2. Hyperbolic Function And Logarithm of Complex Number.

Circular functions of complex number And Hyperbolic Functions. Inverse Circular and Inverse Hyperbolic function. Separation of real and imaginary parts of all types of functions.

I Circular Functions of a Complex No.

We have Euler's Expression

$$e^{i\theta} = \cos\theta + i\sin\theta \quad \text{—(1)}$$

$$e^{-i\theta} = \cos\theta - i\sin\theta \quad \text{—(2)}$$

Add (1) + (2)

$$e^{i\theta} + e^{-i\theta} = 2\cos\theta$$

$$\therefore \boxed{\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}}$$

And By (1) − (2)

$$e^{i\theta} - e^{-i\theta} = 2i\sin\theta$$

$$\therefore \boxed{\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}}$$

called as Euler's exponential forms of circular funⁿ where θ is real number.

Now If $z = x + iy$ is complex No.

Then

$$\boxed{\cos z = \frac{e^{iz} + e^{-iz}}{2}} \quad \text{And} \quad \boxed{\sin z = \frac{e^{iz} - e^{-iz}}{2i}}$$

Known as circular function of complex number.

II Hyperbolic function

If x is real or complex number then $\frac{e^x+e^{-x}}{2}$ is called Hyperbolic cosine of x or cosine hyperbolic of x i and denoted by $\cosh x$

Also

$\frac{e^x-e^{-x}}{2}$ is called Hyperbolic sine of x denoted by $\sinh x$

$$\boxed{\sinh x = \frac{e^x-e^{-x}}{2}} \quad \text{And} \quad \boxed{\cosh x = \frac{e^x+e^{-x}}{2}}$$

$\therefore$

$$\tanh x = \frac{e^x-e^{-x}}{e^x+e^{-x}}$$

$$\coth x = \frac{e^x+e^{-x}}{e^x-e^{-x}}$$

$$\operatorname{sech} x = \frac{2}{e^x+e^{-x}}$$

$$\operatorname{cosech} x = \frac{2}{e^x-e^{-x}}$$

Relationship of circular function into hyperbolic functions :

i) $\sin(ix) = i\sinh x$

ii) $\cos(ix) = \cosh x$

iii) $\tan(ix) = i\tanh x$

iv) $\cot(ix) = -i\coth x$

v) $\sec(ix) = \operatorname{sech} x$

vi) $\operatorname{cosec}(ix) = -i\operatorname{cosech} x$

(1) $\sin(ix) = i\sinh x$

$\sin x = \frac{e^{ix}-e^{-ix}}{2i}$

Put $x = ix$

$\sin(ix) = \frac{e^{i(ix)}-e^{-i(ix)}}{2i}$

$= \frac{e^{-x}-e^{x}}{2i} = \frac{-1}{i}\left[\frac{e^x-e^{-x}}{2}\right]$

$= -(-i)\sinh x$

$\sin(ix) = i\sinh x$

Relationship of Hyperbolic function into Circular functions:

i) $\sinh(ix) = i\sin x$

ii) $\cosh(ix) = \cos x$

iii) $\tanh(ix) = i\tan x$

iv) $\coth(ix) = i\cot x$

v) $\operatorname{sech}(ix) = \sec x$

vi) $\operatorname{cosech}(ix) = -i\operatorname{cosec} x.$

① $\sinh x = \frac{e^x - e^{-x}}{2}$

put $x = ix$

$\sinh(ix) = \frac{e^{ix} - e^{-ix}}{2}$

$= i\left[\frac{e^{ix} - e^{-ix}}{2i}\right]$

$= i\sin x$

* Hyperbolic Identities *

(I) Square Hyperbolic identities

① $\cosh^2 x - \sinh^2 x = 1$

② $1 - \tanh^2 x = \operatorname{sech}^2 x$

③ $1 - \coth^2 x = -\operatorname{cosech}^2 x$

(II) Sum and difference Hyperbolic formulae

① $\sinh(x+y) = \sinh x.\cosh y + \cosh x.\sinh y$

② $\sinh(x-y) = \sinh x.\cosh y - \cosh x.\sinh y$

③ $\cosh(x+y) = \cosh x.\cosh y + \sinh x.\sinh y$

④ $\cosh(x-y) = \cosh x.\cosh y - \sinh x.\sinh y$

⑤ $\tanh(x+y) = \frac{\tanh x + \tanh y}{1 + \tanh x.\tanh y}$

⑥ $\tanh(x-y) = \frac{\tanh x - \tanh y}{1 - \tanh x.\tanh y}$

(III) multiple Angle Hyperbolic formulae

① $\sinh(2x) = 2\sinh x.\cosh x.$

② $\sinh(2x) = \dfrac{2\tanh x}{1-\tanh^2 x}$

③ $\cosh(2x) = \cosh^2 x + \sinh^2 x$

$= 2\cosh^2 x - 1$

$= 1 + 2\sinh^2 x$

$= \dfrac{1+\tanh^2 x}{1-\tanh^2 x}$

④ $\tanh(2x) = \dfrac{2\tanh x}{1+\tanh^2 x}$

⑤ $\sinh(3x) = 3\sinh x + 4\sinh^3 x$

⑥ $\cosh(3x) = 4\cosh^3 x - 3\cosh x$

⑦ $\tanh(3x) = \dfrac{3\tanh x + \tanh^2 x}{1+3\tanh^2 x}$.

