{I}_0$ then $\dot{\psi}$ has the same sign as $\dot{\phi}$; this is called *direct precession.*

2. If $\overline{I} > \overline{I}_0$ then $\dot{\psi}$ has the opposite sign as $\dot{\phi}$, called *retrograde precession.*

3. If $\overline{I} = \overline{I}_0$ then $\dot{\psi}\dot{\phi}\sin\theta\overline{I} = 0$ which implies $\dot{\psi} = 0$; this is *no precession* (for example a homogeneous sphere).

We now consider some examples.

(1) For a spinning football (Fig. 10-13), if aerodynamic forces are ignored there is no moment about the center of mass. For a football, $\overline{I}_0 = 3\overline{I}$, approximately, so that the precession is direct, and Eqn. (10.36) gives $\dot{\psi} = \dfrac{\dot{\phi}}{2\cos\theta}$.

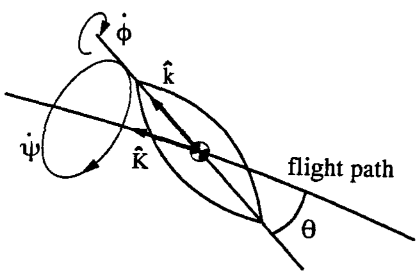


Fig. 10-13

(2) In Section 6.7 we considered a rotating space station. The rotation must be started carefully so that the angle θ is sufficiently small (Fig. 10-14); otherwise there will be an unacceptable precession, either direct or retrograde depending on the sign of $(\overline{I}_0 - \overline{I})$.

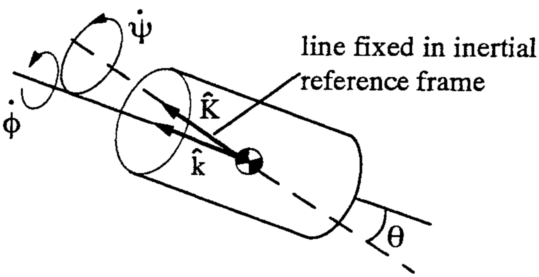


Fig. 10-14

(3) Because the earth is not a radially symmetric sphere (and thus $\overline{I}_0 \neq \overline{I}$) and because the spin axis is not perpendicular to the ecliptic plane,[5] the earth's polar axis precesses as shown in Fig. 10-15. Because $\overline{I}_0 < \overline{I}$, the precession is retrograde. The precessional period is about 26,000 years, or one degree every 80 years. This precession is thought to be a cause of the earth's (very!) long term weather patterns.

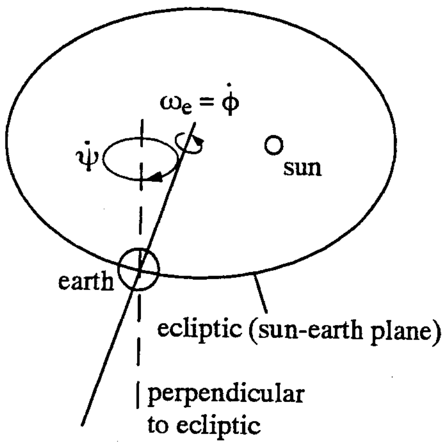


Fig. 10-15

10.9 Steady Precession About An Axis Normal to the Spin Axis

The special case of $\theta = \pi/2$ will now be considered (Fig. 10-16). This is an important case for applications. Equation (10.32) now becomes

$$\boxed{M = \dot{\psi}\dot{\phi}I} \tag{10.37}$$

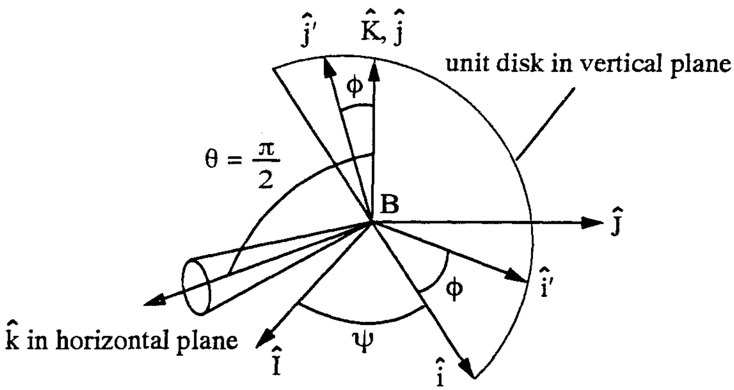


Fig. 10-16

It is useful to write this in vector form

$$\begin{aligned} \underline{M} &= M\hat{\imath} = \dot{\psi}\dot{\phi}I\hat{\imath} = \dot{\psi}\dot{\phi}I(\hat{\jmath} \times \hat{k}) \\ M\hat{\imath} &= I(\dot{\psi}\hat{\jmath} \times \dot{\phi}\hat{k}) \end{aligned} \tag{10.38}$$

This shows that the moment vector, $M\hat{\imath}$, the precession vector, $\dot{\psi}\hat{\jmath}$, and the spin vector, $\dot{\phi}\hat{k}$, are mutually perpendicular (Fig. 10-17). The advantage of using the $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ frame, as opposed to either the ground- or body-fixed frames, is obvious.

In Eqns. (10.37) and (10.38) the moment $\underline{M}$ is the moment exerted *on the body by the surroundings.* The moment exerted by the rigid body on the surroundings is of course equal in magnitude and opposite in direction.

It is instructive to derive Eqn. (10.38) directly from the second of Eqns. (8.15) for this special case:

$$
\begin{aligned}
\underline{\omega} &= \dot{\psi}\hat{j} + \dot{\phi}\hat{k} \\
\underline{\Omega} &= \dot{\psi}\hat{j} \\
\underline{H}_B &= [I]\underline{\omega} = I_0\dot{\psi}\hat{j} + I\dot{\phi}\hat{k} \\
\underline{M}_B &= \frac{D\underline{H}_B}{Dt} = \frac{d\underline{H}_B}{dt} + \underline{\Omega} \times \underline{H}_B = \underline{0} + (\dot{\psi}\hat{j}) \times (I_0\dot{\psi}\hat{j} + I\dot{\phi}\hat{k}) \\
\underline{M} &= I\dot{\psi}\dot{\phi}\hat{i}
\end{aligned}
$$

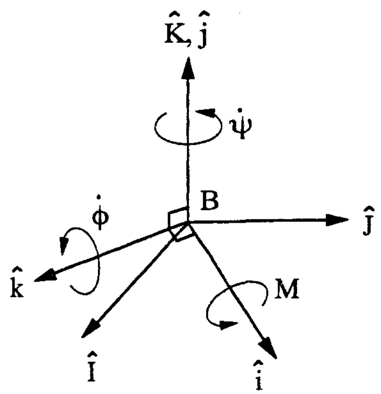


Fig. 10-17

Note that in the computation of $\underline{H}_B$, $\underline{\omega}$ must be used, and in the computation of $D\underline{H}_B/Dt$, $\underline{\Omega}$ must be used, as before.

10.10 Use of a Rotor to Stabilize a Car in Turns

In this example, a spinning rotor is installed in a car so that the gyroscopic moment cancels out the overturning moment in a turn. The car is travelling on a circle of radius r at speed ν (Fig. 10-18). Since the desired moment is about the $\hat{\imath}$ axis and the precession is about $\hat{\jmath}$, the rotor must spin about the $\hat{k}$ axis, that is, about an axis parallel to the axles of the car.

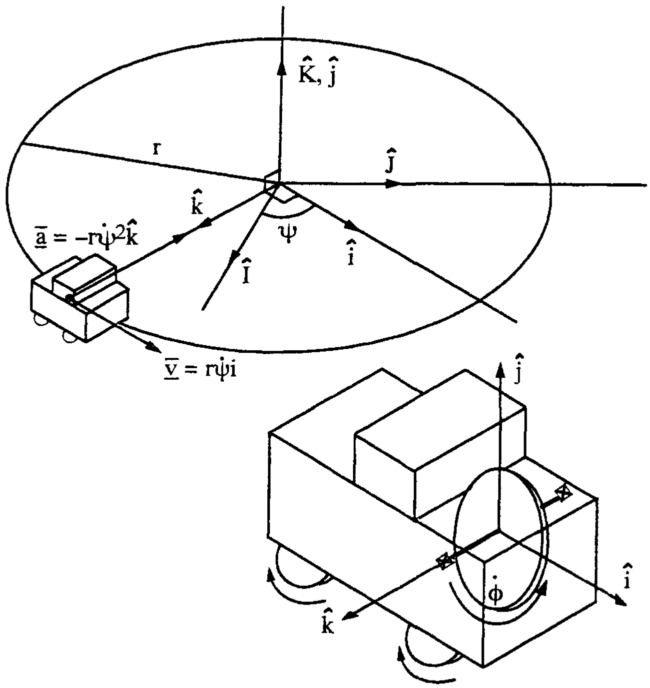


Fig. 10-18

The desired conditions are that the net moment is zero and the forces on the outside and inside wheels are equal (Fig. 10-19); thus

$$\overline{\underline{M}} = \underline{0}$$
$$F_{V_1} = F_{V_2}$$

From the first of Eqns. (8.15),

$$\underline{F}^e = m\underline{\overline{a}}$$
$$-F_{H_1} - F_{H_2} = m(-r\dot{\psi}^2) = -\frac{m\nu^2}{r}$$
$$F_{V_1} + F_{V_2} - W = 0$$

and from the second,

$$\overline{\underline{M}} = \underline{0}$$
$$M_R + hF_{H_1} + hF_{H_2} + \frac{d}{2}F_{V_2} - \frac{d}{2}F_{V_1} = 0$$
$$M_R = -h(F_{H_1} + F_{H_2}) = -\frac{hm\nu^2}{r}$$

This is the moment needed to be exerted by the rotor on the car. The moment in Eqn. (10.37) is the reaction to this moment, that is, the moment exerted on the rotor by the car:

$$M = -M_R = \frac{hm\nu^2}{r}$$

Since $M > 0$ and $\dot{\psi} > 0$, we have $\dot{\phi} > 0$ so that the rotor must turn in the opposite direction to the wheels. Note, therefore, that the tires and wheels produce an overturning moment.

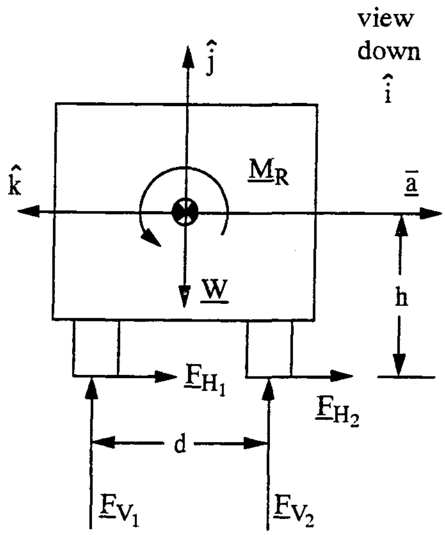


Fig. 10-19

From Eqn. (10.37),

$$M = I\dot{\psi}\dot{\phi}$$

Now let $I = m_R K^2$ where K is the radius of gyration of the rotor and m_R is its mass. Then

$$\frac{hm\nu^2}{r} = \frac{m_R K^2 \nu \dot{\phi}}{r}$$

so that

$$\dot{\phi} = \frac{h\nu m}{K^2 m_R}$$

As long as the car travels at speed ν, it will never tip, independent of the radius of turn r. For a typical case:

$$W = 3000 \text{ lb}\,, \quad W_R = 300 \text{ lb}\,, \quad h = 4 \text{ ft}\,, \quad K = 2 \text{ ft}\,,$$
$$\nu = 100 \text{ ft/sec}$$

which gives

$$\dot{\phi} \doteq 1000 \,\frac{\text{rad}}{\text{sec}} \doteq 10,000 \text{ r.p.m.}$$

The car, of course, will skid at some value of lateral acceleration. Thus it's logical to design for a constant lateral acceleration (namely, the value at which the wheels skid, say a). Then from $a = \nu^2/r$:

$$\dot{\phi} = \frac{hm\sqrt{ar}}{K^2 m_R}$$

so that $\dot{\phi}$ now depends on r. If the car were to loose control and spin, the precession would increase greatly, causing a very large over-turning moment, thus making driving such a car very dangerous.

10.11 Examples and Applications

Turning the handle-bars on a moving bicycle creates a precession of the front wheel that can produce an up-righting moment (Fig. 10-20). How

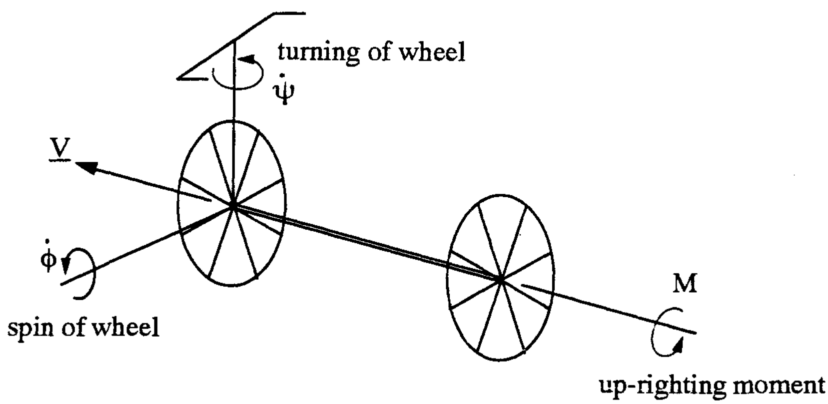


Fig. 10-20

ever, "anti-gyro" bicycles that cancel the gyroscopic moment (Fig. 10-21) have been constructed and it has been found possible to ride such a bike. Therefore factors other than gyroscopic moment contribute to staying up on a bicycle.

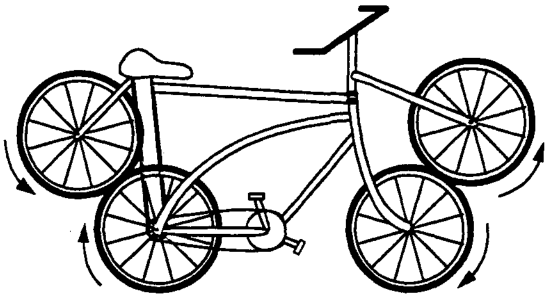


Fig. 10-21

The most wide-spread practical use of the nature of gyroscopic motion is the *gyroscope.* This device is a heavy, symmetrical body (rotor) spinning rapidly about its axis of symmetry, with no gravity torque, as illustrated on Fig. 10-22. Recall that the torque exerted by the bearing is perpendicular to both the spin and precession axes, that is, along the line of nodes. Thus this can be used to detect and measure motion about the $\hat{\jmath}$ axis because such motion causes a moment about the $\hat{\imath}$ axis which can be measured in the bearings. Three such devices mounted in directions perpendicular to each other can detect and measure any rotational motion.

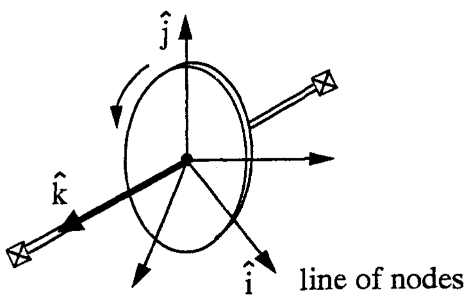


Fig. 10-22

Another use of the gyroscopic is as an angular reference in an "inertial" navigation system. In this application, the gyroscope is mounted in a system of gimbals such that it nominally experiences no moments (Fig. 10-23). As the vehicle to which it is attached moves, the gyroscope remains fixed relative to an inertial frame. Therefore, measuring the orientation of the vehicle relative to the gyroscope gives the orientation of the vehicle relative to an inertial frame.

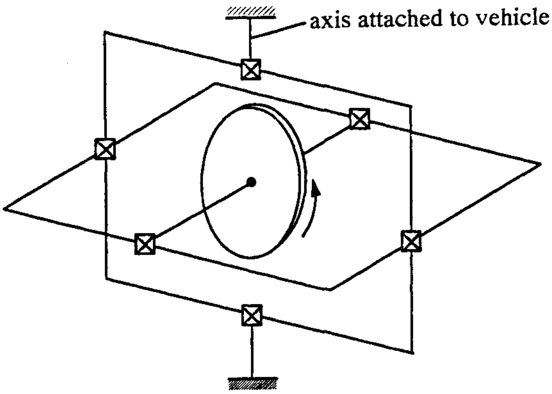


Fig. 10-23

It is important to minimize both friction in the bearings and air resistance on the gyroscope, because these forces produce precession and nutation of the spinning rotor. This is called "drift". The inertial navigation systems of modern commercial aircraft use a "ring-laser" gyroscope. This device has very low friction bearings and uses laser beams and mirrors to measure angles between the rotor and the vehicle.

