



University of Minho
School of Engineering

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A Metric Equational System for Quantum Computation



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Master's Dissertation
Master in Physics Engineering

Work carried out under the supervision of
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Acknowledgements

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University of Minho, Braga, February 2024

Bruna Filipa Martins Salgado

Abstract

Noisy intermediate-scale quantum (NISQ) computers are expected to operate with severely limited hardware resources. Precisely controlling qubits in these systems comes at a high cost, is susceptible to errors, and faces scarcity challenges. Therefore, error analysis is indispensable for the design, optimization, and assessment of NISQ computing. Nevertheless, the analysis of errors in quantum programs poses a significant challenge. The overarching goal of the M.Sc. project is to provide a fully-fledged quantum programming language on which to study metric program equivalence in various scenarios, such as in quantum algorithmics and quantum information theory.

Keywords approximate equivalence, λ -calculus, metric equations

Resumo

Escrever aqui o resumo (pt)

Palavras-chave palavras, chave, aqui, separadas, por, vírgulas

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Part I

Introductory material

Chapter 1

Introduction

1.1 Motivation and Context

Quantum computing dates back to 1982 when Nobel laureate Richard Feynman proposed the idea that constructing computers founded on the principles of quantum mechanics could efficiently simulate quantum systems of interest to physicists, whereas this seemed to be very difficult with classical computers [[Feynman \(1982\)](#)].

This paradigm holds immense promise, as evidenced by several compelling results in computational complexity theory [[Shor \(1994\)](#); [Grover \(1996\)](#)]. While hardware advancements have brought the scientific community closer to realizing this potential, the ultimate goal the ultimate goal is yet to be accomplished. A NISQ quantum computer equipped with 50-100 qubits may surpass the capabilities of current classical computers, yet the impact of quantum noise, such as decoherence in entangled states, imposes limitations on the size of quantum circuits that can be executed reliably [[Preskill \(2018\)](#)]. Unfortunately, general-purpose error correction techniques [[Calderbank and Shor \(1996\)](#); [Gottesman \(1997\)](#); [Steane \(1996\)](#)] consume a substantial number of qubits, making it difficult for NISQ devices to make use of them in the near term. For instance, the implementation of a single logical qubit may require between 10^3 and 10^4 physical qubits [[Fowler et al. \(2012\)](#)].

To reconcile quantum computation with NISQ computers, quantum compilers perform transformations for error mitigation [[Wallman and Emerson \(2016\)](#)] and noise-adaptive optimization [[Murali et al. \(2019\)](#)]. Additionally, current quantum computers only support a restricted, albeit universal, set of quantum operations. As a result, nonnative operations must be decomposed into sequences of native operations before execution [[Harrow et al. \(2002\)](#), [Burgholzer and Wille \(2020\)](#)]. In general, perfect computational universality is not sought, but only the ability to approximate any quantum algorithm, with a preference for minimizing the use of

additional gates beyond the original requirements. The assessment of these compiler transformations necessitates a comparison of the error bounds between the source and compiled quantum programs. Furthermore, in quantum information theory, the concept of an ϵ – approximation channel is fundamental when studying quantum teleportation via noisy channels [Watrous (2018)]. This suggests the development of appropriate notions of approximate program equivalence, *in lieu* of the classical program equivalence and underlying theories that typically hinge on the idea that equivalence is binary, *i.e.* two programs are either equivalent or they are not [Winskel (1993)].

As previously noted, Shor’s and Grover’s algorithms have played a pivotal role in sparking heightened interest within the scientific community toward quantum computing research. On these bases, various endeavors to establish quantum programming languages have surfaced over the past 20 years. These include imperative languages such as Qiskit [Qiskit contributors (2023)] and Silq [Bichsel et al. (2020)], as well as functional languages such as Quipper [Green et al. (2013)] and Q# [Svore et al. (2018)]. On one hand, the design of quantum programming languages is strongly oriented towards implementing quantum algorithms. On the other hand, the definition of functional paradigmatic languages or functional calculi serves as a valuable tool for delving into theoretical aspects of quantum computing, particularly exploring the foundational basis of quantum computation [Zorzi (2016)]. Given the nature of this work, the focus will be on quantum languages designed with this latter aspect in mind.

QPL, a quantum language within the functional programming paradigm, marks a significant milestone in this context [Selinger (2004)]. This is a first-order functional language featuring a static type system based on the idea of classical control and quantum data.

Most of the current research on algorithms and programming languages assumes that addressing the challenge of noise during program execution will be resolved either by the hardware or through the implementation of fault-tolerant protocols designed independently of any specific application [Chong et al. (2017)]. As previously stated, this assumption is not realistic in the NISQ era. Nonetheless, there have been efforts to address the challenge of approximate program equivalence in the quantum setting. [Hung et al. (2019)] and [Tao et al. (2021)] reason about the issue of noise in a quantum while-language by developing a deductive system to determine how similar a quantum program is from its idealised, noise-free version. An alternative approach was explored in [Dahlqvist and Neves (2022)], using linear

λ -calculus as basis – *i.e* programs are written as linear λ -terms – which has deep connections to both logic and category theory [[Girard et al. \(1995\)](#), [Benton \(1994\)](#)]. Some positive results were achieved in this setting, but much remains to be done.

