Reserve Prices in Art Auctions

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Abstract

This paper introduces a new nonparametric approach to the identification of ascending auctions and applies it to a large novel dataset of live art auctions that we constructed. We apply the approach, which requires only a lower bound on the number of bidders and the top two bids, to our dataset comprising complete bids from more than 3500 live auctions by Sotheby's and Christie's. We additionally solve the minimax-regret problem applied to this partially-identified model and use it to propose new reserve prices. Compared to setting the reserve at the low estimate as is common practice today, our proposed reserve increases average expected profits by up to 17% of the high estimate, equivalent to up to US$30,000,000 per auction.

Keywords: Nonparametric identification, minimax regret, ascending auctions, unknown number of bidders, correlated values, art auctions

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1 Introduction

1.1 Motivation

The auction market for art has large economic significance. In 2023, the two largest art auction houses, Sotheby's and Christie's, reported revenues of US$7.9 billion and USD$6.2 billion respectively. On November 9 and 10, 2022 alone, the 155-work collection of deceased technology billionaire Paul Allen sold for US$1,622,249,500 towards philanthropy, with the most expensive piece *Georges Seurat, Les Poseuses, Ensemble (Petite version), 1888-1890* selling for US$149,240,000.

However, art auctions present a unique empirical challenge to researchers because they are notoriously secretive (see Akbarpour and Li (2020), Marra (2020)). On the websites of major art auction houses, one can only find very basic information including the low and high estimate, and the final transaction price. Important information for analysis such as the trajectory of bids and the number of bidders are kept secret. Even if a bidder were to be physically present during a live auction, she would not be able to have a full grasp on the number of bidders because besides live bidding, alternate forms of bidding exist today in the live auction room including telephone bidding, online bidding, and absentee bidding, all of which are opaque to the live bidder.

Therefore, the literature thus far focuses only on small samples of limited auctions, and few discuss the optimal reserve price for maximizing profits. Ashenfelter and Graddy (2003), Ashenfelter (1989), McAndrew, Smith, and Thompson (2012), Beggs and Graddy (2009) all discuss wine and art auctions, their mechanics, and pricing effects. Hortaçsu and Perrigne (2021) reviews empirical auctions and discuss online eBay and wine auctions. Ashenfelter and Graddy (2011) find that the confidential reserve price is commonly thought to be related to an auctioneer's pre-sale estimates, and that the convention is for it to be at or below the auctioneer's low estimate. Such a common rule gives rise to the suspicion that very likely, this blanket rule to setting reserve prices is not optimal towards maximizing expected profit in certain auction categories. On this note, Marra (2020) analyzed Sotheby's auction data to find an improved reserve price 110% of the estimate that increases revenue by 2.5%, but the data was restricted to a small sample of 884 wine lots from only 1 day of online wine auctions. All of the above works are novel in their data collection and application. However, they do not discuss optimal reserve prices for auctions held in the live art auction room, where most of the high value objects are sold[1].

To overcome the limited data, we adopt a novel approach to data collection by analzying livestream video data that Sotheby's and Christie's have uploaded to their YouTube channel over the past few years. By applying suitable algorithms to the frame-by-frame video image data on bids and the audio data in the videos, we are able to generate complete bidding trajectories and a lower bound on the number of bidders systematically across a large number of live auctions. We applied such an approach to 65 Sotheby's live auctions and 39 Christie's auctions, generating data on more than 2500 auction lots. These are the first publicly available data collected on live ascending auctions that include complete bidding trajectories and a lower bound on the number of bidders.

A second major challenge in the analysis of art auctions is an uncertain number of bidders. In addressing an unknown number of bidders, the current literature has formulated several approaches to estimate optimal reserves in ascending auctions. Hernández, Quint, and Turansick (2020) develop a method of point-identification for English auctions that allows for unobserved heterogeneity and no need to observe more than the number of bidders and the winning bid in each auction. Freyberger and Larsen (2022) use the (known) reserve price and two order statistics of bids to estimate optimal reserve prices. Marra (2020) uses the stochastic difference between adjacent order statistics, and requires two losing order statistics besides the winning bid. These papers are all innovative in their econometric use of bid data, but do not apply to the data situation in this paper because only the top two order statistics of bids and a lower bound on the number of bidders are observed in our case.

Our econometric approach to nonparametric identification of ascending auctions overcomes the problem of uncertain number of bidders but requires minimal assumptions that are all intuitively obvious in the context of a Sotheby's or Christie's ascending art auction: bidder symmetry, bidders will not bid more than their willingness to pay, a bidder will outbid an opponent if her valuation is higher than the opponent's bid, and the dependence among bidder values is nonnegative. Our paper extends on the econometric works by Haile and Tamer (2003), Quint (2008), Aradillas-López, Gandhi, and Quint (2013), Chesher and Rosen (2017), Ponomarev (2022) by providing sharp bounds on profits when the econometrician only observes the top two bids

[1] According to Christie's 2023 press release, \$4.6b of their 2023 total \$6.2b sales came from live auction rooms.

and a lower bound on the number of bidders. Even though our bounds are relatively wider with these minimal assumptions, they are sufficiently informative to provide insightful suggestions on a better choice of reserve price. We additionally provide a solution to the minimax regret problem in deciding a single reserve price within the sharp identified set, and advocate for its use to maximize expected profit. When applied to our data subset of Modern Art in New York City, simulations show that our proposed reserve increases expected profits relative to the auction house's typical reserve by up to 17% of the high estimate, equivalent to approximately US$30,000,000 per auction.

1.2 Relationship to Literature

This paper contributes to literature in three ways: the methodological literature on nonparametric estimation of English auctions, the theoretical literature on minimax regret, and the empirical literature on art auctions.

The methodological contribution is a novel nonparametric identification approach that builds from Aradillas-López, Gandhi, and Quint (2013)'s and Haile and Tamer (2003)'s methods. In our approach, the econometrician no longer requires an exact number of bidders in the data as was in Aradillas-López, Gandhi, and Quint (2013) but instead only a lower bound on the number of bidders. Even though our bounds are wider in comparison, they are still sufficiently informative in our application to provide insightful reserve price recommendations.

The theoretical contribution is a solution to the choice of a single optimal reserve price given profit bounds via the minimax regret criterion as first formulated by Savage (1951) and most recently suggested by Manski (2022), within the sharp identified set of profit functions. Such an approach offers an alternate decision criteria to Aryal and Kim (2013)'s and Jun and Pinkse (2019)'s papers, which provide methods to choose a single optimal reserve price. The algorithm we propose uses the upper bound on profit, unlike the maxmin solution, and is computationally feasible in polynomial time. The minimax regret approach is especially relevant to the application to art auctions because it is in line with the goal of profit maximization in the presence of ambiguity.

The empirical contribution is the construction of a large novel dataset from Christie's and Sotheby's live auctions. Our dataset contains complete bids and a lower bound on the number of bidders from more than 3500 live auction lots covering the largest live auctions run by Christie's and Sotheby's in the last few years. While there exists datasets of final transaction prices (for example in Ashenfelter and Graddy (2011)), or small samples of bids from singular auctions (for example online Sotheby's wine auctions in Marra (2020)), this is the first paper to propose a comprehensive and non-tedious method to collect large amounts of auction data. This is also the first paper to collect bids from Christie's and Sotheby's *live* auctions. We further show that adjusting the reserve price can improve expected profits by up to the order of tens of millions of dollars per auction in certain subgroups.

The rest of the paper proceeds as follows. In Section II, data construction is discussed. Section III describes the identification argument and estimation approach. Section IV describes the minimax regret algorithm to choose a single optimal reserve. Section V displays the results. Section VI discusses simulation results to illustrate profit improvements under our suggested reserves. Section VII concludes.

2 Art Auction Data

Despite the large number of public auctions that Christie's and Sotheby's have run, they reveal very little information about their auctions. In particular, their public websites only provide data on lot details (e.g. artist name, period, provenance, condition report), low and high estimates, and the final sale price after the buyer's premium. The most crucial identifying information such as bids, bidder identities, and number of bidders in past auctions are kept private to the firms.

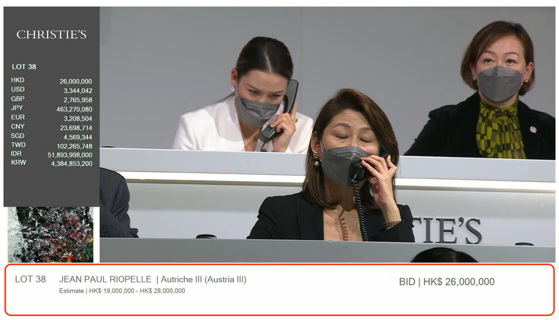
We construct a large novel dataset comprising 3500+ auction lots using live-stream YouTube auction videos held between 2020 and 2024. Our dataset comprises complete bidding trajectory, a lower bound on the number of bidders, and auction lot details such as the low and high estimates, location of auction and category of art. To the best of our knowledge, this is the first large dataset on auctions of modern art with bidding data.

We generate the complete bidding trajectory by applying computer vision techniques to frame-by-frame video image data on bids, as shown in Figure 1. These bid data are paired with scraped data from the auction houses' websites to match the lot and its characteristics such as the estimate and description.

Our dataset additionally provides a lower bound on the number of bidders by

Category	Subcategory	Location	Count
Art	Chinese Art	New York	112
Art	Impressionist/20th/21st Century Art	Hong Kong	636
Art	Impressionist/20th/21st Century Art	Las Vegas	9
Art	Impressionist/20th/21st Century Art	London	708
Art	Impressionist/20th/21st Century Art	New York	1208
Art	Impressionist/20th/21st Century Art	Shanghai	55
Art	Old Masters	London	102
Art	Old Masters	New York	134
Others (e.g. Jewelry)			542
Total			3506

Table 1: Our Art Auction Dataset



Lot 38, Christie's Hong Kong, Nov 2022

Lot 117, Sotheby's New York, Nov 2022

Figure 1: Example screenshots taken from Christie's and Sotheby's YouTube live stream auctions. Red boxes enclose image areas where important data can be collected, such as the current bids.

using the audio transcript from the auction videos. In a live Christie's or Sotheby's auction, there are exactly four sources of bids: (i) telephone bids, (ii) live bids, (iii) absentee bids, and (iv) online bids, as shown in Figure 2. A close lower bound on the true number of bidders can be constructed by accurately capturing a certain unique number of bidders in each of these four sources of bids. For example, when telephone bids are accepted, auctioneers typically call out specific names of colleagues, each of whom corresponds to exactly one bidder per auction lot. In the top-right of Figure 2, the Christie's auctioneer references Olivier Camu, another employee at Christie's who is putting in telephone bids on behalf of an unknown buyer. Similarly, for live bids, we capture positional references such as "to the right", while for absentee bidders

