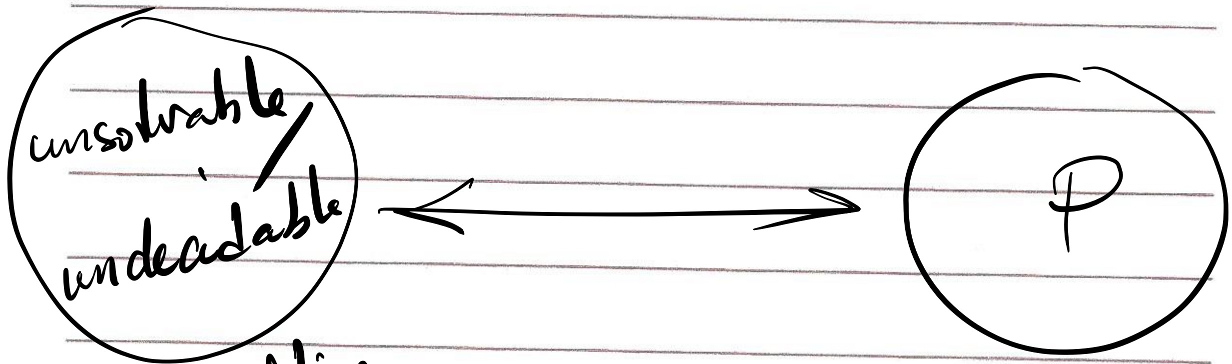
Computational Tractability



ex. Halting Prob.

Plan: Explore the space of computationally hard problems to arrive at a mathematical characterization of a large class of them.

Technique: Compare relative difficulty of different problems.

Loose definition: If problem X is at least as hard as problem Y, it means that if we could solve X, we could also solve Y.

Formal definition:

$Y \leq_p X$ (Y is polynomial time reducible to X)

if Y can be solved using a polynomial number of standard computational steps plus a polynomial number of calls to a blackbox that solves X.

#1

Suppose $Y \leq_p X$, if X can be solved in polynomial time, then Y can be solved in polynomial time.

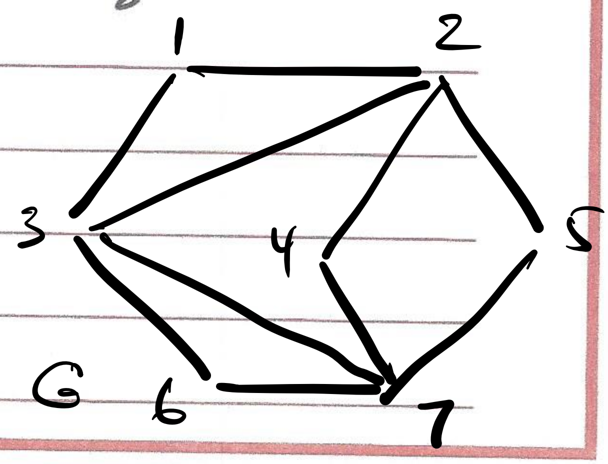
#2

Suppose $Y \leq_p X$, if Y cannot be solved in polynomial time, then X cannot be solved in polyn. time.

Independent Set

Def. In a graph $G=(V,E)$, we say that a set of nodes $\underline{S} \subseteq V$ is "independent" if no two nodes in S are joined by an edge.

$\{3,4,5\}$,
$\{1,4,5,6\}$
$\{1\}$



Independent set problem

- Find the largest independent set in graph G.
(optimization version)

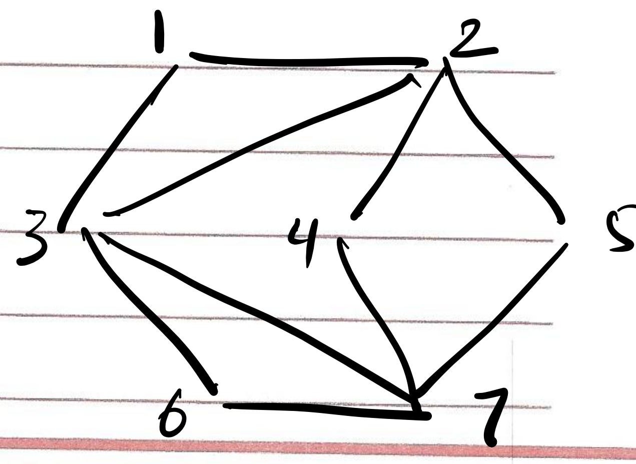
- Given a graph G and a no. $\underline{k}$, does G contain an indep set of size at least k?
(decision version)

Vertex Cover

Def. Given a graph $G=(V,E)$, we say that a set of nodes $S \subseteq V$ is a vertex cover if every edge in E has at least one end in S.

