NOTES ON ATTITUDE TOWARD RISK TAKING AND THE EXPONENTIAL UTILITY FUNCTION

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ABSTRACT

This paper summarizes useful concepts for analyzing attitude toward risk taking in decision analysis practice. Particular attention is given to the exponential utility function which is widely used in applications. Conditions are reviewed under which this utility function form is appropriate. Tables are presented which aid in using the exponential utility function, including finding the value of the risk tolerance. The use of the exponential utility function is considered in analyzing portfolio decisions and determining the value of perfect information. The accuracy is considered of an approximate formula for determining certainty equivalents when the exponential utility function holds. Exercises on this material are also included.

NOTES ON ATTITUDE TOWARD RISK TAKING AND THE EXPONENTIAL UTILITY FUNCTION

This paper summarizes concepts related to analyzing attitude toward risk taking in decision analysis practice, with particular emphasis on the use of the *exponential utility function*. Relevant theorems and empirical observations are presented. Informal "proofs" are presented for several theorems, but these are intended to be plausibility arguments rather than detailed proofs. Technical conditions (for example, requirements that certain functions be continuous or have continuous derivatives) are not presented. The reader is assumed to be familiar with basic probability concepts, including probability distributions, expected values, and variances.

1. The Certainty Equivalent and the Idea Underlying Utility Functions

A difficulty with decision-making under uncertainty is illustrated by the following: You are offered an alternative with equal chances of winning \$10,000 or losing \$1,000. How much are you willing to pay for this? You certainly will not pay more than \$10,000, and you will certainly take the alternative if someone offers to give you \$1,000 in addition to the alternative. How can you settle on a number somewhere between these two extremes? Some thought shows that different individuals might be willing to pay different amounts—If you are a graduate student with just enough money to make it to the end of the school year, you may have a different view of the risks associated with this alternative than if you are a wealthy businessman.

To develop a criterion for making decisions under uncertainty, it makes sense to start with reasonable conditions that we would wish our decision-making to obey, and then to see what criterion we must use to obey these conditions. The *axioms of consistent choice* (also sometimes called the *axioms of rational choice* or the *axioms of decision theory*) provide such a set of conditions. These axioms and two important theorems that result from them are presented in Appendix A. In this section, we will make informal plausibility arguments for the decision analysis procedures which are implied by these axioms.

We will restrict ourselves to situations where the consequences of a decision can be adequately described by a single *evaluation measure* or *evaluation attribute* x. For many business decisions, this will be a monetary measure, such as profit, cost, or assets, possibly discounted to account for the time value of money. We will further assume that preferences for x are either *monotonically increasing* (that is, more of x is always preferred to less) or *monotonically decreasing* (that is, more of x is always less preferred). For a monetary evaluation measure, the monotonically increasing case corresponds to using profit or total assets as an evaluation measure, while the monotonically decreasing case corresponds to using costs or losses as an evaluation measure.

There are situations where preferences over an evaluation measure are neither monotonically increasing nor monotonically decreasing. For example, consider the evaluation measure "blood pressure level" to measure the results of various medical treatments. There is a most preferred level for this evaluation measure, and either greater or smaller levels are less preferred. This type of situation is not common in typical business decisions, and even in such situations it is often possible to use a modified evaluation measure which *is* monotonic. (In the blood pressure example, the attribute "distance from the most preferred level" might be used.)

A decision maker's attitude toward risk taking is addressed with the concept of the *certainty* (or *certain*) *equivalent,* which is the *certain* amount that is equally preferred to an uncertain alternative. If certainty equivalents are known for the alternatives in a decision, then it is easy to find the most preferred alternative: It is the one with the highest (lowest) certainty equivalent if we are considering profit (cost).

The "Weak Law of Large Numbers" (Drake 1967) argues for using expected values as certainty equivalents when the stakes in a decision under uncertainty are small. This Law shows that under general conditions the average outcome for a large number of independent decisions *stochastically converges* to the average of the expected values for the selected alternatives in the decisions. (The term *stochastically converges* means that the probability the actual value will differ from the expected value by any specified amount gets closer to zero as the number of independent decisions increases.) Thus, if you value alternatives at more than their expected values, you will lose money over many decisions since you will only sell such alternatives for more than they will return on average. Similarly, if you value alternatives at less than their expected values, you will lose money because you will sell alternatives for less than they will return on average.

However, additional factors enter when the stakes are high. Most of us would be willing to pay up to the expected value of \$2.50 for a lottery ticket giving us a 50:50 chance of winning \$10.00 or losing \$5.00. On the other hand, most of us would *not* be willing to pay as much as \$25,000 for a lottery ticket with a 50:50 chance of winning \$100,000 or losing \$50,000 even though \$25,000 is the expected value of this lottery. This is because a few \$50,000 losses would leave most of us without the resources to continue. We cannot "play the averages" over a series of decisions where the stakes are this large, and thus considerations of long-run average returns are less relevant to our decision making.

Many conservative business people are averse to taking risks. That is, they attempt to avoid the possibility of large losses. Such individuals have certainty equivalents that are lower than the expected values of uncertain alternatives if we are dealing with profits, or higher than the expected values if we are dealing with costs. That is, these individuals are willing to sell the alternatives for less than these alternatives will yield on average over many such decisions in order to avoid the risk of a loss. Intuitively, we might consider incorporating this aversion toward risk into an analysis by replacing expected value as a decision criterion by something else which weights less desirable outcomes more heavily. Thus, we might replace the expected value of alternative A

$$\mathrm{E}(x|A) = \sum_{i=1}^{n} x_i p(x_i|A)$$

as a decision criterion by the expected value of some *utility function* $u(x)$, that is,

$$\mathrm{E}[u(x)|A] = \sum_{i=1}^{n} u(x_i) p(x_i|A)$$

where $p(x_i|A)$ is the probability of x_i given that A is selected.

If x is total assets in hundreds of thousands of dollars, then we might have $u(x) = \log(x + 1)$. With this utility function, higher asset positions will not receive as much weight as with expected value and very low asset positions will receive large negative weight. This will tend to favor alternatives that have lower risk even if they also have lower expected values.

The certainty equivalent CE can be determined if a utility function $u(x)$ is known using the relationship $u(\mathrm{CE}) = \mathrm{E}[u(x)|A]$ where $\mathrm{E}[u(x)|A]$ is the expectation of the utility for alternative A. As an example, consider again the decision above which has equal chances of either winning \$100,000 or losing \$50,000, and suppose that the decision maker's initial asset position is \$100,000. The expected value of this alternative in terms of total assets is $0.5 \times \$200,000 + 0.5 \times \$50,000 = \$125,000$. Using the logarithmic utility function shown in the preceding paragraph, we can solve for the certainty equivalent from $\log(\mathrm{CE}+1) = 0.5\log(2+1) + 0.5\log(0.5+1)$ which gives $\mathrm{CE} = \$112,000$. Thus, the alternative has a certainty equivalent which is \$13,000 less than the expected value of \$125,000 when it is evaluated with the utility function. This demonstrates the aversion to taking risks that was discussed above.

This is the basic idea underlying utility functions. Appendix A presents the theoretical basis in more detail

2. Attitude Toward Risk Taking and Utility Function Shapes

Someone who prefers to receive the expected value of an uncertain alternative for certain rather than the uncertain alternative is called *risk averse*, while someone who finds receiving the expected value for certain to be equally preferred to the alternative is called *risk neutral*, and someone who prefers to receive the alternative rather than the expected value for certain is called *risk seeking*. The most common attitude toward risk taking in business decision making is to be risk neutral for decisions with small risks and to be risk averse for decisions with larger risks. Deliberate risk seeking behavior is sometimes seen in entrepreneurs ("I can always go back to working for somebody else if it doesn't work out") or in situations where you have to "pray for rain" because the situation is already so desperate that you are going to be in serious trouble if you don't have a miracle. Of course, what constitutes a "small risk" may differ depending on the size of the company. (A vice president of a Fortune 500 company once commented to me, "Most of the decisions we analyze are for a few million dollars. It is adequate to use expected value for these." Whether this is true or not depends on your asset position.)