Another application of the gyroscope is the gyrocompass. This is a gyroscope fixed to the earth in such a way that the rotation of the earth causes the gyroscope to precess with a period of one day. This causes bearing moments to act such that the gyroscope always lines up with the direction of precession, that is in a north-south direction.

Notes

1 See Section 4.4.

2 See Section 3.8.

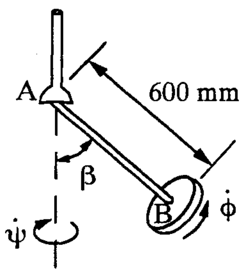
3 Recall that for any matrices A and B, $(AB)^T = B^T A^T$ and that if A is orthogonal, $A^T = A^{-1}$.

4 More generally, has a prescribed motion.

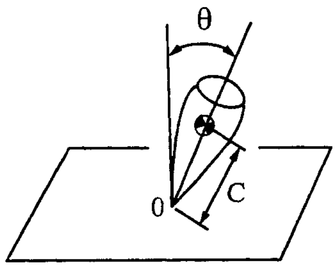
5 Plane in which the earth moves around the sun.

Problems

10/1 A disk of mass 2 kg and diameter 150 mm is attached to a rod AB of negligible mass to a ball-and-socket joint at A. The disk precesses at a steady rate about the vertical axis at $\dot{\psi} = 36$ rpm and the rod makes an angle of $\beta = 60°$ with the vertical. Determine the spin rate of the disk about rod AB.



Problem 10/1



Problem 10/3

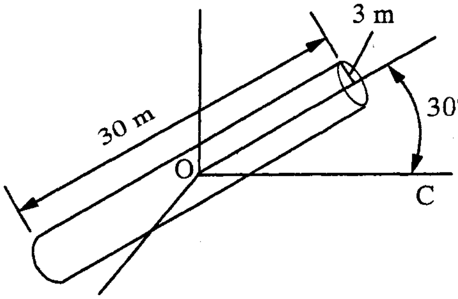
10/2 Same as Problem 10/1 except that $\beta = 30°$.

10/3 The figure shows a top weighing 3 oz. The radii of gyration of the top are 0.84 in. and 1.80 in. about the axis of symmetry and about a perpendicular axis passing through the support point 0, respectively. The length $C = 1.5$ in., the steady spin rate of the top about it's axis is 1800 rpm, and $\theta = 30°$. Use Eqn. (10.33) to determine the two possible rates of precession.

10/4 Derive Eqns. (10.34), starting from Eqn. (10.33).

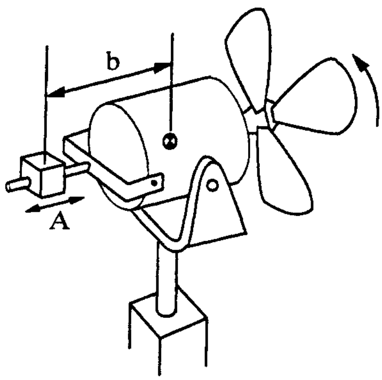
10/5 Use Eqns. (10.34) to obtain approximations for the two possible rates of precession of the top described in Prob. 10/3.

10/6 A space station may be approximated as a homogeneous cylinder as shown. The station precesses at a steady rate of one revolution per hour about axis $0C$. Determine the spin rate of the station about its axis of symmetry. Is the precession direct or retrograde?



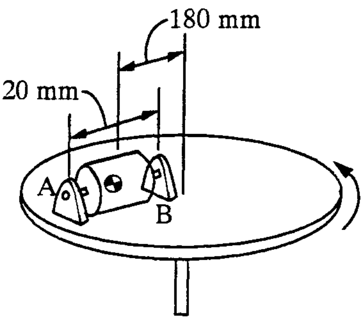
Problem 10/6

10/7 A fan is made to rotate about the vertical axis by using block A to create a moment about a horizontal axis. The parts of the fan that spin when it is turned on have a combined mass of 2.2 kg with a radius of gyration of 60 mm about the spin axis. The position of block A, which has a mass of 0.8 kg, may be adjusted. With the fan turned off, the unit is balanced when $b = 180$ mm. The fan spins at a rate of 1725 rpm with the fan turned on. Find the value of b that will produce a steady precession about the vertical of 0.2 rad/s.



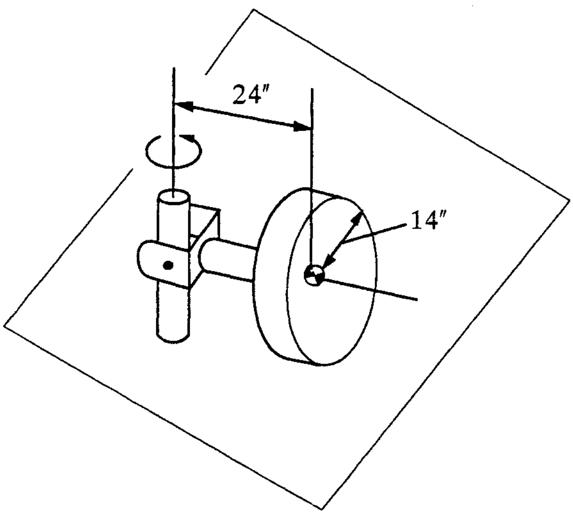
Problem 10/7

10/8 The motor shown has a total mass of 10 Kg and is attached to a rotating disk. The rotating components of the motor have a combined mass of 2.5 Kg and a radius of gyration of 35 mm. The motor rotates with a constant angular speed of 1725 rpm in a counter clockwise direction when viewed from A to B, and the turntable revolves about a vertical axis at a constant rate of 48 rpm in the direction shown. Determine the forces in the bearings A and B.



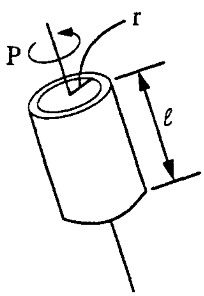
Problem 10/8

10/9 A rigid homogeneous disk of weight 96.6 lb. rolls on a horizontal plane on a circle of radius 2 ft. The steady rate of rotation about the vertical axis is 48 rpm. Determine the normal force between the wheel and the horizontal surface. Neglect the weight of all components except the disk.

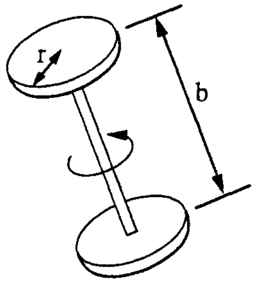


Problem 10/9

10/10 The thin homogeneous cylindrical shell is rotating in space about its axis of symmetry. If the axis has a slight wobble, for what range of the ratio ℓ/r will the motion be direct precession?



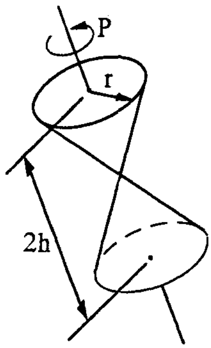
Problem 10/10



Problem 10/11

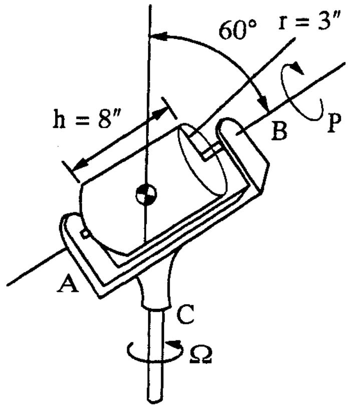
10/11 Two identical thin homogeneous disks, each of mass m and radius r, are connected by a thin rigid rod. If the assembly is spinning about its axis of symmetry, determine the value of b for which no precession will occure.

10/12 Two homogeneous cones, each of mass m, radius r, and altitude h, are rigidly joined at their vertices. If the assembly is made to spin about the axis of symmetry in space, for what value of h/r will precession not occure?



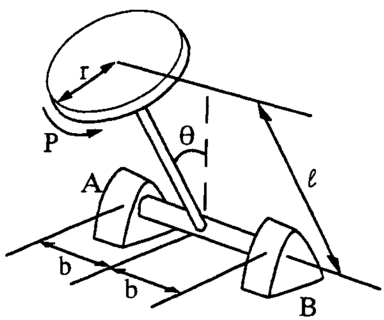
Problem 10/12

10/13 The 64.4 lb homogeneous cylinder is mounted in bearing at A and B to a bracket which rotates about a vertical axis. If the cylinder spins at steady rate $P = 50$ rad/s and the bracket at 30 rad/s, compute the moment that the assembly exerts on the shaft at C. Neglect the mass of everything except the cylinder.



Problem 10/13

10/14 A homogeneous thin disk of mass m and radius r spins on its shaft at a steady rate P. This shaft is rigidly connected to a horizontal shaft that rotates in bearings at A and B. If the assembly is released from rest at the vertical position ($\theta = 0$, $\dot{\theta} = 0$), determine the forces in the bearing at A and B as the horizontal position ($\theta = \pi/2$) is passed. Neglect the masses of all components except that of the disk.



Problem 10/14

Chapter 11

Work and Energy

11.1 Introduction

The concepts of work and energy arose from a desire to put dynamics on a more intuitive basis. These ideas later became the key concepts in many branches of physics and engineering, such as thermodynamics and fluid mechanics. In modern dynamics, work and energy methods are often useful but not essential in the analysis of specific problems.

The work-energy equation, being a scalar relation, provides only one piece of information, and this typically is not enough to solve for all the information required. Use of this equation, however, frequently simplifies the analysis and provides insight regarding the motion under consideration. Further, it is a once-integrated form of the equation of motion (a relation between speeds instead of accelerations). Although work and energy are very general concepts, after introducing these ideas for a particle we will concentrate on developing them only for the special case of a rigid body.

11.2 Work

Suppose a force $\underline{F}$ acts at a point A that moves on a curve C (Fig. 11-1). The velocity of point A relative to a frame $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ is $\underline{v}(t) = d\underline{r}(t)/dt = v(t)\hat{e}_t$. The *work* done by force $\underline{F}$ during displacement of the point A

from position $\underline{r}_0$ to $\underline{r}_1$ along curve C is defined as

$$\boxed{U = \int_{\underline{r}_0}^{\underline{r}_1} \underline{F} \cdot d\underline{r}} \tag{11.1}$$

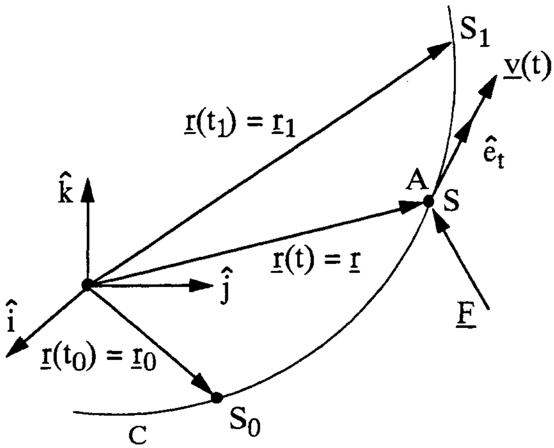


Fig. 11-1

This is a *line integral*; it is meaningless to speak of work at a given instant of time or at a position. We can speak only of work done during a displacement or, equivalently, during an interval of time.

11.3 Forms of the Work Integral

Other forms of the work integral are frequently useful. First, from $\underline{\nu} = d\underline{r}/dt$, $\underline{F} \cdot d\underline{r} = \underline{F} \cdot \underline{\nu} dt$ so that

$$\boxed{U = \int_{t_0}^{t_1} \underline{F} \cdot \underline{\nu} dt} \tag{11.2}$$

Second, resolving the force in the normal and tangential directions (Fig. 11-2),

$$\underline{F} \cdot \underline{\nu} = F\nu \cos\beta = F_t \nu$$

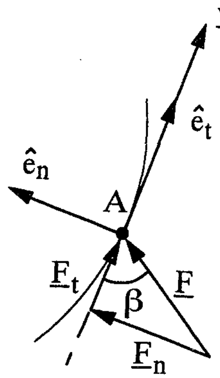


Fig. 11-2

so that

$$\underline{F} \cdot \underline{\nu} = F_t \nu dt = F_t ds$$

and hence

$$\boxed{U = \int_{s_0}^{s_1} F_t ds} \tag{11.3}$$

where F_t is the component of $\underline{F}$ along $\underline{\nu}$.

Alternatively, we may resolve $d\underline{r}$ along directions tangent to, and perpendicular to, force $\underline{F}$ (Fig. 11-3):

$$\underline{F} \cdot \underline{\nu} dt = \underline{F} \cdot d\underline{r} = F(\cos\beta ds) = F\ ds_F$$

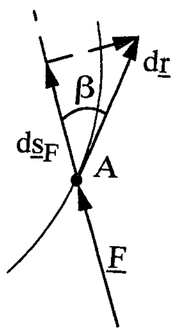


Fig. 11-3

so that

$$\boxed{U = \int_{s_{F_0}}^{s_{F_1}} F \, ds_F} \tag{11.4}$$

where ds_F is the component of $d\underline{r}$ in direction $\underline{F}$.

Finally, if $\underline{r}$ and $\underline{F}$ are expressed in rectangular coordinates,

$$\underline{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$
$$\underline{F} = F_x\hat{\imath} + F_y\hat{\jmath} + F_z\hat{k}$$

then Eqn. (11.1) gives

$$\boxed{U = \int_{x_0}^{x_1} F_x dx + \int_{y_0}^{y_1} F_y dy + \int_{z_0}^{z_1} F_z dz} \tag{11.5}$$

The most advantageous form of the energy integral to be used, Eqns. (11.1) – (11.5), depends on the particular application. It is important to note that only the tangential component of a force does work; in particular if $\underline{F} = F\hat{e}_n$, then $U = 0$.

11.4 Example – Constant Force

A force constant in magnitude and direction moves along a curve relative to a reference frame, Fig. 11-4. Without loss of generality, take the

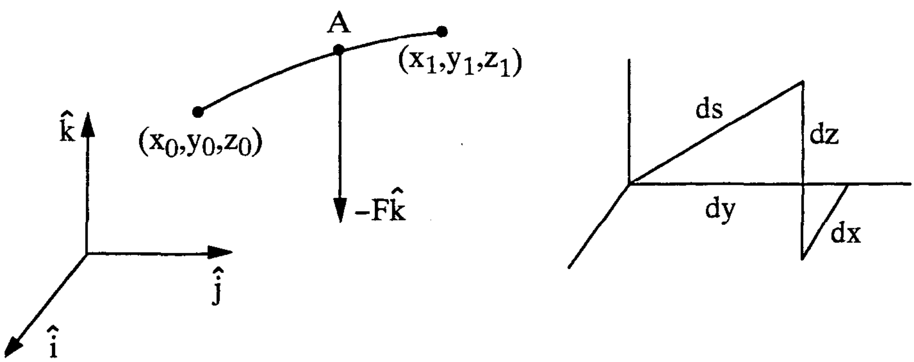


Fig. 11-4

direction of the force to be $-\hat{k}$. We will compute the work done from a point (x_0, y_0, z_0) to another point (x_1, y_1, z_1). The obvious choice for the work integral is Eqn. (11.4). In our case,

$$\underline{F} = -F\hat{k}\ , \qquad ds_F = dz$$

so that

$$U = \int_{s_{F_0}}^{s_{F_1}} F\, ds_F = \int_{z_0}^{z_1} (-F) dz = -F(z_1 - z_0) \tag{11.6}$$

A special case of this result is the gravitational force acting on a particle of mass m in a constant gravitational field; in this case $F = mg$ and

$$U = mg(z_0 - z_1) \tag{11.7}$$

11.5 Power

The rate of doing work is called *power*, that is

$$\boxed{P = \frac{dU}{dt}} \tag{11.8}$$

Using Eqn. (11.2), this can be written as

$$\boxed{P = \underline{F} \cdot \underline{v}} \tag{11.9}$$

Like work, power is a scalar, but unlike work it is an instantaneous quantity, not a line integral.

