SUCCESSIVE DIFFERENTIATIVE

* If we write $y = f(x)$ then the derivatives can also be denoted by $y_1, y_2, y_3 \ldots y_n$ or by $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3} \ldots \frac{d^ny}{dx^n}$.

The values of these derivatives at $x = a$ are denoted by, $f^n(a), y_n(a)$ or $\left[\frac{d^ny}{dx^n}\right]_{x=a}$.

SUCCESSIVE DIFFERENTIAL

" Derivatives of n^{th} order ⇒

① If $y = (ax+b)^m$ then,

$y_n = m(m-1)(m-2)\cdots(m-n+1)a^n(ax+b)^{m-n}$ if $n<m$.

$y_n = m!\,a^n$ if $n=m$

$y_n = 0$ if $n>m$.

② If $y = (ax+b)^{-m}$ Then,

$y_n = (-1)^n m(m+1)(m+n)\cdots(m+n-1)a^n(ax+b)^{-m-n}$

or $y_n = (-1)^n \dfrac{(m+n-1)!}{(m-1)!} \cdot \dfrac{a^n}{(ax+b)^{m+n}}$.

③ If $y = x^m$ Then,

$y_n = m(m-1)(m-2)\cdots(m-n+1)\,x^{m-n}$ if $n<m$

$y_n = m!$ if $n=m$

$y_n = 0$ if $n>m$.

④ If $y = \dfrac{1}{ax+b}$ Then.

$y_n = \dfrac{(-1)^n n!\,a^n}{(ax+b)^{n+1}}$

⑤ If $y = \log(ax+b)$ Then.

$y_n = \dfrac{(-1)^{n-1}(n-1)!\,a^n}{(ax+b)^n}$

⑥ If $y = a^{mx}$ Then,

$y_n = m^n a^{mx}(\log a)^n$.

⑦ If $y = e^{mx}$ Then

$y_n = m^n e^{mx}$

(8) If $y = \sin(ax+b)$ Then,

$$y_n = a^n \sin\left(ax+b+\frac{n\pi}{2}\right)$$

(9) If $y = \cos(ax+b)$ Then,

$$y_n = a^n \cos\left(ax+b+\frac{n\pi}{2}\right)$$

(10) If $y = e^{ax}\sin(bx+c)$ Then,

$$y_n = r^n e^{ax}\sin(bx+c+n\phi) \quad \text{where } \phi = \tan^{-1}\left(\frac{b}{a}\right)$$

(11) If $y = e^{ax}\cos(bx+c)$ Then.

$$y_n = r^n e^{ax}\cos(bx+c+n\phi).$$

(12) If $y = k^x \sin(bx+c)$ Then.

$$y_n = r^n k^x \sin(bx+c+n\phi).$$

(13) If $y = k^x \cos(bx+c)$ Then,

$$y_n = r^n k^x \cos(bx+c+n\phi).$$

* Successive Differentiation *

Defⁿ

Let $y = f(x)$ be given funⁿ

Then $\frac{dy}{dx} = f'(x)$

Again Diff

$$\frac{d^2y}{dx^2} = f''(x) \quad \text{and so on.}$$

The process of finding higher derivatives called successive diff.

Notation :-

Successive derivatives of y w.r.to x are denoted by

1. $y_1, y_2, y_3 \cdots y_n$
2. $y', y'', y''' \cdots y^n$
3. $Dy, D^2y, D^3y \cdots D^ny$ where $D = \frac{d}{dx}$
4. $f', f'', \cdots f^n$.

TYPE I Algebric funⁿ (Partial fraction method)

Note If $y = f(x) = \frac{P(x)}{g(x)}$

where $P(x)$ & $g(x)$ are some polynomials in x then apply partial fraction method

CASE I :- If $\deg P(x) < \deg q(x)$ then use partial fractⁿ

CASE II :- If $\deg P(x) \geq \deg q(x)$ then first take actual div. and express y into

Dividend = (Divisor × quotient) + Remainder

$$\frac{P(x)}{g(x)} = \quad = Q + \frac{R}{D}$$

Examples

(1) find n^{th} derivative of $y = \dfrac{1}{x^2 - 4x + 3}$

→

$$y = \frac{1}{(x-1)(x-3)}$$

$$= \frac{A}{(x-1)} + \frac{B}{(x-3)}$$

$\boxed{A = \frac{-1}{2}}$ $\boxed{B = \frac{1}{2}}$ – By partial fraction method

$$y = \frac{\frac{-1}{2}}{x-1} + \frac{\frac{1}{2}}{x-3}$$

Take n^{th} order derivative

$$y_n = \frac{-1}{2}\frac{d}{dx^n}\left(\frac{1}{x-1}\right) + \frac{1}{2}\frac{d}{dx^n}\left(\frac{1}{x-3}\right)$$

$$y = \frac{1}{x-b}$$

$$y_n = \frac{(-1)^n n!}{(x+b)^{n+1}}$$

$$\boxed{y_n = \frac{1}{2}\left[\frac{(-1)^n n!}{(x-1)^{n+1}}\right] + \frac{1}{2}\left[\frac{(-1)^n n!}{(x-3)^{n+1}}\right]}$$

