计算方法第五次编程作业

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1　题目

"point.txt" 文件中包含了 21 个压铁的位置信息

- (a) 利用大 M 法计算出木条在压铁控制下的曲线，边界条件取自然边界条件，并使用追赶法对得到的线性方程组进行求解。
- (b) 将第 10 个压铁的位置移动至 (0,10)，计算出新的曲线，观察每个区间内的三次函数是否改变。

注：已将第 10 个压铁的位置移动至 (0,10) 后的数据保存在 "$point_b.txt$" 文件中。

2　原理

- 大 M 法计算三次样条插值（自然边界条件）

  给定插值点 $\{(x_i, f(x_i)), i = 0, 1 \ldots, n\}$ ，记插值函数在 $[x_i, x_{i+1}]$ 上的表达式为 $S_i(x)$ ，为三次多项式。

  记 $S''(x_i) = M_i, \quad h_i = x_{i+1} - x_i$, 则

$$
\begin{aligned}
S(x) =& \frac{(x_{i+1}-x)^3 M_i + (x-x_i)^3 M_{i+1}}{6h_i} + \frac{(x_{i+1}-x)y_i + (x-x_i)y_{i+1}}{h_i} \\
& - \frac{h_i}{6}[(x_{i+1}-x)M_i + (x-x_i)M_{i+1}], \quad x \in [x_i, x_{i+1}] \\
=& \frac{M_{i+1}-M_i}{6h_i}x^3 + \frac{x_{i+1}M_i - x_i M_{i+1}}{2}
\end{aligned}
\tag{1}
$$

  展开得

$$
\begin{aligned}
S(x) =& \frac{M_{i+1}-M_i}{6h_i}x^3 + \frac{x_{i+1}M_i - x_i M_{i+1}}{2h_i}x^2 \\
& + \frac{3(x_i^2 M_{i+1} - x_{i+1}^2 M_i) + 6(y_{i+1}-y_i) - h_i^2(M_{i+1}-M_i)}{6h_i}x \\
& + \frac{x_{i+1}^3 M_i - X_i^3 M_{i+1} + 6(x_{i+1}y_i - x_i y_{i+1}) - h_i^2(x_{i+1}M_i - x_i M_{i+1})}{6h_i}
\end{aligned}
\tag{2}
$$

其中 M 满足

$$\mu_i M_{i-1} + 2M_i + \lambda_i M_{i+1} = d_i, \quad i = 1, 2, \ldots, n-1 \tag{3}$$

where

$$\begin{aligned} \lambda_i &= \frac{h_i}{h_i + h_{i-1}} \quad \mu_i = 1 - \lambda_i \\ d_i &= \frac{6}{h\_i + h_{i-1}}\left(\frac{y_{i+1} - y_i}{h_i} - \frac{y_i - y_{i-1}}{h_{i-1}}\right) = 6f[x_{i-1}, x_i, x_{i+1}] \end{aligned} \tag{4}$$

在自然边界条件下 ($M_0 = M_n = 0$)，方程组为

$$\begin{bmatrix} 2 & \lambda_1 & & & \\ \mu_2 & 2 & \lambda_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \mu_{n-2} & 2 & \lambda_{n-2} \\ & & & \mu_{n-1} & 2 \end{bmatrix} \begin{bmatrix} M_1 \\ M2 \\ \vdots \\ M_{n-2} \\ M_{n-1} \end{bmatrix} = \begin{bmatrix} d_1 - \mu_1 M_0 \\ d_2 \\ \vdots \\ d_{n-2} \\ d_{n-1} - \lambda_{n-1} M_n \end{bmatrix}$$

