

Identification and Optimal Reserve Prices in Ascending Auctions, with an Application to Art

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Abstract

This paper proposes a novel non-parametric approach to partial identification of the distribution of the highest bidder's value, seller's expected profit, and optimal reserve price in symmetric ascending auctions. Our approach restricts the distribution of bidders' values away from the pure common- and independent private values settings, leading to substantially tighter bounds than those available in the literature, and accommodates an unknown number of bidders, as long as bounds on it are available. Moreover, we formulate and solve the min-max-regret problem for the seller choosing a reserve price while facing ambiguity about the distribution of bidders values. We apply the proposed methodology to a novel dataset comprising bids from more than 3500 live auctions held by the two largest auction houses, Christie's and Sotheby's. For Modern Art in New York City priced between \$1.0M and \$10.0M, we find that setting reserve prices 26% higher would increase the expected seller's profit by at least \$126K per lot or \$3.8M per auction.

Keywords: Nonparametric identification, ascending auctions, partial identification, mini-max regret, unknown number of bidders, correlated values, art auctions.

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1 Introduction

Open ascending auctions are widely used for selling a variety of goods ranging from used cars and timber to wine and art. The two major challenges in empirical analysis of such auctions are unobserved heterogeneity at the auction level, which makes the bidders' values interdependent, and an unknown number of bidders. In this paper, we propose a new non-parametric partial identification approach that addresses these challenges in a transparent and computationally simple manner, and apply it to art auctions.

We make two main assumptions: (i) the bidders are symmetric, and their values depend on a common component (e.g., projected future resale value of an artwork) and independent private components (e.g., consumption value for art); and (ii) the transaction price in an auction is the larger of the reserve price and second-largest bidder's value. Under Assumption (ii), the only unknown component in the seller's expected profit is the marginal distribution of the bidders' largest value. We note that Assumption (i) allows to express the latter as an expectation of a convex function of the distribution of the second-largest value, conditional on the common component. Bounding the distance between this conditional distribution and its' unconditional counterpart, we restrict the dependence between bidders' values away from the pure common- and independent private value settings. This step further allows us to express the bounds on the distribution of the largest value as solutions to a pair of generalized moment problems minimizing/maximizing an expected value of a convex function subject to a constraint on the mean and the variance of the underlying distribution. We solve these problems analytically, obtaining new point-wise sharp bounds on the distribution of the largest value and the seller's expected profit. Next, given the bounds on expected profit, we consider the problem of a seller selecting a reserve price in the presence of ambiguity. To resolve the ambiguity, we advocate for using the min-max-regret criterion, which is in line with the goal of profit maximization. Since a complete description of the sharp identified set for the expected profit is intractable, we formulate and analytically solve two relaxations of the min-max-regret problem. Using empirically-calibrated simulations, we compare their performance with the pessimistic max-min reserve and the status quo of no reserve. Using empirically-calibrated simulations, we find that the proposed reserve may lead to a sizable increase in profit.

Next, we apply the proposed methodology to obtain optimal reserve prices in art auctions. The auction market for art is large: in 2023, the two major art auction houses, Christie's and Sotheby's, reported revenues of \$7.9B and \$6.2B respectively. On November 9 and 10, 2022 alone, the 155-work collection of a deceased technology billionaire Paul Allen sold for \$1.65M towards philanthropy, with the most expensive piece *Les Poseuses, Ensemble*

(*Petite version*), 1888-1890 by Georges Seurat selling for \$149.24M. Besides the challenges mentioned above, empirical studies of art auctions are limited due to data availability. The auction houses are notoriously secretive; their websites only provide basic information about the lot and the final transaction price. The data on bids, the number of bidders, and bidders' identities in past auctions are kept private to the firms. To this end, we collect a large novel dataset comprising more than 3500 auction lots using live-stream YouTube auction videos held by Sotheby's and Christie's between 2020 and 2024. For each lot, we obtain a complete bidding sequence, lower and upper bounds on the number of bidders, and detailed information about the lot, including the low and high estimates of the lot's value, location of the auction and category of art, artists name, period, provenance, and condition report. Applying the methodology developed in the paper to the auctions for modern art in New York City with price range between \$1.0M and \$10.0M, we find that setting reserve prices 26% higher would increase the expected seller's profit by at least \$126K per lot or \$3.8M per auction.

Related Literature. We refer the reader to [Athey and Haile \(2007\)](#) and [Hortaçsu and Perrigne \(2021\)](#) and references therein for the detailed accounts of the broader auction literature and focus only on ascending auctions below.

Early work on non-parametric identification in ascending auctions focused primarily on the independent private value (IPV) settings with known number of bidders. Assuming equilibrium play, [Athey and Haile \(2002\)](#) established point identification of the marginal distribution of bidders' values in the IPV setting. [Haile and Tamer \(2003\)](#) relaxed the equilibrium play assumption, allowing for minimal bid increments and jump bidding, and provided partial identification results for the marginal distribution of valuations and expected profit. Recently, [Chesher and Rosen \(2017\)](#), [Molinari \(2020\)](#), and [Ponomarev \(2022\)](#) provided sharp identification results in such settings, using random set theory. Their results imply that even point-wise sharp bounds on the distribution of valuations are hard to characterize.

With unobserved heterogeneity and interdependent bidders' values, the analysis is more complicated. [Athey and Haile \(2002\)](#) showed that the joint distribution of bidders' values is not identified in the affiliated private values model because, by design, the value of the largest bidder is never observed. [Aradillas-López, Gandhi, and Quint \(2013\)](#) pointed out that if the transaction price in an auction is the larger of the reserve price and the second-largest bidder's value, then certain functionals of interest, such as seller's expected profit or bidder's expected surplus, depend only on the marginal distributions of the two largest values. Under the stated assumption, the distribution of the second-largest value is observed, so the main identification problem is to obtain bounds on the distribution of the largest value.

Assuming that the values are symmetric and positively dependent, the authors derived simple bounds corresponding to the extreme cases of independent private and pure common values. Observing that the bounds may be very wide, the authors proposed to leverage variation in the number of bidders to tighten the bounds, under additional restrictions. In this paper, we do not rely on the variation the number of bidders and, instead, propose to rule out the extreme cases of independent private and pure common values, which are arguably often implausible. Our bounds are point-wise sharp and straightforward to compute.

Several approaches have been proposed to deal with unknown number of bidders. In IPV settings, a popular approach is due to [Song \(2004\)](#). The author noted that the distribution of one order statistic, conditional on a lower order statistic, does not depend on the number of bidders, and used this fact to develop non-parametric identification arguments for the marginal distribution of bidders' values. In turn, [Marra \(2020\)](#) used results on stochastic ordering between the differences of adjacent order statistics, known as sample spacings, to partially identify the structural features of interest. In settings with unobserved heterogeneity, dealing with unknown number of bidders typically requires some additional information. For example, [Freyberger and Larsen \(2022\)](#) relied on observing the seller's profit, [Mbakop \(2017\)](#) and [Luo and Xiao \(2023\)](#) required observing more than two bids, while [Hernndez, Quint, and Turansick \(2020\)](#) required additional information on the distribution of the number of bidders. In contrast, the approach developed in this paper restricts the "magnitude" of unobserved heterogeneity, and only requires the analyst to observe (or set) a lower and upper bounds on the number of bidders.

A few papers considered the problem of a seller selecting a reserve price in the presence of ambiguity about the distribution of bidders' values. [Aryal and Kim \(2013\)](#) proposed using the max-min criterion, and [Jun and Pinkse \(2024\)](#) suggested augmenting it with an entropy regularization. In this paper, we propose using the min-max-regret criterion, which originated [Savage \(1951\)](#) was popularized in econometrics following the work on statistical treatment rules by [Manski \(2004\)](#). Since the sharp identified set for expected profit functions is intractable, we formulate and analytically solve two relaxations. The first relaxation admits all CDF-s of the largest value between the bounds. The second relaxation imposes an intuitive shape restriction on the distribution of bidder's values, under which the CDF of the largest value is a convex transformation of the CDF of the second-largest value. We solve both problems analytically and show that the proposed regrets perform well in simulations.

This paper also contributes to the literature on the auctions of art and wine, including [Ashenfelter \(1989\)](#); [Ashenfelter and Graddy \(2003\)](#); [Beggs and Graddy \(2009\)](#); [Ashenfelter and Graddy \(2011\)](#); [McAndrew, Smith, and Thompson \(2012\)](#), and [Marra \(2020\)](#). Although the analysis have been somewhat restricted by data availability, a common finding in the

literature is that the reserve prices are generally too low. In this paper, we construct a large dataset using a new source of auction data and develop flexible tools for quantitative analysis under relatively weak assumptions. Our empirical results confirm the common finding regarding the low reserves.

The rest of the paper is organized as follows. Section 2 gives a formal setup and presents main identification results; Section 3 discusses the seller’s problem under ambiguity; Section 4 briefly discusses estimation and inference; Section 5 illustrates the proposed approach with simulated data; Section 6 discusses data collection and applies the proposed methodology to art auctions; and Section 7 concludes.

2 Identification in Ascending Auctions

2.1 Environment and Bidding Behavior

Consider an ascending auction with N bidders with symmetric private values and a secret reserve price. Let $V_1, \dots, V_N \in \mathcal{V}$ denote the bidders’ values, $V_{1:N}, \dots, V_{N:N}$ denote the corresponding order statistics, and $F_{j:N}$ denote the distribution of $V_{j:N}$, conditional on N , for each $j = 1, \dots, N$. Let $v_0 \in \mathcal{V}$ be the value of the unsold good to the seller, and $r \in \mathbb{R}_+$ the reserve price. Following Aradillas-López, Gandhi, and Quint (2013), we impose the following assumption on the bidding behavior.

Assumption 2.1 (Transaction Price). *The transaction price is the greater of the reserve price r and the second-highest bidders value $V_{N-1:N}$.*

If bidders play equilibrium strategies in the standard button auction model, Assumption 2.1 holds exactly. In many empirical settings, it holds approximately if bidders do not “jump bid” at the end of the auction. We find this assumption plausible in our dataset and discuss possible relaxations in Section 7. Under Assumption 2.1, the seller’s realized profit is:

$$(r - v_0)\mathbf{1}(V_{N:N} \geq r, V_{N-1:N} < r) + (V_{N-1:N} - v_0)\mathbf{1}(V_{N-1:N} \geq r),$$

so the expected profit, conditional on N , can be expressed as:

$$\pi_N(r) = \int_0^{+\infty} \max(r, v)dF_{N-1:N}(v) - v_0 - (r - v_0)F_{N:N}(r). \quad (1)$$

Importantly, the expected profit only depends on the marginal distributions of the second-largest and largest bidders’ values.

Since the auction ends before the highest bidder gets to fully reveal their value, the main identification problem is to obtain useful bounds on $F_{N:N}(\cdot)$. We start with the following general setting.

Assumption 2.2 (Conditional Independence). (i) *In an auction with N bidders, the values take the form $V_j = g(U, \varepsilon_j)$, for some i.i.d. random variables $\varepsilon_1, \dots, \varepsilon_N \in \mathbb{R}$, random vector $U \in \mathbb{R}^{d_U}$, independent of the ε_j -s, conditional on N , and a measurable function $g : \mathbb{R}^{d_U} \times \mathbb{R} \rightarrow \mathcal{V}$.* (ii) *The map $e \mapsto g(u, e)$ is weakly increasing, for all u .*

Condition (i) states that bidders' values are symmetric and consist of a common component U and independent private components ε_j , so that V_1, \dots, V_N are i.i.d., conditional on U . In the context of art auctions, U may represent, for example, projected future resale value, and ε_j represents individual consumption value. We do not take a stance on whether the bidders observe U or not. Note that Condition (i) implies positive dependence between the values.¹ Specifically, by the Law of Iterated Expectations and Jensen's inequality, for any measurable function $h : \mathcal{V} \rightarrow \mathbb{R}$ and non-empty subset $J \subseteq \{1, \dots, N\}$:

$$\mathbb{E} \left[\prod_{j \in J} h(V_j) \mid N \right] = \mathbb{E} \left[\mathbb{E}[h(V_j) \mid U, N]^{|J|} \mid N \right] \geq \mathbb{E}[h(V_j) \mid N]^{|J|}.$$

In particular, the above implies $\text{Cov}(h(V_i), h(V_j) \mid N) \geq 0$, for all $i \neq j$, for all $h(\cdot)$. Importantly, Condition (i) restricts the sign of the dependence between the bidders' values but not its degree, allowing for the two extreme cases of independent private values and pure common value. In turn, Condition (ii) will be used for tractability in the sequel.

Assumption 2.2 (i) implies simple bounds on $F_{N:N}(\cdot)$ in terms of the observed $F_{N-1:N}(\cdot)$. Consider a function $\phi_N : [0, 1] \rightarrow [0, 1]$ defined implicitly via the relation:

$$u = N\phi_N(u)^{N-1} - (N-1)\phi_N(u)^N. \quad (2)$$

The function $\phi_N(\cdot)$ maps the CDF of the second-largest order statistic out of N i.i.d. draws to the corresponding marginal CDF. It is strictly increasing and smooth on $(0, 1)$.

Proposition 1 (Bounds on $F_{N:N}$ Under Conditional Independence). *Let Assumption 2.2 (i) hold and the function ϕ_N be defined in (2). Then, for all $v \in \mathcal{V}$,*

$$\phi_N(F_{N-1:N}(v))^N \leq F_{N:N}(v) \leq F_{N-1:N}(v).$$

¹By the de Finetti-Hewitt-Savage Theorem, Condition (i) is equivalent to the values V_1, \dots, V_N being part of an infinitely exchangeable sequence. It is more restrictive than finite exchangeability precisely because of the positive dependence, as explained below.

The equality $F_{N:N}(v) = \phi_N(F_{N-1:N}(v))^N$, for all v , corresponds to independent private values, and the equality $F_{N:N}(v) = F_{N-1:N}(v)$, for all v , corresponds to pure common value.

The bounds turn out to be the same as the ones obtained by Aradillas-López, Gandhi, and Quint (2013) under a weaker notion of positive dependence.² We derive the result directly from conditional independence, using Jensen's inequality and convexity of the function $u \mapsto \phi_N(u)^N$, and take these bounds as a starting point of the analysis.

Since Assumption 2.2 (i) does not restrict the degree of dependence between bidders' values, the bounds in Proposition 1 are typically very wide and have drastically different economic implications. At the same time, allowing for pure common or independent private values may not be realistic in many empirical settings, so it is desirable to control the degree of dependence between the bidder's values more precisely. Under Assumption 2.2 (i), by the Law of Iterated Expectations:

$$F_{N:N}(v) = \mathbb{E}[\phi_N(P(V_{N-1:N} \leq v | U, N))^N | N]$$

$$F_{N-1:N}(v) = \mathbb{E}[P(V_{N-1:N} \leq v | U, N) | N].$$

Denoting $\mu = F_{N-1:N}(v)$ and $X = P(V_{N-1:N} \leq v | U, N)$, for any fixed $v \in \mathcal{V}$, the bounds in Lemma 1 correspond to the solutions of the following generalized moment problems:

$$\min_P \{\mathbb{E}_P[\phi_N(X)^N] : \mathbb{E}_P[X] = \mu, P(X \in [0, 1]) = 1\} = \phi_N(\mu)^N; \quad (3)$$

$$\max_P \{\mathbb{E}_P[\phi_N(X)^N] : \mathbb{E}_P[X] = \mu, P(X \in [0, 1]) = 1\} = \mu.$$

The lower bound corresponds to Jensen's inequality and the upper bound to the so-called Edmundson-Madansky's inequality for the convex function $u \mapsto \phi_N(u)^N$ on $u \in [0, 1]$ (Edmundson, 1957; Madansky, 1959). In the absence of any additional restrictions on the distribution of $P(V_{N-1:N} \leq v | U)$, these bounds are sharp.

To tighten the bounds and control the degree of dependence between the bidder's values, we additionally restrict the variance of $P(V_{N-1:N} \leq v | U, N)$, conditional on N , denoted:

$$D_N(v) = \mathbb{E}[(P(V_{N-1:N} \leq v | U, N) - F_{N-1:N}(v))^2 | N]. \quad (4)$$

The variance corresponds to the squared $L_2(P)$ distance between the conditional and unconditional CDFs of $V_{N-1:N}$, thus providing a measure of the common component in the valuations. Since $P(V_{N-1:N} \leq v | U, N)$ is a random variable supported within the unit interval

²The authors assume that valuations are exchangeable and the function $f(k) = P(V_i \leq v | \#\{j \neq i : V_j \leq v\} = k)$ is non-decreasing in k . Lemma 1 in the aforementioned paper shows that conditional independence implies the stated property.

terval, $D_N(v)$ takes values in $[0, F_{N-1:N}(v)(1 - F_{N-1:N}(v))]$. The marginal cases $D_N(v) = 0$ and $D_N(v) = F_{N-1:N}(v)(1 - F_{N-1:N}(v))$ correspond to independent private values and pure common value correspondingly.³ Thus, bounding the variance function $D_N(v)$ away from the extremes allows to control the degree of dependence in the valuations.