(2) find n^{th} derivative of $y = \dfrac{2x+3}{(x+1)^2(x-3)}$

→ $$\frac{2x+3}{(x+1)^2(x-3)} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{(x-3)}$$

$\rightarrow$ $2x+3 = A(x+1)(x-3) + B(x-3) + C(x+1)^2$

Put $x+1=0 \Rightarrow x=-1$

$2(-1)+3 = B(-1-3) \Rightarrow 1 = -4B \Rightarrow \boxed{B = -1/4}$

Put $x-3=0 \Rightarrow x=3$

$9 = C(4)^2$

$\boxed{C = 9/16}$

equating the coeff. of x^2 on both side.

$0 = A + C \Rightarrow C = -A$

$\boxed{A = -9/16}$

$$y = \frac{-9/16}{(x+1)} - \frac{1/4}{(x+1)^2} + \frac{9/16}{(x-3)}$$

$$y_n = \frac{-9}{16}\left[\frac{(-1)^n n!\, 1^n}{(x+1)^{n+1}}\right] - \frac{1}{4}\left[\frac{(-1)^n 1^n (n+1)!}{(x+1)^{n+2}(2-1)!}\right] + \frac{9}{16}\left[\frac{(-1)^n n!\, 1^n}{(x-3)^{n+1}}\right].$$

(13) $\dfrac{4x}{(x-1)^2(x+1)}$

Sol $\Rightarrow$ $\dfrac{4x}{(x-1)^2(x+1)} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{x+1}$

$\therefore$ From partial fraction.

$4x = A(x-1)(x+1) + B(x+1) + C(x-1)^2$. OR

put $x = 1$, $4 = 2B \Rightarrow B = 2$.

put $x = -1$, $-4 = 4C \Rightarrow C = -1$

put $x = 0$, $0 = -A + B + C$

$A = 2 - 1$

$A = 1$

OR

$4x = A(x^2-1) + B(x+1) + C(x^2 - 2x + 1)$

$\therefore A + C = 0 \Rightarrow A = -C$

$B - 2C = 4 \Rightarrow B = 4+2C$

$-A + B + C = 0$

$\therefore C + 4 + 2C + C = 0$

$4C = -4$

$C = -1$

$\therefore B = 2$

$\therefore A = 1$

$\therefore \dfrac{4x}{(x-1)^2(x+1)} = \dfrac{1}{x-1} + \dfrac{2}{(x-1)^2} - \dfrac{1}{x+1}$

$\therefore y_n = (-1)^n \left[\dfrac{n!}{(x-1)^{n+1}} + \dfrac{2(n+1)!}{(x-1)^{n+2}} - \dfrac{n!}{(x+1)!} \right]$

(14) $\dfrac{x}{(x+1)^5}$

Sol $\Rightarrow$ $\dfrac{x+1-1}{(x+1)^5}$

$= \dfrac{\cancel{x+1}}{(x+1)^{\cancel{5}}} - \dfrac{1}{(x+1)^5}$

$= \dfrac{1}{(x+1)^4} - \dfrac{1}{(x+1)^5}$

$\therefore y_n = \dfrac{(-1)^n\, 4(5)(6)\cdots(4+n-1)}{3!(x+1)^{n+4}} - \dfrac{(-1)^n\, 5\cdot6\cdot7\cdots(5+n-1)}{4!(x+1)^{n+5}}$

$= \dfrac{(-1)^n (n+3)!}{3!(x+1)^{n+4}} - \dfrac{(-1)^n (n+4)!}{4!(x+1)^{n+5}}$

$= \dfrac{(-1)^n (n+3)!}{4!(x+1)^{n+4}} \left[4 - \dfrac{n+4}{x+1} \right]$

(3) find the n^{th} derivative of $\frac{x^2}{(x+2)(2x+3)}$

$(x+2)(2x+3) = 2x^2+7x+6$

We can not express the given expression in term of partial fractions.
since degree of numerator is equal to the degree of denominator we divide numerator by denominator

$$\begin{array}{r|l} & \frac{1}{2} \\ \hline 2x^2+7x+6 & \cancel{x^2} \\ & -\cancel{x^2} - \frac{7}{2}x - 3 \\ \hline & -\left(\frac{7}{2}x+3\right) \end{array}$$

$$\frac{x^2}{2x^2+7x+6} = Q + \frac{R}{D}$$

$$= \frac{1}{2} - \frac{\left(\frac{7}{2}x+3\right)}{2x^2+7x+6} \quad \text{—— (A)}$$