由于矩阵为三对角阵，因此可使用追赶法解方程组。

3 结果

函数结果如图 1 和图 2 所示。

```
[-9, -8]: S{0}=(0.219791)*x^3 + (5.133804)*x^2 + (40.366445)*x + (107.814439)
[-8, -7]: S{1}=(0.248246)*x^3 + (5.816716)*x^2 + (45.016242)*x + (115.875232)
[-7, -6]: S{2}=(-0.242973)*x^3 + (-4.498877)*x^2 + (-28.162706)*x + (-59.401381)
[-6, -5]: S{3}=(0.095147)*x^3 + (1.587286)*x^2 + (8.982766)*x + (17.403564)
[-5, -4]: S{4}=(-0.082815)*x^3 + (-1.082141)*x^2 + (-4.419166)*x + (-5.115657)
[-4, -3]: S{5}=(-0.309288)*x^3 + (-3.799817)*x^2 + (-14.744472)*x + (-17.428331)
[-3, -2]: S{6}=(0.908466)*x^3 + (7.159968)*x^2 + (18.546383)*x + (16.685524)
[-2, -1]: S{7}=(-0.889576)*x^3 + (-3.628288)*x^2 + (-5.465127)*x + (-2.568816)
[-1, 0]: S{8}=(0.203140)*x^3 + (-0.350139)*x^2 + (0.259722)*x + (0.970600)
[0, 1]: S{9}=(0.445017)*x^3 + (-0.350139)*x^2 + (-0.108278)*x + (0.970600)
[1, 2]: S{10}=(-0.738108)*x^3 + (3.199237)*x^2 + (-4.902754)*x + (3.398825)
[2, 3]: S{11}=(0.747415)*x^3 + (-5.713904)*x^2 + (14.683527)*x + (-12.005362)
[3, 4]: S{12}=(-0.339953)*x^3 + (4.072413)*x^2 + (-16.587024)*x + (23.088389)
[4, 5]: S{13}=(-0.111902)*x^3 + (1.335803)*x^2 + (-4.916282)*x + (5.595933)
[5, 6]: S{14}=(0.079663)*x^3 + (-1.537673)*x^2 + (10.158999)*x + (-21.889202)
[6, 7]: S{15}=(0.163651)*x^3 + (-3.049471)*x^2 + (18.859387)*x + (-37.808379)
[7, 8]: S{16}=(-0.319969)*x^3 + (7.106553)*x^2 + (-52.647083)*x + (130.973453)
[8, 9]: S{17}=(0.354323)*x^3 + (-9.076456)*x^2 + (77.578891)*x + (-220.359280)
[9, 10]: S{18}=(-0.163425)*x^3 + (4.902740)*x^2 + (-49.167776)*x + (165.484122)
[10, 11]: S{19}=(0.021167)*x^3 + (-0.635000)*x^2 + (6.487833)*x + (-21.889300)
```

图 1: (a) 中得到的三次样条插值函数

4 结果分析

将两个函数分别绘制出来，如图 3 5 所示。

```
[-9, -8]: S{0}=(0.221230)*x^3 + (5.172672)*x^2 + (40.714818)*x + (108.850922)
[-8, -7]: S{1}=(0.241048)*x^3 + (5.648288)*x^2 + (43.706243)*x + (112.489387)
[-7, -6]: S{2}=(-0.215621)*x^3 + (-3.941767)*x^2 + (-24.393938)*x + (-50.936768)
[-6, -5]: S{3}=(-0.007062)*x^3 + (-0.187692)*x^2 + (-1.240989)*x + (-2.116869)
[-5, -4]: S{4}=(0.298669)*x^3 + (4.398264)*x^2 + (21.633990)*x + (35.825429)
[-4, -3]: S{5}=(-1.733013)*x^3 + (-19.981911)*x^2 + (-75.341310)*x + (-92.020571)
[-3, -2]: S{6}=(6.221882)*x^3 + (51.612140)*x^2 + (139.852344)*x + (123.996083)
[-2, -1]: S{7}=(-11.690115)*x^3 + (-55.859842)*x^2 + (-86.556021)*x + (-42.228694)
[-1, 0]: S{8}=(11.003679)*x^3 + (12.221539)*x^2 + (11.060260)*x + (10.000000)
[0, 1]: S{9}=(-4.868399)*x^3 + (12.221539)*x^2 + (-16.395940)*x + (10.000000)
[1, 2]: S{10}=(0.685617)*x^3 + (-4.440511)*x^2 + (8.050410)*x + (-3.338316)
[2, 3]: S{11}=(0.365929)*x^3 + (-2.522381)*x^2 + (5.974151)*x + (-4.300811)
[3, 4]: S{12}=(-0.237734)*x^3 + (2.910591)*x^2 + (-12.236366)*x + (17.732906)
[4, 5]: S{13}=(-0.139292)*x^3 + (1.729281)*x^2 + (-6.786824)*x + (8.535384)
[5, 6]: S{14}=(0.087002)*x^3 + (-1.665122)*x^2 + (10.893086)*x + (-23.290800)
[6, 7]: S{15}=(0.161685)*x^3 + (-3.009426)*x^2 + (18.588511)*x + (-37.200051)
[7, 8]: S{16}=(-0.319443)*x^3 + (7.094257)*x^2 + (-52.551566)*x + (130.726864)
[8, 9]: S{17}=(0.354185)*x^3 + (-9.072801)*x^2 + (77.546794)*x + (-220.265563)
[9, 10]: S{18}=(-0.163397)*x^3 + (4.901909)*x^2 + (-49.159495)*x + (165.456705)
[10, 11]: S{19}=(0.021167)*x^3 + (-0.635000)*x^2 + (6.487833)*x + (-21.889300)
```

