Exercise

07

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Informatics 3 - Professorship of Data Mining and Analytics

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Constrained Optimization

Problem 1:

Since f_0 is a hyperplane, the min and max values are at the vertices:

$$f_0((9,3)^T) = -2 \qquad f_0((5,7)^T) = -11$$
$$f_0((7,2)^T) = 8 \qquad f_0((9,3)^T) = 10$$

We immediately see that $\theta_{\text{min}} = (5,7)^T$ and $\theta_{\text{max}} = (9,3)^T$ and also $\min(f_0) = -11$ and $\max(f_0) = 10$.

Problem 2:

$$\pi_\chi(p) = \begin{cases} p & \text{if } p \in \chi \\ (1.5, 2.5)^T & \text{if } p \text{ is in the green corridor} \\ (3,1)^T & \text{if } p \text{ is in the red corridor} \\ \begin{pmatrix} \frac{1}{2}p_1 - \frac{1}{2}p_2 + 2 \\ \frac{1}{2} - p_1 + \frac{1}{2}p_2 + 2 \end{pmatrix} & \text{if } p \text{ is in the blue corridor} \\ \begin{pmatrix} \max(0, \min(3, p_1)) \\ \max(0, \min(2.5, p_2)) \end{pmatrix} & \text{if } p \text{ is outside green/red/blue corridor and } p \notin \chi \end{cases}$$

The blue corridor projects down to a line. We can take any two points on the line to calculate the projection vector $\pi_{\chi_{a,b}}$. We choose $\mathbf{a} := (0,4)^T$ and $\mathbf{b} := (4,0)^T$ for simplicity.

$$\begin{aligned} \pi_{\chi_{a,b}} &= a + \frac{(\mathbf{p}-\mathbf{a})^T(\mathbf{b}-\mathbf{a})}{\|\mathbf{b}-\mathbf{a}\|_2^2}(\mathbf{b}-\mathbf{a}) \\ &= \left(\begin{smallmatrix} 0 \\ 4 \end{smallmatrix}\right) + \frac{\left(\begin{smallmatrix} p_1 - 0 \\ p_2 - 4 \end{smallmatrix}\right)^T \left(\begin{smallmatrix} 4 \\ -4 \end{smallmatrix}\right)}{32} \left(\begin{smallmatrix} 4 \\ -4 \end{smallmatrix}\right) \\ &= \left(\begin{smallmatrix} 0 \\ 4 \end{smallmatrix}\right) + \frac{4p_1 - 4p_2 + 16}{32} \left(\begin{smallmatrix} 4 \\ -4 \end{smallmatrix}\right) \\ &= \left(\begin{smallmatrix} 0 \\ 4 \end{smallmatrix}\right) + \left(\frac{1}{8}p_1 - \frac{1}{8}p_2 + \frac{1}{2}\right) \left(\begin{smallmatrix} 4 \\ -4 \end{smallmatrix}\right) \\ &= \begin{pmatrix} \frac{1}{2}p_1 - \frac{1}{2}p_2 + 2 \\ \frac{1}{2} - p_1 + \frac{1}{2}p_2 + 2 \end{pmatrix} \end{aligned}$$

Problem 3:

1. Formulate the Lagrangian:

$$
\begin{aligned}
L(\theta, \alpha) &= f_0(\theta) + \sum_{i=1}^{M} \alpha_i f_i(\theta) \\
&= \theta_1 - \sqrt{3}\theta_2 + \alpha(\theta_1^2 + \theta_2^2 - 4) \\
&= \theta_1 - \sqrt{3}\theta_2 + \alpha\theta_1^2 + \alpha\theta_2^2 - \alpha 4
\end{aligned}
$$

2.1 Solve:

$$
\begin{aligned}
g(\alpha) &= \min_{\theta} \left[L(\theta, \alpha)\right] \\
&= \min_{\theta} \left[\theta_1 - \sqrt{3}\theta_2 + \alpha\theta_1^2 + \alpha\theta_2^2 - \alpha 4\right] \\
&\Rightarrow \left[\theta_1 - \sqrt{3}\theta_2 + \alpha\theta_1^2 + \alpha\theta_2^2 - \alpha 5\right]' \stackrel{!}{=} 0 \\
&\Rightarrow \nabla_\theta L(\theta, \alpha) = \begin{pmatrix} 1 + 2\alpha\theta_1 \\ -\sqrt{3} + 2\alpha\theta_2 \end{pmatrix} = \stackrel{!}{=} 0 \\
&\Rightarrow \begin{cases} 1 + 2\alpha\theta_1 = 0 \\ -\sqrt{3} + 2\alpha\theta_2 = 0 \end{cases} \\
&\Rightarrow \begin{cases} \theta_1 = -\frac{1}{2\alpha} \\ \theta_2 = \frac{\sqrt{3}}{2\alpha} \end{cases}
\end{aligned}
$$

2.2 Get the dual function $g(\alpha)$:

$$
\begin{aligned}
g(\alpha) &= L(\theta, \alpha) \\
&= \theta_1 - \sqrt{3}\theta_2 + \alpha\theta_1^2 + \alpha\theta^2 - \alpha 4 \\
&= -\frac{1}{2\alpha} - \sqrt{3}\frac{\sqrt{3}}{2\alpha} + \alpha\left(-\frac{1}{2\alpha}\right)^2 + \alpha\left(\frac{\sqrt{3}}{2\alpha}\right)^2 - \alpha 4 \\
&= -\frac{1}{2\alpha} - \frac{3}{2\alpha} + \left(\frac{1}{4\alpha}\right) + \left(\frac{3}{4\alpha}\right) - \alpha 4 \\
&= -\frac{1}{\alpha} - \alpha 4
\end{aligned}
$$

3. Solve the dual problem:

$$
[g(\alpha)]' = \frac{1}{\alpha^2} - 4 \Rightarrow \alpha = \frac{1}{2} \text{ (since } \alpha \geq 0)
$$

$$
g\left(\frac{1}{2}\right) = -4
$$

Slater's condition holds.

The minimizer θ becomes:

$$\theta_1 = -\frac{1}{2\alpha} = -\frac{1}{2 \cdot \frac{1}{2}} = -1$$

$$\theta_2 = \frac{\sqrt{3}}{2\alpha} = \frac{\sqrt{3}}{2 \cdot \frac{1}{2}} = \sqrt{3}$$

Problem 4:

Appendix

We confirm that the submitted solution is original work and was written by us without further assistance. Appropriate credit has been given where reference has been made to the work of others.

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