11.6 Work Done by Force Couple

Consider a force couple of moment $\underline{M}$, that is, two forces of equal magnitudes and opposite directions, acting at two points fixed with respect to each other, say C and D (Fig. 11-5). Let $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ be a frame in which C and D are fixed and let $\underline{\omega}$ be the angular velocity of $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ w.r.t. another frame $\{\hat{I}, \hat{J}, \hat{K}\}$.

Since $\underline{v}_r = \underline{0}$, the relative velocity equation gives:

$$\underline{v}_C = \underline{v}_B + \underline{\omega} \times \underline{r}_C$$
$$\underline{v}_D = \underline{v}_B + \underline{\omega} \times \underline{r}_D$$

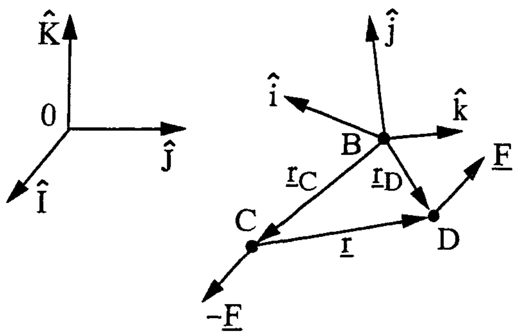


Fig. 11-5

The power of the force system is obtained from Eqn. (11.9) as

$$\begin{aligned} P &= \underline{F} \cdot \underline{v}_D - \underline{F} \cdot \underline{v}_C = \underline{F} \cdot (\underline{v}_D - \underline{v}_C) \\ &= \underline{F} \cdot \underline{\omega} \times (\underline{r}_D - \underline{r}_C) = \underline{F} \cdot \underline{\omega} \times \underline{r} \\ &= \underline{\omega} \cdot \underline{r} \times \underline{F} = \underline{\omega} \cdot \underline{M} \end{aligned} \tag{11.10}$$

where Eqn. (A.15) and the definition of a moment, $\underline{M} = \underline{r} \times \underline{F}$, were used.

For the special case of 2-D motion, say in the (x, y) plane, we have $\underline{M} = M\hat{k}$ and $\underline{\omega} = \omega\hat{k}$ and Eqn. (11.10) reduces to

$$P = M\omega$$

so that

$$U = \int_{t_0}^{t_1} P dt = \int_{t_0}^{t_1} M\omega dt = \int_{\theta_0}^{\theta_1} M d\theta \tag{11.11}$$

11.7 Kinetic Energy and the Energy Equation

Up to now we have considered a force $\underline{F}$ whose point of application moves on a curve. Now suppose, in fact, that $\underline{F}$ is the resultant of all forces

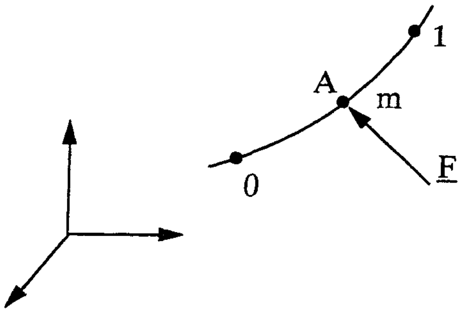


Fig. 11-6

acting on a particle of mass m at point A that moves on a curve C, and let $\underline{\nu}$ be the velocity of the particle relative to an inertial frame $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ (Fig. 11-6). Then Newton's Second Law, $md\underline{\nu}/dt = \underline{F}$, applies; taking the scalar (dot) product of both sides with $\underline{\nu}$, and integrating, we obtain

$$m\frac{d\underline{\nu}}{dt} \cdot \underline{\nu} = \underline{F} \cdot \underline{\nu}$$

$$\int_{\nu_0}^{\nu_1} m\underline{\nu} \cdot d\underline{\nu} = \int_{t_0}^{t_1} \underline{F} \cdot \underline{\nu} dt$$

$$\frac{1}{2}m(\nu_1^2 - \nu_0^2) = U \tag{11.12}$$

where Eqn. (11.2) was used.

Now define the *kinetic energy* of the particle at any time as

$$\boxed{T = \frac{1}{2}m\nu^2} \tag{11.13}$$

Then Eqn. (11.12) becomes

$$\boxed{\Delta T = U} \tag{11.14}$$

where $\Delta T = T_1 - T_0 = \frac{1}{2}m\nu_1^2 - \frac{1}{2}m\nu_0^2$. This is the *energy equation*; it states that *the change in kinetic energy between two positions of the particle is equal to the work done by the resultant force between the two positions.* It is important to note that:

1. U is defined over an interval of motion but T is defined at an instant.
2. Eqn. (11.14) is a once-integrated form of Newton's Second Law.
3. Eqn. (11.14) is a *scalar* relation; it gives changes in speed (magnitude of velocity) and not in velocity.
4. The speed must be measured in an inertial frame.

11.8 Example

A ball on a cord is released at position A with 2 m/s velocity as shown on Fig. 11-7. At the bottom, the cord hits pin B. We want the ball's velocity as it passes the horizontal position at C. We note that the direction of the velocity at C is vertically up, i.e. in the direction $-\hat{\imath}$; thus we need only to determine the speed, which we can get by the energy equation. We will work this problem two ways.

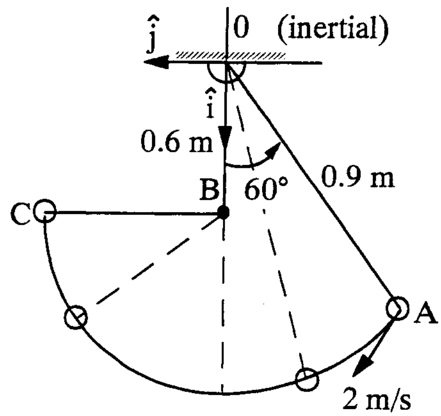


Fig. 11-7

In the first method, the work done by the gravitational force is computed by Eqn. (11.3). In this case the motion must be divided into two phases as shown on Fig. 11-8.

During Phase I:

$$\underline{F}_t = F_t \hat{e}_t \ , \qquad F_t = mg \sin\theta \ , \qquad ds = -\ell_1 d\theta$$

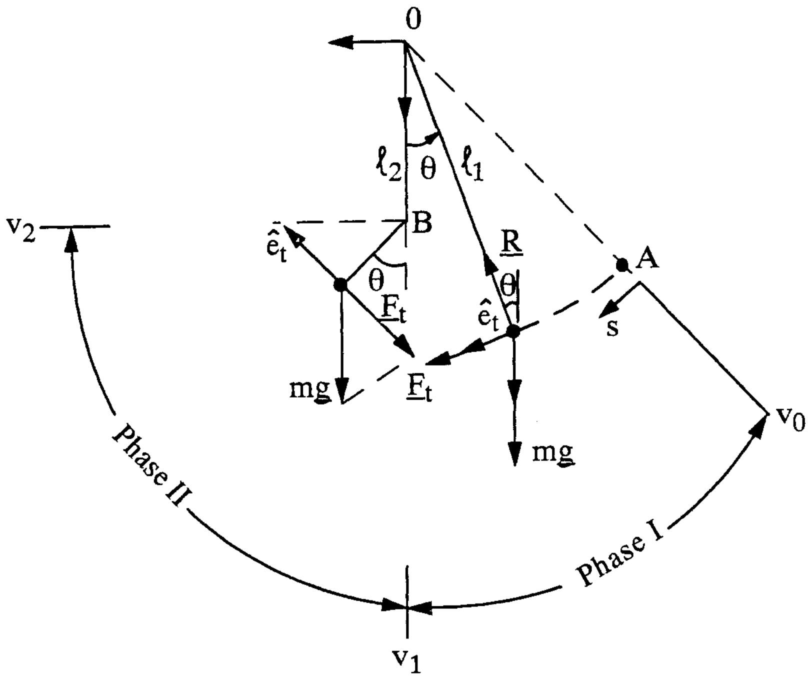


Fig. 11-8

Now applying Eqns. (11.3), (11.13), and (11.14),

$$\Delta T = U$$

$$\frac{1}{2}m(\nu_1^2 - \nu_0^2) = \int_{s_0}^{s_1} F_t\, ds = \int_{60°}^{0°} mg\sin\theta(-\ell_1 d\theta)$$

$$\nu_1^2 - \nu_0^2 = -2g\ell_1 \int_{60°}^{0} \sin\theta\, d\theta = -2g\ell_1\left(-\frac{1}{2}\right) = g\ell_1$$

$$\nu_1^2 = \nu_0^2 + g\ell_1$$

During Phase II:

$$F_t = -mg\sin\theta\,, \qquad ds = (\ell_1 - \ell_2)d\theta$$

$$\frac{1}{2}m(\nu_2^2 - \nu_1^2) = \int_0^{90°} (-mg\sin\theta)(\ell_1 - \ell_2)d\theta$$

$$\nu_2^2 - \nu_1^2 = -2g(\ell_1 - \ell_2)$$

$$\nu_2^2 = \nu_1^2 - 2g(\ell_1 - \ell_2) = \nu_0^2 + g\ell_1 - 2g(\ell_1 - \ell_2)$$

$$= 4 + 9.81(0.3) = 6.943$$

$$\nu_2 = 2.635 \text{ m/s}$$

Note that the work done by $\underline{R}$ and by $F_n\hat{e}_n$ is zero.

In the second method Eqn. (11.4) is used. We have (Fig. 11-9):

$$F = mg\ , \qquad ds_F = dx$$

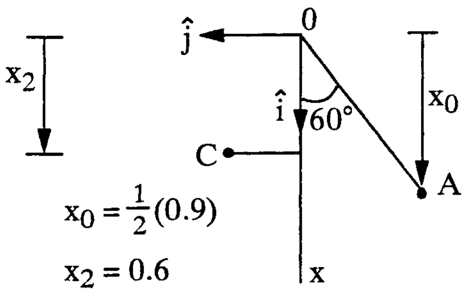


Fig. 11-9

so that

$$\frac{1}{2}m(\nu_2^2 - \nu_0^2) = \int_{x_0}^{x_2} mg\ dx = mg(x_2 - x_0)$$
$$\nu_2^2 = \nu_0^2 + 2g(x_2 - x_1) = 2^2 + (2)(9.81)(0.15) = 6.943$$
$$\nu_2 = 2.635\ \text{m/s}$$

Obviously, the second method is easier.

11.9 Potential Energy

Consider again a force $\underline{F}$ whose point of application A moves on curve C, Fig. 11-10. In rectangular coordinates,

$$\underline{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}\ , \qquad \underline{F} = F_x\,\hat{\imath} + F_y\,\hat{\jmath} + F_z\,\hat{k}$$

Thus the work done by $\underline{F}$ from $\underline{r}_0$ to $\underline{r}_1$ is, by definition,

$$U = \int_{\underline{r}_0}^{\underline{r}_1} \underline{F}\cdot d\underline{r} = \int_{\underline{r}_0}^{\underline{r}_1} (F_x\,dx + F_y\,dy + F_z\,dz) \qquad (11.15)$$

If $\underline{F}$ is a function of position, $\underline{F} = \underline{F}(\underline{r}) = \underline{F}(x, y, z)$, this integral generally depends on the path C.

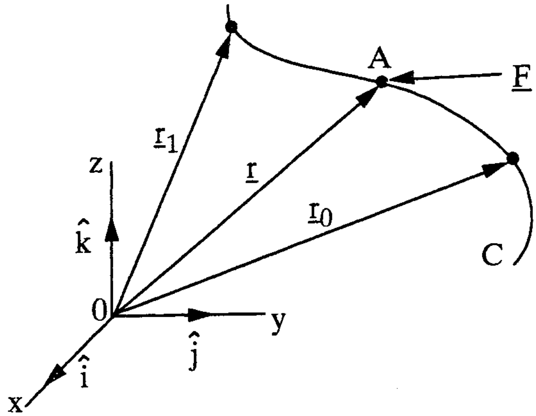


Fig. 11-10

Now suppose $F(\underline{r})$ is such that there exists a function $V(\underline{r})$ such that

$$\boxed{\underline{F}(\underline{r}) = -\nabla V(\underline{r})} \tag{11.16}$$

where from Eqn. (A.22) the gradient operator is defined as

$$\nabla V(\underline{r}) = \frac{\partial V}{\partial x}\hat{\imath} + \frac{\partial V}{\partial y}\hat{\jmath} + \frac{\partial V}{\partial z}\hat{k} \tag{11.17}$$

in rectangular coordinates. Then

$$F_x = -\frac{\partial V}{\partial x}\ , \qquad F_y = -\frac{\partial V}{\partial y}\ , \qquad F_z = -\frac{\partial V}{\partial z} \tag{11.18}$$

and so

$$\underline{F} \cdot d\underline{r} = -\left(\frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz\right) = -dV \tag{11.19}$$

Therefore, the work done by $\underline{F}$ between positions $\underline{r}_0$ and $\underline{r}_1$ is

$$U = \int_{\underline{r}_0}^{\underline{r}_1} \underline{F} \cdot d\underline{r} = \int_{V_0}^{V_1} (-dV) = -(V_1 - V_0) = -\Delta V \tag{11.20}$$

where $V_1 = V(\underline{r}_1)$ and $V_0 = V(\underline{r}_0)$. This shows that now the work done by $\underline{F}$ *depends only on the initial and final points and not on the path.* A force $\underline{F}(\underline{r})$ with this property is called a *conservative force*, and $V(\underline{r})$ is called a *potential energy function.*

Now suppose curve C is a closed path so that $\underline{r}_0 = \underline{r}_1$ (Fig. 11-11). Then $V_0 = V_1$ and $U = -\Delta V = 0$. Thus no work is done along a closed path by a conservative force.

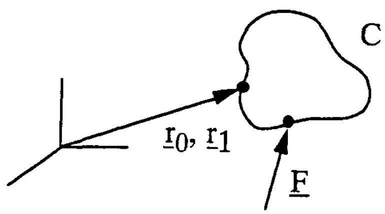


Fig. 11-11

11.10 Example – Gravitational Attraction

Recall from Section 5.10 that the gravitational force on a mass particle is given by (Fig. 11-12):

$$\underline{F} = -\frac{KMm}{r^2}\hat{e}_r \tag{11.21}$$

We have

$$\underline{r} = r\hat{e}_r \ , \qquad \underline{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} \ , \qquad r = (x^2 + y^2 + z^2)^{\frac{1}{2}}$$

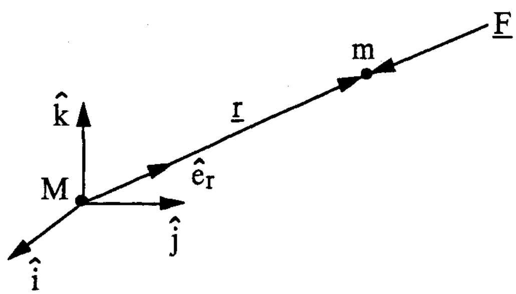


Fig. 11-12

so that

$$\underline{F} = -\frac{KMm}{r^2}\frac{\underline{r}}{r} = -\frac{KMm}{r^3}(x\hat{\imath} + y\hat{\jmath} + z\hat{k}) \tag{11.22}$$

and hence

$$\begin{aligned} F_x &= -KMm\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \\ F_y &= -KMm\frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \\ F_z &= -KMm\frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}} \end{aligned} \tag{11.23}$$

Thus the gravitational potential energy function is

$$\boxed{V = -\frac{KMm}{r} = -\frac{KMm}{(x^2+y^2+z^2)^{\frac{1}{2}}}} \tag{11.24}$$

because

$$-\frac{\partial V}{\partial x} = -KMm\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}} = F_x \tag{11.25}$$

and likewise for the other two components.