1.2 Goals

The notion of approximate equivalence for quantum programming explored in [[Dahlqvist and Neves \(2022\)](#)] does not take important operations into account. Specifically, the corresponding mathematical model does not include measurements, classical control flow, or discard operations. Also, the corresponding typing system is often times too strict and cannot properly handle multiple uses of the same resource, such as sampling exactly n -times from a distribution. The overarching goal of this M.Sc. project is to tackle the aforementioned limitations. A successful completion of this goal will provide a fully-fledged quantum programming language on which to study metric program equivalence in various scenarios. This includes not only quantum algorithmics – where, for example, the number of iterations in Grover’s algorithm involves approximations – but also quantum information theory, where, for instance, quantum teleportation and the problem of the discrimination of quantum states have important roles [[Nielsen and Chuang \(2010\)](#)].

Chapter 2

Background

2.1 Linear Lambda Calculus

The Lambda-Calculus, developed by Church and Curry in the 1930s, serves as a formal language capturing the key attribute of higher-order functional languages, treating functions as first-class citizens, allowing them to be passed as arguments [[Barendregt et al. \(1984\)](#)]. Beyond its foundational aspects, this calculus incorporates extensions for modeling side effects, including probabilistic or non-deterministic behaviors and shared memory. Centered around the expression of higher-order functions, where functions can serve as inputs or outputs, it emerges as a potent computational tool. Higher-order functions form a pivotal abstraction in practical programming languages such as LISP, Scheme, ML, and Haskell.

In quantum information theory, the role of higher-order functions encompasses two fundamental aspects. The first involves the concept of entangled functions and how well-known quantum phenomena find natural descriptions through such functions. The second concerns the interplay between classical objects and quantum objects in a higher-order context. Quantum computation conventionally handles classical and quantum data, while the higher-order context introduces a third data type: functions. These functions fall into two categories - those "quantum-like" (entangled, single-use) and those "classical-like" (duplicable, reusable). Remarkably, this classification transcends input/output types, highlighting the coexistence of quantum-like functions operating on classical data and classical-like functions operating on quantum data. [[Selinger et al. \(2009\)](#)].

2.1.1 Syntax

The grammar and term formation rules of the linear lambda calculus, discussed in [Dahlqvist and Neves (2022)], are presented in this subsection.

The definition of the grammar for linear lambda calculus is as follows, where G represents a set of ground types.

$$\mathbb{A} ::= X \in G \mid \mathbb{I} \mid \mathbb{A} \otimes \mathbb{A} \mid \mathbb{A} \oplus \mathbb{A} \mid \mathbb{A} \multimap \mathbb{A} \quad (2.1)$$

Regarding the term formation rules, Σ corresponds to a class of sorted operation symbols $f : \mathbb{A}_1, \dots, \mathbb{A}_n \rightarrow \mathbb{A}$, where $n \geq 1$. Typing contexts are represented as lists $x_1 : \mathbb{A}_1, \dots, x_n : \mathbb{A}_n$ of typed variables, with each variable x_i (where $1 \leq i \leq n$) occurring at most once in x_1, \dots, x_n . The typing contexts are denoted by greek letters Γ , Δ , and E . The concept of shuffling is employed to construct a linear typing system that ensures the admissibility of the exchange rule and enables unambiguous reference to judgment's denotations $\llbracket \Gamma \triangleright v : \mathbb{A} \rrbracket$. Shuffling is defined as a permutation of typed variables in a sequence of contexts, $\Gamma_1, \dots, \Gamma_n$, preserving the relative order of variables within each Γ_i . For instance, if $\Gamma_1 = x : \mathbb{A}, y : \mathbb{B}$ and $\Gamma_2 = z : \mathbb{C}$, then $z : \mathbb{C}, x : \mathbb{A}, y : \mathbb{B}$ is a valid shuffle of Γ_1, Γ_2 . On the other hand, $y : \mathbb{B}, x : \mathbb{A}, z : \mathbb{C}$ is not a shuffle because it alters the occurrence order of x and y in Γ_1 . The set of shuffles in $\Gamma_1, \dots, \Gamma_n$ is denoted as $\text{Sf}(\Gamma_1, \dots, \Gamma_n)$. The term formation rules of the linear lambda calculus are shown in Figure 1.

