Some Basic Indefinite Integral Formulas
By Bruce Zhou

1. $\int x^{\alpha}dx = \frac{x^{\alpha+1}}{\alpha+1} + c$, $\alpha \neq 1$

2. $\int \frac{1}{x}dx = ln|x| + c$

3. $\int e^{x}dx = e^{x} + c$

4. $\int a^{x}dx = \frac{a^{x}}{lna} + c$, $(a > 0, a \neq 1)$

5. $\int cosxdx = sinx + c$

6. $\int sinxdx = -cosx + c$

7. $\int \frac{1}{cos^2x}dx = tanx + c$

8. $\int \frac{1}{sin^2x}dx = -cotx + c$

9. $\int \frac{1}{\sqrt{1-x^2}}arcsinx + c$

10. $\int \frac{1}{1+x^2}dx = arctanx + c$

11. $\int \frac{1}{a^2+x^2}dx = \frac{1}{a}arctan\frac{x}{a} + c$
Pf: $\int \frac{1}{a^2+x^2}dx = \frac{a}{a^2}\int \frac{1}{(\frac{x}{a})^2+1}d\frac{x}{a}$, use 10.

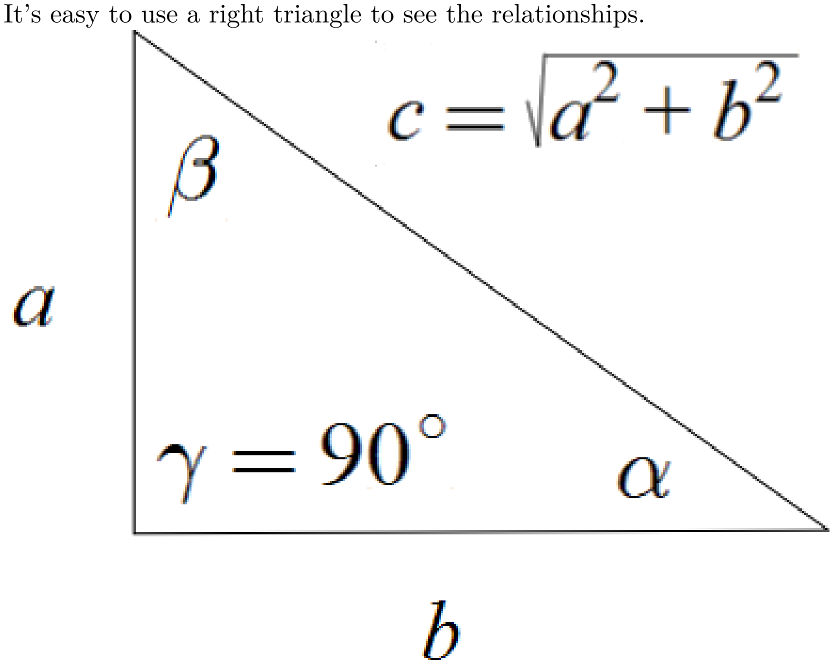
12. $\int \frac{1}{\sqrt{a^2-x^2}}dx = arcsin\frac{x}{a} + c$
Pf: $\int \frac{1}{\sqrt{a^2-x^2}}dx = \int \frac{dx}{a\sqrt{1-(\frac{x}{a})^2}} = \int \frac{d\frac{x}{a}}{\sqrt{1-(\frac{x}{a})^2}}$, use 9.

13(a). $\int \frac{1}{a^2-x^2}dx = \frac{1}{2a}ln|\frac{a+x}{a-x}| + c$
Pf: $\int \frac{1}{a^2-x^2}dx = \int \frac{dx}{(a-x)(a+x)} = \frac{1}{2a}\int(\frac{1}{a-x} + \frac{1}{a+x})dx$
use 2, $= \frac{1}{2a}(-ln|a-x| + ln|a+x|) + c = \frac{1}{2a}ln|\frac{a+x}{a-x}| + c.$
Similarly, we have
13(b). $\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}ln|\frac{x-a}{x+a}| + c$
Pf: $\int \frac{1}{x^2-a^2}dx = \int \frac{dx}{(x-a)(x+a)} = \frac{1}{2a}\int(\frac{1}{x-a} - \frac{1}{x+a})dx$
$= \frac{1}{2a}(ln|x-a| - ln|x+a|) + c = \frac{1}{2a}ln|\frac{x-a}{x+a}| + c.$

14. $\int \frac{1}{\sqrt{x^2+a^2}}dx = ln|x + \sqrt{x^2+a^2}| + c$
Pf: Let $x = arctant$, then $dx = a \times sec^2tdt$, $\sqrt{x^2+a^2} = a \times sect$
so left side $= \int \frac{1}{asect}a \times sec^2tdt = \int \frac{1}{cost}dt$
use 20, $= ln|tan(\frac{t}{2} + \frac{\pi}{4})| + c = ln|sect + tant| + c$
$= ln|x + \sqrt{x^2+a^2}| + c.$

