

计算方法第二次编程作业

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1 题目

通过使用 C/C++ 语言实现下面两种线性方程组求解的算法：

- 列主元 Gauss 消元
- Gauss-Seidel 迭代法

分析比较两种算法的表现。

考虑两点边值问题

$$\begin{cases} \epsilon \frac{d^2 y}{dx^2} + \frac{dy}{dx} = a, & 0 < a < 1 \\ y(0) = 0, y(1) = 1 \end{cases}$$

其精确解为

$$y = \frac{1-a}{1-e^{-1/\epsilon}}(1-e^{-x/\epsilon}) + ax$$

若用差分法，将 $[0,1]$ 区间 n 等分，令 $h = \frac{1}{n}$ ，得差分方程

$$(\epsilon + h)y_{i+1} - (2\epsilon + h)y_i + \epsilon y_{i-1} = ah^2$$

从而离散后的线性方程组的系数矩阵为

$$A = \begin{bmatrix} -(2\epsilon + h) & \epsilon + h & & & & \\ \epsilon & -(2\epsilon + h) & \epsilon + h & & & \\ & \epsilon & -(2\epsilon + h) & \ddots & & \\ & & \ddots & \ddots & \epsilon + h & \\ & & & \epsilon & -(2\epsilon + h) \end{bmatrix}$$

$a = \frac{1}{2}$, $n = 100$, 对 $\epsilon = 1, 0.1, 0.01, 0.0001$ 分别用 Gauss 列主元法和 Gauss-Seidel 迭代法求线性方程组的解。其中 Gauss-Seidel 法要求有 4 位有效数字。然后比较与精确解的误差。

2 算法

- 方法 1: Gauss 列主元消元法

- **具体算法 1:** 由于 \mathbf{A} 为三对角阵, 每次选取列主元时只需要和下一行的元素比较绝对值大小即可。同时考虑到对角线下方元素在消元之后为 0 且不会在算法中再次出现, 因此可用来储存消元的系数。

化为上三角阵之后, 从最右下方的元素开始, 算出解的值后向上进行回代求解。同时考虑到第 i 行的 \mathbf{b} 的元素 b_i 仅用于 y_i 的求解, 因此可将得到的解直接存在 b_i 中并直接当作 y_i 来参与到下一次的计算中。最终得到的 \mathbf{b} 即为所求的解 \mathbf{y} 。

- **方法 2:** Gauss-Seidel 迭代法

- **具体算法 2:**

$$\mathbf{Ax} = \mathbf{b}$$

$$\Rightarrow \mathbf{Dx} = (\mathbf{D} - \mathbf{A})\mathbf{x} + \mathbf{b}$$

$$\Rightarrow \mathbf{x} = (\mathbf{I} - \mathbf{D}^{-1}\mathbf{A})\mathbf{x} + \mathbf{D}^{-1}\mathbf{b}$$

其中

$$(\mathbf{I} - \mathbf{D}^{-1}\mathbf{A}) = \begin{bmatrix} & -\frac{\epsilon+h}{2\epsilon+h} & & & \\ -\frac{\epsilon}{2\epsilon+h} & & -\frac{\epsilon+h}{2\epsilon+h} & & \\ & -\frac{\epsilon}{2\epsilon+h} & & \ddots & \\ & & \ddots & \ddots & -\frac{\epsilon+h}{2\epsilon+h} \\ & & & -\frac{\epsilon}{2\epsilon+h} & -\frac{\epsilon+h}{2\epsilon+h} \end{bmatrix}$$

$$\mathbf{D}^{-1}\mathbf{b} = \begin{pmatrix} -\frac{ah^2}{2\epsilon+h} \\ -\frac{ah^2}{2\epsilon+h} \\ \vdots \\ -\frac{ah^2}{2\epsilon+h} \end{pmatrix}$$

取 $\mathbf{x} = (0, 0, \dots, 0)$ 为初始输入, 代入迭代公式中进行迭代计算。从 y_0 开始求解, 并在得到某个分量的新的解后立即对 \mathbf{y} 向量进行更新。

