3. PARTIAL DIFFERENTIATION

Def^n If $y = f(x)$ then we take derivative by ordinary derivative.

But if $z = f(x, y)$ then we take partial derivative and denoted by $\partial z/\partial x$, $\partial z/\partial y$

To calculate partial derivative with respect to x of z we treat y as constant

And to calculate partial derivate with respect to y of z we treat x as a constant.

It is also denoted by z_x, z_y

And it is 1^{st} order derivative of z.

Second order derivative of z

$$\frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y \partial x}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y^2}.$$

$$\Rightarrow z_{xy}, z_{yx}, z_{xx}, z_{yy}.$$

• Rules of partial Differentiation

Let u & v are fun^n of two indepedant variables x & y.

① $\frac{\partial}{\partial x}(c) = 0$ {where c is const.}

② $\frac{\partial}{\partial x}(u \pm v) = \frac{\partial u}{\partial x} \pm \frac{\partial v}{\partial x}$, $\frac{\partial}{\partial y}(u \pm v) = \frac{\partial u}{\partial y} \pm \frac{\partial v}{\partial y}$.

③ $\frac{\partial}{\partial x}(u \cdot v) = u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x}$, $\frac{\partial}{\partial y}(u \cdot v) = u\frac{\partial v}{\partial y} + v\frac{\partial u}{\partial y}$.

④ $\frac{\partial}{\partial x}\left(\frac{u}{v}\right) = \frac{v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}}{v^2}$, $\frac{\partial}{\partial y}\left(\frac{u}{v}\right) = \frac{v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial x}}{v^2}$.

Ex - I

State the first order partial derivative the following function.

(1) $\log(x^2+y^2)$

Let $z = \log(x^2+y^2)$

$$\therefore \frac{\partial z}{\partial x} = \frac{2x}{x^2+y^2}, \quad \frac{\partial z}{\partial y} = \frac{2y}{x^2+y^2}$$

(2) $(y-ax)^{3/2}$

Let $z = (y-ax)^{3/2}$

$$\therefore \frac{\partial z}{\partial x} = \frac{3}{2}(y-ax)^{1/2} \times -a.$$

$$\frac{\partial z}{\partial y} = \frac{3}{2}(y-ax)^{1/2}$$

H.W

(3) $10^{x}\log$

(4) $\operatorname{Cosec}(ax+by)$

Ex - II

State the first order partial derivatives of the following.

(1) $e^{x+y} \cdot \sin(x-y)$

Sol ⇒ Let $z = e^{x+y} \cdot \sin(x-y)$

$$\frac{\partial z}{\partial x} = e^{x+y}\cos(x-y) + \sin(x-y)e^{x+y}$$

$$\frac{\partial z}{\partial y} = -e^{x+y}\cos(x-y) + \sin(x-y)e^{x+y}$$

Ex ⇒ Find the second order derivatives $\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y^2}$ of the following functions.

(I) $x^3y + e^{xy^2}$

Sol ⇒ Let $z = x^3y + e^{xy^2}$

$$\therefore \frac{\partial z}{\partial x} = 3x^2y + e^{xy^2}(y^2)$$

$$\therefore \frac{\partial^2 z}{\partial x^2} = 6xy + e^{xy^2}(y^4) \quad \text{--- (I)}$$

$$\therefore \frac{\partial z}{\partial y} = x^3 + 2xy\, e^{xy^2}$$

$$\frac{\partial^2 z}{\partial y^2} = 2x e^{xy^2} + 2x^2y^2 2 e^{xy^2}$$

$$= 2xe^{xy^2} + 4x^2y^2e^{xy^2} \quad \text{--- (II)}$$

H.W

(II) $e^x \log y + \sin y \cdot \log x$.