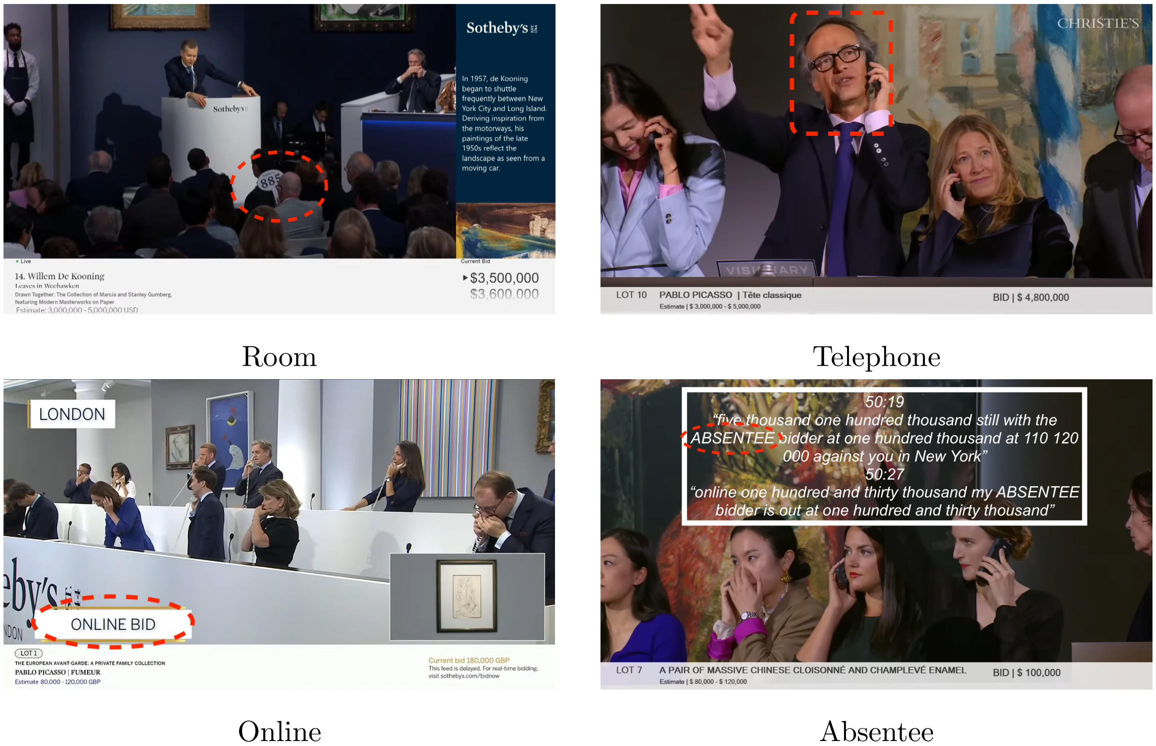


Figure 2: The four types of bidders possible in a Sotheby's or Christie's auction. Dotted red circles indicate the bidder's location. The absentee bidder is captured through the audio transcript.

and online bidders we capture terms such as "absentee" and "online". Our complete methodology is elaborated upon in the Appendix.

Our approach allows us to capture as many unique telephone bidders, one unique online bidder, one unique absentee bidder, and up to 5 unique in-person bidders. However, it is unclear how exactly close the observed lower bound on the number of bidders is to the true N. In 2021, telephone bids made up 42% of winning bids in Christie's live auctions versus 7% in the saleroom[2]. While the winning bid is not completely representative of all bids, the statistic hints at the informativeness of our lower bound. The distribution of the lower bound on the number of bidders is shown in Figure 3.

The final cleaned dataset also scales each recorded bid according to buyer's premiums, which are payable by the successful buyer of an item at an auction based on the hammer price of a lot sold. Table 8 summarizes the Buyer's Premium Schedule,

[2]Source: Christie's website at this link.

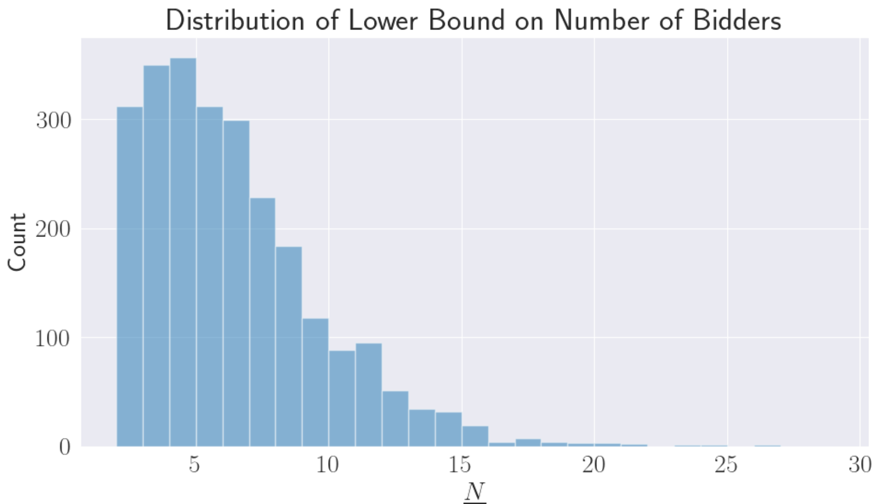


Figure 3: Distribution of lower bound on number of bidders across all auctions.

for both Christie's and Sotheby's, as of March 2023.

2.1 Modern Art Sold in New York City

Our dataset contains auctions of art across multiple subcategories. To take into account how auction characteristics might affect the price, we bin the data into smaller sub-sample groups . Consider the following functional determinants of price:

$$P = f(V, N, \Xi),$$

where V is the vector of bidder valuations, N is the number of bidders, and Ξ is the vector of covariates. Here, Ξ may include the following:

1. **Category of Art**, e.g. Modern Art vs. Chinese Art.
2. **Location of Auction**, e.g. New York, Paris, London, Shanghai.
3. **Final Transaction Price**.

Based on the possible realizations of Ξ, we create sub-sample groups of auctions, grouping the auctions by category and location (as in table 1). We then specifically focus our subsequent econometric analysis on Modern Art sold in New York City, because of the economic significance of the art sold and the largest number auctions held there relative to all other locations, which is also reflected in our dataset. We further filter to the auctions with transaction price range between \$1.0M and \$10.0M.

Summary statistics for these auction lots are shown in Table 2.

Variable	**Median**	**Mean**	**Std**	**Min-Max**
Transaction Price	1.15	1.49	1.37	0.37 - 15.75
2nd-Highest Bid	1.10	1.42	1.31	0.35 - 14.96
Number of Bidders	5.00	5.96	2.99	2.00 - 20.00
Low Est. Relative to High Est.	0.67	0.68	0.05	0.48 - 0.84
Number of Bids	10.00	11.67	8.12	2.00 - 76.00
Number of Auction Lots		685		

Table 2: Summary Statistics for Modern Art Auction Lots in New York City (July 2020 - July 2024), with transaction price range between \$1.0M and \$10.0M.

3 Identification and Estimation

Our main objective is choose the reserve price that maximizes the seller's expected profit. As we discuss below, the expected profit function, or sharp bounds on it, can typically be written as a linear functional of the marginal distributions of the two highest valuations. Since the auction ends before the highest-valuation bidder gets to reveal it, the main identification problem is to obtain a good lower bound on the disribution of the highest valuation.

3.1 Bidding Behavior and Information Structure

Here, the theoretical framework is introduced. Let N (a random variable) denote the number of bidders in an auction and let n denote a value in the support of N. In an auction with N bidders, let $V_1, \ldots, V_N$ denote bidders valuations, and $B_1, \ldots, B_N$ denote their bids. Let $V_{1:N}, \ldots, V_{N:N}$ and $B_{1:N}, \ldots, B_{N:N}$ denote ordered valuations and bids, correspondingly. Finally, let $F_{j:N}$ and $G_{j:N}$ denote the distributions of $V_{j:N}$ and $B_{j:N}$, correspondingly, for each $j = 1, \ldots, N$.

We maintain the following assumptions. Assumption 3.1 is due to Haile and Tamer (2003) while the rest are modifications of assumptions from Aradillas-López, Gandhi, and Quint (2013).

Assumption 3.1 (Bidding strategies)**.**

1. *Bidders do not bid above their valuation:* $V_i \geqslant B_i$.

2. *Bidders do not give up:* $V_{N-1:N} \leqslant B_{N:N}$.

In English auctions, the above are mild and natural restrictions on bidders' behavior. They do not uniquely specify the bidding functions and, for each realization of the valuations, lead to a set of bids compatible with the model.

Assumption 3.2 (Information structure)**.** *The joint distribution of valuations, F, belongs to the set of distributions $\mathcal{F}$ satisfying the following conditions:*

1. *Valuations are symmetric:* $(V_1, \dots, V_N) \overset{d}{=} (V_{\pi(1)}, \dots, V_{\pi(N)})$, *for any permutation* $\pi : \{1, \dots, N\} \to \{1, \dots, N\}$.

2. *Valuations satisfy:* $F_{N:N} \geqslant g(F_{N-1:N})$ *for some known function* $g : [0, 1] \to [0, 1]$ *such that* $g(x) \leqslant x$.

The first part of the assumption implies that bidders' identities are not informative and it suffices to consider ordered valuations $V_{1:N}, \dots, V_{N:N}$. The second part of the assumption addresses the main identification problem in English auctions, which is to lower bound the distribution of the largest valuation, $F_{N:N}$. Since the auction ends before the bidder with the highest valuation exhausts her bidding power, the biding data alone cannot provide a non-trivial lower bound on $F_{N:N}$. Thus, the bound must follow from assumptions on the information structure. The specific form of the function g depends on the type of dependence between valuations. The requirement $g(x) \leqslant x$ simply ensures that the lower bound is valid, since $F_{N-1:N} \geqslant F_{N:N}$ must hold.

For example, Aradillas-López, Gandhi, and Quint (2013) assume that valuations are symmetric and non-negatively dependent in the sense that

$$f(k) = P(V_i \leqslant v \,|\, \#\{j \neq i : V_j \leqslant v\} = k)$$

is non-decreasing in k. Under this assumption, they show that

$$F_{N:N} \geqslant \phi_{N-1:N}(F_{N-1:N})^N,$$

where $\phi_{N-1:N} : [0,1] \to [0,1]$ is a known, strictly increasing function, mapping the distribution of the second largest order statistic of an i.i.d. sample to the parent marginal distribution.

Assumption 3.3. *The transaction price in an auction is the greater of the reserve price and the second-highest bidder's willingness to pay.*

This is the exact outcome when bidders play according to the equilibrium strategy in a theoretical ascending button auction, but also holds approximately if bidders do not "jump bid" at the end of the auction. It is a stricter version of Assumption 3.1, applied only the second highest valuation.

Figure 4 demonstrates that in our sample, 96.2% of final bids are within the stipulated 10% increment from the second highest bid. Figure 5 additionally shows the closeness between the distributions of the highest and second highest bids. In the appendix, we provide an alternative approach to relax this assumption on the second-highest bidder's valuation and instead bound her valuations using Assumption 3.1. The results do not change significantly.