11.11 Energy Equation

Suppose a number of forces act on a particle of mass m, some conservative and some not (Fig. 11-13). Then

$$\underline{F}_i^c = -\nabla V_i \tag{11.26}$$

for all conservative forces. The resultant force is

$$\underline{F} = \sum_i \underline{F}_i^c + \sum_j \underline{F}_j^n = \sum_i(-\nabla V_i) + \sum_j \underline{F}_j^n \tag{11.27}$$

The work done by the resultant force in going from $\underline{r}_0$ to $\underline{r}_1$ along curve C is then

$$\begin{aligned} U = \int_{\underline{r}_0}^{\underline{r}_1} \underline{F}\cdot d\underline{r} &= -\sum_i[V_i(\underline{r}_1) - V_i(\underline{r}_0)] + \int_{\underline{r}_0}^{\underline{r}_1}\sum_j \underline{F}_j^n \cdot d\underline{r} \\ &= -\Delta V + U^n \end{aligned} \tag{11.28}$$

where

$$U^n = \int_{\underline{r}_0}^{\underline{r}_1} \sum_j \underline{F}_j^n \cdot d\underline{r} = \text{work done by non-conservative forces}$$

$$V = \sum_i V_i = \text{sum of potential energies of all conservative forces.}$$

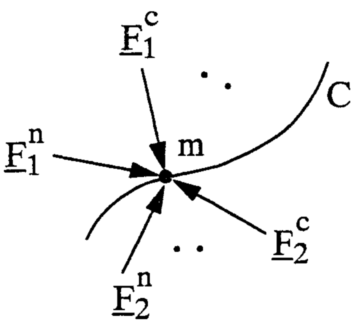


Fig. 11-13

Since $U = \Delta T$, the energy equation becomes

$$\boxed{\Delta T = -\Delta V + U^n\,, \qquad \text{or}\,, \qquad U^n = \Delta(T + V)} \tag{11.29}$$

Now define the *total mechanical energy* by

$$\boxed{E = T + V} \tag{11.30}$$

Then Eqn. (11.29) can be written as

$$\boxed{U^n = \Delta E} \tag{11.31}$$

If all forces are conservative (i.e. constitute a conservative system) and accounted for in ΔV then

$$\boxed{\Delta E = 0} \tag{11.32}$$

Thus *energy is conserved* (remains constant) in a conservative system.

It is clear that use of the work-energy relation requires a classification of the forces acting on the rigid body (Fig. 11-14). The forces that do no work do not enter into the work-energy equation, and need not be considered further. Among the forces doing work, the conservative forces are accounted for by their potential energy functions, while the work integrals must be computed for the non-conservative forces. Because computing work integrals is often tedious, energy methods work best for systems in which all, or most, of the forces either do no work or are conservative. A possible disadvantage of the energy method is that it does not give the values of the forces not doing work; sometimes this is desired information.

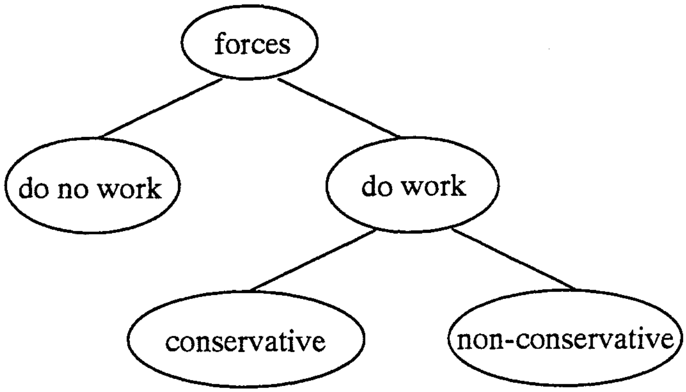


Fig. 11-14

There is an alternative point of view to the energy equation. If we introduce a new kind of energy, say I, such that $\Delta I = -U^n$, then Eqn. (11.29) becomes

$$\Delta T + \Delta V + \Delta I = \Delta E' = 0 \tag{11.33}$$

where $E' = T + V + I$ is the *total energy* of the body. Now, energy is *always* conserved. This new energy I is called *internal energy* and is changed by the flow of heat into and out from the body and is measured by temperature. This viewpoint is extremely important in thermodynamics; Eqn. (11.33) is in fact the First Law of Thermodynamics.

11.12 Examples

First, consider a constant gravitational field. Referring to Fig. 11-15, $\underline{F} = -mg\hat{k} = F_z\hat{k}$ so that $F_z = -mg$ and hence

$$V(z) = mgz \tag{11.34}$$

because $-\dfrac{\partial V}{\partial z} = -mg = F_z$. This result also follows as a special case of Section 11.10. It is in agreement with Eqn. (11.7).

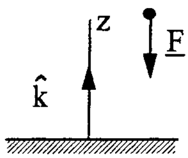


Fig. 11-15

As a second example consider a mass-spring system (Fig. 11-16). For a linear spring, $\underline{F} = -Kx\hat{\imath} = F_x\hat{\imath}$ so that $F_x = -Kx$ and thus

$$V(x) = \frac{1}{2}Kx^2 \tag{11.35}$$

because $-\dfrac{\partial V}{\partial x} = -Kx = F_x$.

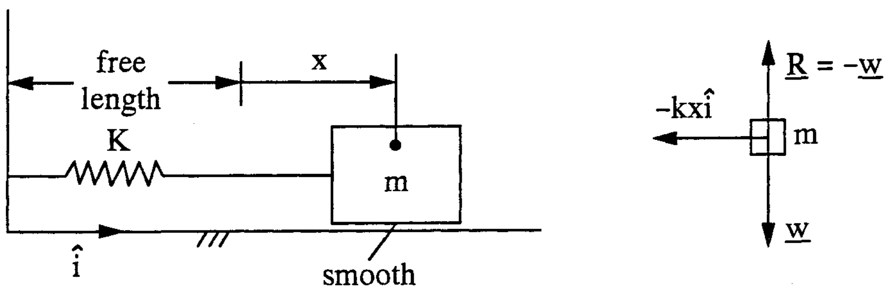


Fig. 11-16

As a final example, we return to the problem of Section 11.8, the pendulum with a pin. Now we solve it using Eqn. (11.32). Since $\hat{\imath}$

is downward, Eqn. (11.34) gives $V = -mgx$. Thus $F_x = -\dfrac{\partial V}{\partial x} = mg$. Referring to Fig. 11-17, we have

$$\begin{aligned} V_0 &= -mgx_0 \ , \qquad V_2 = -mgx_2 \\ T_0 &= \frac{1}{2}m\nu_0^2 \ , \qquad T_2 = \frac{1}{2}m\nu_2^2 \end{aligned}$$

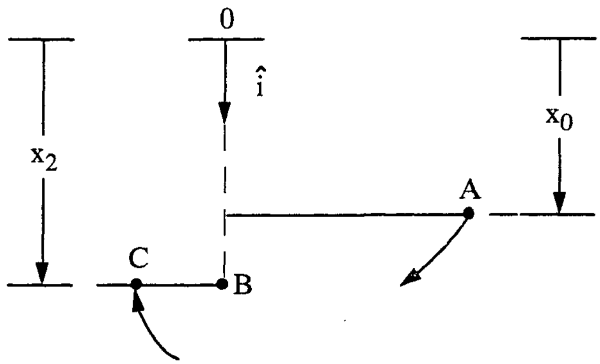


Fig. 11-17

$$\begin{aligned} \Delta E &= \Delta(T+V) = T_2 - T_0 + V_2 - V_0 \\ &= \frac{1}{2}m\nu_2^2 - \frac{1}{2}m\nu_0^2 - mg\ x_2 + mg\ x_0 = 0 \\ \nu_2^2 &= \nu_0^2 + 2g(x_2 - x_0) \\ \nu_2 &= 2.635 \text{ m/s} \end{aligned}$$

Clearly, this is the easiest method of solving this problem.

11.13 Work Done by Internal Forces in a Rigid Body

We now wish to extend the work-energy concepts to the motion of a rigid body. We begin by showing that the net work done by the internal forces in a rigid body is zero.

Let m_i and m_j be the masses of two particles in a rigid body and let $\underline{F}_{ij}$ be the force exerted on particle i by particle j, Fig. 11-18. By

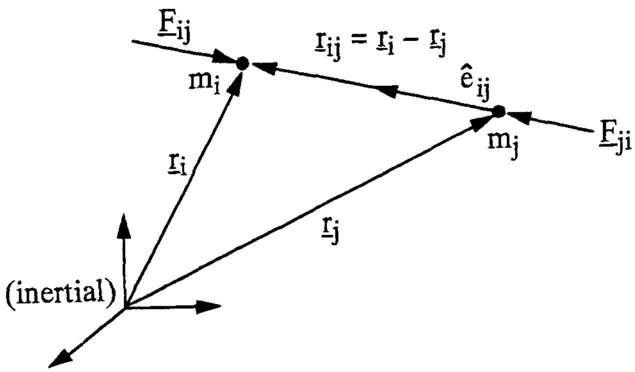


Fig. 11-18

Newton's Third Law, $\underline{F}_{ij} = -\underline{F}_{ji}$ and these two forces act on the line connecting the two particles.

Consider the rate of doing work by the pair of internal forces $\underline{F}_{ij}$ and $\underline{F}_{ji}$:

$$P = \underline{F}_{ij} \cdot \underline{\nu}_i + \underline{F}_{ji} \cdot \underline{\nu}_j = \underline{F}_{ij} \cdot \underline{\nu}_i - \underline{F}_{ij} \cdot \underline{\nu}_j = \underline{F}_{ij} \cdot (\underline{\nu}_i - \underline{\nu}_j)$$

But

$$\underline{\nu}_i - \underline{\nu}_j = \frac{d\underline{r}_i}{dt} - \frac{d\underline{r}_j}{dt} = \frac{d\underline{r}_{ij}}{dt}\,, \qquad \underline{r}_{ij} = r_{ij}\,\hat{e}_{ij}$$

so that

$$\frac{d\underline{r}_{ij}}{dt} = \dot{r}_{ij}\,\hat{e}_{ij} + r_{ij}\frac{d\hat{e}_{ij}}{dt}$$

For a rigid body, the particles remain fixed relative to each other so that r_{ij} = constant and thus $\dot{r}_{ij} = 0$. Also, for a unit vector such as $\hat{e}_{ij}$ (see Fig. 11-19), $\dfrac{d\hat{e}_{ij}}{dt} \perp \hat{e}_{ij}$.

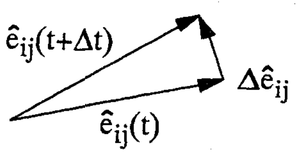


Fig. 11-19

Consequently,

$$\begin{aligned} P = \underline{F}_{ij} \cdot (\underline{v}_i - \underline{v}_j) &= \underline{F}_{ij} \cdot \frac{d\underline{r}_{ij}}{dt} = \underline{F}_{ij} \cdot r_{ij}\frac{d\hat{e}_{ij}}{dt} \\ &= -F_{ij}\,\hat{e}_{ij} \cdot r_{ij}\,\frac{d\hat{e}_{ij}}{dt} = 0 \end{aligned}$$

so that

$$U = \int_{t_0}^{t_1} P\,dt = 0$$

Thus, in a rigid body the work done by internal forces cancels in pairs and the net work done by these forces is zero.

11.14 Work-Energy for a Rigid Body

Let $\underline{F}^e$ be the resultant of all external forces acting on a rigid body and form the dot product of $\overline{\underline{v}}$, the velocity of the body's center of mass with respect to an inertial frame, with the linear momentum equation, Eqn. (8.15):

$$\begin{aligned} \underline{F}^e &= m\frac{D\overline{\underline{v}}}{Dt} \\ \underline{F}^e \cdot \overline{\underline{v}} &= m\frac{D\overline{\underline{v}}}{Dt} \cdot \overline{\underline{v}} \\ &= \frac{1}{2}m\frac{D}{Dt}(\overline{\underline{v}} \cdot \overline{\underline{v}}) = \frac{1}{2}\frac{D}{Dt}(\overline{\underline{v}} \cdot \overline{\underline{L}}) = \frac{1}{2}\frac{d}{dt}(\overline{\underline{v}} \cdot \overline{\underline{L}}) \end{aligned} \tag{11.36}$$

where $\overline{\underline{L}} = m\overline{\underline{v}}$ is the body's linear momentum, d/dt is relative to a body fixed reference frame, and D/Dt is relative to the inertial frame.[1] Integrating this equation between two times gives

$$\int_{t_0}^{t_1} \underline{F}^e \cdot \overline{\underline{v}}dt = \frac{1}{2}\overline{\underline{v}} \cdot \overline{\underline{L}}\Big|_{t_0}^{t_1} \tag{11.37}$$

Next dot $\underline{\omega}$ (the body's angular velocity) with the moment equation, Eqn. (8.15):

$$\begin{aligned} \overline{\underline{M}} &= \frac{D\overline{\underline{H}}}{Dt} \\ \overline{\underline{M}} \cdot \underline{\omega} &= \frac{D\overline{\underline{H}}}{Dt} \cdot \underline{\omega} = \frac{d\overline{\underline{H}}}{dt} \cdot \underline{\omega} + (\underline{\omega} \times \overline{\underline{H}}) \cdot \underline{\omega} \end{aligned}$$

Using the triple scalar product relation, Eqn. (A.15), this becomes

$$\begin{aligned}\underline{\overline{M}} \cdot \underline{\omega} &= \frac{d\underline{\overline{H}}}{dt} \cdot \underline{\omega} + \underline{\omega} \cdot (\underline{\omega} \times \underline{\overline{H}}) \\ &= \frac{d\underline{\overline{H}}}{dt} \cdot \underline{\omega} - \underline{\overline{H}} \cdot (\underline{\omega} \times \underline{\omega}) = \frac{d\underline{\overline{H}}}{dt} \cdot \underline{\omega} \end{aligned} \tag{11.38}$$

Also, using Eqn. (7.12),

$$\underline{\overline{M}} \cdot \underline{\omega} = \frac{d}{dt}([\overline{I}]\underline{\omega}) \cdot \underline{\omega} = [\overline{I}]\underline{\dot{\omega}} \cdot \underline{\omega} = [\overline{I}]\underline{\omega} \cdot \underline{\dot{\omega}} = \underline{\overline{H}} \cdot \underline{\dot{\omega}} = \underline{\dot{\omega}} \cdot \underline{\overline{H}} \tag{11.39}$$

where we have used the fact that $[I]\underline{\dot{\omega}} \cdot \underline{\omega} = [\overline{I}]\underline{\omega} \cdot \underline{\dot{\omega}}$ which may be verified by writing it out in component form. Now consider

$$\frac{d}{dt}(\underline{\omega} \cdot \underline{\overline{H}}) = \underline{\dot{\omega}} \cdot \underline{\overline{H}} + \underline{\omega} \cdot \frac{d\underline{\overline{H}}}{dt} = \underline{\overline{M}} \cdot \underline{\omega} + \underline{\overline{M}} \cdot \underline{\omega} = 2\underline{\overline{M}} \cdot \underline{\omega} \tag{11.40}$$

where Eqns. (11.38) and (11.39) were used. Thus

$$\begin{aligned}\underline{\overline{M}} \cdot \underline{\omega} &= \frac{d}{dt}\left(\frac{1}{2}\underline{\omega} \cdot \underline{\overline{H}}\right) \\ \int_{t_0}^{t_1} \underline{\overline{M}} \cdot \underline{\omega} dt &= \frac{1}{2}\underline{\omega} \cdot \underline{\overline{H}}\Big|_{t_0}^{t_1}\end{aligned} \tag{11.41}$$

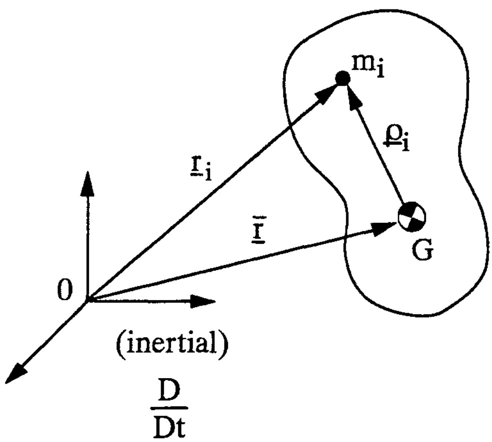


Fig. 11-20

Figure 11-20 shows a rigid body moving with respect to an inertial frame. Let particle i of the rigid body have mass m_i and position $\underline{r}_i$. The kinetic energy of the body is defined as the sum of the kinetic energies of the particles making up the rigid body:

$$T = \sum_i \frac{1}{2} m_i \nu_i^2 = \sum_i \frac{1}{2} m_i \underline{\nu}_i \cdot \underline{\nu}_i = \sum_i \frac{1}{2} m_i \frac{D\underline{r}_i}{Dt} \cdot \frac{D\underline{r}_i}{Dt} \tag{11.42}$$

From Fig. 11-20,

$$\underline{r}_i = \overline{\underline{r}} + \underline{\rho}_i \tag{11.43}$$

Substituting this into Eqn. (11.42),

$$T = \frac{1}{2} \sum_i m_i \frac{D\overline{\underline{r}}}{Dt} \cdot \frac{D\overline{\underline{r}}}{Dt} + \sum_i m_i \frac{D\overline{\underline{r}}}{Dt} \cdot \frac{D\underline{\rho}_i}{Dt} + \frac{1}{2} \sum_i m_i \frac{D\underline{\rho}_i}{Dt} \cdot \frac{D\underline{\rho}_i}{Dt} \tag{11.44}$$

But

$$\sum_i m_i \frac{D\overline{\underline{r}}}{Dt} \cdot \frac{D\underline{\rho}_i}{Dt} = \frac{D\overline{\underline{r}}}{Dt} \cdot \frac{D}{Dt} \left(\sum_i m_i \underline{\rho}_i \right) = 0$$

and

$$\frac{D\underline{\rho}_i}{Dt} = \underline{\omega} \times \underline{\rho}_i$$

so that Eqn. (11.44) becomes, using the triple scalar product again,

$$\begin{aligned} T &= \frac{1}{2} \sum_i m_i \overline{\underline{\nu}} \cdot \overline{\underline{\nu}} + \frac{1}{2} \sum_i m_i (\underline{\omega} \times \underline{\rho}_i) \cdot \frac{D\underline{\rho}_i}{Dt} \\ &= \frac{1}{2} m \overline{\underline{\nu}} \cdot \overline{\underline{\nu}} + \frac{1}{2} \sum_i m_i \underline{\omega} \cdot \left(\underline{\rho}_i \times \frac{D\underline{\rho}_i}{Dt} \right) \\ &= \frac{1}{2} m \overline{\underline{\nu}} \cdot \overline{\underline{\nu}} + \frac{1}{2} \underline{\omega} \cdot \sum_i m_i \underline{\rho}_i \times \frac{D\underline{\rho}_i}{Dt} \end{aligned}$$

$$\boxed{T = \frac{1}{2} \overline{\underline{\nu}} \cdot \overline{\underline{L}} + \frac{1}{2} \underline{\omega} \cdot \overline{\underline{H}}} \tag{11.45}$$

This equation accounts for the kinetic energy of all the mass elements that make up the rigid body.