图 2: (b) 中得到的三次样条插值函数

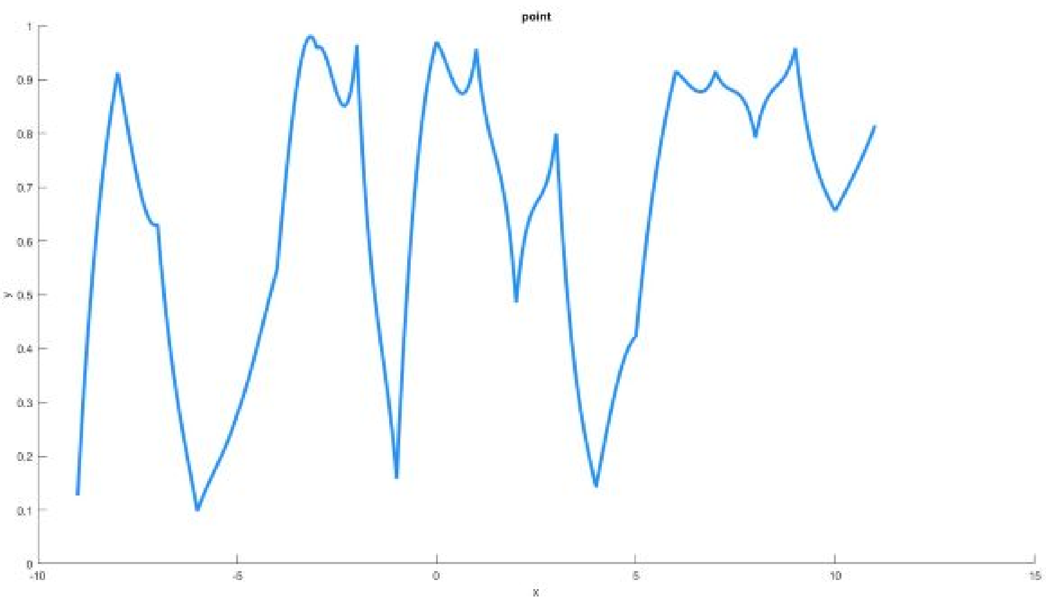

图 3: (a) 中得到的三次样条插值函数图像

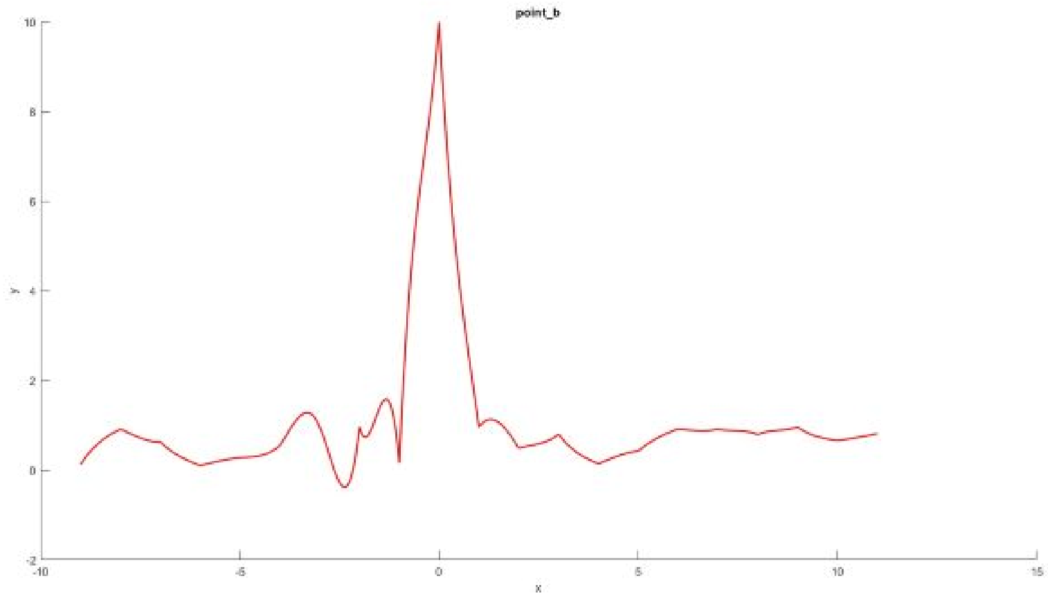

