Contents

CHAPTER ONE

The Limit

The limit is one of the most crucial topics in calculus, because it helps us define the derivative as well as calculate them. The limit is usually applied to a variable, where that variable approaches some number infinitely close.

There is generally one definition of the limit, the *Epsilon Delta Definition of a Limit*, but that is too complicated for our purposes.[1] Our definition is much simpler.

[1] The Epsilon Delta Definition of a Limit appears in year 1 to 2 of university calculus.

> **Definition 1.1** (Definition of a Limit)**.** The limit for a variable x to some value a is denoted by
> $$x \to a$$
> Or
> $$\lim_{x \to a}$$
> Where the lim stands for "limit". Both of these are pronounced "as x approaches a", and that implies "as x becomes infinitely close to a".

What does it mean for x to become "infinitely close to a"? Let's consider for $a = 2$. When we say $x \to 2$, this implies some number infinitely close to 2. This can be 1.0000....0001 or 0.99999....9; we can clearly see that both of these numbers are "infinitely" close to 2 (see Figure 1.1).

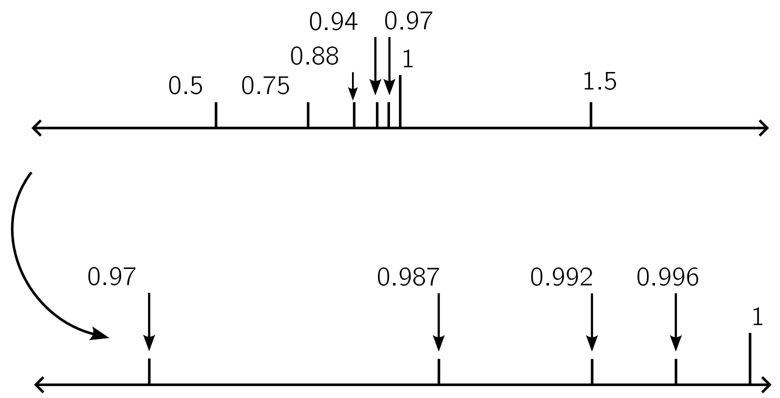

Figure 1.1: We keep on dividing the parts between $\frac{1}{2}$ and 1 by half until infinity. It will never reach 1, but it becomes infinitely close to 1 as we keep on repeating this process.

The most important thing to remember for a limit is that it approaches the value, but is never that value. This becomes important when we are determing the derivative of functions using the First Principle.

It should be noted that the value for the limit of a variable is independent to other variables. For example, if we had $\lim_{x\to 0}(x+y)$, the y is independent of the x, so we can really write $y + \lim_{x\to 0} x$.[2]

[2] The limit rule that we just used is called the sum property for limits, where y is a constant that isn't dependent of x.

1.1 Sum Property

Definition 1.2. For functions $f(x)$ and $g(x)$, the limit of their sum is the sum of their limits

$$\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$

And if there is a constant C

$$\lim_{x\to a}[f(x)+C] = C + \lim_{x\to a} f(x)$$

And we will not dive into the proof for this property.

1.2 Difference Property

Definition 1.3. For functions $f(x)$ and $g(x)$, the limit of their difference is the difference of their limits

$$\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$$

And if there is a constant C

$$\lim_{x\to a}[f(x)-C] = -C + \lim_{x\to a} f(x)$$

Or

$$\lim_{x\to a}[C-f(x)] = C - \lim_{x\to a} f(x)$$

Proof. We will prove this using Definition 1.1

$$\begin{aligned}\lim_{x\to a}[f(x)-g(x)] &= \lim_{x\to a} f(x) + (\lim_{x\to a} -[g(x)]) \\ &= \lim_{x\to a} f(x) - \lim_{x\to a} g(x)\end{aligned}$$

Where it because the negative is independent of the limit (it is a constant multiplier), we can drag it outside of the limit.[3] □

[3] These proofs aren't "mathematical", but these properties are pretty intuitive so this will suffice. You can search up the actual proofs online. They are pretty weird though.

1.3 Product Property

Definition 1.4. For functions $f(x)$ and $g(x)$, the limit of their product is the product of their limits

$$\lim_{x\to a} f(x)g(x) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$$

The proof for this is too obscure, so we won't include that here. Just remember this property.

1.4 Quotient Property

Definition 1.5. For functions $f(x)$ and $g(x)$, the limit of their quotient is the quotient of their limits

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$$

Proof. Using Definition 1.3

$$\begin{aligned}
\lim_{x\to a} \frac{f(x)}{g(x)} &= \lim_{x\to a} f(x)[g(x)]^{-1} \\
&= \lim_{x\to a} f(x) \lim_{x\to a}[g(x)]^{-1} \\
&= \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}
\end{aligned}$$

□

1.5 Using Limit Properties

Now that we have established the properties of limits, we can use them in some examples.

Example 1.6. Evaluate $\lim\limits_{h\to 0}(x+h)$. In this case, we see that the limit is evaluating $h \to 0$. Therefore, x is completely independent in this limit. Hence, we write

$$\begin{aligned}
\lim_{h\to 0}(x+h) &= x + \lim_{h\to 0} h \\
&= x
\end{aligned}$$

Since as $h \to 0$, we approximate it to be 0, leaving us with just x.

Example 1.7. Evaluate $\lim\limits_{x\to 2} \dfrac{x^2}{x+1}$. We can actually just plug in $x = 2$ into the numerator and denominator

$$\begin{aligned}
\lim_{x\to 2} \frac{x^2}{x+1} &= \frac{(2)^2}{(2)+1} \\
&= \frac{4}{3}
\end{aligned}$$

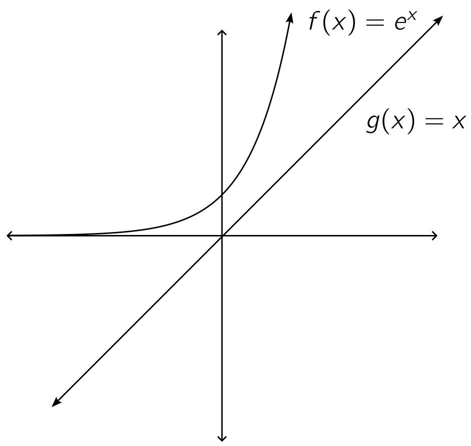

Figure 1.2: $f(x) = e^x$ clearly outgrows $g(x) = x$ for larger values of x.

Example 1.8. Evaluate $\lim_{x \to \infty} \frac{e^x}{x}$. Now to evaluate this, we need to consider which one rises faster, or the same: e^x or x? The reason for this is because if the denominator becomes really big, then this just shrinks to 0, whereas if the numerator is really big, then this grows to infinity. We see that for large values of x, e^x clearly outgrows x, since it is an exponential function, whereas x is a linear function. So for example, if x was say 99, we will get something like

$$\begin{aligned}\lim_{x \to 99} \frac{e^x}{x} &= \frac{e^{99}}{99} \\ &= \text{very large number}\end{aligned}$$

And the result would just evaluate larger and larger for larger values of x. We can also see in Figure 1.2 that e^x undoubtedly clearly outgrows x, so the numertor outgrows the denominator, hence the value shrinks to 0 for $x \to \infty$.