$$\frac{\frac{7}{2}x+3}{(x+2)(2x+3)} = \frac{A}{(x+2)} + \frac{B}{(2x+3)}$$

$$\frac{7}{2}x+3 = A(2x+3) + B(x+2)$$

put $x = -2$

$$\frac{7}{\cancel{2}}(\cancel{-2}) + 3 = A(2(-2)+3)$$

$$-4 = -A$$

$$\boxed{A = 4}$$

put $x = -3/2$

$$\frac{7}{2}\left(\frac{-3}{2}\right) + 3 = B\left(\frac{-3}{2}+2\right)$$

$$-\frac{21}{4} + 3 = \frac{B}{2}$$

$$\frac{-9}{4} = \frac{B}{2}$$

$$\boxed{B = \frac{-9}{2}}$$

put in A

$$y = \frac{1}{2} - \frac{4}{(x+2)} + \frac{9/2}{(2x+3)}$$

$$y = \frac{1}{2} - 4\left[\frac{1}{(x+2)}\right] + \frac{9}{2}\left[\frac{1}{(2x+3)}\right]$$

$$\boxed{y_n = -4\left[\frac{(-1)^n n!\, 1^n}{(x+2)^{n+1}}\right] + \frac{9}{2}\left[\frac{(-1)^n n!\, 2^n}{(2x+3)^{n+1}}\right]}$$

4) find nth derivative of $\frac{x}{x^2+a^2}$

$$y = \frac{x}{x^2+a^2} \Rightarrow x^2+a^2=0 \Rightarrow x = \pm ai$$

$$= \frac{A}{(x+ai)} + \frac{B}{(x-ai)}$$

$$x = A(x-ai) + B(x+ai)$$

put $(x-ai)=0 \Rightarrow x = ai$

$$ai = B(ai+ai)$$

$$ai = B(2ai)$$

$$\boxed{B = 1/2}$$

put $(x+ai)=0 \Rightarrow x = -ai$

$$-ai = A(-2ai)$$

$$\boxed{A = 1/2}$$

$$y = \frac{1}{2}\left[\frac{1}{(x+ai)} + \frac{1}{(x-ai)}\right]$$

$$\boxed{y_n = \frac{1}{2}(-1)^n n! \left[\frac{1}{(x+ai)^{n+1}} + \frac{1}{(x-ai)^{n+1}}\right]}$$

TYPE II Trigonometric functions

① find n^{th} derivative of $y = \cos x \cos 2x \cos 3x$

$y = \cos x \cos 2x \cos 3x$

$= \cos x \left[\frac{1}{2}(\cos(5x) + \cos(x))\right]$

$= \frac{1}{2}\left[\cos x \cdot \cos 5x + \cos^2 x\right]$

$= \frac{1}{2}\left[\frac{1}{2}(\cos 6x + \cos 4x) + \frac{(1+\cos 2x)}{2}\right]$

$= \frac{1}{4}\left[\cos 6x + \cos 4x + 1 + \cos 2x\right]$

$$\boxed{y_n = \frac{1}{4}\left[6^n \cos\left(6x + \frac{n\pi}{2}\right) + 4^n \cos\left(4x + \frac{n\pi}{2}\right) + 2^n \cos\left(2x + \frac{n\pi}{2}\right)\right]}$$

② If $y = e^{x} \sin x$ show that $y_n = 2^{n/2} e^x \sin\left(x + \frac{n\pi}{4}\right)$

$y = e^x \sin x$

$$y_n = r^n e^{ax} \sin(x + n\phi)$$

$$r = \sqrt{a^2+b^2}$$

$$= \sqrt{1+1} = \sqrt{2}$$

$$\phi = \tan^{-1}\left(\frac{b}{a}\right)$$

$$= \tan^{-1}\left(\frac{1}{1}\right) = \pi/4$$

$$\boxed{y_n = 2^{n/2} e^{x} \sin\left(x + \frac{n\pi}{4}\right)}$$

(3) If $y = e^{ax} \cos^2 x \sin x$ find y_n.

$$\cos^2 x \sin x = \cos x \sin x \cos x$$

$$= \frac{1}{2} \cos x \sin 2x \rightarrow \{\sin\theta \cos\theta = \sin 2\theta/2\}$$

$$= \frac{1}{4}[\sin 3x + \sin x] \rightarrow \left\{\begin{array}{l}\cos Ax \sin Bx \\ = \frac{1}{2}(\sin(A+B) \\ + \sin(A-B)\end{array}\right.$$

$$y = \frac{1}{4} e^{ax} [\sin 3x + \sin x]$$

$$= \frac{1}{4}\left[e^{ax} \sin 3x + e^{ax} \sin x\right]$$

$$y_n = \frac{1}{4}\left[r_1^n e^{ax} \sin(a_1 x + b_1 + n\phi_1) + r_2^n e^{ax} \sin(a_2 x + b_2 + n\phi_2)\right]$$

$$y_n = \frac{1}{4}\left[(a^2+9)^{n/2} e^{ax} \sin\left(3x + n\tan^{-1}\left(\frac{3}{a}\right)\right] + \left[(a^2+1)^{n/2} e^{ax} \sin\left(x + n\tan^{-1}\left(\frac{1}{a}\right)\right)\right]$$