IV Product Hyperbolic formulae

① $2\sinh x \cdot \cosh y = \sinh(x+y) + \sinh(x-y)$

② $2\cosh x \cdot \sinh y = \sinh(x+y) - \sinh(x-y)$

③ $2\cosh x \cdot \cosh y = \cosh(x+y) + \cosh(x-y)$

④ $2\sinh x \cdot \sinh y = \cosh(x+y) - \cosh(x-y)$

V Defactorization Hyperbolic formulae

① $\sinh x + \sinh y = 2\sin\left(\frac{x+y}{2}\right) \cdot \cosh\left(\frac{x-y}{2}\right)$

② $\sinh x - \sinh y = 2\cosh\left(\frac{x+y}{2}\right) \cdot \sinh\left(\frac{x-y}{2}\right)$

③ $\cosh x + \cosh y = 2\cosh\left(\frac{x+y}{2}\right) \cdot \cosh\left(\frac{x-y}{2}\right)$

④ $\cosh x - \cosh y = 2\sinh\left(\frac{x+y}{2}\right) \cdot \sinh\left(\frac{x-y}{2}\right)$.

Ex

P.T $\left[\sin(\alpha-\theta)+e^{i\alpha}\sin\theta\right]^n = \sin^n\alpha\, e^{in\theta}$

Sol ⇒ Consider $\left[\sin(\alpha-\theta)+e^{i\alpha}\sin\theta\right]^n$

$$= \left[\sin\alpha\cos\theta - \cos\alpha\sin\theta + (\cos\alpha + i\sin\alpha)\sin\theta\right]^n$$

$$= \left[\sin\alpha\cos\theta - \cancel{\cos\alpha\sin\theta} + \cancel{\cos\alpha\sin\theta} + i\sin\alpha\sin\theta\right]^n$$

$$= \left[\sin\alpha(\cos\theta + i\sin\theta)\right]^n$$

$$= \sin^n\alpha\left[e^{i\theta}\right]^n$$

$$= \sin^n\alpha \cdot e^{in\theta}$$

P.T $\sin^{-1}z = -i\log\left[iz \pm \sqrt{1-z^2}\right]$

Sol ⇒ Let $\sin^{-1}z = u$ ∴ $z = \sin u = \dfrac{e^{iu}-e^{-iu}}{2i}$

$$\therefore 2zi = e^{iu} - e^{-iu}$$

$$e^{2iu} - 2zie^{iu} - 1 = 0$$

Solving this by quadratic,

$$\therefore e^{iu} = \frac{2zi \pm \sqrt{4z^2i^2+4}}{2}$$

$$= zi \pm \sqrt{z^2i^2+1}$$

$$\therefore iu = \log\left(zi \pm \sqrt{z^2i^2+1}\right)$$

$$u = -i\log\left(zi \pm \sqrt{-z^2+1}\right)$$

$$\therefore \sin^{-1}z = -i\log\left(zi \pm \sqrt{-z^2+1}\right)$$

$$\therefore \sin^{-1}z = -i\log\left(zi \pm \sqrt{1-z^2}\right)$$

(iii) P.T. $\left(\dfrac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}\right)^n = e^{in(\pi/2-\theta)}$

Sol ⇒ Consider $\left(\dfrac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}\right)^n$

$$=\left(\frac{1+\cos\left(\frac{\pi}{2}-\theta\right)+i\sin\left(\frac{\pi}{2}-\theta\right)}{1+\cos\left(\frac{\pi}{2}-\theta\right)-i\sin\left(\frac{\pi}{2}-\theta\right)}\right)^n$$

$$=\left(\frac{2\cos^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right)+2i\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}{2\cos^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right)-2i\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}\right)^n$$

$$=\left[\frac{\cancel{2\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}\left(\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)+i\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right)}{\cancel{2\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}\left(\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)-i\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right)}\right]^n$$

$$=\left[\frac{e^{i\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}}{e^{-i\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}}\right]^n$$

$$=\left[e^{2i\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}\right]^n = e^{in\left(\frac{\pi}{2}-\theta\right)}$$

If $\tanh x = \frac{2}{3}$ Find the value of x and then $\cosh 2x$.

Sol: ⇒ We can obtain $\cosh 2x$ from the defn,

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}} = \frac{2}{3}$$

Multiply by e^{x}

$$\therefore \frac{e^{2x}-1}{e^{2x}+1} = \frac{2}{3} \quad \therefore 3e^{2x}-3 = 2e^{2x}+2$$

$$\therefore 3e^{2x}-2e^{2x} = 3+2$$

$$\therefore e^{2x} = 5 \Rightarrow 2x = \log 5 \Rightarrow \boxed{x = \frac{1}{2}\log 5.}$$

$$\therefore \cosh 2x = \frac{e^{2x}-e^{-2x}}{2} = \frac{5-(1/5)}{2}$$

$$= \frac{(25-1)/5}{2} = \frac{\cancel{24}\,12}{\cancel{10}\,5} = \frac{12}{5}$$

$\tanh x = \frac{1}{2}$ Find the value of x & Then $\sinh 2x$

<2> Find the value of $\tanh \log x$ if $x = \sqrt{3}$

So ⇒ $$\tanh \log x = \frac{e^{\log x}-e^{-\log x}}{e^{\log x}+e^{-\log x}} \quad (\text{But } e^{\log x} = x)$$

$$= \frac{x - x^{-1}}{x + x^{-1}} \quad (\times \text{ by } x.)$$

$$= \frac{x^2-1}{x^2+1} \quad \left[\because x = \sqrt{3} \;\therefore x^2 = 3\right]$$

$$= \frac{3-1}{3+1}$$

$$= \frac{2}{4} = \frac{1}{2}.$$

H.W Find the value of $\tanh \log \sqrt{5}$.

