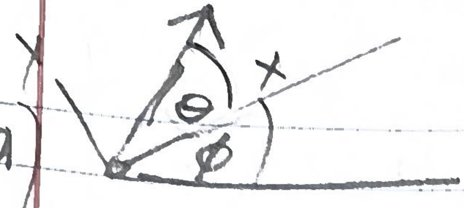
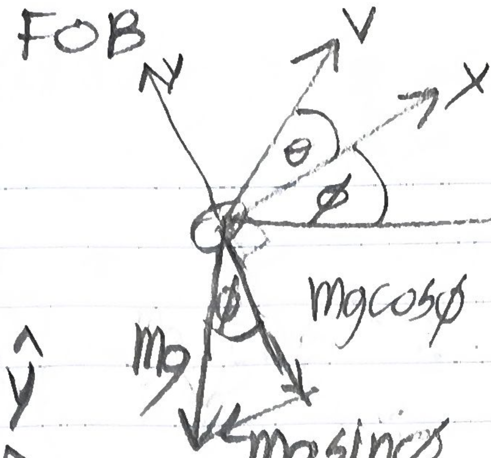
5a)





Assumptions:
- No drag force
- Only weight acts on ball
- Motion is confined to x-y plane (Does not move in z-direction)

$F_y = ma_y = -mg\cos\phi\,\hat{y}$

$F_x = ma_x = -mg\sin\phi\,\hat{x}$

$\frac{d^2y}{dt^2} = -g\cos\phi \quad \frac{d^2x}{dt^2} = -g\sin\phi$

$v_y = \int_0^t -g\cos\phi\, dt = v_0\sin\theta - gt\cos\phi$

$v_x = \int_0^t -g\sin\phi\, dt = v_0\cos\theta - gt\sin\phi$

$y = \int_0^t (v_0\sin\theta - gt\cos\phi)dt = v_0 t\sin\theta - \frac{gt^2}{2}\cos\phi$

$x = \int_0^t (v_0\cos\theta - gt\sin\phi)dt = v_0 t\cos\theta - \frac{gt^2}{2}\sin\phi$

$x(t) = v_0 t\cos\theta - \frac{gt^2}{2}\sin\phi \quad y(t) = v_0 t\sin\theta - \frac{gt^2}{2}\cos\phi$

Position is $r(t) = \langle x(t), y(t), 0\rangle$

$= \left\langle v_0 t\cos\theta - \frac{gt^2}{2}\sin\phi,\ v_0 t\sin\theta - \frac{gt^2}{2}\cos\phi,\ 0\right\rangle$

5b) Set $y=0$ in $y(t)$ to find the times where the ball is in contact with the plane.

$t\left(v_0\sin\theta - \frac{gt}{2}\cos\phi\right) = 0$

$t=0$ is the trivial solution when the ball is initially at the ramp

However, $t = \frac{2v_0 \sin\theta}{g\cos\phi}$ is the time taken for the ball to land on the ramp at the end of its motion.
Substitute this t into $x(t)$ denoting the range as R:

$$R = x\left(\frac{2v_0\sin\theta}{g\cos\phi}\right) = v_0\cos\theta\left(\frac{2v_0\sin\theta}{g\cos\phi}\right) - \frac{g\sin\phi}{2}\left(\frac{4v_0^2\sin^2\theta}{g^2\cos^2\phi}\right)$$

$$= \frac{2v_0^2}{g}\left(\frac{\sin\theta\cos\theta}{\cos\phi} - \frac{\sin\phi\sin^2\theta}{\cos^2\phi}\right)$$

$$= \frac{2v_0^2\sin\theta}{g\cos\phi}\left(\cos\theta - \frac{\sin\phi\sin\theta}{\cos\phi}\right)$$

$$= \frac{2v_0^2\sin\theta}{g\cos^2\phi}\left(\cos\theta\cos\phi - \sin\phi\sin\theta\right)$$

By using the identity $\cos(a+b) = \cos a\cos b - \sin a \sin b$

$$R = \frac{2v_0^2\sin\theta}{g\cos^2\phi}\cos(\theta+\phi)$$

5c) Find $\frac{dR}{d\theta}$ and set derivative to zero so we can find θ that maximises R

$$\frac{dR}{d\theta} = \frac{2v_0^2}{g\cos^2\phi}\frac{d}{d\theta}\left(\sin\theta\cos(\theta+\phi)\right)$$

$$= \frac{2v_0^2}{g\cos^2\phi}\left(\cos\theta\cos(\theta+\phi) - \sin\theta\sin(\theta+\phi)\right)$$ By cosine addition identity

$$= \frac{2v_0^2}{g\cos^2\phi}\cos(2\theta+\phi) = 0 \Rightarrow 2\theta+\phi = \frac{n\pi}{2}$$

where n is an integer

$$\frac{n\pi}{4} - \frac{\phi}{2}$$

Plug θ into R:

$$R = \frac{2v_0^2}{g\cos^2\phi}\sin\left(\frac{n\pi}{4} - \frac{\phi}{2}\right)\cos\left(\frac{n\pi}{4} + \frac{\phi}{2}\right)$$

Use product to sum identity

$$\sin a \cos b = \frac{1}{2}\left(\sin(a+b) + \sin(a-b)\right)$$

$$\sin\left(\frac{n\pi}{4} - \frac{\phi}{2}\right)\cos\left(\frac{n\pi}{4} + \frac{\phi}{2}\right)$$

$$= \frac{1}{2}\left(\sin\left(\frac{n\pi}{4} - \frac{\phi}{2} + \frac{n\pi}{4} + \frac{\phi}{2}\right) + \sin\left(\frac{n\pi}{4} - \frac{\phi}{2} - \frac{n\pi}{4} - \frac{\phi}{2}\right)\right)$$

$$= \frac{1}{2}\left(\sin\left(\frac{n\pi}{2}\right) + \sin(-\phi)\right)$$

But $\sin\left(\frac{n\pi}{2}\right) = 1$ for integers n and $\sin(-\phi) = -\sin\phi$

$$R = \frac{2v_0^2}{g\cos^2\phi}\,\frac{1}{2}(1 - \sin\phi) = \frac{v_0^2(1-\sin\phi)}{g(1-\sin^2\phi)}$$

$$= \frac{v_0^2(1-\sin\phi)}{g(1+\sin\phi)(1-\sin\phi)} = \frac{v_0^2}{g(1+\sin\phi)}$$

Therefore, maximum range for given v_0 and ϕ is $\dfrac{v_0^2}{g(1+\sin\phi)}$