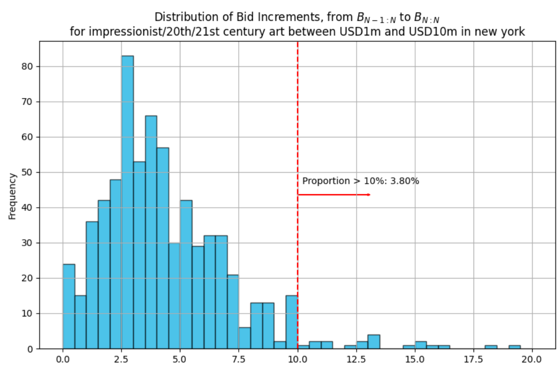


Figure 4: Bid Increments are mostly within Christie's and Sotheby's stipulated 10%.

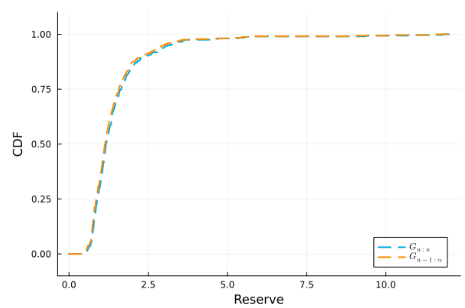


Figure 5: Empirical CDF-s of the highest and second highest bids.

3.2 Identification

In this section, we detail a partially identified model on expected profit that requires only a lower bound on the number of bidders instead of the exact number of bidders.

Letting r denote a reserve price, and v_0 denote the value of the unsold lot to the seller, the profit is given by:

$$\pi(r) = (r - v_0) \cdot \mathbb{1}(V_{N-1:N} \leqslant r, V_{N:N} > r) + (V_{N-1:N} - v_0) \cdot \mathbb{1}(V_{N-1:N} \geqslant r).$$

Taking expectations conditional on $N = n$ and rearranging:

$$\mathbb{E}[\pi(r)|N = n] = \int_0^{+\infty} \max\{r,v\} dF_{n-1:n}(v) - v_0 - F_{n:n}(r)(r - v_0). \tag{1}$$

Therefore, to study optimal reserve prices, it suffices to identify or bound the distributions $F_{n:n}$ and $F_{n-1:n}$.

Identification of $F_{n-1:n}$ follows from assumption 3.3, giving $F_{n-1:n}(v) = G_{n:n}(v)$. For identification of $F_{n:n}$, define the strictly increasing differentiable function $\phi_{i:n}(H)$: $[0,1] \to [0,1]$ as the implicit solution to $H = \frac{n!}{(n-i)!(i-1)!} \int_0^{\phi} s^{i-1}(1-s)^{n-i} ds$. Then by assumption 3.2 and using the Aradillas-Lopez et al (2013) lower bound, the bounds for $F_{n:n}$ are given by:

$$\phi_{n-1:n}(G_{n:n}(v))^n \leqslant F_{n:n}(v) \leqslant G_{n:n}(v).$$

Ponomarev (2022) showed that these bounds for $F_{n:n}(v)$ are sharp. It follows that the sharp bounds[3] for the profit function for $r \geqslant v_0$ conditional on $N = n$ are:

$$\begin{aligned}
\mathbb{E}[\pi(r)|N = n] &\geqslant \int_0^{\infty} \max\{r,v\} dG_{n:n}(v) - v_0 - G_{n:n}(r)(r - v_0),\\
\mathbb{E}[\pi(r)|N = n] &\leqslant \int_0^{\infty} \max\{r,v\} dG_{n:n}(v) - v_0 - \phi_{n-1:n}(G_{n:n}(r))^n(r - v_0).
\end{aligned}$$

We now generalize these bounds so that they are unconditional on the number of bidders. Let $F_{\mathcal{I}}$ and $F_{\mathcal{II}}$ denote the unconditional CDFs of $V_{n:n}$ and $V_{n-1:n}$ correspondingly. Then, the general form of the profit equation in equation 1 unconditional

[3]Note that the bounds only hold for $r \geqslant v_0$. For $r < v_0$, the bounds are switched:

$$\begin{aligned}
\mathbb{E}[\pi(r)|N = n] &\geqslant \int_0^{\infty} \max\{r,v\} dG_{n:n}(v) - v_0 - \phi_{n-1:n}(G_{n:n}(r))^n(r - v_0),\\
\mathbb{E}[\pi(r)|N = n] &\leqslant \int_0^{\infty} \max\{r,v\} dG_{n:n}(v) - v_0 - G_{n:n}(r)(r - v_0).
\end{aligned}$$

A discussion is provided in the Appendix on these bounds.

on N is:

$$\mathbb{E}[\pi(r)] = \int_0^{+\infty} \max\{r,v\} dF_{\mathcal{II}}(v) - v_0 - F_{\mathcal{I}}(r)(r - v_0). \tag{2}$$

We introduce the following theorem to bound the expected profit in equation 2 for a scenario where we only observe a lower bound on the number of bidders.

Theorem 1 (Bounds on Expected Profit Unconditional on N)**.** *Let $\underline{n}$ be the minimum number of bidders in the entire auction population. Further let $G_{\mathcal{I}}$ and $G_{\mathcal{II}}$ denote the unconditional CDFs of $B_{n:n}$ and $B_{n-1:n}$ correspondingly. Under our assumptions, the sharp bounds for the expected profit unconditional on N are:*

$$\mathbb{E}[\pi(r)] \geqslant \int_0^{+\infty} \max\{r,v\} dG_{\mathcal{II}}(v) - v_0 - G_{\mathcal{I}}(r)(r - v_0),$$
$$\mathbb{E}[\pi(r)] \leqslant \int_0^{+\infty} \max\{r,v\} dG_{\mathcal{II}}(v) - v_0 - \phi_{\underline{n}-1:\underline{n}}(G_{\mathcal{I}}(r))^{\underline{n}}(r - v_0).$$

Proof. By our assumptions, for each n and for all v,

$$\begin{aligned} F_{n-1:n}(v) &= G_{n:n}(v), \\ \phi_{n-1:n}(G_{n:n}(v))^n \leqslant F_{n:n}(v) &\leqslant G_{n:n}(v). \end{aligned}$$

Taking expectations in the first line, we obtain:

$$F_{\mathcal{II}}(v) = G_{\mathcal{I}}(v).$$

For the second line, note that the function $f(t,n) = \phi_{n-1:n}(t)^n$ is increasing in n for all t and is convex in t for all n (proofs in Appendix). Therefore, for each $n \geqslant \underline{n}$,

$$\phi_{\underline{n}-1:\underline{n}}(G_{n:n}(v))^{\underline{n}} \leqslant F_{n:n}(v) \leqslant G_{n:n}(v),$$

and by Jensen's inequality,

$$\phi_{\underline{n}-1:\underline{n}}(G_{\mathcal{I}}(v))^{\underline{n}} \leqslant F_{\mathcal{I}}(v) \leqslant G_{\mathcal{I}}(v).$$

■

Theorem 1 is useful because it allows us to bound the profit function while only knowing a lower bound on the number of bidders and not the exact number. An accompanying benefit is that it also pools together a larger amount of data, that is, all auctions with at least $\underline{n}$ bidders. Observe that the lower bound on profit is only changed through pooling together data involving higher number of bidders than $\underline{n}$, while the upper bound is in effect less tight due to the function $f(t,n) = \phi_{n-1:n}(t)^n$ being increasing in n for all t. Theorem 1 is especially relevant to our empirical application because we only observe a close lower bound to the true number of bidders rather than an exact number of bidders from the auction videos.

3.3 Estimation and Inference

Let $T_{\underline{n}}$ be the total number of auctions in the sample with at least $\underline{n}$ bidders, h the smoothing parameter for a Gaussian kernel $\mathcal{K}$, and $b^i_{\mathcal{II},\underline{n}}$, $b^i_{\mathcal{I},\underline{n}}$ the second highest bid and highest bid in the i^{th} auction with at least $\underline{n}$ bidders respectively. Further let $c : \mathbb{R}_+ \to \mathbb{R}_+$ be a scaling function of bids to account for the auction house buyer's premium.

We estimate the profit bounds using a Kernel Density Estimator, with the probability density of the highest order statistics being:

$$\hat{g}_{\mathcal{I},\underline{n}}(v) = \frac{1}{hT_{\underline{n}}} \sum_{i=1}^{T_{\underline{n}}} \mathcal{K}\left(\frac{v - c\left(b^i_{\mathcal{I},\underline{n}}\right)}{h}\right).$$

Further define the CDF $\hat{G}_{\mathcal{I},\underline{n}}$ corresponding to the probability density $\hat{g}_{\mathcal{I},\underline{n}}$. Then, the pointwise estimated bounds are:

$$\begin{aligned}
\hat{\pi}_{\underline{n}}(r) &\geqslant \int_0^\infty \max\{r,v\}\hat{g}_{\mathcal{I},\underline{n}}(v)dv - v_0 - \hat{G}_{\mathcal{I},\underline{n}}(r)(r - v_0), \\
\hat{\pi}_{\underline{n}}(r) &\leqslant \int_0^\infty \max\{r,v\}\hat{g}_{\mathcal{I},\underline{n}}(v)dv - v_0 - \phi_{\underline{n}-1:\underline{n}}\left(\hat{G}_{\mathcal{I},\underline{n}}(r)\right)^{\underline{n}}(r - v_0).
\end{aligned}$$

For kernel density estimation, the bandwidth h is determined by the Improved Sheather-Jones (ISJ) method as in Botev, Grotowski, and Kroese (2010) to reduce out of sample prediction error. It is chosen for its ability to fit multimodal data much better than the original Silverman rule. ϕ is computed using the Powell (1964) conjugate direction numerical optimization method.

95% pointwise confidence intervals (CI) are constructed analytically and applied to the kernel density estimates. Let $\underline{\hat{\pi}}_{\underline{n}}(r)$ and $\hat{\bar{\pi}}_{\underline{n}}(r)$ be the estimated lower and upper bounds on profits respectively. Let $\hat{\underline{\sigma}}_{\underline{n}}(r)$ and $\hat{\bar{\sigma}}_{\underline{n}}(r)$ be the standard deviation of the lower and upper bounds on profit computed using the Delta method. This involves the random vector $\binom{\max\{r, B_{\mathcal{I},\underline{n}}\}}{\mathbb{1}(B_{\mathcal{I},\underline{n}} \leq r)}$ for both the lower and upper bounds on profit.