Combining Eqns. (11.37), (11.41), and (11.45) gives the work-energy relation:

$$T \Big|_{t_0}^{t_1} = \int_{t_0}^{t_1} \underline{F}^e \cdot \overline{\underline{\nu}} dt + \int_{t_0}^{t_1} \overline{\underline{M}} \cdot \underline{\omega} dt$$

$$\Delta T = U \tag{11.46}$$

where

$$U = \int_{t_0}^{t_1} \underline{F}^e \cdot \underline{v} dt + \int_{t_0}^{t_1} \underline{\overline{M}} \cdot \underline{\omega} dt \tag{11.47}$$

is the work done by all external forces and moments.

As before if an external force is conservative then $\underline{F}^e(\underline{r}) = -\nabla V(\underline{r})$ and $U = -\Delta V$. Then Eqn. (11.46) may be written

$$\Delta T = U = U^n + U^c$$

$$\boxed{\Delta T + \Delta V = \Delta E = U^n} \tag{11.48}$$

where ΔV is the change in potential energy of all the conservative external forces and U^n is the work done by all the non-conservative external forces.

If all external forces are conservative and included in a potential energy function, then the work-energy equation becomes, as before,

$$\boxed{\Delta E = 0} \tag{11.49}$$

where $E = T + V$ is the *total mechanical energy.* In this case we say that *energy is conserved.*

11.15 Example – Sphere Rolling on Inclined Plane

A homogeneous sphere of radius R rolls on a plane inclined by an angle α to the horizontal (Fig. 11-21). For a sphere,

$$\overline{I}_{xx} = \overline{I}_{yy} = \overline{I}_{zz} = \frac{2}{5} m R^2 = I$$
$$\overline{I}_{xy} = \overline{I}_{yz} = \overline{I}_{zx} = 0$$

Let $\{\hat{I}, \hat{J}, \hat{K}\}$ be an inertial frame and let $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ be body-fixed, the latter not shown on Fig. 11-21 because any body-fixed frame with origin

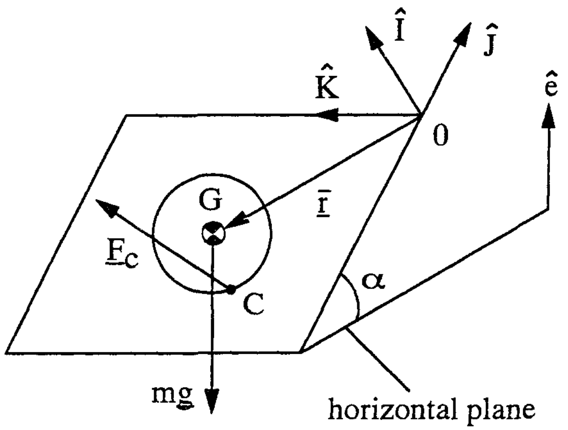


Fig. 11-21

at G will have the same moments and products of inertia. Point C is the contact point, the point where $\underline{F}_c$, the contact force acts.

The velocity of the center of mass is (see Fig. 11-22):

$$\underline{\overline{v}} = \frac{D\underline{\overline{r}}}{Dt} = \frac{D\underline{r}_c}{Dt} + \frac{D}{Dt}(R\overline{I}) = \frac{D}{Dt}(R\overline{i}) = \underline{\omega} \times R\hat{i} = \underline{\omega} \times R\hat{I}$$

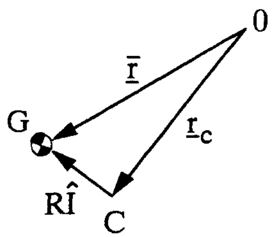


Fig. 11-22

The kinetic energy is obtained from Eqn. (11.45):

$$T = \frac{1}{2}m\underline{\overline{v}} \cdot \underline{\overline{v}} + \frac{1}{2}\underline{\overline{H}} \cdot \underline{\omega}$$

First consider $\underline{\overline{v}} \cdot \underline{\overline{v}}$; because $\underline{\omega}$ cannot have an $\hat{I}$ component,

$$\begin{aligned}
\underline{\overline{v}} \cdot \underline{\overline{v}} &= (\underline{\omega} \times R\hat{I}) \cdot (\underline{\omega} \times R\hat{I}) \\
&= [(\omega_y \hat{J} + \omega_z \hat{K}) \times R\hat{I}] \cdot [(\omega_y \hat{J} + \omega_z \hat{K}) \times R\hat{I}] \\
&= R^2(-\omega_y \hat{K} + \omega_z \hat{J}) \cdot (-\omega_y \hat{K} + \omega_z \hat{J}) = R^2(\omega_y^2 + \omega_z^2) = \omega^2 R^2
\end{aligned}$$

Next consider $\underline{\overline{H}} \cdot \underline{\omega}$:

$$\underline{\overline{H}} \cdot \underline{\omega} = [\overline{I}]\underline{\omega} \cdot \underline{\omega} = I\omega^2$$

Thus

$$\begin{aligned} T &= \frac{1}{2}m\omega^2 R^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}m\omega^2 R^2 + \frac{1}{2}\,\frac{2}{5}mR^2\omega^2 = \frac{7}{10}mR^2\omega^2 \\ &= \frac{7}{10}m\overline{\nu}^2 \end{aligned}$$

Next we compute the work done. The contact force does no work because it acts at the contact point, where the velocity is zero:

$$\int_{t_0}^{t_1} \underline{F}_c \cdot \underline{\nu} dt = 0$$

The only other external force is the weight, which is conservative; therefore

$$\Delta T + \Delta V_g = 0$$

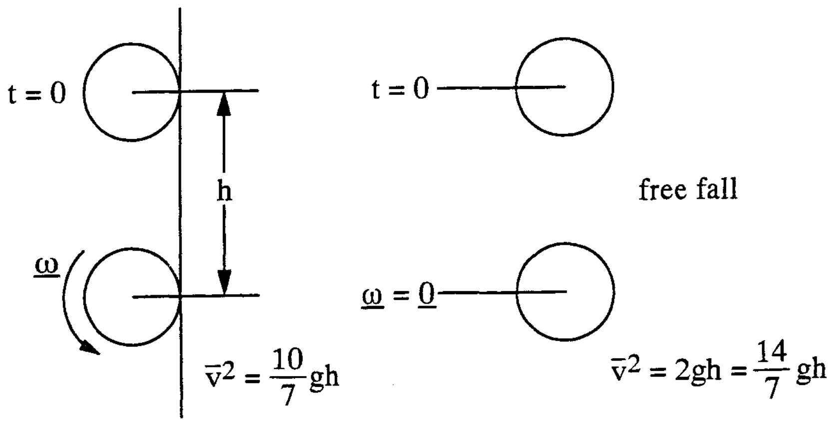


Fig. 11-23

Now suppose the sphere starts from rest at $t = 0$; then $\overline{\underline{\nu}}(0) = \underline{\omega}(0) = \underline{0}$ and

$$\begin{aligned} &T(t) - T(0) + V_g(t) - V_g(0) = 0 \\ &\frac{7}{10}m\overline{\nu}^2 - 0 + mgh_1 - mgh_0 = 0 \\ &\overline{\nu}^2 = \frac{10}{7}gh \end{aligned}$$

where $h = h_0 - h_1$ is the vertical height the sphere has fallen in time t.

As a special case, consider a vertical plane ($\alpha = 90°$), Fig. 11-23, and contrast the rolling sphere with a free-falling one. The body that doesn't roll attains a greater speed for a given vertical drop than the one that does. This is because of the potential energy available at $t = 0$, $5/7^{th}$ has gone in to translational kinetic energy and $2/7^{th}$ into rotational.

11.16 Special Cases

First, suppose the rigid body is in 2-D motion, say in the (x, y) plane. Then $\underline{\omega} = \omega\hat{k}$ and therefore

$$\begin{aligned} \overline{\underline{H}} &= \left[\overline{I}\right]\underline{\omega} = -\overline{I}_{xz}\omega\hat{\imath} - \overline{I}_{yz}\omega\hat{\jmath} + \overline{I}_{zz}\omega\hat{k} \\ \underline{\omega}\cdot\overline{\underline{H}} &= \overline{I}_{zz}\omega^2 \end{aligned} \tag{11.50}$$

so that Eqn. (11.45) becomes

$$\boxed{T = \frac{1}{2}m\overline{v}^2 + \frac{1}{2}\overline{I}_{zz}\omega^2} \tag{11.51}$$

As a second special case, suppose the rigid body moves such that one point, say B, is fixed in the inertial frame and take the origin of inertial- and body-fixed frames at that point. Then $\underline{r}_i$ is fixed in the body-fixed frame and, from Eqn. (11.42),

$$\begin{aligned} T &= \frac{1}{2}\sum_i m_i \frac{D\underline{r}_i}{Dt}\cdot\frac{D\underline{r}_i}{Dt} = \frac{1}{2}\sum_i m_i(\underline{\omega}\times\underline{r}_i)\cdot\frac{D\underline{r}_i}{Dt} \\ &= \frac{1}{2}\sum_i m_i\underline{\omega}\cdot\left(\underline{r}_i\times\frac{D\underline{r}_i}{Dt}\right) = \frac{1}{2}\underline{\omega}\cdot\sum_i m_i\underline{r}_i\times\frac{D\underline{r}_i}{Dt} \end{aligned}$$

$$\boxed{T = \frac{1}{2}\underline{\omega}\cdot\underline{H}_B} \tag{11.52}$$

where $\underline{H}_B = \sum_i m_i\underline{r}_i \times D\underline{r}_i/Dt$ is the angular momentum about origin B. Now let $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ be body-fixed principal axes of inertia with origin at B. Then $I_{xy} = I_{yz} = I_{zx} = 0$ and Eqn. (11.52) becomes

$$\boxed{T = \frac{1}{2}(I_{xx}\omega_x^2 + I_{yy}\omega_y^2 + I_{zz}\omega_z^2)} \tag{11.53}$$

where $\underline{\omega} = \omega_x\hat{\imath} + \omega_y\hat{\jmath} + \omega_z\hat{k}$.

Recall that gyroscopic motion is motion of a radially symmetric rigid body moving such that one point on the axis of symmetry remains fixed. In this case, Eqns. (10.17) apply and the components of $\underline{\omega}$ are given in terms of Euler's angles and their rates by Eqns. (10.19). Consequently Eqn. (11.53) becomes

$$\boxed{T = \frac{1}{2}\left[I_0(\dot{\theta}^2 + \dot{\psi}^2 \sin^2\theta) + I(\dot{\phi} + \dot{\psi}\cos\theta)^2\right]}$$

In the case of the heavy symmetrical top, Fig. 10-11, the contact force at B does no work, and the only other external force, the gravitational attraction, is conservative with potential energy function $V = -mg\overline{r}\cos\theta$. Thus energy is conserved and Eqn. (11.49) applies:

$$T + V = \text{constant}$$

$$\frac{1}{2}\left[I_0\left(\dot{\theta}^2 + \dot{\psi}^2\sin^2\theta\right) + I\left(\dot{\phi} + \dot{\psi}\cos\theta\right)^2\right] - mg\overline{r}\cos\theta = \text{constant}$$

This gives one relation between the Euler angles and their rates.

11.17 Example

Consider a physical pendulum free to rotate in the $\{\hat{\imath}, \hat{\jmath}\}$ plane, as shown on Fig. 11-24. The pendulum has mass m and moment of inertia about its center of mass of $\overline{I}$. The kinetic energy is determined from Eqn. (11.51) as

$$T = \frac{1}{2}m\overline{v}^2 + \frac{1}{2}\overline{I}\omega^2$$

Alternatively, we can use Eqn. (11.53) with $\omega_x = \omega_y = 0$:

$$T = \frac{1}{2}I_B\omega^2$$

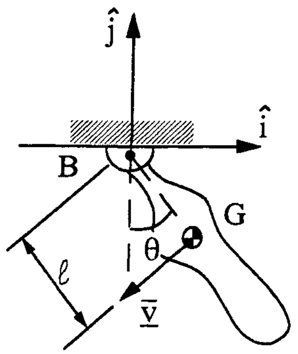


Fig. 11-24

Using the parallel axis theorem, $I_B = \overline{I} + m\ell^2$, and the relation $\overline{v} = \omega\ell$, it may be seen that these two expressions are equivalent:

$$T = \frac{1}{2}I_B\omega^2 = \frac{1}{2}(\overline{I} + m\ell^2)\omega^2 = \frac{1}{2}m\overline{v}^2 + \frac{1}{2}\overline{I}\omega^2$$

Because the gravitational force is the only force doing work, the energy relation gives

$$\frac{1}{2}(\overline{I} + m\ell^2)\omega^2 - mg\ell\cos\theta = \text{constant}$$

11.18 Systems of Rigid Bodies

Consider a system of rigid bodies connected in any way, Fig. 11-25. The energy equation is still Eqn. (11.48),

$$\boxed{\Delta T + \Delta V = U^n}$$

where now T is the sum of the kinetic energies of the individual bodies, V is the total potential energy of all the conservative forces acting on the bodies and U^n is the total work done by all the non-conservative forces. It is important to note that some of these forces way be internal to the system but external to an individual rigid body. For example,

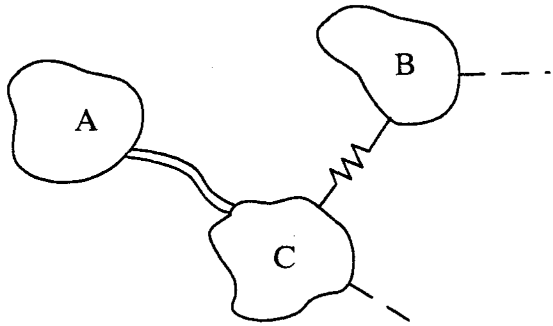


Fig. 11-25

in the three-bar linkage shown on Fig. 11-26 the spring produces forces internal to the system but its potential energy must be included in the energy equation.

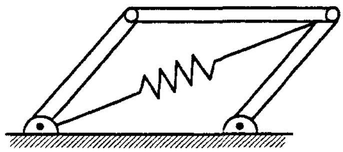


Fig. 11-26

11.19 Example – Car Accelerating Up a Hill

At a certain instant, a car of mass m has a velocity of $\underline{\overline{v}}$ and an acceleration $\underline{\overline{a}}$ while traveling up an incline of angle θ. Each of the four wheels has mass m_w, radius r, and radius of gyration k. We want to determine the power being delivered by the engine to the driving wheels at that instant. Friction is sufficient to prevent the wheels from slipping. This is a system of five rigid bodies.