- **注意事项:** 若直接采用上述方法进行计算会得到错误的数值解。注意到这是一个二阶微分方程, 有多解性, 真正的解需要满足题设条件 $y(0) = 0, y(1) = 1$ 。而对于上述差分方程来说, 当 $i=0$ 和 $i=99$ 时, 由于忽略了 y_{-1} 和 y_{100} , 相当于让这两个值为 0, 即 $y(-0.01) = y(1) = 0$, 与题设的初值条件矛盾。会造成较大的误差。

因此需要调整 b_0 和 b_{99} 的值。令 $y_{-1} = y(-0.01) \approx y(0) = 0$, $y_{100} = y(1) = 1$, 得

$$b_{99} = ah^2 - \epsilon - h$$

修正之后再用上述两种方法进行求解, 得到的数值解会有较小的误差。

在得到求解结果后用 2 范数来衡量数值解与精确解之间的相对误差。

3 结果

修正前的结果如图 1 至图 4 所示，修正后的结果如图 5 至图 8 所示。

```

epsilon is: 1.000000
/*****
Solution by Gauss Elimination with partial pivoting:
x = [-0.002887, -0.005696, -0.008428, -0.011083, -0.013662, -0.016166, -0.018596, -0.020953, -0.023236, -0.025448,
      -0.027588, -0.029657, -0.031657, -0.033587, -0.035448, -0.037242, -0.038968, -0.040628, -0.042221, -0.043750,
      -0.045214, -0.046613, -0.047950, -0.049223, -0.050435, -0.051585, -0.052674, -0.053703, -0.054672, -0.055583,
      -0.056434, -0.057228, -0.057964, -0.058644, -0.059267, -0.059834, -0.060347, -0.060805, -0.061209, -0.061559,
      -0.061856, -0.062101, -0.062294, -0.062436, -0.062527, -0.062567, -0.062557, -0.062498, -0.062390, -0.062234,
      -0.062029, -0.061778, -0.061479, -0.061133, -0.060742, -0.060305, -0.059822, -0.059295, -0.058724, -0.058109,
      -0.057450, -0.056749, -0.056005, -0.055219, -0.054391, -0.053521, -0.052611, -0.051660, -0.050670, -0.049639,
      -0.048569, -0.047461, -0.046314, -0.045128, -0.043905, -0.042645, -0.041347, -0.040013, -0.038642, -0.037236,
      -0.035794, -0.034317, -0.032805, -0.031258, -0.029677, -0.028062, -0.026414, -0.024733, -0.023019, -0.021272,
      -0.019493, -0.017682, -0.015840, -0.013966, -0.012061, -0.010126, -0.008160, -0.006164, -0.004139, -0.002084,
      ]
/*****
The error related to accurate solution is: 106.084831%

/*****
Solution by Gauss Seidel iteration:
x = [-0.0029, -0.0057, -0.0084, -0.0111, -0.0136, -0.0161, -0.0186, -0.0209, -0.0232, -0.0254,
      -0.0275, -0.0296, -0.0316, -0.0335, -0.0354, -0.0372, -0.0389, -0.0405, -0.0421, -0.0437,
      -0.0451, -0.0465, -0.0478, -0.0491, -0.0503, -0.0515, -0.0526, -0.0536, -0.0545, -0.0555,
      -0.0563, -0.0571, -0.0578, -0.0585, -0.0591, -0.0597, -0.0602, -0.0607, -0.0611, -0.0614,
      -0.0617, -0.0620, -0.0622, -0.0623, -0.0624, -0.0624, -0.0624, -0.0624, -0.0622, -0.0621,
      -0.0619, -0.0616, -0.0613, -0.0610, -0.0606, -0.0602, -0.0597, -0.0592, -0.0586, -0.0580,
      -0.0573, -0.0566, -0.0559, -0.0551, -0.0543, -0.0534, -0.0525, -0.0516, -0.0506, -0.0495,
      -0.0485, -0.0474, -0.0462, -0.0450, -0.0438, -0.0426, -0.0413, -0.0399, -0.0386, -0.0372,
      -0.0357, -0.0343, -0.0327, -0.0312, -0.0296, -0.0280, -0.0264, -0.0247, -0.0230, -0.0212,
      -0.0195, -0.0177, -0.0158, -0.0139, -0.0120, -0.0101, -0.0081, -0.0062, -0.0041, -0.0021,
      ]
/*****
The error related to accurate solution is: 106.071664%

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图 1: 修正前 $\varepsilon = 1$ 的计算结果

4 结果分析

从结果中可明显看出修正对减小误差的作用。修正前的结果误差很大，相对误差均在 100% 以上；而修正后的误差较小，相对误差最大为 6.5174%，最小为 1.0319%。同时对比两种求解方法，每次的结果都表明两者的效果十分相近，在保存至两到三个有效数字的情况下，两种方法的得到的解的相对误差相同。因此可得两种方法的运行表现均为良好，且两者之间没有明显的差距。

对于结果的误差，一个最主要的原因就是上述提到的初值条件与题设并不完全一致。尽管对 $y(1)=1$ 处进行了修正，对于 $y(-0.01)$ 处仍然由于取近似结果 $y(-0.01) \approx 0$ 而带来了一定的误差。若将 b_0 取为 $y(-0.01)$ 的真实值，则最终结果的误差将会进一步减小。但是对于使用差分法求解二阶微分方程的情况是默认了不知道 y 的解析解的，否则都知道答案了，还用什麼差分法呢？