Example 1.9. Evaluate $\lim_{x \to 1} \frac{x}{x-1}$. We can actually just see that letting $x \to 1$ makes the denominator approach 0, and so therefore the fraction approaches ∞

$$\begin{aligned}\lim_{x \to 1} \frac{x}{x-1} &\to \frac{1}{1-1} \\ &\to \frac{1}{0} \\ &\to \infty\end{aligned}$$

WHere we write it approaches instead of equals, because we technically cannot have 0 in the denominator. For a more general proof, we will first add $1 - 1$ so that it is 0 in the numerator

$$\lim_{x \to 1} \frac{x}{x-1} = \lim_{x \to 1} \frac{x-1+1}{x-1}$$

Then using the limit of sums property

$$\begin{aligned}\lim_{x \to 1} \frac{x-1+1}{x-1} &= \lim_{x \to 1} \frac{x-1}{x-1} + \lim_{x \to 1} \frac{1}{x-1} \\ &= \lim_{x \to 1}(1) + \lim_{x \to 1} \frac{1}{x-1} \\ &= 1 + \infty \\ &= \infty\end{aligned}$$

Example 1.10. Evaluate $\lim_{x \to -3} \frac{x^3 + 4x^2 + 2x - 3}{(x-2)(x+3)}$. The trick for this problem (and similar problems) is that we can immediately tell that if this limit evaluates to something, then $(x+3)$ must be factor for the numerator as well.[4] And we can immediately see that's the case, since -3 is a solution to the numerator, therefore $(x+3)$ must be a factor.

[4] The reason this must be true is because we see that $x \to -3$ makes the factor $(x+3) = 0$, and so therefore if we want this limit to not be some value divided by 0, then the numerator must have another $(x+3)$ to cancel out with the $(x+3)$ in the denominator.

So if we use synthetic division for the numerator

$$\begin{aligned}\lim_{x\to -3}\frac{x^3+4x^2+2x-3}{(x-2)(x+3)} &= \lim_{x\to -3}\frac{(x+3)(x^2+x-1)}{(x-2)(x+3)}\\ &= \lim_{x\to -3}\frac{x^2+x-1}{x-2}\\ &= \frac{(-3)^2+(-3)-1}{(-3)-2}\\ &= \frac{9-4}{-5}\\ &= -1\end{aligned}$$

Example 1.11. Evaluate $\lim_{x\to 1}\left[\frac{1}{1-x}-\frac{3}{1-x^3}\right]$. To evaluate this, we will first combine fractions. How do we combine fractions? We will first make use of the difference of cubes identity:

$$\begin{aligned}a^3-b^3 &= (a-b)(a^2+b^2+ab)\\ 1-x^3 &= (1-x)(x^2+x+1)\end{aligned}$$

Substituting

$$\begin{aligned}\lim_{x\to 1}\left[\frac{1}{1-x}-\frac{3}{1-x^3}\right] &= \lim_{x\to 1}\left[\frac{1}{1-x}-\frac{3}{(1-x)(x^2+x+1)}\right]\\ &= \lim_{x\to 1}\frac{x^2+x-2}{(1-x)(x^2+x+1)}\end{aligned}$$

Now we still see that we get an undefined value, namely $\frac{0}{0}$. What we can do is we can try and factor the numerator so that the $(1-x)$ term cancels out with the denominator. Luckily, $(x-1)$, which is the negative of the $(1-x)$ is a factor of the numerator

$$\begin{aligned}\lim_{x\to 1}\frac{x^2+x-2}{(1-x)(x^2+x+1)} &= \lim_{x\to 1}\frac{(x-1)(x+2)}{(1-x)(x^2+x+1)}\\ &= \lim_{x\to 1}\frac{-\cancel{(1-x)}(x+2)}{\cancel{(1-x)}(x^2+x+1)}\\ &= -\lim_{x\to 1}\frac{x+2}{x^2+x+1}\end{aligned}$$

And now we can substitute in $x=1$ because it won't give us an undefined answer

$$\begin{aligned}-\lim_{x\to 1}\frac{x+2}{x^2+x+1} &= -\frac{1+2}{1^2+1+1}\\ &= -1\end{aligned}$$

Example 1.12. Evaluate $\lim_{x\to 4} \dfrac{\sqrt{x+5}-3}{x-4}$. We see that if we plug in $x=4$ we get $\frac{0}{0}$. This is an issue, because this is undefined. However, we have a trick: if we multiply the fraction by $\dfrac{\sqrt{x+5}+3}{\sqrt{x+5}+3}$, the radical in the numerator dissapears

$$\begin{aligned}
\lim_{x\to 4} \frac{\sqrt{x+5}-3}{x-4} &= \lim_{x\to 4} \frac{\sqrt{x+5}-3}{x-4} \frac{\sqrt{x+5}+3}{\sqrt{x+5}+3} \\
&= \lim_{x\to 4} \frac{x-4}{(x-4)(\sqrt{x+5}+3)} \\
&= \lim_{x\to 4} \frac{1}{\sqrt{x+5}+3} \\
&= \frac{1}{6}
\end{aligned}$$

Example 1.13. Evaluate $\lim_{x\to 2} \dfrac{x^2-2x}{\sqrt{x+2}-2}$. We will first factor out an x from the numerator

$$\lim_{x\to 2} \frac{x^2-2x}{\sqrt{x+2}-2} = \lim_{x\to 2} \frac{x(x-2)}{\sqrt{x+2}-2}$$

Then will multiply the fraction by $\dfrac{\sqrt{x+2}+2}{\sqrt{x+2}+2}$

$$\begin{aligned}
\lim_{x\to 2} \frac{x(x-2)}{\sqrt{x+2}-2} &= \lim_{x\to 2} \frac{x(x-2)}{\sqrt{x+2}-2} \frac{\sqrt{x+2}+2}{\sqrt{x+2}+2} \\
&= \lim_{x\to 2} \frac{x(x-2)(\sqrt{x+2}+2)}{x+2-4} \\
&= \lim_{x\to 2} \frac{x\cancel{(x-2)}(\sqrt{x+2}+2)}{\cancel{x-2}} \\
&= 2(\sqrt{2+2}+2) \\
&= 8
\end{aligned}$$

1.5.1 Problems

1. Evaluate $\lim_{x \to 1} \left(\frac{1}{x^2 - 1} - \frac{2}{x^4 - 1} \right)$.

2. Evaluate $\lim_{x \to 2} \frac{x^2 - 2x}{\sqrt{2x} - 2}$.

1.5.2 Solutions

1. We know that $x^4 - 1 = (x^2-1)(x^2+1)$ and then we will combine fractions

$$\begin{aligned}
\lim_{x\to 1}\left(\frac{1}{x^2-1} - \frac{2}{x^4-1}\right) &= \lim_{x\to 1}\left(\frac{1}{x^2-1} - \frac{2}{(x^2-1)(x^2+1)}\right)\\
&= \lim_{x\to 1}\left(\frac{x^2-1}{(x^2-1)(x^2+1)}\right)\\
&= \lim_{x\to 1}\frac{1}{x^2+1}\\
&= \frac{1}{2}
\end{aligned}$$

2. We multiply the fraction by $\dfrac{\sqrt{2x}+2}{\sqrt{2x}+2}$ so that the radicals dissapear

$$\begin{aligned}
\lim_{x\to 2}\frac{x^2-2x}{\sqrt{2x-2}} &= \lim_{x\to 2}\frac{x^2-2x}{\sqrt{2x-2}}\frac{\sqrt{2x}+2}{\sqrt{2x}+2}\\
&= \lim_{x\to 2}\frac{x^2\sqrt{2x}+2x^2-2x\sqrt{2x}-4x}{2x-4}\\
&= \frac{1}{2}\lim_{x\to 2}\frac{2x(x-2)+x^2\sqrt{2x}-2x\sqrt{2x}}{x-2}\\
&= \frac{1}{2}\lim_{x\to 2}\frac{2x(x-2)+x\sqrt{2x}(x-2)}{x-2}\\
&= \frac{1}{2}\lim_{x\to 2}(2x+x\sqrt{2x})\\
&= \frac{1}{2}(4+4)\\
&= 4
\end{aligned}$$

CHAPTER TWO

The Derivative

The derivative is the most essential thing in all of calculus, and it deals with the idea of rate of change. What is the velocity of a ball at that exact second? At that exact millisecond? At that exact microsecond? What about at that exact instance? The derivative deals with problems like these, and simplifies them down to a matter of mathematical expressions. Another major thing that calculus discovered is the integral. The integral deals with determining the area under certain curves,[5] but that is for later.