$$\begin{array}{c}
\frac{\Gamma_i \triangleright v_i : \mathbb{A}_i \quad f : \mathbb{A}_1, \dots, \mathbb{A}_n \rightarrow \mathbb{A} \in \Sigma \quad E \in \text{Sf}(\Gamma_1; \dots; \Gamma_n)}{E \triangleright f(v_1, \dots, v_n) : \mathbb{A}} \text{(ax)} \quad \frac{}{x : \mathbb{A} \triangleright x : \mathbb{A}} \text{(hyp)} \\
\\
\frac{}{- \triangleright * : \mathbb{I}} \text{(\mathbb{I}_i)} \quad \frac{\Gamma \triangleright v : \mathbb{A} \otimes \mathbb{B} \quad \Delta, x : \mathbb{A}, y : \mathbb{B} \triangleright w : \mathbb{C} \quad E \in \text{Sf}(\Gamma; \Delta)}{E \triangleright \text{pm } v \text{ to } x \otimes y.w : \mathbb{C}} \text{(\otimes_e)} \\
\\
\frac{\Gamma \triangleright v : \mathbb{A} \quad \Delta \triangleright w : \mathbb{B} \quad E \in \text{Sf}(\Gamma; \Delta)}{E \triangleright v \otimes w : \mathbb{A} \otimes \mathbb{B}} \text{(\otimes_i)} \quad \frac{\Gamma \triangleright v : \mathbb{I} \quad \Delta \triangleright w : \mathbb{A} \quad E \in \text{Sf}(\Gamma; \Delta)}{E \triangleright v \text{ to } *.w : \mathbb{A}} \text{(\mathbb{I}_e)} \\
\\
\frac{\Gamma, x : \mathbb{A} \triangleright v : \mathbb{B}}{\Gamma \triangleright \lambda x : \mathbb{A}.v : \mathbb{A} \multimap \mathbb{B}} \text{(-\multimap_i)} \quad \frac{\Gamma \triangleright v : \mathbb{A} \multimap \mathbb{B} \quad \Delta \triangleright w : \mathbb{A} \quad E \in \text{Sf}(\Gamma; \Delta)}{E \triangleright vw : \mathbb{B}} \text{(-\multimap_e)} \quad \frac{\Gamma \triangleright v : \mathbb{A}}{\Gamma \triangleright \text{dis}(v) : \mathbb{I}} \text{(dis)}
\end{array}$$

Figure 1: Term formation rules of affine lambda calculus.

The no-cloning theorem states that it is impossible to duplicate a quantum bit [Wootters and Zurek (1982)]. This principle is upheld by the type system outlined in Figure 1, which does not

allow the repeated use of a variable (seen as a quantum resource). Nevertheless, the linearity constraint is often deemed too restrictive, prompting research into relaxing it in various computational paradigms. In [Dahlqvist and Neves (2023)], the controlled use of a resource multiple times is explored within approximate program equivalence paradigms. Moreover, the grammar introduced allows the specification of how many times a resource can be used—a notion particularly relevant in quantum computation, especially within the NISQ era where resources are scarce.

2.1.2 Metric equational system

Metric equations [Mardare et al. (2016), Mardare et al. (2017)] are a strong candidate for reasoning about approximate program equivalence. These equations take the form of $t =_\epsilon s$, where ϵ is a non-negative rational representing the “maximum distance” between the two terms t and s . The metric equational system for linear lambda calculus is depicted in Figure 2 [Dahlqvist and Neves (2022)].

$$\begin{array}{c}
\frac{}{v =_0 v} \text{ (refl)} \qquad \frac{v =_q w \quad w =_r u}{v =_{q+r} u} \text{ (trans)} \qquad \frac{v =_q w \quad r \geq q}{v =_r w} \text{ (weak)} \\
\\
\frac{\forall r > q. v =_r w}{v =_q w} \text{ (arch)} \qquad \frac{\forall i \leq n. v =_{q_i} w}{v =_{\vee q_i} w} \text{ (join)} \qquad \frac{v =_q w \quad v' =_r w'}{v \otimes v' =_{q+r} w \otimes w'} \\
\\
\frac{\forall i \leq n. v_i =_{q_i} w_i}{f(v_1, \dots, v_n) =_{\Sigma q_i} f(w_1, \dots, w_n)} \quad \frac{v =_q w \quad v' =_r w'}{v \text{ to } * . v' =_{q+r} w \text{ to } * . w'} \quad \frac{v =_q w}{\lambda x : \mathbb{A}. v =_q \lambda x : \mathbb{A}. w} \\
\\
\frac{v =_q w \quad v' =_r w'}{\text{pm } v \text{ to } x \otimes y. v' =_{q+r} \text{pm } w \text{ to } x \otimes y. w'} \qquad \frac{v =_q w \quad v' =_r w'}{vv' =_{q+r} ww'} \\
\\
\frac{\Gamma \triangleright v =_q w : \mathbb{A} \quad \Delta \in \text{perm}(\Gamma)}{\Delta \triangleright v =_q w : \mathbb{A}} \qquad \frac{v =_q w \quad v' =_r w'}{v[v'/x] =_{q+r} w[w'/x]}
\end{array}$$

Figure 2: Metric equational system

In the quantum paradigm, a potential notion of approximate equivalence arises from the so-called diamond norm [Watrous (2018)], which induces a metric (roughly, a distance function) on the space of quantum programs (seen semantically as completely positive trace-preserving super-operators). This norm relies on another norm known as the trace norm.

