407MonteCarlo Assignments 1

Haohan Yao u2119251

question 1

(a)

1. use the property of p.d.f :

$$\int_0^4 c*(x^3+8x)\,dx = 1$$

   solve the equation and we can get

$$c = 1/128$$

2. as

$$F(x) = \int_{-\infty}^{x} f(t)\,dx$$

   so it's easy to get:

$$F(x) = \begin{cases} 0 & x \le 0 \\ \frac{x^4}{512} + \frac{x^2}{32} & 0 \le x \le 4 \\ 1 & x \ge 1 \end{cases}$$

3.assume $u = F^{-1}(x)$ then we got:

$$\frac{1}{512}(U^4 + 16U^2 + 64) = X + \frac{1}{8}$$

so solve the equation

$$U = \sqrt{\sqrt{512y+64} - 8}$$

(b)

1.

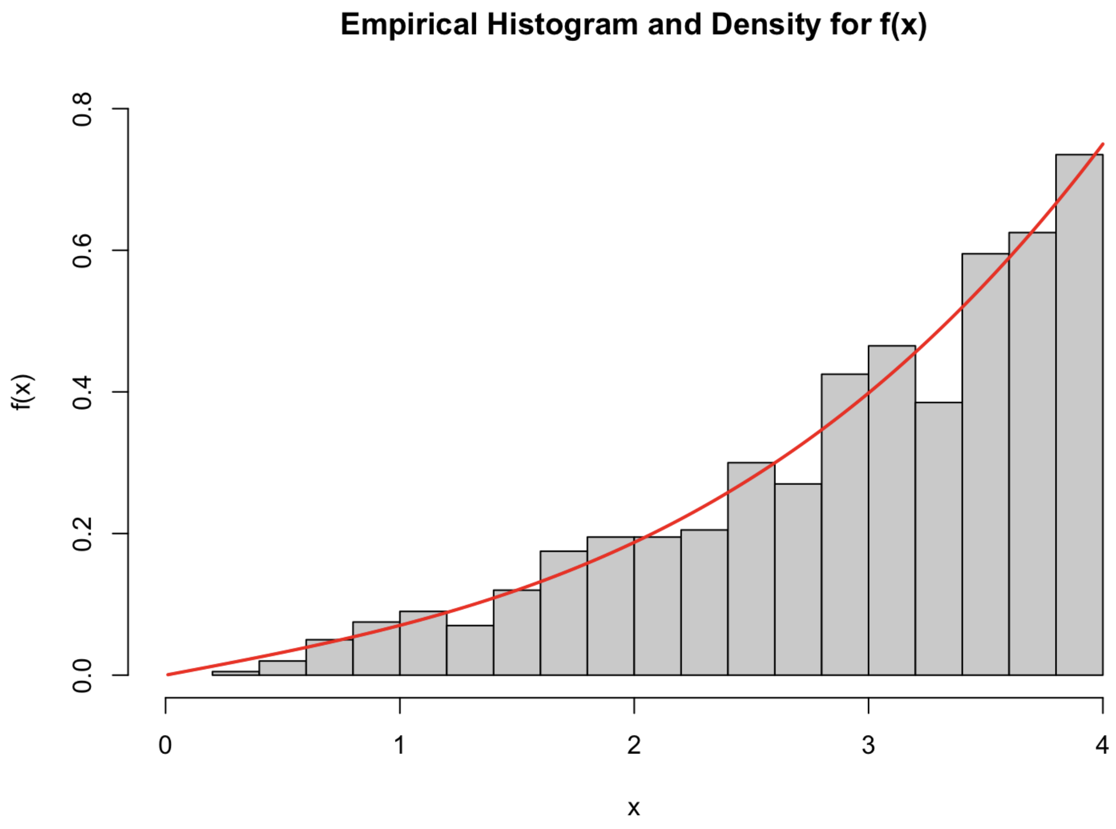
```
F_1 <- function(u){
u <- sqrt(8*sqrt(8*u+1)-8)
return(u)
}
```

2.

```
f <- function(x){
  x <- (x^3+8*x)/128
  return(x)
}
m <- f(4) #max of f(x)
xs <- c()
count <- 0
########  generate 1000 samples ##########
for (i in 1:1000) {
  u <- 2
  x <- 0
  while (u > f(x)/m) {
    u <- runif(1)
    x <- runif(1,min=0,max = 4)
    count <- count + 1
  }
  xs[i] <- x
}
print(xs)
```