(4) If $y = 2^x \cos 9x$ find y_n.

$$y = 2^x \cos 9x$$

If $y = a^x \cos(bx + c)$

Then $y_n = r^n a^x \cos(bx + c + n\alpha)$

$$r = \sqrt{(\log a)^2 + b^2}, \quad \alpha = \tan^{-1}\left(\frac{b}{\log a}\right)$$

$$y_n = ((\log 2)^2 + 9^2)^{n/2}\, 2^x \cos\left(9x + \tan^{-1}\left(\frac{9}{\log 2}\right)\right)$$

(5) If $y = 2^x \sin^2 x \cos^3 x$. find y_n

$$\sin^2 x \cos^3 x = 4\,\frac{\sin^2 x \cos^2 x \cos x}{4}$$

$$= \cancel{4}\left[\frac{\left(\frac{1}{\cancel{2^2}}\right)\sin^2 2x \cdot \cos x}{4}\right]$$

$$= \frac{\sin^2 2x \cdot \cos x}{4}$$

$$= \frac{(1 - \cos 4x)}{8} \cdot \cos x$$

$$= \frac{1}{8}\left[\cos x - \cos 4x \cos x\right]$$

$$= \frac{1}{8}\cos x - \frac{1}{16}\left[\cos 5x + \cos 3x\right]$$

$$y = 2^x\left[\frac{1}{8}\cos x - \frac{1}{16}(\cos 5x + \cos 3x)\right]$$

$$y = \frac{1}{8}\left[2^x \cos x\right] - \frac{1}{16}\left[2^x \cos 5x + 2^x \cos 3x\right]$$

p.110 Ex $y = \frac{8x}{x^3 - 2x^2 - 4x + 8}$ Find y_n

\# Find n^{th} derivatives of the following.

(1) $y = e^x \cos 2x \cos x$.

$$= \frac{1}{2} e^x [\cos 3x + \cos x]$$

$$\therefore y_n = \frac{e^x}{2} \left[(\sqrt{3^2+1^2})^n \cos(3x + n\phi_1) + (\sqrt{1+1})^n \cos(x + n\phi_2) \right]$$ where $\tan\phi_1 = 3/1$

$$\phi_2 = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$$

$$= \frac{e^x}{2} \left[(\sqrt{10})^n \cos(3x + n\tan^{-1}(3)) + (\sqrt{2})^n \cos(x + n\pi/4) \right]$$

TYPE III

Based on De Moivre's Theorem

(1) If $y = \dfrac{1}{x^2+a^2}$ then prove that.

$$y_n = \frac{(-1)^n n!}{a^{n+2}} \sin^{n+1}\theta \sin(n+1)\theta$$

where $\theta = \tan^{-1}\left(\frac{a}{x}\right)$.

$$y = \frac{1}{x^2+a^2}$$

$$x^2+a^2=0 \Rightarrow x = \pm ai$$

$$= \frac{1}{(x+ai)(x-ai)} = \frac{A}{(x+ai)} + \frac{B}{(x-ai)}$$

$$1 = A(x-ai) + B(x+ai)$$

Put $(x-ai)=0$

$$x = ai$$

$$1 = B(2ai)$$

$$\boxed{B = \frac{1}{2ai}}$$

put $(x+ai)=0$

$$x = -ai$$

$$1 = A(-2ai)$$

$$\boxed{A = \frac{-1}{2ai}}$$

$$y = \frac{1}{2ai}\left[\frac{1}{(x-ai)} - \frac{1}{(x+ai)}\right]$$

$$y_n = \frac{1}{2ai}\left[\frac{(-1)^n n!\, 1^n}{(x-ai)^{n+1}} - \frac{(-1)^n n!\, 1^n}{(x+ai)^{n+1}}\right]$$

$$y_n = \frac{(-1)^n n!}{2ai}\left[\frac{1}{(x-ai)^{n+1}} - \frac{1}{(x+ai)^{n+1}}\right] \quad \text{--- } (1)$$