Solve the following eq^n for real values of x
$17\cosh x + 18\sinh x = 1$

$$17\left(\frac{e^x+e^{-x}}{2}\right)+18\left(\frac{e^x-e^{-x}}{2}\right)=1$$

$17e^x + 17e^{-x} + 18e^x - 18e^{-x} = 2$
$35e^x - e^{-x} = 2$ (Multiply by e^x)
$\therefore 35e^{2x} - 1 = 2e^x$ $\quad\therefore 35e^{2x} - 2e^x - 1 = 0$
Solving it as a quadratic eq^n in e^x,

$$e^x = \frac{2 \pm \sqrt{4+140}}{70} = \frac{2 \pm \sqrt{144}}{70}$$

$$= \frac{2 \pm 12}{70} = \frac{\cancel{2}(1\pm6)}{\cancel{70}_{35}} = \frac{1\pm6}{35}$$

$$= \frac{7}{35} \text{ or } \frac{-5}{35} = \frac{1}{5} \text{ or } \frac{-1}{7}$$

$$\therefore e^x = \frac{1}{5} \text{ or } -\frac{1}{7}$$

$$x = \log\left(\frac{1}{5}\right) \text{ or } x = \log\left(\frac{-1}{7}\right)$$

$\because$ Logarithm of a negative number is complex.

$$\therefore x = \log\left(\frac{1}{5}\right) = -\log 5.$$

Solve the equation $7\cosh x + 8\sinh x = 1$ for real values of x.

HW
⟨4⟩ If $\sinh x - \cosh x = 5$ Find $\tanh x$.

Sol ⇒
$$\left(\frac{e^x-e^{-x}}{2}\right)-\left(\frac{e^x+e^{-x}}{2}\right)=5.$$

$$\cancel{e^x} - e^{-x} - \cancel{e^x} - e^{-x} = 10$$

$$-2e^{-x} = 10$$

$$e^{-x} = \frac{-10}{2} \Rightarrow e^{-x} = -5$$

If $\tanh x = \frac{1}{2}$ Find $\sinh 2x$ and $\cosh 2x$.

sol => $\sinh 2x = \frac{2\tanh x}{1-\tanh^2 x} = \frac{2(1/2)}{1-(1/4)}$

$= \frac{1}{3/4} = \frac{4}{3}$.

$\cosh 2x = \frac{1+\tanh^2 x}{1-\tanh^2 x}$.

$= \frac{1+(1/4)}{1-(1/4)}$

$= \frac{5/4}{3/4} = \frac{5}{3}$.

<12> $\sin\alpha \cosh\beta = \frac{x}{2}$, $\cos\alpha \sinh\beta = \frac{y}{2}$

Show that.

(i) $\operatorname{cosec}(\alpha - i\beta) + \operatorname{cosec}(\alpha + i\beta) = \frac{4x}{x^2+y^2}$

(ii) $\operatorname{cosec}(\alpha - i\beta) - \operatorname{cosec}(\alpha + i\beta) = \frac{4iy}{x^2+y^2}$.

(i)

Sol => we have $\operatorname{cosec}(\alpha + i\beta) = \frac{1}{\sin(\alpha + i\beta)}$

$= \frac{1}{\sin\alpha \cos i\beta + \cos\alpha \sin i\beta}$

$= \frac{1}{\sin\alpha \cosh\beta + i\cos\alpha \sinh\beta}$

$= \frac{1}{(x/2) + i(y/2)}$

$= \frac{2}{x+iy}$

(i) LHS $= \frac{2}{x-iy} + \frac{2}{x+iy} = \frac{2x + \cancel{2iy} + 2x - \cancel{2iy}}{x^2 - (iy)^2}$

$= \frac{4x}{x^2 - i^2y^2} = \frac{4x}{x^2+y^2}$ $\because (i^2 = -1)$.

6) If $\cosh^{-1} a + \cosh^{-1} b = \cosh^{-1} c$ P.T

$a^2 + b^2 + c^2 = 2abc + 1$.

Sol ⇒ Let $\cosh^{-1} a = \alpha$, $\cosh^{-1} b = \beta$

& $\cosh^{-1} c = \gamma$

∴ we are given,

$\alpha + \beta = \gamma$

∴ $\cosh(\alpha + \beta) = \cosh\gamma$

∴ $\cosh\alpha \cdot \cosh\beta + \sinh\alpha \cdot \sinh\beta = \cosh\gamma$ —— (1)

