CSCI 570 - Homework 11 Solutions

1 Graded Problems

1. Prove that the following problem is in NPC: Given an undirected graph $G = (V, E)$, determine whether there is a spanning tree whose degree is not greater than k. That is, whether there is a subgraph $G'(E', V)$, $E' \subset E$, $|E'| = |V|-1$, G' is a connected graph and all its node degrees are less than or equal to k.(20pts)

   First we need to prove finding the k-spanning-tree is in NP. Obviously, as long as a subgraph G is given, it can be verified in polynomial time whether the degrees of its vertices are all less equal than k and whether it is a legal spanning tree. Hence the problem of 'Spanning Tree with Bounded Degree'(STBD) $\in$ NP.

   Then we need to prove finding the k-spanning-tree is NP-Hard. We use a Hamiltonian Path to Prove this, which is a NPC problem. A H-Path will visit each vertex exactly once, which will have a max degree of 2 for each vertex in the path. And as the path is opened, $V = E + 1$.

   To prove that H-Path $\leq_p$ STBD, We configure the STBD problem as follows: For an input of H-Path Graph G, we input the exact graph into STBD problem and set K=2. If there is a solution to the STBD problem, that means there is a spanning tree that goes through all vertices with degree less or equal to K=2. Since there are no branches as no point has 3 connections, this tree is a path that goes through all vertices which is a H-Path. Vise-Versa, if there is a H-Path in the graph, the path is also a tree with degree less than or equal to 2, and that satisfies the STBD problem.

   Rubric (20pts):

   - 5 pts for Proving k-spanning-tree is in NP.
   - 15 pts for Proving k-spanning-tree is in NP-Hard.

2. You are given a directed graph G=(V,E) with weights on its edges e∈E. The weights can be negative or positive. The Zero-Weight-Cycle Problem is to decide if there is a simple cycle in G so that the sum of the edge weights on this cycle is exactly 0. Prove that this problem is NP-complete. (20pts)

   Zero-weight-cycle is in NP because we can exhibit a cycle in G, and it can be checked that the sum of the edge weights on this cycle are equal to 0.

   We now show that subset sum $\leq$ Zero-weight-cycle. We are given the number $w_1, ..., w_n$, and we want to know if there is a subset that adds up to exactly W. We construct an instance of the Zero-weight-cycle in which the graph has nodes $0, 1, 2, ..., n, n+1$and an edge (i, j) for all pairs $i < j$. The weight of the edge (i, j) is equal to w_j. Finally, there is an edge $(n+1, 0)$ of weight $-W$. All edges to $n+1$ have capacity 0.

   We claim that there is a subset that adds up to exactly W if and only if G has a zero-weight-cycle. If there is such a subset S, then we define a cycle that starts at 0, goes through the nodes whose indices are in S, and then returns to 0 on the edge $(n+1, 0)$. The weight of $-W$ on the edge $(n+1, 0)$ precisely cancels the sum of the other edge weights. Conversely, all cycles in G must use the edge $(n+1, 0)$, and so if there is a zero-weight-cycle, then the other edges must exactly cancel $-W$, in other words, their indices must form a set that adds up to exactly W.

   Rubric (20pt):

   - 5 pts for Proving Zero-weight-cycle is in NP.
   - 15 pts for Proving Zero-weight-cycle is in NP-Hard.

3. In a certain town, there are many clubs, and every adult belongs to at least one club. The town's people would like to simplify their social life by disbanding as many clubs as possible, but they want to make sure that afterwards everyone will still belong to at least one club.

   Formally the Redundant Clubs problem has the following input and output.

   INPUT: List of people; list of clubs; list of members of each club; number K.

   OUTPUT: Yes if there exists a set of K clubs such that, after disbanding all clubs in this set, each person still belongs to at least one club. No otherwise.