[5] What is funny is that even though the derivative is the basic foundation for calculus, the fathers of calculus actually began calculus on its counterpart; the integral. You will learn soon enough that the derivative and integral are really opposites of each other.

Let us begin with the definition of the derivative

> **Definition 2.1** (The First Principle)**.** For a function $f(x)$ its derivative $f'(x)$ is defined by
> $$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$
> As long as the function $f(x)$ is continuous.

Continuous just means that there are no random jumps in the curve, or in other words, the curve is connected. See Figure 2.1.

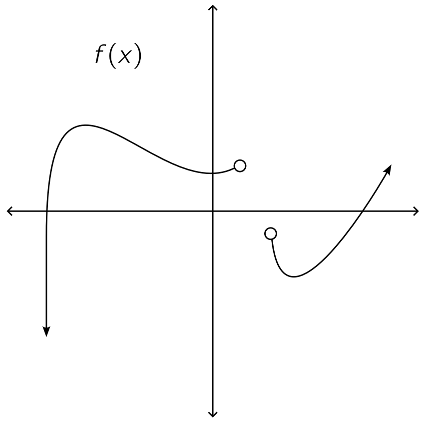

Figure 2.1: A function $f(x)$ that is not continuous because there is a discontintuity.

Derivation. It may not be clear, but $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ actually looks pretty similar to the formula for the slope. The slope formula is

$$m = \frac{y_2 - y_2}{x_2 - x_1}$$

And if we let the two points be $(x,\ f(x))$ and $(x+h,\ f(x+h))$

$$\begin{aligned} m &= \frac{f(x+h)-f(x)}{x+h-x} \\ &= \frac{f(x+h)-f(x)}{h} \end{aligned}$$

Which is what we had earlier. However, what about that $\lim_{h\to 0}$? What does that mean? To gain an understanding of what that means, let us consider the graph of an arbitrary function $f(x)$ and label the two points mentioned on it Then we want $\lim_{h\to 0}$, meaning that we want the distance between the two points as close to each other as humanly possible. If we increment steps as h approaches 0.

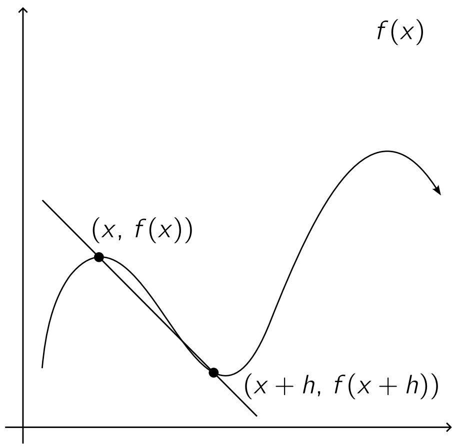

Figure 2.2: Two points $(x,\ f(x))$ and $(x+h,\ f(x+h))$.

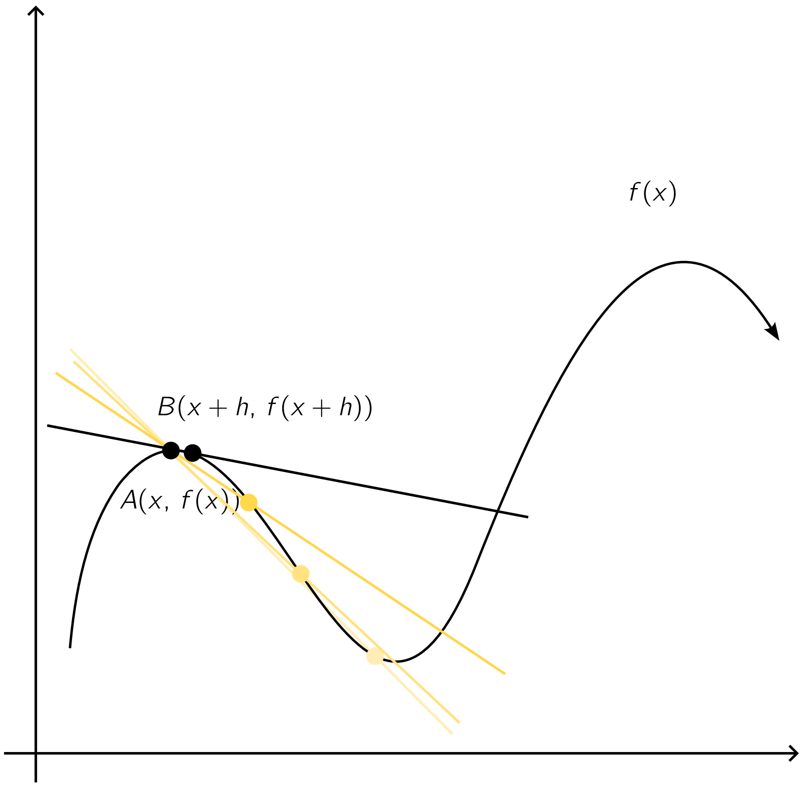

Figure 2.3: The two points $(x, f(x))$ and $(x + h, f(x+h))$ as they get closer and closer $\lim_{h \to 0}$. As you can imagine, it will form a tangent line.

And as you can see from the figure, as $h \to 0$, the point B $\to$ A, and as it B approaches A, it forms a tangent line at that point A.

Therefore, this makes sense that we must take the $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ to get the tangent slope.[6]

[6] Another way to think about this is that we are trying to find the **instantaneous** rate of change. That is, the slope of two points that are infinitely close to each other as possible.

2.1 Derivative Examples

Now that we intuition behind Definition 2.1 we can apply it to some basic functions.

Example 2.2. To determine $f'(x)$ if $f(x) = x^2$, we substitute it into the formula

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^2-x^2}{h} \\
&= \lim_{h\to 0} \frac{x^2+2xh+h^2-x^2}{h} \\
&= \lim_{h\to 0} \frac{h(2x+h)}{h} \\
&= \lim_{h\to 0} 2x+h \\
&= 2x
\end{aligned}
$$

$\therefore$ the derivative of x^2 is $2x$.[7]

[7] Just as a reminder, $f'(x)$ means that it is the **slope** of the tangent line at a certain point; it isn't the equation of the tangent line at that point.