$\{1,2,3,4,5,6,7\}$

$\{2,3,7\}$



Vertex Cover problem

- Find the smallest vertex cover set in G.

(opt. version)

- Given a graph G and a no. k, does G contain a vertex cover of size at most k?

(decision version)

FACT: let $G=(V,E)$ be a graph, then S is an independent set if and only if its complement $V-S$ is a vertex cover set.

Proof: A) First suppose that S is an independent set

1- U is in S and V is not

$\Rightarrow$ $V-S$ will have V and not U.

2- V is in S and U is not

$\Rightarrow$ $V-S$ will have U and not V

3- Neither V nor U is in S

$\Rightarrow$ $V-S$ will have both V & U

B- Suppose that $V-S$ is a vertex cover - -

Claim: Indep. set $\leq_p$ vertex cover

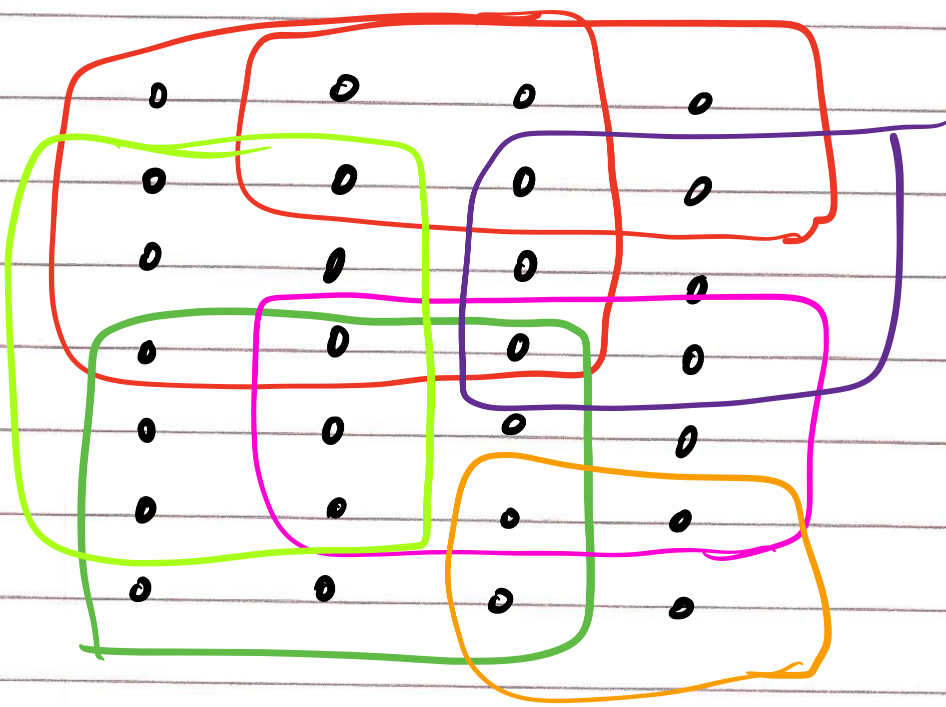
Process ~~Proof~~: If we have a blackbox to solve vertex cover, we can decide if G has an independent set of size at least $\underline{k}$, by asking the blackbox if G has a vertex cover of size at most $\underline{n-k}$.

Claim: Vertex Cover $\leq_p$ Indep set

Process ~~Proof~~: If we have a blackbox to solve independent set, we can decide if G has a vertex cover set of size at most $\underline{k}$, by asking the blackbox if G has an indep set of size at least $\underline{n-k}$

Set Cover Problem

Given a set U of $\underline{n}$ elements, a collection $S_1, S_2, \ldots S_m$ of subsets of U, and a number $\underline{k}$, does there exist a collection of at most $\underline{k}$ of these sets whose union is equal to all of U.



Claim: Vertex Cover $\leq_p$ Set Cover

$S_1 = \{(1,2),(1,3)\}$

$S_2 = \{(1,2),(2,3),(2,4),(2,5)\}$

$S_3 =$

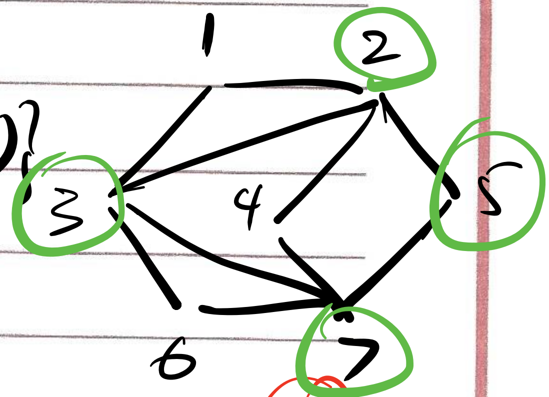
$\vdots$

S_5

$\vdots$

$S_7 =$

$\underline{k}$



G, k

Need to show that G has a vertex cover of size k, iff the corresponding set cover instance has $\underline{k}$ sets whose union ~~is~~ contains ~~equal to~~ all edges in G.

