$$\mathbb{E}[\pi(r) \mid N] = \mathbb{E}\left[\max\{r, V_{N-1:N}\} \mid N\right] - v_0 - \mathbb{E}\left[\mathbb{1}\{V_{N:N} \leq r\} \mid N\right] \cdot (r - v_0)$$

$\therefore$ By tower law,

$$\mathbb{E}[\pi(r)] = \mathbb{E}\left[\max\{r, V_{N-1:N}\}\right] - v_0 - \mathbb{E}\left[\mathbb{1}\{V_{N:N} \leq r\}\right] \cdot (r - v_0)$$

Now, suppose that we only observe the bounds for N, $\underline{N}$ and $\overline{N}$. We do not observe N.

By our assumptions, the following hold:

- $\mathbb{E}\left[\max\{r, B_{N-1:N}\}\right] \leq \mathbb{E}\left[\max\{r, V_{N-1:N}\}\right] \leq \mathbb{E}\left[\max\{r, B_{N:N}\}\right]$

Since $\phi_{N-1:N}(t)^N$ is increasing in N for all $0 \leq t \leq 1$,

$$\mathbb{E}\left[\phi_{N-1:N}(G_{N:N})^N\right] \geq \phi_{N-1:N}\left(\mathbb{E}[G_{N:N}]\right)^N \quad \text{(Jensen, } \phi_N(\cdot)^N \text{ convex)}$$

$$\geq \phi_{\underline{N}-1:\underline{N}}\left(\mathbb{E}[G_{N:N}]\right)^{\underline{N}}$$

$$\leq \mathbb{E}[F_{N:N}]$$

Thus,

$$\mathbb{E}[F_{N:N}] \geq \phi_{\underline{N}-1:\underline{N}}\left(\mathbb{E}[G_{N:N}]\right)^{\underline{N}}$$

We also have, by assumption (Haile-Tamer),

$$\mathbb{E}\left[\mathbb{1}\{V_{N:N} \le r\}\right] \le \mathbb{E}\left[\mathbb{1}\{B_{N:N} \le r\}\right]$$

which means

$$F_{N:N} \le G_{N:N}$$