A decision maker's attitude toward risk taking determines the shape of his or her utility function. The remainder of this paper presents various results relating the utility function shape to attitude toward risk taking. It is not necessary to understand the details of the proofs of these results in order to use utility functions, but the results themselves are important for applying decision analysis. (The proofs are indented slightly and set in smaller type.)

Theorem 1 *(Utility Function Shapes). If risk aversion holds for all alternatives which have outcomes within some range over x then $u(x)$ is concave downward over that range of x. [That is, $d^2u(x)/dx^2 < 0$ over the range.] Similarly, risk neutrality implies that $u(x)$ is linear [that is $d^2u(x)/dx^2 = 0$], and risk seeking behavior implies that $u(x)$ is convex downward [that is, $d^2u(x)/dx^2 > 0$.]*

Proof. The result will be shown for the risk averse case, and the other cases can be proved in a similar way. The demonstration proceeds by showing that for a particular type of uncertain alternative the utility function must be concave to yield the required behavior. Since this particular type of alternative is one of the class of "all alternatives which have outcomes within some range of x," this establishes the desired result.

Consider an alternative with possible outcomes $\bar{x} - \delta$ and $\bar{x} + \delta$ which are equally likely. If the decision maker is risk averse, then this alternative must be less preferred than receiving the expected value $\bar{x}$ of the alternative. Thus, it must be true that

$$u(\bar{x}) > (1/2)[u(\bar{x} - \delta) + u(\bar{x} + \delta)].$$

Write the right hand side of this relation as a Taylor expansion around $\bar{x}$. This leads to

$$\begin{aligned} u(\bar{x}) > (1/2)[u(\bar{x}) - \frac{du(\bar{x})}{dx}\delta + (1/2)\frac{d^2u(\bar{x})}{dx^2}\delta^2 + \cdots \\ + u(\bar{x}) + \frac{du(\bar{x})}{dx}\delta + (1/2)\frac{d^2u(\bar{x})}{dx^2}\delta^2 + \cdots]. \end{aligned}$$

In the limit as δ approaches 0, this becomes

$$u(\bar{x}) > u(\bar{x}) + (1/2)\frac{d^2u(\bar{x})}{dx^2}\delta^2$$

which only holds if $d^2u(\bar{x})/dx^2 < 0$. Since $\bar{x}$ can be any level of x, this establishes the desired result. ■

Knowing that a decision maker is risk averse can substantially restrict the shape of a utility function. For example, suppose a decision maker has monotonically increasing preferences, is risk averse, and has a certainty equivalent of 3 for an alternative with a 50:50 chance of yielding either 0 or 10. Then, the decision maker's utility function is restricted to the dotted region of Figure 1a, since otherwise the utility function would have to be convex downward in some region. This example shows that if the decision maker is not very risk averse (for example, the certainty equivalent for the 50:50 chance of 0 or 10 does not differ too much from the expected value of 5), then the possible region within which the utility function can fall is substantially restricted. This leads naturally to considering whether there is a simple form for the utility function that provides an adequate approximation for such situations—perhaps the curve shown in Figure 1b. In fact, there is such a simple form—the exponential—and we will investigate it in the next section. Keeney and Raiffa (1976, Chapter 4) consider related issues.

3. Constant Risk Aversion and the Exponential Utility Function

The theoretical basis for the exponential utility function is a condition called *constant risk aversion.* This condition holds if it is true that whenever all possible outcomes of any uncertain alternative are changed by the same specified amount the decision maker's certainty equivalent for the alternative also changes by that same amount.

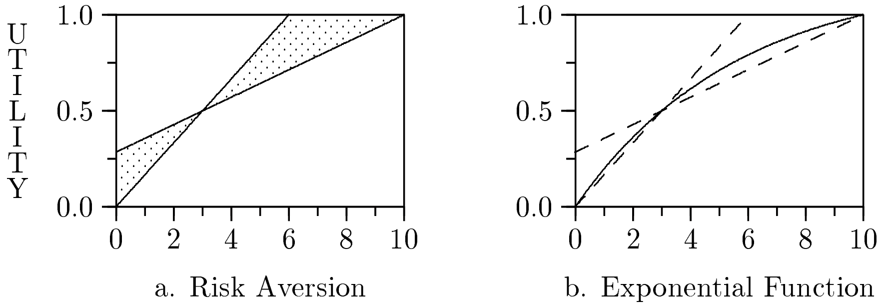

Figure 1. Risk Aversion and Utility Function Shapes

Theorem 2 *(Constant Risk Aversion). Constant risk aversion holds if and only if $u(x)$ has either an exponential or linear shape. (Pratt 1964)*

Proof. Showing that a linear or exponential utility function form implies that constant risk aversion holds is straightforward. We will show that constant risk aversion implies that one of these forms holds, and will only consider the case where preferences are monotonically increasing with respect to the evaluation measure. (The proof for the monotonically decreasing case is analogous.) Additional notation is useful for this proof. The *risk premium* for an alternative is the difference between the expected value $\bar{x}$ for the alternative and its certainty equivalent CE. Specifically, for an evaluation measure with monotonically increasing preferences, the risk premium π is given by $\pi = \bar{x} - \mathrm{CE}$ while with monotonically decreasing preferences it is $\pi = \mathrm{CE} - \bar{x}$. Thus, the risk premium is positive for a risk averse decision maker, 0 for one who is risk neutral, and negative for one who is risk seeking.

Consider two alternatives related in the manner given in the definition of constant risk aversion. That is, the second alternative differs from the first only by having the same amount either added to or subtracted from each outcome. It follows directly from the definition of the risk premium π that the risk premiums must be the same for the two alternatives if constant risk aversion holds. Furthermore, if z is defined by $z = x - \bar{x}$, then z has the same probability distribution for both alternatives.

Suppose that the certainty equivalent of an uncertain alternative is CE. Then it must be true that $u(\mathrm{CE}) = \mathrm{E}[u(x)]$. This can be rewritten in terms of π and z as $u(\bar{x} - \pi) = \mathrm{E}[u(\bar{x} + z)]$.

Now Taylor expand both sides of this equation around $\bar{x}$. The left side becomes

$$u(\bar{x} - \pi) = u(\bar{x}) - \frac{du(\bar{x})}{dx}\pi + (1/2)\frac{d^2u(\bar{x})}{dx^2}\pi^2 + \cdots$$

and the right side becomes

$$\begin{aligned}\mathrm{E}[u(\bar{x} + z)] &= \mathrm{E}[u(\bar{x}) + \frac{du(\bar{x})}{dx}z + (1/2)\frac{d^2u(\bar{x})}{dx^2}z^2 + \cdots] \\ &= u(\bar{x}) + \frac{du(\bar{x})}{dx}\mathrm{E}(z) + (1/2)\frac{d^2u(\bar{x})}{dx^2}\mathrm{E}(z^2) + \cdots\end{aligned}$$

Since $\bar{x}$ is the expected value for the alternative and $z = x - \bar{x}$, then $\mathrm{E}(z) = 0$ and $\mathrm{E}(z^2) = \sigma^2$ where σ^2 is the variance for the alternative.

Using these facts, equating the right-hand and left-hand Taylor expansions, and dropping common terms leads to

$$-\frac{du(\bar{x})}{dx}\pi + (1/2)\frac{d^2u(\bar{x})}{dx^2}\pi^2 + \cdots = (1/2)\frac{d^2u(\bar{x})}{dx^2}\sigma^2 + \cdots$$

Now assume a situation with "small" risk aversion so that $\pi \ll \sigma$ and where the uncertainty is small enough that only terms through second order need be considered in the Taylor expansion. Then the equation above reduces to

$$-\frac{du(\bar{x})}{dx}\pi = (1/2)\frac{d^2u(\bar{x})}{dx^2}\sigma^2$$

or

$$\pi = -(1/2)\frac{d^2u(\bar{x})/dx^2}{du(\bar{x})/dx}\sigma^2 \qquad (1)$$

With constant risk aversion, π and σ will not change when a constant amount is added to each possible outcome of an alternative. However, $\bar{x}$ will change by the constant amount. Thus, for Equation 1 to hold, it must be true that

$$\frac{d^2u(x)/dx^2}{du(x)/dx} = -c$$

for some constant c. (Otherwise the right hand side of Equation 1 will vary as $\bar{x}$ changes, and hence constant risk aversion will not hold.) This is a second-order linear constant-coefficient differential equation, and the solution is

$$u(x) = \begin{cases} a + b\exp(-cx), & c \neq 0 \\ a + bx, & c = 0 \end{cases}$$

where a and b are undetermined constants.

