Theorems on $\bar{n}$ in AL Varying N

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1 Setup

Recall that if valuations are independent of N, the following upper bound holds:

$$F_{n:n}(v) \leq \sum_{m=n+1}^{\bar{n}} \frac{n}{(m-1)m} F_{m-1:m}(v) + \frac{n}{\bar{n}} F_{\bar{n}-1:\bar{n}}(v)$$

And the following lower bound holds:

$$F_{n:n}(v) \geq \sum_{m=n+1}^{\bar{n}} \frac{n}{(m-1)m} F_{m-1:m}(v) + \frac{n}{\bar{n}} \left(\phi_{\bar{n}-1:\bar{n}}\left(F_{\bar{n}-1:\bar{n}}(v)\right)\right)^{\bar{n}}$$

2 Theorems

2.1 Theorem 1

Let $S = \{n+1, n+2, ..., \bar{n}\}$ be the set of all possible upper limits of summation. The upper bound generated with $\max(S) = \bar{n}$ is the tightest upper bound for $F_{n:n,s}$ from all of $s \in S$.

Proof:
Define $\bar{n}_2, \bar{n}_1 \in \mathbb{N}$ and let $n < \bar{n}_1 = \bar{n}_2 - 1 \leq \bar{n}$. Then take the difference of these upper bounds,

$$\begin{aligned}
F_{n:n,\bar{n}_1}(v) - F_{n:n,\bar{n}_2}(v) &= \frac{n}{\bar{n}_1} F_{\bar{n}_1-1:\bar{n}_1}(v) - \frac{n}{\bar{n}_2} F_{\bar{n}_2-1:\bar{n}_2}(v) - \frac{n}{\bar{n}_2(\bar{n}_2-1)} F_{\bar{n}_2-1:\bar{n}_2}(v) \\
&= \frac{n}{\bar{n}_1} F_{\bar{n}_1-1:\bar{n}_1}(v) - F_{\bar{n}_2-1:\bar{n}_2}(v) * \left(\frac{n + n\left(\bar{n}_2 - 1\right)}{\bar{n}_2\left(\bar{n}_2-1\right)}\right) \\
&= \frac{n}{\bar{n}_1} F_{\bar{n}_1-1:\bar{n}_1}(v) - \frac{n}{\bar{n}_2 - 1} F_{\bar{n}_2-1:\bar{n}_2}(v) \\
&= \frac{n}{\bar{n}_1}\left(F_{\bar{n}_1-1:\bar{n}_1}(v) - F_{\bar{n}_2-1:\bar{n}_2}(v)\right)
\end{aligned}$$

Recall that the CDF of the second highest order statistic is $F_{k-1:k}(v) = kF(v)^{k-1} - (k-1)F(v)^k$. So,

$$\begin{aligned}
F_{\bar{n}_1-1:\bar{n}_1}(v) - F_{\bar{n}_2-1:\bar{n}_2}(v) &= \bar{n}_1 F^{\bar{n}_1-1}(v) - \left(\bar{n}_1 - 1\right) F^{\bar{n}_1}(v) - \left[(\bar{n}_1+1)F^{\bar{n}_1}(v) - \bar{n}_1 F^{\bar{n}_1+1}(v)\right] \\
&= \bar{n}_1 F^{\bar{n}_1-1}(v) - 2\bar{n}_1 F^{\bar{n}_1}(v) + \bar{n}_1 F^{\bar{n}_1+1}(v) \\
&= \bar{n}_1\left[F^{\bar{n}_1-1}(v) - 2F^{\bar{n}_1}(v) + F^{\bar{n}_1+1}(v)\right] \\
&= \bar{n}_1 F(v)^{\bar{n}_1-1}\left[1 - 2F(v) + F(v)^2\right] \\
&= \bar{n}_1 F(v)^{\bar{n}_1-1}\left(1 - F(v)\right)\left(1 - F(v)\right) \\
&\geq 0
\end{aligned}$$

So $F_{n:n,\bar{n}_1}(v) \geq F_{n:n,\bar{n}_2}(v)$ for all v. It follows by an inductive argument that $F_{n:n,\bar{n}}(v)$ is indeed the lowest upper bound.

2.2 Theorem 2

Let $S = \{n+1, n+2, ..., \bar{n}\}$ be the set of all possible upper limits of summation. If valuations are *i.i.d.*, the lower bounds for $F_{n:n,s}$ from $s \in S$ are all equally tight.