3.

```
x1 <- 1:400/100
hist(xs, breaks = 20,prob=TRUE, main='Empirical Histogram and Density for f(x)',xlab='x',ylab='f(x)', xlim=c(0,4),ylim=c(0,0.8))
lines(x1, f(x1), lwd=2, col='red')
```



4.

```
ex <- mean(xs)
varx <- var(xs)

#ex[1] 2.942713
#varx[1] 0.7104359
```

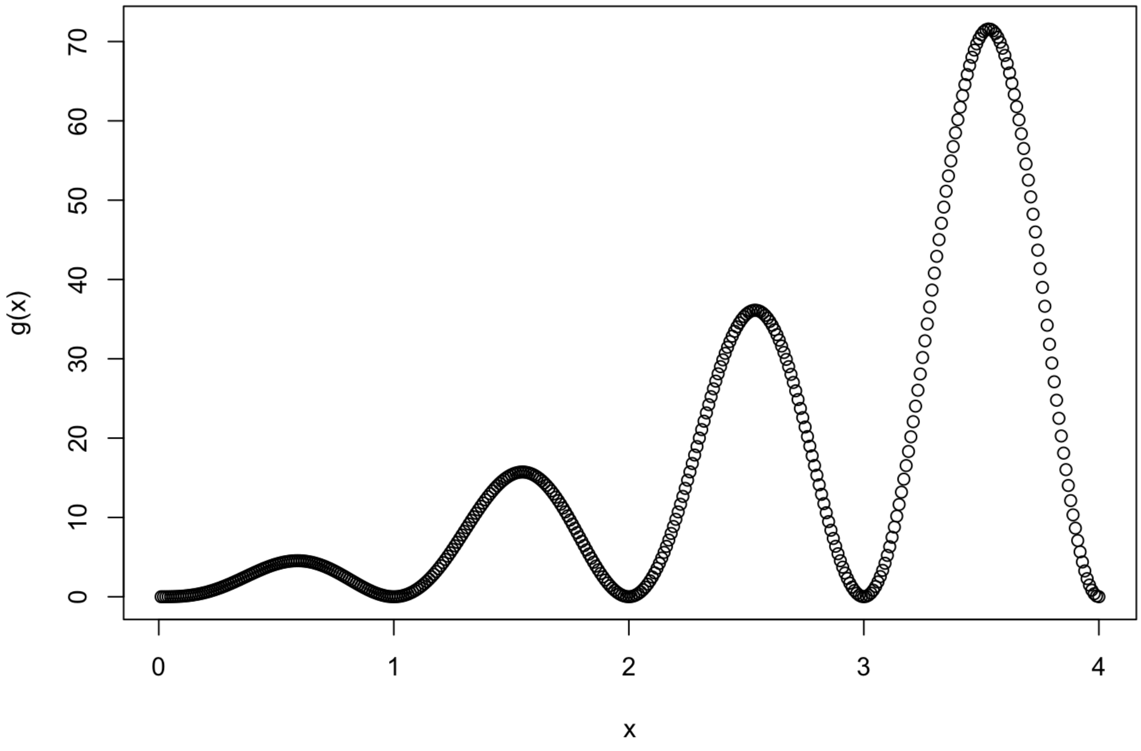
question 2

(a)

1.

```
g <- function(y,c=1){
  y <- c(sin(pi*y))^2*((y^3)+8*y)
  return(y)
}

x <- 1:400/100
plot(x,g(x,c=1))
```

2.



(b)

1. choose uniform distribution as the rejection sampler
2. run this code we got 1600 samples accepted among 10000

```
xs <- c()
u <- 1
x <- 0
for(i in 1:10000){
  u <- runif(1,0,1)
  x <- runif(1,0,4)
  if(u <= g(x)/100){
      xs <- append(xs,x)
    }
}
```

3.

