计算方法第五次编程作业

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1 题目

"point.txt" 文件中包含了 21 个压铁的位置信息

- (a) 利用大 M 法计算出木条在压铁控制下的曲线，边界条件取自然边界条件，并使用追赶法对得到的线性方程组进行求解。
- (b) 将第 10 个压铁的位置移动至 (0,10)，计算出新的曲线，观察每个区间内的三次函数是否改变。

注：已将第 10 个压铁的位置移动至 (0,10) 后的数据保存在 "*point_b.txt*" 文件中。

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2 原理

- 大 M 法计算三次样条插值（自然边界条件）

  给定插值点 $\{(x_i, f(x_i)), i = 0, 1 \ldots, n\}$ ，记插值函数在 $[x_i, x_{i+1}]$ 上的表达式为 $S_i(x)$ ，为三次多项式。

  记 $S''(x_i) = M_i, \quad h_i = x_{i+1} - x_i$, 则

$$\begin{aligned} S(x) =& \frac{(x_{i+1} - x)^3 M_i + (x - x_i)^3 M_{i+1}}{6h_i} + \frac{(x_{i+1} - x)y_i + (x - x_i)y_{i+1}}{h_i} \\ & - \frac{h_i}{6}[(x_{i+1} - x)M_i + (x - x_i)M_{i+1}], \quad x \in [x_i, x_{i+1}] \end{aligned} \tag{1}$$

  展开得

$$\begin{aligned} S(x) =& \frac{M_{i+1} - M_i}{6h_i}x^3 + \frac{x_{i+1}M_i - x_i M_{i+1}}{2h_i}x^2 \\ & + \frac{3(x_i^2 M_{i+1} - x_{i+1}^2 M_i) + 6(y_{i+1} - y_i) - h_i^2(M_{i+1} - M_i)}{6h_i}x \\ & + \frac{x_{i+1}^3 M_i - X_i^3 M_{i+1} + 6(x_{i+1}y_i - x_i y_{i+1}) - h_i^2(x_{i+1}M_i - x_i M_{i+1})}{6h_i} \end{aligned} \tag{2}$$

  其中 M 满足

$$\mu_i M_{i-1} + 2M_i + \lambda_i M_{i+1} = d_i, \quad i = 1, 2, \ldots, n-1 \tag{3}$$

where

$$\begin{aligned}\lambda_i &= \frac{h_i}{h_i + h_{i-1}} \quad \mu_i = 1 - \lambda_i \\ d_i &= \frac{6}{h_i + h_{i-1}}\left(\frac{y_{i+1} - y_i}{h_i} - \frac{y_i - y_{i-1}}{h_{i-1}}\right) = 6f[x_{i-1}, x_i, x_{i+1}]\end{aligned} \tag{4}$$

在自然边界条件下 $(M_0 = M_n = 0)$，方程组为

$$\begin{bmatrix} 2 & \lambda_1 & & & \\ \mu_2 & 2 & \lambda_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \mu_{n-2} & 2 & \lambda_{n-2} \\ & & & \mu_{n-1} & 2 \end{bmatrix} \begin{bmatrix} M_1 \\ M2 \\ \vdots \\ M_{n-2} \\ M_{n-1} \end{bmatrix} = \begin{bmatrix} d_1 - \mu_1 M_0 \\ d_2 \\ \vdots \\ d_{n-2} \\ d_{n-1} - \lambda_{n-1} M_n \end{bmatrix}$$

