Letting r denote a reserve price, and v_0 denote the value of the unsold lot to the seller, the profit is given by:

$$\pi(r) = (r - v_0) \cdot \mathbf{1}(V_{N-1:N} \leqslant r, V_{N:N} > r) + (V_{N-1:N} - v_0) \cdot \mathbf{1}(V_{N-1:N} > r).$$

Taking expectations conditional on N and rearranging:

$$\mathbb{E}[\pi(r)|N] = \int_0^{+\infty} \max(r, v) dF_{N-1:N}(v) - v_0 - (r - v_0)F_{N:N}(r).$$

Further, letting F_1 and F_2 denote the unconditional CDFs of $V_{N-1:N}$ and $V_{N:N}$ correspondingly:

$$\mathbb{E}[\pi(r)] = \int_0^{+\infty} \max(r, v) dF_2(v) - v_0 - (r - v_0)F_1(r).$$

Therefore, to study optimal reserve prices, it suffices to identify or bound the distributions F_1 and F_2.

Theorem 1 (Bounds on the Expected Profit)**.** *Under the above assumptions, the sharp bounds for expected profit are:*

$$\mathbb{E}[\pi(r)] \geqslant \int_0^{+\infty} \max(r, v) dG_2(v) - v_0 - (r - v_0)G_1(r).$$

$$\mathbb{E}[\pi(r)] \geqslant \int_0^{+\infty} \max(r, v) dG_2(v) - v_0 - (r - v_0)\phi_{\underline{N}-1:\underline{N}}(G_1(r))^{\underline{N}}.$$

Proof. Let G_1 and G_2 denote the unconditional CDF's of $B_{N:N}$ and $B_{N-1:N}$ correspondingly. By our assumptions, for each N,

$$\begin{aligned} G_{N:N} &\leqslant F_{N-1:N} &&\leqslant G_{N-1:N}, \\ \phi_{N-1:N}(G_{N:N})^N &\leqslant F_{N:N} &&\leqslant G_{N:N}. \end{aligned}$$

Taking expectations in the first line, we obtain:

$$G_1 \leqslant F_2 \leqslant G_2$$

For the second line, note that the function $f(t, N) = \phi_{N-1:N}(t)^N$ is increasing in N for all t, and convex in t for all N. Therefore, for each $N \geqslant \underline{N}$,

$$\phi_{\underline{N}-1:\underline{N}}(G_{N:N})^{\underline{N}} \leqslant F_{N:N} \leqslant G_{N:N},$$

and, by Jensen's inequality,

$$\phi_{\underline{N}-1:\underline{N}}(G_1)^{\underline{N}} \leqslant F_1 \leqslant G_1.$$

■