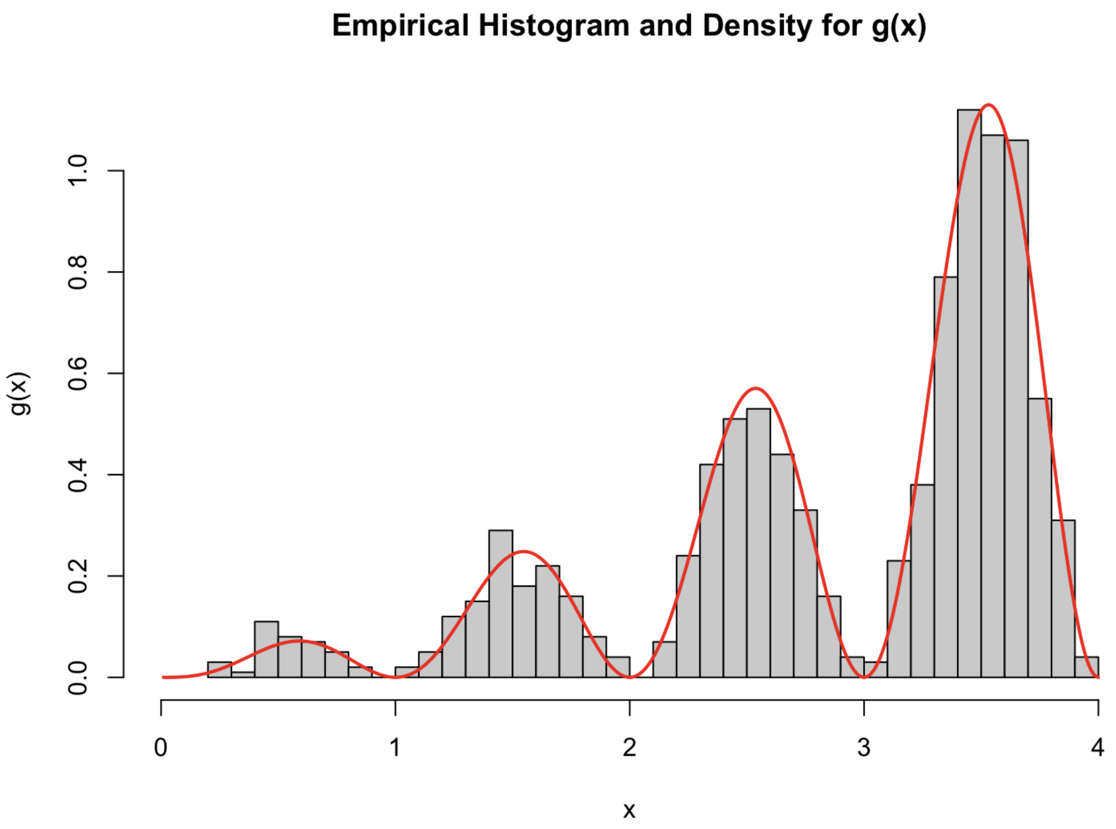
```
ex <- mean(xs)
varx <- var(xs)

#mean(xs)[1] 2.891552
#var(xs)[1] 0.6961375
```

4. the proportion is 1600/10000 (0.16).

calculate c solving equation: (c = 0.0158)

$$C * \int_0^4 g(y)dy = 1$$

```
-propotion <- length(xs)/10000
c <- 1/(propotion*4*100)

#c = 0.01576293
```



question 3

(a)

Q3

(a) $var(w) = E_g(w^2) - E_g^2 w$

$$E_g w = \int \frac{f}{g}\cdot g\,dx = \int f\,dx = 1$$

$$E_g w^2 = \int \frac{f^2}{g}\,dx = E_f w$$

$$= \int \frac{\delta}{\sqrt{2\pi}} e^{-\frac{1}{2}\cdot\frac{x^2}{\frac{\delta^2}{2\delta^2-1}}}\,dx$$

when $\delta^2 > \frac{1}{2}$ $= \frac{\delta^2}{\sqrt{2\delta^2-1}} \int \frac{1}{\sqrt{2\pi}\,\frac{\delta}{\sqrt{2\delta^2-1}}} e^{-\frac{1}{2}\frac{x^2}{\frac{\delta^2}{2\delta^2-1}}}\,dx$

$= \frac{\delta^2}{\sqrt{2\delta^2-1}}$, min $E_g w^2 = 1$, when $\delta^2 = 1$

when $\delta^2 > \frac{1}{2}$, $E_g w^2$ is finite

and when $\sigma^2 = 1$, $var = \frac{\delta^2}{\sqrt{2\delta^2-1}} - 1 = 0$

(min)

b.

$E_f(x) = E_g(w*x)$,so we generate x from g and calculate mean of $w*x$(miu in the figure)

