CHAPTER ONE

DNE - Does Not Exist

Some functions do not have a derivative at a certain point. The reason for this in most cases as we will see is because the slope f' approaches ∞, $-\infty$, or simply doesn't exist.

Some common functions include $x^{\frac{2}{3}}$ at $x = 0$, $x^{\frac{1}{3}}$ at $x = 0$, $|x|$ at $x = 0$, and $\frac{1}{x}$ at $x = 0$.

Definition 1.1 (DNE)**.** The derivative of $f(x)$ at $x = a$ is considered DNE if

$$\lim_{x \to a^+} f'(x) \neq \lim_{x \to a^-} f'(x)$$

This definition is actually expanded upon in first to second year university mathematics [1] by the *Epsilon Delta Definition of a Limit*. Regardless, it is pretty intuitive that Definition 1.1 is true.

[1] We will refer to first year university mathematics at U1 mathematics. This applies to any year as well (example year 2 university mathematics is U2 mathematics).

Example 1.2. Consider the derivative of $f(x) = \frac{1}{x}$ at $x = 0$ (see Figure 1.1). We can immediately see that the value of $f'(x)$ at $x = a$ is unclear. To prove this, differentiate to get $f'(x) = -\frac{1}{x^2}$

$$\lim_{x \to 0^+} = -\infty$$

$$\lim_{x \to 0^-} = \infty$$

Which we can tell from the graph as well. Therefore, we see that we cannot reach a consenus.[2]

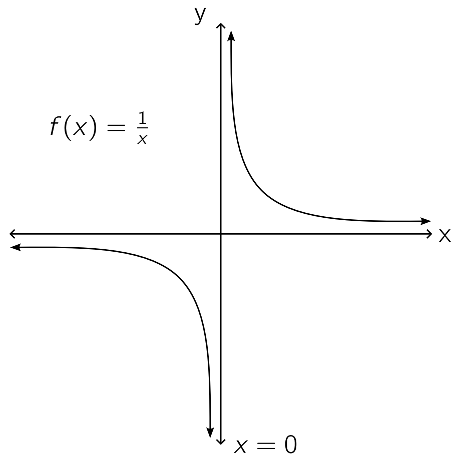

Figure 1.1: Graph of $f(x) = \frac{1}{x}$. There is a V.A at $x = 0$.

[2] It should be noted that this was a bad example, but one that first comes to mind. The reason for this is because even if there weren't two values for $\lim_{x \to 0^+} f'(x)$ and $\lim_{x \to 0^-} f'(x)$, it still wouldn't have mattered, since they both evaluate to $\pm\infty$, which is DNE. However, I hope that it proves the point that if there are two possible values for the limiting case, then the derivative is defined as DNE.

Example 1.3. Consider the derivative of $f(x) = x^3$ at $x = 0$ (see Figure 1.2). We can immediately see $\lim_{x \to 0} f'(x) = \infty$, implying DNE.

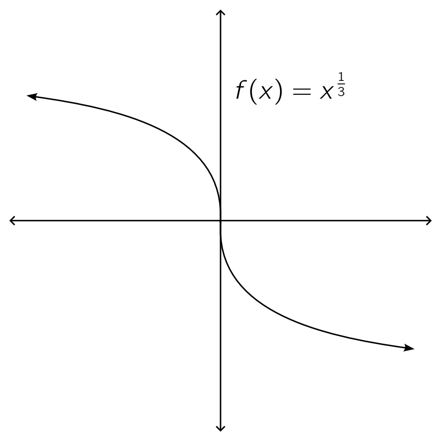

Figure 1.2: Graph of $f(x) = x^{\frac{1}{3}}$. There is a vertical POI at $x = 0$.

Example 1.4. Consider the derivative of $f(x) = |x|$ at $x = 0$ (see Figure 1.3). We apply Definition 1.1 to prove that $\lim_{x \to 0} f'(x) = \text{DNE}$. The derivative of $f(x) = |x|$ is interestingly $f'(x) = \frac{|x|}{x}$ or $f'(x) = \frac{x}{|x|}$. This implies

$$\lim_{x \to 0^+} f'(x) = 1$$

$$\lim_{x \to 0^-} f'(x) = -1$$

Or you can just look at the graph to determine these values. Therefore, according to Definition 1.1, $f'(0)$ is undefined.

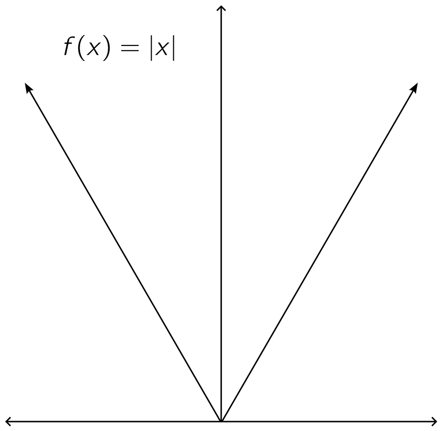

Figure 1.3: Graph of $f(x) = |x|$. The sharp turn at $x = 0$ is what we call a **cusp**.

Proposition 1.5. The derivative of $f'(x)$ at $x = a$ is DNE if:

1. $\lim_{x \to a^+} f'(x) \neq \lim_{x \to a^-} f'(x)$
2. There is a horizontal POI at $x = a$.
3. There is a cusp at $x = a$.