From Theorem A-2 in Appendix A, it follows that the values of a and b do not matter except that b must have the correct sign so that preferences either increase or decrease as is appropriate for the evaluation measure of interest. ■

The usual convention is for a utility function to be scaled so that the least preferred level of the evaluation measure that is being considered has a utility of zero and the most preferred level being considered has a utility of one. With these conventions, if preferences are *monotonically increasing* over x (that is, larger amounts of x are preferred to smaller amounts), then the exponential utility function can be written

$$u(x) = \begin{cases} \dfrac{\exp[-(x - \text{Low})/\rho] - 1}{\exp[-(\text{High} - \text{Low})/\rho] - 1}, & \rho \neq \text{Infinity} \\[2ex] \dfrac{x - \text{Low}}{\text{High} - \text{Low}}, & \text{otherwise,} \end{cases} \qquad (2a)$$

and if preferences are *monotonically decreasing* over x, then

$$u(x) = \begin{cases} \dfrac{\exp[-(\text{High} - x)/\rho] - 1}{\exp[-(\text{High} - \text{Low})/\rho] - 1}, & \rho \neq \text{Infinity} \\[2ex] \dfrac{\text{High} - x}{\text{High} - \text{Low}}, & \text{otherwise,} \end{cases} \qquad (2b)$$

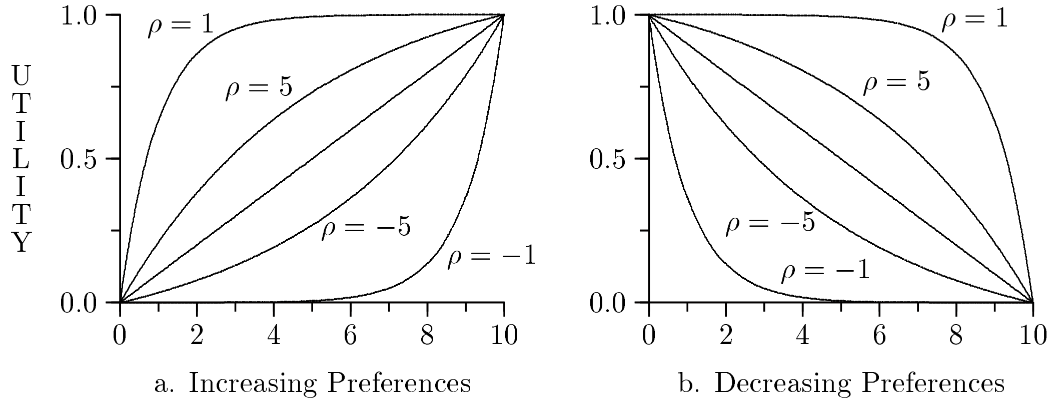

Figure 2. Exponential Utility Functions

where "Low" is the lowest level of x that is of interest, "High" is the highest level of interest, and ρ (rho) is the *risk tolerance* for the utility function. The utility functions in (2a) and (2b) are scaled so that they vary between 0 and 1 over the range from x = Low to x = High. That is, for monotonically increasing preferences, $u(\text{Low}) = 0$ and $u(\text{High}) = 1$, while for monotonically decreasing preferences $u(\text{Low}) = 1$ and $u(\text{High}) = 0$. (The proof of Theorem 1 used the *risk aversion coefficient* c rather than the risk tolerance ρ. The two are related by the equation $c = 1/\rho$.)

The curve in Figure 1b is an exponential utility function of the form of Equation 2a which goes through the three specified values for the utility function. Generally, ρ is of the order of magnitude of the range from Low to High. Specifically, if $\rho < 0.1\times(\text{High}-\text{Low})$ then the utility function displays highly risk averse behavior—so risk averse that the assumption of constant risk aversion is very suspect—while if $\rho > 10 \times (\text{High} - \text{Low})$ the utility function is essentially linear, and you might as well use the linear form (that is, use expected value as a decision criterion).

Figure 2 shows examples of exponential utility functions for different risk tolerances. In both parts of this figure, the range of values for the evaluation measure is from Low = 0 to High = 10. Part a of the figure shows functions with monotonically increasing preferences (corresponding to Equation 2a), and part b shows functions with monotonically decreasing preferences (corresponding to Equation 2b). The unlabeled straight line in the center of each part of the figure corresponds to the case when ρ = Infinity. We see from the figure that as the magnitude of the risk tolerance increases, the utility function becomes more linear.

With some algebraic manipulation, it can be shown that when Equation 2a holds the certainty equivalent CE for an uncertain alternative is

$$\text{CE} = \begin{cases} -\rho \ln \text{E}[\exp(-x/\rho)], & \rho \neq \text{Infinity} \\ \text{E}(x), & \text{otherwise} \end{cases} \tag{3a}$$

and when Equation 2b holds

$$\text{CE} = \begin{cases} \rho \ln \text{E}[\exp(x/\rho)], & \rho \neq \text{Infinity} \\ \text{E}(x), & \text{otherwise} \end{cases} \tag{3b}$$

4. Determining the Risk Tolerance

It is straightforward to show that ρ is approximately equal to the X such that an alternative with equal chances of winning X or losing $X/2$ has a certainty equivalent of zero. It is also true that ρ is approximately equal to the X' such that an alternative with a 0.75 chance of winning X' and a 0.25 chance of losing X' has a certainty equivalent of zero.

Howard (1988) gives some rules of thumb for the size of ρ as a function of certain financial measures of a company. He has found values of ρ which are about six percent of net sales, about 100 to 150 percent of net income, and about one-sixth of equity. (These figures were derived from companies in the oil and chemicals industries.) McNamee and Celona (1990) comment that a ratio of risk tolerance to equity of one-sixth or of risk tolerance to market value of one-fifth seems to translate best across companies in different industries.

The value of ρ can be more precisely determined by eliciting the certainty equivalent for an uncertain alternative and then solving for the ρ which gives this same certainty equivalent. For example, consider a situation with monotonically increasing preferences where the certainty equivalent for an alternative with equal chances of Low or High is CE. From Equation 2a, if CE $=$ (Low $+$ High)/2, then $\rho =$ Infinity; otherwise it is the solution to

$$0.5 = \frac{\exp[-(\mathrm{CE} - \mathrm{Low})/\rho] - 1}{\exp[-(\mathrm{High} - \mathrm{Low})/\rho] - 1} \tag{4}$$

This equation must be solved numerically.

Appendix B includes a table of solution values for Equation 4. To use this table for a situation with monotonically increasing preferences, set $z_{0.5} = (\mathrm{CE} - \mathrm{Low})/(\mathrm{High} - \mathrm{Low})$ and look this up in the table to find a corresponding value for R. Then, $\rho = R \times (\mathrm{High} - \mathrm{Low})$. For monotonically decreasing preferences, the same procedure is followed, except that $z_{0.5}$ is determined by $z_{0.5} = (\mathrm{High} - \mathrm{CE})/(\mathrm{High} - \mathrm{Low})$. For example, suppose that preferences are monotonically decreasing, and that the certainty equivalent for an alternative with equal chances of \$10,000 and \$5,000 is \$8,500. Then $z_{0.5} = (\$10,000 - \$8,500)/(\$10,000 - \$5,000) = 0.3$. Looking this entry up in the Appendix B table, we see that $R = 0.56$, and hence $\rho = 0.56 \times (\$10,000 - \$5,000) = \$2,800$.