Following Imbens and Manski (2004) and Stoye (2009), the CI is computed using:

$$\mathrm{CI}_{1-\alpha}(\pi_{\underline{n}}(r)) = \left[\hat{\underline{\pi}}_{\underline{n}}(r) - c_\alpha \cdot \frac{\hat{\underline{\sigma}}_{\underline{n}}(r)}{\sqrt{T_{\underline{n}}}}, \hat{\bar{\pi}}_{\underline{n}}(r) + c_\alpha \cdot \frac{\hat{\bar{\sigma}}_{\underline{n}}(r)}{\sqrt{T_{\underline{n}}}}\right],$$

where c_α solves

$$\Phi\left(c_\alpha + \frac{\sqrt{T_{\underline{n}}}\left(\hat{\bar{\pi}}_{\underline{n}}(r) - \hat{\underline{\pi}}_{\underline{n}}(r)\right)}{\max\{\hat{\underline{\sigma}}_{\underline{n}}(r), \hat{\bar{\sigma}}_{\underline{n}}(r)\}}\right) - \Phi(-c_\alpha) = 1 - \alpha,$$

and Φ is the standard normal cumulative distribution function, $T_{\underline{n}}$ is the number of auctions in the sample, α is the significance level.

4 Selecting a Single Optimal Reserve Price

The goal of the auction house is to set a reserve price to maximize expected profit from an auction. Under reasonably weak assumptions, the expected profit function, and therefore the optimal reserve price, is only partially identified, and thus the target is ambiguous. We must therefore make a decision amidst this statistical ambiguity.

Formally, let $\pi(\cdot)$ denote the expected profit function, Π denote the sharp identified set for $\pi(\cdot)$, and $\mathcal{R}$ denote the sharp identified set for the optimal reserve price. The three most popular methods for resolving the ambiguity are Bayesian, maxmin, and minimax regret.

The Bayesian approach is to assume a subjective prior Q over Π and solve:

$$\max_{r \in \mathcal{R}} \int_\Pi \pi(r) dQ(\pi).$$

However, due to lack of identifying information, one can hardly formulate a reasonable prior, and it is easy to verify that by trying different priors one can recover any point $r^* \in \mathcal{R}$ as a solution to the above problem. So, conceptually, the Bayesian approach cannot be helpful in the present setting.

The maxmin approach is to solve:

$$\max_{r\in\mathcal{R}} \min_{\pi\in\Pi} \pi(r) = \max_{r\in\mathcal{R}} \underline{\pi}(r)$$

which amounts to maximizing the sharp lower bound on the profit function. Conceptually, this corresponds to setting the reserve price cautiously, which may not align with the goals of the auction house. Indeed, if the lot is unsold, the marginal cost associated with organizing its resale is likely negligible, compared to the selling price. Practically, as discussed already in Aradillas-López, Gandhi, and Quint (2013), in the absense of strong distributional assumptions, the lower bound $\underline{\pi}$ is monotonically decreasing after v_0. In this case, the solution will be to set $r = v_0$, which is not particularly insightful.

The minmax regret approach is to solve:

$$\min_{r\in\mathcal{R}} \max_{\pi\in\Pi} \{\pi(r^*_\pi) - \pi(r)\},$$

where r^*_π denotes the optimal reserve price under profit function π. Defining the functionals $\phi^*(\pi) = \pi(r^*_\pi)$ and $\phi_r(\pi) = \pi(r)$, the above problem can be written as

$$\min_{r\in\mathcal{R}} \max_{\pi\in\Pi} |\phi^*(\pi) - \phi_r(\pi)|.$$

That is, choosing r is equivalent to choosing a functional $\phi_r(\cdot)$ that is as close as possible to the unknown optimal functional $\phi^*(\pi)$ with respect to a maximum distance metric. This is in line with the goal of profit maximization in the presence of ambiguity.

A first intuitive approach to solving for the minimax regret solution may be as follows. Given a particular fixed $r \in \mathcal{R}$, we can attempt to characterize a regret-maximizing profit function that attains its lowest point at r and attains its maximum point at $\operatorname{argmax}_{\mathcal{R}} \bar{\pi}$. The following lemma characterizes such a solution.

Lemma 1 (Naive Minimax-Regret Solution for Profit)**.** *Let $\mathcal{C} = [v_0,\bar{v}]$ and assume that $\overline{\pi}$ and $\underline{\pi}$ are continuous on $\mathcal{C}$. Let $\{\pi : \underline{\pi} \leq \pi \leq \overline{\pi}\}$ be the space of continuous profit functions $\pi(r) : \mathcal{C} \to \mathbb{R}$, and suppose any continuous profit function within the bounds are attainable. Then the solution to the minimax-regret problem is* $\operatorname*{argmax}_{r\in\mathcal{C}} \underline{\pi}(r)$.

Proof. See Appendix A. ■

However, the above solution might utilize profit functions that are not in the sharp identified set. For example, a jump from $\underline{\pi}(r)$ to $\overline{\pi}(r+\delta)$ for some small $\delta > 0$ is not always permissible because the CDF-s of the top two order statistics that correspond to such a profit function may not be in the sharp identified set. To visually see this, in the right display in Figure 6, a jump up from $\underline{\pi}(r)$ where $r = 1.5$ to $\overline{\pi}(r+\delta)$ for some small $\delta > 0$ is not possible due to the constraints on the CDF F_1.

4.1 Minimax Regret Solution

In addition to the fixed CDF F_2 of the second highest valuation, suppose sharp bounds on the CDF of the highest valuation are given:

$$F_1^L \leqslant F_1 \leqslant F_1^U.$$

Every feasible profit function within the bounds corresponds to some F_1 within the bounds. To study maximum regret, profit functions that "jump" up or down are of primary interest. The "jumps" cannot be arbitrary as they must be accommodated by the CDF-s subject to the restrictions above. Moreover, CDFs that correspond to the boundary profit functions are very specific. Indeed, using integration by parts, the profit function given in (1) can be rewritten as:

$$\pi(r; F_1, F_2) = \int_r^{+\infty} (1 - F_2(v))dv + (r - v_0)(1 - F_1(r)).$$

Thus, for example, if $\pi(r) = \overline{\pi}(r)$ for some r, $F_1(v)$ must take a minimum value at r. Similarly, if $\pi(r) = \underline{\pi}(r)$ for some r, then $F_1(v)$ must take a maximum value at r. Together with the fact that CDFs must be weakly increasing, the profit functions within the sharp identified set are restricted.

To proceed, we first need to carefully determine the sharp identified set for which the CDF-s F_1 and F_2 lie in. The following lemma shows that there for any ordered random vector, it is possible to construct an exchangeable random vector that is equal to that ordered random vector almost surely.

Lemma 2 (Symmetrization)**.** *Let $X = (X_1, \ldots, X_n) \in \mathbb{R}^n$ be a random vector with an arbitrary joint distribution. Let $X^{1:n} = (X_{1:n}, \ldots, X_{n:n})$ denote the vector of order* *statistics, where $X_{j:n}$ is the j-th smallest of $(X_1, \ldots, X_n)$. There exists a random vector $Y = (Y_1, \ldots, Y_n) \in \mathbb{R}^n$ such that: (1) Y is exchangeable; and (2) $Y^{1:n} = X^{1:n}$ almost surely.*

Proof. Such Y can be constructed as follows. Let Π denote the set of all permutation functions $p : \{1, \ldots, n\} \to \{1, \ldots, n\}$, and π be a random element distributed uniformly in Π. That is, $P(\pi = p) = 1/n!$ for all $p \in \Pi$. Define $Y_j = X_{\pi(j)}$ for $j = 1, \ldots, n$. Then:

$$P(Y_1 \leqslant y_1, \ldots, Y_n \leqslant y_n) = \frac{1}{n!} \sum_{p \in \Pi} P(X_{p(1)} \leqslant y_1, \ldots, X_{p(n)} \leqslant y_n).$$

Since the summation in the right hand side includes all possible events of the form $\{X_{j_1} \leqslant y_1, \ldots, X_{j_n} \leqslant y_n\}$, an arbitrary permutation of $\{y_1, \ldots, y_n\}$ in the above display changes the order of summands but does not affect the value of the sum. Therefore, for any permutation p,

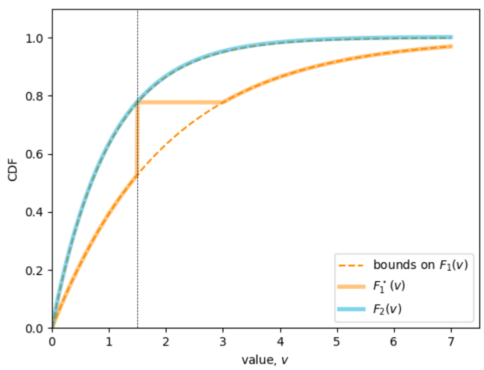
$$P(Y_1 \leqslant y_{p(1)}, \ldots, Y_n \leqslant y_{p(n)}) = P(Y_1 \leqslant y_1, \ldots, Y_n \leqslant y_n).$$

Finally, rearranging the elements of X does not affect the order statistics. That is, for all realizations of π, $Y^{1:n} = X^{1:n}$, so that $P(Y^{1:n} = X^{1:n}) = 1$. ∎

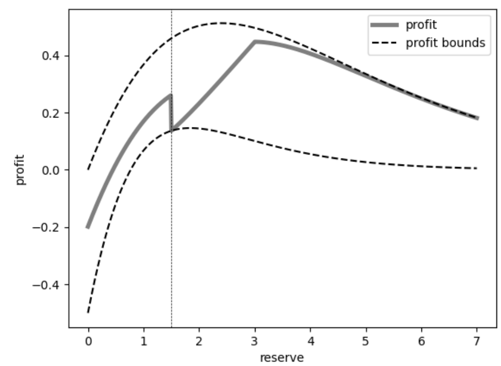
Lemma 2 shows that any CDF F_1 that lies within the bounds can correspond to a vector of valuations that satisfies our Assumption 3.2 on exchangeability, and so it is within the sharp identified set. This leads to the following theorem which characterizes the maximum regret given any choice of reserve r, where the profit function falls within the sharp identified set.

Theorem 2 (Maximum Regret under Equilibrium Play)**.** *Take a reserve $r \geqslant v_0$. Under assumptions 3.1, 3.2, 3.3, the maximum regret at r for any profit function given by F_1 and F_2 contained in the sharp identified set is*

$$\max_v \pi(v; F_1^*, F_2) - \pi(r; F_1^*, F_2)$$



Example CDF-s



Example Corresponding Profits

Figure 6: The pair of CDF-s corresponding to the maximum "Jump Down" (to the left of r) and "Jump Up" (to the right of r) in the profit function at $r = 1.5$.

where F_1^* *is given by*

$$F_1^*(v;r) = \mathbf{1}(v < r)F_1^L(v) + \mathbf{1}(v \in [r, v_1(r)))F_1^U(r) + \mathbf{1}(v \geqslant v_1(r))F_1^L(v);$$

and $v_1(r) = \min\{v : F_1^L(v) = F_1^U(r)\}$.