To obtain an interpretable expression for $D_N(v)$, we invoke Assumption 2.2 (ii). Under this assumption, $V_{N-1:N} = g(U, \varepsilon_{N-1:N})$, so letting $f_{U|N}$ denote the density of U , conditional on N (w.r.t. Lebesgue measure, for simplicity),

$$\begin{aligned}\mathbb{E}[P(V_{N-1:N} \leq v | U, N)^2 | N] &= \int P(g(u, \varepsilon_{N-1:N}) \leq v | N)^2 f_{U|N}(u) du \\ &= P(g(U, \varepsilon_{N-1:N}) \leq v, g(U, \tilde{\varepsilon}_{N-1:N}) \leq v | N) \\ &= C_N(F_{N-1:N}(v), F_{N-1:N}(v)),\end{aligned}$$

where $\tilde{\varepsilon}_{N-1:N}$ is an independent copy of $\varepsilon_{N-1:N}$, and $C_N : [0, 1]^2 \rightarrow [0, 1]$ is the copula function of the vector $(V_{N-1:N}, \tilde{V}_{N-1:N}) = (g(U, \varepsilon_{N-1:N}), g(U, \tilde{\varepsilon}_{N-1:N}))$, conditional on N .⁴ The copula function captures the dependence between the second largest valuation in two identical auctions with the same common component and two independent draws of the private components.⁵ Letting $C_0 : [0, 1]^2 \mapsto [0, 1]$ denote the independence copula, defined as $C_0(u, v) = uv$, the variance $D_N(v)$ can be written as:

$$D_N(v) = C_N(F_{N-1:N}(v), F_{N-1:N}(v)) - C_0(F_{N-1:N}(v), F_{N-1:N}(v)).$$

This expression motivates the following assumption.

Assumption 2.3 (Departure from IPV and Pure Common Values). *Let $C_\rho : [0, 1]^2 \rightarrow [0, 1]$ be a copula function, parametrized by $\rho \in [0, 1]$ such that: (i) $C_0(u_1, u_2) = u_1 u_2$, $C_1(u_1, u_2) = \min(u_1, u_2)$; (ii) for each u , $\rho \mapsto C_\rho(u, u)$ is non-decreasing; (iii) for each ρ ,*

³Notice $D_N(v) = 0$, for all v , implies that $P(V_{N-1:N} \leq v | U, N) = F_{N-1:N}(v)$ almost surely, which, under Assumption 2.2 (i), implies that either U is constant or $V_j = g(\varepsilon_j)$, corresponding to the i.i.d. valuations. On the other hand, $D_N(v) = F_{N-1:N}(v)(1 - F_{N-1:N}(v))$, for all v , can only be attained if $P(V_{N-1:N} \leq v | U, N) \in \{0, 1\}$ such that $P(\{P(V_{N-1:N} \leq v | U, N) = 1\} | N) = F_{N-1:N}(v)$, which can only happen if $V_{N-1:N}$ is a deterministic function of U , corresponding to the pure common values.

⁴The copula function of random variables X and Y is defined as a function $C : [0, 1]^2 \rightarrow [0, 1]$ satisfying $F_{XY}(x, y) = C(F_X(x), F_Y(y))$. If X and Y are continuous, the copula represents the joint distribution of the random variables $F_X(X)$ and $F_Y(Y)$, which are marginally Uniform $[0, 1]$.

⁵If the researcher observes two repeated auctions of identical goods with independent draws of N bidders, and the transaction price reveals second largest valuation in each of the auctions, the copula function is identified from the data. Pursuing this formally is beyond the scope of the paper.

$(C_\rho(u, u))' \geq \max\{2\frac{u-C_\rho(u, u)}{1-u}, \frac{C_\rho(u)}{u}\}$. The variance $D_N(v)$, defined in (4), satisfies:

$$\underline{D}_N(v) \leq D_N(v) \leq \overline{D}_N(v)$$

with

$$\underline{D}_N(v) = C_{\underline{\rho}}(F_{N-1:N}(v), F_{N-1:N}(v)) - C_0(F_{N-1:N}(v), F_{N-1:N}(v))$$

$$\overline{D}_N(v) = C_{\bar{\rho}}(F_{N-1:N}(v), F_{N-1:N}(v)) - C_0(F_{N-1:N}(v), F_{N-1:N}(v)),$$

for some $0 < \underline{\rho} \leq \bar{\rho} < 1$.

Intuitively, ρ is interpreted as the share of the common component in bidders values, and $\underline{\rho}, \bar{\rho}$ are lower and upper bounds on this share. Choosing $\underline{\rho} > 0$ ensures a non-trivial common component, and $\bar{\rho} < 1$ ensures a non-trivial private component. Setting $\underline{\rho} = \bar{\rho}$ fixes $D_N(v)$ but leaves the joint distribution of $(V_{N-1:N}, \tilde{V}_{N-1:N})$ outside of the diagonal (v, v) completely unspecified. In such case, without further assumptions, the CDF of the largest valuation $F_{N:N}(v)$ is not point identified.

We highlight that Assumption 2.3 does not impose a specific dependence structure on the valuations; it simply provides a coherent way to choose the bounding functions $\underline{D}_N(v)$ and $\overline{D}_N(v)$ for the analysis. The formal results below do not rely on the specific parametric form of $C_\rho(u, u)$ in any way.

Condition (i) of the Assumption maintains independent private values and pure common value as the marginal cases; Condition (ii) ensures that the assumed bounds are properly ordered; Condition (iii) is a shape restriction that guarantees that the bounds derived in Theorem 1 below are plausible CDFs. In practice, it suffices to choose a function $C_\rho(u, u)$ that interpolates between u and u^2 depending on the value of $\rho \in [0, 1]$. Lemma A.2 in the Appendix verifies that natural choices, such as $C_\rho(u, u) = \rho u + (1 - \rho)u^2$, $C_\rho(u, u) = u^{2-\rho}$, or $C_\rho(u, u)$ corresponding to the bivariate Gaussian copula with correlation $\rho \in [0, 1]$, satisfy all of the stated assumptions. Any other bivariate copula satisfying the stated assumptions, and appropriately re-parametrized, can be used (see, e.g., Table 4.1. in [Nelsen, 2006](#), for examples).

2.2 Bounding CDF of The Largest Value and Expected Profit

Under Assumptions 2.2 and 2.3, pointwise sharp bounds on $F_{N:N}$ can be obtained by solving generalized moment problems similar to the ones in (3) with an extra constraint of the form $c_1 \leq \mathbb{V}ar(X) \leq c_2$. Using geometric arguments of [Kemperman \(1968\)](#), we show that both bounds can be computed by solving univariate convex optimization problems.

Theorem 1 (CDF Bounds Conditional on N). *Let Assumptions 2.2 and 2.3 hold. Then, for each $v \in \mathcal{V}$,*

$$\underline{\psi}_N(F_{N-1:N}(v)) \leq F_{N:N}(v) \leq \overline{\psi}_N(F_{N-1:N}(v)),$$

where:

$$\begin{aligned}\underline{\psi}_N(F_{N-1:N}(v)) &= \min_{s \in [\frac{c_1}{1-\mu}, \mu]} \left\{ \phi_N(\mu - s)^N \frac{c_1}{c_1 + s^2} + \phi_N \left(\mu + \frac{c_1}{s} \right)^N \frac{s^2}{c_1 + s^2} \right\}; \\ \overline{\psi}_N(F_{N-1:N}(v)) &= \max_{s \in [\mu - \frac{c_2}{1-\mu}, \mu + \frac{c_2}{\mu}]} \left\{ \mu - \frac{\mu(1-\mu)-c_2}{s(1-s)} (s - \phi_N(s)^N) \right\},\end{aligned}$$

with $\mu = F_{N-1:N}(v)$, $c_1 = \underline{D}_N(v)$, and $c_2 = \overline{D}_N(v)$. The minimization problem is strictly convex, and the maximization problem is strictly concave. Moreover, the bounding functions $\underline{\psi}_N(\cdot)$ and $\overline{\psi}_N(\cdot)$ are monotonically increasing and smooth in their argument.

For $\underline{\rho} = 0$ and $\overline{\rho} = 1$, the bounds coincide with those in Lemma 1, because they solve the moment problems in (3). For any $0 < \overline{\rho} \leq \underline{\rho} < 1$, the bounds in Theorem 1 are strictly tighter. If the only information available to the econometrician is $F_{N-1:N}$ (i.e., only the transaction price is observed), the bounds are pointwise sharp. Since the bounds are monotonically increasing, they are plausibly sharp uniformly over $v \in \mathcal{V}$, although it is hard to show formally. Additionally, we note that certain CDFs within the bounds do not belong to the sharp identified set because they cannot correspond to the CDF of order statistics of conditional i.i.d. values. For example, if $F_{N-1:N}(v)$ does not have atoms (“jumps”), $F_{N:N}(v)$ cannot have atoms either, so all such CDFs will be excluded from the identified set. Although the sharp identified set is very hard to describe, the proposed bounds are sufficiently informative, as we show in the sequel.

Theorem 1 implies the following bounds on the expected profit.

Theorem 2 (Bounds on Expected Profit Conditional on N). *Suppose Assumptions 2.1, 2.2, and 2.3 hold. Let $\underline{\psi}_N(F_{N-1:N}(v))$ and $\overline{\psi}_N(F_{N-1:N}(v))$ denote the lower and upper bounds on $F_{N:N}(v)$ in Theorem 1. Then, the expected profit, conditional on N , is bounded by:*

$$\begin{aligned}\pi_N(r) &\geq \int_0^\infty \max\{r, v\} dF_{N-1:N}(v) - v_0 - \overline{\psi}_N(F_{N-1:N}(r))(r - v_0); \\ \pi_N(r) &\leq \int_0^\infty \max\{r, v\} dF_{N-1:N}(v) - v_0 - \underline{\psi}_N(F_{N-1:N}(r))(r - v_0),\end{aligned}$$

for all $r \geq v_0$.

As in Theorem 1, the bounds are sharp point-wise in r and plausibly sharp uniformly. Likewise, many profit functions within the bounds cannot be attained under the stated

assumptions. The bounds on expected profit for r below v_0 can be obtained by switching the roles of $\underline{\psi}_N(\cdot)$ and $\bar{\psi}_N(\cdot)$, but they are irrelevant because it is never optimal to set the reserve price below v_0 .⁶

2.3 Unknown Number of Bidders

A common problem with the empirical analysis of auctions is that the exact number of bidders may be unknown to the econometrician, and potentially even to the seller. Below we show that the preceding analysis can be naturally extended to the case when only bounds on the number of bidders are available.

Assumption 2.4 (Unknown Number of Bidders). *The number of bidders N is unobserved but satisfies $\underline{N} \leq N \leq \bar{N}$ for some known $2 \leq \underline{N} \leq \bar{N}$.*

In such context, a natural objective function is the unconditional expected profit. Taking expectations over N in (1) yields:

$$\mathbb{E}[\pi_N(r)] = \int_0^\infty \max\{r, v\} dF_2(v) - v_0 - F_1(r)(r - v_0), \quad (5)$$

where

$$F_2(v) = \mathbb{E}[F_{N-1:N}(v)] = P(V_{N-1:N} \leq v);$$

$$F_1(v) = \mathbb{E}[F_{N:N}(v)] = P(V_{N:N} \leq v)$$

denote the marginal distributions of $V_{N-1:N}$ and $V_{N:N}$, correspondingly. Although N is unobserved, under Assumption 2.1, the marginal distribution $F_2(v)$ is identified by the data, so the identification problem is to bound $F_1(\cdot)$. To this end, we apply the same ideas as in the preceding subsection.

By Lemma A.1 in Appendix, for each $t \in [0, 1]$, the map $N \mapsto \phi_N(t)^N$, with $\phi_N(\cdot)$ defined in (2), is increasing. Using this fact and the Law of Iterated Expectations, the marginal CDF of $V_{N:N}$ is bounded by:

$$\mathbb{E}[\phi_{\underline{N}}(P(V_{N-1:N} \leq v | U, N))^{\underline{N}}] \leq F_1(v) \leq \mathbb{E}[\phi_{\bar{N}}(P(V_{N-1:N} \leq v | U, N))^{\bar{N}}]. \quad (6)$$

and, additionally:

$$F_2(v) = \mathbb{E}[P(V_{N-1:N} \leq v | U, N)].$$

Let

$$D(v) = \mathbb{E}[(P(V_{N-1:N} \leq v | U, N) - F_2(v))^2] \quad (7)$$

⁶This follows directly from the first-order condition for maximizing $\pi_N(r)$ in (1).

be the unconditional analog of $D_N(v)$ defined in (4). In order to bound $D(v)$ away from independent private and pure common values, we modify Assumptions 2.2 and 2.3 as follows.

Assumption 2.2⁺ (Conditional Independence). *In an auction with N bidders, (i) Bidders' values take the form $V_j = g(U, \varepsilon_j)$, for some i.i.d. random variables $\varepsilon_1, \dots, \varepsilon_N \in \mathbb{R}$, random vector $U \in \mathbb{R}^{d_U}$, and a measurable function $g : \mathbb{R}^{d_U} \times \mathbb{R} \rightarrow \mathcal{V}$. (ii) The map $e \mapsto g(u, e)$ is weakly increasing, for all u . (iii) (U, N) is jointly independent of $(\varepsilon_1, \dots, \varepsilon_N)$.*

Notice that Condition (iii) does not restrict the dependence between the common component and the number of bidders, but requires full independence of the private components. In the context of art auctions, this assumption allows for a situation that artworks by more popular artists attract more bidders, but does not allow those bidders to have a different distribution of tastes. Under Assumption 2.2⁺ (part (iii) in particular), repeating the arguments in the preceding section unconditionally, the variance $D(v)$ can be expressed as:

$$D(v) = C(F_2(v), F_2(v)) - C_0(F_2(v), F_2(v)),$$

where $C : [0, 1]^2 \rightarrow [0, 1]$ denotes the copula of $(V_{N-1:N}, \tilde{V}_{N-1:N}) = (g(U, \varepsilon_{N-1:N}), g(U, \tilde{\varepsilon}_{N-1:N}))$, and C_0 the independence copula.

Assumption 2.3⁺ (Departure from IPV and Pure Common Values). *Let $C_\rho : [0, 1]^2 \rightarrow [0, 1]$ be a copula function, parametrized by $\rho \in [0, 1]$ such that: (i) $C_0(u_1, u_2) = u_1 u_2$, $C_1(u_1, u_2) = \min(u_1, u_2)$; (ii) for each u , $\rho \mapsto C_\rho(u, u)$ is non-decreasing; (iii) for each ρ , $(C_\rho(u, u))' \geq \max\{2\frac{u-C_\rho(u,u)}{1-u}, \frac{C_\rho(u)}{u}\}$. The variance $D(v)$, defined in (7), satisfies:*

$$\underline{D}(v) \leq D(v) \leq \overline{D}(v)$$

with

$$\underline{D}(v) = C_{\underline{\rho}}(F_2(v), F_2(v)) - C_0(F_2(v), F_2(v))$$

$$\overline{D}(v) = C_{\bar{\rho}}(F_2(v), F_2(v)) - C_0(F_2(v), F_2(v)),$$

for some $0 \leq \underline{\rho} \leq \bar{\rho} \leq 1$.

Note that since $C(F_2(v), F_2(v)) = \mathbb{E}[C_N(F_{N-1:N}(v), F_{N-1:N}(v))]$, by the Law of Iterated Expectations, Assumption 2.3 implies Assumption 2.3⁺. The interpretation of $\underline{\rho}, \bar{\rho}$ and the arguments leading to the choice of bounding functions remain the same. The following results are direct analogs of Theorems 1 and 2.

Theorem 3 (CDF Bounds With Unknown N). *Let Assumptions 2.2⁺ and 2.3⁺ hold. Then, for each $v \in \mathcal{V}$,*

$$\underline{\psi}_N(F_2(v)) \leq F_1(v) \leq \overline{\psi}_N(F_2(v)),$$

where:

$$\begin{aligned}\underline{\psi}_N(F_2(v)) &= \min_{s \in [\frac{c_1}{1-\mu}, \mu]} \left\{ \phi_N(\mu - s)^N \frac{c_1}{c_1 + s^2} + \phi_N(\mu + \frac{c_1}{s})^N \frac{s^2}{c_1 + s^2} \right\}; \\ \overline{\psi}_N(F_2(v)) &= \max_{s \in [\mu - \frac{c_2}{1-\mu}, \mu + \frac{c_2}{\mu}]} \left\{ \mu - \frac{\mu(1-\mu)-c_2}{s(1-s)} (s - \phi_N(s)^{\overline{N}}) \right\},\end{aligned}$$

with $\mu = F_2(v)$, $c_1 = \underline{D}(v)$, and $c_2 = \overline{D}(v)$. The minimization problem is strictly convex, and the maximization problem is strictly concave. Moreover, the bounding functions $\underline{\psi}_N(\cdot)$ and $\overline{\psi}_N(\cdot)$ are monotonically increasing and smooth in their argument.

