$f_n(t) : [0,1] \to [0,1]$.
$f_n(t) = \Phi_{n-1:n}(t)^n$.

Thm: Let $n \geq 2$. Then, $f_n(t)$ is increasing in $n \in \mathbb{N}$, for all $0 \leq t \leq 1$.

Proof $\left(\alpha(n)^{\beta(n)}\right)'_n = \alpha(n)^{\beta(n)} \beta'(n) \log \alpha(n) + \alpha(n)^{\beta(n)-1} \beta(n) \alpha'(n)$

The derivative is,

$$\phi^n \log \phi + \phi^{n-1} \cdot n \cdot \underbrace{(\phi(t))'_n}_{(**)} \qquad (*)$$

Need to compute $(\phi(t))'_n$.

Observe that $\Phi_{n-1:n}(t)$ is equivalent to implicitly solving for $t = n\phi^{n-1} - (n-1)\phi^n$.

Again use the $\left(\alpha(n)^{\beta(n)}\right)'_n$ identity to differentiate this,

$$0 = \phi^{n-1} + n \cdot \left[\phi^{n-1} \log \phi + \phi^{n-2} \cdot (n-1) \cdot \phi'\right] - \left\{\phi^n + (n-1) \cdot \left[\phi^n \log \phi + \phi^{n-1} \cdot n \cdot \phi'\right]\right\}$$

$$\Leftrightarrow 0 = \phi^{n-1}\left[1 + n \log \phi\right] - \phi^n\left[1 + (n-1)\log \phi\right] + \phi'\left[n\phi^{n-2}(n-1) - (n-1)n\phi^{n-1}\right]$$

$$\Leftrightarrow (\phi(t,n))'_n = \frac{\phi^n - \phi^{n-1} + \log \phi\left[\phi^n(n-1) - \phi^{n-1} \cdot n\right]}{n(n-1)\phi^{n-2}(1-\phi)}$$

$$= \frac{\phi^n - \phi^{n-1} - t\log\phi}{n(n-1)\phi^{n-2}(1-\phi)} \qquad (**)$$

Put $(**)$ into $(*)$,

$$(*) = \phi^n \log\phi + \phi^{n-1} \cdot n \cdot \frac{\phi^n - \phi^{n-1} - t\log\phi}{n(n-1)\phi^{n-2}(1-\phi)}$$

$$= \phi^n \log\phi + \phi \cdot \frac{\phi^n - \phi^{n-1} - t\log\phi}{(n-1)(1-\phi)}$$

$$= \frac{\phi}{(n-1)(1-\phi)} \left[\phi^{n-1}\log\phi(n-1)(1-\phi) + \phi^n - \phi^{n-1} - t\log\phi \right]$$

$$= \frac{\phi}{(n-1)(1-\phi)} \left[(\phi^{n-1} - \phi^n)\left[\log\phi(n-1) - 1\right] - t\log\phi \right]$$

Recall that $t = n\phi^{n-1} - (n-1)\phi^n$, so,

$$(*) = \frac{\phi}{(n-1)(1-\phi)} \left[(\phi^{n-1} - \phi^n)(\log\phi(n-1) - 1) - n\log\phi \cdot \phi^{n-1} + (n-1)\phi^n \log\phi \right]$$

$$= \frac{\phi}{(n-1)(1-\phi)} \left[-\phi^{n-1}\log\phi - \phi^{n-1} + \phi^n \right]$$

$$= \underbrace{\frac{\phi}{(n-1)(1-\phi)}}_{\geq 0} \left[\phi^n - \phi^{n-1} - \phi^{n-1}\log\phi \right]$$

By Taylor expansion centered at 1,

$$\log x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \dots$$

So $\log(\phi) \leq \phi - 1 \quad \forall \phi \in [0,1]$.

$$\Rightarrow \phi^n - \phi^{n-1} - \phi^{n-1}\log\phi$$

$$\geq \phi^n - \phi^{n-1} - \phi^{n-1}(\phi - 1)$$

$$= 0$$

$\square$