On Borel–Cantelli Lemma

Borel–Cantelli lemma is a theorem about sequences of events. It is named after emile Borel and Francesco Paolo Cantelli ([E. Borel,1909,F.P. Cantelli,1917]). The Borel–Cantelli lemma states that:
(1) If $\Sigma_{n=1}^{\infty} P(A_n) < \infty$, then

$$P\{A_n i.o.\} = P(\limsup A_n) = 0.$$

(2) If A_n are independent and $\Sigma_{n=1}^{\infty} P(A_n) = \infty$, then

$$P\{A_n i.o.\} = P(\limsup A_n) = 1.$$

In this paper, we review some extensions of the original Borel–Cantelli Lemma.

In Barndorff-Nielsen's paper ([Barndorff-Nielsen,1961]), a general form of the Law of the Iterated Logarithm for Maximal Order Statistics is established by means of a generalization of the convergence part of the Borel-Cantelli lemmas. The generalized Lemma is as follows:

Lemma: If

$$\liminf_{n\to\infty} P(A_n) = 0$$

and

$$\sum_{n=1}^{\infty} P(A_n \cap A_{n+1}^c) < \infty,$$

then one has

$$P(\limsup A_n) = 0.$$

Proof: Set $B_n = A_n \cap A_{n+1}^c$, $E = \limsup A_n$ and $F = \limsup A_n^c$. We have

$$P(\limsup B_n) \leq P(F^c) = P(\liminf A_n) \leq \liminf_{n\to\infty} P(A_n) = 0.$$

Observe that $E \cap F \subset \limsup B_n$. To see this, fix $n \geq 1$ and $\omega \in E$, $\exists m \geq n$ such that $\omega \in A_m$. Put $l = inf\{k \geq m : \omega \in A_k^c\}$; by definition of F,
$l < \infty$ and $\omega \in A_l \in A_{l-1}^c$. Since n is arbitrary, $\omega \in \limsup B_n$. Hence

$$P(E) \leq P(F^c) + P(E \cap F) = 0.$$

The Extended Renyi-Lamperti lemma relaxes the assumption of independence.

Theorem: If

$$\sum_{n=1}^{\infty} P(A_n) = \infty$$

and

$$\liminf_{n\to\infty} \frac{\sum_{j=1}^{n}\sum_{k=}^{n} P(A_i \cap A_j)}{(\sum_{j=1}^{n} P(A_j))^2} = c,$$

then

$$P(\limsup A_n) \geq 1/c.$$

Proof: Recall that if $EX > 0$ and $0 \leq \epsilon < 1$, applying Cauchy-Schwarz inequality to $E(XI\{X > \epsilon EX\})$ we have

$$P(X > \epsilon EX) \geq (1-\epsilon)^2 (EX)^2/EX^2.$$

Put $J_n = \Sigma_{j=1}^{n} I(A_j)$ and $s_n = \Sigma_{j=1}^{n} P(A_j) = EJ_n$. Fix $0 < \epsilon < 1$ and put $B_n = \{J_n \geq \epsilon s_n\}$. As $s_n \to \infty$, $\limsup B_n \subset \limsup A_n$. Now $P(B_n) \geq (1-\epsilon)^2 (EJ_n)^2/EJ_n^2$, and so $\limsup_{n\to\infty} P(B_n) \geq (1-\epsilon)^2/c$. As $\epsilon > 0$ is arbitrary, the proof is complete.

T Chandra and S Ghosal ([T. K. Chandra,1993]) gave another extension in 1993.

Theorem: Suppose $\{A_n\}$ is a sequence of events satisfying

$$P(A_i \cap A_j) - P(A_i)P(A_j) \leq q(j-i)P(A_j), i < j,$$

where $q(m) \geq 0, \forall m \geq 1$ and

$$\frac{\sum_{m=1}^{\infty} q(m)}{\sum_{j=1}^{m} P(A_j)} \leq \infty.$$

If

$$\sum_{n=1}^{\infty} P(A_n) = \infty,$$

then

$$P(\limsup A_n) = 1.$$

A Borel-Cantelli lemma for *-mixing sequences was given by J. R. BLC~, D. L. HANSON and L. II. KOOPMANS ([J. R. BLC,1963]).