Theorem 4 (Bounds on Expected Profit With Unknown N). *Let Assumptions 2.1, 2.2⁺, 2.3⁺, and 2.4 hold. Let $F_1(\cdot)$ and $F_2(\cdot)$ denote the marginal distributions of $V_{N:N}$ and $V_{N-1:N}$ correspondingly. Then, $F_2(\cdot)$ is identified by the data, and, for each r :*

$$\begin{aligned}\mathbb{E}[\pi_N(r)] &\geq \int_0^\infty \max\{r, v\} dF_2(v) - v_0 - \overline{\psi}_N(F_2(r))(r - v_0); \\ \mathbb{E}[\pi_N(r)] &\leq \int_0^\infty \max\{r, v\} dF_2(v) - v_0 - \underline{\psi}_N(F_2(r))(r - v_0),\end{aligned}$$

where $\underline{\psi}_N(\cdot)$ and $\overline{\psi}_N(\cdot)$ are the bounding functions defined in Theorem 3.

As in Theorems 1 and 2, the bounds are point-wise sharp. The lower bound may be binding if and only if $N = \overline{N}$, and the upper bound — if and only if $N = \underline{N}$.

3 Optimal Reserve Prices

3.1 Resolving Ambiguity

Consider the problem of the seller (auction house) who aims to set a reserve price to maximize expected profit from an auction. Under our assumptions on information structure and bidding behavior, the expected profit function is only partially identified, so the target is ambiguous. In this section, we discuss existing approaches to resolving the ambiguity and advocate for using the so-called min-max-regret approach.

Let $\pi(\cdot)$ denote the true expected profit function, Π_0 — the set of all plausible profit functions, and \mathcal{R}_0 — the corresponding set of optimal reserve prices. There are three prevalent approaches to resolving ambiguity in the literature: Bayesian, Max-Min, and Min-Max-Regret; see, e.g., [Manski \(2022\)](#). The Bayesian approach is to assume a prior distribution Q over Π_0 and solve:

$$\sup_{r \in \mathcal{R}_0} \int_{\Pi_0} \pi(r) dQ(\pi).$$

In our setting, Π_0 represents the sharp identified set for $\pi(\cdot)$, so any prior within the identified set cannot be updated based on the data. On the other hand, the prior directly affects the proposed solution, and by using different priors (e.g., point masses on the elements of Π) one can recover any point $r \in \mathcal{R}_0$ as the “optimal” solution. Thus, the Bayesian approach directly imposes further restrictions on the model and is conceptually not helpful in the present setting. The Max-Min (MM) approach is to solve:

$$\sup_{r \in \mathcal{R}_0} \inf_{\pi \in \Pi_0} \pi(r),$$

i.e., maximize the lower bound on the expected profit function. Intuitively, this corresponds to setting the reserve price cautiously, which may not align with the goals of the auction house. Indeed, if the lot is unsold, the marginal cost associated with organizing its resale is likely negligible, compared to the selling price.⁷ Nevertheless, as we show in Section 5, the max-min solution may perform well.

The Min-Max-Regret (MMR) approach is to solve:

$$\inf_{r \in \mathcal{R}_0} \sup_{\pi \in \Pi_0} \{\pi(r^*) - \pi(r)\},$$

where r_π^* denotes an optimal reserve price under the profit function π . To gain intuition, let $\phi^* : \Pi_0 \rightarrow \mathbb{R}$, given by $\phi^*(\pi) = \pi(r_\pi^*)$, return the maximum of π , and for each $r \in \mathcal{R}_0$, let $\phi_r : \Pi_0 \rightarrow \mathbb{R}$, given by $\phi_r(\pi) = \pi(r)$, evaluate π at a given reserve price r . Then, the MMR problem can be equivalently stated as:

$$\inf_{r \in \mathcal{R}_0} \sup_{\pi \in \Pi_0} |\phi^*(\pi) - \phi_r(\pi)|. \quad (8)$$

That is, choosing r is equivalent to choosing a functional $\phi_r(\cdot)$ as close as possible to the profit-maximizing functional $\phi^*(\cdot)$ uniformly over unknown $\pi \in \Pi_0$. Conceptually, this is in line with the goal of profit maximization under ambiguity.⁸

The MMR problem is formulated with the sharp identified sets Π_0 and \mathcal{R}_0 . When these are intractable, one may consider a “relaxed” problem with some outer sets Π and \mathcal{R} :

$$\inf_{r \in \mathcal{R}} \sup_{\pi \in \Pi} |\phi^*(\pi) - \phi_r(\pi)|. \quad (9)$$

The interpretation remains the same: one is choosing a functional $\phi_r(\cdot)$ as close as possible to

⁷In our dataset, the auction houses may sell nearly fifty lots in one auction.

⁸In some settings, the MM and MMR solutions may coincide. We find that it is typically not the case in our setting, and both solutions may perform well in practice; see Section 5.

the profit-maximizing functional $\phi^*(\cdot)$ uniformly over unknown $\pi \in \Pi$. Intuitively, if $\Pi \setminus \Pi_0$ is not too large, and $\phi^*(\cdot)$ is well-behaved on $\Pi \setminus \Pi_0$, the solutions of (8) and (9) should be close to each other. Therefore, the relaxed MMR criterion in (9) provides a reasonable way of choosing the reserved price under ambiguity.

3.2 The Min-Max-Regret Problem

Theorem 1 derived lower and upper bounds on $F_{N:N}$ under Assumptions 2.2 and 2.3. While the bounds are point-wise sharp, certain CDFs between the bounds may not be admissible under the conditional independence assumption. For example, if $F_{N-1:N}$ does not have flat regions or jumps, then $F_{N:N}$ cannot have those either. Unfortunately, the sharp identified set for $F_{N:N}$ and thus for the expected profit function is hard to characterize, which makes the MMR problem in (8) intractable. In this section, we consider two relaxations of the MMR problem that can be solved analytically. Our results take bounds on $F_{N:N}$ as given, and hold for any bounds satisfying the stated assumptions going beyond the setting of Theorem 1. The discussion is presented conditional on N , but the results apply similarly to the unconditional case with obvious modifications of the assumptions.

3.2.1 CDFs Between the Bounds

First, we consider the following setting.

Assumption 3.1 (Relaxed MMR). *(i) Bidders' values are exchangeable; (ii) The transaction price is the greater of the reserve price and second-largest bidders value; (iii) The CDF of the largest value satisfies $\underline{F}(v) \leq F_{N:N}(v) \leq \bar{F}(v)$ for some known weakly increasing functions $\underline{F}(v)$, $\bar{F}(v)$ such that $0 \leq \underline{F}(v) \leq \bar{F}(v) \leq F_{N-1:N}(v)$, for all $v \in \mathcal{V}$.*

Condition (i) assumes valuations are symmetric but does not impose conditional i.i.d. structure as in Assumption 2.2; Condition (ii) is a re-statement of Assumption 2.1; and Condition (iii) states that any CDF $F_{N:N}$ between the available bounds is admissible (e.g. for the bounds in Theorem 1). To ensure coherency with Condition (i), Lemma A.5 in the Appendix shows that any pair $(F_{N-1:N}(v), F_{N:N}(v))$ with $F_{N:N}(v)$ between the bounds can be the CDF's of the top two order statistics of exchangeable random variables.

Assumption 3.1 defines a set of expected profit functions for the relaxed MMR problem. Plugging $\bar{F}(r)$ and $\underline{F}(r)$ instead of $F_{N:N}(r)$ in Equation (1) yields the respective lower and upper bounds on the profit, but not all functions between the bounds are admissible profit functions. For example, expected profit cannot arbitrarily "jump" up because $F_{N:N}(r)$ must be non-decreasing. Nevertheless, the relaxed MMR solution is easy to characterize.

Theorem 5 (Relaxed MMR Solution). *Let Assumption 3.1 hold. Denote the expected profit in (1) by $\pi_N(r; F_{N-1:N}, F_{N:N})$. Then, for any $r \geq v_0$, the max-regret is given by $R(r) = \max\{R_1(r), R_2(r)\}$, where:*

$$R_1(r) = \max_{v_0 \leq v \leq r} \pi_N(v; F_{N-1:N}, F_{1,r}^*) - \pi_N(r; F_{N-1:N}, F_{1,r}^*)$$

$$R_2(r) = \max_{v \geq r} \pi_N(v; F_{N-1:N}, F_{2,r}^*) - \pi_N(r; F_{N-1:N}, F_{2,r}^*)$$

with $F_{1,r}^*(v)$ and $F_{2,r}^*(v)$ given by:

$$F_{1,r}^*(v) = \mathbf{1}(v < r)\underline{F}(v) + \mathbf{1}(v \geq r)\overline{F}(v) \quad (10)$$

$$F_{2,r}^*(v) = \mathbf{1}(v < r)\overline{F}(v) + \mathbf{1}\{r \leq v < \bar{v}(r)\}\overline{F}(r) + \mathbf{1}\{v \geq \bar{v}(r)\}\underline{F}(v) \quad (11)$$

where $\bar{v}(r) = \min\{v : \underline{F}(v) \geq \overline{F}(r)\}$. The min-max-regret reserve is $r^* = \operatorname{argmin}_{r \geq v_0} R(r)$.

The proof is straightforward. Given any reserve r , the max-regret is attained either below or above it. We show that the max-regret realizes below r for a profit function that “jumps down” at r , corresponding to the CDF $F_{1,r}^*$, and above r for a profit function that increases “as much as possible” after r , corresponding to the CDF $F_{2,r}^*$. Solving for the MMR optimal reserve r^* numerically is straightforward. In fact, since $R_1(r)$ is increasing and $R_2(r)$ is decreasing (at least locally around the point of their intersection), r^* solves $\Delta(r) = R_1(r) - R_2(r) = 0$, where $\Delta(r)$ is smooth and locally increasing.

3.2.2 CDFs with a Shape Restriction

Next we consider an additional restriction that induces more structure on the problem.

Assumption 3.2 (Convex MMR). *(i) Bidders’ values are exchangeable; (ii) The transaction price is the greater of the reserve price and second-largest bidders value; (iii) The CDF of the largest values satisfies $\underline{\psi}(F_{N-1:N}) \leq F_{N:N}(v) \leq \overline{\psi}(F_{N-1:N})$ for some increasing, twice differentiable, convex functions $\underline{\psi}, \overline{\psi} : [0, 1] \rightarrow [0, 1]$; (iv) Letting $f_{V_{N:N}}(v)$ and $f_{V_{N-1:N}}(v)$ denote the density functions of the largest and second-largest valuations, the likelihood ratio $f_{V_{N:N}}(v)/f_{V_{N-1:N}}(v)$ is increasing in v .*

Conditions (i)–(ii) are the same as before. Condition (iii) imposes a shape restriction on the bounds on $F_{N:N}$, which is satisfied, e.g., by the bounds in Theorems 1 and 3. Condition (iv), known as the *likelihood ratio dominance*, is a strong form of stochastic dominance. Intuitively, it means that if an observer sees a realization $V = v$ of a random variable drawn either from $f_{V_{N:N}}$ or from $f_{V_{N-1:N}}$ but does not know with certainty which one, then the

higher $V = v$ is, the more likely it was drawn from $f_{V_{N:N}}$ rather than $f_{V_{N-1:N}}$. Importantly, Condition (iv) by itself does not restrict the strength of dependence between bidders values, and it is easy to verify that it holds under IPV and pure common values. It also holds for many commonly used copulas, such as Gaussian, Clayton, or Frank copulas.⁹

As shown in Lemma A.6 in the Appendix, exchangeability and likelihood ratio dominance imply that

$$F_{N:N}(v) = g_N(F_{N-1:N}(v)),$$

for some convex function $g_N : [0, 1] \rightarrow [0, 1]$, for all $v \in \mathcal{V}$. This fact substantially refines the set of admissible CDFs between the bounds and allows to solve the corresponding MMR problem analytically, as we do below.

Theorem 6 (Convex MMR Solution). *Let Assumption 3.2 hold. Denote the expected profit in (1) by $\pi_N(r; F_{N-1:N}, F_{N:N})$. Then, for any $r \geq v_0$, the max-regret is given by $R(r) = \max\{R_1(r), R_2(r)\}$, where:*

$$R_1(r) = \max_{v_0 \leq v \leq r} \pi_N(v; F_{N-1:N}, F_{1,r}^*) - \pi_N(r; F_{N-1:N}, F_{1,r}^*)$$

$$R_2(r) = \max_{v \geq r} \pi_N(v; F_{N-1:N}, F_{2,r}^*) - \pi_N(r; F_{N-1:N}, F_{2,r}^*)$$

where $F_{j,r}^*(v) = g_{j,r}(F_{N-1:N}(v))$ for convex functions $g_j : [0, 1] \rightarrow [0, 1]$ defined as follows. Denoting $u_r = F_{N-1:N}(r)$,

$$g_{1,r}(u) = \begin{cases} \underline{\psi}(u) & u \leq \underline{u}_{1,r} \\ a_{1,r}u + b_{1,r} & \underline{u}_{1,r} < u \leq u_r \\ \bar{\psi}(u) & u > u_r, \end{cases}$$

where $\underline{u}_{1,r}$ solves $\bar{\psi}(u_r) = \bar{\psi}'(u_r)(u_r - \underline{u}_{1,r}) + \underline{\psi}(\underline{u}_{1,r})$, $a_{1,r} = \bar{\psi}'(u_r)$, and $b_{1,r} = \bar{\psi}(u_r) - a_{1,r}u_r$. Further:

$$g_{2,r}(u) = \begin{cases} \bar{\psi}(u) & u \leq u_r \\ a_{2,r}u + b_{2,r} & u_r < u \leq \bar{u}_{2,r} \\ \underline{\psi}(u) & u > \bar{u}_{2,r}, \end{cases}$$

where $\bar{u}_{2,r}$ solves $\underline{\psi}(\bar{u}_{2,r}) - \bar{\psi}'(\bar{u}_{2,r})(\bar{u}_{2,r} - u_r) = \bar{\psi}(u_r)$, $a_{2,r} = \bar{\psi}'(u_r)$, and $b_{2,r} = \bar{\psi}(u_r) - a_{2,r}u_r$. The min-max-regret reserve is $r^* = \operatorname{argmin}_{r \geq v_0} R(r)$.

⁹See e.g. Nelsen (2006) or Joe (2006) for reviews of the copula theory. The likelihood ratio dominance is equivalent to $[V_{N:N} | a \leq V_{N:N} \leq b] \geq_{st} [V_{N-1:N} | a \leq V_{N-1:N} \leq b]$, for any $a, b \in \mathcal{V}$ with $a < b$, where \geq_{st} denotes first-order stochastic dominance; see, e.g., Chapter 1.C. in Shaked and Shanthikumar (1994) for more details.

The proof idea is similar to that of Theorem 5, except the “worst-case” CDFs $F_{1,r}^*$ and $F_{2,r}^*$ are now required to be convex transformations of $F_{N-1:N}$. The transformation $g_{1,r}$ corresponds to the CDF that decreases “as much as possible” just below r , and $g_{2,r}$ to the CDF that increases “as much as possible” above r , while preserving convexity. As in Theorem 5, solving for r^* is numerically simple.

4 Estimation and Inference

To highlight the main ideas, the discussion in this section treats the number of bidders N as known and fixed; the extension to varying and/or unknown N is immediate. We assume that the econometrician has access to a sample of n independent auctions, $(P_i, X_i)_{i=1}^n$, where $X_i \in \mathcal{X}$ is vector of auction-specific covariates, and $P_i \in \mathbb{R}_+$ is the transaction price in auction i , which is assumed equal to the second-largest bidders value $V_{N-1:N,i}$.

4.1 Bounds on Expected Profit

The profit bounds in Theorem 2 are known, deterministic functions of the observable distribution $F_{N-1:N}(v) = P(V_{N-1:N} \leq v | N)$. The best approach to estimating the distribution depends on the context. In addition to the number of bidders N , one may want to condition on a vector of covariates X . If the dataset is small, or X is very rich, non-parametric estimation may be imprecise. In such cases, one may impose a flexible parametric model for $P(V_{N-1:N} \leq v | N, X)$, similar to [Athey, Levin, and Seira \(2011\)](#), or assume that $P(V_{N-1:N} \leq v | N, X)$ depends on X only through an index $X'\gamma$ and employ, i.e., a semiparametric maximum likelihood estimator of [Ai \(1997\)](#). When non-parametric estimation is appropriate, the bandwidths may be selected using a reference parametric model; see [Ichimura and Todd \(2007\)](#).