But,

$a = \cosh\alpha$, $b = \cosh\beta$, $c = \cosh\gamma$

∴ $\sinh\alpha = \sqrt{\cosh^2\alpha - 1} = \sqrt{a^2 - 1}$

$\sinh\beta = \sqrt{\cosh^2\beta - 1} = \sqrt{b^2 - 1}$

$\therefore ab + \sqrt{a^2-1}\sqrt{b^2-1} = c$

$\sqrt{a^2-1}\sqrt{b^2-1} = c - ab$

Taking square on B.S

$\therefore (a^2-1)(b^2-1) = (c-ab)^2$

$\cancel{a^2b^2} - a^2 - b^2 + 1 = c^2 - 2abc + \cancel{a^2b^2}$

$a^2 + b^2 + c^2 = 2abc + 1$

H.W Ex $\Rightarrow$ If $\cosh^{-1} a + \cosh^{-1} b = \cosh^{-1} x$ Then P.T

$a\sqrt{b^2-1} + b\sqrt{a^2-1} = \sqrt{x^2-1}$

(6) Ex $\Rightarrow$ If $\cosh^6 x = a\cosh 6x + b\cosh 4x + c\cosh 2x + d$

P.T $5a - 5b + 3c - 2d = 0$

So $\Rightarrow$ LHS $= \cosh^6 x$

$= \left[\dfrac{e^{x} + e^{-x}}{2}\right]^6$

$= \dfrac{1}{64}\left[e^{6x} + 6e^{5x}\cdot e^{-x} + 15e^{4x}\cdot e^{-2x} + 20e^{3x}\cdot e^{-3x} + 15e^{2x}\cdot e^{-4x} + 6e^{x}\cdot e^{-5x} + e^{-6x}\right]$

$= \dfrac{1}{64}\left[(e^{6x} + e^{-6x}) + 6(e^{4x} + e^{-4x}) + 15(e^{2x} + e^{-2x}) + 20\right]$

$= \dfrac{1}{32}\left[\cosh 6x + 6\cosh 4x + 15\cosh 2x + 10\right]$

$= \dfrac{1}{32}\cosh 6x + \dfrac{6}{32}\cosh 4x + \dfrac{15}{32}\cosh 2x + \dfrac{10}{32}$

$\therefore 5a - 5b + 3c - 2d$

$= 5\times\dfrac{1}{32} - 5\times\dfrac{6}{32} + 3\times\dfrac{15}{32} - 2\times\dfrac{10}{32}$

$= \dfrac{5}{32} - \dfrac{30}{32} + \dfrac{45}{32} - \dfrac{20}{32}$

$= \dfrac{5 - 30 + 45 - 20}{32} = 0$

(14) P.T $(\cosh x + \sinh x)^n = \cosh nx + \sinh nx$

Sol ⇒ LHS $= \left(\frac{e^x+e^{-x}}{2} + \frac{e^x-e^{-x}}{2}\right)^n$

$= \left(\frac{e^x + \cancel{e^{-x}} + e^x - \cancel{e^{-x}}}{2}\right)^n$

$= \left(\frac{\cancel{2}e^x}{\cancel{2}}\right)^n = e^{nx}$

RHS $= \left(\frac{e^{nx}+e^{-nx}}{2}\right) + \left(\frac{e^{nx}-e^{-nx}}{2}\right)$

$= \frac{e^{nx} + \cancel{e^{-nx}} + e^{nx} - \cancel{e^{-nx}}}{2}$

$= \frac{\cancel{2}e^{nx}}{\cancel{2}} = e^{nx}$

∴ LHS = RHS

(1) $\left(\frac{1+\tanh x}{1-\tanh x}\right)^n = \cosh 2nx + \sinh 2nx$

(16) P.T $\frac{1}{1-\frac{1}{1-\frac{1}{1+\sinh^2 x}}} = -\sinh^2 x$

LHS ⇒ $\frac{1}{1-\frac{1}{1-\frac{1}{\cosh^2 x}}} = \frac{1}{1-\frac{1}{1-\text{sech}^2 x}} = \frac{1}{1-\frac{1}{\tanh^2 x}}$

$= \frac{1}{1-\frac{\cosh^2 x}{\sinh^2 x}}$

$= \frac{\sinh^2 x}{\sinh^2 x - \cosh^2 x}$

$= \frac{\sinh^2 x}{-(\cosh^2 x - \sinh^2 x)}$

$= -\sinh^2 x$

OR

$= \frac{1}{1-\frac{1}{\tanh^2 x}}$

$= \frac{1}{1-\coth^2 x}$

$= \frac{1}{-\text{cosech}^2 x}$

$= -\sinh^2 x.$

If $u = \log_e \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$ P.T

(i) $\cosh u = \sec\theta$

(ii) $\sinh u = \tan\theta$

(iii) $\tanh u = \sin\theta$

(iv) $\tanh\frac{u}{2} = \tan\frac{\theta}{2}$

Sol ⇒ We have $u = \log \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$

$$\therefore e^u = \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$$

$$= \frac{\tan\frac{\pi}{4} + \tan\frac{\theta}{2}}{1 - \tan\frac{\pi}{4}\tan\frac{\theta}{2}}$$

$$= \frac{1 + \tan\frac{\theta}{2}}{1 - \tan\frac{\theta}{2}}$$

$$= \frac{\cos\frac{\theta}{2} + \sin\frac{\theta}{2}}{\cos\frac{\theta}{2} - \sin\frac{\theta}{2}} \times \frac{\cos\frac{\theta}{2} + \sin\frac{\theta}{2}}{\cos\frac{\theta}{2} + \sin\frac{\theta}{2}}$$

$$= \frac{\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right)^2}{\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}}$$

$$= \frac{\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right)^2}{\cos\theta}$$

$$= \frac{\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cos\theta}$$

$$= \frac{1 + \sin\theta}{\cos\theta}$$

$$e^u = \sec\theta + \tan\theta. \quad \therefore e^{-u} = \sec\theta - \tan\theta.$$

