Optimal Reserve Prices in Art Auctions

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Abstract

This paper introduces a new approach to the identification of ascending auctions by bounding the correlation of valuations across bidders in a precise manner. We apply the approach, which requires only bounds on the number of bidders and the top two bids, to a large novel dataset comprising complete bids from more than 3500 live auctions by Christie's and Sotheby's. We additionally solve the minimax-regret problem applied to this partially-identified model and use it to propose new reserve prices. For modern art sold in New York City, our proposed reserve increases expected profits by 4.7% to 9.9% of the high estimate, equivalent to US$4,900,000 to US$10,400,000 per auction.

Keywords: Nonparametric identification, minimax regret, ascending auctions, unknown number of bidders, correlated values, partial identification, art auctions

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1 Introduction

1.1 Motivation

The auction market for art has large economic significance. In 2023, the two largest art auction houses, Sotheby's and Christie's, reported revenues of US$7.9 billion and USD$6.2 billion respectively. On November 9 and 10, 2022 alone, the 155-work collection of deceased technology billionaire Paul Allen sold for US$1,622,249,500 towards philanthropy, with the most expensive piece *Georges Seurat, Les Poseuses, Ensemble (Petite version), 1888-1890* selling for US$149,240,000.

However, art auctions are challenging to study because they are notoriously secretive (see Akbarpour and Li, 2020; Marra, 2020). On the websites of major art auction houses, one can only find basic information including the low and high estimate, and the final transaction price. Important information for analysis such as the trajectory of bids and the number of bidders are kept secret. Even if a bidder were to be physically present during a live auction, she would not be able to have a full grasp on the number of bidders because besides live bidding, alternate forms of bidding exist today in the live auction room including telephone bidding, online bidding, and absentee bidding, all of which are opaque to the live bidder.

As a result of these difficulties, the empirical study of live art auctions has been very challenging. First, there is limited data available for researchers to study live art auctions. Second, methods must be provided to deal with the lack of identification on the number of bidders, and the inability to fully observe bidder identities. Third, bidder valuations are correlated in a special sense in art auctions. In particular, valuations are unlikely to be i.i.d. due to common components such as investment value and historical value of the well-known pieces of art. On the other hand, pure common values are unlikely too because of differing utility derived from ownership[1], differing levels of knowledge and expertise on specific pieces of art, and financial constraints. Fourth, the auction house must choose a reserve price under statistical ambiguity.

Our paper addresses all of these concerns, extending on the econometric works by Haile and Tamer (2003), Quint (2008), Aradillas-López, Gandhi, and Quint (2013), Chesher and Rosen (2017), Coey, Larsen, Sweeney, and Waisman (2017),Ponomarev

[1] for example, the Saudi Arabian Crown Prince famously jump bid opponents multiple times to purchase the *Salvator Mundi* at US$450m in 2017.

(2022). When applied to our data subset of Modern Art in New York City, our proposed reserve increases expected profits relative to existing realized profits by 6.6% to 11.9% of the high estimate, equivalent to approximately US$6,900,000 to US$12,500,000 per auction.

1.2 Relationship to Literature

This paper contributes to literature in three ways: the methodological literature on nonparametric estimation of English auctions, the theoretical literature on minimax regret, and the empirical literature on art auctions.

1.2.1 Methodological Contribution

The methodological contribution is a novel identification approach that provides sharp bounds on the highest bidder's valuation based on a precise definition of correlation between bidders' valuations. In Haile and Tamer (2003), bidder valuations were assumed to be i.i.d. in order to identify profit bounds. Aradillas-López, Gandhi, and Quint (2013) subsequently showed that these profit bounds often led to optimal reserves that were significantly above the true value when valuations were positively correlated, and proposed an identification argument that can capture any degree of positive correlation between bidders' valuations. However, their method results in profit bounds that are only sharp to the degree of the extreme ends of either i.i.d. valuations or perfectly correlated values, resulting in relatively wide profit bounds that are uninformative, including in the precise sense that the lower bound strictly decreases once the reserve exceeds the auctioneer's valuation. As a result, they use an additional assumption, specifically that valuations are independent of the number of bidders, to obtain informative profit bounds. However, this assumption is difficult to justify in the context of live art auctions, where the auction is highly transparent and bidders' valuations are likely to increase in the number of bidders. Moreover, the precise number of bidders is not observed by the econometrician.

Our method allows the econometrician to intuitively restrict the correlation among bidders' valuations, with both lower and upper bounds, based on her conception on the degree of common component of valuations between the bidders. It is computationally straightforward, requiring the econometrician to only solve a pointwise univariate convex optimization problem.

Another challenge in the identification of ascending auctions is when the data does not provide an exact number of bidders. In addressing an unknown number of bidders, the current literature has formulated several approaches to estimate optimal reserves in ascending auctions. Hernández, Quint, and Turansick (2020) develop a method of point-identification for English auctions that allows for unobserved heterogeneity and no need to observe more than the number of bidders and the winning bid in each auction. Freyberger and Larsen (2022) use the (known) reserve price and two order statistics of bids to estimate optimal reserve prices. Marra (2020) uses the stochastic difference between adjacent order statistics, and requires two losing order statistics besides the winning bid. These papers are all innovative in their econometric use of bid data, but do not apply to the data situation in this paper because only the top two order statistics of bids and bounds on the number of bidders are observed.

In this paper, we generalize our approach so that the econometrician no longer requires an exact number of bidders in the data, but instead only lower and upper bounds on the number of bidders.

1.2.2 Theoretical Contribution

The theoretical contribution is a solution to the choice of a single optimal reserve price given profit bounds via the minimax regret criterion as first formulated by Savage (1951) and most recently suggested by Manski (2022), within a sharp identified set of profit functions. Such an approach offers an alternate decision criteria to Aryal and Kim (2013)'s and Jun and Pinkse (2019)'s papers, which provide methods to choose a single optimal reserve price. The algorithm we propose also uses the upper bound on profit, unlike the maxmin solution, and is computationally feasible. The interpretation of our approach is the minimization of the sup distance between a functional and the true optimal functional on profit. This approach is especially relevant to the application to art auctions because it is in line with the goal of profit maximization in the presence of ambiguity.

1.2.3 Empirical Contribution

The empirical contribution is the construction of a large novel dataset from Christie's and Sotheby's live auctions. Thus far, the literature thus far focuses only on small samples of limited auctions, and few discuss the optimal reserve price for maximizing profits. Ashenfelter and Graddy (2003), Ashenfelter (1989), McAndrew, Smith, and Thompson (2012), Beggs and Graddy (2009) all discuss wine and art auctions, their mechanics, and pricing effects. Hortaçsu and Perrigne (2021) reviews empirical auctions and discuss online eBay and wine auctions. Ashenfelter and Graddy (2011) find that the confidential reserve price is commonly thought to be related to an auctioneer's pre-sale estimates, and that the convention is for it to be at or below the auctioneer's low estimate. Such a common rule gives rise to the suspicion that very likely, this blanket rule to setting reserve prices is not optimal towards maximizing expected profit in certain auction categories. On this note, Marra (2020) analyzed Sotheby's auction data to find an improved reserve price 110% of the estimate that increases revenue by 2.5%, but the data was restricted to a sample of 884 wine lots from only 1 day of online wine auctions. These works are novel in their data collection and application. However, they do not discuss optimal reserve prices for auctions held in the live art auction room, where most of the high value objects are sold[2].

In contrast, our dataset contains complete bids and a lower bound on the number of bidders from more 3500 live auction lots, covering the largest live auctions run by Christie's and Sotheby's in the last five years. While there exists datasets of final transaction prices (for example in Ashenfelter and Graddy (2011)), or small samples of bids from singular auctions (for example online Sotheby's wine auctions in Marra (2020)), this is the first paper to propose a comprehensive and non-tedious method to collect large amounts of auction data on art auctions that include complete bidding trajectories and bounds on the number of bidders. This is also the first paper to collect bids from Christie's and Sotheby's *live* auctions. We further show that adjusting the reserve price can improve expected profits significantly by 4.7% to 9.9% per auction in certain subgroups.

The rest of the paper proceeds as follows. In Section 2, data construction is discussed. Section 3 describes the identification argument and estimation approach. Section 4 describes the minimax regret algorithm to choose a single optimal reserve. Section 5 discusses simulation results to illustrate profit improvements under our suggested reserves. Section 6 displays the results. Section 7 concludes.

[2]According to Christie's 2023 press release, \$4.6b of their 2023 total \$6.2b sales came from live auction rooms.

2 Art Auction Data

Despite the large number of public auctions that Christie's and Sotheby's have run, they reveal very little information about their auctions. In particular, their public websites only provide data on lot details (e.g. artist name, period, provenance, condition report), low and high estimates, and the final sale price after the buyer's premium. The most crucial identifying information such as bids, bidder identities, and number of bidders in past auctions are kept private to the firms.

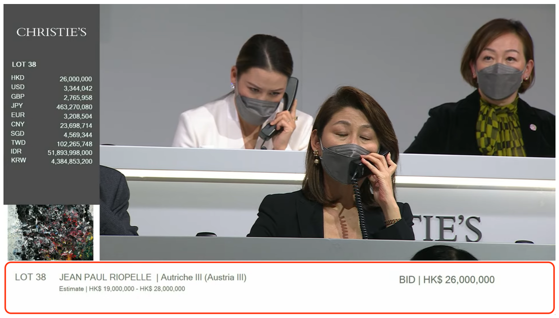
We construct a large novel dataset comprising 3500+ auction lots using live-stream YouTube auction videos held between 2020 and 2024. Our dataset comprises complete bidding trajectory, bounds on the number of bidders, and auction lot details such as the low and high estimates, location of auction and category of art. To the best of our knowledge, this is the first large dataset on auctions of modern art with bidding data.

Category	**Subcategory**	**Location**	**Count**
Art	Chinese Art	New York	112
Art	Impressionist/20th/21st Century Art	Hong Kong	636
Art	Impressionist/20th/21st Century Art	Las Vegas	9
Art	Impressionist/20th/21st Century Art	London	708
Art	Impressionist/20th/21st Century Art	New York	1208
Art	Impressionist/20th/21st Century Art	Shanghai	55
Art	Old Masters	London	102
Art	Old Masters	New York	134
Others (e.g. Jewelry)			542
Total			3506

Table 1: Our Art Auction Dataset

We generate the complete bidding trajectory by applying computer vision techniques to frame-by-frame video image data on bids, as shown in Figure 1. These bid data are paired with scraped data from the auction houses' websites to match the lot and its characteristics such as the estimate and description.

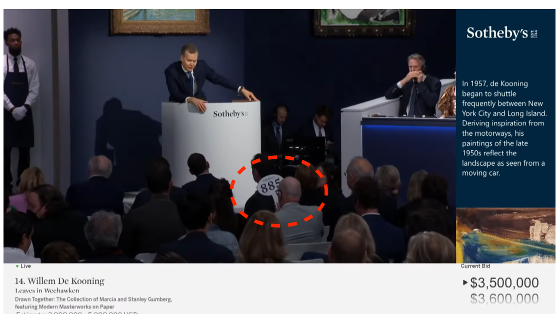
Our dataset additionally provides a lower bound on the number of bidders by using the audio transcript from the auction videos. In a live Christie's or Sotheby's auction, there are exactly four sources of bids: (i) telephone bids, (ii) live bids, (iii) absentee bids, and (iv) online bids, as shown in Figure 2. A close lower bound on the



Lot 38, Christie's Hong Kong, Nov 2022

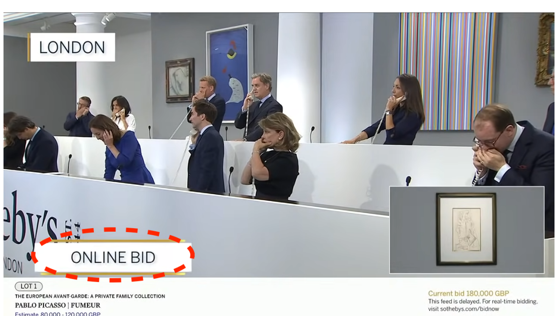
Lot 117, Sotheby's New York, Nov 2022

Figure 1: Example screenshots taken from Christie's and Sotheby's YouTube live stream auctions. Red boxes enclose image areas where important data can be collected, such as the current bids.

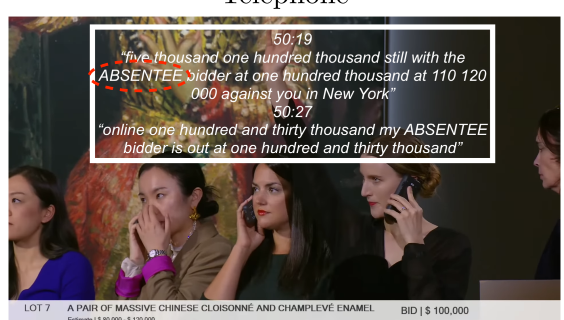


Room

Telephone



Online



Absentee

Figure 2: The four types of bidders possible in a Sotheby's or Christie's auction. Dotted red circles indicate the bidder's location. The absentee bidder is captured through the audio transcript.

true number of bidders can be constructed by accurately capturing a certain unique number of bidders in each of these four sources of bids. For example, when telephone bids are accepted, auctioneers typically call out specific names of colleagues, each of whom corresponds to exactly one bidder per auction lot. In the top-right of Figure 2, the Christie's auctioneer references Olivier Camu, another employee at Christie's who is putting in telephone bids on behalf of an unknown buyer. Similarly, for live bids, we capture positional references such as "to the right", while for absentee bidders and online bidders we capture terms such as "absentee" and "online". Our complete methodology is elaborated upon in the Appendix.

Our approach allows us to capture as many unique telephone bidders, one unique online bidder, one unique absentee bidder, and up to 5 unique in-person bidders. However, it is unclear how exactly close the observed lower bound on the number of bidders is to the true N. In 2021, telephone bids made up 42% of winning bids in Christie's live auctions versus 7% in the saleroom[3]. While the winning bid is not completely representative of all bids, the statistic hints at the informativeness of our lower bound. The distribution of the lower bound on the number of bidders is shown in Figure 3.