Proof:

A) If I have a vertex cover set of size $\underline{k}$ in G, I can find a collection of $\underline{k}$ sets whose union contains all edges in G.

B) If I have k sets whose union contains all edges in G, I can find a vertex cover set of size k in G.

Reduction Using Gadgets

- Given n Boolean variables $x_1 \ldots x_n$, a clause is a disjunction of terms $t_1 \vee t_2 \vee \ldots t_\ell$ where $t_i \in \{x_1 \ldots x_n, \bar{x}_1 \ldots \bar{x}_n\}$

- A truth assignment for X is an assignment of values 0 or 1 to each x_i.

$(x_1 \vee \bar{x}_2)$

$\rightarrow x_1 = 0,\ x_2 = 0$ ✓

$x_1 = 1,\ x_2 = 0$ ✓

$x_1 = 0,\ x_2 = 1$ ✗

- An assignment satisfies a clause C if it causes C to evaluate to 1.

- An assignment satisfies a collection of clauses if

$$C_1 \wedge C_2 \wedge \ldots C_k$$

evaluates to 1.

ex. $(x_1 \vee \bar{x}_2) \wedge (\bar{x}_1 \vee \bar{x}_3) \wedge (x_2 \vee \bar{x}_3)$

$x_1 = 1$, $x_2 = 1$, $x_3 = 1$ X

$x_1 = 0$, $x_2 = 0$, $x_3 = 0$ ✓

$x_1 = 1$, $x_2 = 0$, $x_3 = 0$ ✓

Problem Statement: Given a set of clauses $C_1 .. C_k$ over a set of variables $X = \{x_1 ... x_n\}$ does there exist a satisfying truth assignment?

Satisfiability problem

SAT

general form of

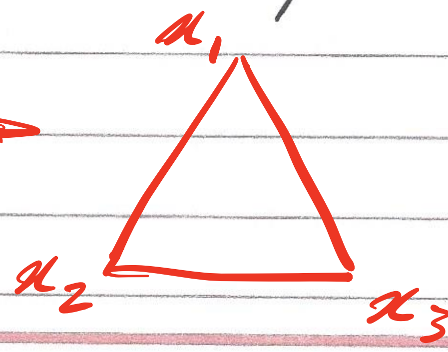
Problem statement: Given a set of clauses $C_1 ... C_k$ each of length 3 over a set of variables $X = \{x_1 .. x_n\}$ does there exist a satisfying truth assignment?

3-SAT

Claim: 3SAT $\leq_p$ Independent Set

Plan: Given an instance of 3SAT with $\underline{k}$ clauses, build a graph G that has an indep. set of size $\underline{k}$ iff the 3SAT instance is satisfiable.

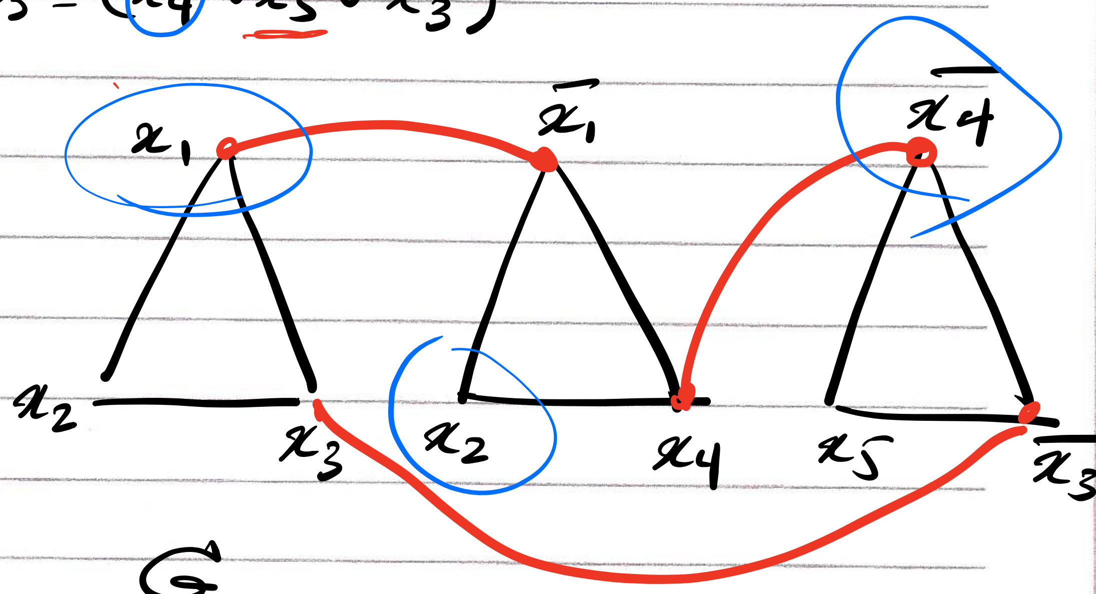
$(x_1 \vee x_2 \vee x_3) \longrightarrow$



