Discussion 5

1. Suppose we have two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, along with T_1 which is a MST of G_1 and T_2 which is a MST of G_2. Now consider a new graph $G = (V, E)$ such that $V = V_1 \cup V_2$ and $E = E_1 \cup E_2 \cup E_3$ where E_3 is a new set of edges that all cross the cut (V_1, V_2).

Consider the following algorithm, which is intended to find a MST of G.

Maybe-MST(T_1, T_2, E_3)

e_{min} = a minimum weight edge in E_3

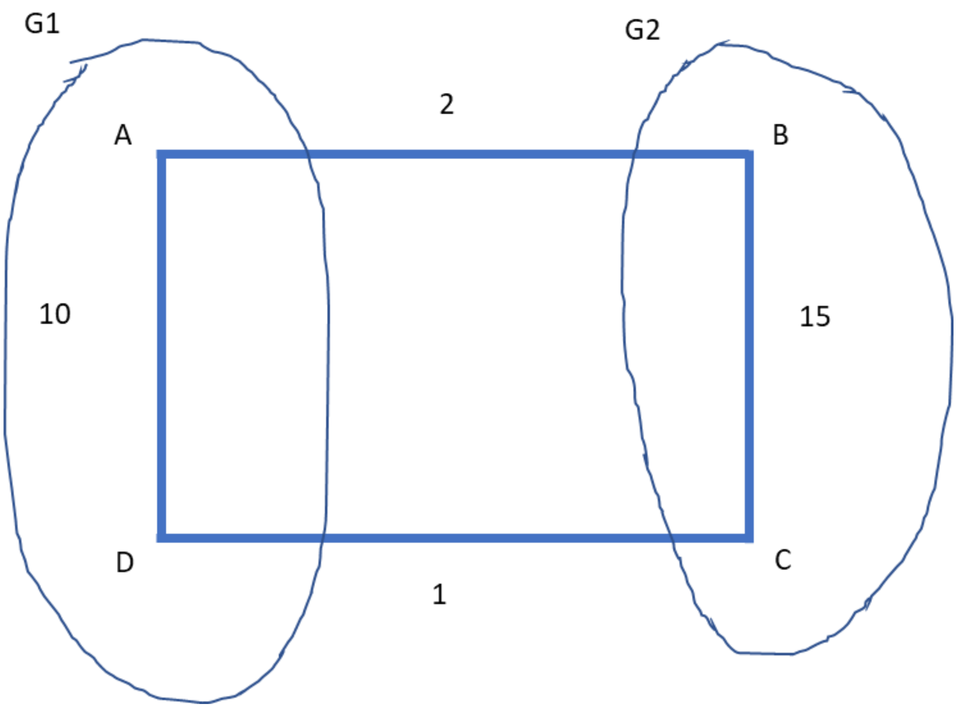
$T = T_1 \cup T_2 \cup \{ e_{min}\}$

return T

Does this algorithm correctly find a MST of G? Either prove it does or prove it does not.

Solution:

No. Here is a counter example.



The correct solution will have edges AB, DC, and AD in the MST. The above solution will result in an incorrect MST with edges AD, BC, and DC.

2. Solve the following recurrences using the Master Method:

a. $A(n) = 3\,A(n/3) + 15$

f(n) = 15 = O(1), $n^{\log_b a} = n^{\log_3 3} = n^1$
This falls under case 1 → A(n) = θ(n)

b. B(n) = 4 B(n/2) + n^3

f(n) = n^3, $n^{\log_b a} = n^{\log_2 4} = n^2$
This can fall under case 3. Now we need to check that
a f(n/b) ≤ c f(n) for some c < 1 :
a f(n/b) = 4 * $(n/2)^3$ = 4 * $n^3/8$ = $n^3/2$ = .5 f(n) , so we have found c=.5 such that
a f(n/b) ≤ c f(n) and the inequality checks out, and case 3 applies → B(n) = θ(n^3)

c. C(n) = 4 C(n/2) + n^2

f(n) = n^2, $n^{\log_b a} = n^{\log_2 4} = n^2$
This falls under case 2 → C(n) = θ($n^2 \log n$)

d. D(n) = 4 D(n/2) + n

f(n) = n, $n^{\log_b a} = n^{\log_2 4} = n^2$
This falls under case 1 → D(n) = θ(n^2)

3. There are 2 sorted arrays A and B of size n each. Design a D&C algorithm to find the median of the array obtained after merging the above 2 arrays (i.e. array of length 2n). Discuss its runtime complexity.