(i) $\cosh u = \frac{e^u + e^{-u}}{2}$

$$= \frac{1}{2}\left[\sec\theta + \cancel{\tan\theta} + \sec\theta - \cancel{\tan\theta}\right]$$

$$= \sec\theta$$

(ii) $\sinh u$ -

(iii) $\tanh u = \frac{\sinh u}{\cosh u} = \frac{\tan\theta}{\sec\theta} = \sin\theta.$

$$\tanh\left(\frac{u}{2}\right) = \frac{\sinh(u/2)}{\cosh(u/2)}$$

$$= \frac{2\sinh(u/2)\cdot\cosh(u/2)}{2\cosh(u/2)\cdot\cosh(u/2)}$$

$$= \frac{\sinh u}{1+\cosh u}$$

$$= \frac{\tan\theta}{1+\sec\theta} \qquad \left[\text{By } \text{(I)} \;\&\; \text{(II)}\right]$$

$$= \frac{\sin\theta/\cos\theta}{1+1/\cos\theta}$$

$$= \frac{\sin\theta/\cos\theta}{(\cos\theta+1)/\cos\theta}$$

$$= \frac{2\sin(\theta/2)\cos(\theta/2)}{2\cos^2(\theta/2)}$$

$$= \frac{\sin\theta/2}{\cos\theta/2}$$

$$= \tan\theta/2$$

If $\log \tan x = y$, prove that

H.W. $\sinh ny = \frac{1}{2}(\tan^n x - \cos^n x)$

$\cosh(n+1)y + \cosh(n-1)y = \cosh ny \operatorname{cosec} 2x$

Separation of Real and Imaginary parts

① $\sin(x+iy) = \sin x \cosh y + i \cos x \sinh y$.

② $\cos(x+iy) = \cos x \cosh y - i \sin x \sinh y$.

③ $\tan(\alpha + i\beta) = \dfrac{\sin(\alpha+i\beta)}{\cos(\alpha+i\beta)}$

$$= \frac{\sin 2\alpha + i \sinh 2\beta}{\cos 2\alpha + \cosh 2\beta}$$

④ $\sinh(x+iy) = \sinh x \cos y + i \cosh x \sin y$.

& $\cosh(x+iy) = \cosh x \cos y + i \sinh x \sin y$.

⑤ $\tanh(x+iy) = \dfrac{\sinh(x+iy)}{\cosh(x+iy)}$

$$= \frac{\sinh(2x) + i\sin(2y)}{\cosh(2x) + \cos(2y)}$$

If $\sin(\theta+i\phi)=\cos\alpha+i\sin\alpha$ (or $e^{i\alpha}$) P.T

$\cos^4\theta=\sin^2\alpha=\sinh^4\phi$

Sol ⇒ Given :- $\sin(\theta+i\phi)=\cos\alpha+i\sin\alpha$

$\therefore \sin\theta\cosh\phi+i\cos\theta\sinh\phi=\cos\alpha+i\sin\alpha$

∴ Comparing & Equating real & imaginary terms,

$\therefore \sin\theta\cosh\phi=\cos\alpha$ —— (I)

$\cos\theta\sinh\phi=\sin\alpha$ —— (II)

But $\cosh^2\phi-\sinh^2\phi=1$

$\therefore \left(\frac{\cos\alpha}{\sin\theta}\right)^2-\left(\frac{\sin\alpha}{\cos\theta}\right)^2=1$

$\therefore \cos^2\alpha\cos^2\theta-\sin^2\alpha\sin^2\theta=\sin^2\theta\cos^2\theta$

$(1-\sin^2\alpha)\cos^2\theta-\sin^2\alpha(1-\cos^2\theta)=(1-\cos^2\theta)\cos^2\theta$

$\cancel{\cos^2\theta}-\cancel{\sin^2\alpha\cos^2\theta}-\sin^2\alpha+\cancel{\sin^2\alpha\cos^2\theta}=\cancel{\cos^2\theta}-\cos^4\theta$

$\cancel{-}\sin^2\alpha=\cancel{-}\cos^4\theta$

$$\boxed{\sin^2\alpha=\cos^4\theta} \quad \text{—— (III)}$$

Again, $\sin^2\theta+\cos^2\theta=1$

$\left(\frac{\cos\alpha}{\cosh\phi}\right)^2+\left(\frac{\sin\alpha}{\sinh\phi}\right)^2=1$

$\cos^2\alpha\sinh^2\phi+\sin^2\alpha\cosh^2\phi=\cosh^2\phi\cdot\sinh^2\phi$

$\therefore (1-\sin^2\alpha)\sinh^2\phi+\sin^2\alpha(1+\sinh^2\phi)=(1+\sinh^2\phi)\sinh^2\phi$