```

epsilon is: 0.100000
/*****
Solution by Gauss Elimination with partial pivoting:
x = [-0.040912, -0.077650, -0.110594, -0.140089, -0.166447, -0.189955, -0.210871, -0.229432, -0.245850, -0.260321,
      -0.273022, -0.284114, -0.293743, -0.302042, -0.309132, -0.315123, -0.320115, -0.324199, -0.327456, -0.329963,
      -0.331788, -0.332992, -0.333632, -0.333760, -0.333421, -0.332658, -0.331511, -0.330013, -0.328196, -0.326091,
      -0.323722, -0.321114, -0.318288, -0.315265, -0.312062, -0.308696, -0.305181, -0.301531, -0.297759, -0.293875,
      -0.289889, -0.285811, -0.281650, -0.277412, -0.273105, -0.268735, -0.264307, -0.259828, -0.255301, -0.250731,
      -0.246122, -0.241478, -0.236801, -0.232095, -0.227362, -0.222605, -0.217826, -0.213026, -0.208209, -0.203375,
      -0.198525, -0.193663, -0.188787, -0.183900, -0.179003, -0.174097, -0.169182, -0.164260, -0.159330, -0.154394,
      -0.149452, -0.144505, -0.139553, -0.134597, -0.129636, -0.124672, -0.119705, -0.114735, -0.109762, -0.104787,
      -0.099809, -0.094830, -0.089848, -0.084865, -0.079880, -0.074894, -0.069907, -0.064918, -0.059929, -0.054938,
      -0.049947, -0.044955, -0.039962, -0.034968, -0.029974, -0.024980, -0.019985, -0.014989, -0.009993, -0.004997,
      ]
/*****
The error related to accurate solution is: 125.634920%

/*****
Solution by Gauss Seidel iteration:
x = [-0.0409, -0.0776, -0.1106, -0.1401, -0.1664, -0.1899, -0.2108, -0.2294, -0.2458, -0.2603,
      -0.2730, -0.2841, -0.2937, -0.3020, -0.3091, -0.3151, -0.3201, -0.3241, -0.3274, -0.3299,
      -0.3317, -0.3329, -0.3336, -0.3337, -0.3334, -0.3326, -0.3315, -0.3300, -0.3281, -0.3260,
      -0.3237, -0.3211, -0.3182, -0.3152, -0.3120, -0.3087, -0.3051, -0.3015, -0.2977, -0.2938,
      -0.2899, -0.2858, -0.2816, -0.2774, -0.2731, -0.2687, -0.2643, -0.2598, -0.2553, -0.2507,
      -0.2461, -0.2415, -0.2368, -0.2321, -0.2273, -0.2226, -0.2178, -0.2130, -0.2082, -0.2034,
      -0.1985, -0.1937, -0.1888, -0.1839, -0.1790, -0.1741, -0.1692, -0.1643, -0.1593, -0.1544,
      -0.1494, -0.1445, -0.1395, -0.1346, -0.1296, -0.1247, -0.1197, -0.1147, -0.1098, -0.1048,
      -0.0998, -0.0948, -0.0898, -0.0849, -0.0799, -0.0749, -0.0699, -0.0649, -0.0599, -0.0549,
      -0.0499, -0.0450, -0.0400, -0.0350, -0.0300, -0.0250, -0.0200, -0.0150, -0.0100, -0.0050,
      ]
/*****
The error related to accurate solution is: 125.631970%