图 4: (b) 中得到的三次样条插值函数图像

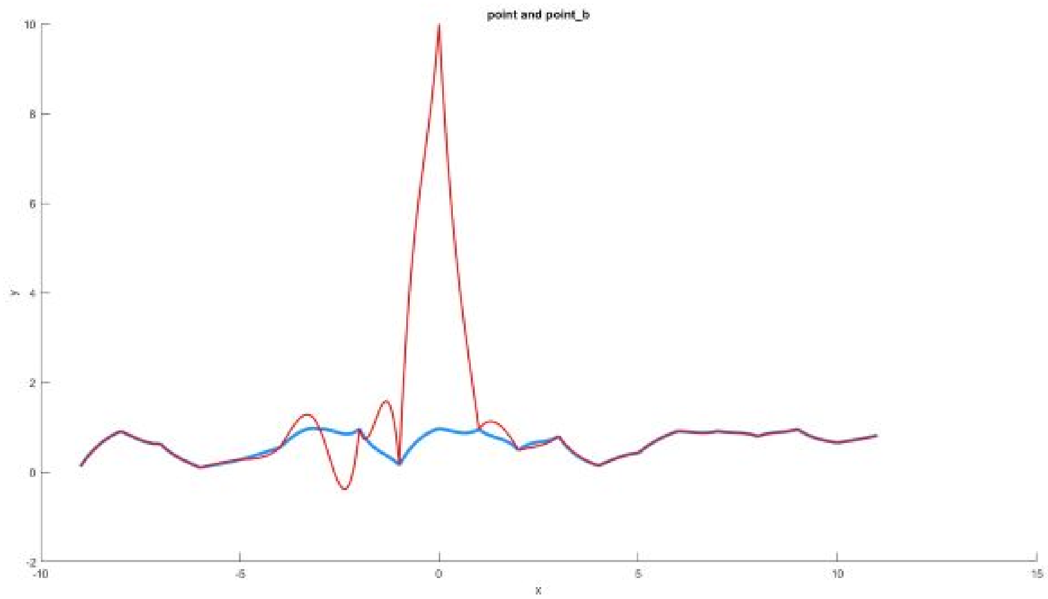

图 5: 两个函数图像对比

由图 5 看出在 (0,10) 附近区间的函数变化明显，区间离 (0,10) 越远，函数变化越小，在两端的函数几乎没什么变化。