Put $x = r\cos\theta$

$a = r\sin\theta$

$r^2 = x^2 + a^2$

$\theta = \tan^{-1}(a/x)$

$$\frac{1}{(x-ai)^{n+1}} = \frac{1}{r^{n+1}((\cos\theta - i\sin\theta)^{n+1}}$$

$$= \frac{1}{r^{n+1}} \cdot \frac{1}{\cos(n+1)\theta - i\sin(n+1)\theta}$$

$$= \frac{1}{r^{n+1}}\left[\cos(n+1)\theta + i\sin(n+1)\theta\right]$$

$$\frac{1}{(x+ai)^{n+1}} = \frac{1}{r^{n+1}(\cos\theta + i\sin\theta)^{n+1}}$$

$$= \frac{1}{r^{n+1}} \frac{1}{\cos(n+1)\theta + i\sin(n+1)\theta}$$

$$= \frac{1}{r^{n+1}}\left[\cos(n+1)\theta - i\sin(n+1)\theta\right]$$

$$\therefore \frac{1}{(x-ai)^{n+1}} - \frac{1}{(x+ai)^{n+1}} = \frac{1}{r^{n+1}} 2i\sin(n+1)\theta$$

Put in (1)

$$y_n = (-1)^n n! \frac{1}{a} \frac{1}{r^{n+1}} \sin(n+1)\theta$$

But $r = \dfrac{a}{\sin\theta}$

$$r^{n+1} = \frac{a^{n+1}}{\sin^{n+1}\theta}$$

$$\boxed{y_n = (-1)^n n! \frac{1}{a^{n+2}} \sin^{(n+1)}\theta \sin(n+1)\theta}$$

(2) find n^{th} derivative of $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$

→ Put $x = \tan\alpha$

$$y = \tan^{-1}\left(\frac{2\tan\alpha}{1-\tan^2\alpha}\right)$$

$$= \tan^{-1}(\tan 2\alpha)$$

$$= 2\alpha$$

$$y = 2\tan^{-1}x \quad \text{——— (1)}$$

diff w.r. to x

$$y_1 = \frac{1}{x^2+1} = \frac{1}{(x+i)(x-i)}$$

$$= \frac{1}{2i}\left[\frac{1}{(x-i)} + \frac{(-1)}{(x+i)}\right]$$

Now diff $(n-1)$ time for finding y_n.

$$y_n = \frac{1}{2i}\left[\frac{(-1)^{n-1}(n-1)!\, 1^{n-1}}{(x-i)^{n-1+1}} - \frac{(-1)^{n-1}(n-1)!\, 1^{n-1}}{(x+i)^{n-1+1}}\right]$$

$$y_n = \frac{(-1)^{n-1}.\,(n-1)!}{2i}\left[\frac{1}{(x-i)^n} - \frac{1}{(x+i)^n}\right] \quad \text{—— (2)}$$

Put $x = r\cos\theta$

$= r\sin\theta$

$r = \sqrt{1+x^2}$

$\theta = \tan^{-1}(1/x)$

$$\therefore \frac{1}{(x-i)^n} = \frac{1}{r^n(\cos\theta - i\sin\theta)^n}$$

$$= \frac{1}{r^n}[\cos n\theta + i\sin n\theta]$$

$$\frac{1}{(x+i)^n} = \frac{1}{r^n(\cos\theta + i\sin\theta)^n}$$

$$= \frac{1}{r^n}[\cos n\theta - i\sin n\theta]$$

$$\frac{1}{(x-i)^n} - \frac{1}{(x+i)^n} = \frac{2i}{r^n}\sin n\theta$$

Put in (2)

$$y_n = \frac{(-1)^{n-1}(n-1)!}{2i}\,\frac{2i}{r^n}\sin n\theta$$

$$= \frac{(-1)^{n-1}(n-1)!\,\sin n\theta}{r^n}$$

$$\boxed{y_n = \frac{(-1)^{n-1}(n-1)!}{(x^2+1)^{n/2}}\sin\left[n\tan^{-1}\left(\frac{1}{x}\right)\right]}$$

(1) If $y = \frac{x}{x^2+1}$ P.T $y_n = (-1)^n n! \sin^{n+1}\theta \cos(n+1)\theta$

Sol ⇒ $\because y = \frac{x}{x^2+1}$

$= \frac{x}{x^2-(i)^2}$

$= \frac{x}{(x-i)(x+i)}$

By partial fraction,

$\frac{x}{(x-i)(x+i)} = \frac{A}{(x-i)} + \frac{B}{(x+i)}$

Solving this, $A = \frac{1}{2}$ & $\frac{B}{1} = \frac{1}{2}$

$\therefore y = \frac{1}{2}\left[\frac{1}{x-i} + \frac{1}{x+i}\right]$

$\therefore y_n = \frac{1}{2}\left[(-1)^n n! \left\{\frac{1}{(x-i)^{n+1}} + \frac{1}{(x+i)^{n+1}}\right\}\right]$

put $x = r\cos\theta$, $1 = r\sin\theta \Rightarrow r^{-1} = \sin\theta$.

$\therefore r^2 = x^2+1$

$\theta = \tan^{-1}\left(\frac{1}{x}\right)$

$\therefore y_n = \frac{(-1)^n n!}{2}\left[\frac{1}{r^{n+1}(\cos\theta - i\sin\theta)^{n+1}} + \frac{1}{r^{n+1}(\cos\theta + i\sin\theta)^{n+1}}\right]$

$\therefore$ By De-Moiver's thm & by cor.

