Exercise 01 - Math Refresher

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Linear Algebra

Problem 1

$$f(x,y,z) = x^T A y + Bx - y^T C z D - y^T E^T y + F$$

$$\mathbb{R}^M, \mathbb{R}^N, \mathbb{R}^{P\times Q} = \mathbb{R}^{1\times M}\mathbb{R}^{M\times N}\mathbb{R}^N \quad \mathbb{R}^{1\times M}\mathbb{R}^M - \mathbb{R}^{1\times N}\mathbb{R}^{N\times P}\mathbb{R}^{P\times Q}\mathbb{R}^Q - \mathbb{R}^{1\times N}\mathbb{R}^{N\times N}\mathbb{R}^N + \mathbb{R}^1$$

$A = \mathbb{R}^{M\times N}$ $\quad D = \mathbb{R}^Q$

$B = \mathbb{R}^{1\times M}$ $\quad E = \mathbb{R}^{N\times N}$

$C = \mathbb{R}^{N\times P}$ $\quad F = \mathbb{R}^1$

Problem 2

$$\sum_i^N \sum_j^N x_i x_j M_{ij} =$$

$$x_1 x_1 M_{11} + x_1 x_2 M_{12} + x_1 x_3 M_{13} + \dots = x_1 (x_1 M_{11} + x_2 M_{12} + x_3 M_{13})$$

$$+ x_2 x_1 M_{21} + x_2 x_2 M_{22} + x_2 x_3 M_{23} + \dots = x_2 (x_1 M_{21} + x_2 M_{22} + x_1 M_{23})$$

$$+ x_3 x_1 M_{31} + x_3 x_2 M_{32} + x_3 x_3 M_{33} + \dots = x_3 (x_1 M_{31} + x_2 M_{32} + x_3 M_{33})$$

$$\Rightarrow \sum_i^N x_i \sum_j^N x_j M_{ij} = \sum_i^N x_i \cdot (M \cdot x) = x^T M x$$

Problem 3

a) - $\mathrm{rank}(A) = M$

- $\det(A) \neq 0$
- $\ker(A) = \{0\}$

b) $|\det(A)| = \prod_i \lambda_i = -5 \cdot 0 \cdot 1 \cdot 1 \cdot 3 = 0$ // $\mathrm{Rank}(A) = \#$ Non 0 EV

$\Rightarrow$ No unique solution

Problem 4

$BA = AB = I$

$\Rightarrow B = A^{-1}$

$\Rightarrow A$ invertierbar $\Rightarrow \det(A) \neq 0 \Rightarrow \forall i: \lambda_i \neq 0$

Problem 5

$$x^T A x \geq 0$$

$$= x^T \lambda x \geq 0$$

$$= \lambda x^T x \geq 0$$

$$= \lambda \sum x_i^2 \geq 0$$

$$= \lambda \geq 0$$

Problem 6

$B = A^T A \Rightarrow Bx = \lambda_B x = A^T A x = \lambda_A \lambda_A x = \lambda_A^2 x$

$\Rightarrow \lambda_B = \lambda_A^2$

$\Rightarrow \lambda_B$ immer ≥ 0

$\Rightarrow B$ PSD

Calculus

Problem 7

a) Unique solution: global minimum $a > 0$

Infinitely many solutions: many local minima $a = b = 0$

No solution: No lower bound $a < 0$

b) $\min_{x \in \mathbb{R}} f(x) = f'(x) \cdot 0$

$$f'(x) = ax + b = 0$$

$$x^* = \underset{x \in \mathbb{R}}{\operatorname{argmin}} f(x) = \frac{-b}{a}$$

Missing

Problem 8

a) $g(x) = \frac{1}{2} x^T A x + b^T x + c = \frac{1}{2} (x_1 x_2 \dots x_n) A \begin{pmatrix} x_1 \\ x_2 \\ x_n \end{pmatrix} + b^T \begin{pmatrix} x_1 \\ x_2 \\ x_n \end{pmatrix} + c$

$$\nabla^2 g(x) = \begin{bmatrix} \frac{\partial^2 g}{\partial x_1 \partial x_1}(x) & \dots & \frac{\partial^2 g}{\partial x_1 \partial x_n}(x) \\ \vdots & & \vdots \\ \frac{\partial^2 g}{\partial x_n \partial x_1}(x) & \dots & \frac{\partial^2 g}{\partial x_n \partial x_n}(x) \end{bmatrix}$$

$$\frac{1}{2} \sum x_i \sum_j A_{ij} x_j + \sum (b^T)_i x_i + c = g(x)$$

$$g'(x) = \begin{cases} A_{ij} & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Missing

$$\begin{bmatrix} A_{11} & & 0 \\ & A_{22} & \\ 0 & & \ddots \; A_{nn} \end{bmatrix}$$

$$\nabla^2_g(x) : \begin{bmatrix} \frac{\partial^2 g}{\partial x_i \partial x_j} \end{bmatrix}$$

$-\forall i: A_{ii} \neq 0 \Rightarrow \det(A) \neq 0$ unique PD

Missing

b) $g(x) = \frac{1}{2} x^T A x + b^T x + c = \frac{1}{2} \lambda \sum x_i^2 + \sum b_i^T x_i + c$

$g''(x) = \lambda \Rightarrow$ Curvature same in all dimensions

$\Rightarrow$ Global minimum $\Rightarrow$ Convex Problem

$\Rightarrow$ Negative EV $\Rightarrow$ Only satle point

Missing

c) $x^* = \underset{x \in \mathbb{R}^N}{\operatorname{argmin}} g(x) \Rightarrow g'(x) \overset{!}{=} 0$

$$g'(x) = \frac{\partial}{\partial x} \left(\frac{1}{2} x^T A x + b^T x + c \right)$$

$$= \frac{1}{2} x^T A \frac{\partial x}{\partial x} + b^T \frac{\partial x}{\partial x} + c \cdot \frac{\partial 1}{\partial x}$$

$$= \frac{1}{2} x^T A + b^T + 0$$

$$= \frac{1}{2} x^T A + b^T \overset{!}{=} 0$$

$$x^T A = -2 b^T$$

$$x^T = -2 b^T A^{-1}$$

$$x = (-2 b^T A^{-1})^T$$

Missing

Formatieren mit & =

$\Rightarrow$ Macht alle = untereinander