It is straightforward to enter Equation 2 into an electronic spreadsheet or a programmable calculator to determine values for $u(x)$ once ρ is known. The table in Appendix C can also be used to find $u(x)$. (A Pascal computer program to solve Equation 4 is given in Appendix D.)

5. Portfolios of Independent Alternatives

Suppose that you have a *portfolio* of n different alternatives where the total outcome s from all the alternatives is the sum of the outcomes $x_1, x_2, \ldots, x_n$ from the different alternatives. When constant risk aversion holds and the alternatives are probabilistically independent, the certainty equivalent CE for the portfolio is the sum of the certainty equivalents $\mathrm{CE}_1, \mathrm{CE}_2, \ldots, \mathrm{CE}_n$ for the individual alternatives.

Proof. Assume that the exponential case of Equation 2a holds. (The proofs for the other cases are analogous.) Then from Equation 3a, $\mathrm{CE} = -\rho \ln \mathrm{E}[\exp(-s/\rho)] = -\rho \ln \mathrm{E}\{\exp[-(\sum_{i=1}^{n} x_i)/\rho]\} = -\rho \ln \mathrm{E}[\prod_{i=1}^{n} \exp(-x_i/\rho)]$. Since the x_i are probabilistically independent, this reduces to $\mathrm{CE} = \sum_{i=1}^{n} -\rho \ln \mathrm{E}[\exp(-x_i/\rho)]$. By similar reasoning, $\mathrm{CE}_i = -\rho \ln \mathrm{E}[\exp(-x_i/\rho)]$. Comparing these two equations, we see that $\mathrm{CE} = \sum_{i=1}^{n} \mathrm{CE}_i$. ■

6. The Value of Perfect Information

A straightforward argument shows that when constant risk aversion holds the value of perfect information is the difference between the certainty equivalent of the perfect information alternative *ignoring the cost of the information* and the certainty equivalent of the best alternative without perfect information. The following argument demonstrates that this is true: The perfect information alternative *taking into account the cost of the information* differs from the perfect information alternative ignoring the cost of the information only by having a constant amount (the cost of the information) subtracted from each possible outcome. Thus, from the definition of constant risk aversion, the certainty equivalent for the perfect information alternative including the cost of information must be equal to the certainty equivalent of the perfect information alternative ignoring the cost of the information minus the cost of the information. Hence, the value of perfect information can be determined by taking the difference between the certainty equivalents of the perfect information alternative ignoring the cost of the information and the best alternative without perfect information.

Note that this is not true for all utility functions. Most introductory decision analysis textbooks restrict themselves to treating the value of information only when expected value is used as a decision criterion. Since this is a special case of constant risk aversion, the value of perfect information can be determined as specified in the preceding paragraph. However, it is easy to construct counterexamples which show that this procedure does not give the correct answer for utility functions that are neither exponential nor linear.

7. Approximations Using Exponential Utility Functions

From Equation 1, it follows that when constant risk aversion holds the certainty equivalent CE for an alternative with monotonically increasing preferences is given approximately by

$$\mathrm{CE} = \bar{x} - \frac{\sigma^2}{2\rho} \tag{5}$$

where $\bar{x}$ is the expected value of the alternative, and σ^2 is the variance of the alternative. (With monotonically decreasing preferences, the equation is the same except that the minus sign is changed to a plus sign.) It is possible to show by direct calculation that Equation 5 is exact for an exponential utility function when an alternative has a Normal (Gaussian) probability distribution for its outcomes. This section presents the results of some empirical studies which show that Equation 5 can be a fairly accurate approximation even when the alternative has a distribution that is not very Normal. However, before presenting these results, note that in situations where Equation 5 is valid, it can be used to make some statements about the types of decisions where considering risk aversion can change the preferred decision.

Howard [1988] notes,

> While the ability to capture risk preference is an important part of our conceptual view of decision-making, I find it is a matter of real practical concern in only 5 percent to 10 percent of business decision analysis. Of course, the situations that require risk preference, such as bidding or portfolio problems, use it very seriously.

Equation 5 gives a basis for this observation. Using this equation, the difference between the expected value and the certainty equivalent is $\sigma^2/(2\rho)$. Hence, the ranking of alternatives will be impacted by risk attitude (which is encoded by ρ) only if σ^2 differs among the alternatives. That is, the *amount* of uncertainty (as measured by σ^2) must differ among the alternatives for risk attitude to impact the ranking of alternatives. Otherwise, the term $\sigma^2/(2\rho)$ merely adds the same constant correction to the expected value for each alternative, and hence expected value will correctly rank alternatives even though it will give incorrect certainty equivalents.

The accuracy of Equation 5 is investigated below for two families of probability distributions over x: asymmetric two-fork lotteries (that is, distributions with two possible outcomes and unequal probabilities for the two outcomes) and beta distributions. These two distribution families were studied because they provide considerable flexibility regarding specific distribution shapes and because together they cover situations that are representative of those seen in practice. The results of the accuracy studies are shown in Table 1 for the asymmetric two-fork lottery, and in Table 2 for the beta distribution.

To study the accuracy of the approximations, it is first necessary to establish scales for measuring errors. For the two fork lottery, two measures are used: the error as a percentage of the range between the upper and lower fork values and the error as a percentage of the standard deviation of the lottery. (The standard deviation is used for comparison because it is a commonly used summary of the uncertainty represented by a probability distribution.)

In Table 1, "Range" is the difference between the values of x for the higher and lower forks of the lottery, $p(x^*)$ is the probability of the higher value, c is the risk aversion coefficient, σ is the standard deviation of x for the two fork lottery, $c\sigma$ is the product of c and the standard deviation (from the Taylor expansion development in Theorem 2, Equation 5 will certainly be an accurate approximation when $c\sigma \ll 1$), x_{rp} is the exact risk premium for the two-fork lottery using Equation 2a, $\sim x_{rp}$ is the approximate risk premium calculated using Equation 5, and the last two columns give the differences between the approximate and exact risk premiums as a percent of the range and standard deviation of the lottery respectively. These two ratios summarize the accuracy of the approximation relative to the uncertainty in the two fork lottery. (For smaller values of the ratios, the approximation is more accurate.)

The upper half of Table 1 gives the maximum values of c for which the error using Equation 5 is less than ten percent of the standard deviation of x for various combinations of Range and $p(x^*)$, and the lower half of the table gives the maximum values of c for which the error is less than twenty percent of the standard deviation. (These maximum values are given in the column labeled "max c.") This table shows that for $c\sigma$ approaching or even exceeding 1 Equation 5 is still a reasonably accurate approximation; the values of c for which Equation 5 is accurate cover many situations of practical interest. Note, in particular, that the approximation is accurate for fairly large values of c even when the two-fork lottery is highly skewed and not at all “normal-like” in shape.

In Table 2, the first column gives the exact expected value of the beta distribution over x, σ is the exact standard deviation, c is the risk aversion coefficient, $c\sigma$ is the product of c and the standard deviation, x_{ce} is the exact certainty equivalent for the beta distribution, $\sim x_{ce}$ is the approximate certainty equivalent calculated using Equation 5, and the last two columns show the differences between the exact and approximate certainty equivalents as a percentage of the range (from 0 to 1) and standard deviation of the beta distribution.

The upper half of Table 2 give the maximum values of c for which the error using Equation 5 is less than ten percent of the standard deviation of the distribution for various combinations of expected values and standard deviations, while the lower half of the table gives the maximum values of c for which the error is less than twenty percent of the standard deviation. (As in Table 1, these maximum values are in the column labeled “max c.”) This table shows that Equation 5 is accurate for many values of c likely to be assessed in practice for a wide range of shapes for the beta distribution. Many of the beta distributions for which results are presented are extreme cases that are not at all “normal-like.” Entries in Table 2 marked with one asterisk are one-tailed distributions with a maximum at either $x = 0$ or $x = 1$, while entries marked with two asterisks have a “bathtub” shape where the distribution increases as $x = 0$ and $x = 1$ are approached. Even with such skewed distributions, Table 2 shows that Equation 5 is accurate for practically useful values for c.