$y_n = \frac{(-1)^n n!}{2}\left[r^{-(n+1)}\left(\cos(n+1)\theta + \cancel{i\sin(n+1)\theta} + \cos(n+1)\theta \cancel{- i\sin(n+1)\theta}\right)\right]$

$\therefore y_n = \frac{(-1)^n n!}{\cancel{2}} \sin^{(n+1)}\theta \left[\cancel{2}\cos(n+1)\theta\right]$

$\boxed{\because r^{-1} = \sin\theta \quad r^{-(n+1)} = \sin^{(n+1)}\theta}$

$= (-1)^n n! \sin^{(n+1)}\theta \cdot \cos(n+1)\theta$

If $y = \frac{1}{x^2+x+1}$ P.T $y_n = \frac{2(-1)^n n!}{\sqrt{3}\, r^{n+1}} \sin(n+1)\theta$

where $\theta = \cot^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$ & $r = \sqrt{x^2+x+1}$

Sol ⇒ $\because y = \frac{1}{x^2+x+1}$

$$= \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$= \frac{1}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

Let $x+\frac{1}{2} = X$

$$\therefore y = \frac{1}{X^2+\left(\frac{\sqrt{3}}{2}\right)^2} = \frac{1}{X^2-\left(\frac{\sqrt{3}i}{2}\right)^2}$$

$$= \frac{1}{\left(X-\frac{\sqrt{3}i}{2}\right)\left(X+\frac{\sqrt{3}i}{2}\right)}$$

$\therefore$ By partial fraction,

$$\frac{1}{\left(X-\frac{\sqrt{3}i}{2}\right)\left(X+\frac{\sqrt{3}i}{2}\right)} = \frac{A}{\left(X+\frac{\sqrt{3}i}{2}\right)} + \frac{B}{\left(X-\frac{\sqrt{3}i}{2}\right)}$$

$\therefore$ By solving $A = \frac{-1}{\sqrt{3}i}$ & $B = \frac{1}{\sqrt{3}i}$

$$\therefore y = \frac{1}{\sqrt{3}i}\left[\frac{1}{\left(X-\frac{\sqrt{3}i}{2}\right)} - \frac{1}{\left(X+\frac{\sqrt{3}i}{2}\right)}\right]$$

$$\therefore y_n = \frac{(-1)^n n!}{\sqrt{3}i}\left[\frac{1}{\left(X-\frac{\sqrt{3}i}{2}\right)^{n+1}} - \frac{1}{\left(X+\frac{\sqrt{3}i}{2}\right)^{n+1}}\right]$$

put $X = r\cos\theta$, $\frac{\sqrt{3}}{2} = r\sin\theta$

$$\therefore r^2 = X^2+\left(\frac{\sqrt{3}}{2}\right)^2 = \left(x+\frac{1}{2}\right)^2+\frac{3}{4} = x^2+x+1$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{3}/2}{X}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{3}/2}{\left(x+\frac{1}{2}\right)}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{3}}{2x+1}\right)$$

$$= \cot^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$$

$$\therefore y_n = \frac{(-1)^n n!}{\sqrt{3}\, i}\left[\frac{1}{r^{n+1}(\cos\theta - i\sin\theta)^{n+1}} - \frac{1}{r^{n+1}(\cos\theta + i\sin\theta)^{n+1}}\right]$$

$$= \frac{(-1)^n n!}{\sqrt{3}\, i}\left[\frac{1}{r^{n+1}}\left(\cancel{\cos(n+1)\theta} + i\sin(n+1)\theta - \cancel{\cos(n+1)\theta} + i\sin(n+1)\theta\right)\right]$$

$$= \frac{(-1)^n n!}{\sqrt{3}\, \cancel{i}}\left[\frac{2\cancel{i}\sin(n+1)\theta}{r^{n+1}}\right]$$

$$y_n = \frac{2(-1)^n n!}{\sqrt{3}\, r^{n+1}} \sin(n+1)\theta.$$

Leibnitz's Theorem ⇒

If $y = u \cdot v$ where u & v are funct^{ns} of x.

$y_n = {}^nC_0 u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \cdots + {}^nC_r$
$u_{n-r} v_r + \cdots + {}^nC_n u v_n$.

$= u_n v + n u_{n-1} v_1 + \frac{n(n-1)}{2!} u_{n-2} v_2 + \frac{n(n-1)(n-2)}{3!}$
$u_{n-3} v_3 + \cdots + u v_n$.

Ex–V

If $y = a \cos \log x + b \sin \log x$ P.T
$x^2 y_{n+2} + (2n+1) x y_{n+1} + (n^2+1) y_n = 0$

Sol ⇒ $\because$ $y = a \cos \log x + b \sin \log x$.