Proof. See Appendix A. ■

Theorem 2 gives rise to the following algorithm to compute the minimax regret choice of reserve. It is polynomial time in the fine-ness of the optimization search grid for the reserve.

Algorithm 1 (Minimax Regret)**.** *In the following algorithm, optimization is done over a desired grid of possible reserve prices such as within* $[v_0, \bar{v}]$.

1. *Setting the jump at* r*, compute the maximum regret at* r *over the grid* $v \in [v_0,\bar{v}]$*:*

$$R^\star(r) = \max_v \pi(v; F_1^\star, F_2) - \pi(r; F_1^\star, F_2)$$

2. *Select the minimax regret choice of reserve price:*

$$r^\star = \operatorname*{argmin}_{r \in [v_0, \bar{v}]} R^\star(r).$$

Notice that the regrets can be computed without numerical integration. This is done by writing $\int_r^{+\infty}(1 - F_2(v))dv$ as $\mathbb{E}[B_1\mathbf{1}(B_1 \geq r) + r\mathbf{1}(B_1 < r)] - r$, where B_1 is the random variable for the highest bids with distribution G_1.

5 Results

The collected data spans many categories of auctions, but we focus on the category of Modern Art sold in New York City, and specifically those with final transaction price between \$1.0M and \$10.0M. This is due to the high monetary significance of these pieces, and the large number of observations (685).

We estimate the bounds on expected seller profit, $\mathbb{E}[\pi(r)|N \geq n]$, and use those to compute the maximum regret at each reserve. We assume that v_0, the seller's valuation, is equal to the midpoint between the low and high estimates provided by the auction house.

Figure 7 shows our estimates of the bounds on profit in blue dashed lines, as well as the pointwise 95% confidence interval in thinner dashed lines, plotted against reserve price for different values of the lower bound on the number of bidders. The maximum regret at each reserve is plotted as a solid orange line. The y-axes (expected profit and maximum regret) are scaled by the average high estimate for the respective auction samples. We display the graphs for at least 2, 3, 4, 5, 6, 7, and 8 bidders.

The following patterns are apparent from these graphs:

- When there are at least a low number of bidders, such as $N \geq 2$ or $N \geq 3$, the choice of reserve that minimizes maximum regret is significantly higher than Christie's or Sotheby's typical reserve set at the low estimate, which is 0.667 of the high estimate.

- Both the upper and lower bounds on expected profit increase as the number of bidders increases.

- The minimax-regret choice on the optimal reserve price decreases as the number of bidders increases, converging to the seller valuation at $\frac{5}{6}$.

We can further quantify the policy significance of setting the reserve price at our proposed minimax-regret choice and compare it with the auctioneer's reserve price,

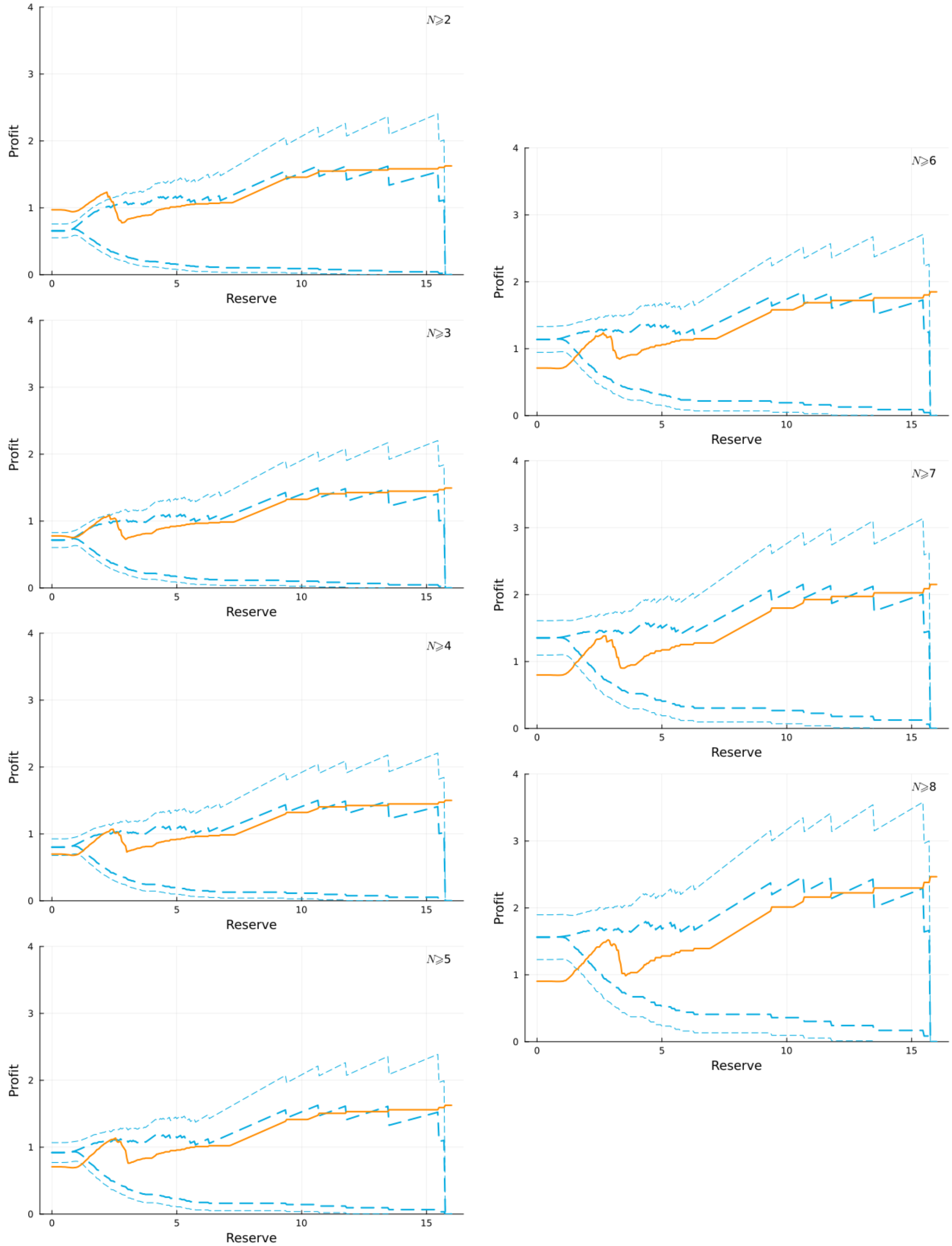


Figure 7: Bounds on expected profit (blue), and maximum regret (orange) against reserve price for Modern Art sold in New York City. 95% CI on the expected profit bounds are shown in light dashed lines.

which cannot exceed the low estimate[4]. Table 3 shows our proposed minimax-regret reserve for each sample, the expected profit range, the expected profit range at the auctioneer's highest possible reserve at the low estimate, and their 95% confidence intervals.

Number of Bidders	$N \geq 2$	$N \geq 3$	$N \geq 4$	$N \geq 5$
Avg High Estimate	\$3.30m	\$3.22m	\$3.11m	\$2.92m
Suggested reserve $r^\star$	2.79	2.95	0.83	0.83
Bounds on $\pi(r^\star)$	[0.30, 1.06]	[0.31, 1.03]	[0.82, 0.82]	[0.93, 0.93]
- 95% CI	[0.21, 1.20]	[0.21, 1.17]	[0.70, 0.94]	[0.79, 1.08]
Bounds on $\pi(r_0)$	[0.66, 0.66]	[0.72, 0.72]	[0.81, 0.81]	[0.92, 0.92]
- 95% CI	[0.56, 0.76]	[0.61, 0.83]	[0.68, 0.93]	[0.78, 1.07]

Number of Bidders	$N \geq 6$	$N \geq 7$	$N \geq 8$
Avg High Estimate	\$2.69m	\$2.61m	\$2.62m
Suggested reserve $r^\star$	0.83	0.83	0.82
Bounds on $\pi(r^\star)$	[1.15, 1.15]	[1.36, 1.36]	[1.57, 1.57]
- 95% CI	[0.95, 1.34]	[1.10, 1.61]	[1.23, 1.90]
Bounds on $\pi(r_0)$	[1.14, 1.14]	[1.35, 1.35]	[1.56, 1.56]
- 95% CI	[0.95, 1.33]	[1.10, 1.61]	[1.23, 1.90]

Table 3: Bounds on expected profit and the 95% CI (in terms of the high estimate) for Modern Art sold in New York City. The suggested reserve $r^\star$ is given by the minimax regret criterion. The reserve r_0 is equal to the low estimate and is the highest reserve an auctioneer will set.

The reserves suggested by the minimax regret criterion for $N \geq 2$ and $N \geq 3$ bidders are significantly higher than the others. To understand why this may be the case, recall that there are two types of jumps in profit when computing the maximum regret: "Jump Up" and "Jump Down". Often, the "Jump Up" regret dominates for smaller r, while the "Jump Down" regret dominates for larger r, and their intersection occurs at the minimax regret. This is exactly the case here. For $N \geq 2$ and $N \geq 3$ bidders, the "Jump Up" regret dominates over a larger domain of r because of the looser upper bound on profit at smaller values of the lower bound on N as discussed in Theorem 1, resulting in larger "Jump Ups" and thus larger regret.

As a note on the tightness of our bounds, relative to previous works such as by

[4]The auctioneer's reserve is at most $\frac{2}{3}$ the high estimate, and we discuss how low it may be in the next section.

Haile and Tamer (2003) and Aradillas-López, Gandhi, and Quint (2013) who applied their identification and estimation approach on timber auctions, our bounds on expected profit are relatively wider, resulting in wider optimal reserve price intervals. While this is a limitation of our approach, our approach requires a much weaker set of assumptions relative to Aradillas-López, Gandhi, and Quint (2013) including not knowing the exact number of bidders and not assuming that the number of bidders and bidders' valuations are independent. Such a tradeoff between the degree of assumptions and width of bounds is typical in partial identifcation. These assumptions would have been otherwise difficult to justify in our context of ascending auctions conducted live in an auction room, and where our data only provides a lower bound on the number of bidders. Despite being wider, our bounds are sufficiently informative to suggest a choice of reserve that guarantees an increase in expected profit in multiple scenarios, such as for the case of $N \geqslant 4$ bidders and above. We will further discuss simulation results in the next section to understand the exact possible increases in profit under the minimax regret criterion.

6 Profit Comparisons with Christie's and Sotheby's Existing Reserve Choice

According to Christie's and Sotheby's terms of sale, the reserve price is confidential and does not exceed a Lot's low estimate. Our data shows that not only is the reserve price bounded above by the low estimate, but the auction houses often set the reserve significantly lower. In Figure 8, we plot all of the auction lots that have final transaction prices less than the low estimate. There are numerous sold lots (in blue) that are significantly below the low estimate. If the auction house's estimates are trusted, such low reserve prices are suboptimal for profit on expectation.