The results in Tables 1 and 2 can be summarized as follows: For the two distributions studied, Equation 5 is accurate even for probability distributions over x which are not very “normal” in shape. Some cases shown in the tables are more skewed than most situations of interest in practice. Therefore, the fact that the approximation is accurate in these cases indicates that it may be accurate in many practical situations.

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Range	$p(x^*)$	max c	σ	$c\sigma$	x_{rp}	$\sim x_{rp}$	$(\sim x_{rp} - x_{rp})/$ Range	σ
Ten Percent Error Limits on c								
0.25	0.2	6.6	0.100	0.66	0.023	0.033	3.9%	9.7%
0.25	0.4	7.7	0.122	0.94	0.046	0.058	4.8%	9.8%
0.25	0.6	12.5	0.122	1.53	0.082	0.094	4.8%	9.8%
0.25	0.8	6.7	0.100	0.67	0.043	0.034	-3.9%	-9.7%
0.50	0.2	3.3	0.200	0.66	0.047	0.066	3.9%	9.7%
0.50	0.4	3.8	0.245	0.93	0.091	0.114	4.7%	9.6%
0.50	0.6	6.2	0.245	1.52	0.163	0.186	4.6%	9.5%
0.50	0.8	3.3	0.200	0.66	0.085	0.066	-3.8%	-9.5%
0.75	0.2	2.2	0.300	0.66	0.070	0.099	3.9%	9.7%
0.75	0.4	2.5	0.367	0.92	0.135	0.169	4.6%	9.3%
0.75	0.6	4.1	0.367	1.51	0.243	0.277	4.5%	9.2%
0.75	0.8	2.2	0.300	0.66	0.128	0.099	-3.8%	-9.5%
Twenty Percent Error Limits on c								
0.25	0.2	9.9	0.100	0.99	0.030	0.050	8.0%	19.9%
0.25	0.4	10.8	0.122	1.32	0.057	0.081	9.7%	19.8%
0.25	0.6	15.7	0.122	1.92	0.093	0.118	9.7%	19.8%
0.25	0.8	11.3	0.100	1.13	0.076	0.057	-8.0%	-19.9%
0.50	0.2	4.9	0.200	0.98	0.059	0.098	7.8%	19.6%
0.50	0.4	5.4	0.245	1.32	0.114	0.162	9.7%	19.8%
0.50	0.6	7.8	0.245	1.91	0.186	0.234	9.5%	19.4%
0.50	0.8	5.6	0.200	1.12	0.151	0.112	-7.9%	-19.7%
0.75	0.2	3.3	0.300	0.99	0.089	0.149	8.0%	19.9%
0.75	0.4	3.6	0.367	1.32	0.170	0.243	9.7%	19.8%
0.75	0.6	5.2	0.367	1.91	0.280	0.351	9.5%	19.4%
0.75	0.8	3.7	0.300	1.11	0.225	0.167	-7.8%	-19.6%

Table 1. Two-Fork Lottery Test of Equation 5

$\bar{x}$	σ	max c	$c\sigma$	x_{ce}	$\sim x_{ce}$	$(\sim x_{ce} - x_{ce})$	$(\sim x_{ce} - x_{ce})/\sigma$
Ten Percent Error Limits on c							
0.10	0.05	18.6	0.93	0.082	0.077	-0.5%	-9.9%
0.10	0.09*	8.1	0.73	0.076	0.067	-0.9%	-9.8%
0.25	0.10	11.3	1.13	0.203	0.194	-1.0%	-9.9%
0.25	0.20*	4.2	0.84	0.185	0.166	-1.9%	-9.6%
0.50	0.20	8.1	1.62	0.358	0.338	-2.0%	-9.8%
0.50	0.30†	4.5	1.35	0.326	0.298	-2.9%	-9.7%
0.50	0.40†	2.8	1.12	0.315	0.276	-3.9%	-9.6%
0.75	0.10	11.0	1.10	0.685	0.695	1.0%	9.8%
0.75	0.20*	4.3	0.86	0.644	0.664	2.0%	9.9%
0.90	0.05	15.2	0.76	0.876	0.881	0.5%	9.9%
0.90	0.09*	6.1	0.55	0.866	0.875	0.9%	9.9%
Twenty Percent Error Limits on c							
0.10	0.05	27.7	1.39	0.075	0.065	-1.0%	-19.9%
0.10	0.09*	12.3	1.11	0.068	0.050	-1.8%	-19.9%
0.25	0.10	16.5	1.65	0.187	0.168	-2.0%	-19.9%
0.25	0.20*	6.3	1.26	0.164	0.124	-4.0%	-19.8%
0.50	0.20	10.8	2.16	0.323	0.284	-3.0%	-19.7%
0.50	0.30†	6.0	1.80	0.288	0.230	-5.8%	-19.3%
0.50	0.40†	3.8	1.52	0.273	0.196	-7.7%	-19.2%
0.75	0.10	16.3	1.63	0.649	0.669	2.0%	19.8%
0.75	0.20*	7.0	1.40	0.570	0.610	4.0%	19.8%
0.90	0.05	20.5	1.03	0.864	0.874	1.0%	19.9%
0.90	0.09*	8.4	0.76	0.848	0.866	1.8%	19.7%

* Single Tailed Distribution

† "Bathtub" Distribution

Table 2. Beta Distribution Test of Equation 5

Pratt, J. W. 1964. Risk Aversion in the Small and in the Large. *Econometrica,* **32**, 122–136.

Pratt, J. W., H. Raiffa and R. O. Schlaifer. 1964. The Foundations of Decision Under Uncertainty: An Elementary Exposition. *Journal of the American Statistical Association* **59**, 353–375.

Exercises

1. An alternative has a probability 0.6 of winning \$25,000, 0.2 of winning \$1,000, and 0.2 of losing \$50,000. Determine the expected profit for this alternative. Determine the certainty equivalent for the alternative using the utility function $u(x) = -e^{-x/10}$, where x is in thousands of dollars.

2. For a decision maker with risk averse, monotonically increasing preferences, determine whether it is possible to select which of the following is more preferred: An alternative with equal chances of yielding 50, 25, 0, or -10; or an alternative that is certain to yield 16.25.

3. For a decision maker with monotonic, risk averse preferences and $u(-100) = 10$, $u(0) = 6$, and $u(100) = 0$, determine the possible expected utilities for an alternative with a 0.6 probability of 0 and a 0.4 probability of 50.

4. Assume the same situation as in problem 3 except that preferences are constantly risk averse. Determine the expected utility of the alternative.

5. Suppose preferences are constantly risk averse, and the certainty equivalent for an alternative with equal chances of winning X or losing $X/2$ is zero. Determine the percentage error from assuming that the risk tolerance is equal to X.

6. Suppose preferences are constantly risk averse, and the certainty equivalent for an alternative with a 0.75 probability of winning X' and a 0.25 probability of losing X' is zero. Determine the percentage error from assuming that the risk tolerance is equal to X'.

7. Show that with constantly risk averse, monotonically increasing preferences the certainty equivalent CE for an alternative is

$$\mathrm{CE} = \begin{cases} -\rho \ln \mathrm{E}[\exp(-x/\rho)], & \rho \neq \text{Infinity} \\ \mathrm{E}(x), & \text{otherwise} \end{cases}$$

and that with constantly risk averse, monotonically decreasing preferences it is

$$\mathrm{CE} = \begin{cases} \rho \ln \mathrm{E}[\exp(x/\rho)], & \rho \neq \text{Infinity} \\ \mathrm{E}(x), & \text{otherwise} \end{cases}$$

8. Assume that a company has annual net sales of \$5 billion. Using Howard's rules-of-thumb, determine the company's risk tolerance. Assuming constant risk aversion and this risk tolerance, determine the certainty equivalent for an alternative with equal chances of winning \$10 million or losing \$5 million. Determine the percent error in the certainty equivalent that results from assuming infinite risk tolerance.