We first conduct an experiment to determine the aerodynamic drag (wind resistance) on the car at speed $\overline{v}$. To this end, we tow the car at constant speed $\overline{v}$ on level ground, Fig. 11-27. For any wheel, $\overline{v} =$ constant so that $\dot{\omega} = 0$. Thus, for any wheel, Eqn. (8.19) gives $\overline{M} = F_i r = 0$ so that $F_i = 0$. Let $\underline{R}$ be the force in the tow rope. Then Newton's Second

Law for the car in the direction of travel is $R - D = m\dot{\overline{v}} = 0$ and $R = D$; thus the force measured in the tow rope is exactly the drag.

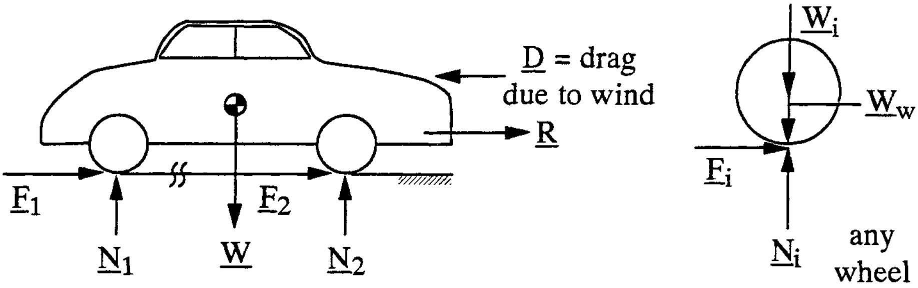


Fig. 11-27

Now consider the car accelerating up the hill (Fig. 11-28). From Eqn. (11.51) the kinetic energy is, using $\overline{v} = r\omega$,

$$T = \frac{1}{2}m\overline{v}^2 + 4\frac{1}{2}\overline{I}\left(\frac{\overline{v}}{r}\right)$$

and the potential energy is

$$V = Ws\sin\theta$$

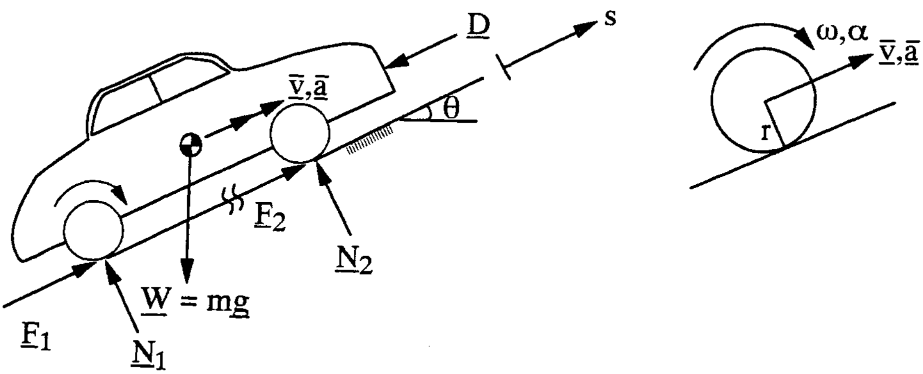


Fig. 11-28

The work done by the drag force is

$$U^D = \int \underline{D} \cdot \underline{v} dt = -\int D\overline{v} dt = -\int R\overline{v} dt$$

Now differentiate the work-energy equation and substitute:

$$\begin{aligned}
\dot{U}^n &= \dot{T} + \dot{V} \\
P_e + \dot{U}^D &= \dot{T} + \dot{V} \\
P_e - R\overline{v} &= \frac{1}{2} m 2\overline{v}\overline{a} + 4\frac{1}{2}\overline{I}\frac{2\overline{v}\overline{a}}{r^2} + mg\sin\theta\overline{v} \\
P_e &= m\overline{v}\overline{a} + \frac{4m_w k^2}{r^2}\overline{v}\overline{a} + mg\sin\theta\overline{v} + R\overline{v}
\end{aligned}$$

where $m = m_{\text{body}} + 4m_w$ and P_e is the power output of the engine. This shows that the power delivered by the engine to the driving wheels[2] is used in four separate ways: (1) to accelerate the car, (2) to spin-up the wheels, (3) to gain altitude, and (4) to overcome air resistance.

We have now seen three reasons why it is particularly important to reduce the weight of wheels and tires on cars and other wheeled vehicles: (1) The forces due to static and dynamic unbalance are proportional to this weight (Chapter 9), (2) The over-turning gyroscopic moment is proportional to this weight (Chapter 10), and (3) It takes energy to spin up this weight and this energy is wasted in braking.[3] (The main reason for reducing this weight, however, is to reduce the "unsprung weight" which improves response and handling.)

Notes

1 Recall again that $DQ/Dt = dQ/dt$ for Q a scalar.

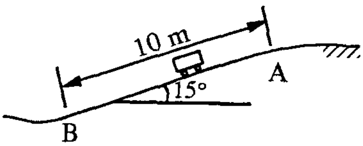
2 This is not, of course, the power available from the fuel. There are significant power losses due to thermal inefficiency, friction losses in the drive train, and power needs for accessories.

3 The current generation of "hybrid" cars have systems that transform the energy of the rotating wheels during braking into electrical energy.

Problems

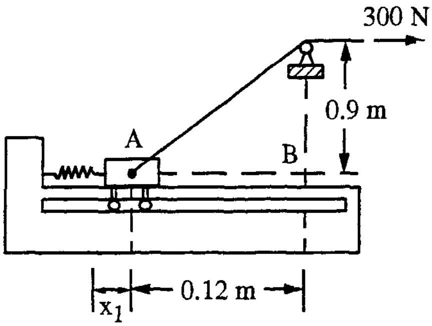
11/1 A 50 kg cart slides down an incline from A to B as shown. What is the speed of the cart at the bottom at B if it starts at the top at

A with a speed of 4 m/s? The coefficient of kinetic friction is 0.30.

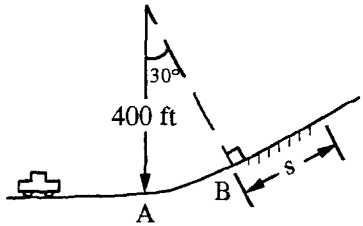


Problem 11/1

11/2 A 50 kg block slides without friction as shown. There is a constant force of 300 N in the cable and the spring attached to the block has stiffness 80 N/m. If the block is released from rest at a position A in which the spring is stretched by amount $x_1 = 0.233$ m, what is the speed when the block reaches position B.



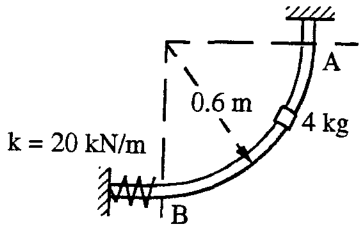
Problem 11/2



Problem 11/3

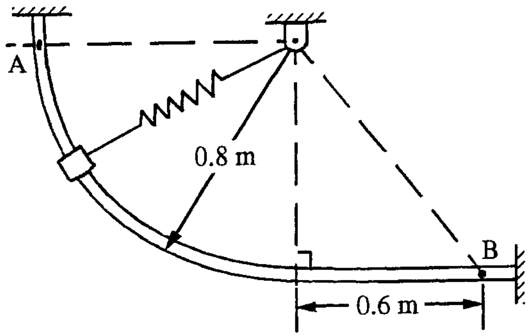
11/3 A 4000 lb car travels up a hill as shown. The car starts from rest at A and the engine exerts a constant force in the direction of travel of 1000 lb until position B is reached, at which time the engine is shut off. How far does the car roll up the hill before stopping? Neglect all friction and air resistance.

11/4 The small 4 kg collar is released from rest at A and slides down the circular rod in the vertical plane. Find the speed of the collar as it reaches the bottom at B and the maximum compression of the spring. Neglect friction.



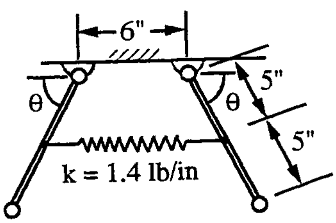
Problem 11/4

11/5 The small 3 kg collar is released from rest at A and slides in the vertical plane to B. The attached spring has stiffness 200 N/m and an unstretched length of 0.4 m. What is the speed of the collar at B? Neglect friction.



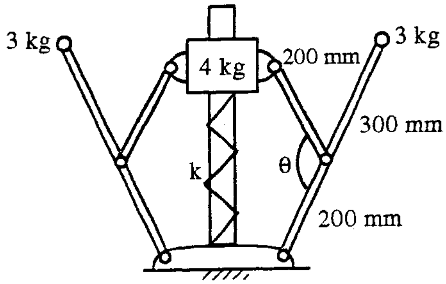
Problem 11/5

11/6 The identical links are released simultaneously from rest at $\theta = 30°$ and rotate in the vertical plane. Find the speed of each 2 lb mass when $\theta = 90°$. The spring is unstretched when $\theta = 90°$. Ignore the mass of the links and model the masses as particles.



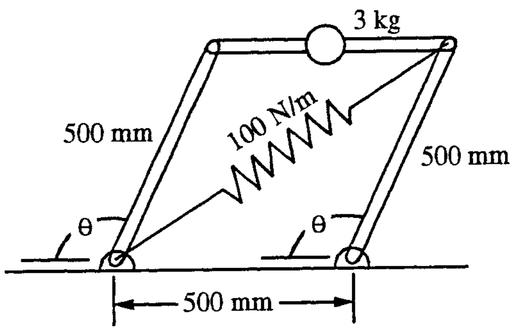
Problem 11/6

11/7 The device shown is released from rest with $\theta = 180°$ and moves in the vertical plane. The spring has stiffness 900 N/m and is just touching the underside of the collar when $\theta = 180°$. Determine the angle θ when the spring reaches maximum compression. Neglect the masses of the links and all friction.



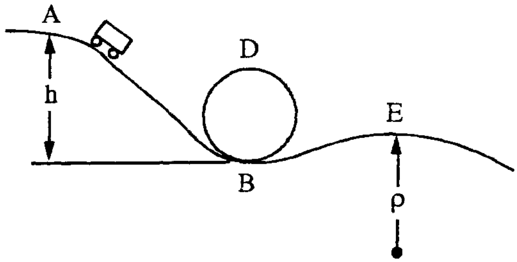
Problem 11/7

11/8 Shown is a frame of negligible weight and friction that rotates in the vertical plane and carries a 3 kg mass. The spring is unstretched when $\theta = 90°$. If the frame is released from rest at $\theta = 90°$, determine the speed of the mass when $\theta = 135°$ is passed.



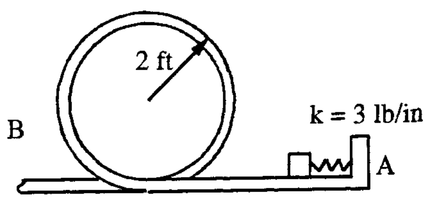
Problem 11/8

11/9 A roller coaster car starts from rest at A, rolls down the track to B, transits a circular loop of 40 ft diameter, and then moves over the hump at E. If $h = 60$ ft, determine (a) the force exerted by his or her seat on a 160 lb rider at both B and D, and (b) the minimum value of the radius of curvature of E if the car is not to leave the track at that point. Neglect all friction and air resistance.

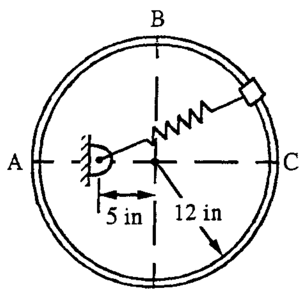


Problem 11/9

11/10 The 0.5 lb pellet is pushed against the spring at A and released from rest. Determine the smallest deflection of the spring for which the pellet will remain in contact with the circular loop at all times. Neglect friction and air resistance.



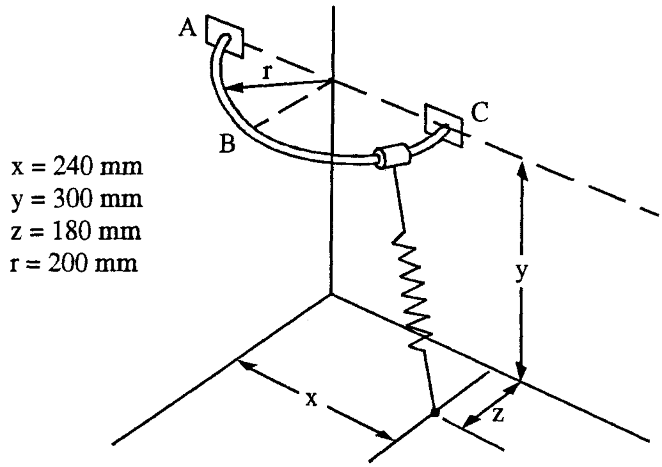
Problem 11/10



Problem 11/11

11/11 A 3 lb collar is attached to a spring and slides without friction on a circular hoop in a horizontal plane. The spring constant is 1.5 lb/in and is undeformed when the collar is at A. If the collar is released at C with speed 6 ft/s, find the speeds of the collar as it passes through points B and A.

11/12 A 600 g collar slides without friction on a horizontal semicircular rod ABC of radius 200 mm and is attached to a spring of spring constant 135 N/m and undeformed length 250 mm. If the collar is released from rest at A, what are the speeds of the collar at B and C?



Problem 11/12

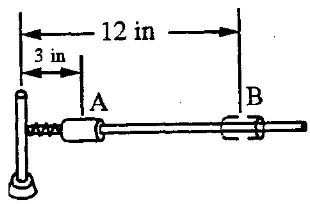
11/13 Prove that a force is conservative if and only if the following relations are satisfied:

$$\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x}, \qquad \frac{\partial F_y}{\partial z} = \frac{\partial F_z}{\partial y}, \qquad \frac{\partial F_z}{\partial x} = \frac{\partial F_x}{\partial z}$$

11/14. Show that the force

$$\underline{F} = (x\hat{\imath} + y\hat{\jmath} + z\hat{k})(x^2 + y^2 + z^2)$$

is conservative by applying the results of Problem 11/13. Also find the potential energy function $V(x, y, z)$ associated with $\underline{F}$.

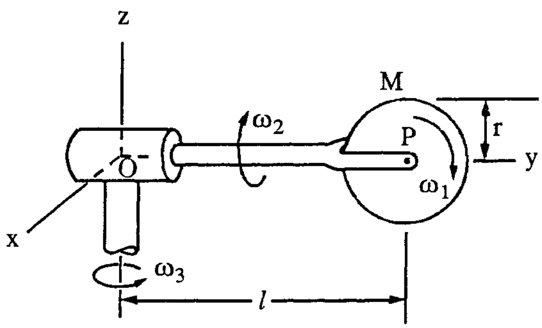


Problem 11/15

11/15 A 1/2 lb collar slides without friction on a horizontal rod which rotates about a vertical shaft. The collar is initially held in position

A against a spring of spring constant 2.5 lb/ft and unstretched length 9 in. As the rod is rotating at angular speed 12 rad/s the cord is cut, releasing the collar to slide along the rod. The spring is attached to the collar and the rod. Find the angular speed of the rod and the radial and transverse components of the velocity of the collar as the rod passes postion B. Also find the maximum distance from the vertical shaft that the collar will reach.

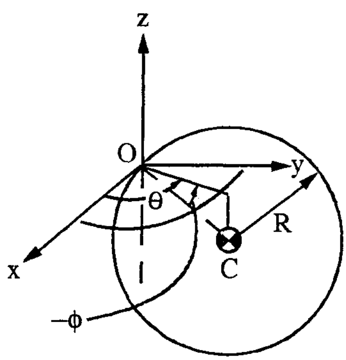
11/16 Determine the kinetic energy of the uniform circular disk of mass M at the instant shown.



Problem 11/16

11/17 Find the kinetic energy of a homogeneous solid disk of mass m and radius r that rolls without slipping along a straight line. The center of the disk moves with constant velocity ν.

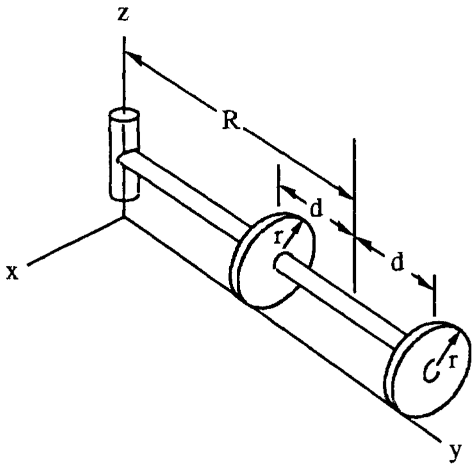
11/18 A homogeneous solid sphere of mass M and radius R is fixed at a point O on its surface by a ball joint. Find the kinetic energy of the sphere for general motion.