Prove that the Redundant Clubs problem is NP-Complete. (20pts)

Solution:

(a) We must show that Redundant Clubs is in NP, but this is easy: if we are given a set of K clubs, it is straightforward to check in polynomial time whether each person is a member of another club outside this set.

(b) We prove Redundant Clubs is in NP-Hard by reducing from a known NP-complete problem, Set Cover, e.g., Set Cover $\leq_p$ Redundant Clubs. We translate inputs of Set Cover to inputs of Redundant Clubs, so we need to specify how each Redundant Clubs input element is formed from the Set Cover instance.

We use the Set Cover's elements as our translated list of people, and make a list of clubs, one for each member of the Set Cover family. The members of each club are just the elements of the corresponding family. To finish specifying the Redundant Clubs input, we need to say what K is:

we let $K = F - K_{SC}$ where F is the number of families in the Set Cover instance and K_{SC} is the value K from the set cover instance. This translation can clearly be done in polynomial time (it just involves copying some lists and a single subtraction).

Finally, for the proof we need to show that we can remove K clubs if and only if we have a set cover of size K_{SC} such that K = F $-$ K_{SC}. If we have a yes-instance of Set Cover, that is, an instance with a cover consisting of K_{SC} subsets, the other K subsets form a solution to the translated Redundant Clubs problem, because each person belongs to a club in the cover. Conversely, if we have K redundant clubs, the remaining K_{SC} clubs form a cover. So the answer to the Set Cover instance is yes if and only if the answer to the translated Redundant Clubs instance is yes. This completes the reduction, and we confirm that the given problem is NP-Hard.

Thus this problem is NP-Complete.

Rubric (20pt):

- 5 pts for Proving Redundant Clubs is in NP.
- 15 pts for Proving Redundant Clubs is in NP-Hard.
  - 5 pts for the claim that Set Cover $\leq_p$ Redundant Clubs.
  - 5 pts for the construction.
  - 4 pts for the reduction proof.
  - 1 pt for the conclusion.

4. Given a graph $G = (V, E)$ with an even number of vertices as the input, the HALF-IS problem is to decide if G has an independent set of size $|V|/2$. Prove that HALF-IS is in NP-Complete. (20pts)

Solution:

(a) Given a graph $G(V, E)$ and a certifier $S \subset V$, $|S| = |V|/2$, we can verify if no two nodes are adjacent in polynomial time $(O(|S|^2) = O(|V|^2))$. Therefore, HALF$-$IS $\in NP$.

(b) We prove HALF-IS is in NP-Hard by using a reduction of the NP-complete problem Independent set problem (IS) to HALF$-$IS, e.g., IS $\leq_p$ HALF-IS. Consider an instance of IS, which asks for an independent set $A \subset V$, $|A| = k$, for a graph $G(V, E)$, such that vertices in A disconnected from each other:

i. If $k = |V|/2$, IS reduces to HALF-IS.

ii. If $k < |V|/2$, then add m new disconnected nodes such that $k + m = (|V| + m)/2$, i.e., $m = |V| - 2k$. Note that the modified set of nodes V' $(= V \cup \{m \text{ new nodes}\})$ has an even number of nodes. Since the additional nodes are all disconnected from each other, they form a subset of the independent set. Therefore, the new graph $G'(V', E')$ where $E' = E$ has an independent-set of size $|V'|/2$ if and only if $G(V, E)$ has an independent set of size k.

iii. If $k > |V|/2$, then again add $m = 2k - |V|$ new nodes to form the modified set of nodes V'. Connect these new nodes to all the other $|V| + m - 1$ nodes. Since these m new nodes are connected to every other, none of them should belong to an independent set. Therefore, the new graph $G'(V', E)$ has an independent-set of size $|V'|/2$ if and only if $G(V, E)$ has an independent set of size k.

Hence, any instance of IS $(G(V, E), k)$, can be reduced to an instance of HALF-IS $(G'(V', E'))$. This completes the reduction, and we confirm that the given problem is NP-Hard.

