Chapter 4
The Equations of Motion

4.1 THE EQUATIONS OF MOTION OF A RIGID SYMMETRIC AIRCRAFT

As stated in Chapter 1, the first formal derivation of the equations of motion for a rigid symmetric aircraft is usually attributed to Bryan (1911). His treatment, with very few changes, remains in use today and provides the basis for the following development. The object is to realise Newton's second law of motion for each of the six degrees of freedom which simply states that,

$$\text{mass} \times \text{acceleration} = \text{disturbing force} \tag{4.1}$$

For the rotary degrees of freedom the mass and acceleration become moment of inertia and angular acceleration respectively whilst the disturbing force becomes the disturbing moment or torque. Thus the derivation of the equations of motion requires that equation (4.1) be expressed in terms of the motion variables defined in Chapter 2. The derivation is *classical* in the sense that the equations of motion are differential equations which are derived from first principles. However, a number of equally valid alternative means for deriving the equations of motion are frequently used, for example, vector methods. The classical approach is retained here since, in the author's opinion, maximum physical visibility is maintained throughout.

4.1.1 The components of inertial acceleration

The first task in realising equation (4.1) is to define the inertial acceleration components resulting from the application of disturbing force components to the aircraft. Consider the motion referred to an orthogonal axis set ($oxyz$) with the origin o coincident with the *cg* of the arbitrary and, in the first instance, not necessarily rigid body shown in Fig. 4.1. The body, and hence the axes, are assumed to be in motion with respect to an external reference frame such as earth (or *inertial*) axes. The components of velocity and force along the axes ox, oy and oz are denoted (U, V, W) and (X, Y, Z) respectively. The components of angular velocity and moment about the same axes are denoted (p, q, r) and (L, M, N) respectively. The point p is an arbitrarily chosen point within the body with coordinates (x, y, z). The local components of velocity and acceleration at p relative to the body axes are denoted (u, v, w) and (a_x, a_y, a_z) respectively.

The velocity components at $p(x, y, z)$ relative to o are given by

$$\begin{aligned} u &= \dot{x} - ry + qz \\ v &= \dot{y} - pz + rx \\ w &= \dot{z} - qx + py \end{aligned} \tag{4.2}$$

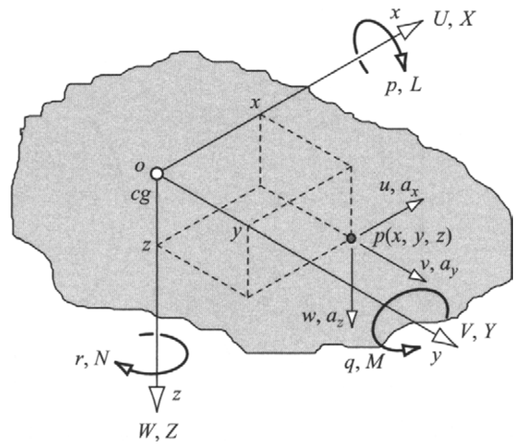

Figure 4.1 *Motion referred to generalised body axes.*

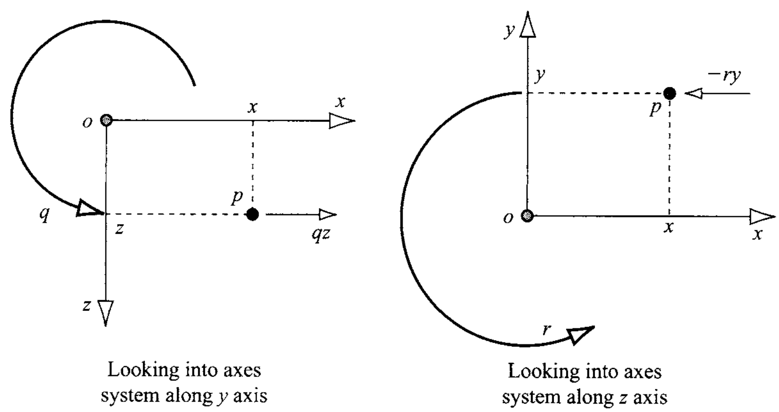

Figure 4.2 *Velocity terms due to rotary motion.*

It will be seen that the velocity components each comprise a linear term and two additional terms due to rotary motion. The origin of the terms due to rotary motion in the component u, for example, is illustrated in Fig. 4.2. Both $-ry$ and qz represent *tangential velocity* components acting along a line through $p(x, y, z)$ parallel to the ox axis. The rotary terms in the remaining two components of velocity are determined in a similar way. Now, since the generalised body shown in Fig. 4.1 represents the aircraft which is assumed to be rigid then

$$\dot{x} = \dot{y} = \dot{z} = 0 \tag{4.3}$$

and equations (4.2) reduce to

$$\begin{aligned} u &= qz - ry \\ v &= rx - pz \\ w &= py - qx \end{aligned} \tag{4.4}$$

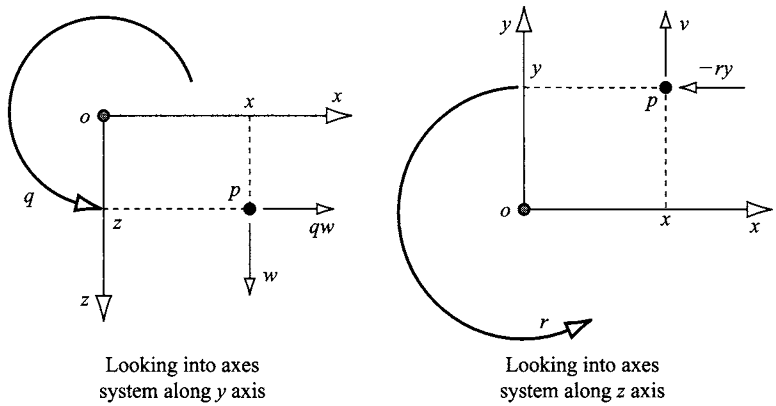

Figure 4.3 *Acceleration terms due to rotary motion*

The corresponding components of acceleration at $p(x, y, z)$ relative to o are given by

$$\begin{aligned} a_x &= \dot{u} - rv + qw \\ a_y &= \dot{v} - pw + ru \\ a_z &= \dot{w} - qu + pv \end{aligned} \tag{4.5}$$

Again, it will be seen that the acceleration components each comprise a linear term and two additional terms due to rotary motion. The origin of the terms due to rotary motion in the component a_x, for example, is illustrated in Fig. 4.3. Both $-rv$ and qw represent *tangential acceleration* components acting along a line through $p(x, y, z)$ parallel to the ox axis. The accelerations arise from the mutual interaction of the linear components of velocity with the components of angular velocity. The acceleration terms due to rotary motion in the remaining two components of acceleration are determined in a similar way.

By superimposing the velocity components of the *cg* (U, V, W) on to the local velocity components (u, v, w) the absolute, or inertial, velocity components (u', v', w') of the point $p(x, y, z)$ are obtained. Thus

$$\begin{aligned} u' &= U + u = U - ry + qz \\ v' &= V + v = V - pz + rx \\ w' &= W + w = W - qx + py \end{aligned} \tag{4.6}$$

where the expressions for (u, v, w) are substituted from equations (4.4). Similarly, the components of inertial acceleration (a'_x, a'_y, a'_z) at the point $p(x, y, z)$ are obtained simply by substituting the expressions for (u', v', w'), equations (4.6), in place of (u, v, w) in equations (4.5). Whence

$$\begin{aligned} a'_x &= \dot{u}' - rv' + qw' \\ a'_y &= \dot{v}' - pw' + ru' \\ a'_z &= \dot{w}' - qu' + pv' \end{aligned} \tag{4.7}$$

Differentiate equations (4.6) with respect to time and note that since a rigid body is assumed equation (4.3) applies then

$$\begin{aligned}
\dot{u}' &= \dot{U} - \dot{r}y + \dot{q}z \\
\dot{v}' &= \dot{V} - \dot{p}z + \dot{r}x \\
\dot{w}' &= \dot{W} - \dot{q}x + \dot{p}y
\end{aligned} \tag{4.8}$$

Thus, by substituting from equations (4.6) and (4.8) into equations (4.7) the inertial acceleration components of the point $p(x, y, z)$ in the rigid body are obtained which, after some rearrangement, may be written,

$$\begin{aligned}
a'_x &= \dot{U} - rV + qW - x(q^2 + r^2) + y(pq - \dot{r}) + z(pr + \dot{q}) \\
a'_y &= \dot{V} - pW + rU + x(pq + \dot{r}) - y(p^2 + r^2) + z(qr - \dot{p}) \\
a'_z &= \dot{W} - qU + pV + x(pr - \dot{q}) + y(qr + \dot{p}) - z(p^2 + q^2)
\end{aligned} \tag{4.9}$$

Example 4.1

To illustrate the usefulness of equations (4.9) consider the following simple example.

A pilot in an aerobatic aircraft performs a loop in 20 s at a steady velocity of 100 m/s. His seat is located 5 m ahead of, and 1 m above the *cg*. What total normal load factor does he experience at the top and at the bottom of the loop?

Assuming the motion is in the plane of symmetry only, then $V = \dot{p} = p = r = 0$ and since the pilot's seat is also in the plane of symmetry $y = 0$ and the expression for normal acceleration is, from equations (4.9):

$$a'_z = \dot{W} - qU + x\dot{q} - zq^2$$

Since the manoeuvre is steady, the further simplification can be made $\dot{W} = \dot{q} = 0$ and the expression for the normal acceleration at the pilots seat reduces to

$$a'_z = -qU - zq^2$$

Now,

$$\begin{aligned}
q &= \frac{2\pi}{20} = 0.314\,\text{rad/s} \\
U &= 100\,\text{m/s} \\
x &= 5\,\text{m} \\
z &= -1\,\text{m (above } cg \text{ hence negative)}
\end{aligned}$$

whence $a'_z = -31.30\,\text{m/s}^2$. Now, by definition, the corresponding incremental normal load factor due to the manoeuvre is given by

$$n' = \frac{-a'_z}{g} = \frac{31.30}{9.81} = 3.19$$

The total normal load factor n comprises that due to the manoeuvre n' plus that due to gravity n_g. At the top of the loop $n_g = -1$, thus the total normal load factor is a given by

$$n = n' + n_g = 3.19 - 1 = 2.19$$

and at the bottom of the loop $n_g = 1$ and in this case the total normal load factor is given by

$$n = n' + n_g = 3.19 + 1 = 4.19$$

It is interesting to note that the normal acceleration measured by an accelerometer mounted at the pilots seat corresponds with the total normal load factor. The accelerometer would therefore give the following readings:

at the top of the loop $\quad a_z = ng = 2.19 \times 9.81 = 21.48\,\mathrm{m/s^2}$
at the bottom of the loop $\quad a_z = ng = 4.19 \times 9.81 = 41.10\,\mathrm{m/s^2}$

Equations (4.9) can therefore be used to determine the accelerations that would be measured by suitably aligned accelerometers located at any point in the airframe and defined by the coordinates (x, y, z).

4.1.2 The generalised force equations

Consider now an incremental mass δm at point $p(x, y, z)$ in the rigid body. Applying Newton's second law, equation (4.1), to the incremental mass the incremental components of force acting on the mass are given by $(\delta ma'_x, \delta ma'_y, \delta ma'_z)$. Thus the total force components (X, Y, Z) acting on the body are given by summing the force increments over the whole body, whence,

$$\begin{aligned} \sum \delta m a'_x &= X \\ \sum \delta m a'_y &= Y \\ \sum \delta m a'_z &= Z \end{aligned} \tag{4.10}$$

Substitute the expressions for the components of inertial acceleration (a'_x, a'_y, a'_z) from equations (4.9) into equations (4.10) and note that since the origin of axes coincides with the *cg*:

$$\sum \delta m x = \sum \delta m y = \sum \delta m z = 0 \tag{4.11}$$

Therefore the resultant components of total force acting on the rigid body are given by

$$\begin{aligned} m(\dot{U} - rV + qW) &= X \\ m(\dot{V} - pW + rU) &= Y \\ m(\dot{W} - qU + pV) &= Z \end{aligned} \tag{4.12}$$

where m is the total mass of the body.

Equations (4.12) represent the force equations of a generalised rigid body and describe the motion of its *cg* since the origin of the axis system is co-located with the *cg* in the body. In some applications, for example the airship, it is often convenient to locate the origin of the axis system at some point other than the *cg*. In such cases the condition described by equation (4.11) does not apply and equations (4.12) would include rather more terms.