Problem 11/18

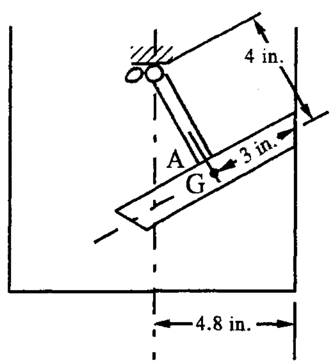
11/19 Two uniform circular disks, each of mass M and radius r, are mounted on the same shaft as shown. The shaft turns about the z-axis, while the two disks roll on the xy-plane without slipping. Prove that the ratio of the kinetic energies of the two disks is

$$\frac{6(R+d)^2+r^2}{6(R-d)^2+r^2}$$



Problem 11/19

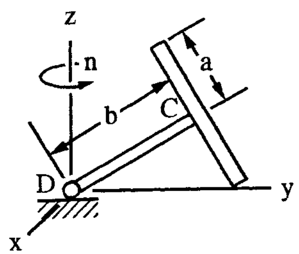
11/20 A disk with arm OA is attached to a socket joint at O. The moment of inertia of the disk and arm about axis OA is I and the total



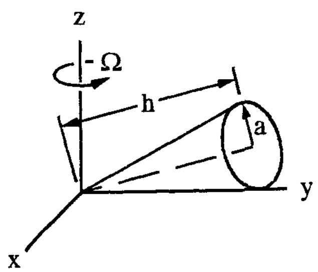
Problem 11/20

mass is M, with the center of gravity at G. The disk rolls inside a cylinder whose radius is 4.8 in. Find the kinetic energy of the disk when the line of contact turns around the cylinder at 10 cycles per second.

11/21 A uniform circular disk of radius a and mass M is mounted on a weightless shaft CD of length b. The shaft is normal to the disk at its center C. The disk rolls on the xy-plane without slipping, with point D remaining at the origin. Determine the kinetic energy of the disk if shaft CD rotates about the z-axis with constant angular speed n.



Problem 11/21



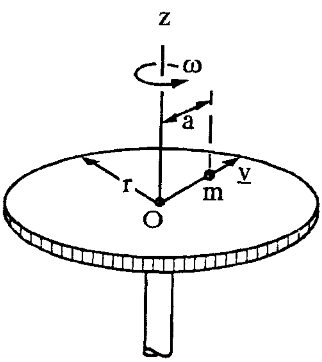
Problem 1/22

11/22 A homogeneous solid right circular cone rolls on a plane without slipping. The line of contact turns at constant angular speed Ω about the z-axis. Find the kinetic energy of the cone.

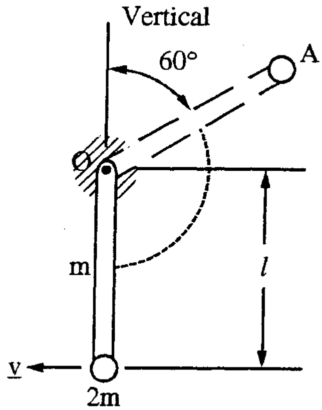
11/23 A particle of mass m slides along one radius of a circular platform of mass M. At the instant shown, the platform has an angular velocity ω and the particle has a speed ν relative to the platform. Determine the kinetic energy and the angular momentum of the system about point O.

11/24 A pendulum consists of a uniform rod of mass m and a bob of mass $2m$. The pendulum is released from rest at position A as shown. What is the kinetic energy of the system at the lowest position? What is the velocity of the bob at the lowest position?

11/25 A particle of mass m is attracted toward the origin by a force with magnitude $(mK)/r^2$ where K is a constant and r is the distance between the particle and the center of attraction. The particle is



Problem 11/23

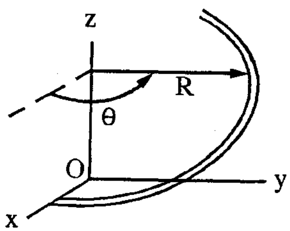


Problem 11/24

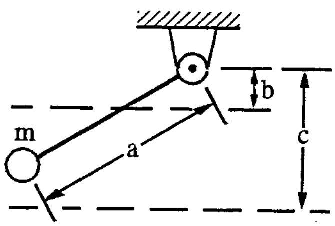
constrained to move in a frictionless tube which lies along the space curve given by

$$\left.\begin{aligned} z &= 5\theta \\ R &= 1 + \frac{1}{2}\theta \end{aligned}\right\} \quad \text{in cylindrical coordinates}$$

If the particle was at rest when $z = 10$, what is the velocity of the particle at $z = 0$?



Problem 11/25

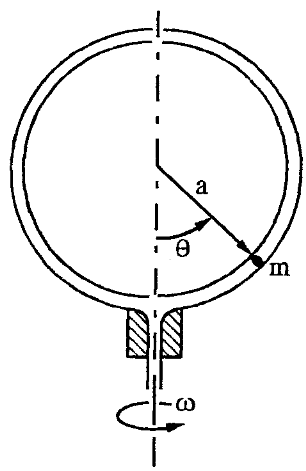


Problem 11/26

11/26 A spherical pendulum of mass m and length a oscillates between levels b and c, located below the support. Find the expression for the total energy of the system in terms of a, b, c, m, and g, taking the horizontal plane passing through the support as the zero potential energy level.

11/27 A spherical pendulum, consisting of a massless rod and a bob of mass m, is initially held at rest in the horizontal plane. A horizontal velocity ν_0 is imparted to the bob normal to the rod. In the resulting motion, what is the angle between the rod and the horizontal plane when the bob is at its lowest position?

11/28 A particle of mass m is placed inside a frictionless tube of negligible mass. The tube is bent into a circular ring with the lowest point left open as shown. The ring is given an initial angular velocity ω about the vertical axis passing through the diameter containing the opening, and simultaneously the particle is released from rest (relative to the tube) at $\theta = \pi/2$. In the subsequent motion, will the particle drop through the opening?



Problem 11/28

Appendix A – Review of Vector Algebra and Derivatives of Vectors

A *physical vector* is a *three-dimensional mathematical object having both magnitude and direction*, and may be conveniently represented as a directed line segment in three-dimensional space. Figure A-1 shows a vector $\underline{P}$ and a reference frame characterized by a set of right-hand orthogonal unit vectors $\{\hat{i}, \hat{j}, \hat{k}\}$.[1] It is clear that the magnitude of $\underline{P}$ (its length) is independent of the choice of reference frame, but that it's direction depends on this choice and will generally be different for different frames.

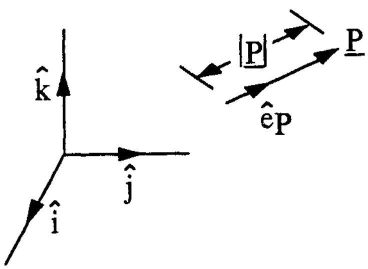


Fig. A-1

Two vectors are equal if and only if they have the same magnitude and direction. This allows us to write

$$\underline{P} = |\underline{P}|\hat{e}_P \qquad (A.1)$$

where $|\underline{P}|$ is the magnitude (length) of $\underline{P}$ and $\hat{e}_P$ is a unit vector in its direction. It is easy to verify that the vectors on both sides of Eqn.

(A.1) have the same magnitude and direction. We can now state that if $\underline{P} = |\underline{P}|\hat{e}_P$ and $\underline{Q} = |\underline{Q}|\hat{e}_Q$ are two vectors, then they are equal if both $|\underline{P}| = |\underline{Q}|$ and $\hat{e}_P = \hat{e}_Q$ (Fig. A-2). Note that the "point of application" of a vector is not a part of its definition. Thus, $\underline{P} = \underline{Q}$ regardless of their points of application.

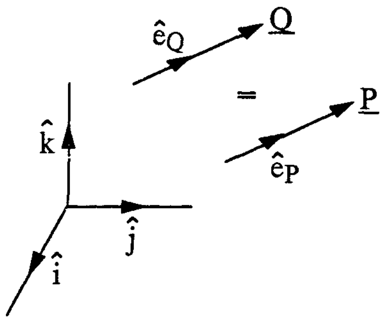


Fig. A-2

The sum of two vectors $\underline{P}$ and $\underline{Q}$, $\underline{R} = \underline{P} + \underline{Q}$, is defined by the parallelogram rule, or equivalently by placing the vectors "tip-to-tail" (Fig. A-3).

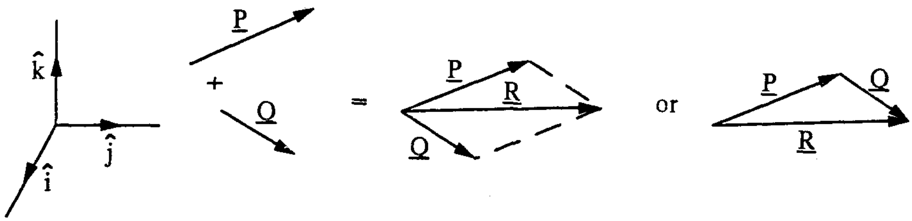


Fig. A-3

Two vectors may be multiplied in two different ways. The *scalar* or *dot product*, denoted $\underline{P} \cdot \underline{Q}$, is defined as the magnitude of $\underline{P}$ times the magnitude of the projection of $\underline{Q}$ along $\underline{P}$, or vice-versa. Thus

$$\underline{P} \cdot \underline{Q} = |\underline{P}|\ |\underline{Q}| \cos\theta \tag{A.2}$$

where θ is the angle between $\underline{P}$ and $\underline{Q}$ (see Fig. A-4). It is obvious that $\underline{P} \cdot \underline{Q} = \underline{Q} \cdot \underline{P}$. If $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ are orthogonal unit vectors,

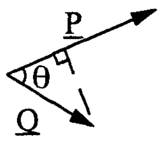


Fig. A-4

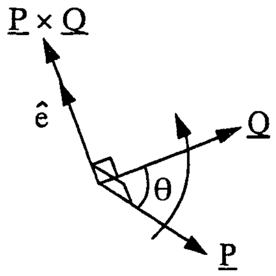


Fig. A-5

$$
\begin{aligned}
\hat{\imath} \cdot \hat{\imath} = \hat{\jmath} \cdot \hat{\jmath} = \hat{k} \cdot \hat{k} = 1 \\
\hat{\imath} \cdot \hat{\jmath} = \hat{\jmath} \cdot \hat{k} = \hat{k} \cdot \hat{\imath} = 0
\end{aligned}
\tag{A.3}
$$

The *vector* or *cross product* is defined as

$$
\underline{P} \times \underline{Q} = \Big(|\underline{P}|\ |\underline{Q}| \sin\theta\Big)\, \hat{e} \tag{A.4}
$$

where $\hat{e}$ is a unit vector perpendicular to both $\underline{P}$ and $\underline{Q}$ and directed in such a way that $\underline{P}$, $\underline{Q}$, and $\hat{e}$ form a right-hand triad in that order (Fig. A-5). It is clear that $\underline{P} \times \underline{Q} = -\underline{Q} \times \underline{P}$. For the right-hand triad of orthogonal unit vectors $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$,

$$
\begin{aligned}
&\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = \underline{0} \\
&\hat{\imath} \times \hat{\jmath} = -\hat{\jmath} \times \hat{\imath} = \hat{k} \\
&\hat{\jmath} \times \hat{k} = -\hat{k} \times \hat{\jmath} = \hat{\imath} \\
&\hat{k} \times \hat{\imath} = -\hat{\imath} \times \hat{k} = \hat{\jmath}
\end{aligned}
\tag{A.5}
$$

For the purpose of calculation, it is convenient to resolve vectors into components in specific directions. The vector $\underline{P}$ resolved into components along $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ is

$$
\underline{P} = P_x \hat{\imath} + P_y \hat{\jmath} + P_z \hat{k} \tag{A.6}
$$

P_x, P_y, and P_z are called the rectangular components of $\underline{P}$. Using Eqns. (A.3), they are

$$\begin{aligned} P_x &= \underline{P} \cdot \hat{\imath} \\ P_y &= \underline{P} \cdot \hat{\jmath} \\ P_z &= \underline{P} \cdot \hat{k} \end{aligned} \tag{A.7}$$

Equation (A.6) represents $\underline{P}$ as the sum of three vectors (Fig. A-6).

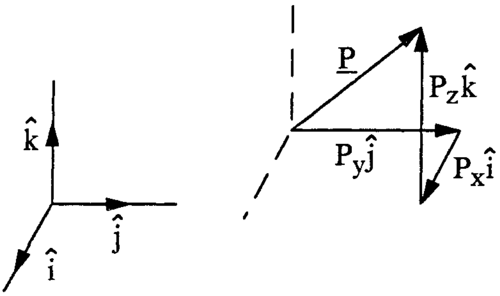


Fig. A-6

Now suppose vector $\underline{Q}$ is also resolved into components along $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$:

$$\underline{Q} = Q_x\hat{\imath} + Q_y\hat{\jmath} + Q_z\hat{k} \tag{A.8}$$

Adding $\underline{Q}$ and $\underline{P}$ gives

$$\begin{aligned} &\underline{R} = \underline{P} + \underline{Q} \\ &R_x\hat{\imath} + R_y\hat{\jmath} + R_z\hat{k} = (P_x + Q_x)\hat{\imath} + (P_y + Q_y)\hat{\jmath} + (P_z + Q_z)\hat{k} \end{aligned} \tag{A.9}$$

which implies

$$R_x = P_x + Q_x\ , \qquad R_y = P_y + Q_y\ , \qquad R_z = P_z + Q_z \tag{A.10}$$

Using Eqns. (A-3), the dot product of two vectors in component form is

$$\underline{P} \cdot \underline{Q} = P_xQ_x + P_yQ_y + P_zQ_z \tag{A.11}$$

In particular,

$$\underline{P} \cdot \underline{P} = P_x^2 + P_y^2 + P_z^2 = |\underline{P}|^2 \tag{A.12}$$

Similarly, using Eqns. (A.5), the cross product of $\underline{P}$ and $\underline{Q}$ is given by

$$\underline{P}\times\underline{Q} = (P_yQ_z - P_zQ_y)\,\hat{\imath}+(P_zQ_x - P_xQ_z)\,\hat{\jmath}+(P_xQ_y - P_yQ_x)\,\hat{k} \quad (A.13)$$

which is conveniently expressed in determinant form:

$$\underline{P}\times\underline{Q} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix} \quad (A.14)$$

Suppose $\underline{P}$, $\underline{Q}$, and $\underline{R}$ are three vectors. The following identities are then valid:

$$\underline{P}\cdot\underline{Q}\times\underline{R} = \underline{Q}\cdot\underline{R}\times\underline{P} = \underline{R}\cdot\underline{P}\times\underline{Q} \quad (A.15)$$

$$(\underline{P}\times\underline{Q})\times\underline{R} = \underline{R}\cdot\underline{P}\underline{Q} - \underline{R}\cdot\underline{Q}\underline{P} \quad (A.16)$$

$\underline{P}\cdot\underline{Q}\times\underline{R}$ is called the *triple scalar product* and $(\underline{P}\times\underline{Q})\times\underline{R}$ is the *triple vector product.*

The derivatives of vectors follow rules analogous to those for scalars. Let $\underline{P}(t)$ and $\underline{Q}(t)$ be vector functions of t and let $u(t)$ be a scalar function of t. Then

$$\begin{aligned} \frac{d\underline{P}(t)}{dt} &= \lim_{\Delta t\to 0}\frac{\underline{P}(t+\Delta t)-\underline{P}(t)}{\Delta t} = \lim_{\Delta t\to 0}\frac{\Delta\underline{P}(t)}{\Delta t} \\ \frac{d\underline{Q}(t)}{dt} &= \lim_{\Delta t\to 0}\frac{\underline{Q}(t+\Delta t)-\underline{Q}(t)}{\Delta t} = \lim_{\Delta t\to 0}\frac{\Delta\underline{Q}(t)}{\Delta t} \end{aligned} \quad (A.17)$$

and the following rules apply

$$\frac{d(\underline{P}+\underline{Q})}{dt} = \frac{d\underline{P}}{dt} + \frac{d\underline{Q}}{dt} \quad (A.18)$$

$$\frac{d(u\underline{P})}{dt} = u\frac{d\underline{P}}{dt} + \underline{P}\frac{du}{dt} \quad (A.19)$$

$$\frac{d(\underline{P}\cdot\underline{Q})}{dt} = \underline{P}\cdot\frac{d\underline{Q}}{dt} + \underline{Q}\cdot\frac{d\underline{P}}{dt} \quad (A.20)$$

$$\frac{d(\underline{P}\times\underline{Q})}{dt} = \underline{P}\times\frac{d\underline{Q}}{dt} + \frac{d\underline{P}}{dt}\times\underline{Q} \quad (A.21)$$

It must be kept in mind that the derivative of a vector is reference frame dependent.[2]

We recall that the value of a scalar function and of its derivative is independent of the choice of reference frame. This is not true, however, for a vector. The derivatives of a vector with respect to two reference frames, say $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ and $\{\hat{\imath}', \hat{\jmath}', \hat{k}'\}$, are equal only if these reference frames are fixed relative to one another, that is, if $\hat{\imath}'$, $\hat{\jmath}'$ and $\hat{k}'$ are constant vectors in $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$.

