LOGARITHM OF COMPLEX NUMBER

Logarithm of complex no. ⇒

Let $z = \log(x+iy)$ & Let $x = r\cos\theta$, $y = r\sin\theta$

$\therefore r = \sqrt{x^2+y^2}$ & $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$\therefore z = \log(re^{i\theta})$

$= \log r + \log e^{i\theta}$

$= \log r + i\theta \quad \text{— (i)}$

$\therefore \log(x+iy) = \log r + i\theta$

$$\boxed{\log(x+iy) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{y}{x}\right)}$$

This is called principal value of $\log(x+iy)$

General value of $\log(x+iy)$ ⇒

$\text{Log}(x+iy) = \log r + i(2n\pi + \theta)$

$$\boxed{\text{Log}(x+iy) = \frac{1}{2}\log(x^2+y^2) + i\left(\tan^{-1}\left(\frac{y}{x}\right) + 2n\pi\right)}$$

Note ⇒ (i) $\log(x-iy) = \frac{1}{2}\log(x^2+y^2) - i\tan^{-1}\left(\frac{y}{x}\right)$

(ii) $\text{Log}(x-iy) = \frac{1}{2}\log(x^2+y^2) - i\left(2n\pi + \tan^{-1}\left(\frac{y}{x}\right)\right)$

(iii) $\log i = \log(0+i) = i\frac{\pi}{2}$

(iv) $\log(i^i) = i\log i = i^2\frac{\pi}{2} = -\frac{\pi}{2}$ & $i^i = e^{-\pi/2}$

(v) $\text{Log}\, i = i\left(2n\pi + \frac{\pi}{2}\right)$

(vi) $\sin(\log i^i) = \sin\left(-\frac{\pi}{2}\right) = -\sin\frac{\pi}{2} = -1$

(vii) $\cos(\log i^i) = \cos\left(-\frac{\pi}{2}\right) = \cos\frac{\pi}{2} = 0$

EX

Express the following in the form $a+ib$ (i) $\text{Log}(1+i)$

Sol ⇒ $\text{Log}(1+i) = \frac{1}{2}\log(1^2+1^2) + i\left(2n\pi + \tan^{-1}\left(\frac{1}{1}\right)\right)$

(as $x=1$ & $y=1$)

$\because \operatorname{Log}(x+iy) = \frac{1}{2}\log(x^2+y^2) + i\left(2n\pi + \tan^{-1}\left(\frac{y}{x}\right)\right)$

$\therefore \operatorname{Log}(1+i) = \frac{1}{2}\log(2) + i\left(2n\pi + \frac{\pi}{4}\right)$

$\therefore \operatorname{Log}(1+i) = \log\sqrt{2} + i\left(2n\pi + \frac{\pi}{4}\right)$

(ii) $\operatorname{Log}(-i)$

(iii) $\operatorname{Log}(3+4i)$ $\Big\}$ H.W

Ex ⇒ Show that $\sin \log_e i^i = -1$

Sol ⇒ Consider $\sin(\log_e i^i) = \sin(i \log i)$

$= \sin\left(i\left(\log 1 + i\frac{\pi}{2}\right)\right)$

$\left(\because r = \sqrt{0+1} = 1,\ \theta = \tan^{-1}\left(\frac{1}{0}\right) = \frac{\pi}{2}\right)$

$= \sin\left(i\left(i\frac{\pi}{2}\right)\right)$

$= \sin\left(-\frac{\pi}{2}\right)$

$= -\sin\left(\frac{\pi}{2}\right)$

$= -1$

Ex ⇒ Prove that $\log\left(\frac{1}{1+e^{i\theta}}\right) = \log\left(\frac{1}{2}\sec\frac{\theta}{2}\right) - \frac{i\theta}{2}$

Sol ⇒ Consider $\log\left(\frac{1}{1+e^{i\theta}}\right) = -\log(1+e^{i\theta})$

$= -\log(1+\cos\theta + i\sin\theta)$

$= -\left[\frac{1}{2}\log\left((1+\cos\theta)^2 + \sin^2\theta\right) + i\tan^{-1}\left(\frac{\sin\theta}{1+\cos\theta}\right)\right]$

$= -\left[\frac{1}{2}\log\left(1+\cos^2\theta + 2\cos\theta + \sin^2\theta\right) + i\tan^{-1}\left(\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}\right)\right]$

$= -\left[\frac{1}{2}\log(2+2\cos\theta) + i\tan^{-1}\left(\tan\frac{\theta}{2}\right)\right]$

$= -\left[\frac{1}{2}\log\left(2(2\cos^2\theta/2)\right) + i\frac{\theta}{2}\right]$