ex. $C_1 = (x_1 \vee x_2 \vee x_3)$

$C_2 = (\bar{x}_1 \vee x_2 \vee x_4)$

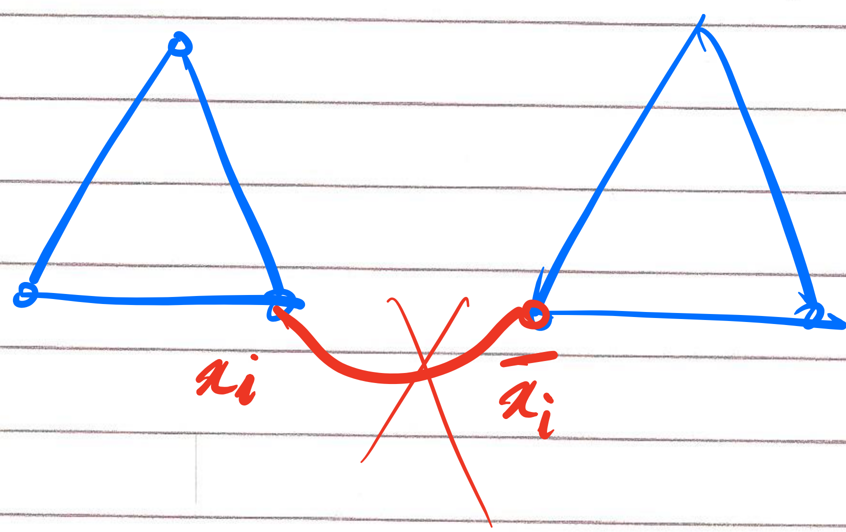
$C_3 = (\bar{x}_4 \vee x_5 \vee \bar{x}_3)$



Claim: The 3-SAT instance is satisfiable iff the graph G has an independent set of size $\underline{k}$.

Proof: A) If the 3-SAT instance is satisfiable, then there is at least one node label per triangle that evaluates to 1.

Let S be a set containing one such node from each triangle



B) Suppose G has an independent set S of size at least k.

if x_i appears as a label in S then set x_i to 1

if $\overline{x_i}$ appears as a label in S then set x_i to 0

if neither x_i nor $\overline{x_i}$ appear as a label in S, then set x_i to either 0 or 1

Efficient Certification

To show efficient certification:

1. Polynomial length certificate

2. Polynomial time certifier

Efficient certification

3-SAT

Certificate t is an assignment of truth values to variables (x_i)

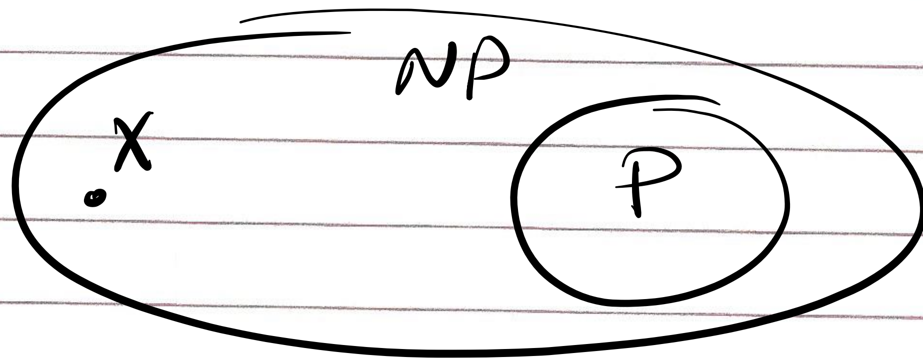
Certifier: evaluate the clauses. if all of them evaluate to 1 then it answers yes.

Indep set

Certificate t is a set of nodes of size at least k in G.

Certifier: check each edge to make sure no edges have both ends in the set
check size of the set $\geq k$
no repeating nodes

class NP is the set of all decision problems for which there exists an efficient certifier



$NP \stackrel{?}{=} P$ we don't know!

if $X \in NP$ and for all $Y \in NP$

$Y \leq_p X$, Then X is the hardest problem in NP.