$y_1 = \frac{-a \sin \log x}{x} + \frac{b \cos \log x}{x}$

$$x^2(y_2)_n + 2xn\,C_1\,(y_2)_{n-1} + 2\frac{n(n-1)}{2!}(y_2)_{n-2}$$

$$xy_1 = -a\sin\log x + b\cos\log x,$$

$$xy_2 + y_1 = \frac{-a\cos\log x}{x} - \frac{b\sin\log x}{x}$$

$$x^2y_2 + xy_1 = -y$$

$$x^2y_2 + xy_1 + y = 0.$$

Applying Leibnitz's thm,

$$x^2y_{n+2} + n(2x)y_{n+1} + \frac{n(n-1)}{\cancel{2}\times 1}\cancel{2}y_n + xy_{n+1} + ny_n + y_n = 0$$

$$\therefore\ x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0.$$

M.Q.

* If $y = \sin(m\sin^{-1}x)$ or if $m\sin^{-1}x = \sin^{-1}y$ P.T

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0.$$

Ex-IV

Find the n^{th} derivative of y if

(3) $y = x^2 \sin x$.

Sol ⇒ $y = \sin x \cdot x^2$

$$y_n = \sin\left(x + \frac{n\pi}{2}\right)\cdot x^2 + 2n\sin\left(x + \overline{n-1}\frac{\pi}{2}\right)\cdot x$$
$$+ n(n-1)\sin\left(x + \overline{n-2}\frac{\pi}{2}\right)$$

(4) $y = x^2 e^{mx}$.

$y = e^{mx} \cdot x^2$

$$\therefore y_n = e^{mx} x^2 \cdot m^n + 2n\, m^{n-1} e^{mx} x + \frac{2n(n-1)}{2!} m^{n-2} e^{mx}.$$

Ex = V

Ex ⇒ If $y = \cos(m \sin^{-1} x)$ P.T

$$(1-x^2) y_{n+2} - (2n+1) x y_{n+1} + (m^2 - n^2) y_n = 0$$

Hence obtain $y_n(0)$.

Sol ⇒ We have $y_1 = -\sin(m\sin^{-1}x) \cdot \dfrac{m}{\sqrt{1-x^2}}$ — (a)

$\therefore \sqrt{1-x^2} \cdot y_1 = -m\sin(m\sin^{-1}x)$

diff. again

$$\sqrt{1-x^2}\, y_2 - \frac{\cancel{2}x y_1}{\cancel{2}\sqrt{1-x^2}} = -m\cos(m\sin^{-1}x)\frac{m}{\sqrt{1-x^2}}$$

$$(1-x^2) y_2 - x y_1 = -m^2 y.$$
$$(1-x^2) y_2 - x y_1 + m^2 y = 0. \quad \text{— (1)}$$

Applying Leibnitz' thm

$$\therefore (1-x^2) y_{n+2} + (-2x) n\, y_{n+1} + (-2)\frac{n(n-1)}{2!} y_n$$
$$- x y_{n+1} - n y_n + m^2 y_n = 0.$$

$$\therefore (1-x^2) y_{n+2} - (2n+1) x y_{n+1} + (m^2 - n^2) y_n = 0 \quad \text{— (H)}$$

Putting $x=0$ in given eqⁿ,

$y = \cos(m\sin^{-1}x)$ we get

$y(0) = \cos(0) = 1$ If $x=0$

$(1-x^2) y_{n+2} = (2n+1) x y_{n+1} - (m^2 - n^2) y_n$ $\therefore \dfrac{y_{n+2}}{y_n} = -(m^2 - n^2)$

put $x=0$ in (a)

$\therefore y_1(0) = \sin(0) \times \frac{1}{\sqrt{1-0}} = 0.$

put $x=0$ in (1)

$\therefore y_2 = -m^2 y(0) + xy_1(0) = -m^2(1) = -m^2.$

put $x=0$ in (2)

$\therefore y_{n+2}(0) = (2n+1)\times 0 \times y_{n+1}(0) + (n^2-m^2)y_n(0)$

$y_{n+2}(0) = (n^2-m^2)y_n(0)$ —— (3)

put $n=1,3,5\ldots$ in (3)

For $n=1$ $\therefore$ $y_{1+2}(0) = (1^2-m^2)y_1(0) = (1-m^2)\times 0 = 0$

For $n=3$ $\therefore$ $y_5(0) = (3^2-m^2)y_3(0) = (9-m^2)\cdot 1 \times 0 = 0$

$$\boxed{\therefore y_n(0) = 0 \quad \text{if } n \text{ is odd.}}$$

putting $n=2,4,6\ldots$ in (3) we got

For $n=2$, $y_{2+2}(0) = y_4(0) = (2^2-m^2)y_2(0) = (2^2-m^2)(-m^2)$

$n=4$, $y_6(0) = (4^2-m^2)(2^2-m^2)(-m^2)$

$\therefore y_n(0) = (n^2-m^2)\cdots(4^2-m^2)(2^2-m^2)(-m^2)$ if n is even.