To examine the profit improvements by setting a reserve price under the minimax-regret approach, we conduct a number of simulation experiments. We generate artificial bidding data for $N = 2,3,...,10$ bidders, drawing their valuations from known lognormal distributions. Bids were then generated via the procedure as follows. Given a set of N bidders, an initial bid by a randomly selected bidder is cast at 5% of her valuation. Subsequently, an active bidder is selected at each step randomly from the pool of standing bidders. Her bid follow an incremental rule, where with probabil-

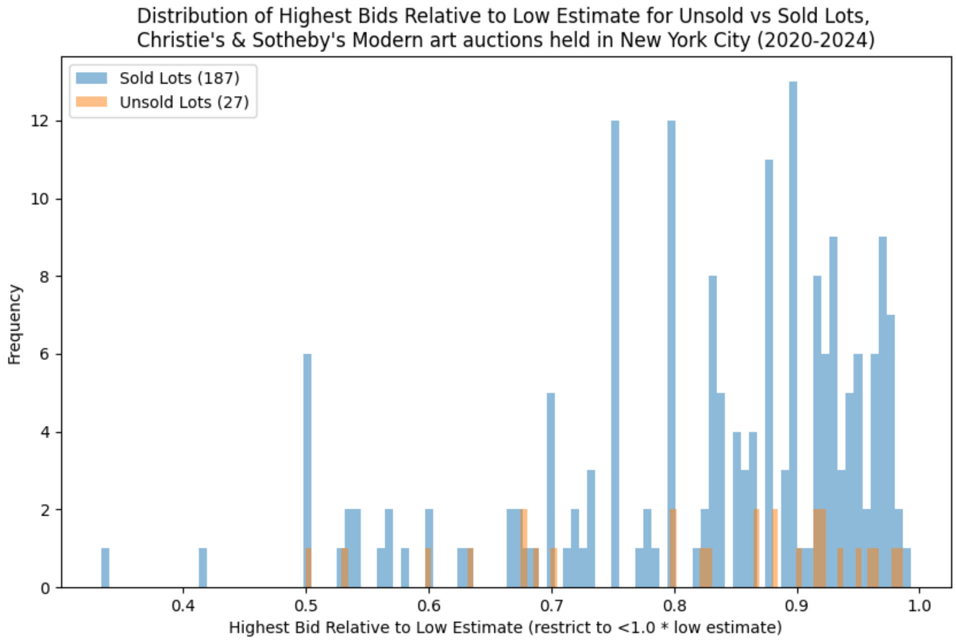


Figure 8: The reserve can be anywhere from 0.3 to 1.0 times the low estimate.

ity $1-\lambda$, the bid is 1.1 of the previous bid. Otherwise, with probability λ the bid is drawn uniformly between the previous bid and 1.1 of the previous bid. This increment pattern is selected following Christie's/Sotheby's auction bidding increment guidelines. If a bidder's valuation is exceeded by the next potential bid, she drops out of the auction. This procedure is repeated until only one bidder remains active.

In our art auction data, the shape of the empirical CDF-s of the highest and second highest bids, as shown in Figure 5, suggest a strong resemblance to a lognormal distribution. We thus use the lognormal distribution as the parent distribution for bidder valuations in our simulations.

In Figure 9, we plot the results of a simulation experiment for bidder valuations drawn from lognormal distribution with mean $\mu = \log\left(\frac{2}{5}\right)$ and standard deviation $\sigma = 1.0$. The parameters are set at $\lambda = 0.2$ with 5,000 auctions, and auctioneer's valuation $v_0 = \frac{5}{6}$, corresponding to the midpoint between the low and high estimates in the typical Christie's and Sotheby's auction. There is an even distribution of the number of bidders across N = 2,3,...,10. We plot the true profit, our bounds on profit, and maximum regrets for $N \geqslant$ 2,3,5,8. Note that the expected value of this lognormal distribution is very close to $\frac{2}{3}$, the low estimate.

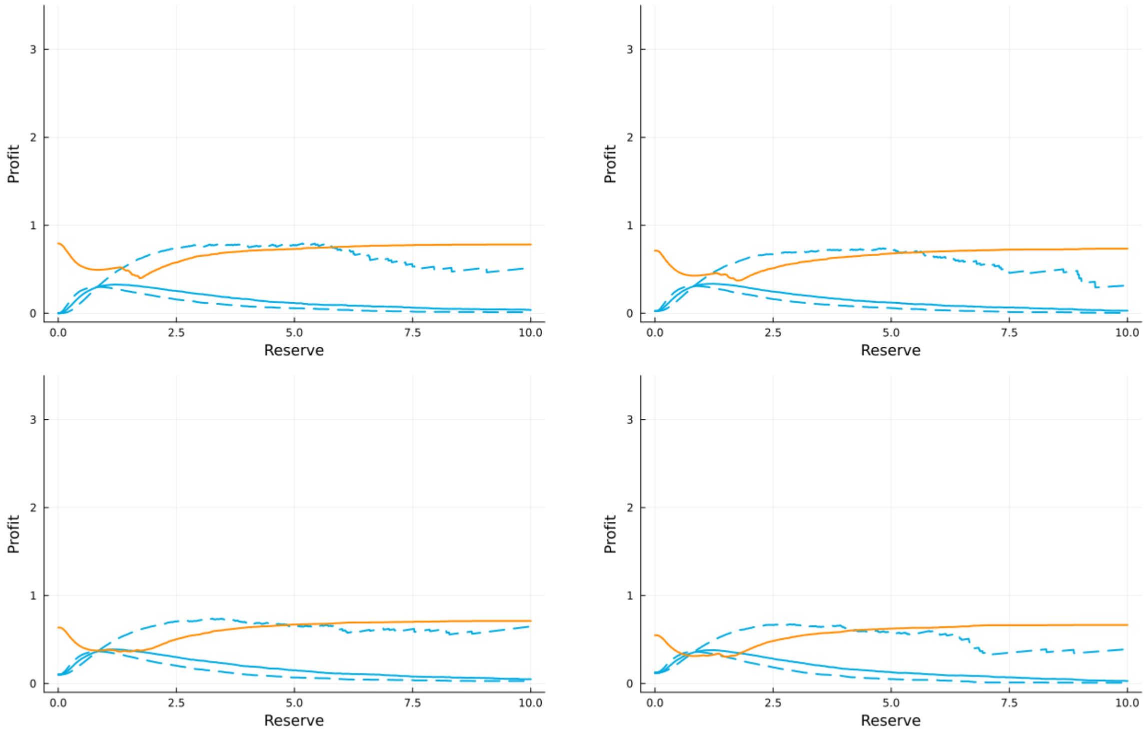


Figure 9: Simulation results of profits where bidder valuations are given by the lognormal distribution with parameters $\mu = \log\left(\frac{2}{5}\right), \sigma = 1.0$. The figures are arranged as $N \geqslant 2,3,5,8$ for top left, top right, bottom left, bottom right respectively.

Table 4 shows the results of the simulation experiments. In the last row of the tables, we compare the profit under the minimax-regret choice of regret with the auctioneer setting a reserve at the low estimate (0.67). Any lower reserve set by them, which the auctioneers sometimes do as shown in Fig 8, results in even lower expected profits. Under these parameters, there is significant improvement to profit when the minimax regret choice is adopted. Our suggested choice of regret, given the parameters, provide significant improvements to profit, from 7.9% to 17.8%. Given that the average high estimate is US$5,670,000, and that there is an average of 32 lots sold in an Impressionist and Modern Art auction in New York City, the expected profit increase per auction is in the range of tens of millions of dollars.

Indeed, the results in Table 4 only hold for one particular type of lognormal distribution. Table 5 displays the percentage increase in profit under a large variety of scenarios, in different lognormal distribution parameters and auTctioneer's original reserve. The range of original reserves from the auctioneer as a base of comparison is chosen based on the fact that it can be anywhere from 0.3 to 1.0 times the low

Number of Bidders	$N \geqslant 2$	$N \geqslant 3$	$N \geqslant 5$	$N \geqslant 8$
Minimax-Regret Reserve	1.72	1.75	1.70	0.83
Expected Profit (Optimal)	0.326	0.337	0.373	0.414
Expected Profit (Minimax-Regret)	0.309	0.312	0.349	0.392
Expected Profit (Max-Min)	0.299	0.311	0.350	0.393
Profit at Low Estimate	0.263	0.276	0.317	0.363
Profit Increase (%)	17.83	12.96	10.04	7.89

Table 4: Simulated Outcomes under Lognormal $\mu = \log\left(\frac{2}{5}\right), \sigma = 1.0$

estimate, as discussed in Figure 8. For all of the figures, we consider the mixture of distributions for auctions with numbers of bidders being $N = 2,3,...,10$, thus setting the minimum number of bidders to 2.

Several insights can be derived from these simulations in Table 5.

1. While the profit increase generally decreases as the μ increases, this decrease is not necessarily monotone. For example, consider the profit increase under minimax regret for $r_0 = 1.0, \mu = 0.325$ is less than for $r_0 = 1.0, \mu = 0.35$.

2. While the minimax regret solution generally results in higher profit than the maxmin solution, it is possible for the converse to hold. One possible reason is that the upper bound on profit is not tight.

3. The reserve price serves as a "guarantee" of sale price. When bidder valuations are low, there is a large increase in profit by increasing the reserve price. At higher bidder valuation (e.g. $\mu = e$), the increase in profit when setting a higher reserve is much smaller.

4. In all scenarios where the lognormal parameters correspond to expected valuation close to the low estimate $\left(\frac{2}{3}\right)$, such as $(\mu = \log(0.4), \sigma = 1.0)$, or $\left(\mu = \log(0.25), \sigma = \sqrt{2}\right)$, the profit increase is positive and notable.