3-SAT has been proven to be the hardest problem in NP

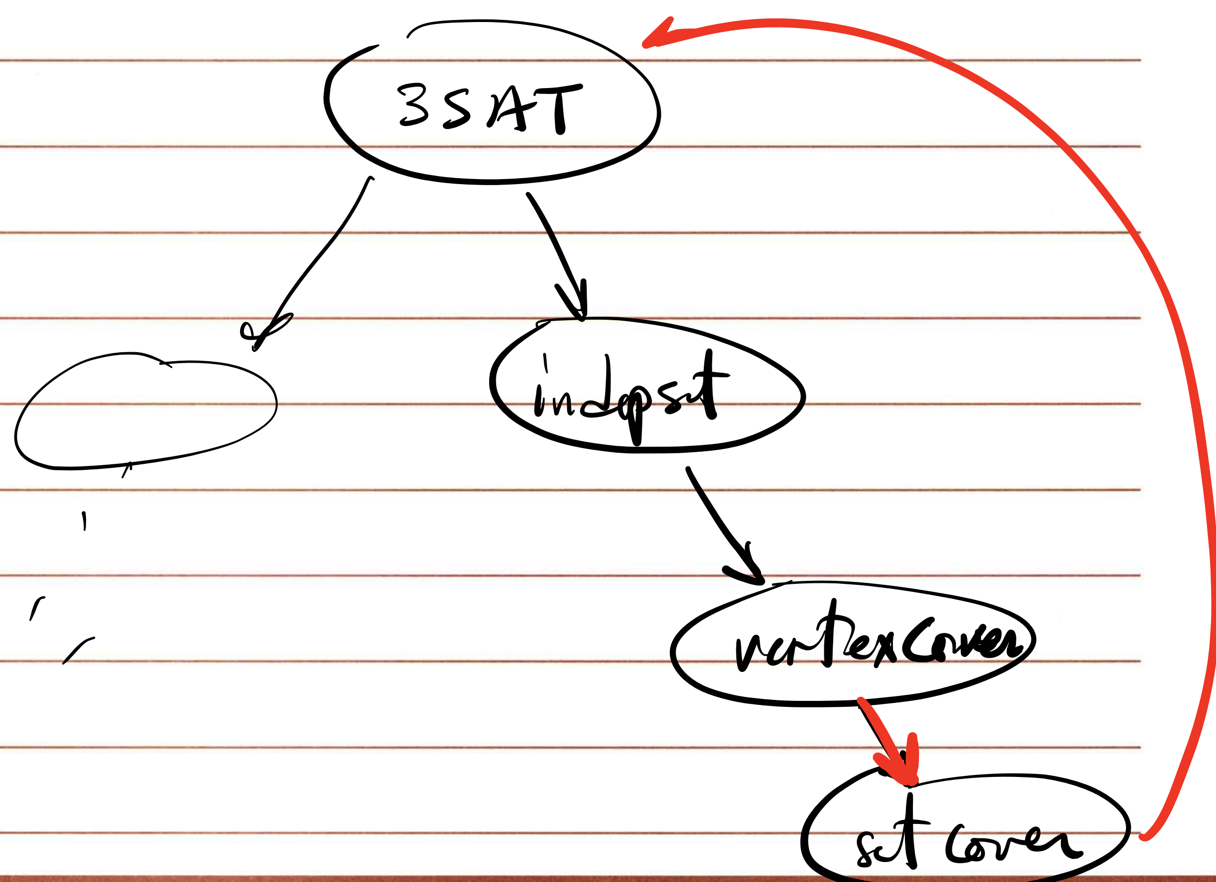
Such a problem is called NP-Complete

Trasitivity

if $Z \leq_p Y$ and $Y \leq_p X$

then $Z \leq_p X$

3SAT $\leq_p$ indepset $\leq_p$ vertexCover $\leq_p$ Set Cover



set Cover $\leq_p$ 3SAT $\leq_p$ indepset $\leq_p$ vertexCover

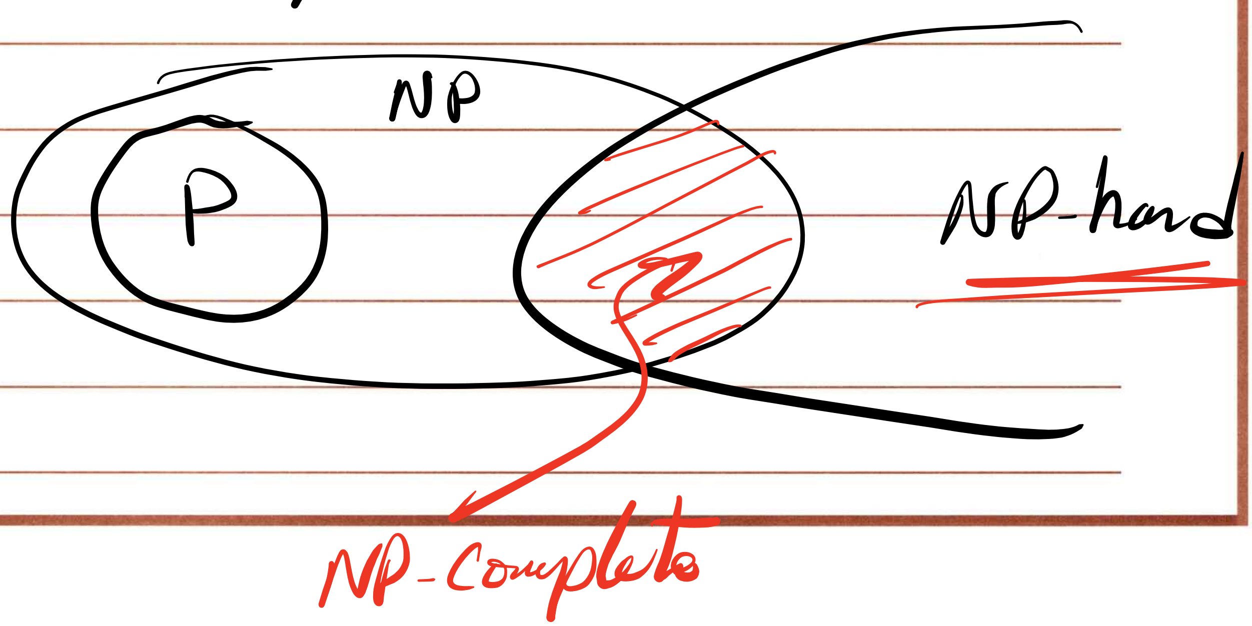
Basic Strategy to prove
a problem X is NP complete

1- Prove $X \in NP$

2- Choose a problem Y that
is known to be NP Complete

3- Prove $Y \leq_p X$

NP-hard is the class of problem
that are at least as hard as
NP-Complete problems.



Discussion 10

1. Given the SAT problem from lecture for a Boolean expression in Conjunctive Normal Form with any number of clauses and any number of literals in each clause. For example,

$$(X_1 \vee \neg X_3) \wedge (X_1 \vee \neg X_2 \vee X_4 \vee X_5) \wedge \ldots$$

Prove that SAT is polynomial time reducible to the 3-SAT problem (in which each clause contains at most 3 literals.)

2. The *Set Packing* problem is as follows. We are given *m* sets $S_1, S_2, \ldots S_m$ and an integer *k*. Our goal is to select *k* of the *m* sets such that no selected pair have any elements in common. Prove that this problem is **NP**-complete.

