Appendix 11

The Root Locus Plot

MATHEMATICAL BACKGROUND

Given the general closed loop system transfer function

$$\frac{r(s)}{c(s)} = \frac{G(s)}{1 + G(s)H(s)} \tag{A11.1}$$

where r is the response to a command input c, $G(s)$ is the transfer function of the open loop system and $H(s)$ is the transfer function of the feedback controller located in the feedback path. The closed loop characteristic equation is given by the denominator of equation (A11.1)

$$1 + G(s)H(s) = 0 \tag{A11.2}$$

Now, in general, the transfer function product $G(s)H(s)$ will itself be a transfer function and may be expressed as the ratio of two polynomials

$$G(s)H(s) = \frac{K_1(1 + sT_1)(1 + sT_3)\cdots}{s^n(1 + sT_2)(1 + sT_4)\cdots} \tag{A11.3}$$

or, alternatively,

$$G(s)H(s) = \frac{K\left(s + \frac{1}{T_1}\right)\left(s + \frac{1}{T_3}\right)\cdots}{S^n\left(s + \frac{1}{T_2}\right)\left(s + \frac{1}{T_4}\right)\cdots} \tag{A11.4}$$

where the gain constant is given by

$$K = \frac{K_1 T_1 T_3 \ldots}{T_2 T_4 \ldots} \tag{A11.5}$$

Each factor in equation (A11.4) may be expressed alternatively in terms of magnitude and phase, assuming sinusoidal command and response such that $s = j\omega$, for example,

$$\left(s + \frac{1}{T_1}\right) = A_1 e^{j\phi_1} \tag{A11.6}$$

whence, equation (A11.4) may be written

$$G(s)H(s) = \frac{KA_1A_3\ldots e^{j((\phi_1+\phi_3+\cdots)-(n\phi_0+\phi_2+\phi_4+\cdots))}}{A_0^nA_2A_4\ldots} \equiv Ae^{j\phi} \tag{A11.7}$$

where, the total magnitude A is given by

$$A = \frac{KA_1A_3\ldots}{A_0^nA_2A_4\ldots} \tag{A11.8}$$

and the total phase is given by

$$\phi = (\phi_1 + \phi_3 + \cdots) - (n\phi_0 + \phi_2 + \phi_4 + \cdots) \tag{A11.9}$$

Thus, the characteristic equation (A11.2) may be written

$$1 + G(s)H(s) \equiv 1 + Ae^{j\phi} = 0 \tag{A11.10}$$

which has solution

$$Ae^{j\phi} = -1 \tag{A11.11}$$

For the solution of equation (A11.11) to exist two conditions must be satisfied:

(i) The *angle condition*

$$\phi = (\phi_1 + \phi_3 + \cdots) - (n\phi_0 + \phi_2 + \phi_4 + \cdots) = (2k+1)180\,\text{deg} \tag{A11.12}$$

where $k = 0, \pm1, \pm2, \pm3, \ldots$

(ii) The *magnitude condition*

$$|G(s)H(s)| = A = \frac{KA_1A_3\ldots}{A_0^nA_2A_4\ldots} = 1 \tag{A11.13}$$

Thus any point in the s-plane where the conditions defined by both equations (A11.12) and (A11.13) are satisfied defines a root of the characteristic equation. By finding all such points in the s-plane a locus of the roots of the characteristic equation may be constructed. In fact the root loci may be identified merely by satisfying the angle condition only, the loci may then be *calibrated* by applying the magnitude condition to selected points of interest on the loci.

THE RULES FOR CONSTRUCTING A ROOT LOCUS PLOT

The simple closed loop system of interest is defined by the structure shown in Fig. A11.1. The object is to establish how the roots of the closed loop transfer function

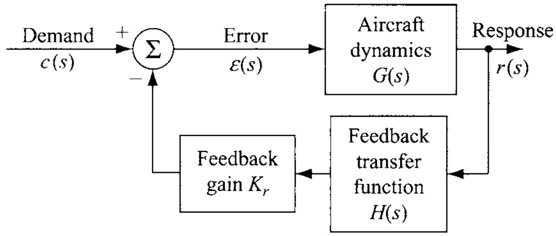

Figure A11.1 *A simple closed loop system.*

are governed by the choice of feedback gain K_r. The *open loop* transfer function of the system is known at the outset and comprises the product of the transfer functions of all the system components in the loop

$$K_r G(s)H(s) \tag{A11.14}$$

The corresponding *closed loop* transfer function is

$$\frac{r(s)}{c(s)} = \frac{G(s)}{1 + K_r G(s)H(s)} \tag{A11.15}$$

The root locus plot is constructed from the open loop transfer function (A11.14) which should be in factorised form for convenience. The *zeros* are the numerator roots and the *poles* are the denominator roots of equation (A11.15). When plotting the root loci on the s-plane it is often convenient to choose the same numerical scales for both the real and imaginary axes.