Note: Some basic triangular transformation for square root type integrals:

(I)sin/cos type $\sqrt{a^2-x^2}$
let $x = a \times sin\theta$, $dx = a \times cos\theta$; or $x = a \times cos\theta$, $dx = -asin\theta d\theta$
$sin^2\theta + cos^2\theta = 1$
(II)sec/csc type $\sqrt{x^2-a^2}$
let $x = a \times sec\theta$, $dx = a \times sec\theta tan\theta d\theta$; or $x = a \times csc\theta$, $dx = -a \times csc\theta cot\theta$
$sec^2\theta = 1 + tan^2\theta$
(III)tan/cot type $\sqrt{x^2+a^2}$
let $x = a \times tan\theta$, $dx = a\frac{1}{cos^2\theta}$; or $x = a \times cot\theta$, $dx = -a\frac{1}{sin^2\theta}$
It's easy to use a right triangle to see the relationships.



$sin\alpha = \frac{a}{c}$, $cos\alpha = \frac{b}{c}$, $tan\alpha = \frac{a}{b}$, $cot\alpha = \frac{b}{c}$;
$sec\alpha = \frac{1}{cos\alpha} = \frac{c}{b}$, $csc\alpha = \frac{1}{sin\alpha} = \frac{c}{a}$.
$(cscx)' = -cscxcotx$, $(secx)' = secxtanx$.
(IV)$^m\sqrt{ax+b} = t$, $x = \frac{t^m - b}{a}$, $dx = \frac{1}{a}mt^{m-1}dt$
(V)$^m\sqrt{x}$, $x = t^m$, $dx = mt^{m-1}$;
$^m\sqrt{x}$ and $^n\sqrt{x}$, let $t =^{mn} \sqrt{x}$, so $^m\sqrt{x} = t^n$, $^n\sqrt{x} = t^m$
$dx = dt^{mn} = (mn)t^{mn-1}dt$

15. $\int \frac{1}{\sqrt{x^2-a^2}}dx = ln|x + \sqrt{x^2-a^2}| + c$
Pf: let $x = a \times sect$, $\sqrt{x^2-a^2} = a \times tant$,
$dx = d(a \times sect) = a\frac{sint}{cos^2t}dt = a \times sect \times tant \times dt$
left side $= \int \frac{a \times sect \times tant}{a \times tant}dt = \int \frac{dt}{cost}$
use 20, $= ln|sect + tant| + c = ln|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}| + c$
$= ln|x + \sqrt{x^2-a^2}| - lna + c$
Similarly, $\int \frac{1}{(x^2-a^2)^{\frac{3}{2}}}dx = -\frac{x}{a^2\sqrt{x^2-a^2}} + c$

Pf: let $x = asec\theta$, $dx = ad\frac{1}{cos\theta} = a\frac{sin\theta}{cos^2\theta}d\theta$,
$(x^2 - a^2)^{\frac{3}{2}} = (a^2tan^2\theta)^{\frac{3}{2}} = a^3tan^3\theta$.
left side $= \frac{a\frac{sin\theta}{cos^2\theta}d\theta}{a^3tan^3\theta} = \frac{1}{a^2}\frac{cos\theta}{sin^2\theta}d\theta = \frac{1}{a^2}\frac{dsin\theta}{sin^2\theta} = -\frac{1}{a^2}\frac{1}{sin\theta} = -\frac{x}{a^2\sqrt{x^2-a^2}} + c$.