Example 2.3. For $f(x) = (x+3)^3$, determine $f'(x)$. We will use the formula for $f'(x)$

2.2 The Second Derivative

The first derivative is the

2.3 Remark on Notation

There are several ways to denote a derivative. The first way is the Newton notation, where we denote the orderWe call the level of the derivative (first, second, third, etc) by "order". So for example, if I were to say a third ordered derivative, we would be talking about the third derivative. of the derivative

CHAPTER THREE

DNE - Does Not Exist

Some functions do not have a derivative at a certain point. The reason for this in most cases as we will see is because the slope f' approaches ∞, $-\infty$, or simply doesn't exist.

Some common functions include $x^{\frac{2}{3}}$ at $x = 0$, $x^{\frac{1}{3}}$ at $x = 0$, $|x|$ at $x = 0$, and $\frac{1}{x}$ at $x = 0$.

Definition 3.1 (DNE)**.** The derivative of $f(x)$ at $x = a$ is considered DNE if

$$\lim_{x \to a^+} f'(x) \neq \lim_{x \to a^-} f'(x)$$

This definition is actually expanded upon in first to second year university mathematics [8] by the *Epsilon Delta Definition of a Limit*. Regardless, it is pretty intuitive that Definition 3.1 is true.

[8] We will refer to first year university mathematics at U1 mathematics. This applies to any year as well (example year 2 university mathematics is U2 mathematics).

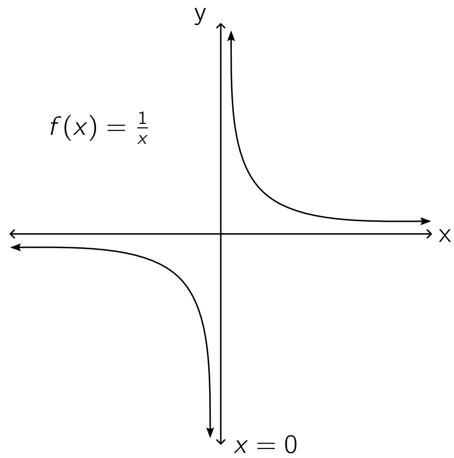

Figure 3.1: Graph of $f(x) = \frac{1}{x}$. There is a V.A at $x = 0$.

Example 3.2. Consider the derivative of $f(x) = \frac{1}{x}$ at $x = 0$ (see Figure 3.1). We can immediately see that the value of $f'(x)$ at $x = a$ is unclear. To prove this, differentiate to get $f'(x) = -\frac{1}{x^2}$

$$\lim_{x \to 0^+} = -\infty$$
$$\lim_{x \to 0^-} = \infty$$

Which we can tell from the graph as well. Therefore, we see that we cannot reach a consenus.[9]

[9] It should be noted that this was a bad example, but one that first comes to mind. The reason for this is because even if there weren't two values for $\lim_{x \to 0^+} f'(x)$ and $\lim_{x \to 0^-} f'(x)$, it still wouldn't have mattered, since they both evaluate to $\pm\infty$, which is DNE. However, I hope that it proves the point that if there are two possible values for the limiting case, then the derivative is defined as DNE.

Example 3.3. Consider the derivative of $f(x) = x^3$ at $x = 0$ (see Figure 3.2). We can immediately see $\lim_{x \to 0} f'(x) = \infty$, implying DNE.

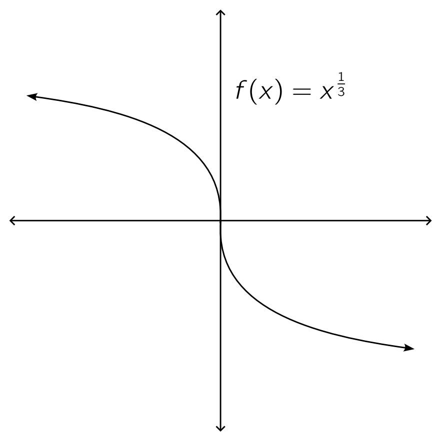

Figure 3.2: Graph of $f(x) = x^{\frac{1}{3}}$. There is a vertical POI at $x = 0$.

Example 3.4. Consider the derivative of $f(x) = |x|$ at $x = 0$ (see Figure 3.3). We apply Definition 3.1 to prove that $\lim_{x \to 0} f'(x) = \text{DNE}$. The derivative of $f(x) = |x|$ is interestingly $f'(x) = \frac{|x|}{x}$ or $f'(x) = \frac{x}{|x|}$. This implies

$$\lim_{x \to 0^+} f'(x) = 1$$
$$\lim_{x \to 0^-} f'(x) = -1$$

Or you can just look at the graph to determine these values. Therefore, according to Definition 3.1, $f'(0)$ is undefined.

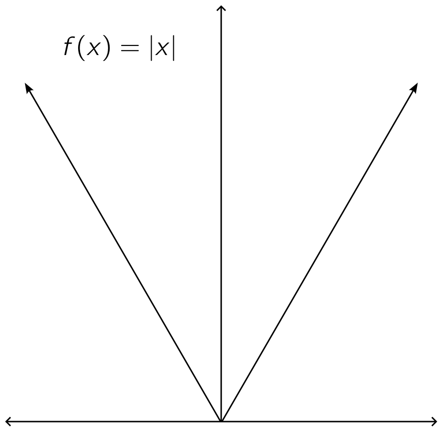

Figure 3.3: Graph of $f(x) = |x|$. The sharp turn at $x = 0$ is what we call a **cusp**.

Proposition 3.5. The derivative of $f'(x)$ at $x = a$ is DNE if:

1. $\lim_{x \to a^+} f'(x) \neq \lim_{x \to a^-} f'(x)$
2. There is a horizontal POI at $x = a$.
3. There is a cusp at $x = a$.

CHAPTER FOUR

Trigonometric Functions

One of the most important applications of calculus is in physics. Calculus helps us understand the motion of a swining pendulum, projectile motion, and countless other applications. Wave functions such as $\sin x$ and $\cos x$ all play a huge role in deriving these motions, making their derivatives essential in physics.

We will begin this section in trigonometric functions by beginning with the basics: the derivative of basic trigonometric functions.

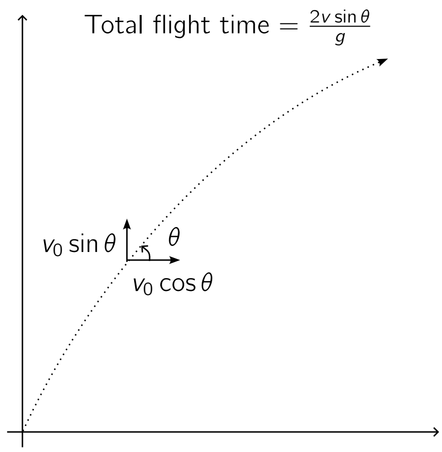

Figure 4.1: Projectile motion: to determine the shortest time, we must differentiate $\sin\theta$.

4.1 The Derivative of sinx

The derivative of trigonometric originates here, in differentiating $\sin x$. This is because once you can differentiate $\sin x$, then you can differentiate all of the other functions, using the rules that we have established and $\cos x = \sin\left(\frac{\pi}{2} - x\right)$.