Differentiation of a Function of a Function $\Rightarrow$

Ex $\Rightarrow$ If $u = \log(\tan x + \tan y)$ P.T $\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} = 2$

Sol $\Rightarrow$ We have $\frac{\partial u}{\partial x} = \frac{1 \times \sec^2 x}{\tan x + \tan y}$

$$\therefore \sin 2x \frac{\partial u}{\partial x} = \frac{(2\sin x \cos x)}{(\tan x + \tan y)} \sec^2 x$$

$$\sin 2x \frac{\partial u}{\partial x} = \frac{2\tan x}{\tan x + \tan y} \quad \text{—— (I)}$$

Now $\frac{\partial u}{\partial y} = \frac{\sec^2 y}{\tan x + \tan y}$

$$\therefore \sin 2y \frac{\partial u}{\partial y} = \frac{2\sin y \cos y \cdot \sec^2 y}{\tan x + \tan y}$$

$$\therefore \sin 2y \frac{\partial u}{\partial y} = \frac{2\tan y}{\tan x + \tan y} \quad \text{—— (II)}$$

Adding (I) & (II)

$$\therefore \sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} = 2\left[\cancel{\frac{\tan x + \tan y}{\tan x + \tan y}}\right]$$

$$\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} = 2$$

Ex $\Rightarrow$ If $u = e^{xyz}$ P.T $\frac{\partial^3 u}{\partial x \partial y \partial z} [1 + 3xyz + x^2y^2z^2) e^{xyz}$

Sol $\Rightarrow$ $\because u = e^{xyz}$

$$\frac{\partial u}{\partial z} = e^{xyz} \cdot xy$$

$$\frac{\partial^2 u}{\partial y \partial z} = e^{xyz} \cdot x + x^2 y z e^{xyz} = e^{xyz}[x + x^2yz]$$

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = e^{xyz}(1 + 2xyz) + (x + x^2yz)(e^{xyz} \cdot yz)$$

$$= e^{xyz}[1 + 2xyz + xyz + x^2y^2z^2]$$

$$= e^{xyz}[1 + 3xyz + x^2y^2z^2)$$

(B) If $u=f(r)$, $r^2=x^2+y^2+z^2$ P.T $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=f''(r)+\frac{2}{r}f'(r)$

Sol $\Rightarrow$ $\because u=f(r)$

$\therefore \frac{\partial u}{\partial r}=f'(r)$, $\frac{\partial^2 u}{\partial r^2}=f''(r)$

$\because r^2=x^2+y^2+z^2$ —— ①

$\therefore \cancel{2}r\frac{\partial r}{\partial x}=\cancel{2}x$

$\therefore \boxed{\frac{\partial r}{\partial x}=\frac{x}{r}}$

Again diff. ① w.r.t y

$\therefore \cancel{2}r\frac{\partial r}{\partial y}=\cancel{2}y$

$\therefore \boxed{\frac{\partial r}{\partial y}=\frac{y}{r}}$

Again diff. w.r.t z

$\therefore \cancel{2}r\frac{\partial r}{\partial z}=\cancel{2}z$

$\boxed{\therefore \frac{\partial r}{\partial z}=\frac{z}{r}}$

$\therefore \frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x}$

$\frac{\partial u}{\partial x}=f'(r)\cdot\left(\frac{x}{r}\right)$

$\therefore \frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left(f'(r)\cdot x\cdot\frac{1}{r}\right)$

$\left[\because \frac{\partial}{\partial x}(u\cdot v\cdot w)=\left(\frac{\partial}{\partial x}u\right)v\cdot w+\left(\frac{\partial}{\partial x}v\right)uw+\left(\frac{\partial}{\partial x}w\right)uv\right]$

$\therefore \frac{\partial^2 u}{\partial x^2}=\left(\frac{\partial}{\partial r}f'(r)\right)\frac{\partial r}{\partial x}\cdot\left(\frac{x}{r}\right)+f'(r)\frac{x(1)}{r}+f'(r)(x)\frac{\partial}{\partial r}(r^{-1})\frac{\partial r}{\partial x}$

$=f''(r)\left(\frac{x}{r}\right)\left(\frac{x}{r}\right)+\frac{f'(r)}{r}+f'(r)x\cdot\frac{(-1)}{r^2}\left(\frac{x}{r}\right)$

$\frac{\partial^2 u}{\partial x^2}=f''(r)\frac{x^2}{r^2}+\frac{f'(r)}{r}-f'(r)\frac{x^2}{r^3}$ —— (II)