4.1.3 The generalised moment equations

Consider now the moments produced by the forces acting on the incremental mass δm at point $p(x, y, z)$ in the rigid body. The incremental force components create an incremental moment component about each of the three body axes and by summing these over the whole body the moment equations are obtained. The moment equations are, of course, the realisation of the rotational form of Newton's second law of motion.

For example, the total moment L about the ox axis is given by summing the incremental moments over the whole body:

$$\sum \delta m(ya'_z - za'_y) = L \tag{4.13}$$

Substituting in equation (4.13) for a'_y and for a'_z obtained from equations (4.9) and noting that equation (4.11) applies then, after some rearrangement, equation (4.13) may be written:

$$\begin{pmatrix} \dot{p}\sum \delta m(y^2+z^2) + qr\sum \delta m(y^2-z^2) \\ +(r^2-q^2)\sum \delta myz - (pq+\dot{r})\sum \delta mxz + (pr-\dot{q})\sum \delta mxy \end{pmatrix} = L \tag{4.14}$$

Terms under the summation sign $\sum$ in equation (4.14) have the units of moment of inertia thus, it is convenient to define the moments and products of inertia as set out in Table 4.1.

Equation (4.14) may therefore be rewritten:

$$I_x\dot{p} - (I_y - I_z)qr + I_{xy}(pr - \dot{q}) - I_{xz}(pq + \dot{r}) + I_{yz}(r^2 - q^2) = L \tag{4.15}$$

In a similar way the total moments M and N about the oy and oz axes respectively are given by summing the incremental moment components over the whole body:

$$\begin{aligned} \sum \delta m(za'_x - xa'_z) &= M \\ \sum \delta m(xa'_y - ya'_x) &= N \end{aligned} \tag{4.16}$$

Table 4.1 *Moments and Products of Inertia*

$I_x = \Sigma\delta m(y^2 + z^2)$	Moment of inertia about *ox* axis
$I_y = \Sigma\delta m(x^2 + z^2)$	Moment of inertia about *oy* axis
$I_z = \Sigma\delta m(x^2 + y^2)$	Moment of inertia about *oz* axis
$I_{xy} = \Sigma\delta mxy$	Product of inertia about *ox* and *oy* axes
$I_{xz} = \Sigma\delta mxz$	Product of inertia about *ox* and *oz* axes
$I_{yz} = \Sigma\delta myz$	Product of inertia about *oy* and *oz* axes

Substituting a'_x, a'_y and a'_z, obtained from equations (4.9), in equations (4.16), noting again that equation (4.11) applies and making use of the inertia definitions given in Table 4.1 then, the moment M about the oy axis is given by

$$I_y\dot{q} + (I_x - I_z)pr + I_{yz}(pq - \dot{r}) + I_{xz}(p^2 - r^2) - I_{xy}(qr + \dot{p}) = M \tag{4.17}$$

and the moment N about the oz axis is given by

$$I_z\dot{r} - (I_x - I_y)pq - I_{yz}(pr + \dot{q}) + I_{xz}(qr - \dot{p}) + I_{xy}(q^2 - p^2) = N \tag{4.18}$$

Equations (4.15), (4.17) and (4.18) represent the moment equations of a generalised rigid body and describe the rotational motion about the orthogonal axes through its cg since the origin of the axis system is co-located with the cg in the body.

When the generalised body represents an aircraft the moment equations may be simplified since it is assumed that the aircraft is symmetric about the oxz plane and that the mass is uniformly distributed. As a result the products of inertia $I_{xy} = I_{yz} = 0$. Thus the moment equations simplify to the following:

$$\begin{aligned}
&I_x\dot{p} - (I_y - I_z)qr - I_{xz}(pq + \dot{r}) = L \\
&I_y\dot{q} + (I_x - I_z)pr + I_{xz}(p^2 - r^2) = M \\
&I_z\dot{r} - (I_x - I_y)pq + I_{xz}(qr - \dot{p}) = N
\end{aligned} \tag{4.19}$$

The equations (4.19), describe rolling motion, pitching motion and yawing motion respectively. A further simplification can be made if it is assumed that the aircraft body axes are aligned to be *principal inertia axes*. In this special case the remaining product of inertia I_{xz} is also zero. This simplification is not often used owing to the difficulty of precisely determining the principal inertia axes. However, the symmetry of the aircraft determines that I_{xz} is generally very much smaller than I_x, I_y and I_z and can often be neglected.

4.1.4 Disturbance forces and moments

Together, equations (4.12) and (4.19) comprise the generalised six degrees of freedom equations of motion of a rigid symmetric airframe having a uniform mass distribution. Further development of the equations of motion requires that the terms on the right hand side of the equations adequately describe the disturbing forces and moments. The traditional approach, after Bryan (1911), is to assume that the disturbing forces and moments are due to *a*erodynamic effects, *g*ravitational effects, movement of aerodynamic *c*ontrols, *p*ower effects and the effects of atmospheric *d*isturbances. Thus bringing together equations (4.12) and (4.19) they may be written to include these contributions as follows:

$$\begin{aligned}
m(\dot{U} - rV + qW) &= X_a + X_g + X_c + X_p + X_d \\
m(\dot{V} - pW + rU) &= Y_a + Y_g + Y_c + Y_p + Y_d \\
m(\dot{W} - qU + pV) &= Z_a + Z_g + Z_c + Z_p + Z_d
\end{aligned} \tag{4.20}$$

$$\begin{aligned}
I_x\dot{p} - (I_y - I_z)qr - I_{xz}(pq + \dot{r}) &= L_a + L_g + L_c + L_p + L_d \\
I_y\dot{q} + (I_x - I_z)pr + I_{xz}(p^2 - r^2) &= M_a + M_g + M_c + M_p + M_d \\
I_z\dot{r} - (I_x - I_y)pq + I_{xz}(qr - \dot{p}) &= N_a + N_g + N_c + N_p + N_d
\end{aligned}$$

Now the equations (4.20) describe the generalised motion of the aeroplane without regard for the magnitude of the motion and subject to the assumptions applying. The equations are *non-linear* and their solution by analytical means is not generally practicable. Further, the terms on the right hand side of the equations must be replaced with suitable expressions which are particularly difficult to determine for the most general motion. Typically, the continued development of the non-linear equations of motion and their solution is most easily accomplished using computer modelling, or simulation techniques which are beyond the scope of this book.

In order to proceed with the development of the equations of motion for analytical purposes, they must be linearised. Linearisation is very simply accomplished by constraining the motion of the aeroplane to small perturbations about the trim condition.

4.2 THE LINEARISED EQUATIONS OF MOTION

Initially the aeroplane is assumed to be flying in steady trimmed rectilinear flight with zero roll, sideslip and yaw angles. Thus, the plane of symmetry of the aeroplane *oxz* is *vertical* with respect to the earth reference frame. At this flight condition the velocity of the aeroplane is V_0, the components of linear velocity are (U_e, V_e, W_e) and the angular velocity components are all zero. Since there is no sideslip $V_e = 0$. A stable undisturbed atmosphere is also assumed such that

$$X_d = Y_d = Z_d = L_d = M_d = N_d = 0 \tag{4.21}$$

If now the aeroplane experiences a small perturbation about trim, the components of the linear disturbance velocities are (u, v, w) and the components of the angular disturbance velocities are (p, q, r) with respect to the undisturbed aeroplane axes $(oxyz)$. Thus the total velocity components of the *cg* in the disturbed motion are given by

$$\begin{aligned}
U &= U_e + u \\
V &= V_e + v = v \\
W &= W_e + w
\end{aligned} \tag{4.22}$$

Now, by definition (u, v, w) and (p, q, r) are small quantities such that terms involving products and squares of these terms are insignificantly small and may be ignored. Thus, substituting equations (4.21) and (4.22) into equations (4.20), note that (U_e, V_e, W_e) are steady and hence constant, and eliminating the insignificantly small terms, the linearised equations of motion are obtained:

$$\begin{aligned}
m(\dot{u} + qW_e) &= X_a + X_g + X_c + X_p \\
m(\dot{v} - pW_e + rU_e) &= Y_a + Y_g + Y_c + Y_p \\
m(\dot{w} - qU_e) &= Z_a + Z_g + Z_c + Z_p
\end{aligned} \tag{4.23}$$

$$\begin{aligned} I_x\dot{p} - I_{xz}\dot{r} &= L_a + L_g + L_c + L_p \\ I_y\dot{q} &= M_a + M_g + M_c + M_p \\ I_z\dot{r} - I_{xz}\dot{p} &= N_a + N_g + N_c + N_p \end{aligned}$$

The development of expressions to replace the terms on the right hand sides of equations (4.23) is now much simpler since it is only necessary to consider small disturbances about trim.

4.2.1 Gravitational terms

The weight force mg acting on the aeroplane may be resolved into components acting in each of the three aeroplane axes. When the aeroplane is disturbed these components will vary according to the perturbations in attitude thereby making a contribution to the disturbed motion. Thus the gravitational contribution to equations (4.23) is obtained by resolving the aeroplane weight into the disturbed body axes. Since the origin of the aeroplane body axes is coincident with the cg there is no weight moment about any of the axes, therefore

$$L_g = M_g = N_g = 0 \tag{4.24}$$

Since the aeroplane is flying wings level in the initial symmetric flight condition, the components of weight only appear in the plane of symmetry as shown in Fig. 4.4. Thus in the steady state the components of weight resolved into aeroplane axes are

$$\begin{bmatrix} X_{g_e} \\ Y_{g_e} \\ Z_{g_e} \end{bmatrix} = \begin{bmatrix} -mg\sin\theta_e \\ 0 \\ mg\cos\theta_e \end{bmatrix} \tag{4.25}$$

During the disturbance the aeroplane attitude perturbation is $(\phi,\ \theta,\ \psi)$ and the components of weight in the disturbed aeroplane axes may be derived with the aid of

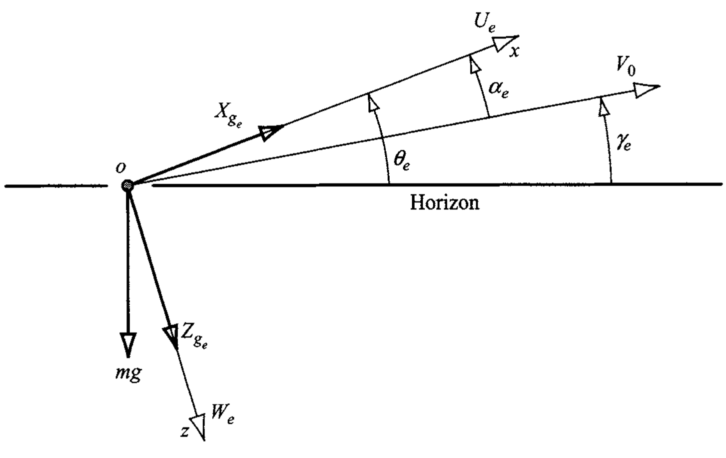

Figure 4.4 *Steady state weight components in the plane of symmetry.*

the transformation equation (2.11). As, by definition, the angular perturbations are small, small angle approximations may be used in the direction cosine matrix to give the following relationship:

$$\begin{bmatrix} X_g \\ Y_g \\ Z_g \end{bmatrix} = \begin{bmatrix} 1 & \psi & -\theta \\ -\psi & 1 & \phi \\ \theta & -\phi & 1 \end{bmatrix} \begin{bmatrix} X_{g_e} \\ Y_{g_e} \\ Z_{g_e} \end{bmatrix} = \begin{bmatrix} 1 & \psi & -\theta \\ -\psi & 1 & \phi \\ \theta & -\phi & 1 \end{bmatrix} \begin{bmatrix} -mg\sin\theta_e \\ 0 \\ mg\cos\theta_e \end{bmatrix} \tag{4.26}$$

And, again, the products of small quantities have been neglected on the grounds that they are insignificantly small. Thus, the gravitational force components in the small perturbation equations of motion are given by

$$\begin{aligned} X_g &= -mg\sin\theta_e - mg\theta\cos\theta_e \\ Y_g &= mg\psi\sin\theta_e + mg\phi\cos\theta_e \\ Z_g &= mg\cos\theta_e - mg\theta\sin\theta_e \end{aligned} \tag{4.27}$$

4.2.2 Aerodynamic terms

Whenever the aeroplane is disturbed from its equilibrium the aerodynamic balance is obviously upset. To describe explicitly the aerodynamic changes occurring during a disturbance provides a considerable challenge in view of the subtle interactions present in the motion. However, although limited in scope, the method first described by Bryan (1911) works extremely well for classical aeroplanes when the motion of interest is limited to (relatively) small perturbations. Although the approach is unchanged the rather more modern notation of Hopkin (1970) is adopted.