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图 2: 修正前 $\varepsilon = 0.1$ 的计算结果

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epsilon is: 0.010000
/*****
Solution by Gauss Elimination with partial pivoting:
x = [-0.247500, -0.368750, -0.426875, -0.453438, -0.464219, -0.467109, -0.466055, -0.463027, -0.459014, -0.454507,
      -0.449753, -0.444877, -0.439938, -0.434969, -0.429985, -0.424992, -0.419996, -0.414998, -0.409999, -0.405000,
      -0.400000, -0.395000, -0.390000, -0.385000, -0.380000, -0.375000, -0.370000, -0.365000, -0.360000, -0.355000,
      -0.350000, -0.345000, -0.340000, -0.335000, -0.330000, -0.325000, -0.320000, -0.315000, -0.310000, -0.305000,
      -0.300000, -0.295000, -0.290000, -0.285000, -0.280000, -0.275000, -0.270000, -0.265000, -0.260000, -0.255000,
      -0.250000, -0.245000, -0.240000, -0.235000, -0.230000, -0.225000, -0.220000, -0.215000, -0.210000, -0.205000,
      -0.200000, -0.195000, -0.190000, -0.185000, -0.180000, -0.175000, -0.170000, -0.165000, -0.160000, -0.155000,
      -0.150000, -0.145000, -0.140000, -0.135000, -0.130000, -0.125000, -0.120000, -0.115000, -0.110000, -0.105000,
      -0.100000, -0.095000, -0.090000, -0.085000, -0.080000, -0.075000, -0.070000, -0.065000, -0.060000, -0.055000,
      -0.050000, -0.045000, -0.040000, -0.035000, -0.030000, -0.025000, -0.020000, -0.015000, -0.010000, -0.005000,
      ]
/*****
The error related to accurate solution is: 130.665878%

/*****
Solution by Gauss Seidel iteration:
x = [-0.2475, -0.3687, -0.4269, -0.4534, -0.4642, -0.4671, -0.4661, -0.4630, -0.4590, -0.4545,
      -0.4498, -0.4449, -0.4399, -0.4350, -0.4300, -0.4250, -0.4200, -0.4150, -0.4100, -0.4050,
      -0.4000, -0.3950, -0.3900, -0.3850, -0.3800, -0.3750, -0.3700, -0.3650, -0.3600, -0.3550,
      -0.3500, -0.3450, -0.3400, -0.3350, -0.3300, -0.3250, -0.3200, -0.3150, -0.3100, -0.3050,
      -0.3000, -0.2950, -0.2900, -0.2850, -0.2800, -0.2750, -0.2700, -0.2650, -0.2600, -0.2550,
      -0.2500, -0.2450, -0.2400, -0.2350, -0.2300, -0.2250, -0.2200, -0.2150, -0.2100, -0.2050,
      -0.2000, -0.1950, -0.1900, -0.1850, -0.1800, -0.1750, -0.1700, -0.1650, -0.1600, -0.1550,
      -0.1500, -0.1450, -0.1400, -0.1350, -0.1300, -0.1250, -0.1200, -0.1150, -0.1100, -0.1050,
      -0.1000, -0.0950, -0.0900, -0.0850, -0.0800, -0.0750, -0.0700, -0.0650, -0.0600, -0.0550,
      -0.0500, -0.0450, -0.0400, -0.0350, -0.0300, -0.0250, -0.0200, -0.0150, -0.0100, -0.0050,
      ]
/*****
The error related to accurate solution is: 130.665841%

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图 3: 修正前 $\varepsilon = 0.01$ 的计算结果


