A Theorem on n

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1 Theorem

From Haile Tamer (2003), define the strictly increasing differentiable function $\phi_{i:n}(H) : [0,1] \to [0,1]$ as the implicit solution to

$$H = \frac{n!}{(n-i)!(i-1)!} \int_0^{\phi} s^{i-1}(1-s)^{n-i} ds$$

Define $G_{n:n}(v)$ as the highest order statistic among a known n-bidder empirical distribution function with support $[\underline{v}, \bar{v}]$.
Define the function, $f_n(v) : [0,1] \to [0,1]$,

$$f_n(v) = \phi_{n:n}^{-1} \phi_{n-1:n} \left(G_{n:n}(v) \right)$$

Theorem 1: Suppose $n \in \mathbb{N}$ and $n > 1$. Then $f_n(v)$ is decreasing in $n \in \mathbb{N}$, for all $\underline{v} \leq v \leq \bar{v}$.

2 Proof

Lemma 0.1: $\phi_{n:n}^{-1}(x) = x^n$.
Proof: See that :

$$\begin{aligned} H &= \frac{n!}{(n-n)!(n-1)!} \int_0^{\phi} s^{n-1}(1-s)^{n-n} ds \\ &= n \int_0^{\phi} s^{n-1} ds \\ &= n \frac{1}{n} \left[s^n \right]_0^{\phi} \\ &= \phi^n \end{aligned}$$

Lemma 1: If $n, n' \in \mathbb{N}$ and $n' > n$, then $G_{n':n'}(v) \succsim_{\text{FOSD}} G_{n:n}(v)$ for all

$v \in [0,1]$.

Proof:

It suffices to show that $G_{n+1:n+1}(v) \succsim_{\text{FOSD}} G_{n:n}(v)$ for all v.

Take a common distribution G where valuations are drawn from. In 1 scenario, $n+1$ valuations are drawn from G, and we can define the CDF of the highest order statistic,$G_{n+1:n+1}(v) = G(v)^{n+1}$. In the other scenario, n valuations are drawn from G, and the CDF for this highest order statistic is, $G_{n:n}(v) = G(v)^n$. Thus,

$$G_{n+1:n+1}(v) \leq G_{n:n}(v)$$

for all v and n, and so $G_{n+1:n+1}(v) \succsim_{\text{FOSD}} G_{n:n}(v)$. ■

Lemma 2: $\phi_{n:n}^{-1}\phi_{n-1:n}\left(G_{n:n}(v)\right)$ is decreasing in $n \in \mathbb{N}, n > 1$ if the following holds: $\phi_{n-1:n}(H)$ is increasing in $H \in [0,1]$ for any $n \geq 2$.

Proof:

We can rewrite $\phi_{n:n}^{-1}\phi_{n-1:n}\left(G_{n:n}(v)\right)$ as $\phi_{n-1:n}\left(v^n\right)^n$ by Lemma 0.1. From Lemma 1, $G_{n:n}(v)$ is decreasing in n for all $v \in [0,1]$. Furthermore, $\phi_{n-1:n}(.)$ maps to $[0,1]$, so the outer exponent is decreasing in n as well. Thus, to show $\phi_{n-1:n}\left(v^n\right)^n$ is decreasing in n, it suffices to show that $\phi_{n-1:n}(H)$ is increasing in $H \in [0,1]$ for any $n \geq 2, n \in \mathbb{N}$. ■

Lemma 3: $\phi_{n-1:n}(H)$ is increasing in $H \in [0,1]$ for any $n \geq 2$.

Proof:

Observe that $\phi_{n-1:n}(H)$ is the implicit solution to the following:

$$\begin{aligned}
H &= \frac{n!}{(n-n+1)!(n-1-1)!}\int_0^{\phi} s^{n-1-1}(1-s)^{n-n+1}ds \\
&= n(n-1)\left[\frac{1}{n-1}s^{n-1} - \frac{1}{n}s^n\right]_0^{\phi} \\
&= n\phi^{n-1} - (n-1)\phi^n
\end{aligned}$$

We will now show that the inverse of $\phi_{n-1:n}(.)$, $H(n,\phi) = n\phi^{n-1} - (n-1)\phi^n$, is increasing in $\phi \in (0,1)$ for all $n \geq 2$. The partial derivative is,

$$\begin{aligned}
\frac{\partial H(n,\phi)}{\partial \phi} &= n(n-1)\phi^{n-2} - n(n-1)\phi^{n-1} \\
&= n(n-1)\left[\phi^{n-2} - \phi^{n-1}\right] \\
&\geq 0
\end{aligned}$$

where the last inequality comes from the fact that $\phi \in [0,1]$ and $n \geq 2$.

Since the inverse of $\phi_{n-1:n}(.)$ is increasing, $\phi_{n-1:n}(.)$ must be increasing as well. ■

The prooof of **Theorem 1** follows from **Lemmas 1,2,3**.

3 Extensions

Lemma 4: $f_n(v) : [0,1] \to [0,1]$, $f_n(v) = \phi_{n:n}^{-1}\phi_{n-1:n}\left(G_{n:n}(v)\right)$ can be rewritten as:

$$t^n = ny^{\frac{n-1}{n}} - (n-1)y$$

Proof:
The function can be rewritten,

$$\begin{aligned}\phi_{n:n}^{-1}\phi_{n-1:n}\left(G_{n:n}(v)\right) &= \phi_{n-1:n}\left(G_{n:n}(v)\right)^n \\ &= \phi_{n-1:n}\left(v^n\right)^n\end{aligned}$$

Let $y \equiv \phi_{n:n}^{-1}\phi_{n-1:n}\left(G_{n:n}(v)\right)$ and $t \equiv v$. Then, $y = \phi_{n-1:n}\left(t^n\right)^n$. And so,

$$y^{\frac{1}{n}} = \phi_{n-1:n}\left(t^n\right)$$

Then,

$$\begin{aligned}t^n &= \frac{n!}{(n-2)!}\int_0^{y^{\frac{1}{n}}} s^{n-2}(1-s)ds \\ &= n(n-1)\left[\frac{1}{n-1}s^{n-1} - \frac{1}{n}s^n\right]_0^{y^{\frac{1}{n}}} \\ &= ny^{\frac{n-1}{n}} - (n-1)y\end{aligned}$$

Theorem 2: Define the implicit equation,

$$t^n = ny^{\frac{n-1}{n}} - (n-1)y$$

For any fixed $t \in [0,1]$, and $n \geq 2$ integers, $y \in [0,1]$ is decreasing in n.

Proof: This follows from **Lemma 4** and **Theorem 1**.