16. $\int\sqrt{a^2 - x^2}dx = \frac{1}{2}x\sqrt{a^2 - x^2} + \frac{a^2}{2}arcsin\frac{x}{a} + c$, $(x^2 \leq a^2 \Rightarrow -a \leq x \leq a)$
Pf(a): let $x = asint$, $dx = acostdt$, $\sqrt{a^2 - x^2} = acost$;
left side $= \int acostacostdt = a^2\int cos^2tdt$
$= \frac{a^2}{2}\int(cos2t + 1)dt = \frac{a^2}{2}(\frac{1}{2}sin2t + t) + c$
$sin2t = 2sintcost = 2\frac{x}{a}\frac{\sqrt{a^2-x^2}}{a}$
so $= \frac{1}{2}x\sqrt{a^2 - x^2} + \frac{a^2}{2}arcsin\frac{x}{a} + c$
Pf(b): integral by parts, left side $= x\sqrt{a^2 - x^2} - \int xd\sqrt{a^2 - x^2}$
$= x\sqrt{a^2 - x^2} - \int\frac{-2x^2\times\frac{1}{2}}{\sqrt{a^2-x^2}}dx = x\sqrt{a^2 - x^2} - \int\frac{-x^2}{\sqrt{a^2-x^2}}dx$
note that for $\frac{-x^2}{\sqrt{a^2-x^2}}$, the degree of numerator is higher than the degree of denominator,
$\int\frac{-x^2}{\sqrt{a^2-x^2}}dx = \int\frac{a^2-x^2-a^2}{\sqrt{a^2-x^2}}dx$, so
$\int\sqrt{a^2 - x^2}dx = x\sqrt{a^2 - x^2} - \int\sqrt{a^2 - x^2}dx + \int\frac{a^2}{\sqrt{a^2-x^2}}dx$
move the term $\int\sqrt{a^2 - x^2}dx$ from right side to left side, we have
$\int\sqrt{a^2 - x^2}dx = \frac{1}{2}x\sqrt{a^2 - x^2} + \frac{a^2}{2}\int\frac{1}{\sqrt{a^2-x^2}}dx$,
now use 12, we get $\int\sqrt{a^2 - x^2}dx = \frac{1}{2}x\sqrt{a^2 - x^2} + \frac{a^2}{2}arcsin\frac{x}{a} + c$.

17. $\int\sqrt{x^2 - a^2}dx = \frac{1}{2}x\sqrt{x^2 - a^2} - \frac{a^2}{2}ln|x + \sqrt{x^2 - a^2}| + c$
Pf: integral by parts, left side $= x\sqrt{x^2 - a^2} - \int\frac{\frac{1}{2}2xx}{\sqrt{x^2-a^2}}dx$
$= x\sqrt{x^2 - a^2} - \int\frac{x^2}{\sqrt{x^2-a^2}}dx = x\sqrt{x^2 - a^2} - \int\frac{x^2-a^2+a^2}{\sqrt{x^2-a^2}}dx$
$= x\sqrt{x^2 - a^2} - \int\sqrt{x^2 - a^2}dx - \int\frac{a^2}{\sqrt{x^2-a^2}}dx$
so $\int\sqrt{x^2 - a^2}dx = x\sqrt{x^2 - a^2} - \int\sqrt{x^2 - a^2}dx - \int\frac{a^2}{\sqrt{x^2-a^2}}dx$
$\int\sqrt{x^2 - a^2}dx = \frac{1}{2}x\sqrt{x^2 - a^2} - \frac{a^2}{2}\int\frac{1}{\sqrt{x^2-a^2}}dx$
use 15.
$= \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}ln|x + \sqrt{x^2 - a^2}| + c$.

18. $\int\sqrt{x^2 + a^2}dx = \frac{1}{2}x\sqrt{x^2 + a^2} + \frac{a^2}{2}ln|x + \sqrt{x^2 + a^2}| + c$
Pf: integral by parts, left side $= x\sqrt{a^2 + x^2} - \int xd\sqrt{a^2 + x^2}$
$= x\sqrt{a^2 + x^2} - \int\frac{x^2}{\sqrt{a^2+x^2}}dx = x\sqrt{a^2 + x^2} - \int\sqrt{a^2 + x^2}dx + \int\frac{a^2}{\sqrt{a^2+x^2}}dx$
so $\int\sqrt{x^2 + a^2}dx = \frac{1}{2}x\sqrt{a^2 + x^2} + \frac{1}{2}\int\frac{a^2}{\sqrt{a^2+x^2}}dx$
use 14.
$= \frac{1}{2}x\sqrt{a^2 + x^2} + \frac{a^2}{2}ln|x + \sqrt{x^2 + a^2}| + c$.

19. $\int cscxdx = \int\frac{1}{sinx}dx = ln|tan\frac{x}{2}| + c = ln|cscx - cotx| + c$
left side $= \int\frac{dx}{2sin\frac{x}{2}cos\frac{x}{2}} = \int\frac{dx}{\frac{sin\frac{x}{2}}{cos\frac{x}{2}}2cos^2\frac{x}{2}} = \int\frac{dtan\frac{x}{2}}{tan\frac{x}{2}} = ln|tan\frac{x}{2}| + c$

and $tan\frac{x}{2} = \frac{sin\frac{x}{2}}{cos\frac{x}{2}} = \frac{2sin^2\frac{x}{2}}{2cos\frac{x}{2}sin\frac{x}{2}} = \frac{1-cosx}{sinx} = cscx - cotx.$
Similarly, we have $\int \frac{1}{sinxcosx}dx = \int \frac{cotx}{cos^2x}dx$
$= \int \frac{1}{tanx}dtanx = ln|tanx| + c.$