Theorem 4.1 (Derivative of $\sin x$)**.** The derivative of the function $\sin x$ is

$$\frac{\mathrm{d}}{\mathrm{d}x} \sin x = \cos x$$

Proof. According to the definition of derivative, if $f(x) = \sin x$

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \\ &= \cos x \lim_{h\to 0} \frac{\sin h}{h} + \sin x \lim_{h\to 0} \frac{\cos h - 1}{h} \end{aligned}$$

But what are $\lim_{h\to 0} \frac{\sin h}{h}$ and $\lim_{h\to 0} \frac{\cos h - 1}{h}$? We will calculate the first limit, and it will becoem apparent that the latter limit can be calculated using the result of the first. To evaluate the first limit, consider Figure 4.2

From Figure 4.2, we will consider and compare the areas of $\triangle OPR$, sector OPR, and lastly $\triangle OQR$.

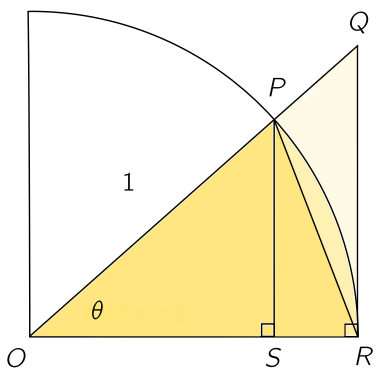

Figure 4.2: We construct a circle with radius of 1. We also form triangles $\triangle OPR$ and $\triangle OQR$.

The area of $\triangle OPR$ is

$$\begin{aligned}\triangle OPR \text{ Area} &= \frac{PS \cdot OR}{2}\\ &= \frac{|\sin\theta \cdot 1|}{2}\\ &= \frac{|\sin\theta|}{2}\end{aligned}$$

The area of sector OPR is

$$\begin{aligned}OPR \text{ Area} &= \frac{|\theta \cdot r^2|}{2}\\ &= \frac{|\theta|}{2}\end{aligned}$$

And lastly the area of $\triangle OQR$ is

$$\begin{aligned}\triangle OQR \text{ Area} &= \frac{|OR \cdot QR|}{2}\\ &= \frac{|1 \cdot \tan\theta|}{2}\end{aligned}$$

And from Figure 4.2, we can see that from comparing the areas

$$\begin{gathered}\text{Areas}\\ \triangle OQR \leq OPR \leq \triangle OQR\\ \frac{|\sin\theta|}{2} \leq \frac{|\theta|}{2} \leq \frac{|\tan\theta|}{2}\\ |\sin\theta| \leq |\theta| \leq |\frac{\sin\theta}{\cos\theta}|\\ 1 \leq |\frac{\theta}{\sin\theta}| \leq \frac{1}{|\cos\theta|}\end{gathered}$$

Then we will take the reciprocal on all sides. When do this we have to remember to flip the inequalities

$$1 \geq |\frac{\sin\theta}{\theta}| \geq |\cos\theta|$$

Next we will take the limit on all sides

$$\lim_{\theta\to 0} 1 \geq \lim_{\theta\to 0} |\frac{\sin\theta}{\theta}| \geq \lim_{\theta\to 0} |\cos\theta|$$

And we can now remove the absolute values, since $\theta > 0$ and $\lim\limits_{\theta\to 0} \sin\theta > 0$ and $\lim_{\theta\to 0} \cos\theta > 0$

$$1 \geq \lim_{\theta\to 0} \frac{\sin\theta}{\theta} \geq \lim_{\theta\to 0} \cos\theta$$

$$1 \geq \lim_{\theta\to 0} \frac{\sin\theta}{\theta} \geq 1$$

And we can readily see from this that because $\lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\theta}$ is in the middle of these inequalities, this limit has to evaluate to 1.[10]

[10] If you would like a more visual way to derive this result, I would recommend checking out Khan Academy's video: https://youtu.be/5xitzTutKqM.

So we determined that $\lim_{h\to 0} \frac{\sin h}{h} = 1$, but what is the value for $\lim_{h\to 0} \frac{\cos h - 1}{h}$? We can actually use the first limit to determine the value of this limit. We will multiply the numerator and denominator by $\cos h + 1$

$$\begin{aligned}
\lim_{h\to 0} \frac{\cos h - 1}{h} &= \lim_{h\to 0} \frac{\cos h - 1}{h} \frac{\cos h + 1}{\cos h + 1} \\
&= -\lim_{h\to 0} \frac{\sin^2 h}{h(\cos h + 1)} \\
&= -\lim_{h\to 0} \frac{\sin h}{h} \lim_{h\to 0} \frac{\sin h}{\cos h + 1} \\
&= -1(0) \qquad\qquad\qquad\qquad = 0
\end{aligned}$$

So now we know that $\lim_{h\to 0} \frac{\sin h}{h} = 1$ and $\lim_{h\to 0} \frac{\cos h - 1}{h} = 0$, we substitute these values in for $f'(x)$

$$\begin{aligned}
f'(x) &= \cos x(1) + \sin x(0) \\
&= \cos x
\end{aligned}$$

$\therefore \frac{\mathrm{d}}{\mathrm{d}x} \sin x = \cos x$. □

4.2 The Derivative of Trigonometric Functions

Theorem 4.2. Below is a list of all trigonometric derivatives

$$\frac{\mathrm{d}}{\mathrm{d}x} \sin x = \cos x \tag{1}$$
$$\frac{\mathrm{d}}{\mathrm{d}x} \cos x = -\sin x \tag{2}$$
$$\frac{\mathrm{d}}{\mathrm{d}x} \tan x = \sec^2 x \tag{3}$$
$$\frac{\mathrm{d}}{\mathrm{d}x} \cot x = -\csc^2 x \tag{4}$$
$$\frac{\mathrm{d}}{\mathrm{d}x} \sec x = \sec x \tan x \tag{5}$$
$$\frac{\mathrm{d}}{\mathrm{d}x} \csc x = -\csc x \cot x \tag{6}$$

A good way to remember this is the derivative of any trigonometry function that starts with the letter "c" is negative. Anything else is positive. Also, the derivatives of $\tan x$, $\sec x$ and $\cot x$, $\csc x$ are basically opposites. You pair $\tan x$ with $\sec x$ and $\cot x$ with $\csc x$.

Proofs.
(2) We use the identity $\cos x = \sin\left(\frac{\pi}{2} - x\right)$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} \cos x &= \frac{\mathrm{d}}{\mathrm{d}x} \sin\left(\frac{\pi}{2} - x\right) \\
&= \cos\left(\frac{\pi}{2} - x\right)(-1) \\
&= -\sin x
\end{aligned}$$

(3) We use the quotient rule

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\tan x &= \frac{\mathrm{d}}{\mathrm{d}x}\frac{\sin x}{\cos x} \\
&= \frac{\cos x(\cos x) - (-\sin x)(\sin x)}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} \\
&= \sec^2 x
\end{aligned}$$

(4) We use the quotient rule

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\cot x &= \frac{\mathrm{d}}{\mathrm{d}x}\frac{\cos x}{\sin x} \\
&= \frac{-\sin x(\sin x) - \cos x(\cos x)}{\sin^2 x} \\
&= -\frac{1}{\sin^2 x} \\
&= -\csc^2 x
\end{aligned}$$

(5) We use the chain rule and power rule

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\sec x &= \frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{\cos x} \\
&= \frac{\mathrm{d}}{\mathrm{d}x}{}^{-1} \\
&= -(\cos x)^{-2}(-\sin x) \\
&= \frac{\sin x}{\cos x} \\
&= \sec x \tan x
\end{aligned}$$

(6) We use the chain rule and the power rule

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\csc x &= \frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{\sin x} \\
&= \frac{\mathrm{d}}{\mathrm{d}x}{}^{-1} \\
&= -(\sin x)^{-2}\cos x \\
&= -\frac{\cos x}{\sin^2 x} \\
&= -\csc x \cot x
\end{aligned}$$

CHAPTER FIVE

Exponential and Logarithmic Functions

Exponential and logarithmic functions are really important within calculus. They have many applications in chemistry, physics, economics, and countless other subjects. One application that wouldn't be possible without calculus is the *half-life equation*.[11]

We will begin this chapter with the derivative of $f(x) = e^x$, and what makes Euler's number e so special. This gives insight as to how we calculate the derivative of other exponential and logarithmic functions.