The usual procedure is to assume that the aerodynamic force and moment terms in equations (4.20) are dependent on the disturbed motion variables and their derivatives only. Mathematically this is conveniently expressed as a function comprising the sum of a number ofTaylor series, each series involving one motion variable or derivative of a motion variable. Since the motion variables are (u, v, w) and (p, q, r) the aerodynamic term X_a in the axial force equation, for example, may be expressed:

$$\begin{aligned} X_a = X_{a_e} &+ \left(\frac{\partial X}{\partial u}u + \frac{\partial^2 X}{\partial u^2}\frac{u^2}{2!} + \frac{\partial^3 X}{\partial u^3}\frac{u^3}{3!} + \frac{\partial^4 X}{\partial u^4}\frac{u^4}{4!} + \cdots\right) \\ &+ \left(\frac{\partial X}{\partial v}v + \frac{\partial^2 X}{\partial v^2}\frac{v^2}{2!} + \frac{\partial^3 X}{\partial v^3}\frac{v^3}{3!} + \frac{\partial^4 X}{\partial v^4}\frac{v^4}{4!} + \cdots\right) \\ &+ \left(\frac{\partial X}{\partial w}w + \frac{\partial^2 X}{\partial w^2}\frac{w^2}{2!} + \frac{\partial^3 X}{\partial w^3}\frac{w^3}{3!} + \frac{\partial^4 X}{\partial w^4}\frac{w^4}{4!} + \cdots\right) \\ &+ \left(\frac{\partial X}{\partial p}p + \frac{\partial^2 X}{\partial p^2}\frac{p^2}{2!} + \frac{\partial^3 X}{\partial p^3}\frac{p^3}{3!} + \frac{\partial^4 X}{\partial p^4}\frac{p^4}{4!} + \cdots\right) \\ &+ \left(\frac{\partial X}{\partial q}q + \frac{\partial^2 X}{\partial q^2}\frac{q^2}{2!} + \frac{\partial^3 X}{\partial q^3}\frac{q^3}{3!} + \frac{\partial^4 X}{\partial q^4}\frac{q^4}{4!} + \cdots\right) \\ &+ \left(\frac{\partial X}{\partial r}r + \frac{\partial^2 X}{\partial r^2}\frac{r^2}{2!} + \frac{\partial^3 X}{\partial r^3}\frac{r^3}{3!} + \frac{\partial^4 X}{\partial r^4}\frac{r^4}{4!} + \cdots\right) \end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{\partial X}{\partial \dot{u}}\dot{u}+\frac{\partial^2 X}{\partial \dot{u}^2}\frac{\dot{u}^2}{2!}+\frac{\partial^3 X}{\partial \dot{u}^3}\frac{\dot{u}^3}{3!}+\cdots\right)\\
&+\left(\frac{\partial X}{\partial \dot{v}}\dot{v}+\frac{\partial^2 X}{\partial \dot{v}^2}\frac{\dot{v}^2}{2!}+\frac{\partial^3 X}{\partial \dot{v}^3}\frac{\dot{v}^3}{3!}+\cdots\right)\\
&+\text{series terms in } \dot{w}, \dot{p}, \dot{q} \text{ and } \dot{r}\\
&+\text{series terms in higher order derivatives}
\end{aligned}
\tag{4.28}
$$

where X_{a_e} is a constant term. Since the motion variables are small, for all practical aeroplanes only the first term in each of the series functions is significant. Further, the only significant higher order derivative terms commonly encountered are those involving $\dot{w}$. Thus equation (4.28) is dramatically simplified to

$$X_a = X_{a_e} + \frac{\partial X}{\partial u}u + \frac{\partial X}{\partial v}v + \frac{\partial X}{\partial w}w + \frac{\partial X}{\partial p}p + \frac{\partial X}{\partial q}q + \frac{\partial X}{\partial r}r + \frac{\partial X}{\partial \dot{w}}\dot{w} \tag{4.29}$$

Using an alternative shorthand notation for the derivatives, equation (4.29) may be written:

$$X_a = X_{a_e} + \mathring{X}_u u + \mathring{X}_v v + \mathring{X}_w w + \mathring{X}_p p + \mathring{X}_q q + \mathring{X}_r r + \mathring{X}_{\dot{w}} \dot{w} \tag{4.30}$$

The coefficients $\mathring{X}_u, \mathring{X}_v, \mathring{X}_w$ etc. are called *aerodynamic stability derivatives* and the dressing (o) denotes the derivatives to be *dimensional*. Since equation (4.30) has the units of force, the units of each of the aerodynamic stability derivatives are self-evident. In a similar way the force and moment terms in the remaining equations (4.20) are determined. For example, the aerodynamic term in the rolling moment equation is given by

$$L_a = L_{a_e} + \mathring{L}_u u + \mathring{L}_v v + \mathring{L}_w w + \mathring{L}_p p + \mathring{L}_q q + \mathring{L}_r r + \mathring{L}_{\dot{w}} \dot{w} \tag{4.31}$$

4.2.3 Aerodynamic control terms

The primary aerodynamic controls are the elevator, ailerons and rudder. Since the forces and moments created by control deflections arise from the changes in aerodynamic conditions, it is usual to quantify their effect in terms of *aerodynamic control derivatives*. The assumptions applied to the aerodynamic terms are also applied to the control terms thus, for example, the pitching moment due to aerodynamic controls may be expressed:

$$M_c = \frac{\partial M}{\partial \xi}\xi + \frac{\partial M}{\partial \eta}\eta + \frac{\partial M}{\partial \zeta}\zeta \tag{4.32}$$

where aileron angle, elevator angle and rudder angle are denoted ξ, η and ζ respectively. Since equation (4.32) describes the effect of the aerodynamic controls with respect to the prevailing trim condition it is important to realise that the control angles,

ξ, η and ζ are measured relative to the trim settings ξ_e, η_e and ζ_e respectively. Again, the shorthand notation may be used and equation (4.32) may be written:

$$M_c = \overset{\circ}{M}_\xi \xi + \overset{\circ}{M}_\eta \eta + \overset{\circ}{M}_\zeta \zeta \tag{4.33}$$

The aerodynamic control terms in the remaining equations of motion are assembled in a similar way. If it is required to study the response of an aeroplane to other aerodynamic controls, for example, flaps, spoilers, leading edge devices, etc. then additional terms may be appended to equation (4.33) and the remaining equations of motion as required.

4.2.4 Power terms

Power, and hence thrust τ, is usually controlled by throttle lever angle ε and the relationship between the two variables is given for a simple turbojet by equation (2.34) in Chapter 2. Movement of the throttle lever causes a thrust change which in turn gives rise to a change in the components of force and moment acting on the aeroplane. It is mathematically convenient to describe these effects in terms of engine thrust derivatives. For example, normal force due to thrust may be expressed in the usual shorthand notation:

$$Z_p = \overset{\circ}{Z}_\tau \tau \tag{4.34}$$

The contributions to the remaining equations of motion are expressed in a similar way. As for the aerodynamic controls, power changes are measured with respect to the prevailing trim setting. Therefore τ quantifies the thrust perturbation relative to the trim setting τ_e.

4.2.5 The equations of motion for small perturbations

To complete the development of the linearised equations of motion it only remains to substitute the appropriate expressions for the aerodynamic, gravitational, aerodynamic control and thrust terms into equations (4.23). The aerodynamic terms are exemplified by expressions like equations (4.30) and (4.31), expressions for the gravitational terms are given in equations (4.27), the aerodynamic control terms are exemplified by expressions like equation (4.33) and the thrust terms are exemplified by expressions like equation (4.34). Bringing all of these together the following equations are obtained:

$$\begin{aligned}
m(\dot{u} + qW_e) &= X_{a_e} + \overset{\circ}{X}_u u + \overset{\circ}{X}_v v + \overset{\circ}{X}_w w + \overset{\circ}{X}_p p + \overset{\circ}{X}_q q + \overset{\circ}{X}_r r + \overset{\circ}{X}_{\dot{w}} \dot{w} \\
&\quad - mg\sin\theta_e - mg\theta\cos\theta_e + \overset{\circ}{X}_\xi \xi + \overset{\circ}{X}_\eta \eta + \overset{\circ}{X}_\zeta \zeta + \overset{\circ}{X}_\tau \tau \\
m(\dot{v} - pW_e + rU_e) &= Y_{a_e} + \overset{\circ}{Y}_u u + \overset{\circ}{Y}_v v + \overset{\circ}{Y}_w w + \overset{\circ}{Y}_p p + \overset{\circ}{Y}_q q + \overset{\circ}{Y}_r r + \overset{\circ}{Y}_{\dot{w}} \dot{w} \\
&\quad + mg\psi\sin\theta_e + mg\phi\cos\theta_e + \overset{\circ}{Y}_\xi \xi + \overset{\circ}{Y}_\eta \eta + \overset{\circ}{Y}_\zeta \zeta + \overset{\circ}{Y}_\tau \tau
\end{aligned}$$

$$\begin{aligned}
m(\dot{w} - qU_e) &= Z_{a_e} + \mathring{Z}_u u + \mathring{Z}_v v + \mathring{Z}_w w + \mathring{Z}_p p + \mathring{Z}_q q + \mathring{Z}_r r + \mathring{Z}_{\dot{w}} \dot{w} \\
&\quad + mg\cos\theta_e - mg\theta\sin\theta_e + \mathring{Z}_\xi \xi + \mathring{Z}_\eta \eta + \mathring{Z}_\zeta \zeta + \mathring{Z}_\tau \tau \\
I_x\dot{p} - I_{xz}\dot{r} &= L_{a_e} + \mathring{L}_u u + \mathring{L}_v v + \mathring{L}_w w + \mathring{L}_p p + \mathring{L}_q q + \mathring{L}_r r \\
&\quad + \mathring{L}_{\dot{w}} \dot{w} + \mathring{L}_\xi \xi + \mathring{L}_\eta \eta + \mathring{L}_\zeta \zeta + \mathring{L}_\tau \tau \\
I_y\dot{q} &= M_{a_e} + \mathring{M}_u u + \mathring{M}_v v + \mathring{M}_w w + \mathring{M}_p p + \mathring{M}_q q + \mathring{M}_r r \\
&\quad + \mathring{M}_{\dot{w}} \dot{w} + \mathring{M}_\xi \xi + \mathring{M}_\eta \eta + \mathring{M}_\zeta \zeta + \mathring{M}_\tau \tau \\
I_z\dot{r} - I_{xz}\dot{p} &= N_{a_e} + \mathring{N}_u u + \mathring{N}_v v + \mathring{N}_w w + \mathring{N}_p p + \mathring{N}_q q + \mathring{N}_r r \\
&\quad + \mathring{N}_{\dot{w}} \dot{w} + \mathring{N}_\xi \xi + \mathring{N}_\eta \eta + \mathring{N}_\zeta \zeta + \mathring{N}_\tau \tau
\end{aligned} \tag{4.35}$$

Now, in the steady trimmed flight condition all of the perturbation variables and their derivatives are, by definition, zero. Thus in the steady state equations (4.35) reduce to

$$\begin{aligned}
X_{a_e} &= mg\sin\theta_e \\
Y_{a_e} &= 0 \\
Z_{a_e} &= -mg\cos\theta_e \\
L_{a_e} &= 0 \\
M_{a_e} &= 0 \\
N_{a_e} &= 0
\end{aligned} \tag{4.36}$$