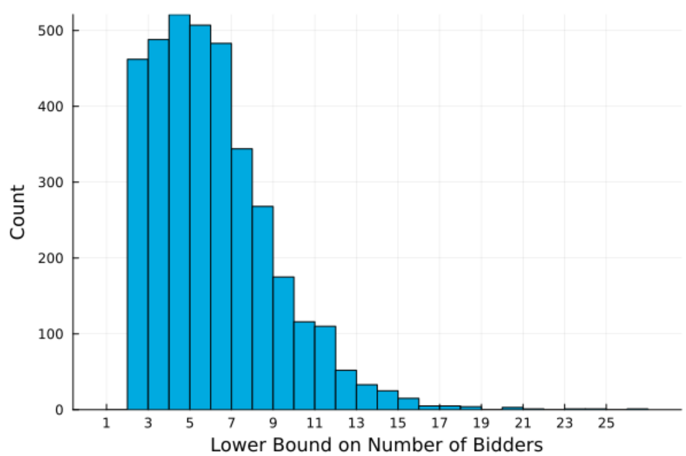


Figure 3: Distribution of lower bound on number of bidders across all auctions.

We conservatively set the upper bound on the number of bidders to be twice that of the lower bound. While an auction may have large number of bidders that participate in the auction but do not submit bids due to the nature of the live ascending auction, we argue that these do not have a significant result on the auction.

[3]Source: Christie's website at this link.

The final cleaned dataset also scales each recorded bid according to buyer's premiums, which are payable by the successful buyer of an item at an auction based on the hammer price of a lot sold. Table 8 summarizes the Buyer's Premium Schedule, for both Christie's and Sotheby's, as of March 2023.

2.1 Modern Art Sold in New York City

Our dataset contXains auctions of art across multiple subcategories. To take into account how auction characteristics might affect the price, we bin the data into smaller sub-sample groups . Consider the following functional determinants of price:

$$P = f(V,N,X),$$

where V is the vector of bidder valuations, N is the number of bidders, and X is the vector of covariates. Here, X may include the following:

1. **Category of Art**, e.g. Modern Art vs. Chinese Art.
2. **Location of Auction**, e.g. New York, Paris, London, Shanghai.
3. **Range of Final Transaction Price**.

Based on the possible realizations of X, we create sub-sample groups of auctions, grouping the auctions by category and location (as in table 1). We then specifically focus our subsequent econometric analysis on Modern Art sold in New York City, because of the economic significance of the art sold and the largest number auctions held there relative to all other locations, which is also reflected in our dataset. We further filter to the auctions with transaction price range between [$100K, $1.0M] and [$1.0M, $10.0M], and remove the small number of auction lots with high transaction prices beyond 7 times the high estimate.

Summary statistics for these auction lots are shown in Table 2.

3 Identification and Estimation

Our main objective is choose the reserve price that maximizes the seller's expected profit. As we discuss below, the expected profit function, or sharp bounds on it, can typically be written as a linear functional of the marginal distributions of the

Variable	Median	Mean	Std	Min-Max
Transaction Price	1.38	1.81	1.29	[0.25, 6.35]
2nd-Highest Bid	1.27	1.70	1.23	[0.24, 6.14]
Number of Bidders (LB)	5.00	5.36	2.52	[2.0, 13.0]
Low Est. Relative to High Est.	0.67	0.69	0.05	[0.60, 0.78]
Number of Bids	10.00	11.20	7.44	[2.0, 39.0]
Number of Auction Lots		294		

Transaction Price $\in$ [$100K, $1.0M]

Variable	Median	Mean	Std	Min-Max
Transaction Price	1.14	1.39	0.85	[0.37, 6.72]
2nd-Highest Bid	1.09	1.33	0.81	[0.35, 6.30]
Number of Bidders (LB)	5.00	5.90	2.92	[2.0, 20.0]
Low Est. Relative to High Est.	0.67	0.68	0.05	[0.48, 0.84]
Number of Bids	10.00	11.55	8.04	[2.0, 76.0]
Number of Auction Lots		679		

Transaction Price $\in$ [$1.0M, $10.0M]

Table 2: Summary Statistics for Modern Art Auction Lots in New York City (July 2020 – July 2024).

two highest valuations. Since the auction ends before the highest-valuation bidder gets to reveal it, the main identification problem is to obtain good bounds on the distribution of the highest valuation.

3.1 Bidding Behavior and Information Structure

Here, the theoretical framework is introduced. Let N (a random variable) denote the number of bidders in an auction and let n denote a value in the support of N. In an auction with N bidders, let $V_1, \ldots, V_N$ denote bidders valuations, and $B_1, \ldots, B_N$ denote their bids. Let $V_{1:N}, \ldots, V_{N:N}$ and $B_{1:N}, \ldots, B_{N:N}$ denote ordered valuations and bids, correspondingly. Finally, let $F_{j:N}$ and $G_{j:N}$ denote the distributions of $V_{j:N}$ and $B_{j:N}$, correspondingly, for each $j = 1, \ldots, N$.

We maintain the following assumptions.

Assumption 3.1 (Bidding strategies)**.**

1. *Bidders do not bid above their valuation:* $V_i \geqslant B_i$.

2. *Bidders do not give up:* $V_{N-1:N} \leqslant B_{N:N}$.

This assumption is due to Haile and Tamer (2003). In English auctions, the above are mild and natural restrictions on bidders' behavior. They do not uniquely specify the bidding functions and, for each realization of the valuations, lead to a set of bids compatible with the model.

Assumption 3.2 (Transaction Price)**.** *The transaction price in an auction is the greater of the reserve price and the second-highest bidder's willingness to pay.*

This assumption is due to Aradillas-López, Gandhi, and Quint (2013). It is the exact outcome when bidders play according to the equilibrium strategy in a theoretical ascending button auction, but also holds approximately if bidders do not "jump bid" at the end of the auction. It is a stricter version of Assumption 3.1, applied only to the second highest valuation.

Figure 4 demonstrates that in our sample, 96.2% of final bids are within the stipulated 10% increment from the second highest bid. Figure 5 additionally shows the closeness between the distributions of the highest and second highest bids.

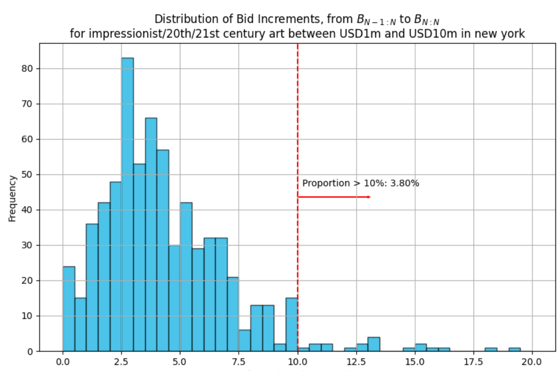


Figure 4: Bid Increments are mostly within Christie's and Sotheby's stipulated 10%.

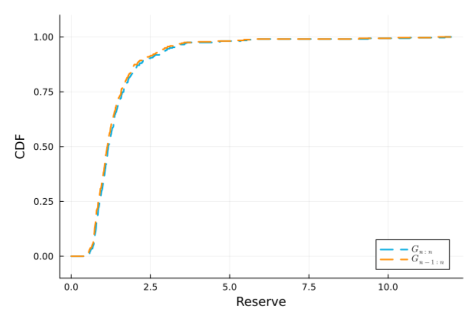


Figure 5: Empirical CDF-s of the highest and second highest bids.

One of the key identification challenges in ascending auctions is that the largest valuation is never fully revealed. As a result, the corresponding marginal CDF $F_{N:N}$ is fundamentally only partially identified. To derive useful bounds on $F_{N:N}$ and therefore expected profit, we start with the following general setting.

Assumption 3.3 (Conditional Independence). *(i) There exist i.i.d. random variables $\varepsilon_1, \ldots, \varepsilon_N \in \mathbb{R}$, a random vector $U \in \mathbb{R}^{d_U}$ independent of the ε_j-s, and a measurable function $g : \mathbb{R}^{d_U} \times \mathbb{R} \to \mathcal{V}$ such that $V_j = g(U, \varepsilon_j)$ for $j = 1, \ldots, N$; (ii) the map $e \mapsto g(u, e)$ is weakly increasing, for all u.*

Under Assumption 3.3 (i),the valuations $V_1, \ldots, V_N$ are i.i.d., conditional on U. Conditional independence of exchangeable random variables implies a certain type of positive dependence between them. Indeed, by the Law of Iterated Expectations and Jensen's inequality, for any measurable function $h : \mathcal{V} \to \mathbb{R}$,

$$\mathbb{E}\left[\prod_{j=1}^{N} h(V_j)\right] = \mathbb{E}\left[\mathbb{E}\left[h(V_j) \,|\, U\right]^N\right] \geqslant \mathbb{E}[h(V_j)]^N = \prod_{i=1}^{N} \mathbb{E}[h(V_i)]$$

for any measurable function $h : \mathcal{V} \to \mathbb{R}$. For example, with $N = 2$, the above is equivalent to $\mathbb{C}ov(h(V_1), h(V_2)) \geqslant 0$ for all $h(\cdot)$. Conditional independence is equivalent to infinite exchangeability (by the de Finetti-Hewitt-Savage Theorem) and is stronger than finite exchangeability precisely because of the induced positive dependence.

If the common component U was fully observed, the CDF of the largest valuation, conditional on U, would be point-identified and rest of the analysis would be straightforward. Since it is likely not the case, the analysis should account for a potentially substantial common component. Conditional independence allows to derive simple bounds on the CDF of the largest valuation. Consider a function $\phi_N : [0, 1] \to [0, 1]$ defined implicitly via the relation:

$$u = N\phi_N(u)^{N-1} - (N-1)\phi_N(u)^N. \tag{1}$$

The function $\phi_N(\cdot)$ maps the CDF of the second-largest order statistic out of N i.i.d. draws to the corresdponding marginal CDF. It is strictly increasing and smooth.

Lemma 1 (Lower Bound on $F_{N:N}$). *Let Assumption 3.3 hold and the function ϕ_N be defined in* (1)*. Then:*

$$\phi_N(F_{N-1:N}(v))^N \leqslant F_{N:N}(v) \leqslant F_{N-1:N}(v).$$

The equality $F_{N:N} = \phi_N(F_{N-1:N}(v))^N$, for all v, corresponds to the i.i.d. valuations and the equality $F_{N:N}(v) = F_{N-1:N}(v)$ corresponds to pure common values.

The same bounds have been obtained in Aradillas-López, Gandhi, and Quint (2013) under a weaker notion of positive dependence.[4] We prove the result directly using Jensen's inequality and convexity of the function $u \mapsto \phi_N(u)^N$ and take these bounds as a starting point of the analysis. Since Assumption 3.3 does not restrict the degree of dependence between valuations, the bounds in Lemma 1 are typically quite wide and have drastically different economic implications. At the same time, allowing for pure common or independent private values may not be realistic in most settings, so it is desirable to control "how common" the values are. Below we propose a way to do so in a computationally simple and theoretically transparent manner.

Under Assumption 3.3, using the Law of Iterated Expectations and the definition of $\phi_N(\cdot)$, the CDF of the largest valuation can be expressed as:

$$F_{N:N}(v) = \mathbb{E}[\phi_N(P(V_{N-1:N} \leqslant v \,|\, U))^N],$$

and it is additionally known that:

$$F_{N-1:N}(v) = \mathbb{E}[P(V_{N-1:N} \leqslant v \,|\, U)].$$

Thus, the bounds on $F_{N:N}(v)$ correspond to the solutions of the following generalized moment problems:

$$\begin{aligned} \min\nolimits_P\{\mathbb{E}_P[\phi_N(X)^N] : \mathbb{E}_P[T] = \mu, P(X \in [0,1]) = 1\} &\quad = \phi_N(\mu)^N; \\ \max\nolimits_P\{\mathbb{E}_P[\phi_N(X)^N] : \mathbb{E}_P[X] = \mu, P(X \in [0,1]) = 1\} &\quad = \mu, \end{aligned} \tag{2}$$

with $\mu = F_{N-1:N}(v)$ and $X = P(V_{N-1:N} \leqslant v \,|\, U)$, for each v. The lower bound corresponds to Jensen's inequality and the upper bound to Edmundson-Mandansky's inequality for the convex function $u \mapsto \phi_N(u)^N$ on $u \in [0,1]$ (see, e.g., Birge and Dula 1991).

To tighten the bounds, we additionally restrict the variance of the conditional distribution function $P(V_{N-1:N} \leqslant v \,|\, U)$, denoted:

$$D_N(v) = \mathbb{E}[(P(V_{N-1:N} \leqslant v \,|\, U) - F_{N-1:N}(v))^2]. \tag{3}$$

[4] Specifically, the authors assume that valuations are exchangeable and the function $f(k) = P(V_i \leqslant v \,|\, \#\{j \neq i : V_j \leqslant v\} = k)$ is non-decreasing in k. Lemma 1 in the aforementioned paper shows that conditional independence implies the stated property.

The above is a squared $L_2(P)$ distance between the conditional and unconditional CDFs of $V_{N-1:N}$, providing a measure of the common component in the valuations. Since $P(V_{N-1:N} \leqslant v \,|\, U)$ is a random variable supported within the unit interval, $D_N(v)$ takes values in $[0, F_{N-1:N}(v)(1 - F_{N-1:N}(v))]$. Notice $D_N(v) = 0$, for all v, implies that $P(V_{N-1:N} \leqslant v \,|\, U) = F_{N-1:N}(v)$ almost surely, which under Assumption 3.3 implies that either U is constant or $V_j = g(\varepsilon_j)$, corresponding to the i.i.d. valuations. On the other hand, $D_N(v) = F_{N-1:N}(v)(1 - F_{N-1:N}(v))$, for all v, can only be attained if $P(V_j \leqslant v \,|\, U) \in \{0, 1\}$ such that $P(P(V_j \leqslant v \,|\, U) = 1) = F_{N-1:N}(v)$, which implies that each V_j is a deterministic function of U, corresponding to pure common values.

To obtain an easily interpretable expression for $D_N(v)$, we invoke Assumption 3.3 (ii). Under this condition, $V_{N-1:N} = g(U, \varepsilon_{N-1:N})$, so letting f_U and $f_{\varepsilon_{N-1:N}}$ denote the densities of U and $\varepsilon_{N-1:N}$ (with respect to Lebesgue measure for simplicity):

$$
\begin{aligned}
\mathbb{E}[P(V_{N-1:N} \leqslant v \,|\, U)^2] &= \int P(g(u, \varepsilon_{N-1:N}) \leqslant v)^2 f_U(u) du \\
&= \int \int \int \mathbf{1} \left\{ \substack{g(u,e_1) \leqslant v; \\ g(u,e_2) \leqslant v} \right\} f_{\varepsilon_{N-1:N}}(e_1) f_{\varepsilon_{N-1:N}}(e_2) f_U(u) de_1 de_2 du \\
&= P(g(U, \varepsilon_{N-1:N}) \leqslant v, g(U, \tilde{\varepsilon}_{N-1:N}) \leqslant v) \\
&= C(F_{N-1:N}(v), F_{N-1:N}(v)),
\end{aligned}
$$

where $\tilde{\varepsilon}_{N-1:N}$ is an independent copy of $\varepsilon_{N-1:N}$, and $C : [0,1]^2 \to [0,1]$ is the copula function of $(g(U, \varepsilon_{N-1:N}), g(U, \tilde{\varepsilon}_{N-1:N}))$. Letting $C_0 : [0,1]^2 \mapsto [0,1]$ denote the independence copula, given by $C_0(u, v) = uv$, the variance $D_N(v)$ can be written as:

$$
D_N(v) = C(F_{N-1:N}(v), F_{N-1:N}(v)) - C_0(F_{N-1:N}(v), F_{N-1:N}(v)). \tag{4}
$$

This expression motivates the following specification for the bounds.