If you do not want to read and would prefer videos instead (personally I would much rather watch videos), check out these two videos in order:

1. 3blue1brown: https://www.youtube.com/watch?v=m2MIpDrF7Es.
2. blackpenredpen: https://www.youtube.com/watch?v=oBlHiX6vrQY.

[11] The half-life equation is a system of equations that shows when the half-life of a quantity will occur. A common fact is that the atoms within the molecule decay at a rate proportional to the number of atoms and the activity measured in terms of atoms per minute. If $N(t)$ is the molecules present at time t, then what we are representing is

$$N(t) \propto t$$

The differential equation obtained from this is

$$\frac{\mathrm{d}N}{\mathrm{d}t} = k \cdot N(t)$$

5.1 The Definition of e

Definition 5.1. The value of e is calculated by evaluating

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$$

And letting $n = \frac{1}{h}$ gives us the second definition

$$e = \lim_{h\to 0} (1 + h)^{\frac{1}{h}}$$

And as a result of this we have the following lemma[12]

Lemma 5.2 (Formula for e^x)**.** The value of e^x can be determined by

$$e^x = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^{nx}$$

or

$$e^x = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$$

[12] You can see the proof for the more general case for e^{abx}: https://youtu.be/HM-kwHR4VO4.

5.2 The Natural Log

Definition 5.3. The natural log is the inverse of $f(x) = e^x$. That is,

$$\ln x = f^{-1}(x)$$

Where we denote the natural logarithm with $\ln x$.

The natural log is key when it comes to determining the derivative of exponential and logarithmic functions, since it is the inverse of e^x.

5.3 Derivative of e

Theorem 5.4. The derivative of the function e^x is

$$\frac{\mathrm{d}}{\mathrm{d}x} e^x = e^x$$

This result is rather shocking–the derivative of a function is itself? An interesting fact is that e^x is the only function where the derivative is itself.

Proof. According to the definition of derivative if $f(x) = \sin x$ then $f'(x)$ is

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} \\ &= \lim_{h \to 0} \frac{e^x(e^h - 1)}{h} \\ &= e^x \lim_{h \to 0} \frac{e^h - 1}{h} \\ &= e^x \lim_{h \to 0} \frac{((1+h)^{\frac{1}{h}})^h - 1}{h} \\ &= e^x \lim_{h \to 0} \frac{h}{h} \\ &= e^x \end{aligned}$$

□

5.4 Derivative of Exponentials

Theorem 5.5. As a result of Theorem 5.4 the derivative of a^x is

$$\frac{\mathrm{d}}{\mathrm{d}x} a^x = a^x \ln a$$

Proof. Now that we have proven $\frac{\mathrm{d}}{\mathrm{d}x} e^x = e^x$, we can proceed with this proof[13]

[13] The reason we have to bring x back into the exponent in step 3 is because if we did not do that, then we could not simplify it to a^x.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} a^x &= \frac{\mathrm{d}}{\mathrm{d}x}(e^{\ln(a^x)}), \quad \text{bring the } x \text{ in the power down} \\
&= \frac{\mathrm{d}}{\mathrm{d}x}(e^{\ln(a)x}), \quad \text{use the chain rule} \\
&= e^{\ln(a)x} \cdot \ln(a)(1), \quad \text{put the } x \text{ back in the power} \\
&= e^{\ln(a^x)} \ln a \\
&= a^x \ln a
\end{aligned}$$

□

5.5 Derivative of the Natural Log

Before we determine $\frac{\mathrm{d}}{\mathrm{d}x}\log_a(x)$, we have to first consider $\frac{\mathrm{d}}{\mathrm{d}x}\ln x$.

Theorem 5.6 (Derivative of the Natural Log)**.**

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln x = \frac{1}{x}$$

For $x > 0$

Proof. We will make use of implicit differentiation. Let $y = \ln x$

$$\begin{aligned}
y &= \ln x \\
e^y &= e^{\ln x} = x \\
e^y \frac{\mathrm{d}y}{\mathrm{d}x} &= 1 \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{e^y} \\
&= \frac{1}{e^{\ln x}} \\
&= \frac{1}{x}
\end{aligned}$$

□

5.6 Derivative of Logarithms

Using what we know from Section 5.5 we can finally determine $\frac{\mathrm{d}}{\mathrm{d}x}\log_a x$.

Theorem 5.7. The derivative of $\log_a x$ is

$$\frac{\mathrm{d}}{\mathrm{d}x}\log_a x = \frac{1}{x \ln a}$$

Proof. We use implicit differentiation. Let $y = \log_a x$

$$\begin{aligned}
y &= \log_a x \\
a^y &= x \\
a^y \ln a \frac{\mathrm{d}y}{\mathrm{d}x} &= 1 \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{a \ln a} \\
&= \frac{1}{x \ln a}
\end{aligned}$$

□

CHAPTER SIX

L'Hopital's Rule

Sometimes, there are limits that we cannot evaluate by simply manipulation. For example, how are we suppose to evaluate

$$\lim_{x\to 0} \frac{3^x - 1}{x}$$

The only way we can do this, is by using something called *L'Hopital's Rule*.

Definition 6.1 (L'Hopital's Rule)**.** Suppose we have two arbitrary functions $f(x)$ and $g(x)$. Then, as $x \to 0$ or $x \to \infty$, if both $f(x) = 0$ and $g(x) = 0$ or $f(x) = \infty$ and $g(x) = \infty$, then

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f'(x)}{g'(x)}$$

Or

$$\lim_{x\to \infty} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f'(x)}{g'(x)}$$

In other words, you differentiate the numerator and denominator.

Example 6.2. Evaluate $\lim\limits_{x\to 0} \frac{3^x - 1}{x}$. Differentiating the numerator and denominator

$$\begin{aligned} \lim_{x\to 0} \frac{3^x - 1}{x} &= \lim_{x\to 0} \frac{3^x \ln 3}{1} \\ &= \ln 3 \frac{1}{1} \\ &= \ln 3 \end{aligned}$$

6.0.1 Challenge Problems

1. Prove that $\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{b}x = e^{ab}$.

2. Evaluate $\lim_{x\to\infty}\left(\frac{2x-1}{2x+1}\right)^{x}$.

6.0.2 Solutions

1.