Equations (4.36) therefore identify the constant trim terms which may be substituted into equations (4.35) and, following rearrangement they may be written:

$$\begin{aligned}
& m\dot{u} - \mathring{X}_u u - \mathring{X}_v v - \mathring{X}_{\dot{w}} \dot{w} - \mathring{X}_w w \\
& - \mathring{X}_p p - \left(\mathring{X}_q - mW_e\right) q - \mathring{X}_r r + mg\theta\cos\theta_e = \mathring{X}_\xi \xi + \mathring{X}_\eta \eta + \mathring{X}_\zeta \zeta + \mathring{X}_\tau \tau \\
& - \mathring{Y}_u u + m\dot{v} - \mathring{Y}_v v - \mathring{Y}_{\dot{w}} \dot{w} - \mathring{Y}_w w - \left(\mathring{Y}_p + mW_e\right) p \\
& - \mathring{Y}_q q - \left(\mathring{Y}_r - mU_e\right) r - mg\phi\cos\theta_e - mg\psi\sin\theta_e = \mathring{Y}_\xi \xi + \mathring{Y}_\eta \eta + \mathring{Y}_\zeta \zeta + \mathring{Y}_\tau \tau \\
& - \mathring{Z}_u u - \mathring{Z}_v v + \left(m - \mathring{Z}_{\dot{w}}\right) \dot{w} - \mathring{Z}_w w \\
& - \mathring{Z}_p p - \left(\mathring{Z}_q + mU_e\right) q - \mathring{Z}_r r + mg\theta\sin\theta_e = \mathring{Z}_\xi \xi + \mathring{Z}_\eta \eta + \mathring{Z}_\zeta \zeta + \mathring{Z}_\tau \tau \\
& - \mathring{L}_u u - \mathring{L}_v v - \mathring{L}_{\dot{w}} \dot{w} - \mathring{L}_w w \\
& + I_x\dot{p} - \mathring{L}_p p - \mathring{L}_q q - I_{xz}\dot{r} - \mathring{L}_r r = \mathring{L}_\xi \xi + \mathring{L}_\eta \eta + \mathring{L}_\zeta \zeta + \mathring{L}_\tau \tau
\end{aligned}$$

$$
\begin{aligned}
&-\overset{\circ}{M}_u u-\overset{\circ}{M}_v v-\overset{\circ}{M}_{\dot{w}}\dot{w}\\
&-\overset{\circ}{M}_w w-\overset{\circ}{M}_p p+I_y\dot{q}-\overset{\circ}{M}_q q-\overset{\circ}{M}_r r=\overset{\circ}{M}_\xi\xi+\overset{\circ}{M}_\eta\eta+\overset{\circ}{M}_\zeta\zeta+\overset{\circ}{M}_\tau\tau\\
&-\overset{\circ}{N}_u u-\overset{\circ}{N}_v v-\overset{\circ}{N}_{\dot{w}}\dot{w}-\overset{\circ}{N}_w w\\
&-I_{xz}\dot{p}-\overset{\circ}{N}_p p-\overset{\circ}{N}_q q+I_z\dot{r}-\overset{\circ}{N}_r r=\overset{\circ}{N}_\xi\xi+\overset{\circ}{N}_\eta\eta+\overset{\circ}{N}_\zeta\zeta+\overset{\circ}{N}_\tau\tau
\end{aligned}
\tag{4.37}
$$

Equations (4.37) are the small perturbation equations of motion, referred to body axes, which describe the transient response of an aeroplane about the trimmed flight condition following a small input disturbance. The equations comprise a set of six simultaneous linear differential equations written in the traditional manner with the forcing, or input, terms on the right hand side. As written, and subject to the assumptions made in their derivation, the equations of motion are perfectly general and describe motion in which longitudinal and lateral dynamics may be fully coupled. However, for the vast majority of aeroplanes when small perturbation transient motion only is considered, as is the case here, longitudinal–lateral coupling is usually negligible. Consequently it is convenient to simplify the equations by assuming that longitudinal and lateral motion is in fact fully decoupled.

4.3 THE DECOUPLED EQUATIONS OF MOTION

4.3.1 The longitudinal equations of motion

Decoupled longitudinal motion is motion in response to a disturbance which is constrained to the longitudinal plane of symmetry, the *oxz* plane, only. The motion is therefore described by the axial force X, the normal force Z and the pitching moment M equations only. Since no lateral motion is involved the lateral motion variables v, p and r and their derivatives are all zero. Also, decoupled longitudinal–lateral motion means that the aerodynamic coupling derivatives are negligibly small and may be taken as zero whence

$$
\overset{\circ}{X}_v=\overset{\circ}{X}_p=\overset{\circ}{X}_r=\overset{\circ}{Z}_v=\overset{\circ}{Z}_p=\overset{\circ}{Z}_r=\overset{\circ}{M}_v=\overset{\circ}{M}_p=\overset{\circ}{M}_r=0 \tag{4.38}
$$

Similarly, since aileron or rudder deflections do not usually cause motion in the longitudinal plane of symmetry the coupling aerodynamic control derivatives may also be taken as zero thus

$$
\overset{\circ}{X}_\xi=\overset{\circ}{X}_\zeta=\overset{\circ}{Z}_\xi=\overset{\circ}{Z}_\zeta=\overset{\circ}{M}_\xi=\overset{\circ}{M}_\zeta=0 \tag{4.39}
$$

The equations of longitudinal symmetric motion are therefore obtained by extracting the axial force, normal force and pitching moment equations from equations (4.37) and substituting equations (4.38) and (4.39) as appropriate. Whence

$$
\begin{aligned}
m\dot{u}-\overset{\circ}{X}_u u-\overset{\circ}{X}_{\dot{w}}\dot{w}-\overset{\circ}{X}_w w-\left(\overset{\circ}{X}_q-mW_e\right)q+mg\theta\cos\theta_e&=\overset{\circ}{X}_\eta\eta+\overset{\circ}{X}_\tau\tau\\
-\overset{\circ}{Z}_u u+\left(m-\overset{\circ}{Z}_{\dot{w}}\right)\dot{w}-\overset{\circ}{Z}_w w-\left(\overset{\circ}{Z}_q+mU_e\right)q+mg\theta\sin\theta_e&=\overset{\circ}{Z}_\eta\eta+\overset{\circ}{Z}_\tau\tau\\
-\overset{\circ}{M}_u u-\overset{\circ}{M}_{\dot{w}}\dot{w}-\overset{\circ}{M}_w w+I_y\dot{q}-\overset{\circ}{M}_q q&=\overset{\circ}{M}_\eta\eta+\overset{\circ}{M}_\tau\tau
\end{aligned}
\tag{4.40}
$$

Equations (4.40) are the most general form of the dimensional decoupled equations of longitudinal symmetric motion referred to aeroplane body axes. If it is assumed that the aeroplane is in level flight and the reference axes are wind or stability axes then

$$\theta_e = W_e = 0 \tag{4.41}$$

and the equations simplify further to

$$\begin{aligned}
m\dot{u} - \overset{\circ}{X}_u u - \overset{\circ}{X}_{\dot{w}} \dot{w} - \overset{\circ}{X}_w w - \overset{\circ}{X}_q q + mg\theta &= \overset{\circ}{X}_\eta \eta + \overset{\circ}{X}_\tau \tau \\
-\overset{\circ}{Z}_u u + \left(m - \overset{\circ}{Z}_{\dot{w}}\right) \dot{w} - \overset{\circ}{Z}_w w - \left(\overset{\circ}{Z}_q + mU_e\right) q &= \overset{\circ}{Z}_\eta \eta + \overset{\circ}{Z}_\tau \tau \\
-\overset{\circ}{M}_u u - \overset{\circ}{M}_{\dot{w}} \dot{w} - \overset{\circ}{M}_w w + I_y \dot{q} - \overset{\circ}{M}_q q &= \overset{\circ}{M}_\eta \eta + \overset{\circ}{M}_\tau \tau
\end{aligned} \tag{4.42}$$

Equations (4.42) represent the simplest possible form of the decoupled longitudinal equations of motion. Further simplification is only generally possible when the numerical values of the coefficients in the equations are known since some coefficients are often negligibly small.

Example 4.2

Longitudinal derivative and other data for the McDonnell F-4C Phantom aeroplane was obtained from Heffley and Jewell (1972) for a flight condition of Mach 0.6 at an altitude of 35000 ft. The original data is presented in imperial units and in a format preferred in the USA. Normally, it is advisable to work with the equations of motion and the data in the format and units as given. Otherwise, conversion to another format can be tedious in the extreme and is easily subject to error. However, for the purposes of illustration, the derivative data has been converted to a form compatible with the equations developed above and the units have been changed to those of the more familiar SI system. The data is quite typical, it would normally be supplied in this, or similar, form by aerodynamicists and as such it represents the starting point in any flight dynamics analysis:

Flight path angle	$\gamma_e = 0°$	Air density	$\rho = 0.3809\,\text{kg/m}^3$
Body incidence	$\alpha_e = 9.4°$	Wing area	$S = 49.239\,\text{m}^2$
Velocity	$V_0 = 178\,\text{m/s}$	Mean aerodynamic chord	$\bar{\bar{c}} = 4.889\,\text{m}$
Mass	$m = 17642\,\text{kg}$		
Pitch moment of inertia	$I_y = 165669\,\text{kgm}^2$	Acceleration due to gravity	$g = 9.81\,\text{m/s}^2$

Since the flight path angle $\gamma_e = 0$ and the body incidence α_e is non-zero it may be deduced that the following derivatives are referred to a body axes system and that $\theta_e \equiv \alpha_e$. The dimensionless longitudinal derivatives are given and any missing aerodynamic derivatives must be assumed insignificant, and hence zero. On the other hand, missing control derivatives may not be assumed insignificant although their absence will prohibit analysis of response to those controls:

$$
\begin{array}{lll}
X_u = 0.0076 & Z_u = -0.7273 & M_u = 0.0340 \\
X_w = 0.0483 & Z_w = -3.1245 & M_w = -0.2169 \\
X_{\dot{w}} = 0 & Z_{\dot{w}} = -0.3997 & M_{\dot{w}} = -0.5910 \\
X_q = 0 & Z_q = -1.2109 & M_q = -1.2732 \\
X_\eta = 0.0618 & Z_\eta = -0.3741 & M_\eta = -0.5581
\end{array}
$$

Equations (4.40) are compatible with the data although the dimensional derivatives must first be calculated according to the definitions given in Appendix 2, Tables A2.1 and A2.2. Thus the dimensional longitudinal equations of motion, referred to body axes, are obtained by substituting the appropriate values into equations (4.40) to give

$$
\begin{aligned}
17642\dot{u} - 12.67u - 80.62w + 512852.94q + 170744.06\theta &= 18362.32\eta \\
1214.01u + 17660.33\dot{w} + 5215.44w - 3088229.7q + 28266.507\theta &= -111154.41\eta \\
-277.47u + 132.47\dot{w} + 1770.07w + 165669\dot{q} + 50798.03q &= -810886.19\eta
\end{aligned}
$$

where $W_e = V_0 \sin\theta_e = 29.07$ m/s and $U_e = V_0 \cos\theta_e = 175.61$ m/s. Note that angular variables in the equations of motion have radian units. Clearly, when written like this the equations of motion are unwieldy. The equations can be simplified a little by dividing through by the mass or inertia as appropriate. Thus the first equation is divided by 17642, the second equation by 17660.33 and the third equation by 165669. After some rearrangement the following rather more convenient version is obtained:

$$
\begin{aligned}
\dot{u} &= 0.0007u + 0.0046w - 29.0700q - 9.6783\theta + 1.0408\eta \\
\dot{w} &= -0.0687u - 0.2953w + 174.8680q - 1.6000\theta - 6.2940\eta \\
\dot{q} + 0.0008\dot{w} &= 0.0017u - 0.0107w - 0.3066q - 4.8946\eta
\end{aligned}
$$

It must be remembered that, when written in this latter form, the equations of motion have the units of acceleration. The most striking feature of these equations, however written, is the large variation in the values of the coefficients. Terms which may, at first sight, appear insignificant are frequently important in the solution of the equations. It is therefore prudent to maintain sensible levels of accuracy when manipulating the equations by hand. Fortunately, this is an activity which is not often required.