$= -\log(2\cos\theta/2) - \frac{i\theta}{2}$

$= \log\left(\frac{1}{2}\sec\frac{\theta}{2}\right) - \frac{i\theta}{2}$

Separate into real & imaginary parts,

(i) $(\sin\theta + i\cos\theta)^i$ (ii) $(\sqrt{i})^{\sqrt{i}}$

Sol $\Rightarrow$ (i) Let $z = (\sin\theta + i\cos\theta)^i$

$$\therefore \log z = i\log(\sin\theta + i\cos\theta)$$
$$= i\log\left(\cos\left(\frac{\pi}{2}-\theta\right) + i\sin\left(\frac{\pi}{2}-\theta\right)\right)$$
$$= i\log\left(e^{i\left(\frac{\pi}{2}-\theta\right)}\right)$$
$$= i\left(i\left(\frac{\pi}{2}-\theta\right)\right) = i^2\left(\frac{\pi}{2}-\theta\right) = \theta - \frac{\pi}{2}$$

$$\therefore z = e^{\theta - \frac{\pi}{2}}$$

Prove that $\log[\cos(x+iy)] = \frac{1}{2}\log\left(\frac{\cosh 2y + \cos 2x}{2}\right) - i\tan^{-1}(\tan x \cdot \tanh y)$

Sol $\Rightarrow$ Let $\log[\cos(x+iy)]$

$$= \log[(\cos x\cosh y - i\sin x\sinh y)]$$
$$= \frac{1}{2}\log\left(\cos^2 x\cosh^2 y + \sin^2 x\sinh^2 y\right) - i\tan^{-1}\left(\frac{\sin x\sinh y}{\cos x\cosh y}\right)$$
$$= \frac{1}{2}\log\left(\cos^2 x\cosh^2 y + (1-\cos^2 x)(\cosh^2 y - 1)\right) - i\tan^{-1}(\tan x\tanh y)$$
$$= \frac{1}{2}\log\left(\cancel{\cos^2 x\cosh^2 y} + \cosh^2 y - \cancel{\cos^2 x\cosh^2 y} - 1 + \cos^2 x\right) - i\tan^{-1}(\tan x \tanh y)$$
$$= \frac{1}{2}\log\left(\cosh^2 y + \cos^2 x - 1\right) - i\tan^{-1}(\tan x \tanh y)$$
$$= \frac{1}{2}\log\left(\frac{1+\cosh 2y}{2} + \frac{1+\cos 2x}{2} - 1\right) - i\tan^{-1}(\tan x \cdot \tanh y)$$
$$= \frac{1}{2}\log\left(\frac{\cancel{1} + \cosh 2y + \cancel{1} + \cos 2x - \cancel{2}}{2}\right) - i\tan^{-1}(\tan x \cdot \tanh y)$$
$$= \frac{1}{2}\log\left(\frac{\cosh 2y + \cos 2x}{2}\right) - i\tan^{-1}(\tan x \cdot \tanh y)$$

MCQ

If $\tan[\log(x+iy)] = a + ib$ Then P.T

$\tan[\log(x^2+y^2)] = \frac{2a}{1-a^2-b^2}$ when $a^2+b^2 \neq 1$

Ex ⇒ If $\log(x+iy) = e^{p+iq}$ P.T $y = x\tan\theta$ where $2\theta = \tan q \cdot \log(x^2+y^2)$

or if $\log\log(x+iy) = p+iq$ P.T $y = x\tan(\tan q \cdot \log\sqrt{x^2+y^2})$.

Sol ⇒ We have $\log(x+iy) = e^p \cdot e^{iq}$

$$= e^p(\cos q + i\sin q)$$

$\therefore \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\frac{y}{x} = e^p(\cos q + i\sin q)$

Equating real & imaginary parts

$$e^p\cos q = \frac{1}{2}\log(x^2+y^2) \quad \text{—— (1)}$$

$$\& \; e^p\sin q = \tan^{-1}\left(\frac{y}{x}\right) \quad \text{—— (11)}$$

$$\therefore \frac{y}{x} = \tan(e^p\sin q)$$

$$y = x\tan(e^p\sin q) \quad \text{where } \theta = e^p\sin q$$

$$y = x\tan\theta.$$

where $\theta = e^p \cdot \sin q$

$= \sin q \cdot e^p$. From (1)

$$= \sin q \cdot \frac{1}{\cos q}\frac{1}{2}\log(x^2+y^2)$$

$$= \frac{1}{2}\tan q \log(x^2+y^2)$$

$$\therefore 2\theta = \tan q \log(x^2+y^2)$$

Or $y = x\tan\left(\frac{1}{2}\tan q \log(x^2+y^2)\right)$

$$= x\tan(\tan q \log\sqrt{x^2+y^2}).$$

H.W

Ex ⇒ Considering only the principle values prove that the real part of $(1+i\sqrt{3})^{(1+i\sqrt{3})}$ is,

$2e^{-\pi/\sqrt{3}}(\cos(\pi/3 + \sqrt{3}\log 2)$.