Assumption 3.4 (Departure from IPV and Common Values)**.** *Let* $C_\rho : [0,1]^2 \to [0,1]$ *be a copula function, parametrized by* $\rho \in [0,1]$ *such that: (i)* $\rho \mapsto C_\rho(u,u)$ *is non-decreasing, for each* u*; (ii)* $C_0(u_1, u_2) = u_1 u_2$*,* $C_1(u_1, u_2) = \min(u_1, u_2)$*; and (iii)* $u \mapsto C_\rho(u,u)/u$ *is non-decreasing, for each* ρ*. Then:*

$$
\underline{D}_N(v) \leqslant D_N(v) \leqslant \overline{D}_N(v)
$$

with

$$\underline{D}_N(v) = C_{\underline{\rho}}(F_{N-1:N}(v), F_{N-1:N}(v)) - C_0(F_{N-1:N}(v), F_{N-1:N}(v))$$

$$\overline{D}_N(v) = C_{\overline{\rho}}(F_{N-1:N}(v), F_{N-1:N}(v)) - C_0(F_{N-1:N}(v), F_{N-1:N}(v)),$$

for some $0 \leqslant \underline{\rho} \leqslant \overline{\rho} \leqslant 1$.

Under Assumption 3.4, the distance $D_N(v)$, defined in (3), is bounded from below by:

$$\underline{D}_N(v) = C_{\underline{\rho}}(F_{N-1:N}(v), F_{N-1:N}(v)) - C_0(F_{N-1:N}(v), F_{N-1:N}(v)). \tag{5}$$

and from above by:

$$\overline{D}_N(v) = C_{\overline{\rho}}(F_{N-1:N}(v), F_{N-1:N}(v)) - C_0(F_{N-1:N}(v), F_{N-1:N}(v)). \tag{6}$$

Choosing $\underline{\rho} > 0$ ensures a non-trivial common component, and $\overline{\rho} < 1$ ensures a non-trivial private component. Any bivariate copula satisfying the stated assumptions can be used (see, e.g., Table 4.1. in Nelsen 2006 for examples). Part (i) of the assumption ensures that the bounds are properly ordered; Part (ii) defines the IPV and pure common values as marginal cases; and Part (iii) is a type of positive dependence condition that will ensure that the bounds derived in Theorem 1 are plausible CDFs. Choosing $\underline{\rho} = \overline{\rho}$ fixes $D_N(v)$ but leaves the joint distribution of the vector $(g(U, \varepsilon_{N-1:N}), g(U, \tilde{\varepsilon}_{N-1:N})$ outside of the diagonal (v, v) completely unspecified. In such case, without further assumptions, the CDF $F_{N:N}(v)$ is still not point identified. We highlight that Assumption 3.4 does not impose a specific dependence structure on the valuations; it merely provides a intuitively simple way to choose bounding functions for the analysis. The formal results below do not rely on the specific parametric form of $C_\rho(u, u)$ in any way.

Under Assumptions 3.3–3.4, sharp bounds on $F_{N:N}$ can be obtained by solving generalized moment problems similar to the ones in (2) with an extra constraint of the form $c_1 \leqslant \mathbb{V}ar(X) \leqslant c_2$. Using geometric arguments of Kemperman (1968), we show that both bounds can be computed by solving univariate convex optimization problems.

Theorem 1 (Sharp Bounds on $F_{N:N}$). *Let Assumptions 3.3 and 3.4 hold. Then:*

$$\begin{aligned} F_{N:N}(v) &\geqslant \min_{s\in[\frac{c_1}{1-\mu},\mu]} \left\{\phi_N(\mu-s)^N \tfrac{c_1}{c_1+s^2} + \phi_N\left(\mu+\tfrac{c_1}{s}\right)^N \tfrac{s^2}{c_1+s^2}\right\} \\ F_{N:N}(v) &\leqslant \max_{s\in(0,1)} \left\{\mu - \tfrac{\mu(1-\mu)-c_2}{s(1-s)}\left(s-\phi_N(s)^N\right)\right\}, \end{aligned}$$

with $\mu = F_{N-1:N}(v)$, $c_1 = \underline{D}_N(v)$, and $c_2 = \overline{D}_N(v)$. The minimization problem is strictly convex, and the maximization problem is strictly concave. Both bounds are monotonically increasing and smooth in $F_{N-1:N}(v)$.

We illustrate the effect of our approach on tightening the bounds on the CDF of the highest valuation in Figure 8. In this example, the valuations are drawn from a Lognormal distribution with mean log(0.4), standard deviation 1 and correlation 0.7. A bivariate Gaussian copula is used.

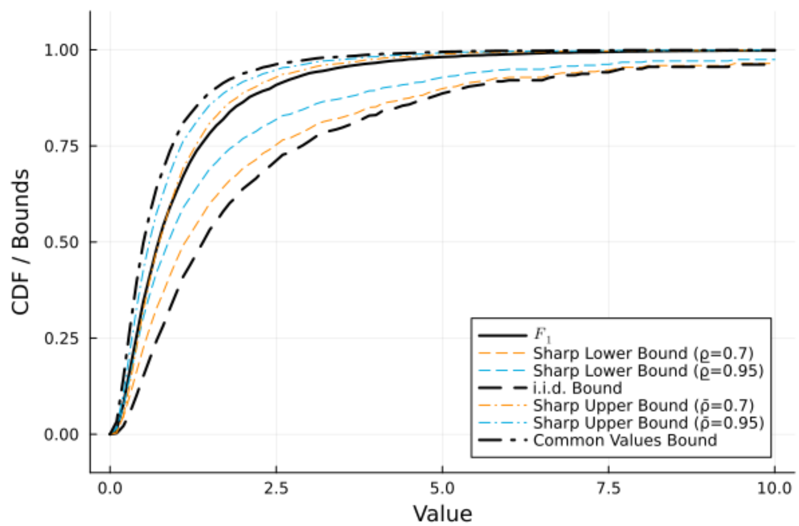


Figure 6: Sharp bounds on the CDF of the highest valuation.

3.2 Identification

In this section, we detail a partially identified model on expected profit that requires only a lower bound on the number of bidders instead of the exact number of bidders.

Letting r denote a reserve price, and v_0 denote the value of the unsold lot to the seller, the profit is given by:

$$\pi(r) = (r - v_0) \cdot \mathbb{1}(V_{N-1:N} \leqslant r, V_{N:N} > r) + (V_{N-1:N} - v_0) \cdot \mathbb{1}(V_{N-1:N} \geqslant r).$$

Taking expectations conditional on $N = n$ and rearranging:

$$\mathbb{E}[\pi(r)|N = n] = \int_0^{+\infty} \max\{r,v\} dF_{n-1:n}(v) - v_0 - F_{n:n}(r)(r - v_0). \tag{7}$$

Therefore, to study optimal reserve prices, it suffices to identify or bound the distributions $F_{n:n}$ and $F_{n-1:n}$, which have been bounded and identified respectively. Specifically, let

$$\begin{aligned}
\underline{\psi}_{\underline{\rho},n}(F_{n-1:n}(v)) \equiv & \quad \min_{s \in [\frac{c_1}{1-\mu}, \mu]} \left\{ \phi_n(\mu - s)^n \frac{c_1}{c_1 + s^2} + \phi_n \left(\mu + \frac{c_1}{s} \right)^n \frac{s^2}{c_1 + s^2} \right\}, \\
\overline{\psi}_{\overline{\rho},n}(F_{n-1:n}(v)) \equiv & \quad \max_{s \in (0,1)} \left\{ \mu - \frac{\mu(1-\mu) - c_2}{s(1-s)} (s - \phi_n(s)^n) \right\}
\end{aligned}$$

be the lower bound and upper bound for $F_{n:n}(v)$ respectively as defined in Theorem 1. In particular, the following ordering holds for all $0 < \underline{\rho} \leq \overline{\rho} < 1$:

$$\phi_n(F_{n-1:n}(v))^n < \underline{\psi}_{\underline{\rho},n}(F_{n-1:n}(v)) \leq F_{n:n}(v) \leq \overline{\psi}_{\overline{\rho},n}(F_{n-1:n}(v)) < F_{n-1:n}(v).$$

Thus, our bounds are a strict tightening of Aradillas-López, Gandhi, and Quint (2013)'s bounds as long as $0 < \underline{\rho} \leq \overline{\rho} < 1$. It follows that the sharp bounds for the profit function for $r \geqslant v_0$ conditional on $N = n$ are:

$$\begin{aligned}
\mathbb{E}[\pi(r)|N = n] &\geqslant \int_0^{\infty} \max\{r,v\} dG_{n:n}(v) - v_0 - \overline{\psi}_{\overline{\rho},n}(G_{n:n}(r))(r - v_0), \\
\mathbb{E}[\pi(r)|N = n] &\leqslant \int_0^{\infty} \max\{r,v\} dG_{n:n}(v) - v_0 - \underline{\psi}_{\underline{\rho},n}(G_{n:n}(r))(r - v_0).
\end{aligned}$$

We now generalize these bounds so that they are unconditional on the number of bidders. Let F_1 and F_2 denote the unconditional CDFs of $V_{n:n}$ and $V_{n-1:n}$ correspondingly. Then, the general form of the profit equation in equation 7 unconditional on N is:

$$\mathbb{E}[\pi(r)] = \int_0^{+\infty} \max\{r,v\} dF_2(v) - v_0 - F_1(r)(r - v_0). \tag{8}$$

We introduce the following theorem to bound the expected profit in equation 8 for a scenario where we only observe a lower bound on the number of bidders.

Theorem 2 (Bounds on Expected Profit Unconditional on N)**.** *Let $\underline{n}$ and $\overline{n}$ be the minimum and maximum number of bidders in the entire auction population respectively. Further let G_1 and G_2 denote the unconditional CDFs of $B_{n:n}$ and $B_{n-1:n}$ correspondingly. Finally, let*

$$\underline{\omega}_n(\mu) \equiv \min_{s\in\left[\frac{c_{\underline{\rho}}}{1-\mu},\mu\right]} \left\{\phi_n(\mu-s)^n \frac{c_{\underline{\rho}}}{c_{\underline{\rho}}+s^2} + \phi_n\left(\mu+\frac{c_{\underline{\rho}}}{s}\right)^n \frac{s^2}{c_{\underline{\rho}}+s^2}\right\},$$
$$\overline{\omega}_n(\mu) \equiv \max_{s\in(0,1)} \left\{\mu - \frac{\mu(1-\mu)-c_{\overline{\rho}}}{s(1-s)}(s-\phi_n(s)^n)\right\}$$

denote the lower and upper bounds for $F_{n:n}$, where $\mu = F_2(v)$ and $c_\rho = C_\rho(\mu,\mu) - C_0(\mu,\mu)$.

Then, under our assumptions, the sharp bounds for the expected profit unconditional on N are:

$$\mathbb{E}[\pi(r)] \geqslant \int_0^{+\infty} \max\{r,v\} dG_2(v) - v_0 - \overline{\omega}_{\overline{n}}(G_1(r))(r-v_0),$$
$$\mathbb{E}[\pi(r)] \leqslant \int_0^{+\infty} \max\{r,v\} dG_2(v) - v_0 - \underline{\omega}_{\underline{n}}(G_1(r))(r-v_0).$$

Theorem 2 is useful because it allows us to bound the profit function while only knowing a lower bound on the number of bidders and not the exact number. An accompanying benefit is that it also pools together a larger amount of data, that is, all auctions with at least $\underline{n}$ and at most $\overline{n}$ bidders. Theorem 2 is especially relevant to our empirical application because we only observe bounds on the true number of bidders rather than an exact number of bidders from the auction videos.

4 An Optimal Reserve Price Under Ambiguity

The goal of the auction house is to set a reserve price to maximize expected profit from an auction. Under reasonably weak assumptions, the expected profit function, and therefore the optimal reserve price, is only partially identified, and thus the target is ambiguous. A decision needs to be made amidst this statistical ambiguity.

Formally, let $\pi(\cdot)$ denote the expected profit function, Π denote the sharp identified set for $\pi(\cdot)$, and $\mathcal{R}$ denote the sharp identified set for the optimal reserve price. The three most popular methods for resolving the ambiguity are Bayesian, maxmin, and minimax regret.

The Bayesian approach is to assume a subjective prior Q over Π and solve:

$$\max_{r\in\mathcal{R}} \int_{\Pi} \pi(r) dQ(\pi).$$

However, due to lack of identifying information, one can hardly formulate a reasonable prior, and it is easy to verify that by trying different priors one can recover any point $r^* \in \mathcal{R}$ as a solution to the above problem. So, conceptually, the Bayesian approach cannot be helpful in the present setting.

The maxmin approach is to solve:

$$\max_{r\in\mathcal{R}} \min_{\pi\in\Pi} \pi(r) = \max_{r\in\mathcal{R}} \underline{\pi}(r)$$

which amounts to maximizing the sharp lower bound on the profit function. Conceptually, this corresponds to setting the reserve price cautiously, which may not align with the goals of the auction house. Indeed, if the lot is unsold, the marginal cost associated with organizing its resale is likely negligible, compared to the selling price. Practically, as discussed already in Aradillas-López, Gandhi, and Quint (2013), in the absense of strong distributional assumptions, the lower bound $\underline{\pi}$ is monotonically decreasing after v_0. In this case, the solution will be to set $r = v_0$, which is not particularly insightful.

