Machine Learning Homework Sheet 2

Parameter Inference

1 Optimizing Likelihoods: Monotonic Transforms

Usually one considers the *log-likelihood*, $\log p(x_1, \ldots, x_n \mid \theta)$. The next problems justify this.

In the lecture, we encountered the likelihood maximization problem

$$\arg\max_{\theta \in [0,1]} \theta^t (1-\theta)^h,$$

where t and h denoted the number of tails and heads in a sequence of coin tosses, respectively.

Problem 1: Compute the first and second derivative of this likelihood w.r.t. θ. Then compute first and second derivative of the log likelihood $\log \theta^t (1-\theta)^h$.

Problem 2: Show that every local maximum of $\log f(\theta)$ is also a local maximum of the differentiable, positive function $f(\theta)$. Considering this and the previous exercise, what is your conclusion?

2 Properties of MLE and MAP

Problem 3: Show that θ_{MLE} can be interpreted as a special case of θ_{MAP} in the sense that there always exists a prior $p(\theta)$ such that $\theta_{\mathrm{MLE}} = \theta_{\mathrm{MAP}}$.

Problem 4: Consider a Bernoulli random variable X and suppose we have observed m occurrences of $X = 1$ and l occurrences of $X = 0$ in a sequence of $N = m + l$ Bernoulli experiments. We are only interested in the number of occurrences of $X = 1$—we will model this with a Binomial distribution with parameter θ. A prior distribution for θ is given by the Beta distribution with parameters a, b. Show that the posterior *mean* value $\mathbb{E}[\theta \mid \mathcal{D}]$ (not the MAP estimate) of θ lies between the prior mean of θ and the maximum likelihood estimate for θ.

To do this, show that the posterior mean can be written as λ times the prior mean plus $(1-\lambda)$ times the maximum likelihood estimate, with $0 \le \lambda \le 1$. This illustrates the concept of the posterior mean being a compromise between the prior distribution and the maximum likelihood solution.

The probability mass function of the Binomial distribution for some $m \in \{0, 1, \ldots, N\}$ is

$$p(x = m \mid N, \theta) = \binom{N}{m} \theta^m (1-\theta)^{N-m}.$$

Hint: Identify the posterior distribution. You may then look up the mean rather than computing it.

3 Poisson Distribution

Problem 5:

(a) The definition of an unbiased estimator is as follows: Let X be a random variable with probability density function $p(X|\lambda)$. Let $\{X_1, ..., X_n\}$ be n i.i.d. samples from X. An estimator λ_{EST} for λ is called unbiased iff

$$\mathbb{E}\Big[\lambda_{EST}\big(X_1, ..., X_n\big)\Big] = \lambda. \tag{1}$$

Note that, as denoted in the above equation, the estimator λ_{EST} is a function of the samples.

Let X be Poisson distributed. For n i.i.d. samples from X, determine the maximum likelihood estimate for λ. Show that this estimate is unbiased!

(b) In class we also talked about avoiding overfitting of parameters via *prior* information. Compute the posterior distribution over λ, assuming a Gamma(α, β) prior for it. Compute the MAP for λ under this prior. Show your work.