Definition: The sequence $\{X_n\}$ will be called *-mixing if there exists a positive integer N and a real-valued function f defined for the integers $n \geq N$ such that (i) f is non-increasing with $lim_{n\to\infty} f(n) = 0$, and (ii) if $n \geq N$,

$$A \in \sum_{1}^{m} (\sigma < x_1, \cdots x_m >),$$

$$B \in \sum_{m+n} (\sigma < x_{m+n} >),$$

then

$$|P(AB) - P(A)P(B)| \leq f(n)P(A)P(B).$$

Theorem: Let $\{A_n\}$ be a sequence of events such that $\{I(A_n)\}$ is *-mixing and

$$\sum_{n=1}^{\infty} P(A_n) = \infty.$$

Then

$$P(\limsup A_n) = 1.$$

Proof: Let $0 < \delta < 1$ and get $k > N$ such that $f_k < \delta$. Get $1 \leq j \leq k$ such that $\Sigma_{n=0}^{\infty} P(A_{nk+j}) = \infty$. Set $B_n = A_{nk+j}$. It suffices to show that $P(\limsup B_n) = 1$. If not, there exists $m \geq 1$ such that $P(\cup_{i=1}^{m} B_i) < 1$. Hence

$$P(\cap_{i=1}^{m} B_i) = P(B_m) + \Sigma_{i=1}^{\infty} P(B_{m+t} \cap_{s=0}^{t-1} B_{m+s}^c)$$

$$\geq (1-\delta)[P(B_m) + \Sigma_{t=1}^{\infty} P(B_{m+t})P(\cap_{s=0}^{t-1} B_{m+s}^c)]$$

$$\geq (1-\delta)P(\cap_{s=0}^{t-1} B_{m+s}^c)\Sigma_{t=0}^{\infty} P(B_{m+t}) = \infty,$$

which is a contradiction.

The following version of Borel-Cantelli lemma is due to R. J. Serfling([R. J. Serfling,1975]), which obtained a new extension for dependent events of the divergent part of the Borel-Cantelli lemma.

Theorem: Let $\{A_n\}$ be a sequence of events and let

$$\mathcal{F}_n = \sigma < A_1, \cdots, A_n > .$$

Then

$$P\{(\limsup A_n \Delta \{\sum_{n=1}^{\infty} P(A_n | \mathcal{F}_{n-1}) = \infty\}\} = 0,$$

where Δ stands for the symmetric difference.

Proof: [R. J. Serfling,1975] has shown that for all $m < N$,

$$P(\cap_{n=m}^{N} A_n^c) \leq exp[-\Sigma_{n=m}^{N} P(A_n)] + \Sigma_{n=m}^{N} E|P(A_n|\mathrm{F}_{n-1}) - P(A_n)|.$$

Let $N \to \infty$ and then $m \to \infty$, the results follows.

In 1970, J. Shuster([Shuster, 1970]) provided a generalized version of the first and the second Borel-Cantelli Lemma through giving lower bound for $p(\cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k)$ The theorem below is the main result of J.Shuster's article.

Theorem:

(a) If there exists an $A \in F$ such that

$$\sum_{k=1}^{\infty} P(A \cap A_k) < \infty$$

then

$$P(A_n i.o.) \leq 1 - P(A)$$

(b) If for every set $A \in F$ such that $P(A) > 0$ it holds that

$$\sum_{k=1}^{\infty} P(A \cap A_k) = \infty$$

then

$$P(A_n i.o.) = 1$$

proof of (a) To see that (a) theorem holds, we can look at the indicator function I_k of the event $A \cap A_k$. Now, define $T = \sum_{k=1}^{\infty} I_k$. Consider the expected value of T. Because $E(T) = \sum_{k=1}^{\infty} P(A \cap A_k)$ and, by the hypothesis of (a),$\sum_{k=1}^{\infty} P(A \cap A_k) = \infty$, we have $E(T) < \infty$. But this means that only a finite number of $A \cap A_k$occur. This means that the probability $P(A_n i.o.)$is at most equal to $P(A^c) = 1 - P(A)$, which is what we wanted to prove. Proof of (b) To see why (b) is true, define the set

$$B_n = \cap_{k=n}^{\infty} A_k^c = A_n^c \cap A_{n+1}^c \cap A_{n+2}^c \cap \ldots$$