Similarly,

$$\frac{\partial^2 u}{\partial y^2} = f''(r)\frac{y^2}{r^2} + \frac{f'(r)}{r} - f'(r)\frac{y^2}{r^3} \longrightarrow (III)$$

$$\frac{\partial^2 u}{\partial z^2} = f''(r)\frac{z^2}{r^2} + \frac{f'(r)}{r} - f'(r)\frac{z^2}{r^3} \longrightarrow (IV)$$

Adding (II) (III) & (IV)

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{f''(r)}{r^2}(x^2+y^2+z^2) + \frac{3f'(r)}{r}$$

$$- \frac{f'(r)}{r^3}(x^2+y^2+z^2)$$

$$= \frac{f''(r)\cancel{r^2}}{\cancel{r^2}} + \frac{3f'(r)}{r} - \frac{f'(r)\cancel{r^2}}{r^{\cancel{3}}}$$

$$= f''(r) + \frac{2f'(r)}{r}.$$

Hence proved.

#(C) If $x = r\cos\theta$, $y = r\sin\theta$ P.T

(I) $\frac{\partial x}{\partial r} = \frac{\partial r}{\partial x}$ (II) $\frac{\partial x}{\partial \theta} = r^2\frac{\partial \theta}{\partial x}$

(III) $\left[x\cdot\left(\frac{\partial x}{\partial r}\right) + y\left(\frac{\partial y}{\partial r}\right)\right]^2 = x^2+y^2$

Sol=> (I) $\because x = r\cos\theta$

$$\therefore \frac{\partial x}{\partial r} = \cos\theta \qquad \because \cos\theta = \frac{x}{r}$$

$$\therefore \boxed{\frac{\partial x}{\partial r} = \frac{x}{r}} \longrightarrow (I)$$

Now $r^2 = x^2 + y^2$

diff. w.r.t x.

$$\therefore \cancel{2}r\frac{\partial r}{\partial x} = \cancel{2}x$$

$$\boxed{\frac{\partial r}{\partial x} = \frac{x}{r}} \longrightarrow (II)$$

From (I) & (II)

$$\frac{\partial x}{\partial r} = \frac{\partial r}{\partial x}.$$

(II) $\because x = r\cos\theta$ $\therefore \frac{\partial x}{\partial \theta} = -r\sin\theta = -y \longrightarrow (III)$

$\because \theta = \tan^{-1}\left(\frac{y}{x}\right)$

$$\therefore \frac{\partial \theta}{\partial x} = \frac{\cancel{x^2}}{x^2+y^2}\left(\frac{-y}{\cancel{x^2}}\right) = \frac{-y}{x^2+y^2} = \frac{-y}{r^2} \quad \therefore r^2\frac{\partial \theta}{\partial x} = -y \longrightarrow (IV)$$

From (III) & (IV) $\frac{\partial x}{\partial \theta} = r^2\frac{\partial \theta}{\partial x}.$

(iii) $\left[x\dfrac{\partial x}{\partial r}+y\dfrac{\partial y}{\partial r}\right]^2$ $\qquad \because \dfrac{\partial x}{\partial r}=\dfrac{x}{r},\ \dfrac{\partial y}{\partial r}=\dfrac{y}{r}$

$=\left[\dfrac{x^2}{r}+\dfrac{y^2}{r}\right]^2$ $\qquad$ As $\dfrac{\partial x}{\partial r}=\cos\theta=\dfrac{x}{r}$

$=\left[\dfrac{x^2+y^2}{r}\right]^2$ $\qquad \dfrac{\partial y}{\partial r}=\sin\theta=\dfrac{y}{r}$

$=\left[\dfrac{r^{\cancel{2}}}{\cancel{r}}\right]^2=(r)^2=r^2=x^2+y^2.$

Ex) If $z=x.\log(x+r)-r$ where $r^2=x^2+y^2$

HW P.T $\dfrac{\partial^2 z}{\partial x^2}+\dfrac{\partial^2 z}{\partial y^2}=\dfrac{1}{x+r},\ \dfrac{\partial^3 z}{\partial x^3}=\dfrac{-x}{r^3}$

HW
Ex ⇒ If $x=e^{r\cos\theta}\cos(r\sin\theta)$,

$y=e^{r\cos\theta}\sin(r\sin\theta)$ P.T

$\dfrac{\partial x}{\partial r}=\dfrac{1}{r}\dfrac{\partial y}{\partial \theta},\ \dfrac{\partial y}{\partial r}=\dfrac{-1}{r}\dfrac{\partial x}{\partial \theta}$