Letting $\underline{\pi}(r; F_{N-1:N})$ and $\bar{\pi}(r, F_{N-1:N})$ denote the profit bounds in Theorem 2 and $\hat{F}_{N-1:N}$ be a suitable estimator for $F_{N-1:N}$, the profit bounds are estimated as:

$$\begin{aligned}\hat{\underline{\pi}}(r) &= \underline{\pi}(r; \hat{F}_{N-1:N}); \\ \hat{\bar{\pi}}(r) &= \bar{\pi}(r; \hat{F}_{N-1:N}).\end{aligned}\tag{12}$$

Since the bounding functions $\underline{\psi}_N$ and $\bar{\psi}_N$ are convex, the plug-in estimators for the bounds are generally biased downward. The bias may be corrected using standard jackknife or bootstrap techniques; see, e.g., [Wasserman \(2006\)](#). In turn, point-wise and uniform confidence bands can be constructed using delta-method.

In our empirical application, there are no continuous covariates, so we use the sample analog estimator $\hat{F}_{N-1:N}$. Additionally, we find that the finite-sample bias is negligible given the sample size and thus do not use any bias-correction methods. Appendix B spells out the asymptotic distributions of $\underline{\pi}(r)$ and $\hat{\pi}(r)$ in that simple case.

4.2 Bounds on Expected Profit under MMR Reserve Price

First, consider estimating the MMR optimal reserve price r^* obtained in Theorems 5 or 6. Letting $R_j(r; F_{N-1:N})$ for $j \in \{1, 2\}$ denote the corresponding maximum regrets, the MMR optimal reserve price r^* solves:¹⁰

$$\Delta(r; F_{N-1:N}) \equiv R_1(r; F_{N-1:N}) - R_2(r; F_{N-1:N}) = 0.$$

Denoting $\hat{\Delta}(r) = \Delta(r; \hat{F}_{N-1:N})$, a natural estimator \hat{r} for r^* solves:

$$\hat{\Delta}(\hat{r}) = 0. \quad (13)$$

Under regularity conditions, consistency of \hat{r} follows from the uniform consistency of $\hat{F}_{N-1:N}$ and continuity of $F \mapsto \Delta(\cdot; F)$ with respect to sup-norms. In Appendix B, we further show:

$$\sqrt{n}(\hat{r} - r^*) \rightarrow_d N(0, V_r) \quad (14)$$

using the mean-value expansion of (13) around r^* and continuity of $F \mapsto \Delta'(\cdot; F)$ with respect to sup-norms.

Next, consider the expected profit under MMR reserve price, $\pi^* = \pi_N(r^*)$. It is bounded by $\pi^* \in [\underline{\pi}^*, \bar{\pi}^*] \equiv [\underline{\pi}_N(r^*), \bar{\pi}_N(r^*)]$. The natural estimators for $\underline{\pi}^*$ and $\hat{\pi}^*$ are:

$$\begin{aligned} \hat{\underline{\pi}}^* &= \hat{\underline{\pi}}(\hat{r}); \\ \hat{\bar{\pi}}^* &= \hat{\bar{\pi}}(\hat{r}), \end{aligned} \quad (15)$$

where $\hat{\underline{\pi}}$ and $\hat{\bar{\pi}}$ are defined in (12), and \hat{r} in (13). Consistency of these estimators follows immediately from consistency of \hat{r} and continuity of the maps $F \mapsto \underline{\pi}(\cdot; F)$ and $F \mapsto \bar{\pi}(\cdot; F)$.

¹⁰That is, the minimizer of $R(r) = \max(R_1(r), R_2(r))$ is given by the intersection of $R_1(r)$ and $R_2(r)$.

with respect to sup-norms. For the asymptotic distribution, consider an expansion:

$$\begin{aligned}\sqrt{n}(\hat{\underline{\pi}}^* - \underline{\pi}^*) &= \{\sqrt{n}(\hat{\underline{\pi}}(\hat{r}) - \underline{\pi}(\hat{r})) - \sqrt{n}(\hat{\underline{\pi}}(r^*) - \underline{\pi}(r^*))\} \quad (I) \\ &\quad + \sqrt{n}(\hat{\underline{\pi}}(r^*) - \underline{\pi}(r^*)) \quad (II) \\ &\quad + \sqrt{n}(\underline{\pi}(\hat{r}) - \underline{\pi}(r^*)). \quad (III)\end{aligned}\tag{16}$$

In Appendix B, using stochastic equicontinuity arguments, we show that term (I) converges in probability to zero. The term (II) + (III) is easily seen to be asymptotically Normal and its limiting distribution can be derived using mean-value theorem and Slutsky's theorem. Applying similar arguments to the upper bound estimator $\hat{\bar{\pi}}^*$, it follows that:

$$\begin{aligned}\sqrt{n}(\hat{\underline{\pi}}^* - \underline{\pi}^*) &\rightarrow_d N(0, V_{\underline{\pi}}); \\ \sqrt{n}(\hat{\bar{\pi}}^* - \bar{\pi}^*) &\rightarrow_d N(0, V_{\bar{\pi}}).\end{aligned}\tag{17}$$

Then, given consistent estimators $\hat{V}_{\underline{\pi}}$ and $\hat{V}_{\bar{\pi}}$, a $(1 - \alpha)$ confidence interval for $\pi^* \in [\underline{\pi}^*, \bar{\pi}^*]$ can be constructed as:

$$\left[\hat{\underline{\pi}}^* - z_{1-\alpha} \sqrt{\frac{\hat{V}_{\underline{\pi}}}{n}}, \quad \hat{\bar{\pi}}^* + z_{1-\alpha} \sqrt{\frac{\hat{V}_{\bar{\pi}}}{n}} \right]. \tag{18}$$

Such confidence interval is uniformly asymptotically valid over the distributions bounded away from point identification; see, e.g., Imbens and Manski (2004); Stoye (2009); and Canay and Shaikh (2017) for related discussions. In our setup, point identification is ruled out by construction of the bounds, so the above confidence interval is appropriate.

The asymptotic variances in (17) are very cumbersome due to the dependence between the terms (II) and (III) in (16). However, the above discussion also shows that the distributions of $\sqrt{n}(\hat{\underline{\pi}}^* - \underline{\pi}^*)$ and $\sqrt{n}(\hat{\bar{\pi}}^* - \bar{\pi}^*)$ are continuous in the underlying distribution of the data, so a valid confidence interval can also be constructed using bootstrap as follows.

Algorithm 1 (Bootstrap CI).

1. Using the original sample, compute $\hat{\underline{\pi}}^*$ and $\hat{\bar{\pi}}^*$ according to (15).
2. Draw a large number of bootstrap samples $(P_{i,b}^*, X_{i,b}^*)_{i=1}^n$, for $b = 1, \dots, B$, and compute $\hat{\underline{\pi}}_b^*$ and $\hat{\bar{\pi}}_b^*$ according to (15), for each b .
3. Letting $\hat{q}_{1-\alpha}$ denote the $(1 - \alpha)$ empirical quantile of $\{\sqrt{n}(\hat{\underline{\pi}}_b^* - \hat{\underline{\pi}}^*)\}_{b=1}^B$, and \hat{q}_α denote the α empirical quantile of $\{\sqrt{n}(\hat{\bar{\pi}}_b^* - \hat{\bar{\pi}}^*)\}_{b=1}^B$, compute the confidence interval:

$$\left[\hat{\underline{\pi}}^* - \frac{\hat{q}_{1-\alpha}}{\sqrt{n}}, \quad \hat{\bar{\pi}}^* - \frac{\hat{q}_\alpha}{\sqrt{n}} \right]. \tag{19}$$

We remark that although Step 2 involves numerically solving the MMR problem for each bootstrap sample, the procedure is computationally simple.

5 Monte Carlo Experiments

In this section, we investigate the performance of the proposed approach in a controlled environment. First, we show that the CDF bounds in Theorems 1 and 3 and the implied profit bounds in Theorems 2 and 4 are substantially tighter than those previously available in the literature. Second, we show that choosing the reserve price using min-max-regret criteria in Theorems 5 and 6, combined with the new tighter identified sets, may lead to significant increase in seller's profit relative to the status quo.

5.1 Simulation Design

The simulation design is calibrated to four stylized facts in our empirical application.

1. *The auction houses report the low and the high estimates of each lot's value. These estimates are crude and not necessarily backed by any market or historical research.* Normalizing the high estimate to 1, the low estimate is around 2/3, on average. We consider the value of unsold good to the seller, v_0 , to be either equal to the low estimate, $v_0 = 2/3$, or the mid point between low and high estimates, $v_0 = 5/6$.
2. *The reserve price is secret and does not the lot's low estimate.* In simulations, we set the reserve price to zero.
3. *There is significant variation in the number of active bidders per auction.* We draw the number of bidders randomly for each auction from the set $N \in \{2, 3, \dots, 10\}$, and use $\underline{N} = 2$ and $\overline{N} = 10$ throughout.
4. *According to the bidding guidelines, the minimal bid increment is 10% of the standing bid, but this rule is not strictly enforced.* We simulate auctions as follows. Given a set of N bidders, the initial bid by a randomly drawn bidder is cast at 5% of their valuation. At each subsequent iteration, an active bidder is drawn at random from the pool of remaining bidders, whose values are above the standing bid. With probability $1 - \lambda$, the new bid is equal to 1.1 of the standing bid, and with probability λ , the new bid is drawn uniformly between the standing bid and 1.1 of the standing bid. If the new bid is above the active bidder's value, the bidder is eliminated, and a new bidder is drawn from the pool of remaining bidders. The iterations proceed until only one bidder remains.

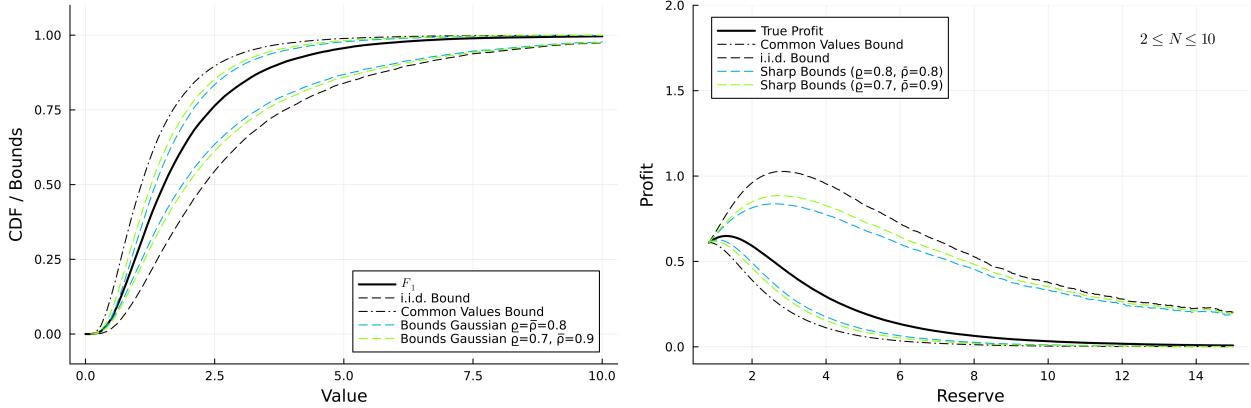


Figure 1: Simulated bounds on the CDF of the largest value and expected profit.

5.2 Simulation Results

5.2.1 The Distribution of Largest Value and Expected Profit

First, we illustrate the bounds from Theorems 3 and 4. The bidders values are generated as $\log(V_j) = U + \varepsilon_j$, where $U \sim N(\mu_U, \sigma_U^2)$ and the distribution of ε_j is such that $\varepsilon_{N-1:N} \sim N(\mu_\varepsilon, \sigma_\varepsilon)$. The parameters $\mu_U, \mu_\varepsilon, \sigma_U$, and σ_ε are chosen to match the average transaction price in the data, $\mathbb{E}[V_{N-1:N}] = 1.35$. The copula function used to define the bounds in Assumption 2.3 matches the above data-generating process and corresponds to $\rho = 0.8$. That is, in the true data-generating process, the common component U accounts for 80% of each bidder's value.

Figure 1 presents bounds on the CDF of $V_{N:N}$ (unconditionally), and the corresponding bounds on expected profit. The black solid line represents the true CDF, and the black dashed lines — the IPV and pure common value bounds. The blue dashed lines represent lower and upper bounds with $\underline{\rho} = \bar{\rho} = 0.8$, and the green lines — with $\underline{\rho} = 0.7$ and $\bar{\rho} = 0.9$. The two main takeaways are: (i) the proposed bounds are substantially tighter than the IPV and common value bounds even for a relatively wide range of $\rho \in [\underline{\rho}, \bar{\rho}] = [0.7, 0.9]$; and (ii) even when correctly specifying $\underline{\rho} = \rho = \bar{\rho} = 0.8$, we are still far from point identification. We find very similar results, conditional on N .

5.2.2 Optimal Reserve Prices

Next, we compare the MMR solutions of Theorems 5 and 6 with the MM solution under different sets of bounds on the profit function. Figure 2 presents the results. The left panel bounds on expected profit under IPV and common value, and the right panel — the bounds in Theorem 2 with $\underline{\rho} = \bar{\rho} = 0.8$. Both panels additionally plot the maximum regret functions $R(r)$ from Theorems 5 and 6, and the reserve prices, corresponding to the MM, relaxed

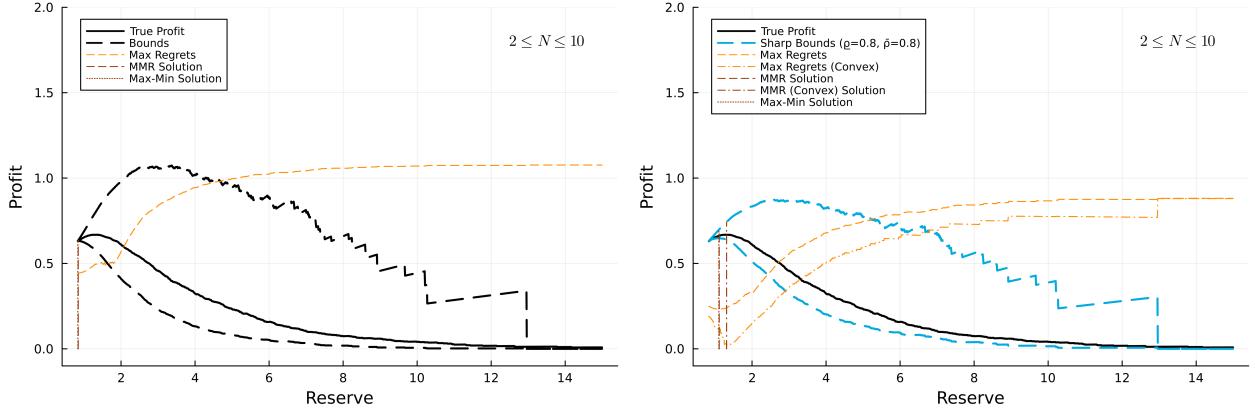


Figure 2: Simulated profit bounds and optimal reserve prices.

| | Reserve | Profit | Increase over $r = v_0$ |
|--|---------|--------|-------------------------|
| True Optimal | 1.33 | 0.668 | 6.03% |
| MM ($\underline{\rho} = 0.8, \bar{\rho} = 0.8$) | 1.10 | 0.662 | 5.01% |
| Relaxed MMR ($\underline{\rho} = 0.0, \bar{\rho} = 1.0$) | 0.84 | 0.632 | 0.32% |
| Relaxed MMR ($\underline{\rho} = 0.8, \bar{\rho} = 0.8$) | 1.11 | 0.663 | 5.24% |
| Convex MMR ($\underline{\rho} = 0.7, \bar{\rho} = 0.9$) | 1.15 | 0.665 | 5.56% |
| Convex MMR ($\underline{\rho} = 0.8, \bar{\rho} = 0.8$) | 1.31 | 0.668 | 6.03% |

Table 1: Simulated profit under suggested reserve prices.

MMR, and convex MMR solutions.

Recall that the MM solution maximizes of the lower bound on the profit. Since the lower bound on the left panel, corresponding to the pure common value setting, is monotonically decreasing, the MM solution is simply $v_0 = 0.84$. It can be formally shown that the convex MMR solution in this case also coincides with v_0 , and we find empirically that the relaxed MMR solution is very close, too. Thus, due to the fact the IPV and common value profit bounds are very loose, the resulting reserves are not particularly insightful. In contrast, on the right panel, the MM and relaxed MMR solutions are both very close to the non-trivial maximizer of the lower bound, while the convex MMR solution essentially coincides with true profit maximizing reserve. Table 1 provides a more detailed comparison.