The $\|\cdot\|_1$ latter is defined by $\|A\|_1 = \text{Tr}\sqrt{A^\dagger A}$ for matrices $A \in \mathbb{C}^{n \times n}$. The trace distance between two super-operators $E, E' : \mathbb{C}^{n \times n} \rightarrow \mathbb{C}^{m \times m}$, denoted as $T(E, E')$, is defined as follows:

$$T(E, E') = \max\{\|(E - E')A\|_1 \mid \|A\|_1 = 1\} \quad (2.2)$$

Unfortunately, this norm is not stable under tensoring [Watrous (2018)], and consequently, the diamond norm, which is based on the trace norm, is used instead. The diamond norm between two super-operators $E, E' : \mathbb{C}^{n \times n} \rightarrow \mathbb{C}^{m \times m}$ is defined as:

$$\|E - E'\|_\diamond = T(E \otimes I_n, E' \otimes I_n) \quad (2.3)$$

where I_n is the identity super-operator over the space $\mathbb{C}^{n \times n}$. The notion of a diamond norm is used in [Dahlqvist and Neves (2022)] which introduces a simple metric theory based on the idea of approximating a quantum operation. The authors argue that their deductive system allows to compute an approximate distance between two quantum programs easily as opposed to computing an exact distance “semantically” which tends to involve quite complex operators. Other works in this spirit include [Hung et al. (2019)] and [Tao et al. (2021)]. They reason about the issue of noise in a quantum while-language by developing a deductive system to determine how similar a quantum program is from its idealised, noise-free version. The former introduces the (Q, λ) -diamond norm which analyzes the output error given that the input quantum state satisfies some quantum predicate Q to degree λ . However, it does not specify any practical method for obtaining non-trivial quantum predicates. In fact, the methods used in [Hung et al. (2019)] cannot produce any post conditions other than $(I, 0)$ (i.e., the identity matrix I to degree 0, analogous to a “true” predicate) for large quantum programs. The latter specifically addresses and delves into this aspect.

2.1.3 Interpretation

In order to define the interpretation of judgments $\Gamma \triangleright v : \mathbb{A}$, it is necessary to establish some notation first. Considering $v \in V, w \in W$, and $u \in U$ where V, W, U represent vector spaces, $\text{sw}_{V,W} : V \otimes W \rightarrow W \otimes V$, denotes the swap operator, defined as $\text{sw}_{V,W} = v \otimes w \mapsto w \otimes v$; $\rho_V : \mathbb{C} \otimes V \rightarrow V$ is the left unitor defined as $\rho_V = 1 \otimes v \mapsto v$; $\lambda_V : V \otimes \mathbb{C} \rightarrow V$ is the right unitor defined as $\lambda_V = v \otimes 1 \mapsto v$; $\alpha_{V,W,U} : V \otimes (W \otimes U) \rightarrow (V \otimes W) \otimes U$ is the left associator, defined as $\alpha_{V,W,U} = v \otimes (w \otimes u) \mapsto (v \otimes w) \otimes u$; and $!_V : V \rightarrow \mathbb{C}$ is the trace operation applied to a vector, defined as $!_V = v \rightarrow \text{Tr} v$. Moreover, for all operators

$f : V \otimes W \rightarrow U$, the operator $\bar{f} : V \rightarrow (W \multimap U)$ denotes the corresponding curried version, defined as $\bar{f}(v) = w \mapsto f(v, w)$. The subscripts in these operators will be omitted unless ambiguity arises.

For all ground types $X \in G$ the interpretation of $\llbracket X \rrbracket$ is postulated as a vector space V . Types are interpreted inductively using the unit \mathbb{I} , the tensor \otimes , and the linear map \multimap . Given a non-empty context $\Gamma = \Gamma', x : \mathbb{A}$, its interpretation is defined by $\llbracket \Gamma', x : \mathbb{A} \rrbracket = \llbracket \Gamma' \rrbracket \otimes \llbracket \mathbb{A} \rrbracket$ if Γ' is non-empty and $\llbracket \Gamma', x : \mathbb{A} \rrbracket = \llbracket \mathbb{A} \rrbracket$ otherwise. The empty context $-$ is interpreted as $\llbracket - \rrbracket = \mathbb{I}$. Given $X_1, \dots, X_n \in V$, the n -tensor $(\dots (X_1 \otimes X_2) \otimes \dots) \otimes X_n$ is denoted as $X_1 \otimes \dots \otimes X_n$, and similarly for operators.