4.3.2 The lateral–directional equations of motion

Decoupled lateral–directional motion involves roll, yaw and sideslip only. The motion is therefore described by the side force Y, the rolling moment L and the yawing moment N equations only. As no longitudinal motion is involved the longitudinal motion variables u, w and q and their derivatives are all zero. Also, decoupled longitudinal–lateral motion means that the aerodynamic coupling derivatives are negligibly small and may be taken as zero whence

$$
\overset{\circ}{Y}_u = \overset{\circ}{Y}_{\dot{w}} = \overset{\circ}{Y}_w = \overset{\circ}{Y}_q = \overset{\circ}{L}_u = \overset{\circ}{L}_{\dot{w}} = \overset{\circ}{L}_w = \overset{\circ}{L}_q = \overset{\circ}{N}_u = \overset{\circ}{N}_{\dot{w}} = \overset{\circ}{N}_w = \overset{\circ}{N}_q = 0 \quad (4.43)
$$

Similarly, since the airframe is symmetric, elevator deflection and thrust variation do not usually cause lateral–directional motion and the coupling aerodynamic control derivatives may also be taken as zero thus

$$\overset{\circ}{Y}_\eta = \overset{\circ}{Y}_\tau = \overset{\circ}{L}_\eta = \overset{\circ}{L}_\tau = \overset{\circ}{N}_\eta = \overset{\circ}{N}_\tau = 0 \tag{4.44}$$

The equations of lateral asymmetric motion are therefore obtained by extracting the side force, rolling moment and yawing moment equations from equations (4.37) and substituting equations (4.43) and (4.44) as appropriate. Whence

$$\begin{aligned}
\left(\begin{array}{c} m\dot{v} - \overset{\circ}{Y}_v v - \left(\overset{\circ}{Y}_p + mW_e\right)p - \left(\overset{\circ}{Y}_r - mU_e\right)r \\ -\, mg\phi\cos\theta_e - mg\psi\sin\theta_e \end{array}\right) &= \overset{\circ}{Y}_\xi \xi + \overset{\circ}{Y}_\zeta \zeta \\
-\overset{\circ}{L}_v v + I_x\dot{p} - \overset{\circ}{L}_p p - I_{xz}\dot{r} - \overset{\circ}{L}_r r &= \overset{\circ}{L}_\xi \xi + \overset{\circ}{L}_\zeta \zeta \\
-\overset{\circ}{N}_v v - I_{xz}\dot{p} - \overset{\circ}{N}_p p + I_z\dot{r} - \overset{\circ}{N}_r r &= \overset{\circ}{N}_\xi \xi + \overset{\circ}{N}_\zeta \zeta
\end{aligned} \tag{4.45}$$

Equations (4.45) are the most general form of the dimensional decoupled equations of lateral–directional asymmetric motion referred to aeroplane body axes. If it is assumed that the aeroplane is in level flight and the reference axes are wind or stability axes then, as before,

$$\theta_e = W_e = 0 \tag{4.46}$$

and the equations simplify further to

$$\begin{aligned}
m\dot{v} - \overset{\circ}{Y}_v v - p\overset{\circ}{Y}_p - \left(\overset{\circ}{Y}_r - mU_e\right)r - mg\phi &= \overset{\circ}{Y}_\xi \xi + \overset{\circ}{Y}_\zeta \zeta \\
-\overset{\circ}{L}_v v + I_x\dot{p} - \overset{\circ}{L}_p p - I_{xz}\dot{r} - \overset{\circ}{L}_r r &= \overset{\circ}{L}_\xi \xi + \overset{\circ}{L}_\zeta \zeta \\
-\overset{\circ}{N}_v v - I_{xz}\dot{p} - \overset{\circ}{N}_p p + I_z\dot{r} - \overset{\circ}{N}_r r &= \overset{\circ}{N}_\xi \xi + \overset{\circ}{N}_\zeta \zeta
\end{aligned} \tag{4.47}$$

Equations (4.47) represent the simplest possible form of the decoupled lateral–directional equations of motion. As for the longitudinal equations of motion, further simplification is only generally possible when the numerical values of the coefficients in the equations are known since some coefficients are often negligibly small.

4.4 ALTERNATIVE FORMS OF THE EQUATIONS OF MOTION

4.4.1 The dimensionless equations of motion

Traditionally the development of the equations of motion and investigations of stability and control involving their use have been securely resident in the domain of the aerodynamicist. Many aerodynamic phenomena are most conveniently explained in terms of *dimensionless aerodynamic coefficients*, for example, lift coefficient, Mach number, Reynolds number, etc., and often this mechanism provides the only practical means for making progress. The advantage of this approach is that the aerodynamic properties of an aeroplane can be completely described in terms of dimensionless parameters which are independent of airframe geometry and flight condition. A lift coefficient of 0.5, for example, has precisely the same meaning whether it applies to a Boeing 747 or to a Cessna 150. It is not surprising therefore, to discover that historically the small perturbation equations of motion of an aeroplane were treated in the same way. This in turn leads to the concept of the *dimensionless derivative* which is just another aerodynamic coefficient and may be interpreted in much the same way. However, the dimensionless equations of motion are of little use to the modern flight dynamicist other than as a means for explaining the origin of the dimensionless derivatives. Thus the development of the dimensionless decoupled small perturbation equations of motion is outlined below solely for this purpose.

As formally described by Hopkin (1970) the equations of motion are rendered dimensionless by dividing each equation by a generalised force or moment parameter as appropriate. Sometimes the dimensionless equations of motion are referred to as the *aero-normalised* equations and the corresponding derivative coefficients are also referred to as *aero-normalised derivatives.* To illustrate the procedure consider the axial force equation taken from the decoupled longitudinal equations of motion (4.42):

$$m\dot{u} - \overset{\circ}{X}_u u - \overset{\circ}{X}_{\dot{w}} \dot{w} - \overset{\circ}{X}_w w - q\overset{\circ}{X}_q + mg\theta = \overset{\circ}{X}_\eta \eta + \overset{\circ}{X}_\tau \tau \tag{4.48}$$

Since equation (4.48) has the units of force it may be rendered dimensionless by dividing, or normalising, each term by the aerodynamic force parameter $\frac{1}{2}\rho V_0^2 S$ where S is the reference wing area. Defining the following parameters:

(i) Dimensionless time

$$\hat{t} = \frac{t}{\sigma} \quad \text{where } \sigma = \frac{m}{\frac{1}{2}\rho V_0 S} \tag{4.49}$$

(ii) The longitudinal relative density factor

$$\mu_1 = \frac{m}{\frac{1}{2}\rho S\bar{\bar{c}}} \tag{4.50}$$

where the longitudinal reference length is $\bar{\bar{c}}$, the mean aerodynamic chord.

(iii) Dimensionless velocities

$$\begin{aligned} \hat{u} &= \frac{u}{V_0} \\ \hat{w} &= \frac{w}{V_0} \\ \hat{q} &= q\sigma = \frac{qm}{\frac{1}{2}\rho V_0 S} \end{aligned} \tag{4.51}$$

(iv) Since level flight is assumed the lift and weight are equal thus

$$mg = \tfrac{1}{2}\rho V_0^2 S C_L \tag{4.52}$$

Thus, dividing equation (4.48) through by the aerodynamic force parameter and making use of the parameters defined in equations (4.49)–(4.52) above, the following is obtained:

$$\begin{pmatrix} \dfrac{\dot{u}}{V_0}\sigma - \left(\dfrac{\overset{\circ}{X}_u}{\frac{1}{2}\rho V_0 S}\right)\dfrac{u}{V_0} - \left(\dfrac{\overset{\circ}{X}_{\dot{w}}}{\frac{1}{2}\rho S\bar{\bar{c}}}\right)\dfrac{\dot{w}\sigma}{V_0\mu_1} \\ - \left(\dfrac{\overset{\circ}{X}_w}{\frac{1}{2}\rho V_0 S}\right)\dfrac{w}{V_0} - \left(\dfrac{\overset{\circ}{X}_q}{\frac{1}{2}\rho V_0 S\bar{\bar{c}}}\right)\dfrac{q\sigma}{\mu_1} + \dfrac{mg}{\frac{1}{2}\rho V_0^2 S}\theta \end{pmatrix}$$

$$= \left(\frac{\overset{\circ}{X}_\eta}{\frac{1}{2}\rho V_0^2 S}\right)\eta + \overset{\circ}{X}_\tau\left(\frac{\tau}{\frac{1}{2}\rho V_0^2 S}\right) \tag{4.53}$$

which is more conveniently written:

$$\hat{\dot{u}} - X_u\hat{u} - X_{\dot{w}}\frac{\hat{\dot{w}}}{\mu_1} - X_w\hat{w} - X_q\frac{\hat{q}}{\mu_1} + C_L\theta = X_\eta\eta + X_\tau\hat{\tau} \tag{4.54}$$

The derivatives denoted $X_u, X_{\dot{w}}, X_w, X_q, X_\eta$ and X_τ are the *dimensionless* or *aero-normalised derivatives* and their definitions follow from equation (4.53). It is in this form that the aerodynamic stability and control derivatives would usually be provided for an aeroplane by the aerodynamicists.

In a similar way the remaining longitudinal equations of motion may be rendered dimensionless. Note that the aerodynamic moment parameter used to divide the pitching moment equation is $\frac{1}{2}\rho V_0^2 S\bar{\bar{c}}$. Whence

$$\begin{aligned} -Z_u\hat{u} + \hat{\dot{w}} - Z_{\dot{w}}\frac{\hat{\dot{w}}}{\mu_1} - Z_w\hat{w} - Z_q\frac{\hat{q}}{\mu_1} - \hat{q} &= Z_\eta\eta + Z_\tau\hat{\tau} \\ -M_u\hat{u} - M_{\dot{w}}\frac{\hat{\dot{w}}}{\mu_1} - M_w\hat{w} + i_y\frac{\hat{\dot{q}}}{\mu_1} - M_q\frac{\hat{q}}{\mu_1} &= M_\eta\eta + M_\tau\hat{\tau} \end{aligned} \tag{4.55}$$

where i_y is the dimensionless pitch inertia and is given by

$$i_y = \frac{I_y}{m\bar{\bar{c}}^2} \tag{4.56}$$

Similarly the lateral equations of motion (4.47) may be rendered dimensionless by dividing the side force equation by the aerodynamic force parameter $\frac{1}{2}\rho V_0^2 S$ and the rolling and yawing moment equations by the aerodynamic moment parameter $\frac{1}{2}\rho V_0^2 Sb$, where for lateral motion the reference length is the wing span b. Additional parameter definitions required to deal with the lateral equations are:

(v) The lateral relative density factor

$$\mu_2 = \frac{m}{\frac{1}{2}\rho Sb} \tag{4.57}$$

(vi) The dimensionless inertias

$$i_x = \frac{I_x}{mb^2}, i_z = \frac{I_z}{mb^2} \text{ and } i_{xz} = \frac{I_{xz}}{mb^2} \tag{4.58}$$

Since the equations of motion are referred to wind axes and since level flight is assumed then equations (4.47) may be written in dimensionless form as follows:

$$\begin{aligned}
\dot{\hat{v}} - Y_v\hat{v} - Y_p\frac{\hat{p}}{\mu_2} - Y_r\frac{\hat{r}}{\mu_2} - \hat{r} - C_L\phi &= Y_\xi\xi + Y_\zeta\zeta \\
-L_v\hat{v} + i_x\frac{\dot{\hat{p}}}{\mu_2} - L_p\frac{\hat{p}}{\mu_2} - i_{xz}\frac{\dot{\hat{r}}}{\mu_2} - L_r\frac{\hat{r}}{\mu_2} &= L_\xi\xi + L_\zeta\zeta \\
-N_v\hat{v} - i_{xz}\frac{\dot{\hat{p}}}{\mu_2} - N_p\frac{\hat{p}}{\mu_2} + i_z\frac{\dot{\hat{r}}}{\mu_2} - N_r\frac{\hat{r}}{\mu_2} &= N_\xi\xi + N_\zeta\zeta
\end{aligned} \tag{4.59}$$

For convenience, the definitions of all of the dimensionless aerodynamic stability and control derivatives are given in Appendix 2.

4.4.2 The equations of motion in state space form

Today the solution of the equations of motion poses few problems since very powerful computational tools are readily available. Since computers are very good at handling numerical matrix calculations the use of matrix methods for solving linear dynamic system problems has become an important topic in modern applied mathematics. In particular, matrix methods together with the digital computer have led to the development of the relatively new field of modern control system theory. For small perturbations, the aeroplane is a classical example of a linear dynamic system and frequently the solution of its equations of motion is a prelude to flight control system design and analysis. It is therefore convenient and straight forward to utilise multi-variable system theory tools in the solution of the equations of motion. However, it is first necessary to arrange the equations of motion in a suitable format.

The motion, or *state*, of any linear dynamic system may be described by a minimum set of variables called the *state variables*. The number of state variables required to completely describe the motion of the system is dependent on the number of degrees of freedom the system has. Thus the motion of the system is described in a multi-dimensional vector space called the *state space*, the number of state variables being equal to the number of dimensions. The equation of motion, or *state equation*, of the *linear time invariant* (LTI) multi-variable system is written:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{4.60}$$

where

$\mathbf{x}(t)$ is the column vector of n state variables called the *state vector*.
$\mathbf{u}(t)$ is the column vector of m input variables called the *input vector*.
$\mathbf{A}$ is the $(n \times n)$ *state matrix*.
$\mathbf{B}$ is the $(n \times m)$ *input matrix*.

Since the system is LTI the matrices **A** and **B** have constant elements. Equation (4.60) is the matrix equivalent of a set of n simultaneous linear differential equations and it is a straightforward matter to configure the small perturbation equations of motion for an aeroplane in this format.

