Exercise

01

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Informatics 3 - Professorship of Data Mining and Analytics

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Optimization

Problem 1:

a)

$h(x) = g_2(g_1(x))$ is not convex in this context:

Since g_2 and g_1 are convex: $g_2'' \geq 0$ and $g_1'' \geq 0$ but g_2 can be decreasing so $g_2'(x)$ does not have to be positive $\forall x$.

$$[h(x)]' = g_2'(g_1(x)) * g_1'(x)$$
$$[h(x)]'' = [g_2'(g_1(x)) * g_1'(x)]'$$
$$[h(x)]'' = g_2''(g_1(x)) * (g_1'(x))^2 + g_2'(g_1(x)) * g_1''(x)$$

So $[h(x)]''$ can be ≤ 0 if $g_2'(g_1(x)) \leq 0$ ($(g_1'(x))^2$ is always ≥ 0)

As an example: $g_2(x) = -\frac{1}{2}x$ and $g_1(x) = x^2$ are convex $\implies h(x) = -\frac{1}{2}x^2$ and should also be convex. But the second derivation of $h(x)$ is -1 and therefore not convex.

b)

$h(x) = g_2(g_1(x))$ is convex in this context:

Since g_2 and g_1 are convex: $g_2'' \geq 0$ and $g_1'' \geq 0$ also g_2 is non-decreasing so $g_2'(x) \geq 0$.

$$[h(x)]' = g_2'(g_1(x)) * g_1'(x)$$
$$[h(x)]'' = [g_2'(g_1(x)) * g_1'(x)]'$$
$$[h(x)]'' = g_2''(g_1(x)) * (g_1'(x))^2 + g_2'(g_1(x)) * g_1''(x)$$

So $[h(x)]''$ can only be ≥ 0. ($(g_1'(x))^2$ is always ≥ 0)

c)

$h(x) = max(g_1(x), ..., g_n(x))$ is always convex:

$$\begin{aligned} h(\lambda x + (1-\lambda)y) &= max(g_1(\lambda x + (1-\lambda)y), ..., g_n(\lambda x + (1-\lambda)y) \\ &\leq max(\lambda g_1(x) + (1-\lambda)g_1(y), ..., \lambda g_n(x) + (1-\lambda)g_n(y)) \\ &\leq max(\lambda g_1(x), ..., \lambda g_n(x)) + max(\lambda g_1(y), ..., \lambda g_n(y)) \\ &= \lambda h(x) + (1-\lambda)h(y) \end{aligned}$$

Problem 2:

a)

Minimum $x*$ of f is the partial derivative wrt. x_1 and x_2:

$$\frac{\partial f}{\partial x_1} = x_1 + 2 \stackrel{!}{=} 0 \implies x_1 = -2$$
$$\frac{\partial f}{\partial x_2} = 2x_2 + 1 \stackrel{!}{=} 0 \implies x_2 = -\frac{1}{2}$$

$x*$ is at $x_1 = -2$ and $x_2 = -\frac{1}{2}$.

b)

2 steps gradient descent with $x^{(0)} = (0, 0)$ and learning rate $\tau = 1$:

Gradient descent in general:

1) Take point $x^{(n)}$

2) compute $f'(x)$

3) $x^{(n+1)} = x^{(n)} - \tau * f'(x)$

First step:

$$\frac{\partial f}{\partial x_1 = 0} = x_1 + 2 = 2 \implies x_1^{(1)} = 0 - 1 * 2 = -2$$
$$\frac{\partial f}{\partial x_2 = 0} = 2x_2 + 1 = 1 \implies x_2^{(1)} = 0 - 1 * 1 = -1$$
$$\implies x^{(1)} = (-2, -1)$$

Second step:

$$\frac{\partial f}{\partial x_1 = -2} = x_1 + 2 = 0 \implies x_1^{(2)} = -2 - 1 * 0 = -2$$
$$\frac{\partial f}{\partial x_2 = -1} = 2x_2 + 1 = -1 \implies x_2^{(2)} = -1 - 1 * -1 = 0$$
$$\implies x^{(2)} = (-2, 0)$$

c)

It will with $x_1 = -2$ but it won't with x_2 because it alternates between -1 and 0.

To solve this problem a learning rate $0 < \tau < 1$ would be needed to stop the alternation of x_2 and help to converge to $x_2 = -\frac{1}{2}$.

Problem 4:

a)

Regin S is not convex.

$$\forall x, y \in S : \lambda x + (1-\lambda)y \in S \qquad \forall \lambda \in [0,1]$$

Let x be $(3.5, 1) \in S$ and let y be $(6, 3.5) \in S$ and let λ be 0.5:

$$0.5 * (3.5, 1) + 0.5 * (6, 3.5) = (4.75, 2.25) \notin S$$

$\implies$ S is not convex

b)

There is no local (or global) maximizer of f, since any such local maximizer would necessarily occur at a critical point, and the only critical point is a local minimizer.

That is because the second derivative of f for x_1 is $\frac{\partial^2 f}{\partial x_1^2} = e^{x_1+x_2}$ and for x_2 is $\frac{\partial^2 f}{\partial x_2^2} = e^{x_1+x_2} - \frac{5}{x_2^2}$

With the domain of S there are no values for x_1 and x_2 so that $\frac{\partial^2 f}{\partial x_{1,2}^2}$ is negativ.

c)

- Subdivide S into convex subregions.
- Use the algorithm to compute the minimum of every subdomain of S.
- Pick the one with the best minimum.

Problem 3:

Appendix

We confirm that the submitted solution is original work and was written by us without further assistance. Appropriate credit has been given where reference has been made to the work of others.

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