由于矩阵为三对角阵，因此可使用追赶法解方程组。

3 结果

函数结果如图 1 和图 2 所示。

```
[-9, -8]: S{0}=(-0.266850)*x^3 + (-7.204950)*x^2 + (-63.791300)*x + (-184.927400)
[-8, -7]: S{1}=(0.219791)*x^3 + (4.474431)*x^2 + (29.690810)*x + (64.609207)
[-7, -6]: S{2}=(0.248246)*x^3 + (5.071979)*x^2 + (33.873647)*x + (74.369161)
[-6, -5]: S{3}=(-0.242973)*x^3 + (-3.769957)*x^2 + (-19.177973)*x + (-31.734078)
[-5, -4]: S{4}=(0.095147)*x^3 + (1.301845)*x^2 + (6.181036)*x + (10.530937)
[-4, -3]: S{5}=(-0.082815)*x^3 + (-0.833697)*x^2 + (-2.361129)*x + (-0.858617)
[-3, -2]: S{6}=(-0.309288)*x^3 + (-2.871954)*x^2 + (-8.475901)*x + (-6.973389)
[-2, -1]: S{7}=(0.908466)*x^3 + (4.434570)*x^2 + (6.137146)*x + (2.768643)
[-1, 0]: S{8}=(-0.889576)*x^3 + (-0.959558)*x^2 + (0.743018)*x + (0.970600)
[0, 1]: S{9}=(0.203140)*x^3 + (-0.959558)*x^2 + (0.743018)*x + (0.970600)
[1, 2]: S{10}=(0.445017)*x^3 + (-1.685190)*x^2 + (1.468650)*x + (0.728723)
[2, 3]: S{11}=(-0.738108)*x^3 + (5.413561)*x^2 + (-12.728852)*x + (10.193724)
[3, 4]: S{12}=(0.747415)*x^3 + (-7.956150)*x^2 + (27.380281)*x + (-29.915409)
[4, 5]: S{13}=(-0.339953)*x^3 + (5.092273)*x^2 + (-24.813411)*x + (39.676180)
[5, 6]: S{14}=(-0.111902)*x^3 + (1.671510)*x^2 + (-7.709596)*x + (11.169821)
[6, 7]: S{15}=(0.079663)*x^3 + (-1.776661)*x^2 + (12.979434)*x + (-30.208238)
[7, 8]: S{16}=(0.163651)*x^3 + (-3.540426)*x^2 + (25.325785)*x + (-59.016389)
[8, 9]: S{17}=(-0.319969)*x^3 + (8.066459)*x^2 + (-67.529296)*x + (188.597158)
[9, 10]: S{18}=(0.354323)*x^3 + (-10.139426)*x^2 + (96.323674)*x + (-302.961751)
[10, 11]: S{19}=(-0.163425)*x^3 + (5.393014)*x^2 + (-59.000730)*x + (214.786263)
```