The minmax regret approach is to solve:

$$\min_{r\in\mathcal{R}} \max_{\pi\in\Pi} \{\pi(r^*_\pi) - \pi(r)\},$$

where r^*_π denotes the optimal reserve price under profit function π. Defining the functionals $\phi^*(\pi) = \pi(r^*_\pi)$ and $\phi_r(\pi) = \pi(r)$, the above problem can be written as

$$\min_{r\in\mathcal{R}} \max_{\pi\in\Pi} |\phi^*(\pi) - \phi_r(\pi)|.$$

That is, choosing r is equivalent to choosing a functional $\phi_r(\cdot)$ that is as close as possible to the unknown optimal functional $\phi^*(\pi)$ with respect to a maximum distance metric. This is in line with the goal of profit maximization in the presence of ambiguity.

A first intuitive approach to solving for the minimax regret solution may be as follows. Given a particular fixed $r \in \mathcal{R}$, we can attempt to characterize a regret-maximizing profit function that attains its lowest point at r and attains its maximum point at $\operatorname{argmax}_{\mathcal{R}} \bar{\pi}$. The following lemma characterizes such a solution.

Lemma 2 (Equivalence of Minimax Regret and Maxmin Solutions)**.** *Let $\mathcal{C} = [v_0, \bar{v}]$ and assume that $\overline{\pi}$ and $\underline{\pi}$ are continuous on $\mathcal{C}$. Let $\{\pi : \underline{\pi} \leq \pi \leq \overline{\pi}\}$ be the space of continuous profit functions $\pi(r) : \mathcal{C} \to \mathbb{R}$, and suppose any continuous profit function within the bounds are attainable. Then the solution to the minimax-regret problem is* $\underset{r \in \mathcal{C}}{\operatorname{argmax}}\, \underline{\pi}(r)$.

Lemma 2 shows that the solution to the minmax-regret problem, without restrictions on the shape of the profit function within its bounds, is equivalent to the maxmin approach. However, the above solution might utilize profit functions that are not in the sharp identified set. For example, a jump from $\underline{\pi}(r)$ to $\overline{\pi}(r + \delta)$ for some small $\delta > 0$ is not always permissible because the CDF-s of the top two order statistics that correspond to such a profit function may not be in the sharp identified set. To visually see this, in the right display in Figure 7, a jump up from $\underline{\pi}(r)$ where $r = 1.5$ to $\overline{\pi}(r + \delta)$ for some small $\delta > 0$ is not possible due to the constraints on the CDF F_1. In such a sense, by restricting the shape of the profit function within the sharp identified set, one can obtain a more informative solution to the choice of reserve price.

4.1 Minimax Regret Solution

In addition to the fixed CDF F_2 of the second highest valuation, suppose sharp bounds on the CDF of the highest valuation are given:

$$F_1^L \leqslant F_1 \leqslant F_1^U.$$

Every feasible profit function within the bounds corresponds to some F_1 within the bounds. To study maximum regret, profit functions that "jump" up or down are of primary interest. The "jumps" cannot be arbitrary as they must be accommodated by the CDF-s subject to the restrictions above. Moreover, CDFs that correspond to the boundary profit functions are very specific. Indeed, using integration by parts, the profit function given in (7) can be rewritten as:

$$\pi(r; F_1, F_2) = \int_r^{+\infty} (1 - F_2(v))dv + (r - v_0)(1 - F_1(r)).$$

Thus, for example, if $\pi(r) = \overline{\pi}(r)$ for some r, $F_1(v)$ must take a minimum value at r. Similarly, if $\pi(r) = \underline{\pi}(r)$ for some r, then $F_1(v)$ must take a maximum value at r. Together with the fact that CDFs must be weakly increasing, the profit functions within the sharp identified set are restricted.

Theorem 1 characterized sharp lower and upper bounds on $F_{N:N}$ under Assumptions 3.3 and 3.4. However, certain CDFs between the bounds may not be feasible under conditional independence, which substantially complicates the minimax regret problem. To this end, we relax Assumptions 3.3 and 3.4 as follows.

Assumption 4.1 (Information structure)**.** *The joint distribution of valuations, F, belongs to the set of distributions $\mathcal{F}$ satisfying the following conditions:*

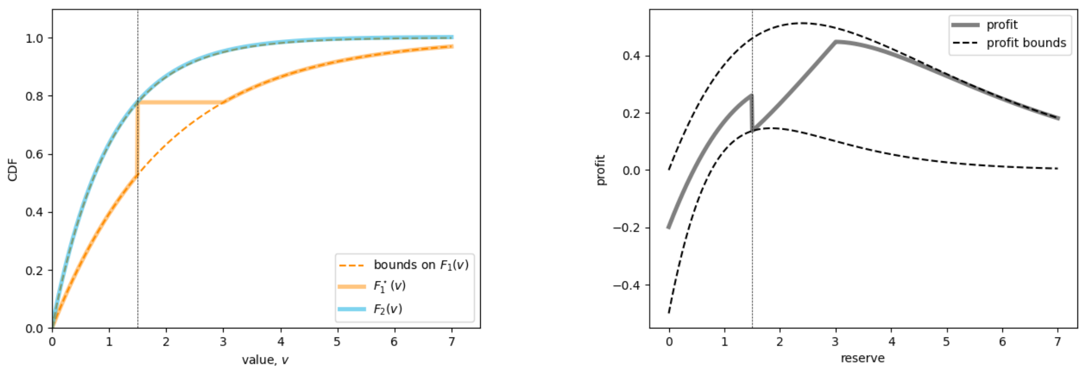
1. *Valuations are symmetric: $(V_1, \ldots, V_N) \stackrel{d}{=} (V_{\pi(1)}, \ldots, V_{\pi(N)})$, for any permutation $\pi : \{1, \ldots, N\} \to \{1, \ldots, N\}$, for each N.*

2. *Valuations satisfy $\overline{\omega}_N(F_{N-1:N}) \geqslant F_{N:N} \geqslant \underline{w}_N(F_{N-1:N})$ for $\underline{\omega}_n$, $\overline{\omega}_n$ defined in Theorem 2.*

3. *The number of participants is bounded by $\underline{n} \leqslant N \leqslant \overline{n}$.*

The first part of the assumption implies that bidders' identities are not informative, and it suffices to consider ordered valuations $V_{1:N}, \ldots, V_{N:N}$, for each N. The second part specifies known bounds on N, and the third part simply states that any CDF between the derived bounds is admissible. Lemma A.4 in the Appendix shows that for any random vector, it is possible to construct an exchangeable random vector with order statistics with the same distribution. This implies that any CDF F_1 within the bounds $[\underline{\omega}_{\underline{n}}(F_2), \overline{\omega}_{\overline{n}}(F_2)]$ can be realized as a mixture (over N) of distributions $F_{N:N}$ supported by an exchangeable vector of valuations for each N.

The following theorem characterizes the maximum regret as a function of the reserve price r.



Example CDF-s Example Corresponding Profits

Figure 7: The pair of CDF-s corresponding to the maximum "Jump Down" (to the left of r) and "Jump Up" (to the right of r) in the profit function at $r = 1.5$.

Theorem 3 (Maximum Regret under Equilibrium Play)**.** *Take a reserve $r \geqslant v_0$. Under assumptions 3.1, 3.2, 4.1, the maximum regret at r for any profit function given by F_1 and F_2 contained in the sharp identified set is*

$$\max_v \pi(v; F_1^*, F_2) - \pi(r; F_1^*, F_2)$$

where F_1^* is given by

$$F_1^*(v; r) = \mathbf{1}(v < r)F_1^L(v) + \mathbf{1}(v \in [r, v_1(r)))F_1^U(r) + \mathbf{1}(v \geqslant v_1(r))F_1^L(v);$$

and $v_1(r) = \min\{v : F_1^L(v) = F_1^U(r)\}$.

Proof. See Appendix A. ■

Theorem 3 gives rise to the following algorithm to compute the minimax regret choice of reserve. It is polynomial time in the fine-ness of the optimization search grid to compute the profit bounds.

Algorithm 1 (Minimax Regret)**.** *In the following algorithm, optimization is done over a desired grid of possible reserve prices such as within $[v_0, \bar{v}]$.*

1. *Setting the jump at r, compute the maximum regret at r over the grid $v \in [v_0, \bar{v}]$:*

$$R^\star(r) = \max_v \pi(v; F_1^\star, F_2) - \pi(r; F_1^\star, F_2)$$

2. Select the minimax regret choice of reserve price:

$$r^\star = \underset{r \in [v_0, \bar{v}]}{\operatorname{argmin}} R^\star(r).$$

Notice that the regrets can be computed without numerical integration. This is done by writing $\int_r^{+\infty}(1 - F_2(v))dv$ as $\mathbb{E}[B_1 \mathbf{1}(B_1 \geq r) + r\mathbf{1}(B_1 < r)] - r$, where B_1 is the random variable for the highest bids with distribution G_1.

5 Simulations

According to Christie's and Sotheby's terms of sale, the reserve price is confidential and does not exceed a Lot's low estimate. Our data shows that not only is the reserve price bounded above by the low estimate, but the auction houses often set the reserve significantly lower. In Figure 8, we plot all of the auction lots that have final transaction prices less than the low estimate. There are numerous sold lots (in blue) that are significantly below the low estimate. If the auction house's estimates are trusted, such low reserve prices are suboptimal for profit on expectation.

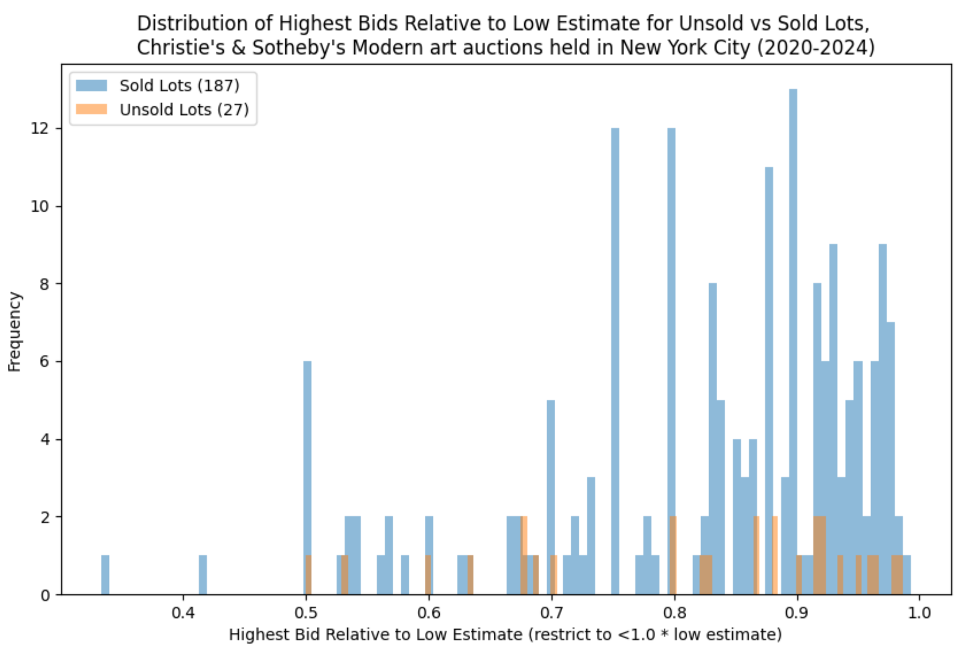


Figure 8: The reserve can be anywhere from 0.3 to 1.0 times the low estimate.

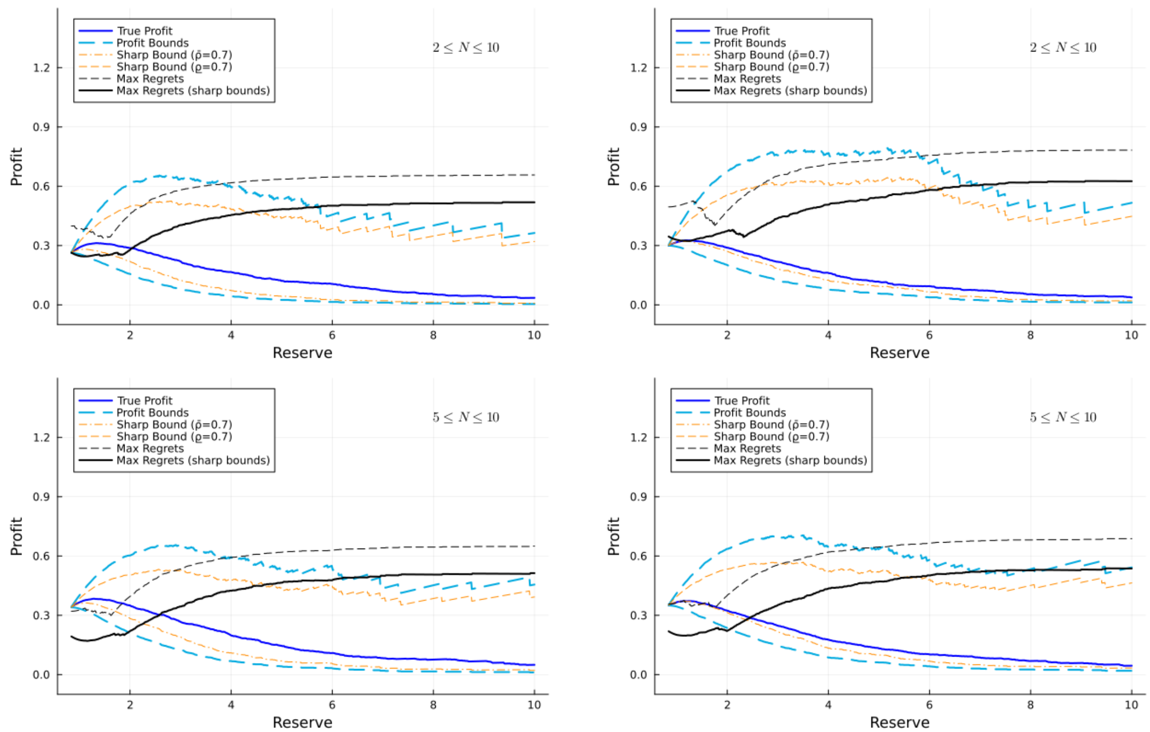
To examine the profit improvements by setting a reserve price under the minimax-regret approach, we conduct a number of simulation experiments. We generate artificial bidding data for N = 2,3,...,10 bidders, drawing their valuations from known lognormal distributions. Bids were then generated via the procedure as follows. Given a set of N bidders, an initial bid by a randomly selected bidder is cast at 5% of her valuation. Subsequently, an active bidder is selected at each step randomly from the pool of standing bidders. Her bid follow an incremental rule, where with probability $1-\lambda$, the bid is 1.1 of the previous bid. Otherwise, with probability λ the bid is drawn uniformly between the previous bid and 1.1 of the previous bid. This increment pattern is selected following Christie's/Sotheby's auction bidding increment guidelines. If a bidder's valuation is exceeded by the next potential bid, she drops out of the auction. This procedure is repeated until only one bidder remains active.