$\therefore \cancel{\sinh^2\phi}-\cancel{\sin^2\alpha\sinh^2\phi}+\sin^2\alpha+\cancel{\sin^2\alpha\sinh^2\phi}=\cancel{\sinh^2\phi}+\sinh^4\phi$

$$\boxed{\therefore \sin^2\alpha=\sinh^4\phi} \quad \text{—— (IV)}$$

∴ From (III) & (IV)

$$\boxed{\cos^4\theta=\sin^2\alpha=\sinh^4\phi}$$

If $\tan(x+iy) = \sin(u+iv)$ Then P.T

$$\frac{\sin 2x}{\sinh 2y} = \frac{\tan u}{\tanh v}$$

Sol ⇒ Given :- $\tan(x+iy) = \sin(u+iv)$

$\therefore \tan(x-iy) = \sin(u-iv)$

Let $A = x+iy$, $B = (x-iy)$, $C = (u+iv)$, $D = (u-iv)$

$\therefore \tan A = \sin C$ & $\tan B = \sin D$ —— (1)

$$\therefore \frac{\tan A + \tan B}{\tan A - \tan B} = \frac{\sin C + \sin D}{\sin C - \sin D}$$

$$\therefore \frac{\left(\frac{\sin A}{\cos A} + \frac{\sin B}{\cos B}\right)}{\left(\frac{\sin A}{\cos A} - \frac{\sin B}{\cos B}\right)} = \frac{\left(2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)\right)}{\left(2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)\right)}$$

$$\therefore \frac{\sin A\cos B + \cos A\sin B}{\sin A\cos B - \cos A\sin B} = \frac{\tan\left(\frac{C+D}{2}\right)}{\tan\left(\frac{C-D}{2}\right)}$$

$$\therefore \frac{\sin(A+B)}{\sin(A-B)} = \frac{\tan\left(\frac{C+D}{2}\right)}{\tan\left(\frac{C-D}{2}\right)}$$

Substituting A, B, C, D values,

$$\therefore \frac{\sin(x+\cancel{iy}+x-\cancel{iy})}{\sin(\cancel{x}+iy-\cancel{x}+iy)} = \frac{\tan\left(\frac{u+\cancel{iv}+u-\cancel{iv}}{2}\right)}{\tan\left(\frac{\cancel{u}+iv-\cancel{u}+iv}{2}\right)}$$

$$\therefore \frac{\sin 2x}{\sin 2iy} = \frac{\tan(2u/\cancel{2})}{\tan(2iv/\cancel{2})}$$

$$\frac{\sin 2x}{\cancel{i}\sinh 2y} = \frac{\tan u}{\cancel{i}\tanh v}$$

$$\boxed{\therefore \frac{\sin 2x}{\sinh 2y} = \frac{\tan u}{\tanh v}}$$

H.W If $\cos(u+iv) = x+iy$ P.T $(1+x^2)+y^2 = (\cosh v + \cos u)^2$

& $(1-x)^2+y^2 = (\cosh v - \cos u)^2$

Hint:- $\cos(u+iv) = x+iy \Rightarrow \cos u\cosh v - i\sin u\sinh v = x+iy$

$\therefore x = \cos u\cosh v$ & $y = -\sin u\sinh v$

Seperate into real & imaginary parts,
① $\tan^{-1}(\alpha+i\beta)$ ② $\tan^{-1}(e^{i\theta})$

Sol ⇒ Let $\tan^{-1}(\alpha+i\beta) = x+iy \Rightarrow \alpha+i\beta = \tan(x+iy)$

$\tan^{-1}(\alpha-i\beta) = x-iy \Rightarrow \alpha-i\beta = \tan(x-iy)$

Consider. $\tan[(x+\cancel{iy})+(x-\cancel{iy})] = \dfrac{\tan(x+iy)+\tan(x-iy)}{1-\tan(x+iy)\cdot\tan(x-iy)}$

$$\therefore \tan(2x) = \frac{(\alpha+\cancel{i\beta})+(\alpha-\cancel{i\beta})}{1-(\alpha+i\beta)(\alpha-i\beta)}$$

$$\therefore \tan 2x = \frac{2\alpha}{1-(\alpha^2-i^2\beta^2)}$$

$$\therefore \tan 2x = \frac{2\alpha}{1-\alpha^2-\beta^2}$$

$$\therefore 2x = \tan^{-1}\left(\frac{2\alpha}{1-\alpha^2-\beta^2}\right)$$

$$\boxed{\therefore x = \frac{1}{2}\tan^{-1}\left(\frac{2\alpha}{1-\alpha^2-\beta^2}\right)}$$

$$\therefore \tan[(\cancel{x}+iy)-\cancel{x}+iy)] = \frac{\tan(x+iy)-\tan(x-iy)}{1+\tan(x+iy)\cdot\tan(x-iy)}$$

$$\tan 2iy = \frac{(\alpha+i\beta)-(\alpha-i\beta)}{1+(\alpha+i\beta)(\alpha-i\beta)}$$

$$i\tanh 2y = \frac{\cancel{\alpha}+i\beta-\cancel{\alpha}+i\beta}{1+(\alpha^2-i^2\beta^2)}$$

$$\cancel{i}\tanh 2y = \frac{2\cancel{i}\beta}{1+\alpha^2-\beta^2}$$

$$\therefore \tanh 2y = \frac{2\beta}{1+\alpha^2-\beta^2} \qquad \therefore 2y = \tanh^{-1}\left(\frac{2\beta}{1+\alpha^2-\beta^2}\right)$$

$$\boxed{\therefore y = \frac{1}{2}\tanh^{-1}\left(\frac{2\beta}{1+\alpha^2-\beta^2}\right)}$$