Now for many systems some of the state variables may be inaccessible or their values may not be determined directly. Thus a second equation is required to determine the system output variables. The output equation is written in the general form

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \tag{4.61}$$

where

$\mathbf{y}(t)$ is the column vector of r output variables called the *output vector*.
$\mathbf{C}$ is the $(r \times n)$ *output matrix*.
$\mathbf{D}$ is the $(r \times m)$ *direct matrix*.

and, typically, $r \leq n$. Again, for a LTI system the matrices **C** and **D** have constant elements. Together equations (4.60) and (4.61) provide a complete description of the system. A complete description of the formulation of the general state model and the mathematics required in its analysis may be found in Barnett (1975).

For most aeroplane problems it is convenient to choose the output variables to be the state variables. Thus

$$\mathbf{y}(t) = \mathbf{x}(t) \quad \text{and} \quad r = n$$

and consequently

$$\mathbf{C} = \mathbf{I}, \text{ the } (n \times n) \textit{ identity} \text{ matrix}$$
$$\mathbf{D} = \mathbf{0}, \text{ the } (n \times m) \textit{ zero} \text{ matrix}$$

As a result the output equation simplifies to

$$\mathbf{y}(t) = \mathbf{I}\mathbf{x}(t) \equiv \mathbf{x}(t) \tag{4.62}$$

and it is only necessary to derive the state equation from the aeroplane equations of motion.

Consider, for example, the longitudinal equations of motion (4.40) referred to aeroplane body axes. These may be rewritten with the acceleration terms on the left hand side as follows:

$$\begin{aligned}
m\dot{u} - \overset{\circ}{X}_{\dot{w}}\dot{w} &= \overset{\circ}{X}_u u + \overset{\circ}{X}_w w + \left(\overset{\circ}{X}_q - mW_e\right)q - mg\theta\cos\theta_e + \overset{\circ}{X}_\eta \eta + \overset{\circ}{X}_\tau \tau \\
m\dot{w} - \overset{\circ}{Z}_{\dot{w}}\dot{w} &= \overset{\circ}{Z}_u u + \overset{\circ}{Z}_w w + \left(\overset{\circ}{Z}_q + mU_e\right)q - mg\theta\sin\theta_e + \overset{\circ}{Z}_\eta \eta + \overset{\circ}{Z}_\tau \tau \\
I_y\dot{q} - \overset{\circ}{M}_{\dot{w}}\dot{w} &= \overset{\circ}{M}_u u + \overset{\circ}{M}_w w + \overset{\circ}{M}_q q + \overset{\circ}{M}_\eta \eta + \overset{\circ}{M}_\tau \tau
\end{aligned} \tag{4.63}$$

Since the longitudinal motion of the aeroplane is described by four state variables u, w, q and θ four differential equations are required. Thus the additional equation is the auxiliary equation relating pitch rate to attitude rate, which for small perturbations is

$$\dot{\theta} = q \tag{4.64}$$

Equations (4.63) and (4.64) may be combined and written in matrix form:

$$\mathbf{M}\dot{\mathbf{x}}(t) = \mathbf{A}'\mathbf{x}(t) + \mathbf{B}'\mathbf{u}(t) \tag{4.65}$$

where

$$\mathbf{x}^T(t) = [u \quad w \quad q \quad \theta] \quad \mathbf{u}^T(t) = [\eta \quad \tau]$$

$$\mathbf{M} = \begin{bmatrix} m & -\overset{\circ}{X}_{\dot{w}} & 0 & 0 \\ 0 & (m - \overset{\circ}{Z}_{\dot{w}}) & 0 & 0 \\ 0 & -\overset{\circ}{M}_{\dot{w}} & I_y & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{A}' = \begin{bmatrix} \overset{\circ}{X}_u & \overset{\circ}{X}_w & (\overset{\circ}{X}_q - mW_e) & -mg\cos\theta_e \\ \overset{\circ}{Z}_u & \overset{\circ}{Z}_w & (\overset{\circ}{Z}_q + mU_e) & -mg\sin\theta_e \\ \overset{\circ}{M}_u & \overset{\circ}{M}_w & \overset{\circ}{M}_q & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \mathbf{B}' = \begin{bmatrix} \overset{\circ}{X}_\eta & \overset{\circ}{X}_\tau \\ \overset{\circ}{Z}_\eta & \overset{\circ}{Z}_\tau \\ \overset{\circ}{M}_\eta & \overset{\circ}{M}_\tau \\ 0 & 0 \end{bmatrix}$$

The longitudinal state equation is derived by pre-multiplying equation (4.65) by the inverse of the *mass matrix* **M** whence

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{4.66}$$

where

$$\mathbf{A} = \mathbf{M}^{-1}\mathbf{A}' = \begin{bmatrix} x_u & x_w & x_q & x_\theta \\ z_u & z_w & z_q & z_\theta \\ m_u & m_w & m_q & m_\theta \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \mathbf{B} = \mathbf{M}^{-1}\mathbf{B}' = \begin{bmatrix} x_\eta & x_\tau \\ z_\eta & z_\tau \\ m_\eta & m_\tau \\ 0 & 0 \end{bmatrix}$$

The coefficients of the state matrix **A** are the aerodynamic stability derivatives, referred to aeroplane body axes, in *concise form* and the coefficients of the input matrix **B** are the control derivatives also in concise form. The definitions of the concise derivatives follow directly from the above relationships and are given in full in Appendix 2. Thus the longitudinal state equation may be written out in full:

$$\begin{bmatrix} \dot{u} \\ \dot{w} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} x_u & x_w & x_q & x_\theta \\ z_u & z_w & z_q & z_\theta \\ m_u & m_w & m_q & m_\theta \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} x_\eta & x_\tau \\ z_\eta & z_\tau \\ m_\eta & m_\tau \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \eta \\ \tau \end{bmatrix} \tag{4.67}$$

and the output equation is, very simply,

$$\mathbf{y}(t) = \mathbf{I}\mathbf{x}(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix} \tag{4.68}$$

Clearly the longitudinal small perturbation motion of the aeroplane is completely described by the four state variables u, w, q and θ. Equation (4.68) determines that, in this instance, the output variables are chosen to be the same as the four state variables.

Example 4.3

Consider the requirement to write the longitudinal equations of motion for the McDonnell F-4C Phantom of Example 4.2 in state space form. As the derivatives are given in dimensionless form it is convenient to express the matrices $\mathbf{M}, \mathbf{A}'$ and $\mathbf{B}'$ in terms of the dimensionless derivatives. Substituting appropriately for the dimensional derivatives and after some rearrangement the matrices may be written:

$$\mathbf{M} = \begin{bmatrix} m' & -\dfrac{X_{\dot{w}}\bar{\bar{c}}}{V_0} & 0 & 0 \\ 0 & \left(m' - \dfrac{Z_{\dot{w}}\bar{\bar{c}}}{V_0}\right) & 0 & 0 \\ 0 & -\dfrac{M_{\dot{w}}\bar{\bar{c}}}{V_0} & I'_y & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{A}' = \begin{bmatrix} X_u & X_w & (X_q\bar{\bar{c}} - m'W_e) & -m'g\cos\theta_e \\ Z_u & Z_w & (Z_q\bar{\bar{c}} + m'U_e) & -m'g\sin\theta_e \\ M_u & M_w & M_q\bar{\bar{c}} & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \mathbf{B}' = \begin{bmatrix} V_0 X_\eta & V_0 X_\tau \\ V_0 Z_\eta & V_0 Z_\tau \\ V_0 M_\eta & V_0 M_\tau \\ 0 & 0 \end{bmatrix}$$

where

$$m' = \frac{m}{\frac{1}{2}\rho V_0 S} \quad \text{and} \quad I'_y = \frac{I_y}{\frac{1}{2}\rho V_0 S\bar{\bar{c}}}$$

and in steady symmetric flight, $U_e = V_0 \cos\theta_e$ and $W_e = V_0 \sin\theta_e$.

Substituting the derivative values given in Example 4.2 the longitudinal state equation (4.65) may be written:

$$\begin{bmatrix} 10.569 & 0 & 0 & 0 \\ 0 & 10.580 & 0 & 0 \\ 0 & 0.0162 & 20.3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{u} \\ \dot{w} \\ \dot{q} \\ \dot{\theta} \end{bmatrix}$$

$$= \begin{bmatrix} 0.0076 & 0.0483 & -307.26 & -102.29 \\ -0.7273 & -3.1245 & 1850.10 & -16.934 \\ 0.034 & -0.2169 & -6.2247 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} 11.00 \\ -66.5898 \\ -99.341 \\ 0 \end{bmatrix} \eta$$

This equation may be reduced to the preferred form by pre-multiplying each term by the inverse of $\mathbf{M}$, as indicated above, to obtain the longitudinal state equation, referred to body axes, in concise form,

$$\begin{bmatrix} \dot{u} \\ \dot{w} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 7.181\times 10^{-4} & 4.570\times 10^{-3} & -29.072 & -9.678 \\ -0.0687 & -0.2953 & 174.868 & -1.601 \\ 1.73\times 10^{-3} & -0.0105 & -0.4462 & 1.277\times 10^{-3} \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} 1.041 \\ -6.294 \\ -4.888 \\ 0 \end{bmatrix} \eta$$

This computation was carried out with the aid of *Program CC* and it should be noted that the resulting equation compares with the final equations given in Example 4.2. The coefficients of the matrices could equally well have been calculated using the concise derivative definitions given in Appendix 2, Tables A2.5 and A2.6. For the purpose of illustration some of the coefficients in the matrices have been rounded to a more manageable number of decimal places. In general this is not good practice since the rounding errors may lead to accumulated computational errors in any subsequent computer analysis involving the use of these equations. However, once the basic matrices have been entered into a computer program at the level of accuracy given, all subsequent computations can be carried out using computer-generated data files. In this way computational errors will be minimised although it is prudent to be aware that not all computer algorithms for handling matrices can cope with poorly conditioned matrices. Occasionally, aeroplane computer models fall into this category.

The lateral small perturbation equations (4.45), referred to body axes, may be treated in exactly the same way to obtain the lateral–directional state equation:

$$\begin{bmatrix} \dot{v} \\ \dot{p} \\ \dot{r} \\ \dot{\phi} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} y_v & y_p & y_r & y_\phi & y_\psi \\ l_v & l_p & l_r & l_\phi & l_\psi \\ n_v & n_p & n_r & n_\phi & n_\psi \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ p \\ r \\ \phi \\ \psi \end{bmatrix} + \begin{bmatrix} y_\xi & y_\zeta \\ l_\xi & l_\zeta \\ n_\xi & n_\zeta \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \xi \\ \zeta \end{bmatrix} \tag{4.69}$$

Note that when the lateral–directional equations of motion are referred to wind axes, equations (4.47), the lateral–directional state equation (4.69) is reduced from fifth order to fourth order to become

$$\begin{bmatrix} \dot{v} \\ \dot{p} \\ \dot{r} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} y_v & y_p & y_r & y_\phi \\ l_v & l_p & l_r & l_\phi \\ n_v & n_p & n_r & n_\phi \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ p \\ r \\ \phi \end{bmatrix} + \begin{bmatrix} y_\xi & y_\zeta \\ l_\xi & l_\zeta \\ n_\xi & n_\zeta \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \xi \\ \zeta \end{bmatrix} \tag{4.70}$$

However, in this case the derivatives are referred to aeroplane wind axes rather than to body axes and will generally have slightly different values. The definitions of the concise lateral stability and control derivatives referred to aeroplane body axes are also given in Appendix 2.

Examples of the more general procedures used to create the state descriptions of various dynamic systems may be found in many books on control systems; for example, Shinners (1980) and Friedland (1987) both contain useful aeronautical examples.