3. The *Steiner Tree* problem is as follows. Given an undirected graph $G=(V,E)$ with nonnegative edge costs and whose vertices are partitioned into two sets, *R* and *S*, find a tree $T \subseteq G$ such that for every *v* in *R*, *v* is in *T* with total cost at most *C*. That is, the tree that contains every vertex in *R* (and possibly some in *S*) with a total edge cost of at most *C*.
Prove that this problem is **NP**-complete.

1. Given the SAT problem from lecture for a Boolean expression in Conjunctive Normal Form with any number of clauses and any number of literals in each clause. For example,

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Prove that SAT is polynomial time reducible to the 3-SAT problem (in which each clause contains at most 3 literals.)

SAT $\leq_p$ 3SAT

$(x_1) \rightarrow (x_1 \vee x_1 \vee x_1)$

$(x_1 \vee x_2) \rightarrow (x_1 \vee x_2 \vee x_1)$

$(x_1 \vee x_2 \vee x_3) \rightarrow (x_1 \vee x_2 \vee x_3)$

$(x_1 \vee x_2 \vee x_3 \vee x_4) \rightarrow (x_1 \vee x_2 \vee S_1) \wedge (\bar{S}_1 \vee x_3 \vee x_4)$

$(x_1 \vee x_2 \vee x_3 \vee x_4 \vee x_5)$

$(x_1 \vee x_2 \vee S_1) \wedge$
$(\bar{S}_1 \vee x_3 \vee S_2) \wedge$
$(\bar{S}_2 \vee x_4 \vee x_5)$

3SAT $\leq_p$ SAT

2. The *Set Packing* problem is as follows. We are given *m* sets $S_1, S_2, \ldots S_m$ and an integer *k*. Our goal is to select *k* of the *m* sets such that no selected pair have any elements in common. Prove that this problem is **NP**-complete.