Proof:
Define $\bar{n}_2, \bar{n}_1 \in \mathbb{N}$ and let $n < \bar{n}_1 = \bar{n}_2 - 1 \leq \bar{n}$. Then take the difference of these upper bounds,

$$\begin{aligned} F_{n:n,\bar{n}_2}(v) - F_{n:n,\bar{n}_1}(v) &= \frac{n}{\bar{n}_2(\bar{n}_2-1)} F_{\bar{n}_2-1:\bar{n}_2}(v) + \frac{n}{\bar{n}_2}\left[\phi_{\bar{n}_2-1:\bar{n}_2}\left(F_{\bar{n}_2-1:\bar{n}_2}(v)\right)\right]^{\bar{n}_2} - \frac{n}{\bar{n}_1}\left[\phi_{\bar{n}_1-1:\bar{n}_1}\left(F_{\bar{n}_1-1:\bar{n}_1}(v)\right)\right]^{\bar{n}_1} \\ &= \frac{n}{\bar{n}_2(\bar{n}_2-1)}\left\{F_{\bar{n}_2-1:\bar{n}_2}(v) + (\bar{n}_2-1)\left[\phi_{\bar{n}_2-1:\bar{n}_2}\left(F_{\bar{n}_2-1:\bar{n}_2}(v)\right)\right]^{\bar{n}_2} - (\bar{n}_2)\left[\phi_{\bar{n}_1-1:\bar{n}_1}\left(F_{\bar{n}_1-1:\bar{n}_1}(v)\right)\right]^{\bar{n}_1}\right\} \end{aligned}$$

Since valuations are *i.i.d.*, $\phi_{k-1:k}\left(F_{k-1:k}(v)\right) = F(v)$, so we have,

$$\begin{aligned} F_{n:n,\bar{n}_2}(v) - F_{n:n,\bar{n}_1}(v) &= \frac{n}{\bar{n}_2(\bar{n}_2-1)}\left\{F_{\bar{n}_2-1:\bar{n}_2}(v) + \bar{n}_1 F(v)^{\bar{n}_2} - \bar{n}_2 F(v)^{\bar{n}_1}\right\} \\ &= \frac{n}{\bar{n}_2(\bar{n}_2-1)}\left\{F_{\bar{n}_2-1:\bar{n}_2}(v) - \left[\bar{n}_2 F(v)^{\bar{n}_1} - \bar{n}_1 F(v)^{\bar{n}_2}\right]\right\} \qquad \text{(i)} \\ &= \frac{n}{\bar{n}_2(\bar{n}_2-1)}\left\{F_{\bar{n}_2-1:\bar{n}_2}(v) - F_{\bar{n}_2-1:\bar{n}_2}(v)\right\} \\ &= 0 \end{aligned}$$

(i): The CDF of the second highest order statistic is $F_{k-1:k}(v) = kF(v)^{k-1} - (k-1)F(v)^k$.

So we have $F_{n:n,\bar{n}_2}(v) = F_{n:n,\bar{n}_1}(v)$. Then, again following an inductive argument, the bounds for $F_{n:n}$ from S are all equally tight.

2.3 Theorem 3

Let $S = \{n+1, n+2, ..., \bar{n}\}$ be the set of all possible upper limits of summation. If valuations are common, the lower bounds for $F_{n:n,s}$ from $s \in S$ are increasing in in s.

Proof:
Define $\bar{n}_2, \bar{n}_1 \in \mathbb{N}$ and let $n < \bar{n}_1 = \bar{n}_2 - 1 \leq \bar{n}$. If valuations are common, $F_{\bar{n}_2-1:\bar{n}_2}(v) = F_{\bar{n}_1-1:\bar{n}_1}(v) \equiv H$. So write,

$$\begin{aligned} F_{n:n,\bar{n}_2}(v) - F_{n:n,\bar{n}_1}(v) &= \frac{n}{\bar{n}_2(\bar{n}_2-1)}\left\{H + \bar{n}_1\left[\phi_{\bar{n}_2-1:\bar{n}_2}(H)\right]^{\bar{n}_2} - \bar{n}_2\left[\phi_{\bar{n}_1-1:\bar{n}_1}(H)\right]^{\bar{n}_1}\right\} \\ &= \frac{n}{\bar{n}_2(\bar{n}_2-1)}\left\{H + \bar{n}_1\left[\phi_{\bar{n}_1:\bar{n}_1+1}(H)\right]^{\bar{n}_1+1} - (\bar{n}_1+1)\left[\phi_{\bar{n}_1-1:\bar{n}_1}(H)\right]^{\bar{n}_1}\right\} \end{aligned}$$

