Problem:

Assume that the number of cases of tetanus reported in the United States during a single month in 2005 has a Poisson distribution with parameter μ. The number of cases reported in January and February are 1 and 3 respectively.

(a) Find and plot the likelihood function over the space of potential values for μ.
(b) What is the maximum likelihood estimate (MLE) of μ?
(c) Give an estimate of the probability that there is no case of tetanus reported for a given month.

Solution:

Let Y be the number of cases of tetanus reported in the United States during a single month in 2005.
Then Y has a Poisson distribution with unknown parameter μ and the probability mass function of Y is $P(Y = y) = e^{-\mu}\frac{\mu^y}{y!}, y = 0, 1, 2, \cdots$.

(a) We have two observations, Y_1 and Y_2 from the above distribution with $Y_1 = 1$ and $Y_2 = 3$.
The likelihood function is then given by
$L(\mu|Y_1 = 1, Y_2 = 3) = P(Y_1 = 1) \times P(Y_2 = 3) = e^{-\mu}\frac{\mu^1}{1!} \times e^{-\mu}\frac{\mu^3}{3!} = e^{-2\mu}\frac{\mu^4}{6}$.
Some simple R codes to make the plot of the likelihood function, $L(\mu)$, are given below:

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> mu<-seq(0,10,by=0.01);
> lik<-exp(-2*mu)*(mu^4)/6;
> plot(mu,lik,type="l",xlab="Mu",ylab="Likelihood Function");
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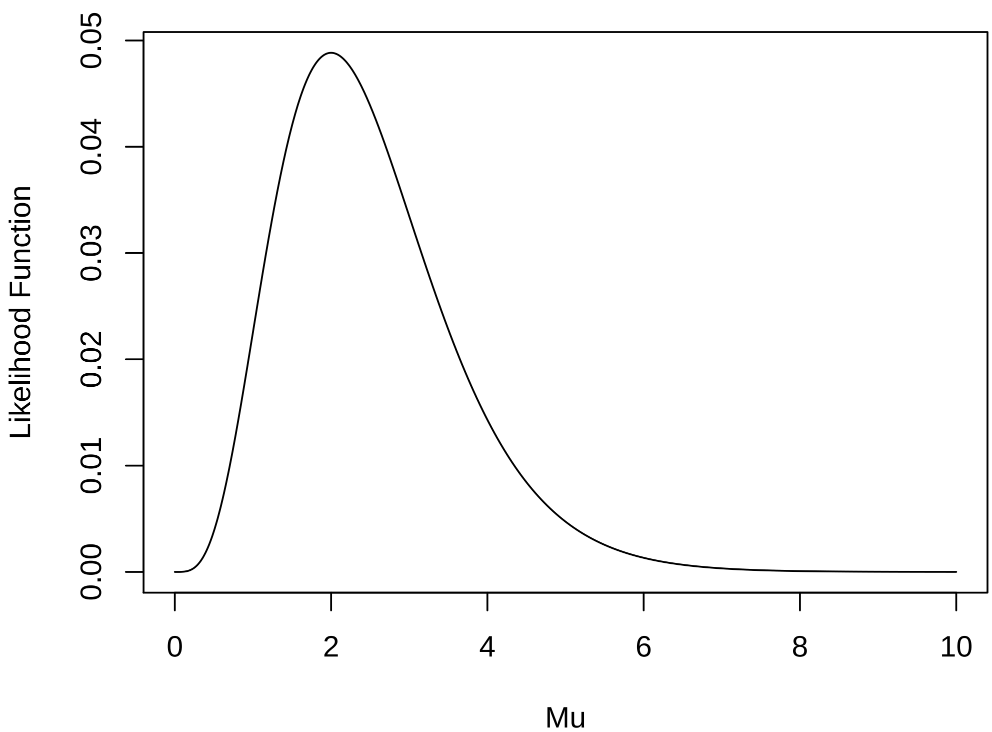

(b) To find the maximum likelihood estimate, take the derivative of $L(\mu|Y_1 = 1, Y_2 = 3)$ with respect to μ, we have $(e^{-2\mu}\frac{\mu^4}{6})' = e^{-2\mu}\frac{4\mu^3}{6} - 2e^{-2\mu}\frac{\mu^4}{6}$.
Let $e^{-2\mu}\frac{4\mu^3}{6} - 2e^{-2\mu}\frac{\mu^4}{6} = 0$, $\frac{4\mu^3}{6} = 2\frac{\mu^4}{6}$, one has the root $\mu = 2$.
So the maximum likelihood estimate of μ is 2.

(c) The probability that there is no case of tetanus reported for a given month is $P(Y = 0) = e^{-\mu}$.
The estimated value of this probability using the MLE of μ is $e^{-\mu} = e^{-2} = 0.13534$.