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epsilon is: 0.000100
/*****
Solution by Gauss Elimination with partial pivoting:
x = [-0.495000, -0.494950, -0.490000, -0.485000, -0.480000, -0.475000, -0.470000, -0.465000, -0.460000, -0.455000,
      -0.450000, -0.445000, -0.440000, -0.435000, -0.430000, -0.425000, -0.420000, -0.415000, -0.410000, -0.405000,
      -0.400000, -0.395000, -0.390000, -0.385000, -0.380000, -0.375000, -0.370000, -0.365000, -0.360000, -0.355000,
      -0.350000, -0.345000, -0.340000, -0.335000, -0.330000, -0.325000, -0.320000, -0.315000, -0.310000, -0.305000,
      -0.300000, -0.295000, -0.290000, -0.285000, -0.280000, -0.275000, -0.270000, -0.265000, -0.260000, -0.255000,
      -0.250000, -0.245000, -0.240000, -0.235000, -0.230000, -0.225000, -0.220000, -0.215000, -0.210000, -0.205000,
      -0.200000, -0.195000, -0.190000, -0.185000, -0.180000, -0.175000, -0.170000, -0.165000, -0.160000, -0.155000,
      -0.150000, -0.145000, -0.140000, -0.135000, -0.130000, -0.125000, -0.120000, -0.115000, -0.110000, -0.105000,
      -0.100000, -0.095000, -0.090000, -0.085000, -0.080000, -0.075000, -0.070000, -0.065000, -0.060000, -0.055000,
      -0.050000, -0.045000, -0.040000, -0.035000, -0.030000, -0.025000, -0.020000, -0.015000, -0.010000, -0.005000,
      ]
/*****
The error related to accurate solution is: 131.139439%

/*****
Solution by Gauss Seidel iteration:
x = [-0.4950, -0.4950, -0.4900, -0.4850, -0.4800, -0.4750, -0.4700, -0.4650, -0.4600, -0.4550,
      -0.4500, -0.4450, -0.4400, -0.4350, -0.4300, -0.4250, -0.4200, -0.4150, -0.4100, -0.4050,
      -0.4000, -0.3950, -0.3900, -0.3850, -0.3800, -0.3750, -0.3700, -0.3650, -0.3600, -0.3550,
      -0.3500, -0.3450, -0.3400, -0.3350, -0.3300, -0.3250, -0.3200, -0.3150, -0.3100, -0.3050,
      -0.3000, -0.2950, -0.2900, -0.2850, -0.2800, -0.2750, -0.2700, -0.2650, -0.2600, -0.2550,
      -0.2500, -0.2450, -0.2400, -0.2350, -0.2300, -0.2250, -0.2200, -0.2150, -0.2100, -0.2050,
      -0.2000, -0.1950, -0.1900, -0.1850, -0.1800, -0.1750, -0.1700, -0.1650, -0.1600, -0.1550,
      -0.1500, -0.1450, -0.1400, -0.1350, -0.1300, -0.1250, -0.1200, -0.1150, -0.1100, -0.1050,
      -0.1000, -0.0950, -0.0900, -0.0850, -0.0800, -0.0750, -0.0700, -0.0650, -0.0600, -0.0550,
      -0.0500, -0.0450, -0.0400, -0.0350, -0.0300, -0.0250, -0.0200, -0.0150, -0.0100, -0.0050,
      ]
/*****
The error related to accurate solution is: 131.139439%

```

图 4: 修正前 $\varepsilon = 0.0001$ 的计算结果

```

epsilon is: 1.000000
/*****
Solution by Gauss Elimination with partial pivoting:
x = [0.012731, 0.025385, 0.037964, 0.050467, 0.062897, 0.075252, 0.087535, 0.099746, 0.111885, 0.123954,
      0.135952, 0.147882, 0.159743, 0.171535, 0.183261, 0.194920, 0.206513, 0.218041, 0.229504, 0.240903,
      0.252239, 0.263512, 0.274723, 0.285873, 0.296961, 0.307989, 0.318958, 0.329868, 0.340719, 0.351512,
      0.362247, 0.372926, 0.383549, 0.394116, 0.404628, 0.415085, 0.425489, 0.435838, 0.446135, 0.456380,
      0.466572, 0.476713, 0.486803, 0.496843, 0.506833, 0.516773, 0.526665, 0.536508, 0.546303, 0.556051,
      0.565752, 0.575406, 0.585014, 0.594576, 0.604094, 0.613566, 0.622994, 0.632379, 0.641720, 0.651018,
      0.660273, 0.669487, 0.678658, 0.687788, 0.696878, 0.705927, 0.714936, 0.723905, 0.732835, 0.741726,
      0.750578, 0.759392, 0.768169, 0.776908, 0.785610, 0.794275, 0.802905, 0.811498, 0.820055, 0.828578,
      0.837065, 0.845518, 0.853937, 0.862322, 0.870674, 0.878992, 0.887277, 0.895530, 0.903751, 0.911940,
      0.920097, 0.928223, 0.936318, 0.944382, 0.952416, 0.960420, 0.968395, 0.976339, 0.984255, 0.992142,
      ]
/*****
The error related to accurate solution is: 1.044196%