HW
Ex ⇒ Find the value of n so that $V=r^n(3\cos^2\theta-1)$

Satisfies the eqn $\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial V}{\partial r}\right)+\dfrac{1}{\sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta\dfrac{\partial V}{\partial\theta}\right)=$

HW
(1)(5) If $x=r\cos\theta$, $y=r\sin\theta$ P.T $\dfrac{\partial^2\theta}{\partial x^2}+\dfrac{\partial^2\theta}{\partial y^2}=0$

(14) If $u=Ae^{-gx}\sin(nt-gx)$ where A, g, n are constants satisfies the equation $\dfrac{\partial u}{\partial t}=a^2\dfrac{\partial^2 u}{\partial x^2}$

S.T $ag=\sqrt{\dfrac{n}{2}}$.

Sol ⇒ $\because u=Ae^{-gx}\sin(nt-gx)$

$\therefore \dfrac{\partial u}{\partial t}=Ae^{-gx}\cos(nt-gx)n$

$=Ane^{-gx}\cos(nt-gx).$

$\therefore \dfrac{\partial u}{\partial x}=\left[A e^{-gx}\cos(nt-gx)\times -g\right]+\left[-Ag.e^{-gx}\sin(nt-gx)\right]$

$\dfrac{\partial^2 u}{\partial x^2}=-Ag^2e^{-gx}\sin(nt-gx)+Ag^2e^{-gx}\cos(nt-gx)$
$+Ag^2e^{-gx}\cos(nt-gx)+Ag^2e^{-gx}\sin(nt-gx)$

$$\frac{\partial^2 y}{\partial x^2} = 2Ag^2 e^{-gx} \cos(nt - gx)$$

$$\therefore \quad a^2 \frac{\partial^2 y}{\partial x^2} = 2a^2 g^2 A e^{-gx} \cos(nt - gx)$$

$$\because \frac{\partial y}{\partial t} = nAe^{-gx} \cos(nt - gx)$$

$$\therefore \quad \frac{\partial y}{\partial t} = a^2 \frac{\partial^2 y}{\partial x^2}$$ where A, g, n are const.

By data $n = 2a^2 g^2$

$$\frac{n}{2} = a^2 g^2$$

$$ag = \sqrt{\frac{n}{2}}$$

Variables to be treated As Constants

Ex – IV

(1) If $x = r\cos\theta$ & $y = r\sin\theta$. P.T

(I) $\left(\frac{\partial r}{\partial x}\right)_y = \left(\frac{\partial x}{\partial r}\right)_\theta$

(II) $\left(\frac{\partial x}{\partial \theta}\right)_r = r^2\left(\frac{\partial \theta}{\partial x}\right)_y$

Sol ⇒ (II) $\because x = r\cos\theta$, $y = r\sin\theta \Rightarrow \sin\theta = \frac{y}{r}$

$\therefore \left(\frac{\partial x}{\partial \theta}\right)_r = -r\sin\theta$

$\therefore \left(\frac{\partial x}{\partial \theta}\right)_r = -y$ ——— (I)

$\because \theta = \tan^{-1}\left(\frac{y}{x}\right)$

$\therefore \frac{\partial \theta}{\partial x} = \frac{1}{1+(y^2/x^2)} \times \left(\frac{-y}{x^2}\right)$

$= \frac{x^2}{x^2+y^2} \times \frac{-y}{x^2} = -\frac{y}{r^2}$

$\therefore r^2\left(\frac{\partial \theta}{\partial x}\right)_y = -y$. ——— (II)

From (I) & (II).

$\left(\frac{\partial x}{\partial \theta}\right)_r = r^2\left(\frac{\partial \theta}{\partial x}\right)_y$.

Ex ⇒ If $x^2 = au + bv$, $y^2 = au - bv$ P.T $\left(\frac{\partial u}{\partial x}\right)_y\left(\frac{\partial x}{\partial u}\right)_v = \frac{1}{2}$ $= \left(\frac{\partial v}{\partial y}\right)_x\left(\frac{\partial y}{\partial v}\right)_u$.

