A Theorem on n - Updated Proof

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1 Theorem

From Haile Tamer (2003), define the strictly increasing differentiable function $\phi_{i:n}(H) : [0,1] \to [0,1]$ as the implicit solution to

$$H = \frac{n!}{(n-i)!(i-1)!} \int_0^{\phi} s^{i-1}(1-s)^{n-i} ds$$

Define $G_{n:n}(t)$ as the highest order statistic among a known n-bidder empirical distribution function with support $[\underline{v}, \bar{v}]$.
Define the function, $f_n(t) : [0,1] \to [0,1]$,

$$f_n(t) = \phi_{n:n}^{-1} \phi_{n-1:n} \left(G_{n:n}(t) \right)$$

Theorem: Suppose $n \in \mathbb{N}$ and $n \geq 2$. Then $f_n(t)$ is decreasing in $n \in \mathbb{N}$, for all $0 \leq t \leq 1$.

2 Proof

First observe that $\phi_{n:n}^{-1}(x) = x^n$ and that the highest order statistic is simply raising to the n^{th} power, so we can write,

$$f_n(t) = \phi_{n-1:n} \left(t^n \right)^n$$

For brevity of notation, from now on let $\phi \equiv \phi_{n-1:n}$.
Consider the identity, $\left(\alpha(n)^{\beta(n)} \right)' = \alpha(n)^{\beta(n)} \beta'(n) \log \alpha(n) + \alpha(n)^{\beta(n)-1} \beta(n) \alpha'(n)$.
Use this to take the derivative with respect to n,

$$\left(f_n(t) \right)_n^{'} = \phi \left(t^n \right)^n \log \phi \left(t^n \right) + \phi \left(t^n \right)^{n-1} n \left(\phi \left(t^n \right) \right)_n^{\prime} \qquad (*)$$

First compute $\left(\phi \left(t^n \right) \right)_n^{\prime}$ in $(*)$, which can be written,

$$\left(\phi \left(t^n, n \right) \right)_n^{'} = \underbrace{\phi_t' \left(t^n, n \right)}_{\text{Term 1}} \cdot t^n \log t + \underbrace{\phi_n' \left(t^n, n \right)}_{\text{Term 2}}$$

Term 1 can be computed as follows. For some $H \in [0,1]$, observe that $\phi_{n-1:n}(H)$ is the implicit solution to the following:

$$\begin{aligned}H &= \frac{n!}{(n-n+1)!(n-1-1)!}\int_0^{\phi} s^{n-1-1}(1-s)^{n-n+1}ds \\ &= n(n-1)\left[\frac{1}{n-1}s^{n-1} - \frac{1}{n}s^n\right]_0^{\phi} \\ &= n\phi^{n-1} - (n-1)\phi^n\end{aligned}$$

And so, the inverse can be written as $\phi^{-1}(H) = nH^{n-1} - (n-1)H^n$. Differentiating this,

$$\begin{aligned}\left(\phi^{-1}\right)^{'}(H) &= n(n-1)H^{n-2} - n(n-1)H^{n-1} \\ &= n(n-1)H^{n-2}(1-H)\end{aligned}$$

And so by the inverse function theorem, we have

$$\phi_t'(t,n) = \frac{1}{n(n-1)\phi(t,n)^{n-2}\left[1-\phi(t,n)\right]}$$

We will now compute **Term 2**.
Again use the identity $\left(\alpha(n)^{\beta(n)}\right)' = \alpha(n)^{\beta(n)}\beta'(n)\log\alpha(n) + \alpha(n)^{\beta(n)-1}\beta(n)\alpha'(n)$ to differentiate $t = n\phi^{n-1}(t,n) - (n-1)\phi^n(t,n)$ implicitly, we get (abbreviating the notation $\phi \equiv \phi(t^n, n)$),

$$\phi_n'(t^n, n) = \frac{\phi^n - \phi^{n-1} - t^n \log\phi}{n(n-1)\phi^{n-2}(1-\phi)}$$

This completes Term 2.
Adding them,

$$\left(\phi(t^n,n)\right)_n^{'} = \frac{t^n\log\frac{t}{\phi} + \phi^n - \phi^{n-1}}{n(n-1)\phi^{n-2}(1-\phi)}$$

Now note that

$$\left(\phi(t^n,n)\right)_n^{'} \geq 0 \;\; \forall n \geq 1 \qquad (**)$$

To see this, first observe that the denominator is positive, so we only need to consider $t^n \log\frac{t}{\phi}$ versus $\phi^n - \phi^{n-1}$. Recall that $t^n = n\phi^{n-1} - (n-1)\phi^n$, so

$$\phi^{n-1} - \phi^n = \frac{t^n - \phi^n}{n} \qquad (***)$$

, so we are comparing $\log\frac{t}{\phi}$ versus $\frac{1}{n}\left(1 - \frac{1}{\left(\frac{t}{\phi}\right)^n}\right)$. But $\log(x) \geq \frac{1}{n}\left(1 - \frac{1}{x^n}\right)$ for all $x \geq 0$, $n \geq 1$.