7 Conclusion

Our paper constructs a large novel dataset on live art auctions from the two largest auction houses in the world and provides a close lower bound on the number of

	$0.3 \times r_0$		$0.5 \times r_0$		$0.8 \times r_0$		$1.0 \times r_0$	
	–	–	–	–	–	–	–	–
$\mu = \log(0.25)$	–	–	–	–	220.6	179.3	56.8	36.6
	173.4	160.3	73.5	65.2	26.6	20.6	14.2	8.7
	–	–	–	–	–	–	715.9	553.9
$\mu = \log(0.275)$	–	–	–	–	147.0	122.0	46.7	31.9
	98.9	92.0	50.3	45.1	20.8	16.6	10.9	7.1
	–	–	–	–	–	–	282.2	291.4
$\mu = \log(0.3)$	–	–	–	–	101.8	84.7	36.8	25.2
	69.7	63.0	39.6	34.1	17.6	13.0	9.9	5.6
	–	–	–	–	–	–	194.6	150.1
$\mu = \log(0.325)$	–	–	1164.5	1093.9	73.2	63.5	26.9	19.8
	63.9	58.0	38.1	33.1	17.4	13.1	9.6	5.6
	–	–	–	–	–	–	141.4	108.5
$\mu = \log(0.35)$	–	–	492.9	442.1	70.8	56.2	29.8	18.7
	50.9	44.7	31.9	26.4	15.6	10.8	9.5	5.0
	–	–	–	–	–	–	106.4	79.9
$\mu = \log(0.375)$	–	–	196.1	215.0	33.7	42.3	8.8	15.8
	40.2	36.0	25.9	22.1	12.5	9.1	7.6	4.4
	–	–	–	–	2611.6	2336.1	86.5	67.6
$\mu = \log(0.4)$	439.1	421.8	141.8	134.0	39.0	34.5	17.7	13.9
	37.8	35.1	24.5	22.0	11.7	9.4	6.5	4.4
	10.2	10.4	9.1	9.3	5.8	5.9	3.0	3.2
$\mu = \log(1.0)$	−0.2	9.0	−1.7	7.3	−4.7	4.1	−6.6	2.1
	5.3	4.4	4.3	3.3	2.7	1.8	1.8	0.9

Table 5: Percentage increase in profit under the minimax-regret reserve versus setting the reserve at X times the low estimate, r_0. Each cell in the table contains a 3x2 grid of % increase in profits, where the left column corresponds to that under minimax-regret solution, and the right column corresponds to that under the maxmin solution. The rows correspond to setting the lognormal distribution parameter $\sigma = \sqrt{0.5}, \sqrt{1}, \sqrt{2}$ respectively (in downward order). Sometimes, the profit under a reserve at $X * r_0$ is negative – in these scenarios, the table entry is marked with "-".

bidders. We use the top two bids and the lower bound on the number of bidders to estimate profit bounds non-parametrically, and propose an algorithm to solve the minimax-regret decision problem applied to this partially-identified model. Using this approach, we find that the auction houses' practice of setting the reserve price at the low estimate is suboptimal for Modern Art auctions in New York City, and propose the minimax-regret choice of reserve price which can significantly increase profit. According to our simulations, we expect a profit change of at least 6.5% and up to 39.0% when choosing the minimax-regret reserve price instead of the auctioneer's typical reserve price of the low estimate.

A key innovation in our econometric analysis was freeing the need to know the exact number of bidders, instead only relying on a lower bound on the number of bidders. In our data where we do not have information on the exact number of bidders, this weaker form of partial identification was critical to provide a justified estimate on profit bounds. We also proposed a novel approach to solving the minimax regret decision problem and advocate for this approach in the art auction application due to its interpretation aligning with the goal of profit maximization.

One possible fruitful area of future research is to consider the effect of tightening the lower bound on the CDF of the highest valuation by imposing a minimum dependence assumption among bidders' valuations. Our current lower bound corresponds to the case where bidder valuations are i.i.d., which is highly unlikely in the context of art auctions. Doing so will also result in a smaller sharp identified set, improving the minimax regret results and lead to a more informed reserve price recommendation.

Appendix A. Proofs

Proof that the function $f(t) = \phi_{n-1:n}(t)^n$ is convex in t. Observe that $\phi_{n-1:n}^{-1}(t) = nt^{n-1} - (n-1)t^n$, and the inverse $f^{-1}(t) = \phi_{n-1:n}^{-1}(t^{1/n})$. Thus, the inverse $f^{-1}(t) = nt^{\frac{n-1}{n}} - (n-1)t$. It is straightforward to verify that $f^{-1}(t)$ is increasing and strictly concave on [0,1], so that $f(t)$ is strictly convex in t. ■

Proof that the function $f(t,n) = \phi_{n-1:n}(t)^n$ is increasing in n for $n \geq 2$. Write $\phi_{n-1:n}(t) = \phi$ in this proof for brevity. Consider the identity for differentiation, $\left(\alpha(n)^{\beta(n)}\right)'_n = \alpha(n)^{\beta(n)}\beta'(n)\log\alpha(n) + \alpha(n)^{\beta(n)-1}\beta(n)\alpha'(n)$. Using the identity, the derivative is $(\phi^n)'_n = \phi^n \log\phi + n\phi^{n-1}\phi'_n$. For the derivative ϕ'_n, first recall that $t = n\phi^{n-1} - (n-1)\phi^n$. Implicitly differentiating using the identity and then rearranging gives $\phi'_n = \frac{\phi^n - \phi^{n-1} - t\log\phi}{n(n-1)\phi^{n-2}(1-\phi)}$. Plugging this in and some algebra then gives

$$(\phi^n)'_n = \frac{\phi}{(n-1)(1-\phi)}\left(\phi^n - \phi^{n-1} - \phi^{n-1}\log\phi\right).$$

The fraction is clearly positive, while it is easy to verify that $(\phi^n - \phi^{n-1} - \phi^{n-1}\log\phi)$ is positive by Taylor expansion of $\log\phi$. ■

Proof of Lemma 1. First, it is obvious that taking any $\hat{r} \in \underset{r\in\mathcal{R}}{\operatorname{argmax}}\,\overline{\pi}(r)$, for some given $p \in \mathcal{R}$, where $p \neq \hat{r}$, a function $\pi^*(r) : \mathcal{R} \to \mathbb{R}$ with $\pi^*(\hat{r}) = \underline{\pi}(\hat{r})$ and $\pi^*(p) = \overline{\pi}(p)$ maximizes regret. It then follows that the max regret for some $p \in \mathcal{R}$ where $p \neq \hat{r}$ is $\overline{\pi}(\hat{r}) - \underline{\pi}(p)$.

Next, consider the scenario of choosing a point $p \in \mathcal{R}$ such that $p = \hat{r} \in \operatorname{argmax}_{\mathcal{R}}\overline{\pi}(r)$. In this scenario, the sup regret at $\hat{r}$ is

$$\sup_{\pi\in\Pi}\left\{\max_{d\in\mathcal{R}}\{\pi(d)\} - \pi(\hat{r})\right\} = \overline{\pi}\left(\hat{r}\right) - \underline{\pi}\left(\hat{r}\right).$$

To see this, consider the following 2 cases:

1. **Case 1** is when there is only one max point for both $\overline{\pi}$ and $\underline{\pi}$ (i.e. $|\operatorname{argmax}_{r\in\mathcal{R}}\underline{\pi}(r)| = |\operatorname{argmax}_{r\in\mathcal{R}}\overline{\pi}(r)| = 1$) and their argument $\hat{r}$ happens to be the same. First we show that $\forall\epsilon > 0$, $\exists\pi^* \in \Pi$ such that $\text{regret}_{\pi^*}(\hat{r}) \equiv \max_{d\in\mathcal{R}}\pi^*(d) - \pi^*(\hat{r}) > \overline{\pi}(\hat{r}) - \underline{\pi}(\hat{r}) - \epsilon$.
   Pick any $\epsilon > 0$. By the continuity of $\overline{\pi}(.)$, there exists a $\delta > 0$ such that for some $t \in \mathcal{R}$, $|\hat{r}-t| < \delta \implies \overline{\pi}(\hat{r})-\overline{\pi}(t) < \epsilon$. Pick such a δ and a corresponding $t \neq \hat{r}$ where $|\hat{r}-t| < \delta$. It is possible to construct a $\pi^* : \mathcal{R} \to \mathbb{R}$ such that $\pi^*(\hat{r}) = \underline{\pi}(\hat{r})$ and $\pi^*(t) = \overline{\pi}(t)$. Then the regret at $\hat{r}$ is,

$$\begin{aligned}\text{regret}_{\pi^*}(\hat{r}) &= \overline{\pi}(t) - \underline{\pi}(\hat{r})\\ &> \overline{\pi}(\hat{r}) - \underline{\pi}(\hat{r}) - \epsilon\end{aligned}$$

   At the same time, it is impossible for the regret to be $\geq \overline{\pi}(\hat{r}) - \underline{\pi}(\hat{r})$ since it is impossible to construct a continuous π^* to satisfy this. So the sup regret at $\hat{r}$ is $\overline{\pi}(\hat{r}) - \underline{\pi}(\hat{r})$.

2. **Case 2** is any other situation than Case 1. An alternate way to interpret this is that for any point $r \in \mathcal{R}$, it is always possible to pick another $\hat{r} \neq r$, where $\hat{r} \in \operatorname{argmax}_{r\in\mathcal{R}}\overline{\pi}(r)$. Then, as shown previously, the max regret follows to be $\overline{\pi}(\hat{r}) - \underline{\pi}(\hat{r})$.

For this scenario, see that at any $p \in \mathcal{R}$ where $p \neq \hat{r}$, the max regret is $\overline{\pi}(\hat{r})-\underline{\pi}(p)$. But $\underline{\pi}(p) \leq \underline{\pi}(\hat{r})$, so

$$\begin{aligned}\max_{\pi\in\Pi}\left\{\max_{d\in\mathcal{R}}\{\pi(d)\} - \pi(p)\right\} &= \overline{\pi}(\hat{r}) - \underline{\pi}(p)\\ &\geq \overline{\pi}(\hat{r}) - \underline{\pi}(\hat{r})\end{aligned}$$

It is clear that in either scenario, the maximum regret is minimized at some $p \in \mathcal{R}$ where p minimizes $\overline{\pi}(\hat{r}) - \underline{\pi}(p)$. This is precisely at $\operatorname{argmax}_{r\in\mathcal{R}}\underline{\pi}(r)$.

■

Proof of Theorem 2. Fix some reserve r. Let $\tilde{r} \neq r$ be some other reserve in the support, and assume that v_0 is always less than both $\tilde{r}$ and r. Let $D(\tilde{r}, r; F_1) = \pi(\tilde{r}; F_1) - \pi(r; F_1)$ be the difference between profits at $\tilde{r}$ and r given fixed a CDF F_1. Observe that this can be written as

$$D(\tilde{r}, r; F_1) = \int_{\tilde{r}}^{r} 1 - F_2(v)dv + \tilde{r} - r + F_1(r)(r - v_0) - F_1(\tilde{r})(\tilde{r} - v_0).$$

Now, the maximum regret optimization problem is

$$\max_{F_1} \max_{\tilde{r} \neq r} D(\tilde{r}, r; F_1)$$

subject to the restrictions on the CDF-s. The objective can be written as the max across $\tilde{r} < r$ and $\tilde{r} > r$:

$$\max \left\{ \max_{F_1} \max_{\tilde{r} < r} D(\tilde{r}, r; F_1), \max_{F_1} \max_{\tilde{r} > r} D(\tilde{r}, r; F_1) \right\}.$$

By direct examination of $D(\tilde{r}, r; F_1)$, the first term $\max_{F_1} \max_{\tilde{r} < r} D(\tilde{r}, r; F_1)$ is attained by

1. maximizing $F_1(r)$,
2. minimizing $F_1(v)$ for all $v < r$.

Both conditions above are satisfied by the CDF F_1^*.