If we look at the event $A_k \cap B_n$, we can see that if $k \geq n$then $P(A_k \cap B_n) = 0$ This means that

$$\sum_{k=1}^{\infty} P(A_k \cap B_n) = \sum_{k=1}^{n-1} P(A_k \cap B_n)$$

Note that the sum on the right-hand side is finite. This implies that we must have that $P(B_n) = 0$ and this means that

$$P(\cup_{n=1}^{\infty} B_n) = P(\cup_{n=1}^{\infty} A_k^c = 0) \rightrightarrows P(\cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k) = 1$$

and this is what we wanted to show.

Clearly, this theorem is a generalization of the original Borel-Cantelli Lemmas. When we set $A = \Omega$, (a) is precisely what the original first Borel-Cantelli Lemma stated. Meanwhile, if we compare (b) with the original second Borel-Cantelli Lemma, we can see that the independence criterion is dropped altogether.

A generalization of the Erdos–Renyi formulation of the Borel–Cantelli lemma is obtained by Valentin V.Petrov ([Valentin V.Petrov,2002] and [Valentin V.Petrov,2004]).

Theorem A: Let $A_1, A_2 \cdots$ be a sequence of events satisfying conditions $\sum_{n=1}^{\infty} P(A_n) = \infty$ and $P(A_k A_j) \leq CP(A_k)P(A_j)$ for all $k, j > L$ such that $k = j$ and for some constants $C \geq 1$ and L. Then $P(\limsup A_n) \geq 1/C$.

Theorem B: Let $A_1, A_2 \cdots$ be a sequence of events satisfying conditions

$$\sum_{n=1}^{\infty} P(A_n) = \infty.$$

Let H be an arbitrary real constant. Put

$$\alpha_H = \liminf \frac{\sum_{1\leq i<k\leq n}(P(A_iA_k) - HP(A_i)P(A_k))}{(\sum_{k=1}^{n} P(A_k))^2}.$$

Then

$$P(\limsup A_n) \leq \frac{1}{H + 2\alpha_H}.$$

Remark: $P(A_n i.o.) = \alpha$ and equivalent statements

Theorem: Let $0 < \alpha \leq 1$. The following statements are equivalent:

1. $P(A_n i.o.) \geq \alpha$
2. $\Sigma_{n=1}^{\infty} P(A_n \cap B) = \infty$ for any $B \in \mathcal{F}$ with $P(B) > 1 - \alpha$
3. for any $B \in \mathcal{F}$ with $P(B) > 1 - \alpha$, the sequence $\{P(A_n \cap B)\}$ contains an infinite number of positive numbers.

Proof: First note that (2)⇒(3),

if not, only a finite number of $P(A_n \cap B) > 0$ so $\Sigma_{n=1}^{\infty} P(A_n \cap B_n) \neq \infty$.

(1)⇒(2):

Suppose $P(A_n i.o.) \geq \alpha$ but assume that $\Sigma_{n=1}^{\infty} P(A_n \cap B) < \infty$. Then according to Borel-Cantelli lemma. $P(A_n i.o.) + P(B) - 1 \leq P(A_n \cap B i.o.) = 0$. This implies $P(B) \leq 1 - \alpha$.