Our main takeaway is that the MM, relaxed MMR, and especially convex MMR, when combined with the new tight bounds, are able to deliver sizable increase in expected profit.

| Category | Subcategory | Location | Count |
|-----------------------|-------------------------------------|-----------|-------------|
| Art | Chinese Art | New York | 112 |
| Art | Impressionist/20th/21st Century Art | Hong Kong | 636 |
| Art | Impressionist/20th/21st Century Art | Las Vegas | 9 |
| Art | Impressionist/20th/21st Century Art | London | 708 |
| Art | Impressionist/20th/21st Century Art | New York | 1208 |
| Art | Impressionist/20th/21st Century Art | Shanghai | 55 |
| Art | Old Masters | London | 102 |
| Art | Old Masters | New York | 134 |
| Others (e.g. Jewelry) | | | 542 |
| Total | | | 3506 |

Table 2: The Art Auction Dataset

6 Application: Reserve Prices in Art Auctions

6.1 Data

The auctions are held in an open ascending format, and, on average, thirty lots are sold in each auction. Prior to the auction, besides the basic information about each lot, the auction houses publish a low and high estimates of the lot’s value. According to the auction houses, these estimates may not be accurate or backed by any market or historical research. The reserve price is kept secret, but is known to be less than the lower estimate of the lot’s value. The auctioneer may bid on behalf of the seller to ensure the reserve is met.

6.1.1 Bids and Lot Information

For each auction lot, we obtain a complete bidding trajectory by applying computer vision techniques to frame-by-frame video data. Figure 3 presents examples of the frames. We crop the bottom sections, highlighted in red, and use an optical character recognition package Tesseract¹¹ to extract the lot’s information and the bidding sequence. Then, we match the bids for each lot with the lot’s characteristics scraped from the auction houses’ websites. We use the Ratcliff-Obershelp strings matching algorithm¹² to accommodate any errors or misspellings in Tesseract’s output. Table 2 summarizes the auctions across art categories and locations. Due to the nature of the data, we cannot match the bids with the bidders. Thus, we only use the two largest bids, which arguably belong to different bidders.

¹¹Available at <https://github.com/tesseract-ocr/tesseract>.

¹²Available in a Python package `difflib` command `SequenceMatcher`.

6.1.2 Number of Bidders

The exact number of bidders in each auction cannot be reliably identified from the auction videos, so we obtain bounds on it using the audio transcripts.¹³ In each auction, there are four types of bidders in each auction: (i) telephone, (ii) live, (iii) absentee, and (iv) online; see Figure 4. We obtain a lower bound on the true number of bidders by counting the number of unique bidders of each type.

Each telephone bidder is represented by an employee of the auction house, whom the auctioneer calls by their names. For example, in the top-right panel of Figure 4, the Christie’s auctioneer references Olivier Camu, who is putting a telephone bid on behalf of an unknown buyer. To identify the names from the transcript, we use the so-called Named Entity Recognition language models. To minimize errors, we apply two state-of-the-art language models, RoBERTa (Liu, Ott, Goyal, Du, Joshi, Chen, Levy, Lewis, Zettlemoyer, and Stoyanov, 2019) and Contextual String Embeddings (Akbik, Blythe, and Vollgraf, 2018), and combine their output matching similar strings and removing duplicates. To capture the identities of the unique live, absentee, and online bidders, we look for keywords $\{$ “online bidder” = “online,” “sir” = “gentleman,” “madam” = “lady,” “back of the room,” “to the right”, “to the left,” “absentee” $\}$. Our approach allows to capture all of the telephone bidders, up to one unique online or absentee bidder, and up to five unique live bidders. The distribution of the lower bound on the number of bidders is shown in Figure 5. Although it is unclear how tight the resulting lower bound is, the existing statistics are reassuring: in 2021, 42% of winning bids in Christie’s live auctions were made over the telephone and only 7% in the salesroom,¹⁴ meaning that at least some of the “missing” bidders may not have affected the auction’s outcome anyway.

We conservatively set the upper bound on the number of bidders to two times the lower bound. An even more conservative alternative would be the total number of bids. However, the function $\phi_N(t)^N$, which is used to construct the bounds, converges point-wise as N increases, so we find that for $N, N' \geq 10$ the difference between $\phi_N(t)^N$ and $\phi_{N'}(t)^{N'}$ becomes negligible. Thus, the choice between conservative upper bounds does not affect the results.

6.1.3 Auction House Fees

The auction houses charge around 25% buyer’s premium and 10% seller’s commission, with some variance depending on location and lot type. Table 7 in the Appendix summarizes the Buyer’s Premium Schedule, for both Christie’s and Sotheby’s, as of March 2023. In our

¹³The transcripts are generated using Google Cloud Platform Speech-to-Text Recognition model. See <https://cloud.google.com/speech-to-text> for the details.

¹⁴See Christie’s website at this [link](#).



Lot 38, Christie's Hong Kong, Nov 2022

Lot 117, Sotheby's New York, Nov 2022



Room



Online

Telephone



Absentee

Figure 4: The four types of bidders in Sotheby's or Christie's live auctions.

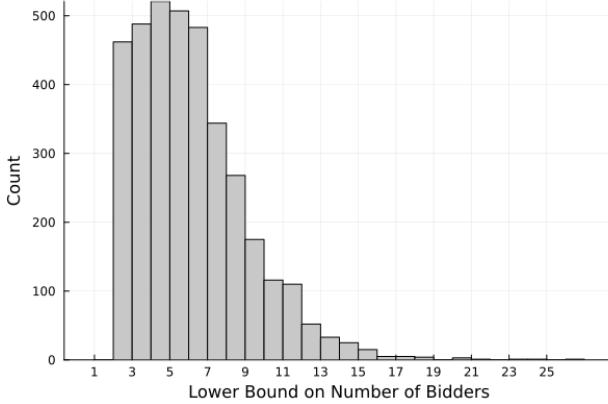


Figure 5: Distribution of lower bound on number of bidders across all auctions.

dataset, the bids are scaled accordingly.

6.2 Modern Art Sold in New York City

Our empirical analysis focuses on modern art sold in New York City. This is the largest subsample in our data both in terms of the number of auctions and generated revenue. To avoid any outliers, we focus on the lots with transaction prices between $\$100K$ and $\$10.0M$, which leaves us with $n = 949$ observations. We further condition on the price range, $[\$100K, \$1.0M]$ vs $[\$1.0M, \$10.0M]$, roughly splitting the sample into one-third and two-thirds, and the number of bidders $[2, 10]$ vs $[6, 40]$ (i.e., $\underline{N} \leq 5$ vs $\underline{N} \geq 6$), splitting each of the subsamples approximately in half. Table 3 displays summary statistics based on the selected price ranges. It is worth noting that the median and average selling price are noticeably larger than the high estimates of the lot’s value provided by the auction houses, confirming that the estimates are imprecise.

Examining the bidding data, we find that (i) there is no evidence of jump bids towards the end of the auctions; (ii) the difference between the two largest bids is small, relative to the size of the bids; and (iii) the auctioneers do not enforce the minimal bid increment guideline towards the end of the auction. Thus, the assumption that the transaction price reveals second-largest value appears plausible in our dataset. Each bidder’s value for a given lot consists of a common component, representing the resale value, and a private component, representing consumption value. We assume that even after controlling for the art category, location, price range, and the number of bidders, the common component in the bidders values is prevalent. Thus, we consider the bounding parameters $\underline{\rho}$ and $\bar{\rho}$ to values in the range $[0.75, 0.85]$. Normalizing the transaction price by the lot’s high estimate, so that the high estimate for each lot is equal to one, we consider two possible choices for the value of

| Variable | Median | Mean | Std | [Min, Max] |
|--|--------|-------|------|---------------|
| Transaction Price | 1.39 | 1.83 | 1.29 | [0.25, 6.35] |
| 2nd-Highest Bid | 1.31 | 1.72 | 1.23 | [0.24, 6.14] |
| Number of Bids | 10.00 | 11.42 | 7.40 | [2.00, 39.00] |
| Number of Bidders | 5.00 | 5.44 | 2.50 | [2.00, 13.00] |
| Low Estimate Relative to High Estimate | 0.67 | 0.69 | 0.05 | [0.60, 0.78] |
| Number of Auction Lots | 287 | | | |

Transaction price $\in [\text{\$}100\text{K}, \text{\$}1.0\text{M}]$

| Variable | Median | Mean | Std | [Min,Max] |
|--|--------|-------|------|---------------|
| Transaction Price | 1.15 | 1.41 | 0.85 | [0.37, 6.72] |
| 2nd-Highest Bid | 1.11 | 1.34 | 0.82 | [0.35, 6.30] |
| Number of Bids | 10.00 | 11.75 | 8.04 | [2.00, 76.00] |
| Number of Bidders | 5.00 | 5.98 | 2.91 | [2.00, 20.00] |
| Low Estimate Relative to High Estimate | 0.67 | 0.68 | 0.05 | [0.48, 0.84] |
| Number of Auction Lots | 662 | | | |

Transaction price $\in [\text{\$}1.0\text{M}, \text{\$}10.0\text{M}]$

Table 3: Summary statistics for modern art auction lots in New York City.

unsold good to the seller: $v_0 = 2/3$, corresponding to the low estimate, on average, and $v_0 = 5/6$, corresponding to the midpoint between the low and high estimates.

Table 4 presents the results. First, with $v_0 = 5/6$, we find that even the lower bounds on expected profit under convex MMR reserves with restricted $\underline{\rho}, \bar{\rho}$ are consistently higher than the average realized profit, and the projected profit gains appear to be higher for pricier lots with fewer bidders. In particular, for the lots within $[\text{\$}1.0\text{M}, \text{\$}10.0\text{M}]$ price range with $N \in [2, 10]$, the lower bound on the profit gain under the Convex MMR reserve with $\underline{\rho} = \bar{\rho} = 0.8$ is 15.35%, which amounts to \$572K per lot, or \$17M per auction (the auctions consist of 30 lots, on average). Pooling together all of the lots in that price range yields a more conservative lower bound on the profit gain of 3.85%, which amounts to \$127K per lot or \$3.8M per auction. With unrestricted $\underline{\rho} = 0, \bar{\rho} = 1$, however, the lower bound on the profit under convex MMR reserve may be substantially lower than the realized profit. Second, with $v_0 = 2/3$ and restricted $\underline{\rho}, \bar{\rho}$, we find that the projected profit gains to be more modest with lower bounds occasionally being slightly below the realized profit. However, even in this arguably conservative scenario, the profit gains are projected to be positive.

| Price Range | [\$100K, \$1.0M] | | | [\$1.0M, \$10.0M] | | |
|---|------------------|-----------------|-----------------|-------------------|-----------------|-----------------|
| | $N \in [2, 10]$ | $N \in [6, 40]$ | $N \in [2, 40]$ | $N \in [2, 10]$ | $N \in [6, 40]$ | $N \in [2, 40]$ |
| $v_0 = 5/6$ | | | | | | |
| Avg High Estimate | \$541,688 | \$316,470 | \$434,965 | \$3,811,117 | \$2,741,925 | \$3,291,057 |
| Avg Realized Profit | 0.545 | 1.502 | 0.998 | 0.228 | 0.937 | 0.573 |
| Convex MMR Reserve | | | | | | |
| $\underline{\rho} = 0.8, \bar{\rho} = 0.8$ | 1.20 | 1.36 | 1.25 | 1.12 | 1.05 | 1.20 |
| – Bounds on profit | [0.587, 0.665] | [1.508, 1.530] | [1.035, 1.094] | [0.263, 0.337] | [0.951, 0.956] | [0.595, 0.666] |
| Convex MMR reserve | | | | | | |
| $\underline{\rho} = 0.75, \bar{\rho} = 0.85$ | 1.14 | 1.32 | 1.14 | 1.31 | 1.02 | 1.06 |
| – Bounds on profit | [0.592, 0.661] | [1.509, 1.534] | [1.034, 1.094] | [0.222, 0.368] | [0.949, 0.955] | [0.720, 0.788] |
| Convex MMR reserve | | | | | | |
| with $\underline{\rho} = 0.0, \bar{\rho} = 1$ | 1.73 | 0.84 | 2.02 | 0.84 | 0.85 | 0.84 |
| – Bounds on profit | [0.414, 0.859] | [1.513, 1.514] | [0.759, 1.352] | [0.273, 0.277] | [0.944, 0.945] | [0.600, 0.602] |
| $v_0 = 2/3$ | | | | | | |
| Avg High Estimate | \$541,688 | \$316,470 | \$434,965 | \$3,811,117 | \$2,741,925 | \$3,291,057 |
| Avg Realized Profit | 0.711 | 1.669 | 1.165 | 0.394 | 1.104 | 0.739 |
| Convex MMR Reserve | | | | | | |
| $\underline{\rho} = 0.8, \bar{\rho} = 0.8$ | 1.11 | 1.32 | 1.11 | 1.13 | 0.85 | 1.09 |
| – Bounds on profit | [0.709, 0.786] | [1.661, 1.684] | [1.174, 1.221] | [0.328, 0.445] | [1.107, 1.108] | [0.722, 0.787] |
| Convex MMR Reserve | | | | | | |
| $\underline{\rho} = 0.75, \bar{\rho} = 0.85$ | 1.05 | 1.26 | 1.00 | 1.09 | 0.80 | 1.06 |
| – Bounds on profit | [0.702, 0.777] | [1.668, 1.691] | [1.169, 1.206] | [0.342, 0.456] | [1.107, 1.108] | [0.720, 0.788] |
| Convex MMR Reserve | | | | | | |
| $\underline{\rho} = 0.0, \bar{\rho} = 1$ | 1.71 | 0.73 | 1.91 | 0.77 | 0.69 | 0.68 |
| – Bounds on profit | [0.447, 0.961] | [1.674, 1.676] | [0.829, 1.444] | [0.393, 0.417] | [1.105, 1.106] | [0.745, 0.745] |
| Observations | 151 | 136 | 287 | 340 | 322 | 662 |

Table 4: Bounds on expected profit for Modern Art sold in New York City. Figures are scaled by the high estimate.

7 Conclusion and Further Research

In this paper, we proposed a novel approach to partial identification in ascending auctions. We derived new bounds on the distribution of the largest value and seller’s profit, transparently bounding the degree of dependence between bidders’ values, and proposed a way to select the optimal reserve price under ambiguity using min-max-regret criterion. The proposed approach is computationally simple and readily extends to settings with unknown number of bidders, as long as bounds on the number of bidders are available. We demonstrated that the resulting bounds are substantially tighter than those previously available in the literature, and the proposed reserve price performs well in practice. We applied the proposed methodology to a novel dataset of art auctions held by the two major auction houses, Christie’s and Sotheby’s, and argued that higher reserve prices would lead to a sizable increase in profit.

Our analysis can be advanced in several directions. First, one may be interested in welfare aspects of the chosen auction format. For example, under the stated assumptions, bidders surplus is also identified from the marginal distributions of the two largest valuations and can be studied similarly to the expected profit. Second, the assumption that the transaction price reveals the second-largest bidder’s value in the observed data can be relaxed with weaker bidding assumptions in the spirit of [Haile and Tamer \(2003\)](#). In such setting, the bounds on the distribution of the largest value follow directly from the arguments above, but the analysis of optimal reserve prices becomes more nuanced: a coherent formulation of expected profit requires additional assumptions on bidding behavior in counterfactual auctions, and the min-max-regret problems for selecting the reserve price become more complex. Finally, an interesting direction for further research is allowing for asymmetric bidders. While our bounding approach critically relies on the bidders’ values being conditionally i.i.d., it appears possible to accommodate several “types” of symmetric bidders using similar ideas.

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A Proofs

A.1 Auxiliary Lemmas

Lemma A.1 (Properties of $\phi_N(\cdot)^N$). *Consider a function $h_N : [0, 1] \rightarrow [0, 1]$ given by:*

$$h_N(t) = \phi_N(t)^N,$$

where $\phi_N : [0, 1] \rightarrow [0, 1]$ is defined implicitly via $t = N\phi_N(t)^{N-1} - (N-1)\phi_N(t)^N$. Then:

1. $h'_N(t) = \frac{1}{N-1} \cdot \frac{\phi_N(t)}{1-\phi_N(t)}$, so h is strictly increasing.
2. $h''_N(t) = \frac{2}{N-1} \cdot \frac{1}{\binom{N}{2}\phi_N(t)^{N-2}(1-\phi_N(t))^2}$, so h is strictly convex, $h''(0) = h''(1) = \infty$.
3. $h'''_N(t) = \frac{N\phi_N(t)-(N-2)}{N(N-1)\phi_N(t)^{N-1}(1-\phi_N(t))} \cdot h''_N(t)$, so $h'_N(\cdot)$ changes curvature exactly once.
4. $f(t) = h_N(t)/t$ satisfies $\lim_{t \rightarrow 0} f(t) = 0$ and is strictly increasing on $(0, 1)$.
5. $h_N(t)$ is increasing in N for all $t \in (0, 1)$ and $N \geq 2$.

Proof. Parts 1–3 follow immediately from taking derivatives. In Part 4, the first claim follows from the L'Hôpital's rule. For the second claim, notice that by definition of $\phi_N(t)$,

$$\frac{h(t)}{t} = \frac{\phi_N(t)^N}{N\phi_N(t) - (N-1)\phi_N(t)^N} = \frac{1}{\frac{N}{\phi_N(t)} - (N-1)},$$

which is a strictly increasing function of t .