Sol ⇒ $\because x^2 = au + bv$.

$2x\left(\frac{\partial x}{\partial u}\right)_v = a \Rightarrow \left(\frac{\partial x}{\partial u}\right)_v = \frac{a}{2x}$. ——— (I)

$y^2 = au - bv$.

$2y\left(\frac{\partial y}{\partial v}\right)_u = -b \Rightarrow \left(\frac{\partial y}{\partial v}\right)_u = \frac{-b}{2y}$ ——— (II)

$\because x^2 = au + bv$ $\qquad \because y^2 = au - bv$

$= au + (au - y^2)$ $\qquad bv = au - y^2$

$= 2au - y^2$

$2au = x^2 + y^2$

$2a\left(\frac{\partial u}{\partial x}\right)_y = 2x \Rightarrow \left(\frac{\partial u}{\partial x}\right)_y = \frac{x}{a}$ — (III)

$y^2 = au - bv$ $\qquad \because x^2 = au + bv$

$\therefore y^2 = x^2 - bv - bv$ $\qquad au = x^2 - bv$

$\Rightarrow 2bv = x^2 - y^2$

$2b\left(\frac{\partial v}{\partial y}\right)_x = -2y \Rightarrow \left(\frac{\partial v}{\partial y}\right)_x = \frac{-y}{b}$ — (IV)

Multiplying (I) × (III) & (II) × (IV)

$\therefore \left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v = \frac{\cancel{a}}{2\cancel{x}} \times \frac{\cancel{x}}{\cancel{a}} = \frac{1}{2}$

& $\left(\frac{\partial v}{\partial y}\right)_x \left(\frac{\partial y}{\partial v}\right)_u = \frac{+\cancel{y}}{\cancel{b}} \times \frac{+\cancel{b}}{2\cancel{y}} = \frac{1}{2}$

$\therefore \left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v = \frac{1}{2} = \left(\frac{\partial v}{\partial y}\right)_x \left(\frac{\partial y}{\partial v}\right)_u$

(H) If $x^2 = a\sqrt{u} + b\sqrt{v}$, $y^2 = a\sqrt{u} - b\sqrt{v}$. P.T

$\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v = \frac{1}{2} = \left(\frac{\partial v}{\partial y}\right)_x \left(\frac{\partial y}{\partial v}\right)_u$.

If $x = \cos\theta - r\sin\theta$, $y = \sin\theta + r\cos\theta$

(I) & (II) H.00

P.T (I) $\left(\frac{\partial \theta}{\partial x}\right)_y = \frac{-\cos\theta}{r}$ (II) $\left(\frac{\partial r}{\partial x}\right)_y = \frac{-x}{r}$

(III) $\left(\frac{\partial^2 \theta}{\partial x^2}\right)_y = \frac{-\cos\theta}{r^3}(\cos\theta - 2r\sin\theta)$

Sol $\Rightarrow$ (III) $\because x\cos\theta + y\sin\theta = \cos^2\theta - r\cancel{\sin\theta\cos\theta} + \sin^2\theta + r\cancel{\sin\theta\cos\theta}$

$x\cos\theta + y\sin\theta = 1$.

$\because\ x\cos\theta + y\sin\theta = 1$

$\therefore\ x\left(-\sin\theta\dfrac{\partial\theta}{\partial x}\right) + (1)\cos\theta + y\cos\theta\dfrac{\partial\theta}{\partial x} = 0$

$\therefore \left(\dfrac{\partial\theta}{\partial x}\right)_y (y\cos\theta - x\sin\theta) = -\cos\theta$

$\therefore \left(\dfrac{\partial\theta}{\partial x}\right)_y = \dfrac{-\cos\theta}{y\cos\theta - x\sin\theta}$

$\left[\because\ y\cos\theta - x\sin\theta = (\sin\theta + r\cos\theta)\cos\theta - (\cos\theta - r\sin\theta)\sin\theta = r(\cos^2\theta + \sin^2\theta) = r\right]$

$\therefore \left(\dfrac{\partial\theta}{\partial x}\right)_y = \dfrac{-\cos\theta}{r}$