In our art auction data, the shape of the empirical CDF-s of the highest and second highest bids, as shown in Figure 5, suggest a strong resemblance to a lognormal distribution. It is also the common distribution used to model bidders' valuations in an auction. We thus use the lognormal distribution as the parent distribution for bidder valuations in our simulations.

In Figure 9, we plot the results of a simulation experiment for bidder valuations drawn fro a lognormal distribution with mean $\mu = \log\left(\frac{2}{5}\right)$ and standard deviation $\sigma = 1.0$. Note that the expected value of this lognormal distribution is very close to $\frac{2}{3}$, the low estimate. The parameters are set at $\lambda = 0.2$ with 5,000 auctions, and auctioneer's valuation $v_0 = \frac{5}{6}$, corresponding to the midpoint between the low and high estimates in the typical Christie's and Sotheby's auction. There is an even distribution of the number of bidders across N = 2,3,...,10. We plot the true profit, our bounds on profit, and maximum regrets for $N \geqslant 2$ and $N \geqslant 5$, for different values of correlation in the lognormal distribution at ρ = 0.5,0.7. We set the lower bound and upper bound on the common component of bidder valuations at 0.7, with the intuition that the common component of valuations is greater than the individual component for art objects.

Observe the following from Figure 9:

- The sharp bounds when fixing $\underline{\rho}$ and $\overline{\rho}$ is significantly tighter than without specifying bounds on the common component of valuations. Since the original upper bound on profit is based on i.i.d. valuations, it is inaccurately claiming a much higher upper bound on profit than the true value. Similarly, since



$\rho = 0.5$ $\rho = 0.7$

Figure 9: Simulation results of profits where bidder valuations are given by the lognormal distribution with parameters $\mu = \log\left(\frac{2}{5}\right), \sigma = 1.0$, and correlation $\rho = 0.5$,0.7.

the original lower bound on profit is based on pure common values, it is also inaccurately claiming a significantly lower lower bound on profit than the true value. In fact, when the lognormal distribution has correlation parameter $\rho = 0.7$ and matches our assumed correlation parameter $\underline{\rho} = \overline{\rho} = 0.7$, our sharp lower bound on profit is very close to the true profit function.

- By obtaining the sharp bounds on profit, the minimax regret reserve choice is much closer to the optimal profit than without the sharp bounds. The extents of the "jumps" in profit when computing maximum regret are restricted due to the sharp bounds on profit.

Table 3 shows the results of the simulation experiments. We display the expected profit under a variety of reserve choices, including the optimal, the minimax regret reserve without bounding the common component of valuations, the minimax regret reserve when bounding it, and the max-min solution when bounding the common

	$\rho = 0.5$		$\rho = 0.7$	
	$2 \leqslant N$	$5 \leqslant N$	$2 \leqslant N$	$5 \leqslant N$
Minimax-Regret Reserve ($\underline{\rho} = 0.7, \overline{\rho} = 0.7$)	1.09	1.15	1.25	1.18
Expected Profit (Optimal)	0.312	0.384	0.326	0.373
Expected Profit (Minimax-Regret, $\underline{\rho} = 0.0, \overline{\rho} = 1.0$)	0.310	0.375	0.308	0.347
Expected Profit (Minimax-Regret, $\underline{\rho} = 0.7, \overline{\rho} = 0.7$)	0.301	0.378	0.326	0.372
Expected Profit (Max-Min, $\underline{\rho} = 0.7, \overline{\rho} = 0.7$)	0.304	0.378	0.326	0.372
Profit at Low Estimate	0.219	0.300	0.262	0.316
Profit Increase (%)	37.4	26.0	24.4	17.7

Table 3: Simulated Outcomes under Lognormal $\mu = \log\left(\frac{2}{5}\right), \sigma = 1.0$ and correlation ρ.

component. In the last row of the table, we compare the profit increase under the minimax-regret choice of regret with the auctioneer setting a reserve at the low estimate (0.67). Any lower reserve set by them, which the auctioneers sometimes do as shown in Fig 8, results in even larger expected profit increase. Under these parameters, there is significant improvement to profit when the minimax regret choice is adopted. Our suggested choice of reserve, given the parameters, provide significant improvements to profit, from 17.7% to 42.0%.

Indeed, the results in Table 3 only hold for one particular type of lognormal distribution. In Table 4, we display the increase in profit under a wider variety of scenarios, in different lognormal distribution parameters $\mu = \log(0.35)$, $\log(0.40)$, $\log(0.45)$, $\log(0.50)$, $\log(1.0)$, $\sigma = \sqrt{0.7}$,$\sqrt{1.0}$, $\rho = 0.5$,0.7,0.9, and assuming different auctioneer's original reserves (0.8 or 1.0 $\times$ the low estimate). In the top table we restrict the sharp bounds on profit by modeling the common component of valuations at $\underline{\rho} = \overline{\rho} = 0.7$, while in the bottom table we loosen this restriction by setting the common component of valuations to between $\underline{\rho} = 0.5$, $\overline{\rho} = 0.9$. In these simulations, we consider the mixture of distributions for auctions with numbers of bidders being $N = 2$,3,...,10, and set the minimum number of bidders to 2, and maximum number to 10.

Several insights can be derived from the simulations in Table 4.

	$\rho = 0.5$		$\rho = 0.7$		$\rho = 0.9$	
$\mu = \log(0.35)$	0.13(1083%)	0.07(114%)	0.08(230%)	0.04(46%)	0.04(79%)	0.00(5%)
	0.14(141%)	0.08(55%)	0.09(64%)	0.04(24%)	0.01(8%)	−0.02(−11%)
$\mu = \log(0.4)$	0.13(243%)	0.08(73%)	0.10(124%)	0.05(43%)	0.02(22%)	−0.01(−11%)
	0.13(77%)	0.08(37%)	0.10(47%)	0.06(24%)	−0.01(−5%)	−0.04(−17%)
$\mu = \log(0.45)$	0.13(129%)	0.08(52%)	0.10(93%)	0.06(37%)	0.06(45%)	0.02(12%)
	0.13(54%)	0.09(29%)	0.10(37%)	0.06(20%)	−0.02(−9%)	−0.05(−19%)
$\mu = \log(0.5)$	0.12(75%)	0.07(35%)	0.10(57%)	0.06(27%)	0.02(10%)	−0.02(−8%)
	0.13(38%)	0.09(22%)	0.10(30%)	0.06(17%)	−0.04(−11%)	−0.07(−19%)
$\mu = \log(1.0)$	0.09(9%)	0.07(7%)	0.07(8%)	0.05(5%)	0.04(5%)	0.02(2%)
	0.10(8%)	0.08(6%)	0.08(5%)	0.06(5%)	−0.19(−16%)	−0.21(−18%)

$\underline{\rho} = \overline{\rho} = 0.7$

	$\rho = 0.5$		$\rho = 0.7$		$\rho = 0.9$	
$\mu = \log(0.35)$	0.13(1102%)	0.08(117%)	0.09(241%)	0.04(51%)	0.03(70%)	−0.00(−0.02%)
	0.14(141%)	0.08(55%)	0.08(60%)	0.04(21%)	0.02(13%)	−0.01(−7%)
$\mu = \log(0.4)$	0.13(242%)	0.08(73%)	0.08(102%)	0.04(29%)	0.03(36%)	−0.00(−1%)
	0.13(77%)	0.08(37%)	0.07(30%)	0.03(10%)	0.01(3%)	−0.02(−10%)
$\mu = \log(0.45)$	0.13(130%)	0.08(52%)	0.08(71%)	0.03(21%)	0.06(52%)	0.03(18%)
	0.11(46%)	0.07(23%)	0.06(24%)	0.03(8%)	−0.00(−2%)	−0.04(−12%)
$\mu = \log(0.5)$	0.13(79%)	0.08(39%)	0.07(40%)	0.03(13%)	0.02(13%)	−0.01(−6%)
	0.12(35%)	0.08(20%)	0.05(14%)	0.01(3%)	−0.01(−2%)	−0.04(−11%)
$\mu = \log(1.0)$	0.07(7%)	0.05(5%)	0.07(7%)	0.04(5%)	0.05(6%)	0.02(3%)
	0.09(6%)	0.07(5%)	−0.03(−2%)	−0.05(−4%)	−0.12(−10%)	−0.14(−12%)

$\underline{\rho} = 0.5, \overline{\rho} = 0.9$

Table 4: Profit difference under the minimax-regret reserve versus setting the reserve at X times the low estimate, r_0. The two tables differ by the bounds on the common component of valuations, defined by $\underline{\rho}$ and $\overline{\rho}$. Each cell in the tables contains a 2x2 grid of profit differences, with each entry displaying the profit difference and the corresponding percentage increase in brackets. % changes are indicated with a dash when original profits are negative. Rows in each cell correspond to $\sigma = \sqrt{0.7}, \sqrt{1}$ (top to bottom), and columns correspond to $x = 0.8 \times r_0, 1.0 \times r_0$ (left to right).

- The reserve price serves as a "guarantee" of sale price. Observe that the percentage profit increase generally decreases as the mean of the lognormal distribution increases, holding all other parameters constant. When bidder valuations are low, there is a large increase in profit by increasing the reserve price. At higher bidder valuation (e.g. $\mu = e$), the percentage increase in profit when setting a higher reserve is much smaller.

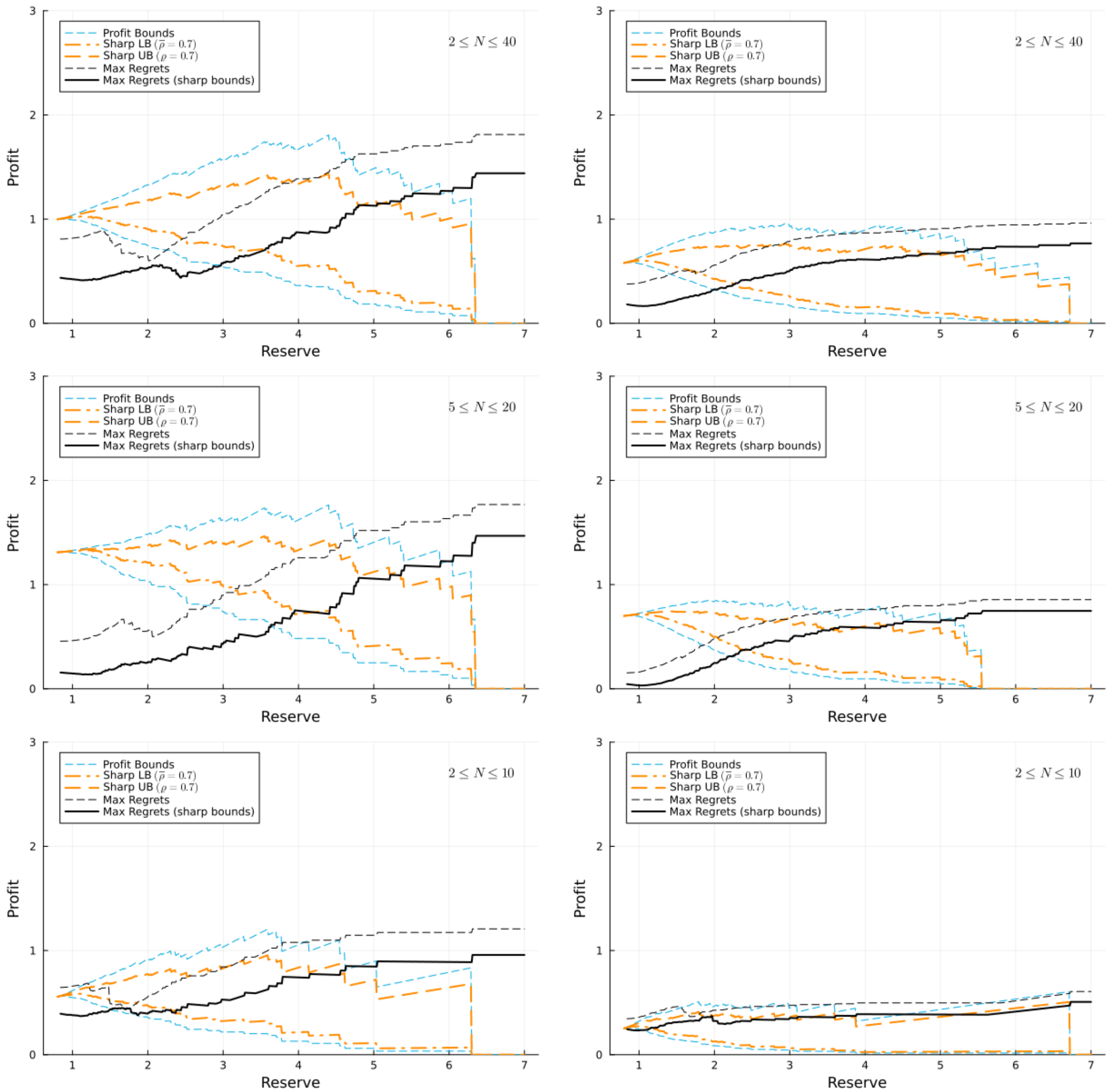
- Consider the lognormal distribution with $\mu = \log(0.4)$, $\sigma = 1.0$, corresponding to an expected value of approximately $\frac{2}{3}$. For correlation $\rho = 0.7$, the profit increase when our model restricts the common component of valuations to exactly 0.7 is higher than the profit increase when we loosen this restriction to $\underline{\rho} = 0.5$, $\overline{\rho} = 0.9$. However, if the true correlation were higher, such as $\rho = 0.9$, the profit increase when we restrict the common component of valuations to 0.7 is less.

6 Results

The collected data spans many categories of auctions, but we focus on Modern Art from the Impressionist, 20th Century, and 21st Century eras sold in New York City, and specifically those with final transaction price either in [\$100K, \$1.0M] or [\$1.0M, \$10.0M].

We estimate the bounds on expected seller profit, $\mathbb{E}[\pi(r)|\underline{n} \leqslant N \leqslant \overline{n}]$, and use these to compute the maximum regret at each reserve. We assume that v_0, the seller's valuation, is equal to the midpoint between the low and high estimates provided by the auction house.