H.W

Ex ⇒ If $\tan[\log(x+iy)] = a+ib$ P.T $\tan[\log(x^2+y^2)] = \frac{2a}{1-a^2-b^2}$ when $a^2+b^2 \neq 1$.

Ex ⇒ If $i^{i^{i \cdots \infty}} = A+iB$ considering the principal value P.T $\tan\left(\frac{\pi A}{2}\right) = \frac{B}{A}$ and $A^2+B^2 = e^{-\pi B}$.

Sol ⇒ $\because i^{i^{i \cdots \infty}} = A+iB$

$$i^{A+iB} = A+iB$$

$$(A+iB)\log i = \log(A+iB)$$

$$(A+iB)\log e^{i\pi/2} = \frac{1}{2}\log\{A^2+B^2\} + i\tan^{-1}\left(\frac{B}{A}\right)$$

$$(A+iB)\frac{i\pi}{2} = \frac{1}{2}\log(A^2+B^2) + i\tan^{-1}\left(\frac{B}{A}\right).$$

$$\frac{i\pi A}{2} - \frac{\pi B}{2} = \frac{1}{2}\log(A^2+B^2) + i\tan^{-1}\left(\frac{B}{A}\right)$$

$$\frac{1}{2}\log(A^2+B^2) = -\frac{\pi B}{2}.$$

$$\therefore \log(A^2+B^2) = -\pi B \qquad \Rightarrow A^2+B^2 = e^{-\pi B}$$

$$\& \tan^{-1}\left(\frac{B}{A}\right) = \frac{\pi A}{2}. \quad \text{(ii)}$$

$$\therefore \tan\frac{\pi A}{2} = \frac{B}{A}.$$

If $\dfrac{(1+i)^{x+iy}}{(1-i)^{x-iy}} = \alpha + i\beta$ Find α & β.

Sol ⇒ Taking log on B.S.

$$\log(\alpha+i\beta) = (x+iy)\log(1+i) - (x-iy)\log(x-iy)$$

$$= (x+iy)\left[\log\sqrt{2} + \frac{i\pi}{4}\right] - (x-iy)\left[\log\sqrt{2} - \frac{i\pi}{4}\right]$$

$$= (x+iy)\left[\frac{1}{2}\log 2 + \frac{i\pi}{4}\right] - (x-iy)\left[\frac{1}{2}\log 2 - \frac{i\pi}{4}\right]$$

$$= 2i\left(\frac{x\pi}{4} + \frac{y}{2}\log 2\right)$$

$$= i\left(\frac{x\pi}{2} + y\log 2\right)$$

$$= iK$$

$$\therefore (\alpha+i\beta) = e^{iK}$$

$$= \cos K + i\sin K.$$

where $K = \left(\frac{x\pi}{2} + y\log 2\right)$

$$\therefore \alpha = \cos\left(\frac{x\pi}{2} + y\log 2\right)$$

$$\& \ \beta = \sin\left(\frac{x\pi}{2} + y\log 2\right)$$

Considering only the principle values P.T real part of $(1+i\sqrt{3})^{(1+i\sqrt{3})}$ is $2e^{-\pi/\sqrt{3}}\left(\cos\frac{\pi}{3}+\sqrt{3}\log 2\right)$

Sol => Let $x+iy = (1+i\sqrt{3})^{(1+i\sqrt{3})}$

$$\log(x+iy) = (1+i\sqrt{3})\log(1+i\sqrt{3})$$

$$= (1+i\sqrt{3})\left(\log(\sqrt{1+3}) + i\tan^{-1}(\sqrt{3})\right)$$

$$= (1+i\sqrt{3})\left(\log 2 + i\frac{\pi}{3}\right)$$

$$= \left(\log 2 - \frac{\pi}{\sqrt{3}}\right) + i\left(\frac{\pi}{3} + \sqrt{3}\log 2\right)$$

$$x+iy = e^{\log 2} \cdot e^{-\pi/\sqrt{3}} \cdot e^{i(\pi/3 + \sqrt{3}\log 2)}$$

$$= 2e^{-\pi/\sqrt{3}} \cdot \left[\cos\left(\frac{\pi}{3} + \sqrt{3}\log 2\right) + i\sin\left(\frac{\pi}{3} + \sqrt{3}\log 2\right)\right]$$

$$\therefore x = 2e^{-\pi/\sqrt{3}}\cos\left(\frac{\pi}{3} + \sqrt{3}\log 2\right)$$