Rule 1
Continuous curves which comprise the branches of the locus start at the poles of $G(s)H(s)$ where the gain $K_r = 0$. The branches of the locus terminate at the zeros of $G(s)H(s)$, or at infinity, where the gain $K_r = \infty$.

Rule 2
The locus includes all points on the real axis to the left of an odd number of poles plus zeros.

Rule 3
As $K_r \to \infty$, the branches of the locus become asymptotic to straight lines with angles,

$$\frac{(2k+1)180}{n_p - n_z} \text{ deg} \quad k = 0, \pm 1, \pm 2, \pm 3, \ldots$$

where
n_p is the number of poles and n_z is the number of zeros.

Rule 4
The asymptotes radiate from a point on the real axis called the *centre of gravity* (*cg*) of the plot and is determined by

$$cg = \frac{\sum poles - \sum zeros}{n_p - n_z}$$

Rule 5
The loci *break in* to or *break away* from points on the real axis located between pairs of zeros or pairs of poles respectively. Two methods may be used to estimate the locations of the break-in or break-away points on the real axis. The first method is approximate and gives results of acceptable accuracy for the majority of cases. The second method is exact and may be used when the first method gives unsatisfactory results:

(i) *Method 1*

- Select a test point on the real axis in the vicinity of a known break-in or break-away point.
- Measure the distances from the test point to each real axis pole and zero. Assign a negative sign to the pole distances, a positive sign to the zero distances and calculate the reciprocals of the distances.
- Calculate the sum of the reciprocals for all poles and zeros to the left of the test point and calculate the sum of the reciprocals for all poles and zeros to the right of the test point.
- The test point is a break-in or break-away point when the left and right reciprocal sums are equal.
- Choose a new test point and iterate until the break-in or break-away point is obtained with acceptable accuracy.
- Note that this method may give inaccurate results when complex poles and zeros lie close to the real axis.

(ii) *Method 2*

- Denote the open loop transfer function

$$G(s)H(s) = \frac{A(s)}{B(s)} \tag{A11.16}$$

- Define a function $F(s)$

$$F(s) = B(s)\frac{\mathrm{d}A(s)}{\mathrm{d}s} - A(s)\frac{\mathrm{d}B(s)}{\mathrm{d}s} \tag{A11.17}$$

- The roots of $F(s)$ include all the break-in or break-away points.

Rule 6
Loci branching in to, or away from the real axis do so at 90° to the real axis.

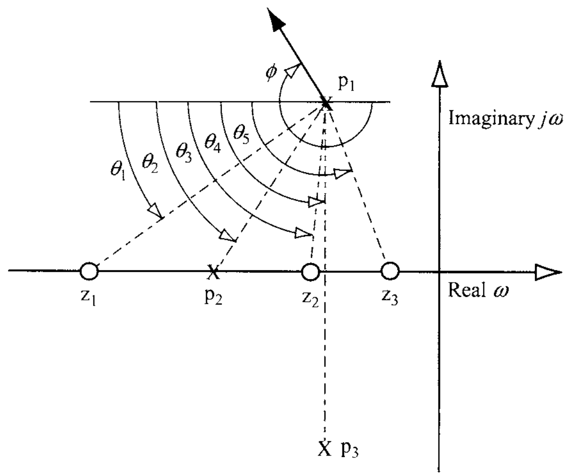

Figure A11.2 *Example of the locus departure angle from a complex pole.*

Rule 7

The angle of departure of a locus from a complex pole, or the angle of arrival at a complex zero, is given by

$$\phi - \sum(\text{angles to all other zeros}) - \sum(\text{angles to all other poles}) - 180\,\text{deg} \tag{A11.18}$$

An example is illustrated in Fig. A11.2.

Thus with reference to Fig. A11.2 the angle of departure of the locus from complex pole p_1 is given by

$$\phi = (\theta_1 + \theta_3 + \theta_5) - (\theta_2 + \theta_4) - 180\,\text{deg} \tag{A11.19}$$

Rule 8

The total loop gain at any point on a locus is given by

$$\text{Gain} = \frac{\Pi(\text{distances from test point to poles})}{\Pi(\text{distances from test point to zeros})} \tag{A11.20}$$

Note that if the system under investigation has no zeros then the denominator of expression (A11.20) is taken to be unity.