“Housekeeping” operators are employed to handle interactions between context interpretation and the vectorial model. Given $\Gamma_1, \dots, \Gamma_n$, the operator that splits $\llbracket \Gamma_1, \dots, \Gamma_n \rrbracket$ into $\llbracket \Gamma_1 \rrbracket \otimes \dots \otimes \llbracket \Gamma_n \rrbracket$ is denoted by $\text{sp}_{\Gamma_1; \dots; \Gamma_n} : \llbracket \Gamma_1, \dots, \Gamma_n \rrbracket \rightarrow \llbracket \Gamma_1 \rrbracket \otimes \dots \otimes \llbracket \Gamma_n \rrbracket$. On the other hand, $\text{jn}_{\Gamma_1; \dots; \Gamma_n}$ denotes the inverse of $\text{sp}_{\Gamma_1; \dots; \Gamma_n}$. Next, given $\Gamma, x : \mathbb{A}, y : \mathbb{B}, \Delta$, the operator permuting x and y is denoted by $\text{exch}_{\Gamma, x : \mathbb{A}, y : \mathbb{B}, \Delta} : \llbracket \Gamma, x : \mathbb{A}, y : \mathbb{B}, \Delta \rrbracket \rightarrow \llbracket \Gamma, y : \mathbb{B}, x : \mathbb{A}, \Delta \rrbracket$. The shuffling operator $\text{sh}_E : \llbracket E \rrbracket \rightarrow \llbracket \Gamma_1, \dots, \Gamma_n \rrbracket$ is defined as a suitable composition of exchange operators.

For every operation symbol $f : \mathbb{A}_1, \dots, \mathbb{A}_n \rightarrow \mathbb{A}$ we assume the existence of an operator $\llbracket f \rrbracket : \llbracket \mathbb{A}_1 \rrbracket \otimes \dots \otimes \llbracket \mathbb{A}_n \rrbracket \rightarrow \llbracket \mathbb{A} \rrbracket$. The interpretation of judgments is defined by induction over derivations according to the rules in [Figure 3 \[Dahlqvist and Neves \(2022\)\]](#).

$$\begin{array}{c}
\frac{\llbracket \Gamma_i \triangleright v_i : \mathbb{A}_i \rrbracket = m_i \quad f : \mathbb{A}_1, \dots, \mathbb{A}_n \in \Sigma \quad E \in \text{Sf}(\Gamma_1; \dots; \Gamma_n)}{\llbracket E \triangleright f(v_1, \dots, v_n) : \mathbb{A} \rrbracket = \llbracket f \rrbracket \cdot (m_1 \otimes \dots \otimes m_n) \cdot \text{sp}_{\Gamma_1; \dots; \Gamma_n} \cdot \text{sh}_E} \quad \frac{}{\llbracket x : \mathbb{A} \triangleright x : \mathbb{A} \rrbracket = \text{id}_{\llbracket \mathbb{A} \rrbracket}} \\
\\
\frac{}{\llbracket - \triangleright * : \mathbb{I} \rrbracket = \text{id}_{\llbracket \mathbb{I} \rrbracket}} \quad \frac{\llbracket \Gamma \triangleright v : \mathbb{A} \otimes \mathbb{B} \rrbracket = m \quad \llbracket \Delta, x : \mathbb{A}, y : \mathbb{B} \triangleright w : \mathbb{C} \rrbracket = n \quad E \in \text{Sf}(\Gamma; \Delta)}{\llbracket E \triangleright \text{pm } v \text{ to } x \otimes y.w : \mathbb{C} \rrbracket = n \cdot \text{jn}_{\Delta; \mathbb{A}; \mathbb{B}} \cdot \alpha \cdot \text{sw} \cdot (m \otimes \text{id}) \cdot \text{sp}_{\Gamma; \Delta} \cdot \text{sh}_E} \\
\\
\frac{\llbracket \Gamma \triangleright v : \mathbb{A} \rrbracket = m \quad \llbracket \Delta \triangleright w : \mathbb{B} \rrbracket = n \quad E \in \text{Sf}(\Gamma; \Delta)}{\llbracket E \triangleright v \otimes w : \mathbb{A} \otimes \mathbb{B} \rrbracket = (m \otimes n) \cdot \text{sp}_{\Gamma; \Delta} \cdot \text{sh}_E} \\
\\
\frac{\llbracket \Gamma \triangleright v : \mathbb{I} \rrbracket = m \quad \llbracket \Delta \triangleright w : \mathbb{A} \rrbracket = n \quad E \in \text{Sf}(\Gamma; \Delta)}{\llbracket E \triangleright v \text{ to } * .w : \mathbb{A} \rrbracket = n \cdot \lambda \cdot (m \otimes \text{id}) \cdot \text{sp}_{\Gamma; \Delta} \cdot \text{sh}_E} \quad \frac{\llbracket \Gamma, x : \mathbb{A} \triangleright v : \mathbb{B} \rrbracket = m}{\llbracket \Gamma \triangleright \lambda x : \mathbb{A}. v : \mathbb{A} \multimap \mathbb{B} \rrbracket = \overline{m} \cdot \text{jn}_{\Gamma; \mathbb{A}}} \\
\\
\frac{\llbracket \Gamma \triangleright v : \mathbb{A} \multimap \mathbb{B} \rrbracket = m \quad \llbracket \Delta \triangleright w : \mathbb{A} \rrbracket = n \quad E \in \text{Sf}(\Gamma; \Delta)}{\llbracket E \triangleright vw : \mathbb{A} \rrbracket = \text{app} \cdot (m \otimes n) \cdot \text{sp}_{\Gamma; \Delta} \cdot \text{sh}_E} \quad \frac{\llbracket \Gamma \triangleright v : \mathbb{A} \rrbracket = f}{\llbracket \Gamma \triangleright \text{dis}(v) : \mathbb{I} \rrbracket = !_\llbracket \mathbb{A} \rrbracket \cdot f}
\end{array}$$

Figure 3: Judgment interpretation

2.2 Quantum Computing Preliminaries

This section presents background on quantum information and quantum computation [Nielsen and Chuang (2010)].