#

If $\tan(\alpha+i\beta) = x+iy$ P.T

$x^2+y^2+2x\cot 2\alpha = 1$, $x^2+y^2-2y\coth 2\beta+1=0$

Sol $\Rightarrow$ Given:- $\tan(\alpha+i\beta) = x+iy$

$$\tan(\alpha-i\beta) = x-iy$$

$$\tan((\alpha+\cancel{i\beta})+(\alpha-\cancel{i\beta})) = \frac{\tan(\alpha+i\beta)+\tan(\alpha-i\beta)}{1-\tan(\alpha+i\beta)\tan(\alpha-i\beta)}$$

$$\tan 2\alpha = \frac{x+\cancel{iy}+x-\cancel{iy}}{1-(x+iy)(x-iy)}$$

$$\tan 2\alpha = \frac{2x}{1-(x^2+y^2)}$$

$$\tan 2\alpha = \frac{2x}{1-x^2-y^2}$$

$$1-x^2-y^2 = 2x\cot 2\alpha$$

$$\therefore 1 = x^2+y^2+2x\cot 2\alpha$$

$$\therefore \boxed{x^2+y^2+2x\cot 2\alpha = 1}$$

$$\tan((\cancel{\alpha}+i\beta)-\cancel{\alpha}+i\beta) = \frac{\tan(\alpha+i\beta)-\tan(\alpha-i\beta)}{1+\tan(\alpha+i\beta)\tan(\alpha-i\beta)}$$

$$\tan 2i\beta = \frac{(x+iy)-(x-iy)}{1+(x+iy)(x-iy)}$$

$$i\tanh 2\beta = \frac{\cancel{x}+iy-\cancel{x}+iy}{1+x^2+y^2}$$

$$\cancel{i}\tanh 2\beta = \frac{2\cancel{i}y}{1+x^2+y^2}$$

$$1+x^2+y^2 = 2y\coth 2\beta$$

$$\boxed{\therefore x^2+y^2-2y\coth 2\beta+1=0}$$

Ex ⇒ If $x+iy = 2\cosh\left(\alpha + \frac{i\pi}{4}\right)$ P.T $x^2 - y^2 = 2$.

Sol ⇒ ∴ $x+iy = 2\cosh(\alpha + i\pi/4)$

$= 2[\cosh\alpha \cos i\pi/4 + \sinh\alpha \sin i\pi/4]$

$= 2[\cosh\alpha \cosh i\pi/4 + i\sinh\alpha \sin h i\pi/4]$

$= 2\left[\cosh\alpha \cos\frac{\pi}{4} + i\sinh\alpha \sin\frac{\pi}{4}\right]$

$= 2\left[\cosh\alpha \times \frac{1}{\sqrt{2}} + i\sinh\alpha \times \frac{1}{\sqrt{2}}\right]$

$= \sqrt{2}[\cosh\alpha + i\sinh\alpha]$

$x = \sqrt{2}\cosh\alpha$, $y = \sqrt{2}\sinh\alpha$

$x^2 - y^2 = 2(\cosh^2\alpha - \sinh^2\alpha) = 2$

$\boxed{x^2 - y^2 = 2.}$

Inverse Hyperbolic Functions ⇒

If $\sinh u = z$ then u is called inverse hyperbolic sine of z & is denoted by $u = \sinh^{-1} z$ ||ly, $\cosh^{-1} z$, $\tanh^{-1} z$,

* If z is real,

(1) $\sinh^{-1} z = \log(z + \sqrt{z^2+1})$

(2) $\cosh^{-1} z = \log(z + \sqrt{z^2-1})$

(3) $\tanh^{-1} z = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$

Integration Formula ⇒

(1) $\int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}\left(\frac{x}{a}\right)$

(2) $\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\left(\frac{x}{a}\right)$

(3) $\int \frac{dx}{a^2-x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$.

Ex

Prove that (i) $\cosh^{-1}(\sqrt{1+x^2}) = \sinh^{-1}x$

(ii) $\cosh^{-1}(\sqrt{1+x^2}) = \tanh^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$

Sol ⇒ (i) let $\cosh^{-1}\sqrt{1+x^2} = y$

$\sqrt{1+x^2} = \cosh y$ —— (1)

$1+x^2 = \cosh^2 y$

$x^2 = \cosh^2 y - 1$

$x^2 = \sinh^2 y$

$\therefore x = \sinh y$ —— (ii)

$y = \sinh^{-1} x$

$\therefore \cosh^{-1}\sqrt{1+x^2} = \sinh^{-1}x$

(ii) Hint: $\frac{x}{\sqrt{1+x^2}} = \sinh y \cosh^{-1} y$

$\frac{x}{\sqrt{1+x^2}} = \frac{\sinh y}{\cosh y}$

$\frac{x}{\sqrt{1+x^2}} = \tanh y$

$y = \tanh^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$

$\therefore \cosh^{-1}\sqrt{1+x^2} = \tanh^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$

Ex Prove that (i) $\tanh^{-1}\cos\theta = \cosh^{-1}\operatorname{cosec}\theta$

Sol ⇒ $\because \tanh^{-1}x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$

$\therefore \tanh^{-1}(\cos\theta) = \frac{1}{2}\log\left(\frac{1+\cos\theta}{1-\cos\theta}\right)$ —— (1)

$\because \cosh^{-1}x = \log\left(x+\sqrt{x^2-1}\right)$

$\cosh^{-1}(\operatorname{cosec}\theta) = \log\left(\operatorname{cosec}\theta + \sqrt{\operatorname{cosec}^2\theta - 1}\right)$

$= \log(\operatorname{cosec}\theta + \cot\theta)$

$= \log\left(\frac{1}{\sin\theta} + \cos\theta\right)$

$= \frac{1}{2}\log\left(\frac{(1+\cos\theta)^2}{\sin^2\theta}\right)$

$= \frac{1}{2}\log\left(\frac{(1+\cos\theta)^2}{1-\cos^2\theta}\right)$ | $= \frac{1}{2}\log\left(\frac{1+\cos\theta}{1-\cos\theta}\right)$ —— (II)

$\therefore$ From (I) & (II)

RHS = LHS.