/*****
Solution by Gauss Seidel iteration:
x = [0.0127, 0.0254, 0.0379, 0.0504, 0.0629, 0.0752, 0.0875, 0.0997, 0.1118, 0.1239,
      0.1359, 0.1478, 0.1597, 0.1715, 0.1832, 0.1948, 0.2064, 0.2180, 0.2294, 0.2408,
      0.2521, 0.2634, 0.2746, 0.2858, 0.2968, 0.3079, 0.3188, 0.3297, 0.3406, 0.3514,
      0.3621, 0.3728, 0.3834, 0.3940, 0.4045, 0.4149, 0.4254, 0.4357, 0.4460, 0.4562,
      0.4664, 0.4766, 0.4867, 0.4967, 0.5067, 0.5166, 0.5265, 0.5364, 0.5462, 0.5559,
      0.5656, 0.5753, 0.5849, 0.5944, 0.6040, 0.6134, 0.6229, 0.6322, 0.6416, 0.6509,
      0.6601, 0.6694, 0.6785, 0.6877, 0.6968, 0.7058, 0.7148, 0.7238, 0.7327, 0.7416,
      0.7505, 0.7593, 0.7681, 0.7768, 0.7855, 0.7942, 0.8028, 0.8114, 0.8200, 0.8285,
      0.8370, 0.8455, 0.8539, 0.8623, 0.8706, 0.8789, 0.8872, 0.8955, 0.9037, 0.9119,
      0.9201, 0.9282, 0.9363, 0.9444, 0.9524, 0.9604, 0.9684, 0.9763, 0.9842, 0.9921,
      ]
/*****
The error related to accurate solution is: 1.031878%

```

图 5: 修正后 $\varepsilon = 1$ 的计算结果


```

epsilon is: 0.100000
/*****
Solution by Gauss Elimination with partial pivoting:
x = [0.050003, 0.095915, 0.138107, 0.176919, 0.212656, 0.245600, 0.276003, 0.304096, 0.330090, 0.354176,
      0.376527, 0.397300, 0.416639, 0.434675, 0.451526, 0.467299, 0.482093, 0.495997, 0.509091, 0.521449,
      0.533139, 0.544220, 0.554748, 0.564774, 0.574343, 0.583497, 0.592273, 0.600705, 0.608826, 0.616663,
      0.624242, 0.631587, 0.638718, 0.645656, 0.652417, 0.659019, 0.665475, 0.671798, 0.678001, 0.684095,
      0.690089, 0.695993, 0.701815, 0.707562, 0.713241, 0.718859, 0.724420, 0.729930, 0.735394, 0.740816,
      0.746199, 0.751548, 0.756864, 0.762152, 0.767414, 0.772652, 0.777869, 0.783065, 0.788244, 0.793407,
      0.798555, 0.803689, 0.808811, 0.813922, 0.819023, 0.824115, 0.829198, 0.834274, 0.839343, 0.844406,
      0.849463, 0.854515, 0.859562, 0.864605, 0.869643, 0.874679, 0.879711, 0.884740, 0.889767, 0.894791,
      0.899813, 0.904833, 0.909851, 0.914868, 0.919883, 0.924896, 0.929909, 0.934920, 0.939930, 0.944939,
      0.949948, 0.954956, 0.959963, 0.964969, 0.969975, 0.974980, 0.979985, 0.984989, 0.989993, 0.994997,
      ]
/*****
The error related to accurate solution is: 1.338838%