1- Set packing $\in NP$

Certificate: a set of $\underline{k}$ sets which have no elements in common

Certifier: form the intersection of all pairs of sets and check to see if the intersection is Null. ✓
Check the count of the sets to be equal to $\underline{k}$. ✓

2- Choose indep set

3- Show $\underline{\text{indep set}} \le_p \underline{\text{set packing}}$

$S_1 = \{(1,2),(1,3)\}$

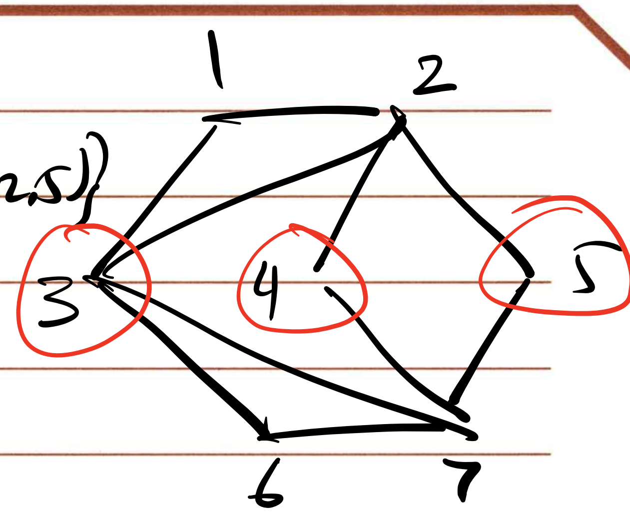
$S_2 = \{(1,2),(2,3),(2,4),(2,5)\}$

$S_3 =$

S_4

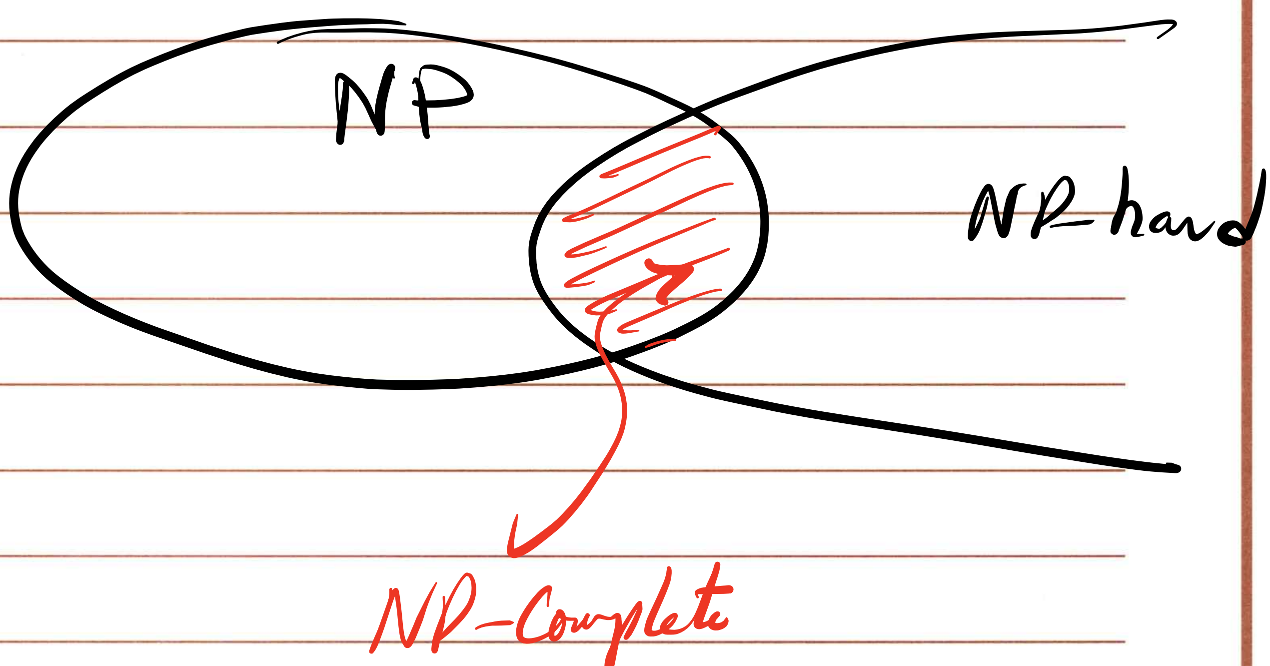