Find the median of the two arrays. Say the medians are m_A and m_B. If m_A=m_B, then this is our median.
Otherwise, say m_A < m_B then throw away all terms lower than m_A in A, and all terms greater than m_B in B. Solve the resulting subproblem recursively.

Complexity analysis:

- Divide step takes O(1). This includes finding medians in A and B and throwing away half of A and B.
- There is no combine step
- Number of subproblems (a) at each step is 1. The size of the subproblem (n/b) is n/2

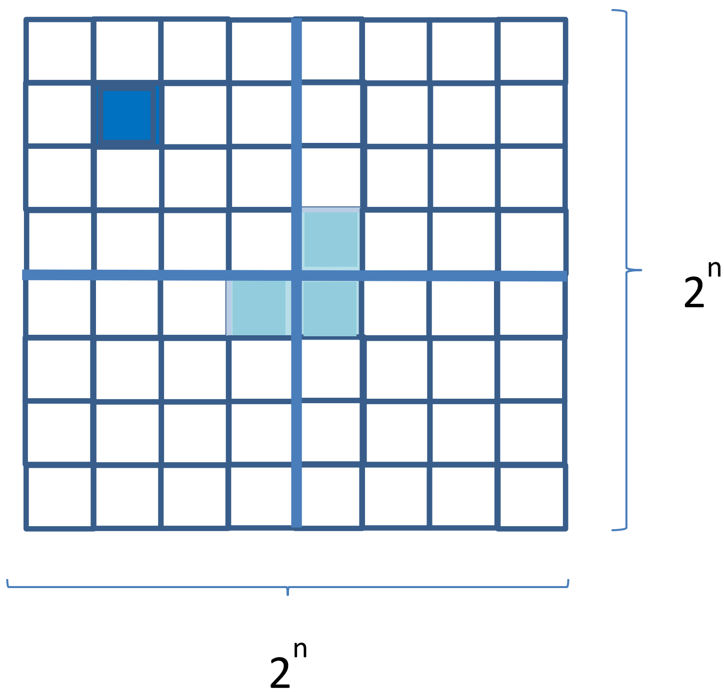
So we can apply the Master Method:
f(n) = O(1), $n^{\log_b a} = n^{\log_2 1} = n^0$= O(1)
This falls under case 2 → T(n) = θ($\log n$)

4. A tromino is a figure composed of three 1x1 squares in the shape of an L.
Given a 2^nx2^n checkerboard with 1 missing square, tile it with trominoes.
Design a D&C algorithm and discuss its runtime complexity.

Solution: Here is a tromino: 

The figure below shows a 2^nx2^n checkerboard with a hole highlighted in dark blue. Here is a divide and conquer solution:

- Divide the grid into 4 equal grids of size 2^{n-1}x2^{n-1}
- Add holes to the three grids that do not have a hole in them. The position of the new holes should be at the center of the grid so that when each region is solved/tiled recursively we can cover the remaining three holes with one tromino.
- Solve all 4 subproblems recursively
- When combining the solutions, place a tile over the three holes in the center to complete the tiling.
- During recursion, when we reach 2x2 grids, we can solve the problem directly by placing a single tromino to tile the grid (that has a single hole).



Complexity analysis:
Divide steps takes O(1) time. So does the combine step.
Number of subproblems (a) = 4, size of each subproblem is half the size of the original problem, so b=2.

f(n) = O(1), $n^{\log_b a} = n^{\log_2 4} = n^2$

This falls under case 1 → T(n) = $\theta(n^2)$