/*****
Solution by Gauss Seidel iteration:
x = [0.0500, 0.0959, 0.1381, 0.1769, 0.2126, 0.2456, 0.2760, 0.3041, 0.3300, 0.3541,
      0.3765, 0.3972, 0.4166, 0.4346, 0.4515, 0.4672, 0.4820, 0.4959, 0.5090, 0.5214,
      0.5331, 0.5442, 0.5547, 0.5647, 0.5743, 0.5834, 0.5922, 0.6007, 0.6088, 0.6166,
      0.6242, 0.6315, 0.6387, 0.6456, 0.6524, 0.6590, 0.6654, 0.6718, 0.6780, 0.6841,
      0.6901, 0.6960, 0.7018, 0.7075, 0.7132, 0.7188, 0.7244, 0.7299, 0.7354, 0.7408,
      0.7462, 0.7515, 0.7568, 0.7621, 0.7674, 0.7726, 0.7779, 0.7831, 0.7882, 0.7934,
      0.7985, 0.8037, 0.8088, 0.8139, 0.8190, 0.8241, 0.8292, 0.8343, 0.8393, 0.8444,
      0.8495, 0.8545, 0.8596, 0.8646, 0.8696, 0.8747, 0.8797, 0.8847, 0.8898, 0.8948,
      0.8998, 0.9048, 0.9098, 0.9149, 0.9199, 0.9249, 0.9299, 0.9349, 0.9399, 0.9449,
      0.9499, 0.9550, 0.9600, 0.9650, 0.9700, 0.9750, 0.9800, 0.9850, 0.9900, 0.9950,
      ]
/*****
The error related to accurate solution is: 1.337586%

```

图 6: 修正后 $\varepsilon = 0.1$ 的计算结果

```

epsilon is: 0.010000
/*****
Solution by Gauss Elimination with partial pivoting:
x = [0.252500, 0.381250, 0.448125, 0.484063, 0.504531, 0.517266, 0.526133, 0.533066, 0.539033, 0.544517,
      0.549758, 0.554879, 0.559940, 0.564970, 0.569985, 0.574992, 0.579996, 0.584998, 0.589999, 0.595000,
      0.600000, 0.605000, 0.610000, 0.615000, 0.620000, 0.625000, 0.630000, 0.635000, 0.640000, 0.645000,
      0.650000, 0.655000, 0.660000, 0.665000, 0.670000, 0.675000, 0.680000, 0.685000, 0.690000, 0.695000,
      0.700000, 0.705000, 0.710000, 0.715000, 0.720000, 0.725000, 0.730000, 0.735000, 0.740000, 0.745000,
      0.750000, 0.755000, 0.760000, 0.765000, 0.770000, 0.775000, 0.780000, 0.785000, 0.790000, 0.795000,
      0.800000, 0.805000, 0.810000, 0.815000, 0.820000, 0.825000, 0.830000, 0.835000, 0.840000, 0.845000,
      0.850000, 0.855000, 0.860000, 0.865000, 0.870000, 0.875000, 0.880000, 0.885000, 0.890000, 0.895000,
      0.900000, 0.905000, 0.910000, 0.915000, 0.920000, 0.925000, 0.930000, 0.935000, 0.940000, 0.945000,
      0.950000, 0.955000, 0.960000, 0.965000, 0.970000, 0.975000, 0.980000, 0.985000, 0.990000, 0.995000,
      ]
/*****
The error related to accurate solution is: 3.428121%

/*****
Solution by Gauss Seidel iteration:
x = [0.2525, 0.3812, 0.4481, 0.4841, 0.5045, 0.5173, 0.5261, 0.5331, 0.5390, 0.5445,
      0.5498, 0.5549, 0.5599, 0.5650, 0.5700, 0.5750, 0.5800, 0.5850, 0.5900, 0.5950,
      0.6000, 0.6050, 0.6100, 0.6150, 0.6200, 0.6250, 0.6300, 0.6350, 0.6400, 0.6450,
      0.6500, 0.6550, 0.6600, 0.6650, 0.6700, 0.6750, 0.6800, 0.6850, 0.6900, 0.6950,
      0.7000, 0.7050, 0.7100, 0.7150, 0.7200, 0.7250, 0.7300, 0.7350, 0.7400, 0.7450,
      0.7500, 0.7550, 0.7600, 0.7650, 0.7700, 0.7750, 0.7800, 0.7850, 0.7900, 0.7950,
      0.8000, 0.8050, 0.8100, 0.8150, 0.8200, 0.8250, 0.8300, 0.8350, 0.8400, 0.8450,
      0.8500, 0.8550, 0.8600, 0.8650, 0.8700, 0.8750, 0.8800, 0.8850, 0.8900, 0.8950,
      0.9000, 0.9050, 0.9100, 0.9150, 0.9200, 0.9250, 0.9300, 0.9350, 0.9400, 0.9450,
      0.9500, 0.9550, 0.9600, 0.9650, 0.9700, 0.9750, 0.9800, 0.9850, 0.9900, 0.9950,
      ]
/*****
The error related to accurate solution is: 3.428072%

```

图 7: 修正后 $\varepsilon = 0.01$ 的计算结果