图 1: (a) 中得到的三次样条插值函数

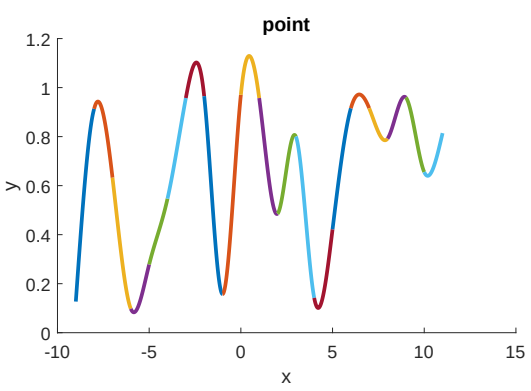
4 结果分析

将两个函数分别绘制出来，如图 3 5 所示。

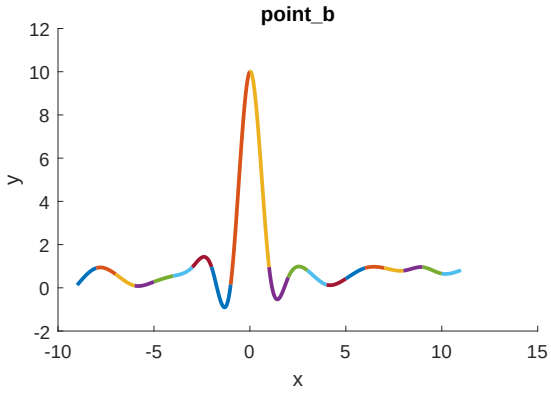
由图 4 看出在 (0,10) 附近区间的函数变化明显，区间离 (0,10) 越远，函数变化越小，在两端的函数几乎没什么变化。

```
[-9, -8]: S{0}=(-0.266850)*x^3 + (-7.204950)*x^2 + (-63.791300)*x + (-184.927400)
[-8, -7]: S{1}=(0.221230)*x^3 + (4.508981)*x^2 + (29.965765)*x + (65.334746)
[-7, -6]: S{2}=(0.241048)*x^3 + (4.925144)*x^2 + (32.878911)*x + (72.132084)
[-6, -5]: S{3}=(-0.215621)*x^3 + (-3.294902)*x^2 + (-16.441369)*x + (-26.508475)
[-5, -4]: S{4}=(-0.007062)*x^3 + (-0.166507)*x^2 + (-0.799390)*x + (-0.438510)
[-4, -3]: S{5}=(0.298669)*x^3 + (3.502258)*x^2 + (13.875668)*x + (19.128234)
[-3, -2]: S{6}=(-1.733013)*x^3 + (-14.782873)*x^2 + (-40.979725)*x + (-35.727160)
[-2, -1]: S{7}=(6.221882)*x^3 + (32.946494)*x^2 + (54.479009)*x + (27.911997)
[-1, 0]: S{8}=(-11.690115)*x^3 + (-20.789497)*x^2 + (0.743018)*x + (10.000000)
[0, 1]: S{9}=(11.003679)*x^3 + (-20.789497)*x^2 + (0.743018)*x + (10.000000)
[1, 2]: S{10}=(-4.868399)*x^3 + (26.826736)*x^2 + (-46.873214)*x + (25.872078)
[2, 3]: S{11}=(0.685617)*x^3 + (-6.497363)*x^2 + (19.774983)*x + (-18.560054)
[3, 4]: S{12}=(0.365929)*x^3 + (-3.620169)*x^2 + (11.143401)*x + (-9.928472)
[4, 5]: S{13}=(-0.237734)*x^3 + (3.623794)*x^2 + (-17.832451)*x + (28.705997)
[5, 6]: S{14}=(-0.139292)*x^3 + (2.147156)*x^2 + (-10.449261)*x + (16.400681)
[6, 7]: S{15}=(0.087002)*x^3 + (-1.926126)*x^2 + (13.990434)*x + (-32.478710)
[7, 8]: S{16}=(0.161685)*x^3 + (-3.494481)*x^2 + (24.968919)*x + (-58.095173)
[8, 9]: S{17}=(-0.319443)*x^3 + (8.052584)*x^2 + (-67.407608)*x + (188.242230)
[9, 10]: S{18}=(0.354185)*x^3 + (-10.135355)*x^2 + (96.283850)*x + (-302.832143)
[10, 11]: S{19}=(-0.163397)*x^3 + (5.392100)*x^2 + (-58.990705)*x + (214.749707)
```

图 2: (b) 中得到的三次样条插值函数



(a) (a) 中得到的三次样条插值函数图像



(b) (b) 中得到的三次样条插值函数图像

图 3: (a) 和 (b) 的结果

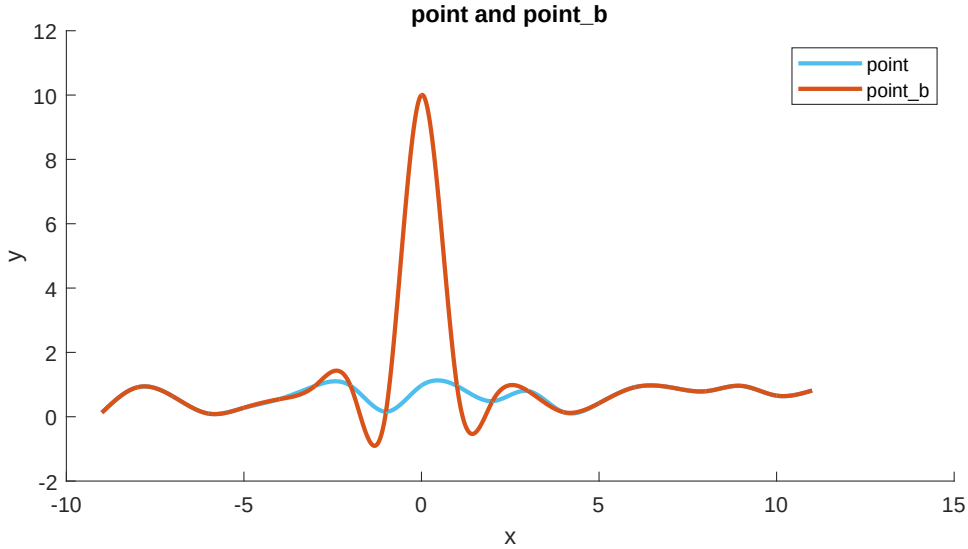


图 4: 两个函数图像对比