Example 4.4

Lateral derivative data for the McDonnell F-4C Phantom, referred to body axes, were also obtained from Heffley and Jewell (1972) and are used to illustrate the formulation of the lateral state equation. The data relate to the same flight condition, namely Mach 0.6 and an altitude of 35000 ft. As before the leading aerodynamic variables have the following values:

Flight path angle	$\gamma_e = 0°$	Inertia product	$I_{xz} = 2952\,\text{kgm}^2$
Body incidence	$\alpha_e = 9.4°$	Air density	$\rho = 0.3809\,\text{kg/m}^3$
Velocity	$V_0 = 178\,\text{m/s}$	Wing area	$S = 49.239\,\text{m}^2$
Mass	$m = 17642\,\text{kg}$	Wing span	$b = 11.787\,\text{m}$
Roll moment of inertia	$I_x = 33898\,\text{kg m}^2$	Acceleration due	
Yaw moment of inertia	$I_z = 189496\,\text{kg m}^2$	to gravity	$g = 9.81\,\text{m/s}^2$

The dimensionless lateral derivatives, referred to body axes, are given and, as before, any missing aerodynamic derivatives must be assumed insignificant, and hence zero. Note that, in accordance with American notation the roll control derivative L_ξ is positive:

$Y_v = -0.5974$	$L_v = -0.1048$	$N_v = 0.0987$
$Y_p = 0$	$L_p = -0.1164$	$N_p = -0.0045$
$Y_r = 0$	$L_r = 0.0455$	$N_r = -0.1132$
$Y_\xi = -0.0159$	$L_\xi = 0.0454$	$N_\xi = 0.00084$
$Y_\zeta = 0.1193$	$L_\zeta = 0.0086$	$N_\zeta = -0.0741$

As for the longitudinal equations of motion, the lateral state equation (4.65) may be written in terms of the more convenient lateral dimensionless derivatives:

$$\mathbf{M}\dot{\mathbf{x}}(t) = \mathbf{A}'\mathbf{x}(t) + \mathbf{B}'\mathbf{u}(t)$$

where

$$\mathbf{x}^{\mathrm{T}}(t) = [v \quad p \quad r \quad \phi \quad \psi] \quad \mathbf{u}^{\mathrm{T}}(t) = [\xi \quad \zeta]$$

$$\mathbf{M} = \begin{bmatrix} m' & 0 & 0 & 0 & 0 \\ 0 & I'_x & -I'_{xz} & 0 & 0 \\ 0 & -I'_{xz} & I'_z & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{A}' = \begin{bmatrix} Y_v & (Y_p b + m'W_e) & (Y_r b - m'U_e) & m'g\cos\theta_e & m'g\sin\theta_e \\ L_v & L_p b & L_r b & 0 & 0 \\ N_v & N_p b & N_r b & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\mathbf{B}' = \begin{bmatrix} V_0 Y_\xi & V_0 Y_\zeta \\ V_0 L_\xi & V_0 L_\zeta \\ V_0 N_\xi & V_0 N_\zeta \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

where

$$m' = \frac{m}{\frac{1}{2}\rho V_0 S}, \quad I'_x = \frac{I_x}{\frac{1}{2}\rho V_0 Sb}, \quad I'_z = \frac{I_z}{\frac{1}{2}\rho V_0 Sb} \quad \text{and} \quad I'_{xz} = \frac{I_{xz}}{\frac{1}{2}\rho V_0 Sb}$$

and, as before, in steady symmetric flight, $U_e = V_0 \cos\theta_e$ and $W_e = V_0 \sin\theta_e$.

Substituting the appropriate values into the above matrices and pre-multiplying the matrices $\mathbf{A}'$ and $\mathbf{B}'$ by the inverse of the mass matrix $\mathbf{M}$ the concise lateral state equation (4.69), referred to body axes, is obtained:

$$\begin{bmatrix} \dot{v} \\ \dot{p} \\ \dot{r} \\ \dot{\phi} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} -0.0565 & 29.072 & -175.610 & 9.6783 & 1.6022 \\ -0.0601 & -0.7979 & -0.2996 & 0 & 0 \\ 9.218\times 10^{-3} & -0.0179 & -0.1339 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ p \\ r \\ \phi \\ \psi \end{bmatrix} + \begin{bmatrix} -0.2678 & 2.0092 \\ 4.6982 & 0.7703 \\ 0.0887 & -1.3575 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \xi \\ \zeta \end{bmatrix}$$

Again, the matrix computation was undertaken with the aid of *Program CC*. However, the coefficients of the matrices could equally well have been calculated using the expressions for the concise derivatives given in Appendix 2, Tables A2.7 and A2.8.

4.4.3 The equations of motion in American normalised form

The preferred North American form of the equations of motion expresses the axial equations of motion in units of linear acceleration, rather than force, and the angular equations of motion in terms of angular acceleration, rather than moment. This is easily achieved by *normalising* the force and moment equations, by dividing by mass or moment of inertia as appropriate. Re-stating the linear equations of motion (4.23):

$$\begin{aligned} m(\dot{u} + qW_e) &= X \\ m(\dot{v} - pW_e + rU_e) &= Y \\ m(\dot{w} - qU_e) &= Z \end{aligned} \tag{4.71}$$

$$\begin{aligned} I_x\dot{p} - I_{xz}\dot{r} &= L \\ I_y\dot{q} &= M \\ I_z\dot{r} - I_{xz}\dot{p} &= N \end{aligned}$$

the normalised form of the decoupled longitudinal equations of motion from equations (4.71) are written:

$$\begin{aligned} \dot{u} + qW_e &= \frac{X}{m} \\ \dot{w} - qU_e &= \frac{Z}{m} \\ \dot{q} &= \frac{M}{I_y} \end{aligned} \tag{4.72}$$

and the normalised form of the decoupled lateral–directional equations of motion may also be extracted from equations (4.71):

$$\begin{aligned} \dot{v} - pW_e + rU_e &= \frac{Y}{m} \\ \dot{p} - \frac{I_{xz}}{I_x}\dot{r} &= \frac{L}{I_x} \\ \dot{r} - \frac{I_{xz}}{I_z}\dot{p} &= \frac{N}{I_z} \end{aligned} \tag{4.73}$$

Further, both the rolling and yawing moment equations in (4.73) include roll and yaw acceleration terms, $\dot{p}$ and $\dot{r}$ respectively, and it is usual to eliminate $\dot{r}$ from the rolling moment equation and $\dot{p}$ from the yawing moment equation. This reduces equations (4.73) to the alternative form:

$$\begin{aligned} \dot{v} - pW_e + rU_e &= \frac{Y}{m} \\ \dot{p} &= \left(\frac{L}{I_x} + \frac{N}{I_z}\frac{I_{xz}}{I_x}\right)\left(\frac{1}{1 - I_{xz}^2/I_xI_z}\right) \\ \dot{r} &= \left(\frac{N}{I_z} + \frac{L}{I_x}\frac{I_{xz}}{I_z}\right)\left(\frac{1}{1 - I_{xz}^2/I_xI_z}\right) \end{aligned} \tag{4.74}$$

Now the decoupled longitudinal force and moment expressions as derived in Section 4.2, may be obtained from equations (4.40):

$$\begin{aligned} X &= \overset{\circ}{X}_u u + \overset{\circ}{X}_{\dot{w}}\dot{w} + \overset{\circ}{X}_w w + \overset{\circ}{X}_q q + \overset{\circ}{X}_\eta \eta + \overset{\circ}{X}_\tau \tau - mg\theta\cos\theta_e \\ Z &= \overset{\circ}{Z}_u u + \overset{\circ}{Z}_{\dot{w}}\dot{w} + \overset{\circ}{Z}_w w + \overset{\circ}{Z}_q q + \overset{\circ}{Z}_\eta \eta + \overset{\circ}{Z}_\tau \tau - mg\theta\sin\theta_e \\ M &= \overset{\circ}{M}_u u + \overset{\circ}{M}_{\dot{w}}\dot{w} + \overset{\circ}{M}_w w + \overset{\circ}{M}_q q + \overset{\circ}{M}_\eta \eta + \overset{\circ}{M}_\tau \tau \end{aligned} \tag{4.75}$$

Substituting equations (4.75) into equations (4.72), and after some rearrangement the longitudinal equations of motion may be written:

$$
\begin{aligned}
\dot{u} &= \frac{\mathring{X}_u}{m}u + \frac{\mathring{X}_{\dot{w}}}{m}\dot{w} + \frac{\mathring{X}_w}{m}w + \left(\frac{\mathring{X}_q}{m} - W_e\right)q - g\theta\cos\theta_e + \frac{\mathring{X}_\eta}{m}\eta + \frac{\mathring{X}_\tau}{m}\tau \\
\dot{w} &= \frac{\mathring{Z}_u}{m}u + \frac{\mathring{Z}_{\dot{w}}}{m}\dot{w} + \frac{\mathring{Z}_w}{m}w + \left(\frac{\mathring{Z}_q}{m} - U_e\right)q - g\theta\sin\theta_e + \frac{\mathring{Z}_\eta}{m}\eta + \frac{\mathring{Z}_\tau}{m}\tau \quad (4.76) \\
\dot{q} &= \frac{\mathring{M}_u}{I_y}u + \frac{\mathring{M}_{\dot{w}}}{I_y}\dot{w} + \frac{\mathring{M}_w}{I_y}w + \frac{\mathring{M}_q}{I_y}q + \frac{\mathring{M}_\eta}{I_y}\eta + \frac{\mathring{M}_\tau}{I_y}\tau
\end{aligned}
$$

Alternatively, equations (4.76) may be expressed in terms of American normalised derivatives as follows:

$$
\begin{aligned}
\dot{u} &= \mathrm{X}_u u + \mathrm{X}_{\dot{w}}\dot{w} + \mathrm{X}_w w + (\mathrm{X}_q - W_e)q - g\theta\cos\theta_e + \mathrm{X}_{\delta_e}\delta_e + \mathrm{X}_{\delta_{th}}\delta_{th} \\
\dot{w} &= \mathrm{Z}_u u + \mathrm{Z}_{\dot{w}}\dot{w} + \mathrm{Z}_w w + (\mathrm{Z}_q + U_e)q - g\theta\sin\theta_e + \mathrm{Z}_{\delta_e}\delta_e + \mathrm{Z}_{\delta_{th}}\delta_{th} \quad (4.77) \\
\dot{q} &= \mathrm{M}_u u + \mathrm{M}_{\dot{w}}\dot{w} + \mathrm{M}_w w + \mathrm{M}_q q + \mathrm{M}_{\delta_e}\delta_e + \mathrm{M}_{\delta_{th}}\delta_{th}
\end{aligned}
$$

and the control inputs are stated in American notation, elevator angle $\delta_e \equiv \eta$ and thrust $\delta_{th} \equiv \tau$.

In a similar way, the decoupled lateral–directional force and moment expressions may be obtained from equations (4.45):

$$
\begin{aligned}
Y &= \mathring{Y}_v v + \mathring{Y}_p p + \mathring{Y}_r r + \mathring{Y}_\xi \xi + \mathring{Y}_\zeta \zeta + mg\phi\cos\theta_e + mg\psi\sin\theta_e \\
L &= \mathring{L}_v v + \mathring{L}_p p + \mathring{L}_r r + \mathring{L}_\xi \xi + \mathring{L}_\zeta \zeta \quad (4.78) \\
N &= \mathring{N}_v v + \mathring{N}_p p + \mathring{N}_r r + \mathring{N}_\xi \xi + \mathring{N}_\zeta \zeta
\end{aligned}
$$

Substituting equations (4.78) into equations (4.74), and after some rearrangement the lateral–directional equations of motion may be written:

$$
\dot{v} = \frac{\mathring{Y}_v}{m}v + \left(\frac{\mathring{Y}_p}{m} + W_e\right)p + \left(\frac{\mathring{Y}_r}{m} - U_e\right)r + \frac{\mathring{Y}_\xi}{m}\xi + \frac{\mathring{Y}_\zeta}{m}\zeta + g\phi\cos\theta_e + g\psi\sin\theta_e
$$

$$
\dot{p} = \left(\begin{array}{l}
\left(\dfrac{\mathring{L}_v}{I_x} + \dfrac{\mathring{N}_v}{I_z}\dfrac{I_{xz}}{I_x}\right)v + \left(\dfrac{\mathring{L}_p}{I_x} + \dfrac{\mathring{N}_p}{I_z}\dfrac{I_{xz}}{I_x}\right)p + \left(\dfrac{\mathring{L}_r}{I_x} + \dfrac{\mathring{N}_r}{I_z}\dfrac{I_{xz}}{I_x}\right)r \\
+ \left(\dfrac{\mathring{L}_\xi}{I_x} + \dfrac{\mathring{N}_\xi}{I_z}\dfrac{I_{xz}}{I_x}\right)\xi + \left(\dfrac{\mathring{L}_\zeta}{I_x} + \dfrac{\mathring{N}_\zeta}{I_z}\dfrac{I_{xz}}{I_x}\right)\zeta
\end{array}\right)\left(\frac{1}{1 - I_{xz}^2/I_x I_z}\right)
$$

$$\dot{r} = \begin{pmatrix} \left(\dfrac{\overset{\circ}{N}_v}{I_z} + \dfrac{\overset{\circ}{L}_v}{I_x}\dfrac{I_{xz}}{I_z}\right)v + \left(\dfrac{\overset{\circ}{N}_p}{I_z} + \dfrac{\overset{\circ}{L}_p}{I_x}\dfrac{I_{xz}}{I_z}\right)p + \left(\dfrac{\overset{\circ}{N}_r}{I_z} + \dfrac{\overset{\circ}{L}_r}{I_x}\dfrac{I_{xz}}{I_z}\right)r \\ + \left(\dfrac{\overset{\circ}{N}_\xi}{I_z} + \dfrac{\overset{\circ}{L}_\xi}{I_x}\dfrac{I_{xz}}{I_z}\right)\xi + \left(\dfrac{\overset{\circ}{N}_\zeta}{I_z} + \dfrac{\overset{\circ}{L}_\zeta}{I_x}\dfrac{I_{xz}}{I_z}\right)\zeta \end{pmatrix} \left(\frac{1}{1 - I_{xz}^2/I_xI_z}\right) \tag{4.79}$$