I choose 230 σ from $\frac{\sqrt{2}}{2}$ to 3

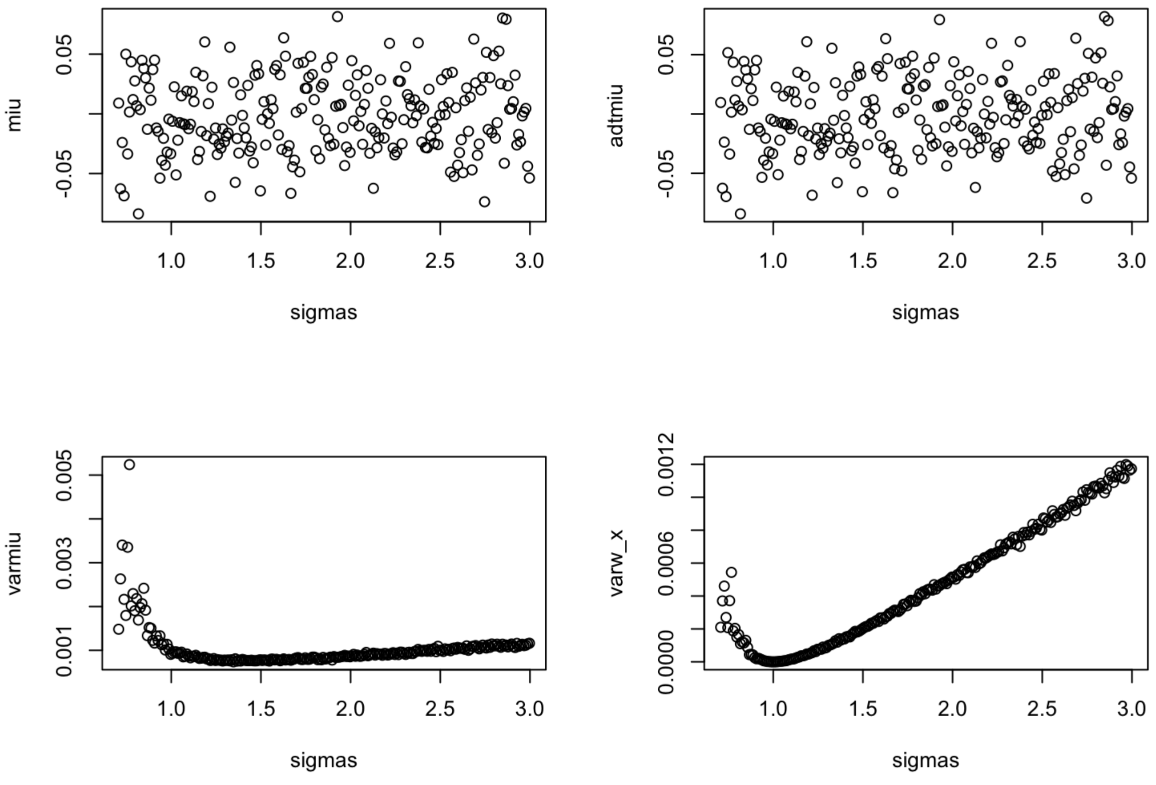
```
g = function(x,v=1){
  p = dnorm(x,0,v)
  return(p)
}
f = function(x){
  p = dnorm(x,0,1)
  return(p)
}
w_x = function(x,v=1){
  return(f(x)/g(x,v))
```

```
}

#choosing sigma for g
#in part a we know that sigma need to be larger than sqrt(2)
#and the var get its min when sigma equal to 1
#so let sigma in (0.707,1.414)
sigmas <- 70.7:300/100
xs <- c()
varmiu <- c()
varw_x <- c()
x <- matrix(0,1000,length(sigmas))
w <- matrix(0,1000,length(sigmas))
#generate x from g
#then generate w, miu = mean(w*x)
miu <- c()
adtmiu <- c()
for ( i in 1:length(sigmas)) {
  xs <- rnorm(1000,0,sigmas[i])
  x[,i] <- xs
  w[,i] <- w_x(xs,sigmas[i])
  ex <- mean(x)
  miu[i] <- mean(x[,i]*w[,i])
  adtmiu[i] <- sum(x[,i]*w[,i])/sum(w[,i]) #adjusted miu
  varmiu[i] <- var(x[,i]*w[,i])/1000
  varw_x[i] <- var(w[,i])/1000
}

#show the relation between sigma and mean/var
par(mfrow =c(2,2))
plot(x=sigmas,y=miu)
plot(x=sigmas,y=adtmiu)
plot(x=sigmas,y=varmiu)
plot(x=sigmas,y=varw_x)
```

plot them in a figure:

top two plots show the μ and self-normalised μ(adtmiu)

bottom two plots show how the var of μ and var of $w(x)$ changes with sigmas



c.

the variance of mean shows in the left bottom of the figure above.

d.

from the plot above we can easily find that the σ has influence on the variance of the integral that we calculated and the $w(x)$ of course. So from this example we can find that when $\sigma = 1$, variance of w reach its min and $f(x) = g(x)$, so the Optimal proposal distribution $g(x)$ should be similar to the