Figure 10 shows our estimates of the bounds on profit and maximum regret, plotted against reserve price for different values of the lower bound on the number of bidders. We plot the profit bounds without restrictions on the extent of the common component of bidders' valuations in blue dashed lines. The corresponding maximum regret is plotted as a dotted black line. We also plot the sharp bounds on profit when bidders' common component of valuations are restricted to $\underline{\rho} = \overline{\rho} = 0.7$ under a bivariate Gaussian copula, in dashed orange lines. The solid black line corresponds to the maximum regret given these sharp bounds on profit. The y-axes (expected profit and maximum regret) are scaled by the average high estimate for the respective



Transaction Price $\in$ [\$100K, \$1.0M] Transaction Price $\in$ [\$1.0M, \$10.0M]

Figure 10: Bounds on expected profit (blue), and maximum regret (black) against reserve price for Modern Art sold in New York City. The orange bounds on profit are sharp for $\rho = 0.7$ under a bivariate Gaussian copula. Here, $v_0 = 5/6$, the middle of the low estimate and high estimate.

auction samples. We display the graphs for at least two, five, and eight bidders.

The following patterns are apparent from these graphs:

- By restricting the degree of correlation among bidder valuations through restricting $\overline{\rho} = 0.7 < 1$, the sharp lower bound on profit is more informative, increasing for a considerable interval beyond v_0. This allows for a more informative choice of minimax regret reserve greater than v_0, which is often the solution under minimax regret without restricting $\overline{\rho}$.

- The average realized profit corresponding to the auction lots in the lower transaction price range of \$100K to \$1.0M is higher than the average realized profit corresponding to the auction lots in the higher transaction price range of \$1.0M to \$10.0M. Correspondingly, the minimax regret reserve is also higher in the lower transaction price range.

- As the number of bidders increases, the profit bounds increase. Furthermore, the profit bounds at low reserves (e.g. below 1.5) are tighter.

Our plots show a significant tightening to both the lower and upper bounds on profit by fixing the degree of correlation among bidders' valuations. Even if the common component parameters were modified slightly, a minimax regret reserve notably higher than $v_0 = \frac{5}{6}$ will be proposed. For example, if $\overline{\rho}$ is set to 0.9 instead of 0.7, for all values of $\underline{\rho}$, the minimax regret reserve is 1.0 for the data with at least five bidders. In Table 9 in the Appendix, we show the minimax regret reserve choice under different choices of $\underline{\rho}$ and $\overline{\rho}$, for the case of at least five bidders and the higher transaction price range of \$1.0M to \$10.0M as an illustrative example. We note that the minimax regret reserve is generally unaffected by the parameter $\underline{\rho}$, the lower bound on the common component of bidders' valuations. However, restricting the $\overline{\rho}$ has affects the minimax regret reserve choice, though in sensible values of $0.5 \leq \overline{\rho} \leq 0.9$, the minimax regret reserve is in the range of [1.0,1.2].

Table 5 shows our proposed minimax regret reserve price by setting $\underline{\rho} = \overline{\rho} = 0.7$, and without restrictions on correlation, i.e. $\underline{\rho} = 0$ and $\overline{\rho} = 1$, as well as their corresponding profit bounds. We make the following observations:

- The average realized profit, scaled by the high estimate, is higher for auction lots with price range \$100K to \$1.0M. Correspondingly, our minimax regret reserve proposed is higher as well.

Price Range	[$100K, $1.0M]			[$1.0M, $10.0M]		
Number of Bidders	$N \in$[2, 10]	$N \in$[5, 20]	$N \in$[2, 40]	$N \in$[2, 10]	$N \in$[5, 20]	$N \in$[2, 40]
$v_0 = 5/6$						
Avg High Estimate	$542k	$342k	$435k	$3.81m	$2.94m	$3.29m
Avg Realized Profit	0.545	1.298	0.998	0.228	0.688	0.573
Minimax regret reserve						
with $\underline{\rho} = 0.7, \overline{\rho} = 0.7$	1.10	1.26	1.14	1.00	1.06	1.06
– Bounds on profit	[0.585, 0.628]	[1.323, 1.340]	[1.025, 1.056]	[0.271, 0.304]	[0.713, 0.723]	[0.600, 0.630]
Minimax regret reserve						
with $\underline{\rho} = 0.0, \overline{\rho} = 1$	1.71	0.84	2.02	0.84	0.84	0.84
– Bounds on profit	[0.396, 0.827]	[1.312, 1.312]	[0.741, 1.333]	[0.261, 0.263]	[0.702, 0.703]	[0.585, 0.586]
$v_0 = 2/3$						
Avg High Estimate	$542k	$342k	$435k	$3.81m	$2.94m	$3.29m
Avg Realized Profit	0.711	1.463	1.164	0.394	0.854	0.739
Minimax regret reserve						
with $\underline{\rho} = 0.7, \overline{\rho} = 0.7$	0.8	1.14	1.14	0.74	1.0	0.84
– Bounds on profit	[0.705, 0.712]	[1.480, 1.490]	[1.156, 1.205]	[0.392, 0.395]	[0.863, 0.872]	[0.732, 0.743]
Minimax regret reserve						
with $\underline{\rho} = 0.0, \overline{\rho} = 1$	1.55	0.72	1.89	0.68	0.66	0.68
– Bounds on profit	[0.459, 0.899]	[1.471, 1.473]	[0.810, 1.417]	[0.388, 0.390]	[0.857, 0.857]	[0.728, 0.729]
Observations	158	161	294	357	381	679

Table 5: Bounds on expected profit for Modern Art sold in New York City. Figures are scaled by the high estimate, and v_0 is set at 5/6 and 2/3.

- Without restricting the common component of bidders' valuations, the minimax regret reserve may be significantly higher than $v_0 = \frac{5}{6}$, such as 2.02 for the auction lots with price range $100K to $1.0M with at least two and up to 40 bidders. However, by restricting the common component, the minimax regret reserve is within a much more reasonable range relative to v_0. Tightening the profit bounds by restricting the common component of bidders' valuations thus has significant empirical relevance, corresponding to the true correlation of valuations among bidders.

When setting $v_0 = 5/6$, our choice of reserve demonstrate a notable improvement over the average realized profit, at 7.3% to 15.2% for lots with price range $100K to $1.0M with between 2 and 10 bidders, and 18.9% to 33.3% for lots with price range $1.0M to $10.0M with between 2 and 10 bidders. When grouping for all bidders, i.e.

between 2 and 40 bidders, the improvement is 2.7% to 5.8% for lots with price range \$100K to \$1.0M, and 4.7% to 9.9% for lots with price range \$1.0M to \$10.0M. There are also profit increases when setting $v_0 = 2/3$, specifically for the case of between 5 and 20 bidders across both auction samples.

7 Conclusion

Our paper presents a novel technique to estimate profit bounds in ascending auctions while precisely providing a measure of correlation among bidders' valuations. The central idea of the method is to adopt a model of conditional independence in bidder valuations, and then carefully quantify the measure on the common component in them by solving a simple univariate convex optimization problem. We also propose a novel approach to solving the minimax regret decision problem and advocate for this approach in the art auction application due to its interpretation aligning with the goal of profit maximization.

We also construct a large novel dataset on live art auctions from the two largest auction houses in the world and provide bounds on the number of bidders. We use the top two bids and the bounds on the number of bidders to estimate profit bounds, and propose an algorithm to solve the minimax-regret decision problem applied to this partially-identified model.

Using this approach, we find that the auction houses' practice of setting the reserve price at or below the low estimate is suboptimal for Modern Art auctions in New York City, and propose minimax-regret reserve prices which can significantly increase profit. We expect a profit change of at least 4.7% and up to 9.9% when choosing the minimax-regret reserve price compared to existing profits, for lots with price range \$1.0M to \$10.0M with at least 2 bidders.

One possible area of future research is the minimax regret solution under the sharp identified set within the model of conditional independence in bidder valuations.

A Proofs

Lemma A.1 (Properties of $\phi_N(\cdot)^N$). *Consider a function $h : [0,1] \to [0,1]$ given by:*

$$h(t) = \phi_N(t)^N,$$

where $\phi_N : [0,1] \to [0,1]$ is defined implicitly via $t = N\phi_N(t)^{N-1} - (N-1)\phi_N(t)^N$. Then:

1. *$h'(t) = \frac{1}{N-1} \cdot \frac{\phi_N(t)}{1-\phi_N(t)}$, so h is strictly increasing.*
2. *$h''(t) = \frac{2}{N-1} \cdot \frac{1}{\binom{N}{2}\phi_N(t)^{N-2}(1-\phi_N(t))^2}$, so h is strictly convex, $h''(0) = h''(1) = \infty$.*
3. *$h'''(t) = \frac{N\phi_N(t)-(N-2)}{N(N-1)\phi_N(t)^{N-1}(1-\phi_N(t))} \cdot h''(t)$, so $h'(\cdot)$ changes curvature exactly once.*
4. *$h(t)$ is increasing in N for all t and $N \geq 2$.*

Proof that the function $h(t)$ is increasing in N for $n \geq 2$. Consider the identity for differentiation, $\left(\alpha(N)^{\beta(N)}\right)'_N = \alpha(N)^{\beta(N)}\beta'(N)\log\alpha(N) + \alpha(N)^{\beta(N)-1}\beta(N)\alpha'(N)$. Using the identity, the derivative is $\left(\phi^N\right)'_N = \phi^N \log\phi + N\phi^{N-1}\phi'_N$. For the derivative ϕ'_N, implicitly differentiating and then rearranging gives $\phi'_N = \frac{\phi^N - \phi^{N-1} - t\log\phi}{N(N-1)\phi^{N-2}(1-\phi)}$. Plugging this in and some algebra then gives

$$\left(\phi^N\right)'_N = \frac{\phi}{(N-1)(1-\phi)}\left(\phi^N - \phi^{N-1} - \phi^{N-1}\log\phi\right).$$

The fraction is clearly positive, while it is easy to verify that $\left(\phi^N - \phi^{N-1} - \phi^{N-1}\log\phi\right)$ is positive by Taylor expansion of $\log\phi$. ■

A.1 Lemma 1

Proof. The upper bound on $F_{N:N}(v)$ is trivial and binds in the case of pure common values. For the lower bound, Lemma A.1 establishes that $u \mapsto \phi_N(u)^N$ is strictly convex. Thus:

$$\begin{aligned} P(V_{N:N} \leqslant v) &= \mathbb{E}[P(V_{N:N} \leqslant v \,|\, U)] \\ &= \mathbb{E}[P(V_i \leqslant v \,|\, U)^N] \\ &\overset{(a)}{=} \mathbb{E}[\phi_N(P(V_{N-1:N} \leqslant v \,|\, U))^N] \\ &\overset{(b)}{\geqslant} \phi_N(F_{N-1:N}(v))^N, \end{aligned}$$

where (a) follows from the definition of $\phi_N(\cdot)$, and (b) from convexity of $t \mapsto \phi_N(t)^N$, Jensen's inequality, and the law of iterated expectations. ■

A.2 Theorem 1

Proof. In the main part of the proof, we show that the stated optimization problems lead to pointwise sharp bounds on $F_{N:N}(v)$. Lemmas A.2 and A.3 show that the bounds are monotone in $F_{N-1:N}(v)$ and thus are plausibly sharp in the functional sense.

For the lower bound, consider the generalized moment problem:

$$\inf_{P \in \mathcal{M}[0,1]} \left\{ \mathbb{E}_P[\phi_N(T)^N] \,\middle|\, \mathbb{E}_P[T] = \mu, c_1 \leqslant \mathbb{V}ar_P(T) \leqslant c_2 \right\} \tag{A.1}$$

where $\mathcal{M}[0,1]$ denotes the set of all probability measures supported on $[0,1]$, $\mu \in [0,1]$ and $0 \leqslant c_1 \leqslant c_2 \leqslant \mu(1-\mu)$. Denote $h(t) = \phi_N(t)^N$. Lemma A.1 establishes that $h(t)$ is strictly increasing, strictly convex, and that $h'(t)$ changes curvature exactly once and satisfies $h''(0) = h''(1) = \infty$. Since $h(t)$ is convex, the variance constraint $\mathbb{V}ar_P(T) = c_1$ must be binding, so the problem is:

$$\inf_{P \in \mathcal{M}[0,1]} \left\{ \mathbb{E}_P[h(T)] \,|\, \mathbb{E}_P[T] = \mu, \mathbb{E}_P[T^2] = \mu^2 + c_1 \right\}. \tag{A.2}$$

When $c_1 = 0$, by Jensen's inequality, the infimum is attained by the distribution $P(T = \mu) = 1$ and equals $h(\mu)$. When $c_1 = \mu(1-\mu)$, the only feasible distribution on $[0,1]$ is the Bernoulli distribution with $P(T = 1) = \mu$. In this case, the infimum is equal to μ. It remains to consider $c_1 \in (0, \mu(1-\mu))$.

It is well known that it suffices to consider distributions P with at most three support points (e.g., Theorem 1 in Kemperman 1968). We first show that, due to the specific shape of the function $h(\cdot)$ and its derivative, it suffices to consider distributions with two support points. Denote $g_1(t) = t$, $g_2(t) = t^2$, and $g(t) = (g_1(t), g_2(t))$ all defined on $\mathcal{T} = [0,1]$. Note that $V = \text{Conv}(g(\mathcal{T})) = \{(z_1, z_2) \in [0,1]^2 : z_1^2 \leqslant z_2 \leqslant z_1\}$, and the point $y = (\mu, \mu^2 + c_1) \in \text{Int(V)}$. Denote $D^* = \{d^* = (d_0, d_1, d_2) \in \mathbb{R}^3 : d_0 + d_1 g_1(t) + d_2 g_2(t) \leqslant h(t) \text{ for all } t \in [0,1]\}$, and, for a given $d^* \in D^*$, let $B(d^*) = \{z = g(t) : d_0 + d_1 g_1(t) + d_2 g_2(t) = h(t) \text{ for some } t \in T\}$. By Theorem 5 and the following remark in Kemperman 1968, for every $y \in \text{Int}(V)$, there exists a d^* for which $y \in \text{Conv}(B(d^*))$. By Theorem 4 of the same paper, such y can be expressed as $y = \sum_{j=1}^{m} p_j g(t_j)$ for some $g(t_j) \in B(d^*)$, and the minimal value of the moment problem is given by $\sum_{j=1}^{m} p_j h(t_j)$. In the problem under consideration, $B(d^*)$ can contain at most two points. Suppose $k(t) = d_0 + d_1 t + d_2 t^2 \leqslant h(t)$, for all $t \in \mathcal{T}$, with equality for some $t_1 < \ldots < t_m$. Then: (i) if $t_1 = 0$, there can be at most one other $t_2 \in (0,1)$; (ii) if all $t_j \in (0,1)$, it must be the case that the line $k'(t) = d_1 + 2d_2 t$ intersects the curve $h'(t)$ from above at each t_j.[5] Indeed, case (i) follows from direct computation, and case (ii) from a simple geometric fact that since $h'(t)$ changes curvature only once (concave then convex) and satisfies $h''(0) = h''(1) = \infty$, there are at most two interior intersections of $d_1 + 2d_2 t$ and $h'(t)$. Thus, the set $B(d^*)$ contains at most two poins, so to solve (A.2), it suffices to consider distributions P with two support points.