```

epsilon is: 0.000100
/*****/
Solution by Gauss Elimination with partial pivoting:
x = [0.495099, 0.504951, 0.510000, 0.515000, 0.520000, 0.525000, 0.530000, 0.535000, 0.540000, 0.545000,
      0.550000, 0.555000, 0.560000, 0.565000, 0.570000, 0.575000, 0.580000, 0.585000, 0.590000, 0.595000,
      0.600000, 0.605000, 0.610000, 0.615000, 0.620000, 0.625000, 0.630000, 0.635000, 0.640000, 0.645000,
      0.650000, 0.655000, 0.660000, 0.665000, 0.670000, 0.675000, 0.680000, 0.685000, 0.690000, 0.695000,
      0.700000, 0.705000, 0.710000, 0.715000, 0.720000, 0.725000, 0.730000, 0.735000, 0.740000, 0.745000,
      0.750000, 0.755000, 0.760000, 0.765000, 0.770000, 0.775000, 0.780000, 0.785000, 0.790000, 0.795000,
      0.800000, 0.805000, 0.810000, 0.815000, 0.820000, 0.825000, 0.830000, 0.835000, 0.840000, 0.845000,
      0.850000, 0.855000, 0.860000, 0.865000, 0.870000, 0.875000, 0.880000, 0.885000, 0.890000, 0.895000,
      0.900000, 0.905000, 0.910000, 0.915000, 0.920000, 0.925000, 0.930000, 0.935000, 0.940000, 0.945000,
      0.950000, 0.955000, 0.960000, 0.965000, 0.970000, 0.975000, 0.980000, 0.985000, 0.990000, 0.995000,
      ]
/*****/
The error related to accurate solution is: 6.517353%

/*****/
Solution by Gauss Seidel iteration:
x = [0.4951, 0.5050, 0.5100, 0.5150, 0.5200, 0.5250, 0.5300, 0.5350, 0.5400, 0.5450,
      0.5500, 0.5550, 0.5600, 0.5650, 0.5700, 0.5750, 0.5800, 0.5850, 0.5900, 0.5950,
      0.6000, 0.6050, 0.6100, 0.6150, 0.6200, 0.6250, 0.6300, 0.6350, 0.6400, 0.6450,
      0.6500, 0.6550, 0.6600, 0.6650, 0.6700, 0.6750, 0.6800, 0.6850, 0.6900, 0.6950,
      0.7000, 0.7050, 0.7100, 0.7150, 0.7200, 0.7250, 0.7300, 0.7350, 0.7400, 0.7450,
      0.7500, 0.7550, 0.7600, 0.7650, 0.7700, 0.7750, 0.7800, 0.7850, 0.7900, 0.7950,
      0.8000, 0.8050, 0.8100, 0.8150, 0.8200, 0.8250, 0.8300, 0.8350, 0.8400, 0.8450,
      0.8500, 0.8550, 0.8600, 0.8650, 0.8700, 0.8750, 0.8800, 0.8850, 0.8900, 0.8950,
      0.9000, 0.9050, 0.9100, 0.9150, 0.9200, 0.9250, 0.9300, 0.9350, 0.9400, 0.9450,
      0.9500, 0.9550, 0.9600, 0.9650, 0.9700, 0.9750, 0.9800, 0.9850, 0.9900, 0.9950,
      ]
/*****/
The error related to accurate solution is: 6.517352%

```

图 8: 修正后 $\varepsilon = 0.0001$ 的计算结果