The basic unit of information in quantum computation is a quantum bit or qubit [Perdrix (2008)]. The state of a single qubit is described by a normalized vector of the 2-dimensional Hilbert space \mathbb{C}^2 . An n -qubit state can be represented by a unit vector in 2^n -dimensional Hilbert space \mathbb{C}^{2^n} . An n -qubit mixed state can be represented by a density operator $\mathbb{C}^{2^n} \rightarrow \mathbb{C}^{2^n}$, whose matrix representation is $\rho = \sum_i p_i |\phi_i\rangle\langle\phi_i|$. A density operator encodes uncertainty about the current state of the quantum system at hand. For example, a mixed state with half probability of $|0\rangle$ and $|1\rangle$ can be represented by $\frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2} = I/2$, where I is the identity matrix. One usually denotes density matrices by the greek letters ρ , σ , and so forth. The set of density operators is denoted by $\mathcal{D}_n \subseteq \mathbb{C}^{2^n \times 2^n}$.

Measurements extract classical information from quantum states. If a measurement M_m is performed on a state ρ , the outcome m is observed with probability $p_m = \text{Tr}(M_m \rho M_m^\dagger)$ for each m . Moreover, after a measurement yielding outcome m , the state collapses to $M_m \rho M_m^\dagger / p_m$. Operations on quantum systems can be described using unitary operators. An operator, U , is unitary if its Hermitian conjugate is its own inverse, i.e., $U^\dagger U = U U^\dagger = I$. For a pure state $|\psi\rangle$, a unitary operator U describes an evolution from $|\psi\rangle$ to $U|\psi\rangle$. Similarly, for a density operator ρ , the corresponding evolution is $\rho \mapsto U \rho U^\dagger$. For example, the bit flip gate $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ maps $|0\rangle$ to $|1\rangle$ and $|1\rangle$ to $|0\rangle$. On the other hand, the Hadamard gate $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ maps $|0\rangle$ to $\frac{|0\rangle + |1\rangle}{\sqrt{2}}$ (denoted as $|+\rangle$) and $|1\rangle$ to $\frac{|0\rangle - |1\rangle}{\sqrt{2}}$ (denoted as $|-\rangle$). There are also multi-qubit gates, such as *CNOT*, which leaves the states $|00\rangle$ and $|01\rangle$ unchanged, and maps $|10\rangle$ and $|11\rangle$ to each other.

More broadly, the evolution of a quantum system can be defined by a super-operator E , which is a completely-positive and trace-preserving linear map from $\mathcal{D}(n)$ to $\mathcal{D}(m)$. A super-operator E is called positive if it sends positive matrices to positive matrices, i.e. $A \geq 0 \Rightarrow EA \geq 0$. A super-operator is said to be completely positive if, for any positive integer k and any k -dimensional Hilbert space \mathbb{C}^{2^k} , the super-operator $E \otimes I_{\mathbb{C}^{2^k}}$ is a positive map on $\mathcal{D}(n \times k)$. Finally, a super-operator E is called trace-preserving if $\text{Tr} EA = \text{Tr} A$ [Watrous (2018)]. Completely-positive, trace-preserving super-operators are traditionally called quantum channels.

For every super-operator $E : \mathcal{D}(n) \rightarrow \mathcal{D}(m)$, there exists a set of Kraus operators $\{\epsilon_k\}_k$ such that $E(\rho) = \sum_k \epsilon_k \rho \epsilon_k^\dagger$ for any input $\rho \in \mathcal{D}(n)$. Note that the set of Kraus operators is finite if the Hilbert space is finite-dimensional. The Kraus form of E is written as $E = \sum_k \epsilon_k \circ \epsilon_k^\dagger$.

2.3 Quantum Lambda Calculus

In the case of quantum lambda calculus, which combines classical and quantum features, it is natural to consider two distinct basic data types: a type *bit* of classical bits and a type *qbit* of quantum bits. The interpretation of these types is defined as $\llbracket bit \rrbracket = \mathbb{C} \oplus \mathbb{C}$ and $\llbracket qbit \rrbracket = \mathbb{C}^{2 \times 2}$. The type \mathbb{I} is interpreted as $\llbracket \mathbb{I} \rrbracket = \mathbb{C}$.

The following operations are considered: $new\ 0 : \mathbb{I} \multimap bit$, $new\ 1 : \mathbb{I} \multimap bit$, $q : bit \multimap qbit$, $meas : qbit \rightarrow bit$, and $U : qbit, \dots, qbit \rightarrow qbit^{\otimes n}$. Their correspondent judgment interpretation is shown in [Figure 4](#).