For Part 5, note that ϕ_N is well-defined for any real $N \geq 2$, so we can rely on differentiation. Recall the identity:

$$(\alpha(N)^{\beta(N)})' = \alpha(N)^{\beta(N)}\beta'(N) \log \alpha(N) + \alpha(N)^{\beta(N)-1}\beta(N)\alpha'(N).$$

Fix any $t \in (0, 1)$ and suppress it from the notation for simplicity. Using the above identity:

$$(h_N)' = (\phi_N)^N \log \phi_N + N(\phi_N)^{N-1}(\phi_N)'. \quad (\text{A.1})$$

To compute the derivative $(\phi_N)'$, recall the definition of ϕ_N :

$$t = N(\phi_N)^{N-1} - (N-1)(\phi_N)^N. \quad (\text{A.2})$$

Differentiation both sides using the above identity and rearranging yields:

$$(\phi_N)' = \frac{(\phi_N)^N - (\phi_N)^{N-1} - t \log \phi_N}{N(N-1)(\phi_N)^{N-2}(1-\phi_N)}.$$

Plugging this in (A.1) and using (A.2), with some algebra:

$$(\phi^N)'_N = \frac{\phi_N}{(N-1)(1-\phi_N)} ((\phi_N)^N - (\phi_N)^{N-1} - (\phi_N)^{N-1} \log \phi_N).$$

The right-hand side is non-negative since $\log(x) \leq x - 1$. ■

Lemma A.2 (Bounding Functions C_ρ Satisfying Assumption 2.3). *The functions:*

1. $C_\rho(u) = \rho u + (1-\rho)u^2$
2. $C_\rho(u) = u^{2-\rho}$
3. $C_\rho(u) = P(\Phi(Z_1) \leq u, \Phi(Z_2) \leq u)$ where (Z_1, Z_2) are jointly normal with mean zero, variances one, and covariance ρ ;

satisfy all requirements of Assumption 2.3.

Proof. Notice that $(C_\rho(u))' \geq \frac{C(u)}{u}$ is equivalent to $\frac{C_\rho(u)}{u}$ being increasing in u .

1. $C_0(u) = u^2$ and $C_1(u) = u$ is immediate. Further, $\frac{C_\rho(u)}{u} = \rho + (1-\rho)u$, which is increasing in u . Finally,

$$2 \frac{u - C_\rho(u)}{1-u} = 2u(1-\rho) < \rho + 2u(1-\rho) = (C_\rho(u))',$$

2. $C_0(u) = u^2$ and $C_1(u) = u$ is immediate. Further, $\frac{C_\rho(u)}{u} = u^{1-\rho}$, which is increasing in u . Finally, consider:

$$\begin{aligned} g(u) &= (C_\rho(u))' - 2 \frac{u - C_\rho(u)}{1-u} \\ &= (2-\rho)u^{1-\rho} - 2 \frac{u - u^{2-\rho}}{1-u} \\ &= \frac{u^{1-\rho}}{1-u} \{2 - \rho + \rho u - 2u^\rho\}. \end{aligned}$$

It is straightforward to verify that the function in the curly brackets is non-negative.

3. As is well-known:

$$C_\rho(u) = 2 \int_0^u \Phi \left(\sqrt{\frac{1-\rho}{1+\rho}} \Phi^{-1}(t) \right) dt.$$

Hence, $C_0(u) = u^2$ and $C_1(u) = u$ is immediate. Further, since the integrand in the above display is non-negative and increasing in t ,

$$C_\rho(u) \leq 2\Phi \left(\sqrt{\frac{1-\rho}{1+\rho}} \Phi^{-1}(u) \right) u$$

By the Leibniz rule,

$$(C_\rho(u))' = 2\Phi \left(\sqrt{\frac{1-\rho}{1+\rho}} \Phi^{-1}(u) \right),$$

so the inequality in the preceding display states $(C_\rho(u))' \geq \frac{C_\rho(u)}{u}$. It remains to show $(C_\rho(u))' \geq 2\frac{u-C_\rho(u)}{1-u}$. Consider two cases.

(i) Suppose $u \leq \frac{1}{2}$ so that $\Phi^{-1}(u) \leq 0$. Then,

$$(C_\rho(u))' \geq 2u = 2\frac{u-u^2}{1-u} \geq 2\frac{u-C_\rho(u)}{1-u},$$

where the first inequality follows from $\sqrt{\frac{1-\rho}{1+\rho}} \Phi^{-1}(u) \geq \Phi^{-1}(u)$ and the second from $C_\rho(u) \geq u^2$.

(ii) Suppose $u \geq \frac{1}{2}$ so that $\Phi^{-1}(u) \geq 0$. Then, $(C_\rho(u))' \geq 2\Phi(0) = 1$, and

$$\begin{aligned} C_\rho(u) &= \int_0^{1/2} (C_\rho(t))' dt + \int_{1/2}^u (C_\rho(t))' dt \\ &\geq \int_0^{1/2} 2tdt + \int_{1/2}^u 1dt \\ &= u - \frac{1}{4}. \end{aligned}$$

Therefore:

$$2\frac{u-C_\rho(u)}{1-u} \leq \frac{1}{2} \cdot \frac{1}{1-u} \leq 1 = (C_\rho(u))'$$

where the last inequality holds since $u \geq \frac{1}{2}$.

■

A.2 Lemma 1

Proof of Lemma 1. The upper bound on $F_{N:N}(v)$ is trivial and binds in the case of pure common values. For the lower bound, Lemma A.1 establishes that $u \mapsto \phi_N(u)^N$ is strictly convex. Thus:

$$\begin{aligned} P(V_{N:N} \leq v) &= \mathbb{E}[P(V_{N:N} \leq v | U)] \\ &= \mathbb{E}[P(V_i \leq v | U)^N] \\ &\stackrel{(a)}{=} \mathbb{E}[\phi_N(P(V_{N-1:N} \leq v | U))^N] \\ &\stackrel{(b)}{\geq} \phi_N(F_{N-1:N}(v))^N, \end{aligned}$$

where (a) follows from the definition of $\phi_N(\cdot)$, and (b) from convexity of $t \mapsto \phi_N(t)^N$, Jensen's inequality, and the law of iterated expectations. \blacksquare

A.3 Theorem 1

Proof of Theorem 1. In the main part of the proof, we show that the stated optimization problems lead to pointwise sharp bounds on $F_{N:N}(v)$. Lemmas A.3 and A.4 show that the bounds are monotone in $F_{N-1:N}(v)$ and thus are plausibly sharp in the functional sense.

For the lower bound, consider the generalized moment problem:

$$\inf_{P \in \mathcal{M}[0,1]} \left\{ \mathbb{E}_P[\phi_N(T)^N] \mid \mathbb{E}_P[T] = \mu, c_1 \leq \text{Var}_P(T) \leq c_2 \right\} \quad (\text{A.3})$$

where $\mathcal{M}[0,1]$ denotes the set of all probability measures supported on $[0,1]$, $\mu \in [0,1]$ and $0 \leq c_1 \leq c_2 \leq \mu(1-\mu)$. Denote $h(t) = \phi_N(t)^N$. Lemma A.1 establishes that $h(t)$ is strictly increasing, strictly convex, and that $h'(t)$ changes curvature exactly once and satisfies $h''(0) = h''(1) = \infty$. Since $h(t)$ is convex, the variance constraint $\text{Var}_P(T) = c_1$ must be binding, so the problem is:

$$\inf_{P \in \mathcal{M}[0,1]} \left\{ \mathbb{E}_P[h(T)] \mid \mathbb{E}_P[T] = \mu, \mathbb{E}_P[T^2] = \mu^2 + c_1 \right\}. \quad (\text{A.4})$$

When $c_1 = 0$, by Jensen's inequality, the infimum is attained by the distribution $P(T = \mu) = 1$ and equals $h(\mu)$. When $c_1 = \mu(1-\mu)$, the only feasible distribution on $[0,1]$ is the Bernoulli distribution with $P(T = 1) = \mu$. In this case, the infimum is equal to μ . It remains to consider $c_1 \in (0, \mu(1-\mu))$.

It is well known that it suffices to consider distributions P with at most three support points (e.g., Theorem 1 in [Kemperman, 1968](#)). We first show that, due to the specific shape of the function $h(\cdot)$ and its derivative, it suffices to consider distributions with two support

points. Denote $g_1(t) = t$, $g_2(t) = t^2$, and $g(t) = (g_1(t), g_2(t))$ all defined on $\mathcal{T} = [0, 1]$. Note that $V = \text{Conv}(g(\mathcal{T})) = \{(z_1, z_2) \in [0, 1]^2 : z_1^2 \leq z_2 \leq z_1\}$, and the point $y = (\mu, \mu^2 + c_1) \in \text{Int}(V)$. Denote $D^* = \{d^* = (d_0, d_1, d_2) \in \mathbb{R}^3 : d_0 + d_1 g_1(t) + d_2 g_2(t) \leq h(t) \text{ for all } t \in [0, 1]\}$, and, for a given $d^* \in D^*$, let $B(d^*) = \{z = g(t) : d_0 + d_1 g_1(t) + d_2 g_2(t) = h(t) \text{ for some } t \in T\}$. By Theorem 5 and the following remark in Kemperman (1968), for every $y \in \text{Int}(V)$, there exists a d^* for which $y \in \text{Conv}(B(d^*))$. By Theorem 4 of the same paper, such y can be expressed as $y = \sum_{j=1}^m p_j g(t_j)$ for some $g(t_j) \in B(d^*)$, and the minimal value of the moment problem is given by $\sum_{j=1}^m p_j h(t_j)$. In the problem under consideration, $B(d^*)$ can contain at most two points. Suppose $k(t) = d_0 + d_1 t + d_2 t^2 \leq h(t)$, for all $t \in \mathcal{T}$, with equality for some $t_1 < \dots < t_m$. Then: (i) if $t_1 = 0$, there can be at most one other $t_2 \in (0, 1)$; (ii) if all $t_j \in (0, 1)$, it must be the case that the line $k'(t) = d_1 + 2d_2 t$ intersects the curve $h'(t)$ from above at each t_j .¹⁵ Indeed, case (i) follows from direct computation, and case (ii) from a simple geometric fact that since $h'(t)$ changes curvature only once (concave then convex) and satisfies $h''(0) = h''(1) = \infty$, there are at most two interior intersections of $d_1 + 2d_2 t$ and $h'(t)$. Thus, the set $B(d^*)$ contains at most two points, so to solve (A.4), it suffices to consider distributions P with two support points.

For some $0 \leq a < b \leq 1$ and $p \in [0, 1]$, consider the distribution P with $P(X = a) = p$ and $P(X = b) = 1 - p$. The constraints are:

$$\begin{cases} ap + b(1-p) = \mu \\ a^2 p + b^2(1-p) = \mu^2 + c_1 \end{cases} \implies \begin{cases} a(p) = \mu - \sqrt{\frac{1-p}{p}c_1} \\ b(p) = \mu + \sqrt{\frac{p}{1-p}c_1}, \end{cases}$$

and p must be subject to $a, b \in [0, 1]$. When $p = 0$ or $p = 1$, $a = b = \mu$ which violates the variance constraint for $c_1 > 0$. Thus, the problem is:

$$\begin{aligned} & \min_{p \in (0,1)} \left\{ h\left(\mu - \sqrt{\frac{1-p}{p}c_1}\right)p + h\left(\mu + \sqrt{\frac{p}{1-p}c_1}\right)(1-p) \right\} \\ & \text{s.t. } a(p), b(p) \in [0, 1] \end{aligned}$$

Letting $s = \sqrt{\frac{1-p}{p}c_1}$ so that $p = \frac{c_1}{c_1+s^2}$ yields an equivalent formulation:

$$\min_{s \in [\frac{c_1}{1-\mu}, \mu]} \left\{ h(\mu - s) \frac{c_1}{c_1 + s^2} + h\left(\mu + \frac{c_1}{s}\right) \frac{s^2}{c_1 + s^2} \right\}.$$

By direct computation, the objective function is seen to be strictly convex on the feasible

¹⁵For $t_j \in (0, 1)$ it means that $k'(t) > h'(t)$ immediately before t_j and $k'(t) < h'(t)$ immediately after. For $t_j \in \{0, 1\}$ only one of the two preceding inequalities is required to hold.

set.

Next, consider the generalized moment problem:

$$\sup_{P \in \mathcal{M}[0,1]} \left\{ \mathbb{E}_P[h(T)] \mid \mathbb{E}_P[T] = \mu, c_1 \leq \text{Var}_P(T) \leq c_2 \right\},$$

or, equivalently,

$$-\inf_{P \in \mathcal{M}[0,1]} \left\{ \mathbb{E}_P[-h(T)] \mid \mathbb{E}_P[T] = \mu, c_1 \leq \text{Var}_P(T) \leq c_2 \right\}.$$

Since $f(t) = -h(t)$ is concave, the variance constraint $\text{Var}_P(T) = c_2$ must be binding, so the problem is:

$$-\inf_{P \in \mathcal{M}[0,1]} \left\{ \mathbb{E}_P[f(T)] \mid \mathbb{E}_P[T] = \mu, \mathbb{E}_P[T^2] = \mu^2 + c_2 \right\}, \quad (\text{A.5})$$

with $\mu \in [0, 1]$ and $c_2 \in [0, \mu(1-\mu)]$. When $c_2 = 0$, by the Edmundson-Mandansky inequality, the infimum is attained by the distribution $P(T = \mu) = 1$ and equals $f(\mu)$. When $c_2 = \mu(1-\mu)$, the only feasible distribution on $[0, 1]$ is the Bernoulli distribution with $P(T = 1) = \mu$. In this case, the infimum is equal to μ . It remains to consider $c_2 \in (0, \mu(1-\mu))$.

Using the idea and notation from the first part of the proof, suppose $k(t) = d_0 + d_1t + d_2t^2 \leq f(t)$ for all $t \in \mathcal{T}$ with equality for some $t_1 < \dots < t_m$. Then: at each $t_j \in (0, 1)$, the line $k'(t)$ must intersect $f'(t)$ from above; if $t_1 = 0$, it must be $k'(0) \leq f'(0)$; and if $t_m = 1$, it must be that $k'(1) \geq f'(1)$. The function $f'(t)$ changes curvature exactly once (convex then concave) and satisfies $f'(0) = 0$ and $f'(1) = -\infty$. Thus, there can be at most one interior t_j satisfying the requirement above. By Theorem 4 of [Kemperman \(1968\)](#), it suffices to consider distributions with tree support points: 0, 1, and $t \in (0, 1)$. Letting $P(T = t) = q$, $P(T = 1) = p$, and $P(T = 0) = 1 - p - q$, the constraints in (A.5) become:

$$\begin{cases} p + qt = \mu \\ p + qt^2 = \mu^2 + c_2 \end{cases} \implies \begin{cases} q(t) = \frac{\mu(1-\mu)-c_2}{t(1-t)} \\ p(t) = \mu - q(t) \cdot t \end{cases},$$

subject to $q(t), p(t) \geq 0$ and $q(t) + p(t) \leq 1$. Plugging this into (A.5) and rearranging yields:

$$\max_{t \in [\mu - \frac{c_2}{1-\mu}, \mu + \frac{c_2}{\mu}]} \left\{ \mu - \frac{\mu(1-\mu)-c_2}{t(1-t)}(t - h(t)) \right\}.$$

■

A.3.1 Properties of Bounds in Theorem 1

Lemma A.3 (Properties of the Lower Bound in Theorem 1). *Consider the function:*

$$f(\mu) = \min_{s \in [\frac{c_1}{1-\mu}, \mu]} \left\{ \phi_N(\mu - s)^N \frac{c_1}{c_1 + s^2} + \phi_N \left(\mu + \frac{c_1}{s} \right)^N \frac{s^2}{c_1 + s^2} \right\}$$

with $c_1 = C_\rho(\mu, \mu) - \mu^2$ with C_ρ satisfying the requirements in Assumption 2.3. Then, $\mu \mapsto f(\mu)$ is non-decreasing.