20. $\int secxdx = \int \frac{1}{cos}dx = ln|tan(\frac{x}{2} + \frac{\pi}{4})| + c = ln|secx + tanx| + c$
Pf: left side $= \int \frac{d(x+\frac{\pi}{2})}{sin(x+\frac{\pi}{2})}$
use 19. $= ln|tan(\frac{x}{2} + \frac{\pi}{4})| + c.$
$tan(\frac{x}{2} + \frac{\pi}{4}) = \frac{tan\frac{x}{2}+1}{1-tan\frac{x}{2}} = \frac{sin\frac{x}{2}+cos\frac{x}{2}}{cos\frac{x}{2}-sin\frac{x}{2}} = \frac{(sin\frac{x}{2}+cos\frac{x}{2})^2}{cos^2\frac{x}{2}-sin^2\frac{x}{2}} = \frac{1+sinx}{cosx}.$

21. $\int tanxdx = \int \frac{sinx}{cosx}dx = \int \frac{-dcosx}{cosx} = -ln|cosx| + c$

22. $\int cotxdx = \int \frac{cosx}{sinx}dx = \int \frac{dsinx}{sinx} = ln|sinx| + c$

23. $\int tan^2xdx = \int(sec^2x - 1)dx = tanx - x + c$

24. $\int cot^2xdx = \int(csc^2x - 1)dx = -cotx - x + c$

25. $\int e^{ax}sin(bx)dx = \frac{e^{ax}(asin(bx)-bcos(bx))}{a^2+b^2} + c$
Pf: integral by parts, left side
$= \frac{1}{a}\int sin(bx)de^{ax} = \frac{1}{a}(sin(bx)e^{ax} - \int be^{ax}cos(bx)dx)$
and $\int be^{ax}cos(bx)dx = \frac{b}{a}\int cos(bx)de^{ax} = \frac{b}{a}(cos(bx)e^{ax} - \int e^{ax}dcos(bx))$
$= \frac{b}{a}(cos(bx)e^{ax} + b\int e^{ax}sin(bx)dx)$
so $\int e^{ax}sin(bx)dx = \frac{1}{a}(sin(bx)e^{ax} - \frac{b}{a}(cos(bx)e^{ax} + b\int e^{ax}sin(bx)dx))$
$a\int e^{ax}sin(bx)dx = sin(bx)e^{ax} - \frac{b}{a}cos(bx)e^{ax} - \frac{b^2}{a}\int e^{ax}sin(bx)dx$
$(a + \frac{b^2}{a})\int e^{ax}sin(bx)dx = sin(bx)e^{ax} - \frac{b}{a}cos(bx)e^{ax}$
$\int e^{ax}sin(bx)dx = \frac{1}{a^2+b^2}(asin(bx)e^{ax} - bcos(bx)e^{ax}) + c.$

26. $\int e^{ax}cos(bx)dx = \frac{e^{ax}(bsin(bx)+acos(bx))}{a^2+b^2} + c$
Pf: left side $= \frac{1}{a}\int cos(bx)de^{ax} = \frac{1}{a}(cos(bx)e^{ax} + \int be^{ax}sin(bx)dx)$
$\int be^{ax}sin(bx)dx = \frac{b}{a}\int sin(bx)de^{ax} = \frac{b}{a}(sin(bx)e^{ax} - \int e^{ax}dsin(bx))$
$= \frac{b}{a}(sin(bx)e^{ax} - b\int e^{ax}cos(bx)dx)$
so $a\int e^{ax}cos(bx)dx = cos(bx)e^{ax} + \frac{b}{a}sin(bx)e^{ax}$
$\int e^{ax}cos(bx)dx = \frac{acos(bx)e^{ax}+bsin(bx)e^{ax}}{a^2+b^2}.$

27. $\int \frac{1}{(x^2+a^2)}dx = \frac{1}{2a^2}\frac{x}{x^2+a^2} + \frac{1}{2a^3}arctan\frac{x}{a} + c$
Pf: let $x = a \times tan\theta$, $x^2 + a^2 = \frac{1}{cos^2\theta}a^2$,
left side $= \int \frac{\frac{a}{cos^2\theta}}{\frac{1}{cos^4\theta}a^4}d\theta = \frac{1}{a^3}\int cos^2\theta d\theta = \frac{1}{2a^3}\int \frac{1+cos2\theta}{2}d2\theta$
$= \frac{1}{4a^3}(2\theta + sin2\theta)$
note that $\theta = arctan\frac{x}{a}$, $sin2\theta = 2sin\theta cos\theta = \frac{2ax}{a^2+x^2}$
so left side $= \frac{1}{4a^3}(2arctan\frac{x}{a} + \frac{2ax}{a^2+x^2}) + c.$