Thus this problem is NP-Complete.

Rubric (20pt):

- 5 pts for Proving Redundant Clubs is in NP.

- 15 pts for Proving Redundant Clubs is in NP-Hard.
  - 5 pts for the claim that IS $\le_p$ HALF-IS.
  - 3 pts for each the construction + proof (9 pts in total)
  - 1 pt for the conclusion.

5. There are n courses at USC, each of them requires multiple disjoint time intervals. For example, a course may require the time from 9am to 11am and 2pm to 3pm and 4pm to 5pm (you can assume the number of intervals of a course is at least 1, at most n). You want to know, given a number K, if it's possible to take at least K courses. You cannot choose any two overlapping courses. Prove that the problem is NP-complete, which means that choosing courses is indeed a difficult thing in our life. Use a reduction from the Independent Set problem. (20pts)

   Solution:

   (a) (Showing Problem in NP) The solution of the problem can be verified in polynomial time (just check the number of the courses in the solution is larger or equal to K, and they don't have time overlap), thus it is in NP.

   (b) (Showing Problem in NP-Hard) Given an independent set problem, suppose the graph has n nodes and asks if it has an independent set of size at least K.

   Now we can construct an instance of the course choosing problem: each course corresponds to a vertex of the graph, and if there exists an edge (v_i, v_j) in the original graph, we let the i-th and the j-th courses require the same 1-hour interval. The problem is to determine whether we can choose K courses. Notice that, if there exists an edge (v_i, v_j) in the original graph, then the i-th course and the j-th course will jointly require the same 1-hour interval, which means that we can't choose these two courses at the same time.
   If the Independent Set problem is a "yes" instance (has an independent set of size at least K), then we can choose the corresponding courses, and they don't overlap. For the other direction, if we can choose the corresponding courses, then it follows that the independent set problem is a "yes" instance.

   Thus we can reduce the independent set problem to the course choosing problem in polynomial time. Since the independent set problem is NP-Complete, the course choosing problem is in NP-Hard.

   Thus the course choosing problem is NP-Complete.

   Rubric (20pt):

   - 5 pts for Proving Redundant Clubs is in NP.
   - 15 pts for Proving Redundant Clubs is in NP-Hard.
     - 8 pts for the construction + explanation (can be any other constructions).
     - 6 pts for the proof
     - 1 pt for the conclusion.

2 Ungraded Problems

1. Suppose we have a variation on the 3-SAT problem called Min-3-SAT, where the literals are never negated. Of course, in this case it is possible to satisfy all clauses by simply setting all literals to true. But, we are additionally given a number k, and are asked to determine whether we can satisfy all clauses while setting at most k literals to be true. Prove that Min-3-SAT is NP-Complete.

   Solution:

   (a) For a truth assignment, we can simply count the number of literals set to true. Then evaluate each clause with the truth assignment. If all clauses equal to true and at most k literals are set to true, then answer yes. So Min-3-SAT $\in$ NP.

   (b) We reduce from vertex cover to Min-3-SAT. For any given instance of the vertex cover problem, we can construct an equivalent Min-3-SAT problem with variables for each vertex of a graph. Each edge (u, v) of the graph can be represented by a clause $(u \vee u \vee v)$ or $(u \vee v \vee v)$ which can be satisfied only by including either u or v among the true variables of the solution. For the constructed Min-3-SAT problem, there is a satisfying assignment within k true variables if and only if there is a vertex cover within k vertices to the corresponding vertex cover problem.

   If we have a satisfiable assignment of k variables to true, we have a corresponding solution to the vertex cover problem by selecting those vertices in the vertex cover set. Conversely, if we have k vertices corresponding to the vertex cover problem then we can assign true values to exactly k variables and solve the Min-3-SAT problem.

   Therefore, Min-3-SAT is NP-Hard.
   Thus this problem is NP-Complete.