$\therefore \left(\dfrac{\partial^2\theta}{\partial x^2}\right)_y = \dfrac{r\left(\sin\theta\dfrac{\partial\theta}{\partial x}\right) + \cos\theta\dfrac{\partial r}{\partial x}}{r^2}$

$\left[\because\ 1 + r^2 = x^2 + y^2\right.$

$2r\dfrac{\partial r}{\partial x} = 2x$

$\left.\therefore \dfrac{\partial r}{\partial x} = \dfrac{x}{r}\right]$

$\therefore \left(\dfrac{\partial^2\theta}{\partial x^2}\right)_y = \dfrac{r\sin\theta\left(\dfrac{-\cos\theta}{r}\right) + \cos\theta\dfrac{x}{r}}{r^2}$

$= \dfrac{-r\sin\theta\cos\theta + (r\sin\theta\cos\theta + \cos^2\theta)}{r^3}$

$= \dfrac{\cos\theta(\cos\theta - 2r\sin\theta)}{r^3}$

Composite Functions ⇒

Let $z = f(x,y)$ & $x = \phi(t)$, $y = \psi(t)$ Then z is called a composite function of t.

eg ⇒ $z = x^2 + y^2$, $x = at^2$, $y = 2at$.

Differentiation ⇒

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}.$$

Ex - IV

① If $u = lx + my$ & $v = mx - ly$ P.T

(i) $\left(\frac{\partial u}{\partial x}\right)_y \cdot \left(\frac{\partial x}{\partial u}\right)_v = \frac{l^2}{l^2+m^2}$ (ii) $\left(\frac{\partial y}{\partial v}\right)_x \cdot \left(\frac{\partial v}{\partial y}\right)_u = \frac{l^2+m^2}{l^2}$

Sol ⇒ (i) $u = lx + my$ $\therefore \left(\frac{\partial u}{\partial x}\right)_y = l$. — (i)

$\therefore y = \frac{mx - v}{l}$.

$\therefore lx + m\left(\frac{mx - v}{l}\right) = u$ $\therefore (l^2 + m^2)x - mv = lu$.

$\therefore x = \frac{lu + mv}{l^2 + m^2}$ $\therefore \left(\frac{\partial x}{\partial u}\right)_v = \frac{l}{l^2+m^2}$ — (ii)

$\therefore \left(\frac{\partial u}{\partial x}\right)_y \cdot \left(\frac{\partial x}{\partial u}\right)_v = \frac{l^2}{l^2+m^2}$.

(ii) $ly = -v + mx$ $\therefore y = \frac{-v + mx}{l}$ $\therefore \left(\frac{\partial y}{\partial v}\right)_x = \frac{-1}{l}$. — (i)

$\therefore x = \frac{u - my}{l}$. $\therefore v = m\left(\frac{u - my}{l}\right) - ly$.

$= \frac{mu - m^2y - l^2y}{l}$.

$v = \frac{mu - (m^2y + l^2y)}{l}$.

$\therefore \left(\frac{\partial v}{\partial y}\right)_u = \frac{-(m^2 + l^2)}{l}$. — (ii)

$\therefore \left(\frac{\partial y}{\partial v}\right)_x \cdot \left(\frac{\partial v}{\partial y}\right)_u = \frac{(m^2 + l^2)}{l^2}$.

EX-V

1) If $z = x^2 + y^2$, $x = at^2$, $y = 2at$ verify that $\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$

Sol ⇒ $z = a^2t^4 + 4a^2t^2$.

$$\frac{dz}{dt} = 4a^2t^3 + 8a^2t . \quad \text{—— (1)}$$

$z = x^2 + y^2$.

$$\frac{\partial z}{\partial x} = 2x \quad , \quad \frac{\partial z}{\partial y} = 2y .$$

$$x = at^2 \Rightarrow \frac{dx}{dt} = 2at \qquad y = 2at \Rightarrow \frac{dy}{dt} = 2a .$$

RHS $\frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \times \frac{dy}{dt}$

$$= 2(at^2) \times 2at + 2(2at)(2a)$$

$$= 4a^2t^3 + 8a^2t \quad \text{—— (11)}$$

∴ LHS = RHS .