For some $0 \leqslant a < b \leqslant 1$ and $p \in [0,1]$, consider the distribution P with $P(X = a) = p$ and $P(X = b) = 1 - p$. The constraints are:

$$\begin{cases} ap + b(1-p) = \mu \\ a^2 p + b^2(1-p) = \mu^2 + c_1 \end{cases} \implies \begin{cases} a(p) = \mu - \sqrt{\frac{1-p}{p} c_1} \\ b(p) = \mu + \sqrt{\frac{p}{1-p} c_1}, \end{cases}$$

and p must be subject to $a, b \in [0,1]$. When $p = 0$ or $p = 1$, $a = b = \mu$ which violates the variance constraint for $c_1 > 0$. Thus, the problem is:

$$\min_{p \in (0,1)} \left\{ h\left(\mu - \sqrt{\tfrac{1-p}{p} c_1}\right) p + h\left(\mu + \sqrt{\tfrac{p}{1-p} c_1}\right)(1-p) \right\}$$
$$\text{s.t. } a(p), b(p) \in [0,1]$$

Leting $s = \sqrt{\frac{1-p}{p} c_1}$ so that $p = \frac{c_1}{c_1 + s^2}$ yields an equivalent formulation:

$$\min_{s \in [\frac{c_1}{1-\mu}, \mu]} \left\{ h(\mu - s) \frac{c_1}{c_1 + s^2} + h\left(\mu + \frac{c_1}{s}\right) \frac{s^2}{c_1 + s^2} \right\}.$$

By direct computation, the objective function is seen to be strictly convex on the feasible set.

[5]For $t_j \in (0,1)$ it means that $k'(t) > h'(t)$ immediately before t_j and $k'(t) < h'(t)$ immediately after. For $t_j \in \{0,1\}$ only one of the two preceding inequalities is required to hold.

Next, consider the generalized moment problem:

$$\sup_{P \in \mathcal{M}[0,1]} \left\{ \mathbb{E}_P[h(T)] \,\middle|\, \mathbb{E}_P[T] = \mu, c_1 \leqslant \mathbb{V}ar_P(T) \leqslant c_2 \right\},$$

or, equivalently,

$$-\inf_{P \in \mathcal{M}[0,1]} \left\{ \mathbb{E}_P[-h(T)] \,\middle|\, \mathbb{E}_P[T] = \mu, c_1 \leqslant \mathbb{V}ar_P(T) \leqslant c_2 \right\}.$$

Since $f(t) = -h(t)$ is concave, the variance constraint $\mathbb{V}ar_P(T) = c_2$ must be binding, so the problem is:

$$-\inf_{P \in \mathcal{M}[0,1]} \left\{ \mathbb{E}_P[f(T)] \,|\, \mathbb{E}_P[T] = \mu, \mathbb{E}_P[T^2] = \mu^2 + c_2 \right\}, \tag{A.3}$$

with $\mu \in [0,1]$ and $c_2 \in [0, \mu(1-\mu)]$. When $c_2 = 0$, by the Edmundson-Mandansky inequality, the infimum is attained by the distribution $P(T = \mu) = 1$ and equals $f(\mu)$. When $c_2 = \mu(1-\mu)$, the only feasible distribution on $[0,1]$ is the Bernoulli distribution with $P(T = 1) = \mu$. In this case, the infimum is equal to μ. It remains to consider $c_2 \in (0, \mu(1-\mu))$.

Using the idea and notation from the first part of the proof, suppose $k(t) = d_0 + d_1 t + d_2 t^2 \leqslant f(t)$ for all $t \in \mathcal{T}$ with equality for some $t_1 < \ldots < t_m$. Then: at each $t_j \in (0,1)$, the line $k'(t)$ must intersect $f'(t)$ from below; if $t_1 = 0$, it must be $k'(0) \leqslant f'(0)$; and if $t_m = 1$, it must be that $k'(1) \geqslant f'(1)$. The function $f'(t)$ changes curvature exactly once (convex then concave) and satisfies $f'(0) = 0$ and $f'(1) = -\infty$. Thus, there can be at most one interior t_j satisfying the requirement above. By Theorem 4 of Kemperman (1968), it suffices to consider distributions with tree support points: 0, 1, and $t \in (0,1)$. Letting $P(T = t) = q$, $P(T = 1) = p$, and $P(T = 0) = 1 - p - q$, the constraints in (A.3) become:

$$\begin{cases} p + qt = \mu \\ p + qt^2 = \mu^2 + c_2 \end{cases} \implies \begin{cases} q = \frac{\mu(1-\mu)-c}{t(1-t)} \\ p = \mu - qt \end{cases}.$$

Plugging this into (A.3) and rearranging yields:

$$\max_{t \in (0,1)} \left\{ \mu - \frac{\mu(1-\mu) - c_2}{t(1-t)} (t - h(t)) \right\}.$$

$\blacksquare$

Lemma A.2 (Properties of the Lower Bound in Theorem 1)**.** *Consider the function:*

$$f(\mu) = \min_{s\in[\frac{c_1}{1-\mu},\mu]} \left\{ \phi_N(\mu-s)^N \frac{c_1}{c_1+s^2} + \phi_N\left(\mu+\frac{c_1}{s}\right)^N \frac{s^2}{c_1+s^2} \right\}$$

with $c_1 = C_\rho(\mu,\mu)-\mu^2$ with C_ρ satisfying the requirements in Assumption 3.4. Then, $\mu \mapsto f(\mu)$ is non-decreasing.

Proof. Denoting the objective function in the optimization problem by $r(s;\mu,c_1)$,

$$\frac{\partial r(s;\mu,c_1)}{\partial s} = \frac{2c_1 s}{(c_1+s^2)}\left\{h\left(\mu+\frac{c_1}{s}\right) - h(\mu-s)\right\} - \frac{c_1}{c_1+s^2}\left\{h'(\mu-s) + h'\left(\mu+\frac{c_1}{s}\right)\right\}.$$

First, note that as $s \to \frac{c_1}{1-\mu}$, $\mu - c_1/s \to 1$, so $h'(\mu - c_1/s) \to +\infty$, by Lemma A.1, and all other terms in the above expression stay finite, so $r'(s;\mu,c_1) \to -\infty$. Thus, the constraint $s \geqslant c_1/(1-\mu)$ never binds, and it suffices to consider two cases: (i) $s = \mu$ and (ii) $s \in (c_1/(1-\mu),\mu)$.

In case (i) (when $C(\mu,\mu) \leqslant c_N\mu$ for a constant c_N solving $2h(u)/u = h'(u)$.)

$$f(\mu) = \frac{\mu^2}{c_1+\mu^2} h\left(\mu+\frac{c_1}{\mu}\right) = \mu \frac{h(\frac{C_\rho(\mu,\mu)}{\mu})}{\frac{C_\rho(\mu,\mu)}{\mu}}.$$

The function $u \mapsto h(u)/u$ is non-decreasing, by Lemma A.1, and $C_\rho(\mu,\mu)/\mu$ is non-decreasing by Assumption 3.4, so $f(\mu)$ is non-decreasing.

In case (ii), by the Envelope Theorem:

$$f'(\mu) = \left.\frac{\partial r(s,\mu,c_1)}{\partial \mu}\right|_{s=s^*(\mu)} + \left.\frac{\partial r(s;\mu,c_1)}{\partial c_1}\right|_{s=s^*(\mu)} \cdot \frac{\partial (C(\mu,\mu)-\mu^2)}{\partial \mu}$$

it can be verified that $\mu \mapsto f'(\mu)$ is non-decreasing. $\blacksquare$

Lemma A.3 (Properties of the Upper Bound in Theorem 1)**.** *Consider the function:*

$$f(\mu) = \max_{s\in(0,1)} \left\{ \mu - \frac{\mu(1-\mu)-c_2}{s(1-s)}(s-\phi_N(s)^N) \right\}$$

with $c_2 = C_\rho(\mu,\mu)-\mu^2$ with C_ρ satisfying the requirements in Assumption 3.4. Then, $\mu \mapsto f(\mu)$ is non-decreasing.

Proof. Denoting the objective function in the optimization problem by $r(s;\mu,c_2)$, by the Envelope Theorem:

$$\begin{aligned}
f'(\mu) &= \left.\frac{\partial r(s,\mu,c_2)}{\partial \mu}\right|_{s=s^*(\mu)} + \left.\frac{\partial r(s;\mu,c_2)}{\partial c_2}\right|_{s=s^*(\mu)} \cdot \frac{\partial (C(\mu,\mu)-\mu^2)}{\partial \mu} \\
&= 1-(1-2\mu)\,\frac{s^*-\phi(s)^N}{s^*-s^{*2}} + \frac{s^*-\phi(s^*)^N}{s^*-s^{*2}} * \left(\frac{\partial (C(\mu,\mu))}{\partial \mu} - 2\mu\right) \\
&= 1+t(s^*)\left(\frac{\partial (C(\mu,\mu))}{\partial \mu} - 1\right)
\end{aligned}$$

where $t(s^*) \equiv \frac{s^*-\phi(s^*)^N}{s^*-s^{*2}}$. By computing the derivative of f with respect to s, it can be verified that $0 < t(s^*) < 1$ for all $N \geqslant 2$, so that a sufficient condition for $f'(\mu) \geqslant 0$ is $\frac{\partial (C(\mu,\mu))}{\partial \mu} \geqslant 0$. But this holds by the 2-increasing nature of copulas. ∎

A.3 Theorem 2

Proof. By our assumptions, for each n and for all v,

$$\begin{aligned}
F_{n-1:n}(v) &= G_{n:n}(v), \\
\underline{\omega}_n(G_{n:n}(v)) \leqslant F_{n:n}(v) &\leqslant \overline{\omega}_n(G_{n:n}(v)).
\end{aligned}$$

Taking expectations in the first line, we obtain:

$$F_{\mathcal{II}}(v) = G_{\mathcal{I}}(v).$$

For the second line, we showed within the proof of Theorem 1 that the functions $\underline{\omega}_n(t)$ and $\overline{\omega}_n(t)$ are convex in t for all n. In addition, both functions are increasing in n for all t since $\phi_n(t)^n$ is increasing in n as shown in Lemma A.1. Therefore, for each $\underline{n} \leqslant n \leqslant \overline{n}$,

$$\underline{\omega}_{\underline{n}}(G_{n:n}(v)) \leqslant F_{n:n}(v) \leqslant \overline{\omega}_{\overline{n}}(G_{n:n}(v)),$$

and by Jensen's inequality,

$$\underline{\omega}_{\underline{n}}(G_{\mathcal{I}}(v)) \leqslant F_{\mathcal{I}}(v) \leqslant \overline{\omega}_{\overline{n}}(G_{\mathcal{I}}(v)).$$

∎

A.4 Lemma 2

Proof. First, it is obvious that taking any $\hat{r} \in \underset{r \in \mathcal{R}}{\operatorname{argmax}}\, \overline{\pi}(r)$, for some given $p \in \mathcal{R}$, where $p \neq \hat{r}$, a function $\pi^*(r) : \mathcal{R} \to \mathbb{R}$ with $\pi^*(\hat{r}) = \underline{\pi}(\hat{r})$ and $\pi^*(p) = \overline{\pi}(p)$ maximizes regret. It then follows that the max regret for some $p \in \mathcal{R}$ where $p \neq \hat{r}$ is $\overline{\pi}(\hat{r}) - \underline{\pi}(p)$.

Next, consider the scenario of choosing a point $p \in \mathcal{R}$ such that $p = \hat{r} \in \operatorname{argmax}_{\mathcal{R}} \overline{\pi}(r)$. In this scenario, the sup regret at $\hat{r}$ is

$$\sup_{\pi \in \Pi} \left\{ \max_{d \in \mathcal{R}} \{\pi(d)\} - \pi(\hat{r}) \right\} = \overline{\pi}\left(\hat{r}\right) - \underline{\pi}\left(\hat{r}\right).$$

To see this, consider the following 2 cases:

1. **Case 1** is when there is only one max point for both $\overline{\pi}$ and $\underline{\pi}$ (i.e. $|\operatorname{argmax}_{r \in \mathcal{R}} \underline{\pi}(r)| = |\operatorname{argmax}_{r \in \mathcal{R}} \overline{\pi}(r)| = 1$) and their argument $\hat{r}$ happens to be the same. First we show that $\forall \epsilon > 0$, $\exists \pi^* \in \Pi$ such that $\text{regret}_{\pi^*}(\hat{r}) \equiv \underset{d \in \mathcal{R}}{\max} \pi^*(d) - \pi^*(\hat{r}) > \overline{\pi}\left(\hat{r}\right) - \underline{\pi}\left(\hat{r}\right) - \epsilon$.
   Pick any $\epsilon > 0$. By the continuity of $\overline{\pi}(.)$, there exists a $\delta > 0$ such that for some $t \in \mathcal{R}$, $|\hat{r} - t| < \delta \implies \overline{\pi}(\hat{r}) - \overline{\pi}(t) < \epsilon$. Pick such a δ and a corresponding $t \neq \hat{r}$ where $|\hat{r} - t| < \delta$. It is possible to construct a $\pi^* : \mathcal{R} \to \mathbb{R}$ such that $\pi^*(\hat{r}) = \underline{\pi}(\hat{r})$ and $\pi^*(t) = \overline{\pi}(t)$. Then the regret at $\hat{r}$ is,

$$\begin{aligned} \text{regret}_{\pi^*}(\hat{r}) &= \overline{\pi}\left(t\right) - \underline{\pi}\left(\hat{r}\right) \\ &> \overline{\pi}\left(\hat{r}\right) - \underline{\pi}\left(\hat{r}\right) - \epsilon \end{aligned}$$

   At the same time, it is impossible for the regret to be $\geq \overline{\pi}\left(\hat{r}\right) - \underline{\pi}\left(\hat{r}\right)$ since it is impossible to construct a continuous π^* to satisfy this. So the sup regret at $\hat{r}$ is $\overline{\pi}\left(\hat{r}\right) - \underline{\pi}\left(\hat{r}\right)$.