Plugging $\left(\phi\left(t^n,n\right)\right)_n'$ back into $(*)$, we have,

$$\begin{aligned}
\left(f_n(t)\right)_n' &= \phi^n \log\phi + \phi^{n-1} n \left(\phi\left(t^n,n\right)\right)_n' \\
&= \phi^n \log\phi + \phi^{n-1} \frac{t^n \log\frac{t}{\phi} + \phi^n - \phi^{n-1}}{(n-1)\phi^{n-2}(1-\phi)} \\
&= \frac{\phi}{(n-1)(1-\phi)} \left[\left(\phi^{n-1}-\phi^n\right)\left((n-1)\log\phi - 1\right) + t^n \log\left(\frac{t}{\phi}\right)\right] \\
&= \frac{t^n\phi}{(n-1)(1-\phi)} \underbrace{\left[\frac{1}{n}\left(1-\frac{1}{\left(\frac{t}{\phi}\right)^n}\right)\left((n-1)\log\phi - 1\right) + \log\left(\frac{t}{\phi}\right)\right]}_{(****)}
\end{aligned}$$

We need to show that $(****) \leq 0$ for all $n \geq 2$. Observe that,

$$(****) \leq 0 \iff \log\phi \leq \frac{1}{n-1} \cdot \frac{\frac{1}{n}\left(1-\frac{1}{\left(\frac{t}{\phi}\right)^n}\right) - \log\frac{t}{\phi}}{\frac{1}{n}\left(1-\frac{1}{\left(\frac{t}{\phi}\right)^n}\right)}$$

Let $z \equiv \frac{t}{\phi}$. Then,

$$(****) \leq 0 \iff \log t \leq \underbrace{\log z + \frac{1}{n-1} \cdot \frac{\frac{1}{n}\left(1-\frac{1}{z^n}\right) - \log z}{\frac{1}{n}\left(1-\frac{1}{z^n}\right)}}_{g(z,n)}$$

Now, see that $\log t$ does not depend on n, while in the RHS, the following holds:

1. $g(z,n)$ is increasing in n for a fixed z.
2. $g(z,n)$ is decreasing in z for a fixed n.

But recall that $z = z_n = \frac{t}{\phi_{n-1:n}(t^n)}$, which is decreasing in n for a fixed t from $(**)$. So $g(z_n,n)$ is increasing in n, and thus it suffices to check $(****)$ for $n=2$.

For the case of $n=2$, consider the following properties:

1. $\frac{z^2}{z^2-1} = \frac{1}{2}\left(\frac{1}{1-\phi}+1\right)$ by substituting $z = \frac{t}{\phi}$ and using $(***)$ for $n=2$, $2\phi - \phi^2 = t^2$.
2. $\log z = \frac{1}{2}\left(\log(2-\phi) - \log\phi\right)$ by again using $2\phi-\phi^2 = t^2$ and taking the natural log.

Then,

$$
\begin{aligned}
\log t - g(z_2, 2) &= \log t - \log z - 1 + \frac{z^2}{z^2 - 1} \cdot 2 \log z \\
&= \log \phi - 1 + \frac{1}{2} \left(\frac{1}{1 - \phi} + 1 \right) (\log(2 - \phi) - \log \phi) \\
&= \log \phi - 1 + \frac{\log(2 - \phi) - \log \phi}{2(1 - \phi)} + \frac{\log(2 - \phi) - \log \phi}{2} \\
&= \frac{\log(2 - \phi) - \log \phi}{2(1 - \phi)} + \frac{\log(2 - \phi) + \log \phi}{2} - 1 \\
&= \log(\phi) + \frac{(\phi - 2) \tanh^{-1}(1 - \phi)}{\phi - 1} - 1 \\
&\leq 0 \qquad \forall \phi \in (0, 1)
\end{aligned}
$$

So $(****)$ is true for $n = 2$, which proves that $(****) \leq 0$.
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