Seperate into real & imaginary,

1) $\sin^{-1}(e^{i\theta})$

Sol => Let $\sin^{-1}(e^{i\theta}) = x+iy$

$\cos\theta + i\sin\theta = \sin(x+iy)$

$\cos\theta + i\sin\theta = \sin x \cosh y + i \cos x \sinh y$

Equating real & img.

$\therefore \cos\theta = \sin x \cosh y$ — (I)

$\sin\theta = \cos x \sinh y$ — (II)

$\because \sin^2\theta + \cos^2\theta = 1$

$\therefore \cos^2 x \sinh^2 y + \sin^2 x \cosh^2 y = 1$

$\cos^2 x \sinh^2 y + \sin^2 x(1+\sinh^2 y) = 1$

$\cos^2 x \sinh^2 y + \sin^2 x + \sin^2 x \sinh^2 y = 1$

$\sinh^2 y(\sin^2 x + \cos^2 x) + \sin^2 x = 1$

$\sinh^2 y + \sin^2 x = 1$

$\sinh^2 y = 1 - \sin^2 x$

$\sinh^2 y = \cos^2 x.$

$\sinh y = \cos x.$ — (III)

put (III) in (II)

$\therefore \sin\theta = \cos x \cdot \cos x$

$= \cos^2 x$

$\therefore \cos x = \sqrt{\sin\theta}$

$x = \cos^{-1}(\sqrt{\sin\theta})$

Again put (III) in (II)

$\therefore \sin\theta = \sinh y \cdot \sinh y$

$= \sinh^2 y$

$\therefore \sinh y = \sqrt{\sin\theta}$

$y = \sinh^{-1}\sqrt{\sin\theta}$ $\quad$ [$\because$ As $\sinh^{-1} z = \log(z+\sqrt{z^2+1})$]

$= \log[\sqrt{\sin\theta} + \sqrt{\sin\theta+1}]$

Ex ⇒ Prove that $\cos^{-1} ix = \frac{\pi}{2} - i \log\left(x + \sqrt{x^2+1}\right)$

Sol ⇒ Let $\cos^{-1} ix = \alpha + i\beta$

$ix = \cos(\alpha + i\beta)$

$ix = \cos\alpha \cosh\beta - i \sin\alpha \sinh\beta.$

$\therefore \cos\alpha \cosh\beta = 0 \Rightarrow \cos\alpha = 0 \Rightarrow \alpha = \cos^{-1}(0)$

$\alpha = \pi/2.$

$\therefore x = -\sin\alpha \sinh\beta$

$\therefore x = -\sin(\pi/2) \sinh\beta$

$x = -\sinh\beta$

$\therefore \beta = \sinh^{-1} -x$

$\beta = \log\left(-x + \sqrt{x^2+1}\right)$

$= \log\left(\sqrt{x^2+1} - x \times \frac{\sqrt{x^2+1} + x}{\sqrt{x^2+1} + x}\right)$

$= \log\left(\frac{1}{x + \sqrt{x^2+1}}\right)$

$= -\log\left(x + \sqrt{x^2+1}\right)$

$\therefore \cos^{-1} ix = \frac{\pi}{2} - i\log\left(x + \sqrt{x^2+1}\right)$

HW

Ex Show that $\sin^{-1}(ix) = 2n\pi + i\log\left(x + \sqrt{1+x^2}\right)$

Seperate into real & imaginary parts

$\cos^{-1}\left(\frac{3i}{4}\right)$

Sol ⇒ Let $\cos^{-1}\left(\frac{3i}{4}\right) = x + iy$

$\frac{3i}{4} = \cos(x + iy)$

$$\frac{3i}{4} = \cos x \cosh y - i \sin x \sinh y$$

$\therefore \cos x \cosh y = 0 \quad \therefore \cos x = 0$

$\therefore x = \cos^{-1}(0)$

$\boxed{x = \frac{\pi}{2}}$ $\quad (\because \cosh y \neq 0)$

And, $-\sin x \sinh y = \frac{3}{4}$

But $\sin x = \sin\frac{\pi}{2} = 1$

$\therefore -\sinh y = \frac{3}{4}$

$\sinh y = -\frac{3}{4}$

$y = \log\left(-\frac{3}{4} + \sqrt{1+\frac{9}{16}}\right)$ $\quad \left(\because \sinh^{-1}(z) = \log\left(z + \sqrt{z^2+1}\right)\right)$

$= \log\left(-\frac{3}{4} + \frac{5}{4}\right)$

$= \log\left(\frac{1}{2}\right)$

$y = -\log 2$

$\therefore x = \frac{\pi}{2}$ & $y = -\log 2$

$\therefore \cos^{-1}\left(\frac{3i}{4}\right) = \frac{\pi}{2} - i\log 2$ //