$\llbracket new\ 0 \rrbracket : \mathbb{C} \multimap \llbracket bit \rrbracket$ $1 \mapsto (1, 0)$	$\llbracket new\ 1 \rrbracket : \mathbb{C} \multimap \llbracket bit \rrbracket$ $1 \mapsto (0, 1)$	$\llbracket q \rrbracket : \llbracket bit \rrbracket \multimap \llbracket qbit \rrbracket$ $(a, b) \mapsto \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$
$\llbracket meas \rrbracket : \llbracket qbit \rrbracket \rightarrow \llbracket bit \rrbracket$ $\rho \mapsto (\text{Tr}(M_0 \rho M_0^\dagger), \text{Tr}(M_1 \rho M_1^\dagger))$	$\llbracket U \rrbracket : \llbracket qbit \rrbracket^{\otimes n} \rightarrow \llbracket qbit \rrbracket^{\otimes n}$ $\rho \mapsto U \rho U^\dagger$	

Figure 4: Judgment interpretation of the operations in quantum lambda calculus.

Chapter 3

The problem and its challenges

The problem and its challenges.

3.1 Images

Example of inserting an image as displayed text,



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— or as a floating body.



Figure 5: Caption

Part II

Core of the Dissertation

Chapter 4

Contribution

Main result(s) and their scientific evidence

4.1 Introduction

4.2 Summary

4.3 Integration of conditionals

The notion of approximate equivalence for quantum programming explored in [Dahlqvist and Neves (2022)] does not encompass classical control flow. As a result, preliminary work based on [Crole (1993); Selinger (2013)] has been undertaken to address the integration of conditionals.

4.3.1 Integration of conditionals

The term formation rules for conditionals are depicted in Figure 6.

$$\begin{array}{c} \frac{\Gamma \triangleright v : \mathbb{A}}{\Gamma \triangleright \text{inl}(v) : \mathbb{A} \oplus \mathbb{B}} (\text{inl}) \quad \frac{\Gamma \triangleright v : \mathbb{B}}{\Gamma \triangleright \text{inr}(v) : \mathbb{A} \oplus \mathbb{B}} (\text{inr}) \\[10pt] \frac{\Gamma \triangleright v : \mathbb{A} \oplus \mathbb{B} \quad \Delta, x : \mathbb{A} \triangleright w : \mathbb{C} \quad \Delta, y : \mathbb{B} \triangleright u : \mathbb{C} \quad E \in \text{Sf}(\Gamma; \Delta)}{E \triangleright \text{cond } v \{ \text{inl}(x) \Rightarrow w; \text{inr}(y) \Rightarrow u \} : \mathbb{C}} (\text{case}) \end{array}$$

Figure 6: Term formation rules for conditionals

Considering $v \in V, w \in W$, and $u \in U$ where V, W, U represent vector spaces, $\text{ll}_V : V \rightarrow$

$V \oplus W$, denotes the left injection operator, defined as $\text{IL}_V = v \mapsto (v, 0)$; $\text{IR}_V : V \rightarrow W \oplus V$, denotes the right injection operator, defined as $\text{IR}_V = v \mapsto (0, v)$; and $\text{dist}_{V,W,U} : V \otimes (W \oplus U) \rightarrow (V \otimes W) \oplus (V \otimes U)$, denotes the distributive property of the tensor product over the direct sum, defined as $\text{dist}_{V,W,U} = v \otimes (w, u) \mapsto (v \otimes w, v \otimes u)$. The subscripts in these operators will be omitted unless ambiguity arises. Moreover, the operation either corresponds to:

$$\frac{\begin{array}{c} V \rightarrow U \\ W \rightarrow U \end{array}}{[T, S] : V \oplus W \rightarrow U} \quad (4.1)$$

$$[T, S] = (v, w) \mapsto T(v) + S(w)$$

The interpretation of conditionals is illustrated in [Figure 7](#).

$$\frac{\frac{\llbracket \Gamma \triangleright v : \mathbb{A} \rrbracket = m}{\llbracket \Gamma \triangleright \text{inl}(v) : \mathbb{A} \oplus \mathbb{B} \rrbracket = \text{IL} \cdot m} \quad \frac{\llbracket \Gamma \triangleright v : \mathbb{B} \rrbracket = m}{\llbracket \Gamma \triangleright \text{inr}(v) : \mathbb{A} \oplus \mathbb{B} \rrbracket = \text{IR} \cdot m}}{\llbracket \Gamma \triangleright v : \mathbb{A} \oplus \mathbb{B} \rrbracket = b \quad \llbracket \Delta, x : \mathbb{A} \triangleright w : \mathbb{C} \rrbracket = p \quad \llbracket \Delta, x : \mathbb{B} \triangleright w_2 : \mathbb{C} \rrbracket = q \quad E \in \text{Sf}(\Gamma; \Delta)} \quad (4.2)$$

$$\frac{}{\llbracket E \triangleright \text{case } v \{ \text{inl}(x) \Rightarrow w; \text{inr}(y) \Rightarrow u \} : \mathbb{C} \rrbracket = \text{either}(p, q) \cdot \text{dist} \cdot \text{sw} \cdot (b \otimes \text{id}) \cdot \text{sp}_{\Gamma; \Delta} \cdot \text{sh}_E}$$

Figure 7: Judgment interpretation for conditionals

Proof In order to validate the judgment interpretation for conditionals, it is necessary to demonstrate its correctness.

