CHAPTER ONE

L'Hopital's Rule

Sometimes, there are limits that we cannot evaluate by simply manipulation. For example, how are we suppose to evaluate

$$\lim_{x \to 0} \frac{3^x - 1}{x}$$

The only way we can do this, is by using something called *L'Hopital's Rule*.

Definition 1.1 (L'Hopital's Rule)**.** Suppose we have two arbitrary functions $f(x)$ and $g(x)$. Then, as $x \to 0$ or $x \to \infty$, if both $f(x) = 0$ and $g(x) = 0$ or $f(x) = \infty$ and $g(x) = \infty$, then

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{f'(x)}{g'(x)}$$

Or

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{f'(x)}{g'(x)}$$

In other words, you differentiate the numerator and denominator.

Example 1.2. Evaluate $\lim_{x \to 0} \frac{3^x - 1}{x}$. Differentiating the numerator and denominator

$$\begin{aligned} \lim_{x \to 0} \frac{3^x - 1}{x} &= \lim_{x \to 0} \frac{3^x \ln 3}{1} \\ &= \ln 3 \frac{1}{1} \\ &= \ln 3 \end{aligned}$$

1.0.1 Challenge Problems

1. Prove that $\lim\limits_{x\to\infty}\left(1+\dfrac{a}{x}\right)^{b}x = e^{ab}$.

2. Evaluate $\lim\limits_{x\to\infty}\left(\dfrac{2x-1}{2x+1}\right)^{x}$.

1.0.2 Solutions

1.

2. We plug in $x=\infty$ we get $(\frac{\infty}{\infty})^{\infty}$, which doesn't work. Instead, we will first separate the fraction

$$\begin{aligned}\lim_{x\to\infty}\left(\frac{2x-1}{2x+1}\right)^{x} &= \lim_{x\to\infty}\left(\frac{2x+1-2}{2x+1}\right)^{x}\\ &= \lim_{x\to\infty}\left(1-\frac{2}{2x+1}\right)^{x}\end{aligned}$$

Then we make the substitution $u=2x+1$, $x=\frac{u-1}{2}$, and limits change from $\lim\limits_{x\to\infty}\to\lim\limits_{u\to\infty}$

$$\begin{aligned}\lim_{x\to\infty}\left(1-\frac{2}{2x+1}\right)^{x} &= \lim_{u\to\infty}\left(1-\frac{2}{u}\right)^{\frac{u-1}{2}}\\ &= \lim_{u\to\infty}\left[\frac{\left(1-\frac{2}{u}\right)^{u}}{\left(1-\frac{2}{u}\right)}\right]^{\frac{1}{2}}\end{aligned}$$

Then, the denominator evaluates to $\lim\limits_{u\to\infty}\left(1-\dfrac{2}{u}\right)=0$

$$\lim_{u\to\infty}\left[\frac{\left(1-\frac{2}{u}\right)^{u}}{\left(1-\frac{2}{u}\right)}\right]^{\frac{1}{2}} = \lim_{u\to\infty}\left[\left(1-\frac{2}{u}\right)^{u}\right]^{\frac{1}{2}}$$

And from the result obtained in the first challenge problem, we know that $\lim\limits_{u\to\infty}\left(1-\dfrac{2}{u}\right)^{u}=\lim\limits_{u\to\infty}\left(1+\dfrac{-2}{u}\right)^{u}=e^{-2}$

$$\begin{aligned}\lim_{u\to\infty}\left[\left(1-\frac{2}{u}\right)\right]^{\frac{1}{2}} &= \left[e^{-2}\right]^{\frac{1}{2}}\\ &= \frac{1}{e}\end{aligned}$$