As before, equations (4.79) may be expressed in terms of American normalised derivatives as follows:

$$\begin{aligned} \dot{v} &= \mathrm{Y}_v v + (\mathrm{Y}_p + W_e)p + (\mathrm{Y}_r - U_e)r + \mathrm{Y}_{\delta_a}\delta_a + \mathrm{Y}_{\delta_r}\delta_r + g\phi\cos\theta_e + g\psi\sin\theta_e \\ \dot{p} &= \begin{pmatrix} \left(\mathrm{L}_v + \mathrm{N}_v\dfrac{I_{xz}}{I_x}\right)v + \left(\mathrm{L}_p + \mathrm{N}_p\dfrac{I_{xz}}{I_x}\right)p + \left(\mathrm{L}_r + \mathrm{N}_r\dfrac{I_{xz}}{I_x}\right)r \\ + \left(\mathrm{L}_{\delta_a} + \mathrm{N}_{\delta_a}\dfrac{I_{xz}}{I_x}\right)\delta_a + \left(\mathrm{L}_{\delta_r} + \mathrm{N}_{\delta_r}\dfrac{I_{xz}}{I_x}\right)\delta_r \end{pmatrix} \left(\frac{1}{1 - I_{xz}^2/I_xI_z}\right) \\ \dot{r} &= \begin{pmatrix} \left(\mathrm{N}_v + \mathrm{L}_v\dfrac{I_{xz}}{I_z}\right)v + \left(\mathrm{N}_p + \mathrm{L}_p\dfrac{I_{xz}}{I_z}\right)p + \left(\mathrm{N}_r + \mathrm{L}_r\dfrac{I_{xz}}{I_z}\right)r \\ + \left(\mathrm{N}_{\delta_a} + \mathrm{L}_{\delta_a}\dfrac{I_{xz}}{I_z}\right)\delta_a + \left(\mathrm{N}_{\delta_r} + \mathrm{L}_{\delta_r}\dfrac{I_{xz}}{I_z}\right)\delta_r \end{pmatrix} \left(\frac{1}{1 - I_{xz}^2/I_xI_z}\right) \end{aligned} \tag{4.80}$$

and the control inputs are stated in American notation, aileron angle $\delta_a \equiv \xi$ and rudder angle $\delta_r \equiv \zeta$.

Clearly, the formulation of the rolling and yawing moment equations in (4.80) is very cumbersome, so it is usual to modify the definitions of the rolling and yawing moment derivatives to reduce equations (4.80) to the more manageable format:

$$\begin{aligned} \dot{v} &= \mathrm{Y}_v v + (\mathrm{Y}_p + W_e)p + (\mathrm{Y}_r - U_e)r + \mathrm{Y}_{\delta_a}\delta_a + \mathrm{Y}_{\delta_r}\delta_r + g\phi\cos\theta_e + g\psi\sin\theta_e \\ \dot{p} &= \mathrm{L}'_v v + \mathrm{L}'_p p + \mathrm{L}'_r r + \mathrm{L}'_{\delta_a}\delta_a + \mathrm{L}'_{\delta_r}\delta_r \\ \dot{r} &= \mathrm{N}'_v v + \mathrm{N}'_p p + \mathrm{N}'_r r + \mathrm{N}'_{\delta_a}\delta_a + \mathrm{N}'_{\delta_r}\delta_r \end{aligned} \tag{4.81}$$

where, for example, the modified normalised derivatives are given by expressions like

$$\begin{aligned} \mathrm{L}'_v &= \left(\mathrm{L}_v + \mathrm{N}_v\frac{I_{xz}}{I_x}\right)\left(\frac{1}{1 - I_{xz}^2/I_xI_z}\right) \equiv \left(\frac{\overset{\circ}{L}_v}{I_x} + \frac{\overset{\circ}{N}_v}{I_z}\frac{I_{xz}}{I_x}\right)\left(\frac{1}{1 - I_{xz}^2/I_xI_z}\right) \\ \mathrm{N}'_r &= \left(\mathrm{N}_r + \mathrm{L}_r\frac{I_{xz}}{I_z}\right)\left(\frac{1}{1 - I_{xz}^2/I_xI_z}\right) \equiv \left(\frac{\overset{\circ}{N}_r}{I_z} + \frac{\overset{\circ}{L}_r}{I_x}\frac{I_{xz}}{I_z}\right)\left(\frac{1}{1 - I_{xz}^2/I_xI_z}\right) \end{aligned} \tag{4.82}$$

and the remaining modified derivatives are defined in a similar way with reference to equations (4.79), (4.80) and (4.81). Thus the small perturbation equations of motion in American normalised notation, referred to aircraft body axes, are given by equations (4.77) and (4.81). A full list of the American normalised derivatives and their British equivalents is given in Appendix 7.

A common alternative formulation of the longitudinal equations of motion (4.77) is frequently used when the thrust is assumed to have a velocity or Mach number dependency. The normalised derivatives X_u, Z_u and M_u, as stated in equations (4.77), denote the aerodynamic derivatives only and the thrust is assumed to remain constant for small perturbations in velocity or Mach number. However, the notation X_u^*, Z_u^* and M_u^*, as shown in equations (4.83), denotes that the normalised derivatives include both the aerodynamic and thrust dependencies on small perturbations in velocity or Mach number.

$$\begin{aligned}
\dot{u} &= X_u^* u + X_{\dot{w}}\dot{w} + X_w w + (X_q - W_e)q - g\theta\cos\theta_e + X_{\delta_e}\delta_e + X_{\delta_{th}}\delta_{th} \\
\dot{w} &= Z_u^* u + Z_{\dot{w}}\dot{w} + Z_w w + (Z_q + U_e)q - g\theta\sin\theta_e + Z_{\delta_e}\delta_e + Z_{\delta_{th}}\delta_{th} \\
\dot{q} &= M_u^* u + M_{\dot{w}}\dot{w} + M_w w + M_q q + M_{\delta_e}\delta_e + M_{\delta_{th}}\delta_{th}
\end{aligned} \tag{4.83}$$

It is also common to express the lateral velocity perturbation v in equations (4.81) in terms of sideslip angle β, since for small disturbances $v = \beta V_0$:

$$\begin{aligned}
\dot{\beta} &= Y_v\beta + \frac{1}{V_0}(Y_p + W_e)p + \frac{1}{V_0}(Y_r - U_e)r + Y_{\delta_a}^*\delta_a + Y_{\delta_r}^*\delta_r \\
&\quad + \frac{g}{V_0}(\phi\cos\theta_e + \psi\sin\theta_e) \\
\dot{p} &= L'_\beta\beta + L'_p p + L'_r r + L'_{\delta_a}\delta_a + L'_{\delta_r}\delta_r \\
\dot{r} &= N'_\beta\beta + N'_p p + N'_r r + N'_{\delta_a}\delta_a + N'_{\delta_r}\delta_r
\end{aligned} \tag{4.84}$$

where

$$\begin{aligned}
Y_{\delta_a}^* &= \frac{Y_{\delta_a}}{V_0} \quad Y_{\delta_r}^* = \frac{Y_{\delta_r}}{V_0} \\
L'_\beta &= L'_v V_0 \quad N'_\beta = N'_v V_0
\end{aligned}$$

Equations (4.83) and (4.84) probably represent the most commonly encountered form of the American normalised equations of motion.

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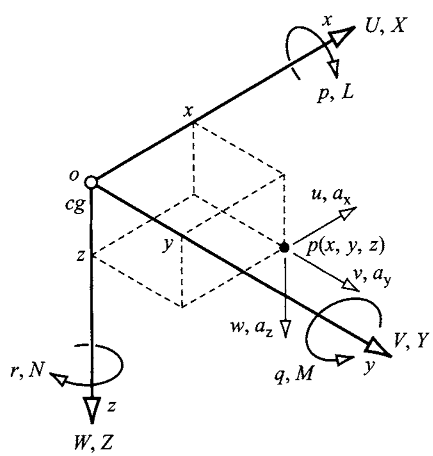
PROBLEMS

1. Given the dimensional longitudinal equations of motion of an aircraft in the following format

$$
\begin{aligned}
m\dot{u} - \mathring{X}_u u - \mathring{X}_w w - (\mathring{X}_q - mW_e)q + mg\theta\cos\theta_e &= \mathring{X}_\eta \eta \\
-\mathring{Z}_u u + m\dot{w} - \mathring{Z}_w w - (\mathring{Z}_q + mU_e)q + mg\theta\sin\theta_e &= \mathring{Z}_\eta \eta \\
-\mathring{M}_u u - \mathring{M}_{\dot{w}}\dot{w} - \mathring{M}_w w + I_y\dot{q} - \mathring{M}_q q &= \mathring{M}_\eta \eta
\end{aligned}
$$

rearrange them in dimensionless form referred to wind axes. Discuss the relative merits of using the equations of motion in dimensional, dimensionless and concise forms. (CU 1982)

2. The right handed orthogonal axis system (*oxyz*) shown in the figure below is fixed in a rigid airframe such that *o* is coincident with the centre of gravity.



The components of velocity and force along *ox*, *oy*, and *oz* are *U*, *V*, *W*, and *X*, *Y*, *Z* respectively. The components of angular velocity about *ox*, *oy*, and *oz* are *p*, *q*, *r* respectively. The point *p*(*x*, *y*, *z*) in the airframe has local velocity and acceleration components *u*, *v*, *w*, and a_x, a_y, a_z respectively. Show that by superimposing the motion of the axes (*oxyz*) on to the motion of the point *p*(*x*, *y*, *z*), the absolute acceleration components of *p*(*x*, *y*, *z*) are given by

$$
\begin{aligned}
a'_x &= \dot{U} - rV + qW - x(q^2 + r^2) + y(pq - \dot{r}) + z(pr + \dot{q}) \\
a'_y &= \dot{V} - pW + rU + x(pq + \dot{r}) - y(p^2 + r^2) + z(qr - \dot{p}) \\
a'_z &= \dot{W} - qU + pV + x(pr - \dot{q}) + y(qr + \dot{p}) - z(p^2 + q^2)
\end{aligned}
$$

Further, assuming the mass of the aircraft to be uniformly distributed show that the total body force components are given by

$$
\begin{aligned}
X &= m(\dot{U} - rV + qW) \\
Y &= m(\dot{V} - pW + rU) \\
Z &= m(\dot{W} - qU + pV)
\end{aligned}
$$

where m is the mass of the aircraft. (CU 1986)

3. The linearised longitudinal equations of motion of an aircraft describing small perturbations about a steady trimmed rectilinear flight condition are given by

$$
\begin{aligned}
m(\dot{u}(t) + q(t)W_e) &= X(t) \\
m(\dot{w}(t) - q(t)U_e) &= Z(t) \\
I_y\dot{q}(t) &= M(t)
\end{aligned}
$$

Develop expressions for $X(t)$, $Z(t)$ and $M(t)$ and hence complete the equations of motion referred to generalised aircraft body axes. What simplifications may be made if a wind axes reference and level flight are assumed? (CU 1987)

4. State the assumptions made in deriving the small perturbation longitudinal equations of motion for an aircraft. For each assumption give a realistic example of an aircraft type, or configuration, which may make the assumption invalid. (LU 2002)

5. Show that when the product of inertia I_{xz} is much smaller than the moments of inertia in roll and yaw, I_x and I_z respectively, the lateral–directional derivatives in modified American normalised form may be approximated by the American normalised form.