If $u = x^2 + y^2 + z^2$, $x = e^t$, $y = e^t \sin t$, $z = e^t \cos t$

Find du/dt.

Sol ⇒ $\because \frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt}$.

$$\because \frac{\partial u}{\partial x} = 2x = 2e^t$$

$$\frac{\partial u}{\partial y} = 2y = 2e^t \sin t.$$

$$\frac{\partial u}{\partial z} = 2z = 2e^t \cos t.$$

$$\frac{dx}{dt} = e^t, \quad \frac{dy}{dt} = e^t \cos t + e^t \sin t$$

$$\frac{dz}{dt} = -e^t \sin t + \cos t\, e^t.$$

$$\text{RHS} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt}$$

$$= 2e^t e^t + 2e^t \sin t\,(e^t \cos t + e^t \sin t)$$
$$+ 2e^t \cos t\,(-e^t \sin t + \cos t\, e^t)$$
$$= 2e^{2t} + 2e^{2t}\cancel{\sin t \cos t} + 2e^{2t}\sin^2 t$$
$$- 2e^{2t}\cancel{\sin t}\cos t + 2e^{2t}\cos^2 t$$
$$= 2e^{2t}(1+1) = 4e^{2t}$$

LHS = RHS.

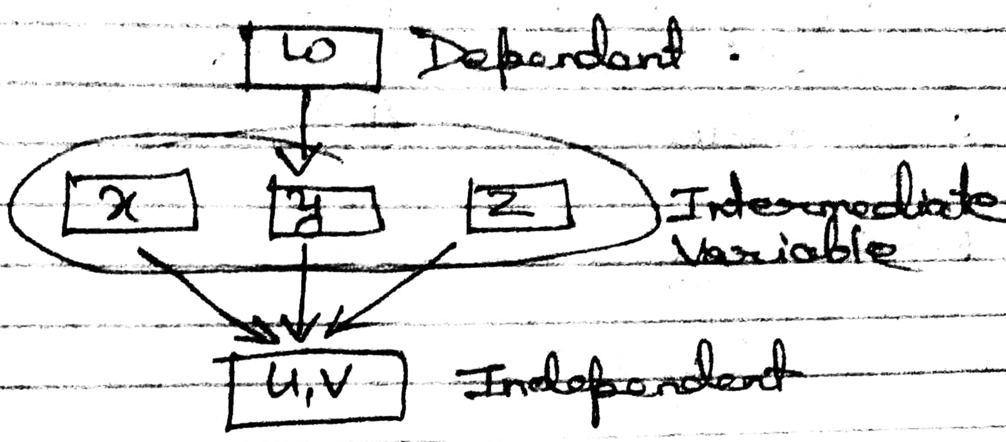
(11)

Partial differential of composite f^n =>

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$$

& $$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}.$$

In general



Ex – VI

(A) If $z = f(u,v)$, $u = x^2 - y^2$, $v = y^2 - x^2$

P.T $y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = 0$.

Sol =>
$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}.$$
$$= \frac{\partial z}{\partial u}(2x) + \frac{\partial z}{\partial v}(-2x)$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$
$$= \frac{\partial z}{\partial u}(-2y) + \frac{\partial z}{\partial v}(2y).$$

$$LHS = y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y}$$
$$2xy\left(\cancel{\frac{\partial z}{\partial u}}\right) \cancel{-2xy\frac{\partial z}{\partial v}} \cancel{-2xy\frac{\partial z}{\partial u}} + \cancel{2xy\frac{\partial z}{\partial v}} = 0. \quad \boxed{\therefore y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = 0}$$

(12) If $u = f(x^2-y^2, y^2-z^2, z^2-x^2)$ P.T

$$\frac{1}{x}\frac{\partial u}{\partial x} + \frac{1}{y}\frac{\partial u}{\partial y} + \frac{1}{z}\frac{\partial u}{\partial z} = 0.$$