Proof. We suppress the dependence of c_1 on μ to simplify the notation. Denoting the objective function in the optimization problem by $r(s; \mu, c_1)$,

$$\frac{\partial r(s; \mu, c_1)}{\partial s} = \frac{2c_1 s}{(c_1 + s^2)} \left\{ h \left(\mu + \frac{c_1}{s} \right) - h(\mu - s) \right\} - \frac{c_1}{c_1 + s^2} \left\{ h'(\mu - s) + h' \left(\mu + \frac{c_1}{s} \right) \right\}.$$

First, note that as $s \rightarrow \frac{c_1}{1-\mu}$, we have $\mu - c_1/s \rightarrow 1$ and thus $h'(\mu - c_1/s) \rightarrow +\infty$, by Lemma A.1. Since all other terms in the above expression stay finite, $r'(s; \mu, c_1) \rightarrow -\infty$. Thus, the constraint $s \geq c_1/(1 - \mu)$ never binds, and it suffices to consider two cases: (i) $s = \mu$ and (ii) $s \in (c_1/(1 - \mu), \mu)$. In case (i), the lower bound is given by:

$$f(\mu) = \frac{\mu^2}{c_1 + \mu^2} h \left(\mu + \frac{c_1}{\mu} \right) = \mu \frac{h \left(\frac{C_\rho(\mu, \mu)}{\mu} \right)}{\frac{C_\rho(\mu, \mu)}{\mu}}.$$

The function $u \mapsto h(u)/u$ is increasing, by Lemma A.1, and $C_\rho(\mu, \mu)/\mu$ is increasing by Assumption 2.3, so $f(\mu)$ is increasing. For case (ii), by the Envelope Theorem:

$$f'(\mu) = \frac{\partial r(s, \mu, c_1)}{\partial \mu} \Big|_{s=s^*(\mu)} + \frac{\partial r(s; \mu, c_1)}{\partial c_1} \Big|_{s=s^*(\mu)} \cdot \frac{\partial c_1}{\partial \mu},$$

where $s^*(\mu)$ solves the first order condition $\frac{\partial r(s; \mu, c_1)}{\partial s} = 0$,

$$\frac{\partial r(s, \mu, c_1)}{\partial \mu} = \frac{c_1}{c_1 + s^2} h'(\mu - s) + \frac{s^2}{c_1 + s^2} h' \left(\mu + \frac{c_1}{s} \right),$$

and

$$\frac{\partial r(s; \mu, c_1)}{\partial c_1} = \frac{s^2}{(c_1 + s^2)^2} \left\{ h(\mu - s) - h \left(\mu + \frac{c_1}{s} \right) \right\} + \frac{s}{c_1 + s^2} h' \left(\mu + \frac{c_1}{s} \right). \quad (\text{A.6})$$

Thus, the condition $f'(\mu) \geq 0$ is equivalent to:

$$\frac{\partial c_1(\mu)}{\partial \mu} \geq -\left. \frac{\frac{\partial r(s, \mu, c_1)}{\partial \mu}}{\frac{\partial r(s; \mu, c_1)}{\partial c_1}} \right|_{s=s^*(\mu)} \quad (\text{A.7})$$

From the first order condition, at $s = s^*(\mu)$,

$$h\left(\mu + \frac{c_1}{s}\right) - h(\mu - s) = \frac{c_1 + s^2}{2} \left\{ h'(\mu - s) + h'\left(\mu + \frac{c_1}{s}\right) \right\}.$$

Plugging the above in (A.6) and the result in (A.7), and applying the Mean Value Theorem yields:

$$\begin{aligned} & -\left. \frac{\frac{\partial r(s, \mu, c_1)}{\partial \mu}}{\frac{\partial r(s; \mu, c_1)}{\partial c_1}} \right|_{s=s^*(\mu)} = -2 \left. \frac{h'(\mu - s) + \frac{s^2}{c_1 + s^2} \{h'(\mu + \frac{c_1}{s}) - h'(\mu - s)\}}{\frac{s}{c_1 + s^2} \{h'(\mu + \frac{c_1}{s}) - h'(\mu - s)\}} \right|_{s=s^*(\mu)} \\ & = -2 \left. \frac{h'(\mu - s) + sh''(\tilde{\mu})}{h''(\tilde{\mu})} \right|_{s=s^*(\mu)} \\ & = -2(s^*(\mu) + \frac{h'(\mu - s^*(\mu))}{h''(\tilde{\mu})}), \end{aligned}$$

for some $\tilde{\mu} \in [\mu - s^*(\mu), \mu + \frac{c_1}{s^*(\mu)}]$. Since both $h'(\cdot)$ and $h''(\cdot)$ are positive, and $s^*(\mu) \geq c_1/(1 - \mu)$ a sufficient condition for the lower bound to increase in μ is:

$$\frac{\partial c_1(\mu)}{\partial \mu} \geq -2 \frac{c_1(\mu)}{1 - \mu}.$$

Plugging in the expression for $c_1(\mu)$ and simplifying yields:

$$(C_\rho(\mu, \mu))' \geq 2 \cdot \frac{\mu - C_\rho(\mu, \mu)}{1 - \mu}.$$

■

Lemma A.4 (Properties of the Upper Bound in Theorem 1). *Consider the function:*

$$f(\mu) = \max_{s \in [\mu - \frac{c_2}{1-\mu}, \mu + \frac{c_2}{\mu}]} \left\{ \mu - \frac{\mu(1 - \mu) - c_2}{s(1 - s)} (s - h(s)) \right\}.$$

where $h(s) = \phi_N(s)^N$ and $c_2 = C_\rho(\mu, \mu) - \mu^2$ with C_ρ satisfying the requirements in Assumption 2.3. Then, $\mu \mapsto f(\mu)$ is non-decreasing.

Proof. Let $s^*(\mu)$ denote the argmax. There are three possible cases.

- (i) $s^*(\mu) = \mu - \frac{c_2}{1-\mu} = \frac{\mu - C_\rho(\mu, \mu)}{1-\mu}$. Plugging this in the objective function and simplifying yields:

$$f(\mu) = \underbrace{\frac{C_\rho(\mu, \mu) - \mu^2}{C_\rho(\mu, \mu) - \mu^2 + (1-\mu)^2}}_{\alpha(\mu)} \cdot 1 + \underbrace{\frac{(1-\mu)^2}{C_\rho(\mu, \mu) - \mu^2 + (1-\mu)^2}}_{1-\alpha(\mu)} h(s^*(\mu)).$$

Since $h(s^*(\mu)) \leq 1$, for all μ , it suffices to show that $\alpha(\mu)$ is increasing. In turn, this is equivalent to

$$\beta(\mu) = \frac{(1-\mu)^2}{C_\rho(\mu, \mu) - \mu^2}$$

being decreasing. By direct calculation, $\beta'(\mu) \leq 0$ is equivalent to:

$$(C_\rho(\mu, \mu))' \geq 2 \frac{\mu - C_\rho(\mu, \mu)}{1-\mu},$$

which is one of the conditions in Assumption 2.3.

- (ii) $s^*(\mu)$ is interior. Denoting the objective function in the optimization problem by $r(s; \mu, c_2)$, by the Envelope Theorem:

$$\begin{aligned} f'(\mu) &= \left. \frac{\partial r(s, \mu, c_2)}{\partial \mu} \right|_{s=s^*(\mu)} + \left. \frac{\partial r(s; \mu, c_2)}{\partial c_2} \right|_{s=s^*(\mu)} \cdot \frac{\partial (C_\rho(\mu, \mu) - \mu^2)}{\partial \mu} \\ &= 1 - (1-2\mu) \frac{s^* - h(s^*)}{s^* - s^{*2}} + \frac{s^* - h(s^*)}{s^* - s^{*2}} \cdot \left(\frac{\partial C_\rho(\mu, \mu)}{\partial \mu} - 2\mu \right) \\ &= 1 + t(s^*) \left(\frac{\partial C_\rho(\mu, \mu)}{\partial \mu} - 1 \right), \end{aligned}$$

where $t(s^*) \equiv \frac{s^* - h(s^*)}{s^* - s^{*2}}$. Notice the solution s^* does not depend on μ and corresponds to the minimum of the function $t(s)$. Since $\frac{\partial C_\rho(\mu, \mu)}{\partial \mu} \geq 0$, to conclude that $f'(\mu) \geq 0$ it suffices to show that $t(s^*) \in [0, 1]$. Since $h(s) \leq s$, we have that $t(s) \geq 0$ for all s . Writing $t(s) = 1 + \frac{s^2 - h(s)}{s - s^2} \leq 1$, it suffices to show $s^2 - h(s) \leq 0$ for some s . By the Taylor expansion, $h(s) = \frac{h''(\tilde{s})}{2}s^2$, for some $\tilde{s} \in (0, s)$, and, by Lemma A.1, $h''(\tilde{s}) \rightarrow +\infty$ as $\tilde{s} \rightarrow 0$. Thus, for s close to zero, $s^2 - h(s) < 0$, and the result follows.

(iii) $s^*(\mu) = \mu + \frac{c_2}{\mu} = \frac{C_\rho(\mu, \mu)}{\mu}$. Plugging this in the objective function and simplifying yields:

$$f(\mu) = \mu \frac{h\left(\frac{C(\mu, \mu)}{\mu}\right)}{\left(\frac{C(\mu, \mu)}{\mu}\right)}.$$

Since $\mu \mapsto \frac{C(\mu, \mu)}{\mu}$ is increasing, by Assumption 2.3, and $t \mapsto \frac{h(t)}{t}$ is increasing, by Lemma A.1, it follows that $f(\mu)$ is increasing. ■

A.4 Theorem 3

Proof of Theorem 3. Starting with Equation (6) and treating the lower and the upper bounds separately, the rest of the proof proceeds exactly like the proof of Theorem 1. ■

A.5 Min-Max-Regret Theorems 5 and 6

The following lemma establishes that, given any vector of order statistics, it is without loss of generality to assume that the underlying distribution is exchangeable.

Lemma A.5 (Symmetrization). *Let $X = (X_1, \dots, X_n) \in \mathbb{R}^n$ be a random vector with an arbitrary joint distribution. Let $X^{1:n} = (X_{1:n}, \dots, X_{n:n})$ denote the vector of order statistics, where $X_{j:n}$ is the j -th smallest of (X_1, \dots, X_n) . There exists a random vector $Y = (Y_1, \dots, Y_n) \in \mathbb{R}^n$ such that: (1) Y is exchangeable; and (2) $Y^{1:n} = X^{1:n}$ almost surely.*

Proof. Let Π denote the set of all permutations $p : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$, and π be drawn uniformly in Π , i.e., $P(\pi = p) = 1/n!$ for all $p \in \Pi$. Define $(Y_j)_{j=1}^n = (X_{\pi(j)})_{j=1}^n$. Then:

$$P(Y_1 \leq y_1, \dots, Y_n \leq y_n) = \frac{1}{n!} \sum_{p \in \Pi} P(X_{p(1)} \leq y_1, \dots, X_{p(n)} \leq y_n).$$

The summation in the right hand side includes all possible events of the form $\{X_{j_1} \leq y_1, \dots, X_{j_n} \leq y_n\}$, so an arbitrary permutation of $\{y_1, \dots, y_n\}$ changes the order of summands but not the value of the sum in the above display. Therefore, for any permutation p ,

$$P(Y_1 \leq y_{p(1)}, \dots, Y_n \leq y_{p(n)}) = P(Y_1 \leq y_1, \dots, Y_n \leq y_n).$$

Since rearranging the elements of X does not affect the order statistics, we have $Y^{1:n} = X^{1:n}$ for all realizations of π , so that $P(Y^{1:n} = X^{1:n}) = 1$. ■

A.5.1 Theorem 5

Proof of Theorem 5. Let F denote $F_{N-1:N}$ and F^* denote $F_{N:N}$, for simplicity. Let $\pi(r; F^*)$ denote the expected profit function in (1), and \mathcal{F} the set of all admissible F^* . Fix some reserve $r \geq v_0$ and consider an alternative $\tilde{r} \geq v_0$. Denote $D(\tilde{r}, r; F^*) = \pi(\tilde{r}; F^*) - \pi(r; F^*)$, which is

$$D(\tilde{r}, r; F^*) = \int_{\tilde{r}}^r (1 - F(v))dv + \tilde{r} - r + \{F^*(r)(r - v_0) - F^*(\tilde{r})(\tilde{r} - v_0)\}.$$

The maximum regret is given by $R(r) = \max\{R_1(r), R_2(r)\}$, where

$$R_1(r) = \sup_{F_1} \sup_{\tilde{r} < r} D(\tilde{r}, r; F^*).$$

$$R_2(r) = \sup_{F_1} \sup_{\tilde{r} > r} D(\tilde{r}, r; F^*).$$

Considering $D(\tilde{r}, r; F^*)$, the supremum $R_1(r)$ is attained by (i) maximizing $F^*(r)$; and (ii) minimizing $F^*(\tilde{r})$ for all $\tilde{r} < r$. Evidently, both conditions are satisfied by the CDF $F_{1,r}^*(\cdot)$ in the statement of the theorem. Similarly, the supremum $R_2(r)$ is attained by: (i) maximizing $F^*(r)$; and (ii) minimizing $F^*(\tilde{r})$ for all $\tilde{r} > r$. Condition (i) is achieved by setting $F^*(r) = \bar{F}(r)$. Since F^* must be non-decreasing, condition (ii) is achieved by setting $F^*(v) = \bar{F}(r)$ for all $v \in [r, \underline{F}^{-1}(\bar{F}(r))]$ and to $F^*(v) = \underline{F}(v)$ afterwards. Both of these conditions are satisfied by the CDF $F_{2,r}^*(\cdot)$ in the statement of the theorem. Hence, the result follows. ■

A.5.2 Theorem 6

Proof of Theorem 6. The argument is essentially the same as in Theorem 5. The supremum $R_1(r)$ is attained by (i) maximizing $F_1(r)$; and (ii) minimizing $F_1(\tilde{r})$ for all $\tilde{r} < r$. Since it must be that $F_1(v) = g_1(F_2(v))$ for a convex function g_1 , Conditions (i) and (ii) can only be attained by defining g_1 as in the statement of the Theorem. The middle part of g_1 is restricted to be linear precisely to preserve convexity while minimizing $F_1(\tilde{r})$ for all $r \leq \tilde{r}$. A similar argument follows for the supremum $R_2(r)$ and g_2 . ■

Finally, we show the following auxiliary result.

Lemma A.6 (CDFs of Order Statistics under Monotone Likelihood Ratio). *Let Assumptions 3.2 (i) and (iv) hold. Then,*

$$F_{N:N} = g_N(F_{N-1:N})$$

for some increasing convex function $g_N : [0, 1] \rightarrow [0, 1]$.

Proof. By exchangeability,

$$\begin{aligned} F_{N-1:N}(v) &= N \cdot F_{N-1:N-1}(v) - (N-1) \cdot F_{N:N}(v) \\ &= N \cdot C_{N-1}(F(v)) - (N-1) \cdot C_N(F(v)), \end{aligned}$$

and

$$F_{N:N}(v) = C_N(F(v)),$$

where $C_J : [0, 1] \rightarrow [0, 1]$ is defined as $C_J(u) = \tilde{C}_J(u, \dots, u)$ where \tilde{C}_J is the copula function of (V_1, \dots, V_J) , for $J \in \{N-1, N\}$. Denote $f(u) = NC_{N-1}(u) - (N-1)C_N(u)$ and notice this function is strictly increasing. It follows that:

$$F_{N:N}(v) = C_N(f^{-1}(F_{N-1:N}(v))) \equiv g(F_{N-1:N}(v)).$$

By direct calculation:

$$g''(u) = \frac{C_N''(s_u)f'(s_u) - C_N'(s_u)f''(s_u)}{f'(s_u)^2},$$

where $s_u = f^{-1}(u)$. Plugging in the expressions for $f'(s_u)$ and $f''(s_u)$ and simplifying yields that $g''(u) \geq 0$ for all $u \in [0, 1]$ holds if and only if:

$$\frac{C_N''(s)}{C_N'(s)} \geq \frac{C_{N-1}''(s)}{C_{N-1}'(s)},$$

for all $s \in [0, 1]$. Denoting $c_J(s) = C'_J(s)$ for $J \in \{N-1, N\}$, the above is equivalent to:

$$\frac{\partial}{\partial s} \left(\frac{c_N(s)}{c_{N-1}(s)} \right) \geq 0.$$

Since $c_N(v) = f_{V_{N:N}}(v)$ and $f_{V_{N-1:N}}(v) = Nc_{N-1}(v) - (N-1)f_{V_{N:N}}(v)$, it follows that:

$$\frac{f_{V_{N-1:N}}(v)}{f_{V_{N:N}}(v)} = N \frac{c_{N-1}(v)}{c_N(v)} - (N-1),$$

so the ratio $\frac{f_{V_{N:N}}(v)}{f_{V_{N-1:N}}(v)}$ is increasing if and only if $\frac{c_N(s)}{c_{N-1}(s)}$ is, which concludes the proof. ■

B Estimation

To simplify notation, we denote $F(v) = F_{N-1:N}(v)$ throughout. The exposition below assumes that N is known and fixed and $X = \emptyset$. To account for varying N and discrete X , it suffices to modify the estimator \hat{F} and its asymptotic distribution.