2. **Case 2** is any other situation than Case 1. An alternate way to interpret this is that for any point $r \in \mathcal{R}$, it is always possible to pick another $\hat{r} \neq r$, where $\hat{r} \in \operatorname{argmax}_{r \in \mathcal{R}} \overline{\pi}(r)$. Then, as shown previously, the max regret follows to be $\overline{\pi}\left(\hat{r}\right) - \underline{\pi}\left(\hat{r}\right)$.

For this scenario, see that at any $p \in \mathcal{R}$ where $p \neq \hat{r}$, the max regret is $\overline{\pi}(\hat{r}) - \underline{\pi}(p)$.

But $\underline{\pi}(p) \leq \underline{\pi}(\hat{r})$, so

$$\begin{aligned}\max_{\pi \in \Pi} \left\{ \max_{d \in \mathcal{R}} \{\pi(d)\} - \pi(p) \right\} &= \overline{\pi}(\hat{r}) - \underline{\pi}(p) \\ &\geq \overline{\pi}(\hat{r}) - \underline{\pi}(\hat{r})\end{aligned}$$

It is clear that in either scenario, the maximum regret is minimized at some $p \in \mathcal{R}$ where p minimizes $\overline{\pi}(\hat{r}) - \underline{\pi}(p)$. This is precisely at $\underset{r \in \mathcal{R}}{\operatorname{argmax}} \, \underline{\pi}(r)$.

■

A.5 Theorem 3

Proof. Fix some reserve r. Let $\tilde{r} \neq r$ be some other reserve in the support, and assume that v_0 is always less than both $\tilde{r}$ and r. Let $D(\tilde{r}, r; F_1) = \pi(\tilde{r}; F_1) - \pi(r; F_1)$ be the difference between profits at $\tilde{r}$ and r given fixed a CDF F_1. Observe that this can be written as

$$D(\tilde{r}, r; F_1) = \int_{\tilde{r}}^{r} 1 - F_2(v) dv + \tilde{r} - r + F_1(r)(r - v_0) - F_1(\tilde{r})(\tilde{r} - v_0).$$

Now, the maximum regret optimization problem is

$$\max_{F_1} \max_{\tilde{r} \neq r} D(\tilde{r}, r; F_1)$$

subject to the restrictions on the CDF-s. The objective can be written as the max across $\tilde{r} < r$ and $\tilde{r} > r$:

$$\max \left\{ \max_{F_1} \max_{\tilde{r} < r} D(\tilde{r}, r; F_1), \max_{F_1} \max_{\tilde{r} > r} D(\tilde{r}, r; F_1) \right\}.$$

By direct examination of $D(\tilde{r}, r; F_1)$, the first term $\max_{F_1} \max_{\tilde{r} < r} D(\tilde{r}, r; F_1)$ is attained by

1. maximizing $F_1(r)$,
2. minimizing $F_1(v)$ for all $v < r$.

Both conditions above are satisfied by the CDF F_1^*.

Similarly, by direct examination, the second term $\max_{F_1} \max_{\tilde{r} > r} D(\tilde{r}, r; F_1)$ is attained by:

1. setting $F_1(v) = F_1^U(r)$ for all $r \leq v < v_1(r)$,

2. minimizing $F_1(v)$ for all $v \geq v_1(r)$.

To see why condition 1 holds, suppose to the contrary that $F_1(v)$ is set to a value x between $F_1^L(r)$ and $F_1^U(r)$ for all $r \leqslant v < \min\{v : F_1^L(v) = x\}$. Notice that because CDF-s are bounded by 1, $|\tilde{r} - r| \geqslant |F_1(r)(r - v_0) - F_1(\tilde{r})(\tilde{r} - v_0)|$ in the domain $r \leqslant v < \min\{v : F_1^L(v) = x\}$. The maximum regret under this cannot exceed that under condition 1.

Both of these conditions are also satisfied by the proposed CDF F_1^*. ■

Lemma A.4 (Symmetrization)**.** *Let $X = (X_1, \dots, X_n) \in \mathbb{R}^n$ be a random vector with an arbitrary joint distribution. Let $X^{1:n} = (X_{1:n}, \dots, X_{n:n})$ denote the vector of order statistics, where $X_{j:n}$ is the j-th smallest of $(X_1, \dots, X_n)$. There exists a random vector $Y = (Y_1, \dots, Y_n) \in \mathbb{R}^n$ such that: (1) Y is exchangeable; and (2) $Y^{1:n} = X^{1:n}$ almost surely.*

Proof. Such Y can be constructed as follows. Let Π denote the set of all permutation functions $p : \{1, \dots, n\} \to \{1, \dots, n\}$, and π be a random element distributed uniformly in Π. That is, $P(\pi = p) = 1/n!$ for all $p \in \Pi$. Define $Y_j = X_{\pi(j)}$ for $j = 1, \dots, n$. Then:

$$P(Y_1 \leqslant y_1, \dots, Y_n \leqslant y_n) = \frac{1}{n!} \sum_{p \in \Pi} P(X_{p(1)} \leqslant y_1, \dots, X_{p(n)} \leqslant y_n).$$

Since the summation in the right hand side includes all possible events of the form $\{X_{j_1} \leqslant y_1, \dots, X_{j_n} \leqslant y_n\}$, an arbitrary permutation of $\{y_1, \dots, y_n\}$ in the above display changes the order of summands but does not affect the value of the sum. Therefore, for any permutation p,

$$P(Y_1 \leqslant y_{p(1)}, \dots, Y_n \leqslant y_{p(n)}) = P(Y_1 \leqslant y_1, \dots, Y_n \leqslant y_n).$$

Finally, rearranging the elements of X does not affect the order statistics. That is, for all realizations of π, $Y^{1:n} = X^{1:n}$, so that $P(Y^{1:n} = X^{1:n}) = 1$. ■

B Tables

Auction Title	URL
Impressionist and Modern Art Evening Sale — New York	link
20th/21st Century Art Evening Sales — Hong Kong	link
20th Century Evening Sale — New York	link
The Collection of Thomas and Doris Ammann Evening Sale — New York	link
The Collection of Anne H. Bass and 20th Century Evening Sale — New York	link
20th/21st Century: London Evening Sale followed by The Art of the Surreal Evening Sale	link
21st Century Evening Sale — New York	link
A Century of Art: The Gerald Fineberg Collection Part I — New York	link
Post-Millennium Evening Sale — Hong Kong	link
20th/21st Century Art Evening Sales — London	link
Rare Watches Including the Property of Michael Schumacher — Geneva	link
Magnificent Jewels — Geneva	link
20th/21st Century Evening Sales — Hong Kong	link
20th/21st Century: London to Paris — Christie's	link
21st Century Evening Sale — New York	link
20th/21st Century Art Auctions — Christie's Hong Kong	link
20th/21st Century: Evening Sale Including Thinking Italian, London — Christie's	link
The Cox Collection and 20th Century Evening Sale — New York	link
20th/21st Century: Shanghai to London	link
The Collection of Thomas and Doris Ammann Evening Sale — New York	link
20th/21st Century Art Evening Sales — Hong Kong	link
20th/21st Century: London to Paris Evening Sales	link
20th/21st Century: London	link
The Ann & Gordon Getty Evening Sale — New York	link

Table 6: Christie's YouTube Data Sources

Auction Title	URL
Hong Kong Contemporary Art Evening Sale (LIVE)	link
LIVE from New York — Modern Evening Auction	link

LIVE from New York — The Now and Contemporary Evening Auctions	link
LIVE from Hong Kong — The Now and Modern & Contemporary Evening Auctions	link
LIVE from London — Modern & Contemporary Evening Auction featuring The Now	link
LIVE from New York — The Now and Contemporary Evening Auctions	link
LIVE from New York — The Modern Evening Auction	link
LIVE from New York — The Emily Fisher Landau Collection: An Era Defined Evening Auction	link
LIVE from London — The Now & Contemporary Evening Auctions	link
LIVE from Hong Kong — The Autumn Sales	link
LIVE from London — Freddie Mercury: A World of His Own Evening Sale	link
LIVE from London — Old Master & 19th Century Paintings Evening Auction	link
LIVE — The Now & Modern and Contemporary Auctions, ft. Face to Face: A Celebration of Portraiture	link
The Mo Ostin Collection Evening Auction & The Modern Evening Auction	link
LIVE from New York — The Now & Contemporary Evening Auctions	link
LIVE from London — The Now and Modern & Contemporary Evening Auctions	link
LIVE from New York — The Masters Week Auctions	link
LIVE from New York — Master Paintings & Sculpture Part I	link
LIVE from New York — The David M. Solinger Collection & Modern Evening Auctions	link
LIVE from Paris — Modernités	link
LIVE from London — The Now & Contemporary Evening Auctions	link
LIVE from Paris — Hôtel Lambert, The Illustrious Collection, Volume I: Chefs-d'oeuvre	link
LIVE from Hong Kong — Modern, Williamson Pink Star & Contemporary Auctions	link
LIVE from London — Old Masters Evening Auction	link
LIVE from London — The Jubilee Auction and Modern & Contemporary Evening Auction	link
LIVE from New York — The Now & Contemporary Evening Auctions	link
LIVE from New York — Modern Evening Auction	link
LIVE from New York — The Macklowe Collection	link
LIVE from New York — Important Watches	link
LIVE from London — Old Masters Evening Sale	link
LIVE From New York — PROUVÉ x BASQUIAT: The Collection of Peter M. Brant and Stephanie Seymour	link
LIVE from New York — Magnificent Jewels	link
LIVE from London — Treasures	link
LIVE from Monaco — KARL, Karl Lagerfeld's Estate Part I	link
LIVE from Edinburgh — The Distillers One of One Whisky Auction	link
LIVE from Paris — Art Contemporain Evening Sale	link

LIVE from Sotheby's New York — The Now & Contemporary Evening Auctions With U.S. Constitution Sale	link
LIVE from Sotheby's New York — Modern Evening Auction	link
LIVE from Sotheby's New York — The Macklowe Collection	link
LIVE from Paris — Past/Forward and Modernités	link
LIVE from Las Vegas: Icons of Excellence & Haute Luxury	link
LIVE from Las Vegas — Picasso: Masterworks from the MGM Resorts Fine Art Collection	link
LIVE from New York — Collector, Dealer, Connoisseur: The Vision of Richard L. Feigen	link
LIVE From Sotheby's London — Richter, Banksy and Twombley lead the Contemporary Art Evening Auction	link
LIVE From Sotheby's Hong Kong — Modern and Contemporary Art Evening Sales	link
LIVE from London — Old Masters Evening Sale	link
LIVE from London: British Art + Modern & Contemporary Auctions	link
LIVE from Hong Kong: Jay Chou x Sotheby's — Evening Sale	link
LIVE From Sotheby's New York — Important Watches	link
LIVE from Sotheby's New York — Magnificent Jewels	link
LIVE from Sotheby's Paris — Important Design: from Noguchi to Lalanne	link
LIVE from Sotheby's New York — Monet, Warhol and Basquiat Lead Marquee Evening Sales	link
LIVE From Sotheby's Hong Kong — Contemporary Art Evening Sale	link
LIVE From Sotheby's Hong Kong — Icons and Beyond Legends: Modern Art Evening Sale	link
LIVE from Sotheby's Impressionist & Modern Art + Modern Renaissance Auctions	link
LIVE from Sotheby's Sales of Important Chinese Art and Chinese Art from the Brooklyn Museum	link
LIVE from Sotheby's: The Collection of Hester Diamond Auction in New York	link
LIVE from Sotheby's Master Paintings & Sculpture Auction in New York	link
LIVE from Sotheby's London Old Masters Evening Sale	link
LIVE from Sotheby's marquee Evening Sales of Contemporary and Impressionist & Modern Art	link

Table 7: Sotheby's YouTube Data Sources

Saleroom Location	Christie's		Sotheby's	
	Threshold	Rate	Threshold	Rate
Hong Kong	$\leq$HK\$7.5M	26.0%	$\leq$HK\$7,500,000	26.0%
	$>$HK\$7.5M and $\leq$ HK\$50M	20.0%	$>$HK\$7.5M and $\leq$HK\$40M	20.0%
	$>$HK\$50M	14.5%	$>$HK\$40M	13.9%
London	$\leq$£700k	26.0%	$\leq$£800k	26.0%
	$>$£700,000 and $\leq$£4.5M	20.0%	$>$£800k and $\leq$£3.8M	20.0%
	$>$£4.5M	14.5%	$>$£3.8M	13.9%
Paris	$\leq$€700k	26.0%	$\leq$€800k	26.0%
	$>$€700,000 and $\leq$€4M	20.0%	$>$€800k and $\leq$€3.5M	20.0%
	$>$€4M	14.5%	$>$€3.5M	13.9%
New York	$\leq$\$1M	26.0%	$\leq$\$1M	26.0%
	$>$\$1M and $\leq$\$6M	20.0%	$>$\$1M and $\leq$\$4.5M	20.0%
	$>$\$6M	14.5%	$>$\$4.5M	13.9%
Shanghai	$\leq$¥6M	26.0%	-	-
	$>$¥6M and $\leq$¥40M	20.0%	-	-
	$>$¥40M	14.5%	-	-

Note: This table is accurate as of February 7 2022 for Christie's and February 1 2023 for Sotheby's. In the last 10 years, there are only minor changes to the base rate (i.e. lowest threshold category). These buyer premium thresholds are additive, so final transaction amounts are strictly increasing.
Source: Christie's and Sotheby's Websites.

Table 8: Buyer's Premiums in Christie's and Sotheby's Auctions

		$\overline{\rho}$										
		0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
	0.0	1.37	1.34	1.30	1.30	1.22	1.22	1.12	1.06	1.02	1.00	0.84
	0.1		1.34	1.26	1.30	1.22	1.22	1.12	1.06	1.02	1.00	0.84
	0.2			1.30	1.30	1.22	1.22	1.12	1.06	1.02	1.00	0.84
	0.3				1.30	1.22	1.22	1.12	1.06	1.02	1.00	0.84
	0.4					1.22	1.22	1.12	1.06	1.02	1.00	0.84
$\underline{\rho}$	0.5						1.22	1.12	1.06	1.02	1.00	0.84
	0.6							1.12	1.06	1.02	1.00	0.84
	0.7								1.06	1.02	1.00	0.84
	0.8									1.02	1.00	0.84
	0.9										1.00	0.84
	1.0											0.84

Table 9: Minimax regret choice reserve for different $\underline{\rho}$ and $\overline{\rho}$ values

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