9. For the risk tolerance and alternative in problem 8, use the equation $\mathrm{CE} = \bar{x} - \sigma^2/(2\rho)$ to determine an approximate certainty equivalent. Compare this approximate certainty equivalent to the exact certainty equivalent determined in problem 8.

10. Suppose a decision maker is constantly risk averse with monotonically increasing preferences and a risk tolerance of 20. Consider an alternative with a 0.8

probability of winning 50 and a 0.2 probability of losing 10. Consider another alternative with equal chances of winning 10 or 25. Show by direct calculation that if the two alternatives are probabilistically independent, then the certainty equivalent for the sum of these two alternatives is equal to the sum of the certainty equivalents for the alternatives.

11. For the decision in problem 10, find the error in the certainty equivalent for the sum of the alternatives when using the approximation $\mathrm{CE} = \bar{x} - \sigma^2/(2\rho)$ for each alternative. Specifically, find the percent error from using the approximation relative to the exact certainty equivalent determined in problem 10 and also relative to the standard deviation for the sum of the two alternatives.

12. For the decision in problem 10, now assume that $u(x) = \sqrt{x + 10}$. Show that the certainty equivalent for the sum of the two alternatives is *not* equal to the sum of the certainty equivalents for the alternatives.

13. A decision problem has two alternatives, one of which yields 10 for certain and one of which has equal chances of yielding 5 or 15. The decision maker has constantly risk averse, monotonically decreasing preferences with a risk tolerance of 10. Show by direct calculation that the value of perfect information about the outcome of the uncertain alternative is the difference between the certainty equivalent for the perfect information alternative *ignoring the cost of the information* and the certainty equivalent of the preferred alternative without perfect information.

14. For the decision problem in problem 13, show that the stated procedure for determining the value of perfect information does not give the correct result if $u(x) = \ln(20 - x)$.

15. Show that if an alternative has a Normal probability distribution and preferences are risk averse and monotonically increasing, then the certainty equivalent is given by $\mathrm{CE} = \bar{x} - \sigma^2/(2\rho)$.

16. The *selling price* of an alternative is the minimum amount for which a decision maker who owns the alternative will sell it, while the *buying price* is the maximum amount which a decision maker who does not own the alternative will pay to buy it. From the definitions, it follows that the selling price is equal to the certainty equivalent and that the buying price is the amount which, when subtracted from each outcome of the alternative, yields a certainty equivalent of zero for the alternative. Show that when constant risk aversion holds the buying price for any alternative is equal to the selling price.

17. Consider an alternative with equal chances of yielding 0 and 10. Show that if $u(x) = \sqrt[3]{x/10}$ then the buying price for this alternative is not equal to the selling price.

1/23/97

Appendix A. Two Fundamental Theorems

This appendix presents the axioms of consistent choice and two fundamental theorems which result from these axioms. In this appendix, the symbol "$\succ$" means "is preferred to," and the *consequences* of a decision are designated $c_1, c_2, \ldots, c_n$. Note that these consequences may themselves be uncertain alternatives. These axioms assume that probabilities exist and that the rules of probability apply. Pratt, Raiffa, and Schlaifer (1964) present a more extensive set of axioms which develops probability from first principles. Here are the axioms of consistent choice:

1. (Transitivity) If $c_i \succ c_j$ and $c_j \succ c_k$, then $c_i \succ c_k$.
2. (Reduction) If the standard rules of probability can be used to show that two alternatives have the same probability for each c_i, then the two alternatives are equally preferred.
3. (Continuity) If $c_i \succ c_j \succ c_k$, then there is a p such that an alternative with a probability p of yielding c_i and a probability $1-p$ of yielding c_k is equally preferred to c_j.
4. (Substitution) If two consequences are equally preferred, then one can be substituted for the other in any decision without changing the preference ordering of alternatives.
5. (Monotonicity) For two alternatives which each yield either c_i or c_j where $c_i \succ c_j$, then the first alternative is preferred to the second if it has a higher probability of yielding c_i.

If these conditions hold, then it is possible to prove the following theorem.

Theorem A-1 *(Expected Utility). If the axioms of consistent choice hold, then there exists a function $u(c_i)$ such that alternative A is preferred to alternative B if*

$$\sum_{i=1}^{n} p(c_i|A)u(c_i) > \sum_{i=1}^{n} p(c_i|B)u(c_i) \qquad (A-1)$$

where $p(c_i|A)$ is the probability of c_i if A is selected, and $p(c_i|B)$ is the probability of c_i if B is selected.

Proof. The following steps demonstrate the desired result:

1. Using the Transitivity Axiom, the consequences can be rank-ordered in terms of preferability. Suppose the consequences are labeled so that $c_1 \succ c_2 \succ \cdots \succ c_n$.
2. By the Reduction Axiom, any uncertain alternative has an equally preferred alternative which directly yields the outcomes $c_1, c_2, \ldots, c_n$. Suppose the equally preferred alternative for A has probabilities $p(c_1|A), p(c_2|A), \ldots, p(c_n|A)$ of yielding $c_1, c_2, \ldots, c_n$ respectively, and the equally preferred alternative for B has probabilities $p(c_1|B), p(c_2|B), \ldots, p(c_n|B)$ of yielding $c_1, c_2, \ldots, c_n$. Then by the Substitution Axiom, the original alternatives can be replaced by their equally preferred reduced equivalents. Make this replacement.
3. By the Continuity Axiom, there is a number $u(c_i)$ such that c_i is equally preferred to an alternative with a probability $u(c_i)$ of yielding c_1 and a probability

$1 - u(c_i)$ of yielding c_n. Thus, by the Substitution Axiom, each c_i can be replaced by the equally preferred alternative which has a probability $u(c_i)$ of yielding c_1 and a probability $1 - u(c_i)$ of yielding c_n. Make this substitution.

5. By the Reduction Axiom, A is equally preferred to an alternative with a probability $\sum_{i=1}^{n} p(c_i|A)u(c_i)$ of yielding c_1 and a probability $1 - \sum_{i=1}^{n} p(c_i|A)u(c_i)$ of yielding c_n. Similarly, B is equally preferred to an alternative with a probability $\sum_{i=1}^{n} p(c_i|B)u(c_i)$ of yielding c_1 and a probability $1 - \sum_{i=1}^{n} p(c_i|B)u(c_i)$ of yielding c_n. Thus, by the Substitution Axiom, A and B can be replaced by these alternatives which have outcomes which only include c_1 and c_n. Make this substitution.

6. Thus, by the Monotonicity Axiom, $A \succ B$ if

$$\sum_{i=1}^{n} p(c_i|A)u(c_i) > \sum_{i=1}^{n} p(c_i|B)u(c_i).$$

   This is the relationship in Equation A-1. ■

The function $u(c_i)$ is called a *utility function,* and the decision criterion in Theorem A-1 says that expected utility must be used as a decision criterion if the axioms of consistent choice are to be obeyed. These axioms were originally postulated as a model of unaided human decision making behavior. Many experiments have been done to test whether unaided human decision making naturally obeys the axioms. The results have shown that unaided human decision making does *not* obey the axioms of consistent choice. This has led to some questioning of whether the axioms are a good basis for decision analysis procedures, and a number of alternative axiom sets have been developed to better describe unaided human decision making.

However, there is a difference between describing how unaided decision making processes work and using analysis to make better decisions. Our focus here is on making better decisions. From this perspective, it is difficult to argue with the axioms of consistent choice. Each is reasonable, and it is hard to give any of them up as logical principles that we would want our reasoning to obey.

On a more practical level, we are all aware of limitations in human reasoning. Very few of us would trust ourselves to accurately add up a column of 100 numbers in our head. Many decisions under uncertainty are more complex than adding up 100 numbers. Why should we trust our unaided reasoning processes to be more accurate at analyzing these decisions than at adding 100 numbers? Thus, the fact that unaided decision making does not obey the axioms of consistent choice is not a convincing argument that these axioms should not be used as a basis for decision making.