Also needed is the *gradient vector.* If $f(x, y, z)$ is any function depending on the rectangular coordinates x, y and z, then the gradient of f is given in rectangular components as

$$\nabla f = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k} \qquad (A.22)$$

We conclude this Appendix with three examples.

Example 1. A unit vector $\hat{e}$ rotates in a plane with rate $\dot{\theta}$ relative to orthogonal unit vectors $\{\hat{\imath}, \hat{\jmath}\}$. We wish to show that $\dot{\hat{e}}$ is perpendicular to $\hat{e}$. From Fig. A-7,

$$\hat{e} = \cos\theta\hat{\imath} + \sin\theta\hat{\jmath}$$

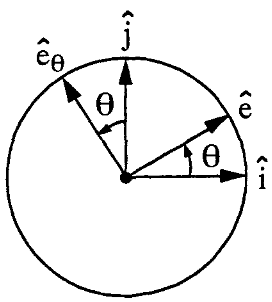


Fig. A-7

Differentiating,

$$\dot{\hat{e}} = (-\sin\theta\hat{\imath} + \cos\theta\hat{\jmath})\dot{\theta}$$

We have

$$\hat{e} \cdot \dot{\hat{e}} = 0$$

so that $\dot{\hat{e}}$ is perpendicular to $\hat{e}$. Furthermore, the vector $\hat{e}_\theta = (-\sin\theta\hat{\imath} + \cos\theta\hat{\jmath})$ is a unit vector; thus we may write

$$\dot{\hat{e}} = \dot{\theta}\hat{e}_\theta$$

This relation is used many times in this book.

Example 2. Referring to Fig. A-8, we wish to find the unit vectors along $\vec{AB}$ and $\vec{AC}$, and the unit vector perpendicular to the plane containing A, B, and C. We have

$$\vec{AB} = (4-1)\hat{\imath} + (5-1)\hat{\jmath} + (6-1)\hat{k} = 3\hat{\imath} + 4\hat{\jmath} + 5\hat{k}$$

$$\vec{AC} = (2-1)\hat{\imath} + (3-1)\hat{\jmath} + (7-1)\hat{k} = \hat{\imath} + 2\hat{\jmath} + 6\hat{k}$$

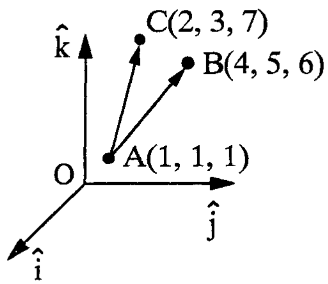


Fig. A-8

The unit vectors along $\vec{AB}$ and $\vec{AC}$ are then, respectively,

$$\frac{\vec{AB}}{|\vec{AB}|} = \frac{3\hat{\imath} + 4\hat{\jmath} + 5\hat{k}}{\sqrt{3^2+4^2+5^2}} = \frac{1}{5\sqrt{2}}\left(3\hat{\imath} + 4\hat{\jmath} + 5\hat{k}\right)$$

$$\frac{\vec{AC}}{|\vec{AC}|} = \frac{\hat{\imath} + 2\hat{\jmath} + 6\hat{k}}{\sqrt{1+4+36}} = \frac{1}{\sqrt{41}}\left(\hat{\imath} + 2\hat{\jmath} + 6\hat{k}\right)$$

A vector perpendicular to $\vec{AB}$ and $\vec{AC}$ is

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & 4 & 5 \\ 1 & 2 & 6 \end{vmatrix} = 14\hat{\imath} - 13\hat{\jmath} + 2\hat{k}$$

Thus the unit vector perpendicular to $\vec{AB}$ and $\vec{AC}$ is

$$\frac{\vec{AB} \times \vec{AC}}{|\vec{AB} \times \vec{AC}|} = \frac{1}{\sqrt{369}}(14\hat{\imath} - 13\hat{\jmath} + 2\hat{k})$$

or

$$-\frac{1}{\sqrt{369}}(14\hat{\imath} - 13\hat{\jmath} + 2\hat{k})$$

Example 3. We wish to find the equation of the plane containing points A(1,1,1), B(4,5,6), and C(2,3,7). Two vectors that lie in this plane are

$$\vec{AB} = (4-1)\hat{\imath} + (5-1)\hat{\jmath} + (6-1)\hat{k} = 3\hat{\imath} + 4\hat{\jmath} + 5\hat{k}$$
$$\vec{AC} = \hat{\imath} + 2\hat{\jmath} + 6\hat{k}$$

A vector normal to this plane is

$$\underline{N} = \vec{AB} \times \vec{AC} = 14\hat{\imath} - 13\hat{\jmath} + 2\hat{k}$$

For any point $P(x, y, z)$ in the plane,

$$\vec{AP} \cdot \underline{N} = 0$$
$$\left[(x-1)\hat{\imath} + (y-1)\hat{\jmath} + (z-1)\hat{k}\right] \cdot \left(14\hat{\imath} + 13\hat{\jmath} + 2\hat{k}\right) = 0$$
$$14x - 13y + 2z - 3 = 0$$

which is the equation of the plane.

Notes

1 In this Appendix, as in the rest of the book, vectors will underlined except that unit vectors will be denoted by (̂).

2 Section 1.2 gives several examples in which the velocity and acceleration vectors of a point are different with respect to different frames.

Problems

A/1 Given an orthogonal triad of unit vectors $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$, express the position vectors $\underline{r}_A$ and $\underline{r}_B$ with respect to the origin for the points $A(3, -2, 5)$ and $B(1, 4, -2)$. What is the position of B with respect to A?

A/2 A vector starts at the origin and extends outward through the point $(4, -1, 3)$ to a total length of 10 units. Give an expression for the vector.

A/3 The sum of the two vectors $\underline{P}$ and $\underline{Q}$ is given by

$$\underline{R} = \underline{P} + \underline{Q}$$

If $\underline{P} = 30\hat{\imath} + 40\hat{\jmath}$ and $\underline{Q} = 20\hat{\imath} - 20\hat{\jmath}$, find the vector $\underline{R}$. What angle does $\underline{R}$ make with the direction of the unit vector $\hat{\imath}$?

A/4 Given: $\underline{P} = 30\hat{\imath} + 40\hat{\jmath} - 7\hat{k}$

$\underline{Q} = 6\hat{\imath} - 5\hat{\jmath}$

Determine: (a) $\underline{P} + \underline{Q}$

(b) $\underline{P} - \underline{Q}$

(c) $\underline{P} \cdot \underline{Q}$

(d) $(\underline{P} + \underline{Q}) \cdot (\underline{P} - \underline{Q})$

A/5 Given a vector $\vec{PQ}$ defined by the line segment beginning at point $P(-3, 1, 5)$ and terminating at $Q(4, 0, -2)$, express $\vec{PQ}$ in terms of the rectangular unit vectors. Find the commponent of $\vec{PQ}$ in the direction defined by a line passing through the points $(3, 1, -1)$ and $(-1, 2, 7)$.

A/6 For the right-handed rectangular triad $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$, show that

$$\underline{A} \times \underline{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

where $\underline{A} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and $\underline{B} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$.

A/7 For the right-handed rectangular triad $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$, show that $\hat{\imath} \times (\hat{\imath} \times \hat{\jmath}) = -\hat{\jmath}$.

A/8 Show that the time derivative of the vector $\dot{\hat{e}}$ of Example 1 can be represented by the vector $-\omega^2\hat{e}$, if $\dot{\theta} = \omega$ is constant.

A/9 A vector is given by the following expression:

$$\underline{v}(t) = 2t^2\hat{\imath} + (3t^3 - 1)\hat{\jmath} - (t - t^4)\hat{k}$$

Find the time derivative of this vector as a function of time t. What is the value of this derivative at $t = 2$?

A/10 Verify Eqns. (A.18) – (A.21) where $\underline{A} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and $\underline{B} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$, and where u, a_i, $b_i (i = 1, 2, 3)$ are functions of t.

A/11 Find the gradient of each of the following scalar functions of x, y, and z:
(a) $V = 6x + 9y - 6z$
(b) $V = -Ax^4y$
(c) $V = Axy + Byz + Cxz$
where A, B, and C are constants.

A/12 Find the gradient of the function

$$V = (\underline{A} \cdot \underline{B}) \quad \text{at the point}(1, -1, 2)$$

where $\underline{A} = xy\hat{\imath} + yz\hat{\jmath} + zx\hat{k}$ and $\underline{B} = y\hat{\imath} + z\hat{\jmath} + x\hat{k}$.

A/13 Find the vector projection of $\underline{B}$ onto $\underline{A}$ if $\underline{A} = 2\hat{\imath} + 3\hat{\jmath} - \hat{k}$ and $\underline{B} = 3\hat{\imath} - \hat{\jmath} + 2\hat{k}$.

A/14 Given vectors $\underline{A}(a_1, a_2, a_3)$ and $\underline{B}(b_1, b_2, b_3)$, find the magnitude of vector $\underline{C} = \underline{B} - \underline{A}$.

A/15 Do the following vectors $\underline{A}$, $\underline{B}$, and $\underline{C}$ form a right-handed triad?

$$\begin{aligned} \underline{A} &= 2\hat{\imath} - \hat{\jmath} + \hat{k} \\ \underline{B} &= -\hat{\imath} + 2\hat{\jmath} + 2\hat{k} \\ \underline{C} &= \hat{\imath} + \hat{\jmath} - \hat{k} \end{aligned}$$

A/16 Choose x, y, and z such that $\hat{\imath} + \hat{\jmath} + 2\hat{k}$, $-\hat{\imath} + z\hat{k}$, and $2\hat{\imath} + x\hat{\jmath} + y\hat{k}$ are mutually orthogonal.

A/17 If $\underline{\nu} = \underline{\omega} \times \underline{R}$, where $\underline{\omega} = t\hat{\imath} + t^2\hat{\jmath} + 3\hat{k}$ and $\underline{R} = 3t\hat{\imath} + t^3\hat{\jmath} + t^5\hat{k}$, find $d\underline{\nu}/dt$.

A/18 For any vector $\underline{R}$, prove that

$$\frac{d}{dt}\left(\underline{R} \times \frac{d\underline{R}}{dt}\right) = \underline{R} \times \frac{d^2\underline{R}}{dt^2}$$

A/19 Find the unit vector which is perpendicular to both $3\hat{\imath} - 2\hat{\jmath} + \hat{k}$ and $\hat{\imath} + \hat{\jmath} - 2\hat{k}$.

A/20 Prove that $\underline{A} \times (\underline{B} \times \underline{C}) + \underline{B} \times (\underline{C} \times \underline{A}) + \underline{C} \times (\underline{A} \times \underline{B}) = \underline{0}$ for any three vectors $\underline{A}$, $\underline{B}$, and $\underline{C}$.

A/21 If $\underline{A} = \hat{\imath} + \hat{\jmath} - \hat{k}$, $\underline{B} = 2\hat{\imath} + \hat{\jmath} + \hat{k}$, and $\underline{C} = -\hat{\imath} - 2\hat{\jmath} + 3\hat{k}$, find

(a) $\underline{A} \times (\underline{B} \times \underline{C})$
(b) $\underline{A} \cdot (\underline{B} \times \underline{C})$
(c) $\underline{B} \cdot (\underline{A} \times \underline{C})$

A/22 Show that

$$(\underline{A} \times \underline{B}) \cdot (\underline{C} \times \underline{D}) = \begin{vmatrix} \underline{A} \cdot \underline{C} & \underline{B} \cdot \underline{C} \\ \underline{A} \cdot \underline{D} & \underline{B} \cdot \underline{D} \end{vmatrix}$$

A/23 Given $\underline{A} = 2t^2\hat{\imath} + 5t\hat{\jmath} - 8\hat{k}$ and $\underline{B} = 3\hat{\imath} + 6\hat{\jmath}$, find $\dfrac{d}{dt}(\underline{B} \cdot \underline{A})$ and $\dfrac{d}{dt}(\underline{B} \times \underline{A})$.

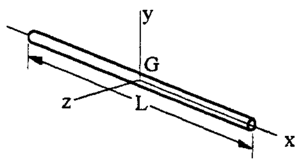
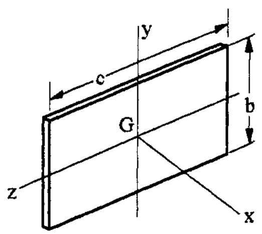
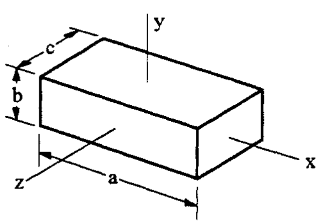
A/24 Integrate $\underline{A} \times \underline{B}$ with respect to time where $\underline{A} = 5t\hat{\imath} - (4t^3 + 6)\hat{\jmath}$ and $\underline{B} = (-6t^2\hat{\imath} + 12\hat{k})$.

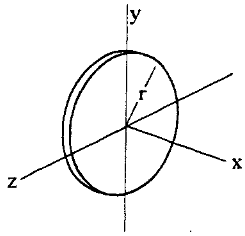
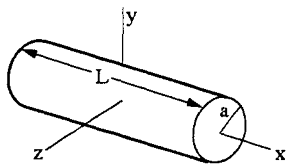
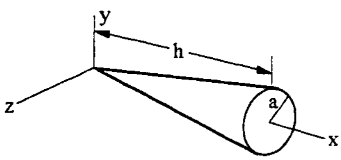
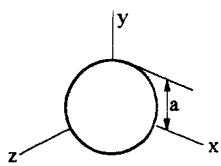
A/25 Integrate the vector $\underline{A} = (10 + 12t^2)\hat{\jmath} + 8\hat{k}$ from time $t = 3$ to time $t = 10$.

A/26 Find the equation of the line which is parallel to the vector $\underline{R} = 2\hat{\imath} + 3\hat{\jmath} + 5\hat{k}$ and passes through the point $Q(3, 4, 5)$.

A/27 Find the equation of the plane which is normal to the vector $\underline{N} = 14\hat{\imath} - 13\hat{\jmath} + 2\hat{k}$ and contains the origin.

Appendix B – Mass Moments of Inertia of Selected Homogeneous Solids

Thin rod		$I_{xx} = 0$ $I_{yy} = I_{zz} = \frac{1}{12} mL^2$
Thin rectangular plate		$I_{xx} = \frac{1}{12} m(b^2 + c^2)$ $I_{yy} = \frac{1}{12} mc^2$ $I_{zz} = \frac{1}{12} mb^2$
Rectangular prism		$I_{xx} = \frac{1}{12} m(b^2 + c^2)$ $I_{yy} = \frac{1}{12} m(c^2 + a^2)$ $I_{zz} = \frac{1}{12} m(a^2 + b^2)$

Thin disk		$I_{xx} = \frac{1}{2}ma^2$ $I_{zz} = I_{yy} = \frac{1}{2}mr^2$
Circular cylinder		$I_{xy} = \frac{1}{2}ma^2$ $I_y = I_z = \frac{1}{12}m(3a^2 + L^2)$
Circular cone		$I_{xx} = \frac{3}{10}ma^2$ $I_{yy} = I_{zz} = \frac{3}{5}m\left(\frac{1}{4}a^2 + h^2\right)$
Sphere		$I_{xx} = I_{yy} = I_{zz} = \frac{2}{5}ma^2$

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