2. We plug in $x=\infty$ we get $(\frac{\infty}{\infty})^{\infty}$, which doesn't work. Instead, we will first separate the fraction

$$\begin{aligned}\lim_{x\to\infty}\left(\frac{2x-1}{2x+1}\right)^{x} &= \lim_{x\to\infty}\left(\frac{2x+1-2}{2x+1}\right)^{x}\\ &= \lim_{x\to\infty}\left(1-\frac{2}{2x+1}\right)^{x}\end{aligned}$$

Then we make the substitution $u=2x+1$, $x=\frac{u-1}{2}$, and limits change from $\lim_{x\to\infty}\to\lim_{u\to\infty}$

$$\begin{aligned}\lim_{x\to\infty}\left(1-\frac{2}{2x+1}\right)^{x} &= \lim_{u\to\infty}\left(1-\frac{2}{u}\right)^{\frac{u-1}{2}}\\ &= \lim_{u\to\infty}\left[\frac{\left(1-\frac{2}{u}\right)^{u}}{\left(1-\frac{2}{u}\right)}\right]^{\frac{1}{2}}\end{aligned}$$

Then, the denominator evaluates to $\lim_{u\to\infty}\left(1-\frac{2}{u}\right)=0$

$$\lim_{u\to\infty}\left[\frac{\left(1-\frac{2}{u}\right)^{u}}{\left(1-\frac{2}{u}\right)}\right]^{\frac{1}{2}} = \lim_{u\to\infty}\left[\left(1-\frac{2}{u}\right)^{u}\right]^{\frac{1}{2}}$$

And from the result obtained in the first challenge problem, we know that $\lim_{u\to\infty}\left(1-\frac{2}{u}\right)^{u} = \lim_{u\to\infty}\left(1+\frac{-2}{u}\right)^{u} = e^{-2}$

$$\begin{aligned}\lim_{u\to\infty}\left[\left(1-\frac{2}{u}\right)\right]^{\frac{1}{2}} &= \left[e^{-2}\right]^{\frac{1}{2}}\\ &= \frac{1}{e}\end{aligned}$$

CHAPTER SEVEN

Related Rates and Optimization

One thing that the derivative is known for is calculating rates. Because of this, we can use calculus to determine the rate at which something will be optimized. For example,what is the fuel efficient way to get from Point A to Point B? What is the best way to create tissue boxes such that the surface area is minimized? The answer to these questions lie in the heart of optimization.

> **Definition 7.1** (Optimization)**.** In determining the best way to optimize something, we think of it as a function. Thus, when we are optimizing a function, we are looking for the largest or smallest value that a function can take.

Example 7.2. A manufacturer needs to make a cylindrical can that will hold 1.5 litres of liquid. Determine the dimensions of the can that will minimize the amount of material used in its construction (see Figure 7.1).

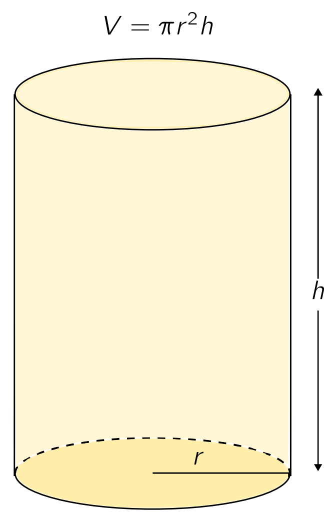

Figure 7.1: Cylinder visual

We first convert 1.5 litres into 1500cm^3. We also know the volume of a cylinder is

$$\begin{aligned} A = \pi r^2 h &= 1500 \quad \text{solve for } h \\ h &= \frac{1500}{\pi r^2} \end{aligned}$$

Next we will substitute this into the formula for the surface area of the cylinder

$$\begin{aligned} \mathrm{S} &= 2\pi r^2 + 2\pi r h \\ &= 2\pi r^2 + 2\pi r \frac{1500}{\pi r^2} \\ &= 2\pi r^2 + \frac{3000}{r} \end{aligned}$$

Then differentiate with respect to r

$$\begin{aligned} \frac{\mathrm{d}S}{\mathrm{d}r} &= 4\pi r - \frac{3000}{r^2} \\ &= \frac{4\pi r^3 - 3000}{r^2} \end{aligned}$$

And set $\frac{\mathrm{d}S}{\mathrm{d}r} = 0$

$$\begin{aligned}\frac{\mathrm{d}S}{\mathrm{d}r} = \frac{4\pi r^3 - 3000}{r^2} &= 0\\ 4\pi r^3 &= 3000\\ r &= \sqrt[3]{\frac{3000}{4\pi}}\\ &\approx 6.2035\end{aligned}$$

Then the height of the cylinder has to be

$$\begin{aligned}h &= \frac{1500}{\pi\sqrt[3]{\frac{3000}{4\pi}}}\\ &\approx 12.407\end{aligned}$$

$\therefore$, the manufacturer should make the cylindrical can to have a radius of 6.2035cm and a height of 12.407cm.

Example 7.3. A particle is moving along the hyperbola $x^2 - y^2 = 5$. As it reaches the point $(3, -2)$, the y-coordinate is decreasing at a rate of 0.9cm/s. How fast is the x-coordinate of the point changing, at this instance? See Figure 7.2

We must write down what we know and what the question is asking for. We know that the hyperbola is $x^2 - y^2 = 5$. We also know that at the point $(3, -2)$ the y-coordinate is decreasing at a rate of 0.9cm/s. From that second piece of information, we can derive $\frac{\mathrm{d}y}{\mathrm{d}t} = -0.9$. This is because when $y = -2$, corresponding to the point $(3, -2)$, the y-value is decreasing at a rate of -0.9. Now, the question is asking for the rate of change of the x-coordinate at that point, which corresponds to determining $\frac{\mathrm{d}x}{\mathrm{d}t}$ at $x = 3$. So in other words, $\frac{\mathrm{d}x}{\mathrm{d}t} = ?$

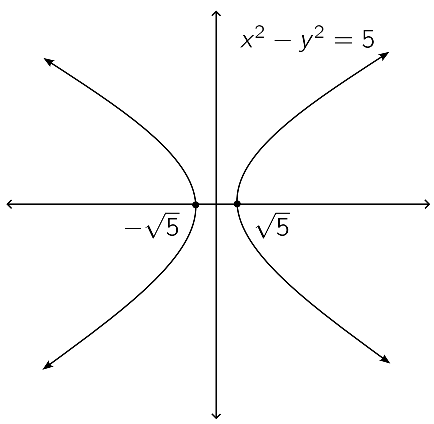

Figure 7.2: Hyperbola $x^2 - y^2 = 5$. It has roots at $x = \pm\sqrt{5}$.

From the equation $x^2 - y^2 = 5$, we will use implicit differentiation to differentiate both sides with respect to t

$$2x\frac{\mathrm{d}x}{\mathrm{d}t} - 2y\frac{\mathrm{d}y}{\mathrm{d}t} = 0$$

Then substituting in the point $(3, -2)$ as well as $\frac{\mathrm{d}y}{\mathrm{d}t} = -0.9$ at that point

$$\begin{aligned}2(3)\frac{\mathrm{d}x}{\mathrm{d}t} - 2(-2)(-0.9) &= 0\\ \frac{\mathrm{d}x}{\mathrm{d}t} &= \frac{4(0.9)}{6}\\ &= 0.6\text{m/s}\end{aligned}$$

$\therefore$, the x-coordinate is changing at a rate of 0.6m/s.

7.0.1 Problems

The first 7 problems are from this video: https://www.youtube.com/watch?v=S2Ab5Euo0Tk

1. Water enters a conical tank at a rate of 9ft^3/min. The tank stands pointing down and has a height of 15ft and a base radius of 5ft. How fast is the water rising when the water is 6 feet deep?