B.1 Profit Bounds

Let $\hat{F}(v) = \frac{1}{n} \sum_{i=1}^n \mathbf{1}(P_i \leq v)$ denote the sample analog estimator for F . Its pointwise asymptotic distribution is:

$$\sqrt{n}(\hat{F}(v) - F(v)) \rightarrow_d N(0, V_F(v)),$$

where $V_F(v) = F(v)(1 - F(v))$. The profit bounds from Theorem 2 are estimated as:

$$\begin{aligned}\hat{\underline{\pi}}(r) &= \frac{1}{n} \sum_{i=1}^n \max(r, P_i) - v_0 - (r - v_0)\bar{\psi}(\hat{F}(r)) \\ \hat{\bar{\pi}}(r) &= \frac{1}{n} \sum_{i=1}^n \max(r, P_i) - v_0 - (r - v_0)\underline{\psi}(\hat{F}(r)).\end{aligned}\tag{B.1}$$

Using delta-method, the asymptotic distributions are:

$$\begin{aligned}\sqrt{n}(\hat{\underline{\pi}}(r) - \underline{\pi}(r)) &\rightarrow_d N(0, V_{\underline{\pi}}(r)); \\ \sqrt{n}(\hat{\bar{\pi}}(r) - \bar{\pi}(r)) &\rightarrow_d N(0, V_{\bar{\pi}}(r)),\end{aligned}$$

where:

$$\begin{aligned}V_{\underline{\pi}}(r) &= \text{Var}(\max(r, P)) + (r - v_0)^2 \bar{\psi}'(F(r))^2 V_F(r) - 2(r - v_0) \bar{\psi}'(F(r))(r - \mathbb{E}[\max(r, P)]) F(r); \\ V_{\bar{\pi}}(r) &= \text{Var}(\max(r, P)) + (r - v_0)^2 \underline{\psi}'(F(r))^2 V_F(r) - 2(r - v_0) \underline{\psi}'(F(r))(r - \mathbb{E}[\max(r, P)]) F(r).\end{aligned}$$

Under standard regularity conditions, the plug-in variance estimators $\hat{V}_{\underline{\pi}}(r)$ and $\hat{V}_{\bar{\pi}}(r)$ are consistent.

B.2 MMR Reserve Price

Let \hat{r} be defined in Equation (13). Suppose (i) The set of possible reserve prices \mathcal{R} is compact; (ii) $\hat{F}_{N-1:N}$ is consistent for $F_{N-1:N}$ uniformly over \mathcal{R} ; (ii) the MMR solution $r^* \in \mathcal{R}$ is unique. Then, by the standard arguments, \hat{r} is consistent for r . To obtain the limiting distribution, suppose, for concreteness, that $\hat{F}_{N-1:N}$ is \sqrt{n} -consistent for $F_{N-1:N}$.

By the mean-value theorem,

$$\hat{\Delta}(\hat{r}) - \hat{\Delta}(r^*) = \hat{\Delta}'(\tilde{r})(\hat{r} - r^*)$$

for some \tilde{r} between \hat{r} and r^* . Suppose that: (i) $\sqrt{n}\hat{\Delta}(\hat{r}) = o_P(1)$; (ii) $\sqrt{n}\hat{\Delta}(r^*) \rightarrow_d N(0, V_\Delta)$; (iii) $\sup_{r \in \mathcal{R}} |\hat{\Delta}'(r) - \Delta'(r)| = o_P(1)$; and (iv) $\Delta'(r^*) > 0$. Condition (i) allows for approximate solution $\hat{\Delta}(\hat{r}) \approx 0$; Condition (ii) can be established using functional delta-method, viewing Δ as a functional of F , although the expression for V_Δ is cumbersome; Condition (iii) follows from continuity of $F \mapsto \Delta'(r; F)$; and Condition (iv) makes first-order mean-value expansion of $\hat{\Delta}(\cdot)$ sufficient. Then, by Slutsky's theorem,

$$\sqrt{n}(\hat{r} - r^*) \rightarrow_d N\left(0, \frac{V_\Delta}{\Delta'(r^*)^2}\right). \quad (\text{B.2})$$

B.3 Expected Profit Under MMR Optimal Reserve Price

Consider the expansion from equation (16):

$$\begin{aligned} \sqrt{n}(\hat{\underline{\pi}}^* - \underline{\pi}^*) &= \{\sqrt{n}(\hat{\underline{\pi}}(\hat{r}) - \underline{\pi}(\hat{r})) - \sqrt{n}(\hat{\underline{\pi}}(r^*) - \underline{\pi}(r^*))\} && (\text{I}) \\ &\quad + \sqrt{n}(\hat{\underline{\pi}}(r^*) - \underline{\pi}(r^*)) && (\text{II}) \\ &\quad + \sqrt{n}(\underline{\pi}(\hat{r}) - \underline{\pi}(r^*)). && (\text{III}) \end{aligned} \quad (\text{B.3})$$

Term (I) can be written as:

$$\begin{aligned} (\text{I}) &= \frac{1}{\sqrt{n}} \sum_{i=1}^n \{\max(\hat{r}, P_i) - \mathbb{E}_P[\max(\hat{r}, P)]\} - \frac{1}{\sqrt{n}} \sum_{i=1}^n \{\max(r^*, P_i) - \mathbb{E}_P[\max(r^*, P)]\} && (\text{A}) \\ &\quad + \sqrt{n} \left(\{\underline{\psi}(\hat{F}(\hat{r})) - \underline{\psi}(F(\hat{r}))\} - \{\underline{\psi}(\hat{F}(r^*)) - \underline{\psi}(F(r^*))\} \right). && (\text{B}) \end{aligned}$$

The term (A) takes the form:

$$(A) = \mathbb{G}_n(\hat{f} - f)$$

where $\mathbb{G}_n(h) = \frac{1}{\sqrt{n}} \sum_{i=1}^n h(P_i) - \mathbb{E}[h(P)]$ denotes the empirical process, $f(P) = \max(r, P)$, and $\hat{f}(P) = \max(\hat{r}, P)$. Notice $\mathcal{F} = \{f(P) = \max(r, P) : r \in \mathcal{R}\}$ is a VC class of functions with a square-integrable envelope $\mathbb{E}[\max(\bar{r}, P)^2] < \infty$. Thus, \mathcal{F} is Donsker, so in particular, $\mathbb{G}_n(\cdot)$ is asymptotically equicontinuous, i.e., for every $\varepsilon, \eta > 0$ there is a $\delta > 0$ such that:

$$\limsup_{n \rightarrow \infty} P\left(\sup_{\rho(f,g) < \delta} |\mathbb{G}_n(f - g)| > \varepsilon\right) < \eta$$

where $\rho(f, g) = \mathbb{E}[(f(P) - g(P))^2]^{1/2}$. Since $\rho(\hat{f}_n, f) \rightarrow_p 0$, the above immediately implies $(A) = o_P(1)$. This result can also be shown directly using symmetrization lemma and a suitable maximal inequality. Further, by the mean-value theorem:

$$(B) = \sqrt{n}\{\underline{\psi}'(\tilde{F})(\hat{F}(\hat{r}) - F(\hat{r})) - \underline{\psi}'(\bar{F})(\hat{F}(r^*) - F(r^*))\} \\ = \underline{\psi}'(\tilde{F}) \cdot \sqrt{n}(\{\hat{F}(\hat{r}) - F(\hat{r})\} - \{\hat{F}(r^*) - F(r^*)\}) \quad (B.1)$$

$$+ (\underline{\psi}(\tilde{F}) - \underline{\psi}(\bar{F})) \cdot \sqrt{n}\{\hat{F}(r^*) - F(r^*)\} \quad (B.2)$$

for some \tilde{F} between $\hat{F}(\hat{r})$ and $F(\hat{r})$ and \bar{F} between $\hat{F}(r^*)$ and $F(r^*)$. The term $(B.1)$ takes the form:

$$(B.1) = \mathbb{G}(\hat{g} - g)$$

where $g(P) = \mathbf{1}(P \leq r)$ and $\hat{g}(P) = \mathbf{1}(P \leq \hat{r})$, so using the same arguments as for (A) , it follows that $(B.1) = o_P(1)$. Finally, since each of $\hat{F}(\hat{r})$, $F(\hat{r})$, and $\hat{F}(r^*)$ converges in probability to $F(r^*)$, we have

$$(B.2) = o_P(1) \cdot \sqrt{n}(\hat{F}(r^*) - F(r^*)) = o_P(1).$$

It follows that $(I) = o_P(1)$. By the mean-value theorem and Slutsky's theorem,

$$\sqrt{n}(\hat{\pi}^* - \underline{\pi}^*) = \sqrt{n}(\hat{\pi}(r^*) - \underline{\pi}(r^*)) + \underline{\pi}'(r^*)\sqrt{n}(\hat{r} - r^*) + o_P(1),$$

so the limiting distribution of $\sqrt{n}(\hat{\pi}^* - \underline{\pi}^*)$ is Gaussian. The asymptotic variance takes a very complicated form due to the dependence between the summands in the above display. However, the above arguments immediately imply that a bootstrap approximation of the distribution of $\sqrt{n}(\hat{\pi}^* - \underline{\pi}^*)$, as described in Algorithm 1 is consistent.

C Tables

| Auction Title | URL |
|--|----------------------|
| Impressionist and Modern Art Evening Sale — New York | link |
| 20th/21st Century Art Evening Sales — Hong Kong | link |
| 20th Century Evening Sale — New York | link |
| The Collection of Thomas and Doris Ammann Evening Sale — New York | link |
| The Collection of Anne H. Bass and 20th Century Evening Sale — New York | link |
| 20th/21st Century: London Evening Sale followed by The Art of the Surreal Evening Sale | link |
| 21st Century Evening Sale — New York | link |
| A Century of Art: The Gerald Fineberg Collection Part I — New York | link |
| Post-Millennium Evening Sale — Hong Kong | link |
| 20th/21st Century Art Evening Sales — London | link |
| Rare Watches Including the Property of Michael Schumacher — Geneva | link |
| Magnificent Jewels — Geneva | link |
| 20th/21st Century Evening Sales — Hong Kong | link |
| 20th/21st Century: London to Paris — Christies | link |
| 21st Century Evening Sale — New York | link |
| 20th/21st Century Art Auctions — Christie's Hong Kong | link |
| 20th/21st Century: Evening Sale Including Thinking Italian, London — Christies | link |
| The Cox Collection and 20th Century Evening Sale — New York | link |
| 20th/21st Century: Shanghai to London | link |
| The Collection of Thomas and Doris Ammann Evening Sale — New York | link |
| 20th/21st Century Art Evening Sales — Hong Kong | link |
| 20th/21st Century: London to Paris Evening Sales | link |
| 20th/21st Century: London | link |
| The Ann & Gordon Getty Evening Sale — New York | link |

Table 5: Christie's YouTube Data Sources

| Auction Title | URL |
|---|----------------------|
| Hong Kong Contemporary Art Evening Sale (LIVE) | link |
| LIVE from New York — Modern Evening Auction | link |
| LIVE from New York — The Now and Contemporary Evening Auctions | link |
| LIVE from Hong Kong — The Now and Modern & Contemporary Evening Auctions | link |
| LIVE from London — Modern & Contemporary Evening Auction featuring The Now | link |
| LIVE from New York — The Now and Contemporary Evening Auctions | link |
| LIVE from New York — The Modern Evening Auction | link |
| LIVE from New York — The Emily Fisher Landau Collection: An Era Defined Evening Auction | link |
| LIVE from London — The Now & Contemporary Evening Auctions | link |
| LIVE from Hong Kong — The Autumn Sales | link |
| LIVE from London — Freddie Mercury: A World of His Own Evening Sale | link |
| LIVE from London — Old Master & 19th Century Paintings Evening Auction | link |
| LIVE — The Now & Modern and Contemporary Auctions, ft. Face to Face: A Celebration of Portraiture | link |
| The Mo Ostin Collection Evening Auction & The Modern Evening Auction | link |
| LIVE from New York — The Now & Contemporary Evening Auctions | link |
| LIVE from London — The Now and Modern & Contemporary Evening Auctions | link |
| LIVE from New York — The Masters Week Auctions | link |
| LIVE from New York — Master Paintings & Sculpture Part I | link |
| LIVE from New York — The David M. Solinger Collection & Modern Evening Auctions | link |
| LIVE from Paris — Modernists | link |
| LIVE from London — The Now & Contemporary Evening Auctions | link |
| LIVE from Paris — Htel Lambert, The Illustrious Collection, Volume I: Chefs-d'oeuvre | link |
| LIVE from Hong Kong — Modern, Williamson Pink Star & Contemporary Auctions | link |
| LIVE from London — Old Masters Evening Auction | link |
| LIVE from London — The Jubilee Auction and Modern & Contemporary Evening Auction | link |
| LIVE from New York — The Now & Contemporary Evening Auctions | link |
| LIVE from New York — Modern Evening Auction | link |
| LIVE from New York — The Macklowe Collection | link |
| LIVE from New York — Important Watches | link |
| LIVE from London — Old Masters Evening Sale | link |
| LIVE From New York — PROUV x BASQUIAT: The Collection of Peter M. Brant and Stephanie Seymour | link |
| LIVE from New York — Magnificent Jewels | link |
| LIVE from London — Treasures | link |
| LIVE from Monaco — KARL, Karl Lagerfelds Estate Part I | link |

| | |
|--|----------------------|
| LIVE from Edinburgh — The Distillers One of One Whisky Auction | link |
| LIVE from Paris — Art Contemporain Evening Sale | link |
| LIVE from Sotheby's New York — The Now & Contemporary Evening Auctions With U.S. Constitution Sale | link |
| LIVE from Sotheby's New York — Modern Evening Auction | link |
| LIVE from Sotheby's New York — The Macklowe Collection | link |
| LIVE from Paris — Past/Forward and Modernists | link |
| LIVE from Las Vegas: Icons of Excellence & Haute Luxury | link |
| LIVE from Las Vegas — Picasso: Masterworks from the MGM Resorts Fine Art Collection | link |
| LIVE from New York — Collector, Dealer, Connoisseur: The Vision of Richard L. Feigen | link |
| LIVE From Sothebys London — Richter, Banksy and Twombly lead the Contemporary Art Evening Auction | link |
| LIVE From Sothebys Hong Kong — Modern and Contemporary Art Evening Sales | link |
| LIVE from London — Old Masters Evening Sale | link |
| LIVE from London: British Art + Modern & Contemporary Auctions | link |
| LIVE from Hong Kong: Jay Chou x Sothebys — Evening Sale | link |
| LIVE From Sothebys New York — Important Watches | link |
| LIVE from Sothebys New York — Magnificent Jewels | link |
| LIVE from Sothebys Paris — Important Design: from Noguchi to Lalanne | link |
| LIVE from Sothebys New York — Monet, Warhol and Basquiat Lead Marquee Evening Sales | link |
| LIVE From Sothebys Hong Kong — Contemporary Art Evening Sale | link |
| LIVE From Sothebys Hong Kong — Icons and Beyond Legends: Modern Art Evening Sale | link |
| LIVE from Sotheby's Impressionist & Modern Art + Modern Renaissance Auctions | link |
| LIVE from Sotheby's Sales of Important Chinese Art and Chinese Art from the Brooklyn Museum | link |
| LIVE from Sotheby's: The Collection of Hester Diamond Auction in New York | link |
| LIVE from Sotheby's Master Paintings & Sculpture Auction in New York | link |
| LIVE from Sotheby's London Old Masters Evening Sale | link |
| LIVE from Sotheby's marquee Evening Sales of Contemporary and Impressionist & Modern Art | link |

Table 6: Sotheby's YouTube Data Sources

| Saleroom Location | Christie's | | Sotheby's | |
|-------------------|---------------------------------|-------|---------------------------------|-------|
| | Threshold | Rate | Threshold | Rate |
| Hong Kong | \leq HK\$7.5M | 26.0% | \leq HK\$7,500,000 | 26.0% |
| | $>$ HK\$7.5M and \leq HK\$50M | 20.0% | $>$ HK\$7.5M and \leq HK\$40M | 20.0% |
| | $>$ HK\$50M | 14.5% | $>$ HK\$40M | 13.9% |
| London | \leq 700k | 26.0% | \leq 800k | 26.0% |
| | $>$ 700,000 and \leq 4.5M | 20.0% | $>$ 800k and \leq 3.8M | 20.0% |
| | $>$ 4.5M | 14.5% | $>$ 3.8M | 13.9% |
| Paris | \leq 700k | 26.0% | \leq 800k | 26.0% |
| | $>$ 700,000 and \leq 4M | 20.0% | $>$ 800k and \leq 3.5M | 20.0% |
| | $>$ 4M | 14.5% | $>$ 3.5M | 13.9% |
| New York | \leq \$1M | 26.0% | \leq \$1M | 26.0% |
| | $>$ \$1M and \leq \$6M | 20.0% | $>$ \$1M and \leq \$4.5M | 20.0% |
| | $>$ \$6M | 14.5% | $>$ \$4.5M | 13.9% |
| Shanghai | \leq 6M | 26.0% | - | - |
| | $>$ 6M and \leq 40M | 20.0% | - | - |
| | $>$ 40M | 14.5% | - | - |

This table is accurate as of February 7 2022 for Christie's and February 1 2023 for Sotheby's. In the last 10 years, there are only minor changes to the base rate (i.e. lowest threshold category). These buyer premium thresholds are additive, so final transaction amounts are strictly increasing.

Source: Christie's and Sotheby's Websites.

Table 7: Buyer's Premiums in Christie's and Sotheby's Auctions