HW

Ex ⇒ If $y = e^{m\sin^{-1}x}$ [or $x = \sin\left(\frac{1}{m}\log y\right)$] P.T

$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0.$

HW

Ex ⇒ If $y^{1/m} + y^{-1/m} = 2x$ P.T

$(x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0$

Sol ⇒ $y^{1/m} + \frac{1}{y^{1/m}} = 2x \therefore y^{2/m} - 2xy^{1/m} + 1 = 0$

$\therefore y^{1/m} = \frac{2x \pm \sqrt{4x^2-4}}{2} = x \pm \sqrt{x^2-1}$

$\therefore y = (x \pm \sqrt{x^2-1})^m$ Taking positive

H.W. $\therefore y = (x+\sqrt{x^2-1})^m.$

\# If $y = \sin \log(x^2+2x+1)$ P.T

$(x+1)^2 y_{n+2} + (2n+1)(x+1) y_{n+1} + (n^2+4) y_n = 0$

Sol $\Rightarrow$ $y = \sin \log(x^2+2x+1)$

$= \sin \log(x+1)^2$

$= \sin[2 \log(x+1)]$

H.W

$$y_6(0) = (4^2-m^2)\,y_4(0) = (4^2-m^2)(2^2-m^2)(-m^2)$$

$\vdots$

$$y_n(0) = (n^2-m^2)\cdots(4^2-m^2)(2^2-m^2)(-m^2)$$

if n is even

(4) If $y = \left[x+\sqrt{x^2-1}\right]^m$ prove that $(x^2-1)y_{n+2}+(2n+1)xy_{n+1}+(n^2-m^2)y_n=0$

$$y_1 = m\left(x+\sqrt{x^2-1}\right)^{m-1}\left[1+\frac{x}{\sqrt{x^2-1}}\right]$$

$$= m\left(x+\sqrt{x^2-1}\right)^{m-1}\frac{\left(x+\sqrt{x^2-1}\right)}{\sqrt{x^2-1}}$$

$$= \frac{m\left(x+\sqrt{x^2-1}\right)^m}{\sqrt{x^2-1}}$$

$$y_1 = \frac{my}{\sqrt{x^2-1}}$$

$$y_1\sqrt{x^2-1} = my$$

Diff again

$$\sqrt{x^2-1}\,y_2 + y_1\frac{x}{\sqrt{x^2-1}} = my_1 = m\cdot\frac{my}{\sqrt{x^2-1}}$$

mul by $\sqrt{x^2-1}$ & By Leibnitz's thm

$$\left[(x^2-1)y_{n+2}+n(2x)y_{n+1}+\frac{n(n-1)}{2!}2y_n\right] + \left[xy_{n+1}+n\cdot1\cdot y_n\right] = m^2y_n$$

$$\boxed{(x^2-1)y_{n+2}+(2n+1)xy_{n+1}+(n^2-m^2)y_n=0}$$

⑤ If $y^{1/m} + y^{-1/m} = 2x$

Prove that

$(x^2-1)y_{n+2} + (2n+1)x\,y_{n+1} + (n^2-m^2)y_n = 0$

⇒

$$y^{1/m} + \frac{1}{y^{1/m}} = 2x$$

$$y^{2/m} - 2x\,y^{1/m} + 1 = 0$$

This is quadratic in $y^{1/m}$

$$y^{1/m} = \frac{2x \pm \sqrt{4x^2-4}}{2}$$

$$y^{1/m} = x \pm \sqrt{x^2-1}$$

$$\boxed{y = (x \pm \sqrt{x^2-1})^m}$$

consider

$$y = (x + \sqrt{x^2-1})^m$$

$$y_1 = m(x+\sqrt{x^2-1})^{m-1} \cdot \left(1 + \frac{x}{\sqrt{x^2-1}}\right)$$

$$= m(x+\sqrt{x^2-1})^m \cdot \frac{1}{\sqrt{x^2-1}}$$

$$= \frac{my}{\sqrt{x^2-1}}$$

$$\sqrt{x^2-1}\,y_1 = my$$

Diff Again w.r. to x

$$\sqrt{x^2-1}\,y_2 + \frac{x}{\sqrt{x^2-1}}\,y_1 = m y_1$$

$$(x^2-1)y_2 + xy_1 = m\sqrt{x^2-1}\,y_1$$
$$= m \cdot my$$
$$= m^2 y$$

$$(x^2-1)y_2 + xy_1 - m^2 y = 0$$

Applying Leibnitz's thm.

$$(x^2-1)y_{n+2} + n(2x)y_{n+1} + \frac{n(n-1)}{2!}(2)y_n + xy_{n+1}$$
$$+ n(1)y_n - m^2 y_n = 0$$

$$\therefore \boxed{(x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0}$$