2. Car A is travelling west at 50mph and car B is travelling north at 60mph. They are heading towards the same intersection. At what rate are the cars approaching each other when car A is 0.3 miles and car B is 0.4 miles from the intersection?

3. If $y^2 = 2x$ and x is increasing at a rate of $\frac{1}{2}$ units/time, how fast is the slope of the curve changing when $x = 32$?

7.0.2 Solutions

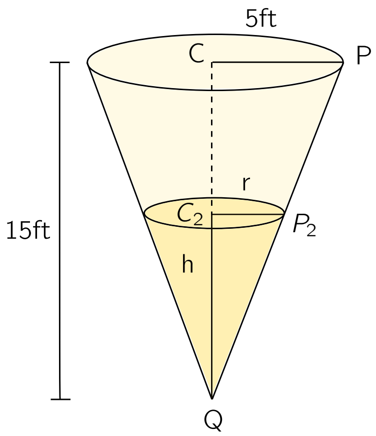

Figure 7.3: A conical tank. Volume $= \frac{1}{3}\pi r^2 h$. Not drawn to scale.

1. From the problem, we draw out Figure 7.3. In the figure, the larger cylinder is the cylindrical tank and the darker cylinder is occupied by the water. Then, we would like to know what $\frac{dh}{dt}$ is. Using the chain rule
$$\frac{dh}{dt} = \frac{dv}{dt}\frac{dh}{dv}$$
And we know that $\frac{dv}{dt} = 9$, since water enters the conical tank at a rate of $9\text{ft}^3/\text{min}$
$$\frac{dh}{dt} = 9\frac{dh}{dv}$$
All we have to solve for now is $\frac{dh}{dv}$. This part is a little bit tricky, since we still have the variable r, so we cannot just differentiate the volume of the cylinder, which is $v = \frac{1}{3}\pi r^2 h$. Therefore, consider similar triangles $\triangle PCQ$ and $\triangle P_2C_2Q$. We see that
$$\begin{aligned} \frac{CQ}{PC} &= \frac{QC_2}{P_2C_2} \\ \frac{15}{5} &= \frac{h}{r} \\ \frac{1}{3} &= \frac{h}{r} \\ r &= \frac{h}{3} \end{aligned}$$
And so the volume of the water at height h is
$$\begin{aligned} v &= \frac{1}{3}\pi(\frac{h}{3})^2 h \\ &= \frac{\pi}{27}h^3 \end{aligned}$$
And now we can differentiate both sides with respect to v
$$\begin{aligned} \frac{d}{dv} &= \frac{\pi}{81}\frac{d}{dv}h^3 \\ 1 &= \frac{\pi}{27}3h^2\frac{dh}{dv} \\ \frac{dh}{dv} &= \frac{9}{\pi h^2} \end{aligned}$$
Substituting this back into our equation for $\frac{dh}{dt}$
$$\begin{aligned} \frac{dh}{dt} &= 9(\frac{9}{\pi h^2}) \\ &= \frac{81}{36\pi} \end{aligned}$$
$\therefore$ the height of the water is changing at a rate of $\frac{81}{36}\pi\text{ft/s}$.

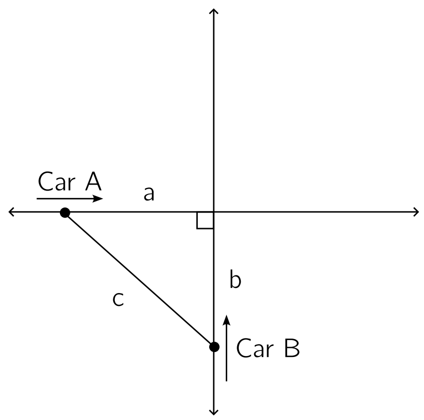

Figure 7.4: The distance between car A and the intersection is a, the distance between car B and the intersection is b, and the distance between the two cars is c.

2. In Figure 7.4, we will label the center to be where the cars will intersect. Then approaching from the left will be car A and approaching from the bottom will be car B. We would like to know the rate at which the

distance AB decreases. To get started, we will calculate the length of c using pythagorean's theorem

$$c^2 = a^2 + b^2$$

Then, we also know that $\frac{da}{dt} = -50$ and $\frac{db}{dt} = -60$[14]. Using implicit differentiation

$$\begin{aligned}2c\frac{dc}{dt} &= 2a\frac{da}{dt} + 2b\frac{db}{dt} \\ \frac{dc}{dt} &= \frac{-2a(50) - 2b(60)}{2c} \\ &= \frac{-100a - 120b}{2(a^2 + b^2)}\end{aligned}$$

Then we plug in for when $a = 0.3$ and $b = 0.4$

$$\frac{dc}{dt} = \frac{-50(0.3) - 120(0.4)}{2(0.3^2 + 0.4^2)} \qquad = -78$$

$\therefore$ the cars are approaching each other at a rate of 78mph.

3. To determine how fast the slope of a curve is changing, we need to calculate $\frac{d^2y}{dx^2}$. If we use implicit differentiation ocne

$$\begin{aligned}2y\frac{dy}{dx} &= 2 \\ \frac{dy}{dx} &= \frac{1}{y} \\ &= \frac{1}{\sqrt{2x}}\end{aligned}$$

And differentiate once more

$$\begin{aligned}\frac{d^2y}{dx^2} &= -\frac{1}{2}(2)(2x)^{-\frac{3}{2}} \\ &= -\frac{1}{(2x)^{\frac{3}{2}}}\end{aligned}$$

And now substitute $x = 32$

$$\begin{aligned}\frac{d^2y}{dx^2} &= -\frac{1}{(2 \cdot 32)^{\frac{3}{2}}} \\ &= -\frac{1}{512}\end{aligned}$$

$\therefore$ the slope of the curve at $x = 32$ is changing at a rate of $\frac{1}{512}$[15].

[14] We use -50 and -60 because we see that the lengths of a and b are decreasing.

[15] We do not care about the negative in this case.

CHAPTER EIGHT

Taylor's Theorem and Power Series

Some functions are impossible to integrate with elementary techniques. Some examples are

$$\frac{\sin x}{x}, \quad e^{-x^2}, \text{ and } \quad \sqrt{1-k^2 sin^2 x}$$

Where k is some constant less than 1. The first function is used as a common example to demonstrate the *Feynman Technique*. The second function is a bell curve (see Figure 8.1), which is very important in statistics. The third function comes up in trying to find the arc length of an ellipse. This is one of the reasons why there is a need to simplify these functions intomuch simpler ones so that we can easily integrate them.

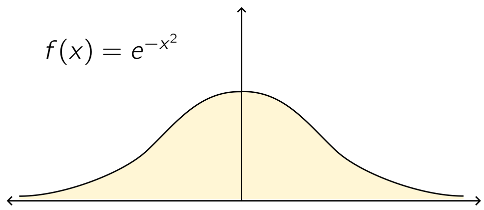

Figure 8.1: A bell curve

Another use is for calculating the values for special functions, such as the trigonometry functions. That is, calculating values such as $\sin 56°$, for example.

8.1 Approximating Functions using Polynomials

One way to approximate functions is by using polynomials.

Theorem 8.1 (Taylor's Theorem)**.** Any function satisfying certain conditions may be represented by Taylor Series

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a) + ... + \frac{f^{(n)}(a)}{n!}(x-a)^n$$