So we want to show that $H + \bar{n}_1\left[\phi_{\bar{n}_1:\bar{n}_1+1}(H)\right]^{\bar{n}_1+1} - (\bar{n}_1+1)\left[\phi_{\bar{n}_1-1:\bar{n}_1}(H)\right]^{\bar{n}_1} \geq 0$.
Lemma: $\phi_{n:n+1}(H) \geq \phi_{n-1:n}(H)$ for all $n \geq 2$ and $H \in [0,1]$.
Proof: For reduced notation let $\phi \equiv \phi_{n-1:n}$ in this proof. Recall that $H = n\phi^{n-1} - (n-1)\phi^n$. Now differentiate this with respect to n. To do so, we need to use the identity $\left((\alpha_n)^{\beta(n)}\right)' = \alpha(n)^{\beta(n)}\beta'(n)\log\alpha(n) + \alpha(n)^{\beta(n)-1}$. So,

$$\begin{aligned} \left(n\phi^{n-1}\right)'_n &= \phi^{n-1} + n\left[\phi^{n-1}\log\phi + \phi^{n-2}(n-1)\phi'\right] \\ \left((n-1)\phi^n\right)'_n &= \phi^n + (n-1)\left[\phi^n\log\phi + \phi^{n-1}n\phi'\right] \end{aligned}$$

Then it follows that,

$$\phi^{n-1} + n\left[\phi^{n-1}\log\phi + \phi^{n-2}(n-1)\phi'\right] = \phi^n + (n-1)\left[\phi^n\log\phi + \phi^{n-1}n\phi'\right]$$

Rearranging,

$$\phi' = \frac{\phi - \phi^2 + n\phi\log\phi - (n-1)\phi^2\log\phi}{n(n-1)(\phi-1)}$$

Recall that $\phi \in [0,1]$, so to show this derivative is ≥ 0, we only need to show that the numerator ≤ 0. We can write,

$$\begin{aligned} 1-\phi+n\log\phi-(n-1)\phi\log\phi &= 1-\phi+[n-(n-1)\phi]\log\phi \\ &\leq 1-\phi+[n-(n-1)\phi](\phi-1) \qquad \text{(i)} \\ &= (1-\phi)^2[1-n] \\ &\leq 0 \end{aligned}$$

(i): This follows from the Taylor expansion centered at 1: $\log x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \ldots$. So $\log(x) \leq x-1$ for all $x \in [0,1]$. ■

Now, again recall that $H = k\phi_{k-1:k}^{k-1} - (k-1)\phi_{k-1:k}^{k}$. Substitute H where $k = \bar{n}_1 + 1$, then

$$\begin{aligned} &H + \bar{n}_1\left[\phi_{\bar{n}_1:\bar{n}_1+1}(H)\right]^{\bar{n}_1+1} - (\bar{n}_1+1)\left[\phi_{\bar{n}_1-1:\bar{n}_1}(H)\right]^{\bar{n}_1} \\ =&(\bar{n}_1+1)\left[\phi_{\bar{n}_1:\bar{n}_1+1}(H)\right]^{\bar{n}_1} - (\bar{n}_1+1)\left[\phi_{\bar{n}_1-1:\bar{n}_1}(H)\right]^{\bar{n}_1} \\ =&(\bar{n}_1+1)\left[\phi_{\bar{n}_1:\bar{n}_1+1}(H)^{\bar{n}_1} - \phi_{\bar{n}_1-1:\bar{n}_1}(H)^{\bar{n}_1}\right] \end{aligned}$$

But from the Lemma, $\phi_{\bar{n}_1:\bar{n}_1+1}(H) \geq \phi_{\bar{n}_1-1:\bar{n}_1}(H)$. So $F_{n:n,\bar{n}_2}(v) - F_{n:n,\bar{n}_1}(v) \geq 0$. It follows by an inductive argument that the lower bounds for $F_{n:n,s}$ from $s \in S$ are increasing in in s, and thus $F_{n:n,\bar{n}}(v)$ is indeed the greatest lower bound.