$\phi(t,n)$: $t = n \cdot \phi^{n-1}(t,n) - (n-1)\cdot \phi^{n}(t,n)$

$$(\phi(t^n, n))'_n = \underbrace{\phi'_t(t^n, n)}_{\text{easy}} \cdot t^n \log(t) + \underbrace{\phi'_n(t^n, n)}_{\text{a bit tricky}}$$

① $\phi^{-1}(a) = n \cdot a^{n-1} - (n-1)\cdot a^{n} \Rightarrow (\phi^{-1})'(a) = n(n-1)a^{n-2} - n(n-1)a^{n-1}$

$$= n(n-1)a^{n-2}(1-a)$$

$$\Rightarrow \boxed{\phi'_t(t,n) = \frac{1}{n(n-1)\cdot\phi(t,n)^{n-2}(1-\phi(t,n))}}$$

② recall $(\log f(x))' = \dfrac{f'(x)}{f(x)} \Rightarrow f'(x) = f(x)\cdot(\log f(x))'$

so: $$\left(a(n)^{b(n)}\right)' = a(n)^{b(n)} \cdot \left(b(n)\cdot \log(a(n))\right)'$$

$$= a(n)^{b(n)}\left(b'(n)\cdot\log(a(n)) + b(n)\cdot\frac{a'(n)}{a(n)}\right)$$

$$= a(n)^{b(n)}\cdot b'(n)\log a(n) + a(n)^{b(n)-1}\cdot b(n)\cdot a'(n)$$

use this to differentiate : $\underline{t} = n\,\phi^{n-1}(t,n) - (n-1)\,\phi^{n}(t,n)$

$$0 = \phi^{n-1} + n\left(\phi^{n-1}\cdot 1\cdot \log\phi + \phi^{n-2}(n-1)\cdot\phi'(n)\right)$$
$$-\left(\phi^{n} + (n-1)\left(\phi^{n}\cdot 1\cdot\log\phi + \phi^{n-1}\, n\cdot\phi'(n)\right)\right)$$

$$\Rightarrow \quad \phi'(n) = \frac{t^n - \phi^{n-1} + \log\phi\left((n-1)\phi^n - n\cdot\phi^{n-1}\right)}{n(n-1)\cdot\phi^{n-2}(1-\phi)}$$

$$\Rightarrow \quad \boxed{\phi_n'(t,n) = \frac{\phi^n(t,n) - \phi^{n-1}(t,n) - \log\phi(t,n)\cdot t}{n(n-1)\,\phi^{n-2}(t,n)\,(1-\phi(t,n))}}$$

$$\Rightarrow \quad \boxed{\left(\phi(t^n,n)\right)_n' = \frac{t^n\cdot\log(t) + \phi^n - \phi^{n-1} - \log\phi\cdot t^n}{n(n-1)\,\phi^{n-2}(1-\phi)}}$$

$\phi = \phi(t^n, n)$

$$= \frac{t^n\cdot\log\left(\frac{t}{\phi}\right) + \phi^n - \phi^{n-1}}{n(n-1)\,\phi^{n-2}(1-\phi)}$$

So, need to compare: $t^n\cdot\log(t/\phi)$ vs. $\phi^{n-1} - \phi^n$

recall $t^n = n\phi^{n-1} - (n-1)\phi^n \Rightarrow \phi^{n-1} - \phi^n = \dfrac{t^n - \phi^n}{n}$

So $t^n\cdot\log\left(\frac{t}{\phi}\right)$ vs $\dfrac{t^n - \phi^n}{n}$

OR $\log\left(\frac{t}{\phi}\right)$ vs $\dfrac{1}{n}\left(1 - \dfrac{1}{(t/\phi)^n}\right)$!!

But: $\log(x) \geq \dfrac{1}{n}\left(1 - \dfrac{1}{x^n}\right) \quad \forall\, x \geq 0 \quad \forall n \geq 1.$

et voilà!