三. proof: 设 $y_k = -\frac{1}{a}\ln(\xi_k) \Rightarrow \xi_k = e^{-ay_k}$ $\xi_k \in (0,1] \Rightarrow y_k \in [0,+\infty)$

$f(\xi_k)|d\xi_k| = g(y_k)|dy_k| \Rightarrow g(y_k) = \left|\frac{d\xi_k}{dy_k}\right| = ae^{-ay_k}$

令: $z = -\frac{1}{a}\ln(\xi_1\cdots\xi_n) = y_1+y_2+\cdots+y_k$

∴ $h(z) = g(y_1) * g(y_2) \cdots * g(y_n)$

$= \iint\cdots\int_0^{+\infty} a^n e^{-ay_1} e^{-a(y_2-y_1)} \cdots e^{-a(z-y_{n-1})} dy_1 dy_2 \cdots dy_n$

∵ $y_2-y_1>0$ $y_3-y_2>0$ $\cdots$ $z-y_{n-1}>0 \Rightarrow z>y_{n-1}>\cdots>y_2>y_1\geq 0$

∴ $h(z) = a^n e^{-az} \iint\cdots\int_0^{y_2} dy_1 \cdots dy_n$

$= a^n e^{-az} \iint\cdots\int_0^{y_3} y_2 dy_2 \cdots dy_n = a^n e^{-az} \int\cdots\int_0^{y_4} \frac{y_3^2}{2!} dy_3 \cdots dy_n$

$= \cdots = a^n e^{-az} \frac{z^{n-1}}{(n-1)!}$

故 Γ分布抽样方式可为 $z = -\frac{1}{a}\ln(\xi_1\cdots\xi_n)$

四. proof: 令 $y_k = x_k^2 \Rightarrow x_k = \pm\sqrt{y_k} = \begin{cases} \sqrt{y_k}, & x_k \geq 0 \\ -\sqrt{y_k}, & x_k < 0 \end{cases}$

$g(y_k)|dy_k| = \sum_{i=0} f(x_{ki})|dx_{ki}|$

$\Rightarrow g(y_k) = f(x_{k1})\left|\frac{dx_{k1}}{dy_k}\right| + f(x_{k2})\left|\frac{dx_{k2}}{dy_k}\right| = \frac{1}{\sqrt{2\pi}} e^{-\frac{y_k}{2}} y_k^{-\frac{1}{2}}$

$= \frac{1}{2^{\frac{1}{2}}\Gamma(\frac{1}{2})} y_k^{(\frac{1}{2}-1)} e^{-\frac{y_k}{2}}$ 即 $Y_k \sim \chi^2(1)$

若 $Y_1 \sim \chi^2(m)$ $Y_2 \sim \chi^2(n)$, 则

$f_{Y_1+Y_2}(z) = f_{Y_1}(y) * f_{Y_2}(y)$

$= \int_0^{+\infty} \frac{1}{2^{\frac{m}{2}}\Gamma(\frac{m}{2})} y^{\frac{m}{2}-1} e^{-\frac{y}{2}} \frac{1}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})} (z-y)^{\frac{n}{2}-1} e^{-\frac{z-y}{2}} dy$

$= \frac{e^{-\frac{z}{2}}}{2^{\frac{m+n}{2}}\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})} \int_0^z y^{\frac{m}{2}-1}(z-y)^{\frac{n}{2}-1} dy$

$= \frac{z^{\frac{m+n}{2}-1} e^{-\frac{z}{2}}}{2^{\frac{m+n}{2}}\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})} \int_0^z \left(\frac{y}{z}\right)^{\frac{m}{2}-1}\left(1-\frac{y}{z}\right)^{\frac{n}{2}-1} d\left(\frac{y}{z}\right)$

$= \frac{z^{\frac{m+n}{2}-1} e^{-\frac{z}{2}}}{2^{\frac{m+n}{2}}\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})} \frac{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})}{\Gamma(\frac{m+n}{2})}$

$= \frac{1}{2^{\frac{m+n}{2}}\Gamma(\frac{m+n}{2})} z^{\frac{m+n}{2}-1} e^{-\frac{z}{2}}$

即 $Y_1+Y_2 \sim \chi^2(m+n)$ 故 $\sum_{k=1}^{n} y_k \sim \chi^2(n)$

即 $z = \sum_{i=1}^{n} x_i^2 = \sum_{k=1}^{n} y_k \sim \chi^2(n)$ $f(z) = \frac{1}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})} z^{\frac{n}{2}-1} e^{-\frac{z}{2}}$

上述过程中, $\int_0^z \left(\frac{y}{z}\right)^{\frac{m}{2}-1}\left(1-\frac{y}{z}\right)^{\frac{n}{2}-1} d\left(\frac{y}{z}\right) = B\left(\frac{m}{2},\frac{n}{2}\right)$

$= \frac{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})}{\Gamma(\frac{m}{2}+\frac{n}{2})}$ $\Gamma(\frac{1}{2}) = \sqrt{\pi}$