(3)⇒(1): Suppose that (3) holds but $P(A_n i.o.) < \alpha$, we have $P(\cup_{k=n}^{\infty} A_n) < \alpha$ for some n. Define $C = \cup_{k=n}^{\infty} A_k$ and $B = C^c$, then $P(B) > 1 - \alpha$ and $B \cap C = B \cap \cup_{k=n}^{\infty} A_k = \phi$, which means $P(A_k \cap B) = 0$ for all $k \geq n$. This is a contradiction. WenLiu, Jia-anYan and Weiguo Yangc generalized the extended Borel–Cantelli lemma as corollaries([WenLiu,2003]).

Corollary: Let $(\mathcal{F}_n, n \geq 0)$ be an increasing sequence of σ-algebras and let$B_n \in \mathcal{F}_n$. Put

$$A = \{\sum_{i=1}^{\infty} I_{B_i} = \infty\}; B = \{\sum_{i=1}^{\infty} P(B_i|\mathcal{F}_{i-1}) = \infty\}.$$

Then we have $A = B$ a.s., and

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} I_{Bi}}{\sum_{i=1}^{n} P(B_i|\mathcal{F}_{i-1})} = 1$$

a.s. on B.

The first part of this corollary is the Extended Borel–Cantelli Lemma (Chow,1988), and the second part of this corollary is the sharper form of the Borel–Cantelli lemma (Dubins,1965).

Generalizations of the second Borel–Cantelli lemma are obtained under very weak dependence conditions by Tapas Kumar Chandra(Tapas Kumar Chandra,2008), subsuming several earlier results as special cases.

Let $\{A_n\} n \geq 1$, satisfy

$$\Sigma_{n=1}^{\infty} P(A_n) = 0$$

and
$$\liminf_{n\to\infty} \frac{\Sigma_{1\leq j<k\leq n}(P(A_j\cap A_k)-a_{ij})}{(\Sigma_{1\leq k\leq n}P(A_k))^2} = L,$$

where a_{ij} satisfy

$$\Sigma_{1\leq j<m\leq k\leq n}|a_{ij}| = o((\Sigma_{m\leq k\leq n}P(A_k))^2)$$

$\forall m \geq 1$ and
$$\liminf_{m\to\infty}\limsup_{n\to\infty} \frac{\Sigma_{1\leq j<k\leq n}a_{jk}}{(\Sigma_{m\leq k\leq n}P(A_k))^2} \leq d.$$

Assume that L and d are finite. Then

$$d+L\geq \frac{1}{2}$$

and
$$P(\limsup A_n) \geq (2d+2L)^{-1}.$$

Borel-Cantelli lemma when the condition of independent events is replaced by negative quadrant dependent condition was discussed by Gholam Hossein Yari and Farhad Hossein Zadeh([Gholam Hossein Yari,2010]).

Definition: A sequence of random variables $X_n; n\geq 1$ is said to be pairwise negative quadrant dependent (NQD) if,

$$P(X_i\leq x, Y_j\leq y)\leq P(X_i\leq x)P(Y_j\leq y)$$

for all $x, y\in R$ and for all $i, j\geq 1$, $i{\neq}j$.

Theorem: Let $A_{n}{}_{n=1}^{\infty}$ be a sequence of events in a probability space

$$(\Omega, \mathrm{A}, \mathrm{P})$$

and set
$$\limsup_{n\to\infty} A_n = A.$$

If
$$\Sigma_{n=1}^{\infty}P(A_n)=\infty,$$

and
$$\limsup_{n\to\infty} \frac{(\Sigma_{i=1}^{\infty}P(A_i))^2}{\Sigma_{i=1}^{n}\Sigma_{j=1}^{n}P[A_i\cap A_j]} = \alpha > 0,$$

then, (a)$P[A]\geq \alpha$. (b) If the events are pairwise (NQD) then $P[A]=1$.

From those theorems above, we can see extended Borel–Cantelli lemmas can relax parts of the assumptions in the original one such as independence. However, the drawback is obvious that we either need some more complicated conditions or can only get some weaker results. The beauty in the original Borel–Cantelli lemma's simplicity and interpretability is lost. Many of those theorem are derived as lemmas for varied purposes or specified problems. Although it seems that most of those lemmas are more general, they're not as popular as the original one because those extended lemmas are more difficult to explain and remember.

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