Similarly, by direct examination, the second term $\max_{F_1} \max_{\tilde{r} > r} D(\tilde{r}, r; F_1)$ is attained by:

1. setting $F_1(v) = F_1^U(r)$ for all $r \leq v < v_1(r)$,
2. minimizing $F_1(v)$ for all $v \geq v_1(r)$.

To see why condition 1 holds, suppose to the contrary that $F_1(v)$ is set to a value x between $F_1^L(r)$ and $F_1^U(r)$ for all $r \leqslant v < \min\{v : F_1^L(v) = x\}$. Notice that because CDF-s are bounded by 1, $|\tilde{r} - r| \geqslant |F_1(r)(r - v_0) - F_1(\tilde{r})(\tilde{r} - v_0)|$ in the domain $r \leqslant v < \min\{v : F_1^L(v) = x\}$. The maximum regret under this cannot exceed that under condition 1.

Both of these conditions are also satisfied by the proposed CDF F_1^*. ■

Tables and Figures

Auction Title	URL
Impressionist and Modern Art Evening Sale — New York	link
20th/21st Century Art Evening Sales — Hong Kong	link
20th Century Evening Sale — New York	link
The Collection of Thomas and Doris Ammann Evening Sale — New York	link
The Collection of Anne H. Bass and 20th Century Evening Sale — New York	link
20th/21st Century: London Evening Sale followed by The Art of the Surreal Evening Sale	link
21st Century Evening Sale — New York	link
A Century of Art: The Gerald Fineberg Collection Part I — New York	link
Post-Millennium Evening Sale — Hong Kong	link
20th/21st Century Art Evening Sales — London	link
Rare Watches Including the Property of Michael Schumacher — Geneva	link
Magnificent Jewels — Geneva	link
20th/21st Century Evening Sales — Hong Kong	link
20th/21st Century: London to Paris — Christie's	link
21st Century Evening Sale — New York	link
20th/21st Century Art Auctions — Christie's Hong Kong	link
20th/21st Century: Evening Sale Including Thinking Italian, London — Christie's	link
The Cox Collection and 20th Century Evening Sale — New York	link
20th/21st Century: Shanghai to London	link
The Collection of Thomas and Doris Ammann Evening Sale — New York	link
20th/21st Century Art Evening Sales — Hong Kong	link
20th/21st Century: London to Paris Evening Sales	link
20th/21st Century: London	link
The Ann & Gordon Getty Evening Sale — New York	link

Table 6: Christie's YouTube Data Sources

Auction Title	URL
Hong Kong Contemporary Art Evening Sale (LIVE)	link
LIVE from New York — Modern Evening Auction	link

LIVE from New York — The Now and Contemporary Evening Auctions	link
LIVE from Hong Kong — The Now and Modern & Contemporary Evening Auctions	link
LIVE from London — Modern & Contemporary Evening Auction featuring The Now	link
LIVE from New York — The Now and Contemporary Evening Auctions	link
LIVE from New York — The Modern Evening Auction	link
LIVE from New York — The Emily Fisher Landau Collection: An Era Defined Evening Auction	link
LIVE from London — The Now & Contemporary Evening Auctions	link
LIVE from Hong Kong — The Autumn Sales	link
LIVE from London — Freddie Mercury: A World of His Own Evening Sale	link
LIVE from London — Old Master & 19th Century Paintings Evening Auction	link
LIVE — The Now & Modern and Contemporary Auctions, ft. Face to Face: A Celebration of Portraiture	link
The Mo Ostin Collection Evening Auction & The Modern Evening Auction	link
LIVE from New York — The Now & Contemporary Evening Auctions	link
LIVE from London — The Now and Modern & Contemporary Evening Auctions	link
LIVE from New York — The Masters Week Auctions	link
LIVE from New York — Master Paintings & Sculpture Part I	link
LIVE from New York — The David M. Solinger Collection & Modern Evening Auctions	link
LIVE from Paris — Modernités	link
LIVE from London — The Now & Contemporary Evening Auctions	link
LIVE from Paris — Hôtel Lambert, The Illustrious Collection, Volume I: Chefs-d'oeuvre	link
LIVE from Hong Kong — Modern, Williamson Pink Star & Contemporary Auctions	link
LIVE from London — Old Masters Evening Auction	link
LIVE from London — The Jubilee Auction and Modern & Contemporary Evening Auction	link
LIVE from New York — The Now & Contemporary Evening Auctions	link
LIVE from New York — Modern Evening Auction	link
LIVE from New York — The Macklowe Collection	link
LIVE from New York — Important Watches	link
LIVE from London — Old Masters Evening Sale	link
LIVE From New York — PROUVÉ x BASQUIAT: The Collection of Peter M. Brant and Stephanie Seymour	link
LIVE from New York — Magnificent Jewels	link
LIVE from London — Treasures	link
LIVE from Monaco — KARL, Karl Lagerfeld's Estate Part I	link
LIVE from Edinburgh — The Distillers One of One Whisky Auction	link
LIVE from Paris — Art Contemporain Evening Sale	link

LIVE from Sotheby's New York — The Now & Contemporary Evening Auctions With U.S. Constitution Sale	link
LIVE from Sotheby's New York — Modern Evening Auction	link
LIVE from Sotheby's New York — The Macklowe Collection	link
LIVE from Paris — Past/Forward and Modernités	link
LIVE from Las Vegas: Icons of Excellence & Haute Luxury	link
LIVE from Las Vegas — Picasso: Masterworks from the MGM Resorts Fine Art Collection	link
LIVE from New York — Collector, Dealer, Connoisseur: The Vision of Richard L. Feigen	link
LIVE From Sotheby's London — Richter, Banksy and Twombley lead the Contemporary Art Evening Auction	link
LIVE From Sotheby's Hong Kong — Modern and Contemporary Art Evening Sales	link
LIVE from London — Old Masters Evening Sale	link
LIVE from London: British Art + Modern & Contemporary Auctions	link
LIVE from Hong Kong: Jay Chou x Sotheby's — Evening Sale	link
LIVE From Sotheby's New York — Important Watches	link
LIVE from Sotheby's New York — Magnificent Jewels	link
LIVE from Sotheby's Paris — Important Design: from Noguchi to Lalanne	link
LIVE from Sotheby's New York — Monet, Warhol and Basquiat Lead Marquee Evening Sales	link
LIVE From Sotheby's Hong Kong — Contemporary Art Evening Sale	link
LIVE From Sotheby's Hong Kong — Icons and Beyond Legends: Modern Art Evening Sale	link
LIVE from Sotheby's Impressionist & Modern Art + Modern Renaissance Auctions	link
LIVE from Sotheby's Sales of Important Chinese Art and Chinese Art from the Brooklyn Museum	link
LIVE from Sotheby's: The Collection of Hester Diamond Auction in New York	link
LIVE from Sotheby's Master Paintings & Sculpture Auction in New York	link
LIVE from Sotheby's London Old Masters Evening Sale	link
LIVE from Sotheby's marquee Evening Sales of Contemporary and Impressionist & Modern Art	link

Table 7: Sotheby's YouTube Data Sources

Saleroom Location	Christie's		Sotheby's	
	Threshold	Rate	Threshold	Rate
Hong Kong	≤HK$7.5M	26.0%	≤HK$7,500,000	26.0%
	>HK$7.5M and ≤ HK$50M	20.0%	>HK$7.5M and ≤HK$40M	20.0%
	>HK$50M	14.5%	>HK$40M	13.9%
London	≤£700k	26.0%	≤£800k	26.0%
	>£700,000 and ≤£4.5M	20.0%	>£800k and ≤£3.8M	20.0%
	>£4.5M	14.5%	>£3.8M	13.9%
Paris	≤€700k	26.0%	≤€800k	26.0%
	>€700,000 and ≤€4M	20.0%	>€800k and ≤€3.5M	20.0%
	>€4M	14.5%	>€3.5M	13.9%
New York	≤$1M	26.0%	≤$1M	26.0%
	>$1M and ≤$6M	20.0%	>$1M and ≤$4.5M	20.0%
	>$6M	14.5%	>$4.5M	13.9%
Shanghai	≤¥6M	26.0%	-	-
	>¥6M and ≤¥40M	20.0%	-	-
	>¥40M	14.5%	-	-

Note: This table is accurate as of February 7 2022 for Christie's and February 1 2023 for Sotheby's. In the last 10 years, there are only minor changes to the base rate (i.e. lowest threshold category). These buyer premium thresholds are additive, so final transaction amounts are strictly increasing.
Source: Christie's and Sotheby's Websites.

Table 8: Buyer's Premiums in Christie's and Sotheby's Auctions

Bounds on the profit function below v_0

According to our assumptions, $F_{N-1:N}$ is fixed, while the sharp bounds on the highest valuation $F_{N:N}$ is given by:

$$\phi_{N-1:N}(G_{N:N})^N \leqslant F_{N:N} \leqslant G_{N:N}.$$

Recall that the profit function can be written as:

$$\pi(r) = \int_r^{+\infty} (1 - F_{N-1:N}(v))dv + (r - v_0)(1 - F_{N:N}(r)), \tag{3}$$

so it depends on $F_{N:N}$ through the value of $F_{N:N}(r)$.

For $r > v_0$, uniform bounds on the profit follow easily from stochastic dominance:

$$\underline{\pi}(r) = \int_r^{+\infty} (1 - G_{N:N}(v))dv + (r - v_0)(1 - G_{N:N}(r)), \tag{4}$$

and

$$\overline{\pi}(r) = \int_r^{+\infty} (1 - G_{N:N}(v))dv + (r - v_0)(1 - \phi_{N-1:N}(G_{N:N}(r))^N), \tag{5}$$

For $r < v_0$, first consider pointwise bounds. Equation (3) suggests that to obtain a lower bound, $F_{N:N}(r)$ needs to be minimized. Thus, a pointwise lower bound is given by:

$$\underline{\pi}(r) = \int_r^{+\infty} (1 - G_{N:N}(v))dv + (r - v_0)(1 - \phi_{N-1:N}(G_{N:N}(r))^N). \tag{6}$$

In fact, the lower bounds in (4) and (6) can be "glued" together using a CDF $F_{N:N}$ that jumps up from $\phi_{N-1:N}(G_{N:N})^N$ to $G_{N:N}$ at v_0, thus delivering a *uniform lower bound* on the profit function.

To obtain an upper bound on $\pi(r)$ for $r < v_0$, $F_{N:N}$ should be chosen to maximize $F_{N:N}(r)$. Notice this implies the solution must satisfy $F_{N-1:N} = F_{N:N}$ for all $r \leqslant v_0$. Thus, the uniform upper bound on the profit for $r < v_0$ is given by:

$$\overline{\pi}(r) = \int_r^{+\infty} (1 - G_{N:N}(v))dv + (r - v_0)(1 - G_{N:N}(r)). \tag{7}$$

Notice, however, that the upper bounds in (5) and (7) are incompatible and achieved by different CDF-s. In particular, (7) $<$ (5) for all $r > v_0$ and vice versa for all $r < v_0$.

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