There is another theorem related to utility functions that is useful. The proof of Theorem A-1 shows that $u(c_i)$ must be between 0 and 1 since it is defined as a probability. However, it is not necessary that all utility functions be between 0 and 1, as the following theorem demonstrates.

Theorem A-2 *(Linear Transformation). Given a utility function $u(c_i)$, then another function $u'(c_i)$ is guaranteed to give the same ranking of alternatives if and only if*

$$u'(c_i) = au(c_i) + b \qquad (A-2)$$

for some constants $a > 0$ and b.

Proof. Suppose that $\sum_{i=1}^{n} p(c_i|A)u(c_i) > \sum_{i=1}^{n} p(c_i|B)u(c_i)$. Then, it is also true that $a\sum_{i=1}^{n} p(c_i|A)u(c_i) > a\sum_{i=1}^{n} p(c_i|B)u(c_i)$ for any constant $a > 0$, and hence $a\sum_{i=1}^{n} p(c_i|A)u(c_i) + b > a\sum_{i=1}^{n} p(c_i|B)u(c_i) + b$ for any constant b.

However, since $a\sum_{i=1}^{n} p(c_i|A)u(c_i) + b = \sum_{i=1}^{n} p(c_i|A)[au(c_i) + b]$ and $a\sum_{i=1}^{n} p(c_i|B)u(c_i) + b = \sum_{i=1}^{n} p(c_i|B)[au(c_i) + b]$, which is the same as

$$\sum_{i=1}^{n} p(c_i|A)u'(c_i) > \sum_{i=1}^{n} p(c_i|B)u'(c_i),$$

then the desired result is demonstrated. ■

Appendix B. Calculating Rho

This table presents pairs of numbers $z_{0.5}$ and R which solve the equation

$$0.5 = \frac{\exp(-z_{0.5}/R) - 1}{\exp(-1/R) - 1}$$

This table can be used to solve for the risk tolerance ρ in equation 2 given the certainty equivalent for an alternative with equal chances of yielding either of two specified values. Suppose preferences are monotonically increasing and the certainty equivalent for an alternative with equal chances of yielding 100 or 200 is 140. Then $z_{0.5} = (140 - 100)/(200 - 100) = 0.40$, and the table below shows that $R = 1.22$. Hence, the risk tolerance is $\rho = 1.22 \times (200 - 100) = 122$.

If preferences are monotonically decreasing with the same certainty equivalent, then $z_{0.5} = (200 - 140)/(200 - 100) = 0.60$, and the table shows that $R = -1.22$, so that $\rho = -1.22 \times (200 - 100) = -122$.

$z_{0.5}$	R	$z_{0.5}$	R	$z_{0.5}$	R	$z_{0.5}$	R
0.00	—	0.25	0.41	0.50	Infinity	0.75	-0.41
0.01	0.01	0.26	0.44	0.51	-12.50	0.76	-0.39
0.02	0.03	0.27	0.46	0.52	-6.24	0.77	-0.36
0.03	0.04	0.28	0.49	0.53	-4.16	0.78	-0.34
0.04	0.06	0.29	0.52	0.54	-3.11	0.79	-0.32
0.05	0.07	0.30	0.56	0.55	-2.48	0.80	-0.30
0.06	0.09	0.31	0.59	0.56	-2.06	0.81	-0.29
0.07	0.10	0.32	0.63	0.57	-1.76	0.82	-0.27
0.08	0.12	0.33	0.68	0.58	-1.54	0.83	-0.25
0.09	0.13	0.34	0.73	0.59	-1.36	0.84	-0.24
0.10	0.14	0.35	0.78	0.60	-1.22	0.85	-0.22
0.11	0.16	0.36	0.85	0.61	-1.10	0.86	-0.20
0.12	0.17	0.37	0.92	0.62	-1.00	0.87	-0.19
0.13	0.19	0.38	1.00	0.63	-0.92	0.88	-0.17
0.14	0.20	0.39	1.10	0.64	-0.85	0.89	-0.16
0.15	0.22	0.40	1.22	0.65	-0.78	0.90	-0.14
0.16	0.24	0.41	1.36	0.66	-0.73	0.91	-0.13
0.17	0.25	0.42	1.54	0.67	-0.68	0.92	-0.12
0.18	0.27	0.43	1.76	0.68	-0.63	0.93	-0.10
0.19	0.29	0.44	2.06	0.69	-0.59	0.94	-0.09
0.20	0.30	0.45	2.48	0.70	-0.56	0.95	-0.07
0.21	0.32	0.46	3.11	0.71	-0.52	0.96	-0.06
0.22	0.34	0.47	4.16	0.72	-0.49	0.97	-0.04
0.23	0.36	0.48	6.24	0.73	-0.46	0.98	-0.03
0.24	0.39	0.49	12.50	0.74	-0.44	0.99	-0.01

Appendix C. Normalized Exponential Utility Function

The table below gives values of the function

$$f(z|R) = \frac{\exp(-z/R) - 1}{\exp(-1/R) - 1}$$

for various combinations of z and R. The primary use of this is to determine values for the exponential utility function in equation 2. Suppose we wish to determine $u(150)$ in equation 2 with monotonically decreasing preferences, a range from 100 to 300, and $\rho = 200$. Then set $z = (300 - 150)/(300 - 100) = 0.75$ and $R = 200/(300 - 100) = 1$. From the table below, $u_i(150) = 0.83$.