S_5

$S_7 =$



$G, \underline{k}$

$\underline{k}$



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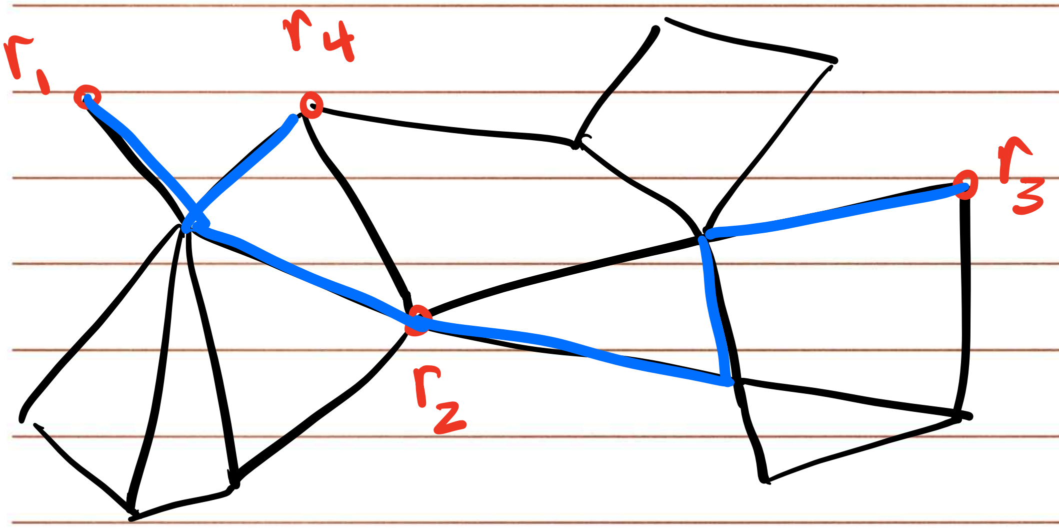
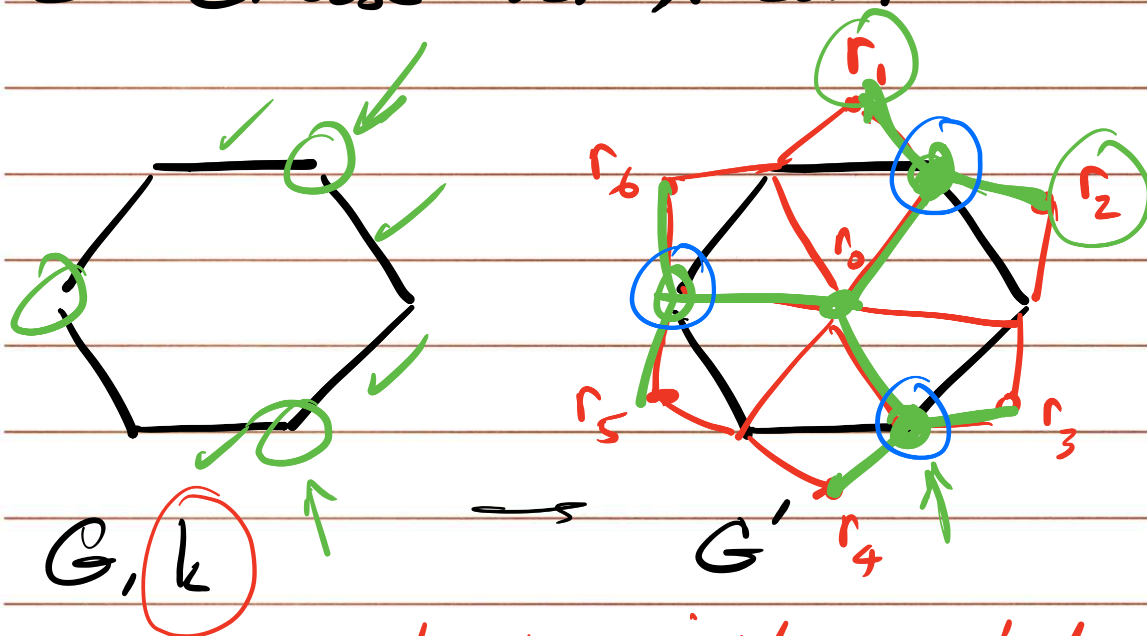


1- Show Steiner Tree $\in NP$

a- Certificate : Tree T
Spanning across all nodes
in R (and maybe some nodes in S)
w/ Cost $\leq C$

b- Certifier :
- Cost of $T \leq C$
- Cover all nodes in R
- T is a tree

2- Choose Vertex Cover



set edge weights in G' to 1

Q: Is there a steiner tree of
Cost $\leq m+k$ in G'?