Statistical Learning Homework 3 Solution (Problem 1)

1. Consider the *piecewise linear model*

   (a) The model is:

   $$y = \alpha_0 + \alpha_1 x I\{x \le x_0\} + \beta_1 x I\{x > x_0\} + (\beta_0 - \alpha_0) I\{x > x_0\} + \epsilon$$

   (b) Here, define $\beta = (\alpha_0, \alpha_1, \beta_1, \gamma)'$, and

   $$y = \begin{pmatrix} 4.533 \\ 4.613 \\ \vdots \\ 5.284 \\ 5.367 \\ 5.450 \\ \vdots \\ 6.080 \end{pmatrix}, \quad X = \begin{pmatrix} 1 & 1981 & 0 & 0 \\ 1 & 1982 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1990 & 0 & 0 \\ 1 & 0 & 1991 & 1 \\ 1 & 0 & 1992 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 2000 & 1 \end{pmatrix}$$

   (c) We have the constraint:

   $$\beta_0 - \alpha_0 = (\alpha_1 - \beta_1) x_0$$

   (d) The model can be re-expressed as:

   $$y = (\alpha_0 + \alpha_1 x_0) + \alpha_1 \big((x - x_0) I\{x \le x_0\}\big) + \beta_1 \big((x - x_0) I\{x > x_0\}\big) + \epsilon$$

(e) In this case, define $\beta = (\gamma_0, \alpha_1, \beta_1)'$, and

$$y = \begin{pmatrix} 4.533 \\ 4.613 \\ \vdots \\ 5.284 \\ 5.367 \\ 5.450 \\ \vdots \\ 6.080 \end{pmatrix}, \quad X = \begin{pmatrix} 1 & -9 & 0 \\ 1 & -8 & 0 \\ \vdots & \vdots & \vdots \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 2 \\ \vdots & \vdots & \vdots \\ 1 & 0 & 10 \end{pmatrix}$$