Sol ⇒ $u = f(X, Y, Z)$

$X = x^2-y^2$; $Y = y^2-z^2$, $Z = z^2-x^2$.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial X}\frac{\partial X}{\partial x} + \frac{\partial u}{\partial Z}\cdot\frac{\partial Z}{\partial x}$$

$$= \frac{\partial u}{\partial X}(2x) + \frac{\partial u}{\partial Z}(-2x)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial X}\cdot\frac{\partial X}{\partial y} + \frac{\partial u}{\partial Y}\cdot\frac{\partial Y}{\partial y}$$

$$= \frac{\partial u}{\partial X}(-2y) + \frac{\partial u}{\partial Y}(2y)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial Y}\cdot\frac{\partial Y}{\partial z} + \frac{\partial u}{\partial Z}\cdot\frac{\partial Z}{\partial z}$$

$$= \frac{\partial u}{\partial Y}(-2z) + \frac{\partial u}{\partial Z}(2z).$$

$\therefore \frac{1}{x}\frac{\partial u}{\partial x} + \frac{1}{y}\frac{\partial u}{\partial y} + \frac{1}{z}\frac{\partial u}{\partial z}$

$$= \frac{1}{x}\left(2x\frac{\partial u}{\partial x} - 2x\frac{\partial u}{\partial z}\right) + \frac{1}{y}\left(-2y\frac{\partial u}{\partial x} + 2y\frac{\partial u}{\partial y}\right)$$

$$+ \frac{1}{z}\left(-2z\frac{\partial u}{\partial y} + 2z\frac{\partial u}{\partial z}\right)$$

$$= 2\left[\cancel{\frac{\partial u}{\partial x}} - \cancel{\frac{\partial u}{\partial z}} - \cancel{\frac{\partial u}{\partial x}} + \cancel{\frac{\partial u}{\partial y}} - \cancel{\frac{\partial u}{\partial y}} + \cancel{\frac{\partial u}{\partial z}}\right]$$

$= 2[0] = 0$ // $\boxed{LHS = RHS}$

(13) If $u = f(x^n - y^n, y^n - z^n, z^n - x^n)$ P.T

$$\frac{1}{x^{n-1}} \cdot \frac{\partial u}{\partial x} + \frac{1}{y^{n-1}} \cdot \frac{\partial u}{\partial y} + \frac{1}{z^{n-1}} \frac{\partial u}{\partial z} = 0.$$

(9) $u = x^2 - y^2$, $v = 2xy$ and $z = f(u, v)$ P.T

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 4\sqrt{u^2+v^2}\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right]$$

Sol $\Rightarrow$ $$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$= \frac{\partial z}{\partial u}(2x) + \frac{\partial z}{\partial v}(2y).$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y}$$

$$= \frac{\partial z}{\partial u} \cdot (-2y) + \frac{\partial z}{\partial v} \cdot (2x)$$

LHS $$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$$

$$= \left[\left(2x\frac{\partial z}{\partial u} + 2y\frac{\partial z}{\partial v}\right)^2 + \left(-2y\frac{\partial z}{\partial u} + 2x\frac{\partial z}{\partial v}\right)^2\right]$$

$$= 4x^2\left(\frac{\partial z}{\partial u}\right)^2 + \cancel{8xy\frac{\partial z}{\partial u} \cdot \frac{\partial z}{\partial v}} + 4y^2\left(\frac{\partial z}{\partial v}\right)^2$$

$$+ 4y^2\left(\frac{\partial z}{\partial u}\right)^2 - \cancel{8xy\frac{\partial z}{\partial u} \cdot \frac{\partial z}{\partial v}} + 4x^2\left(\frac{\partial z}{\partial v}\right)^2$$

$$= 4x^2\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right] + 4y^2\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right]$$

$$= (4x^2 + 4y^2)\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right]$$

$$= 4(x^2+y^2)\left[\text{——} \,{''}\, \text{——}\right]$$

RHS $\Rightarrow$ $$4\sqrt{u^2+v^2}\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right]$$

$$= 4\sqrt{(x^2-y^2)^2 + 4x^2y^2}\left[\text{——} \,{''}\, \text{——}\right]$$

$$= 4\sqrt{x^4+y^4-2x^2y^2+4x^2y^2}\left[\text{——} \,{''}\, \text{——}\right]$$

$$= 4\sqrt{(x^2+y^2)^2}\left[\text{——} \,{''}\, \text{——}\right]$$

$$= 4(x^2+y^2)\left[\text{——} \,{''}\, \text{——}\right]$$