R	z																		
	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
0.10	0.39	0.63	0.78	0.86	0.92	0.95	0.97	0.98	0.99	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.20	0.22	0.40	0.53	0.64	0.72	0.78	0.83	0.87	0.90	0.92	0.94	0.96	0.97	0.98	0.98	0.99	0.99	1.00	1.00
0.30	0.16	0.29	0.41	0.50	0.59	0.66	0.71	0.76	0.81	0.84	0.87	0.90	0.92	0.94	0.95	0.96	0.98	0.99	0.99
0.40	0.13	0.24	0.34	0.43	0.51	0.57	0.64	0.69	0.74	0.78	0.81	0.85	0.87	0.90	0.92	0.94	0.96	0.97	0.99
0.50	0.11	0.21	0.30	0.38	0.46	0.52	0.58	0.64	0.69	0.73	0.77	0.81	0.84	0.87	0.90	0.92	0.95	0.97	0.98
0.60	0.10	0.19	0.27	0.35	0.42	0.49	0.54	0.60	0.65	0.70	0.74	0.78	0.82	0.85	0.88	0.91	0.93	0.96	0.98
0.70	0.09	0.18	0.25	0.33	0.39	0.46	0.52	0.57	0.62	0.67	0.72	0.76	0.80	0.83	0.86	0.90	0.92	0.95	0.98
0.80	0.08	0.16	0.24	0.31	0.38	0.44	0.50	0.55	0.60	0.65	0.70	0.74	0.78	0.82	0.85	0.89	0.92	0.95	0.97
0.90	0.08	0.16	0.23	0.30	0.36	0.42	0.48	0.53	0.59	0.64	0.68	0.73	0.77	0.81	0.84	0.88	0.91	0.94	0.97
1.00	0.08	0.15	0.22	0.29	0.35	0.41	0.47	0.52	0.57	0.62	0.67	0.71	0.76	0.80	0.83	0.87	0.91	0.94	0.97
2.00	0.06	0.12	0.18	0.24	0.30	0.35	0.41	0.46	0.51	0.56	0.61	0.66	0.71	0.75	0.79	0.84	0.88	0.92	0.96
3.00	0.06	0.12	0.17	0.23	0.28	0.34	0.39	0.44	0.49	0.54	0.59	0.64	0.69	0.73	0.78	0.83	0.87	0.91	0.96
4.00	0.06	0.11	0.17	0.22	0.27	0.33	0.38	0.43	0.48	0.53	0.58	0.63	0.68	0.73	0.77	0.82	0.87	0.91	0.96
5.00	0.05	0.11	0.16	0.22	0.27	0.32	0.37	0.42	0.47	0.52	0.57	0.62	0.67	0.72	0.77	0.82	0.86	0.91	0.95
6.00	0.05	0.11	0.16	0.21	0.27	0.32	0.37	0.42	0.47	0.52	0.57	0.62	0.67	0.72	0.77	0.81	0.86	0.91	0.95
7.00	0.05	0.11	0.16	0.21	0.26	0.32	0.37	0.42	0.47	0.52	0.57	0.62	0.67	0.71	0.76	0.81	0.86	0.91	0.95
8.00	0.05	0.11	0.16	0.21	0.26	0.31	0.36	0.42	0.47	0.52	0.57	0.61	0.66	0.71	0.76	0.81	0.86	0.91	0.95
9.00	0.05	0.11	0.16	0.21	0.26	0.31	0.36	0.41	0.46	0.51	0.56	0.61	0.66	0.71	0.76	0.81	0.86	0.90	0.95
10.00	0.05	0.10	0.16	0.21	0.26	0.31	0.36	0.41	0.46	0.51	0.56	0.61	0.66	0.71	0.76	0.81	0.86	0.90	0.95
Infinity	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
-10.00	0.05	0.10	0.14	0.19	0.24	0.29	0.34	0.39	0.44	0.49	0.54	0.59	0.64	0.69	0.74	0.79	0.84	0.90	0.95
-9.00	0.05	0.10	0.14	0.19	0.24	0.29	0.34	0.39	0.44	0.49	0.54	0.59	0.64	0.69	0.74	0.79	0.84	0.89	0.95
-8.00	0.05	0.09	0.14	0.19	0.24	0.29	0.34	0.39	0.43	0.48	0.53	0.58	0.64	0.69	0.74	0.79	0.84	0.89	0.95
-7.00	0.05	0.09	0.14	0.19	0.24	0.29	0.33	0.38	0.43	0.48	0.53	0.58	0.63	0.68	0.74	0.79	0.84	0.89	0.95
-6.00	0.05	0.09	0.14	0.19	0.23	0.28	0.33	0.38	0.43	0.48	0.53	0.58	0.63	0.68	0.73	0.79	0.84	0.89	0.95
-5.00	0.05	0.09	0.14	0.18	0.23	0.28	0.33	0.38	0.43	0.48	0.53	0.58	0.63	0.68	0.73	0.78	0.84	0.89	0.95
-4.00	0.04	0.09	0.13	0.18	0.23	0.27	0.32	0.37	0.42	0.47	0.52	0.57	0.62	0.67	0.73	0.78	0.83	0.89	0.94
-3.00	0.04	0.09	0.13	0.17	0.22	0.27	0.31	0.36	0.41	0.46	0.51	0.56	0.61	0.66	0.72	0.77	0.83	0.88	0.94
-2.00	0.04	0.08	0.12	0.16	0.21	0.25	0.29	0.34	0.39	0.44	0.49	0.54	0.59	0.65	0.70	0.76	0.82	0.88	0.94
-1.00	0.03	0.06	0.09	0.13	0.17	0.20	0.24	0.29	0.33	0.38	0.43	0.48	0.53	0.59	0.65	0.71	0.78	0.85	0.92
-0.90	0.03	0.06	0.09	0.12	0.16	0.19	0.23	0.27	0.32	0.36	0.41	0.47	0.52	0.58	0.64	0.70	0.77	0.84	0.92
-0.80	0.03	0.05	0.08	0.11	0.15	0.18	0.22	0.26	0.30	0.35	0.40	0.45	0.50	0.56	0.62	0.69	0.76	0.84	0.92
-0.70	0.02	0.05	0.08	0.10	0.14	0.17	0.20	0.24	0.28	0.33	0.38	0.43	0.48	0.54	0.61	0.67	0.75	0.82	0.91
-0.60	0.02	0.04	0.07	0.09	0.12	0.15	0.18	0.22	0.26	0.30	0.35	0.40	0.46	0.51	0.58	0.65	0.73	0.81	0.90
-0.50	0.02	0.03	0.05	0.08	0.10	0.13	0.16	0.19	0.23	0.27	0.31	0.36	0.42	0.48	0.54	0.62	0.70	0.79	0.89
-0.40	0.01	0.03	0.04	0.06	0.08	0.10	0.13	0.15	0.19	0.22	0.26	0.31	0.36	0.43	0.49	0.57	0.66	0.76	0.87
-0.30	0.01	0.01	0.02	0.04	0.05	0.06	0.08	0.10	0.13	0.16	0.19	0.24	0.29	0.34	0.41	0.50	0.59	0.71	0.84
-0.20	0.00	0.00	0.01	0.01	0.02	0.02	0.03	0.04	0.06	0.08	0.10	0.13	0.17	0.22	0.28	0.36	0.47	0.60	0.78
-0.10	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.01	0.01	0.02	0.03	0.05	0.08	0.14	0.22	0.37	0.61

Appendix D. Computer Program to Solve for the Risk Tolerance

The Pascal computer program below solves for the risk tolerance in Equation 2 given the certainty equivalent for an alternative with equal chances of yielding either of two specified values. Thus, this is a computerized equivalent to the table in Appendix B. The program is written in Standard Pascal and should compile correctly with virtually any Pascal compiler. Some compilers do not produce output from a **write** statement until an input line is received. For such compilers, the **write** statements which prompt for input should be replaced with **writeln** statements.

```
program findrho(input, output);

label 100, 9999;

const
  CELIMIT = 0.05; {Limit on how close CE can be to Low or High;
                   avoids exponential function crashing;
                   probably overly conservative}
  RHOLIMIT = 0.05; {Limit on how small abs(Rho/(High-Low)) can be;
                    avoids exponential function crashing;
                    probably overly conservative}
  DELTA = 1.0e-6; {Limit on accuracy of Rho/(High-Low)}

var
  Low, High : real; {Branches of uncertain alternative}
  Mono : char; {Monotonicity of preferences}
  CE : real; {Certainty equivalent of uncertain alternative}
  NearRho, MidRho, FarRho, ZHalf : real; {For finding rho}

{Find sign of x}
function sgn(x : real) : integer; begin
  if x > 0.0 then sgn := 1
  else sgn := -1
end;

{Find normalized utility of z given normalized risk tolerance}
function u(z, R : real) : real; begin
  u := (exp(-z/R)-1) / (exp(-1/R)-1)
end;

begin
  {Data input and checking}
  write('Low = '); readln(Low);
  write('High = '); readln(High);
  if Low >= High then goto 100;
  write('Monotonicity (I/D)? '); readln(Mono);
  if not((Mono = 'i') or (Mono = 'I')
         or (Mono = 'd') or (Mono = 'D')) then goto 100;
  write('Certainty Equivalent = '); readln(CE);
```

```
if (CE < Low) or (CE > High) then goto 100;
if (Mono = 'i') or (Mono = 'I')
  then ZHalf := (CE - Low) / (High - Low)
else ZHalf := (High - CE) / (High - Low);
if ZHalf < CELIMIT then goto 100;
if abs(ZHalf - 0.5) < DELTA then begin
  writeln('Rho = Infinity');
  goto 9999;
end;

{Restrict search region for solution so you don't get Rho = 0}
if ZHalf < 0.5 then begin
  NearRho := RHOLIMIT; FarRho := 10.0 end
else begin NearRho := -RHOLIMIT; FarRho := -10.0 end;

{If necessary, increasing the search range for the solution}
while sgn(u(ZHalf, FarRho)-0.5) = sgn(u(ZHalf, NearRho)-0.5)
  do FarRho := FarRho * 10.0;

{Use method of bisection to find solution}
while abs(FarRho - NearRho) > DELTA do begin
  MidRho := (NearRho + FarRho)/2.0;
  if sgn(u(ZHalf, FarRho)-0.5) = sgn(u(ZHalf, MidRho)-0.5)
    then FarRho := MidRho
  else NearRho := MidRho
end;
writeln('Rho = ', MidRho*(High-Low):10:3);
goto 9999;

{Error message}
100 : writeln('Invalid input.');
9999 : end.
```