For the booleans:

$$\begin{array}{c} \llbracket \Gamma \rrbracket \xrightarrow{m} \llbracket \mathbb{A} \rrbracket \xrightarrow{\text{IL}} \llbracket \mathbb{A} \oplus \mathbb{B} \rrbracket \\ \llbracket \Gamma \rrbracket \xrightarrow{m} \llbracket \mathbb{B} \rrbracket \xrightarrow{\text{IR}} \llbracket \mathbb{A} \oplus \mathbb{B} \rrbracket \end{array} \quad (4.3)$$

Now, for the conditional statement:

$$\begin{array}{c} \llbracket E \rrbracket \xrightarrow{\text{sh}_E} \llbracket \Gamma, \Delta \rrbracket \xrightarrow{\text{sp}_{\Gamma; \Delta}} \llbracket \Gamma \rrbracket \otimes \llbracket \Delta \rrbracket \xrightarrow{b \otimes \text{id}} (\llbracket \mathbb{A} \rrbracket \oplus \llbracket \mathbb{B} \rrbracket) \otimes \llbracket \Delta \rrbracket \xrightarrow{\text{sw}} \llbracket \Delta \rrbracket \otimes (\llbracket \mathbb{A} \rrbracket \oplus \llbracket \mathbb{B} \rrbracket) \\ \xrightarrow{\text{dist}} (\llbracket \Delta \rrbracket \otimes \llbracket \mathbb{A} \rrbracket) \oplus (\llbracket \Delta \rrbracket \otimes \llbracket \mathbb{B} \rrbracket) \xrightarrow{\text{either}(p, q)} \llbracket \mathbb{C} \rrbracket \end{array} \quad (4.4)$$

The quantum lambda calculus with conditionals is illustrated with an example—the quantum teleportation protocol—in [subsection 4.3.2](#).

The metric equations for conditionals are presented in [Figure 8](#). Note that the first two equations are redundant.

$$\begin{array}{c}
\frac{v =_q w}{\text{inl}(v) =_q \text{inl}(w)} \quad \frac{v =_q w}{\text{inr}(v) =_q \text{inr}(w)} \\
\hline
\frac{v =_q v' \quad w =_r w' \quad u =_s u'}{\text{case } v \{ \text{inl}(x) \Rightarrow w; \text{inr}(y) \Rightarrow u \} =_{q+\max(r,s)} \text{case } v' \{ \text{inl}(x) \Rightarrow w'; \text{inr}(y) \Rightarrow u' \}}
\end{array}$$

Figure 8: Metric equational system for conditionals

Proof In order to validate the metric equational system for conditionals, it is necessary to demonstrate its correctness.

The diamond norm is a particular instance of the operator norm. The operator norm [[Guide \(2006\)](#)] for a super-operator E is defined as:

$$\|E\|_\sigma = \sup\{\|E(v)\| \mid \|v\| = 1\} \quad (4.5)$$

For the **injections**:

Firstly, it is necessary to prove that the identity operator I has a norm equal to 1.

Lemma 4.3.1. $\|I\|_\sigma = 1$

Proof. Using the definition of operator norm in [Equation 4.5](#), it follows that:

$$\|I\|_\sigma = \sup\{\|I(v)\| \mid \|v\| = 1\} = \sup\{\|v\| \mid \|v\| = 1\} = 1 \quad (4.6)$$

Thereafter, it is imperative to show that the injection operators IL and IR are have a norm equal to 1.

Lemma 4.3.2. $\|\text{IL}\|_\sigma = 1$

Lemma 4.3.3. $\|\text{IR}\|_\sigma = 1$

Proof. Employing the definition of operator norm as defined in [Equation 4.5](#), it ensues that:

$$\begin{aligned}
\|\text{IL}\|_\sigma &= \sup\{\|\text{IL}(v)\| \mid \|v\| = 1\} = \sup\{\|(v, 0)\| \mid \|v\| = 1\} = \sup\{\|v\| + \|0\| \mid \|v\| = 1\} \\
&= \sup\{\|v\| + 0 \mid \|v\| = 1\} \quad \{\text{Positive definiteness}\} \\
&= \sup\{\|v\| \mid \|v\| = 1\} = 1
\end{aligned}$$

(4.7)

The proof for [Lemma 4.3.3](#) is analogous to the proof for [Lemma 4.3.2](#).

$$\begin{aligned}
\|\mathbf{I}\mathbf{R}\|_{\sigma} &= \sup\{\|\mathbf{I}\mathbf{R}(v)\| \mid \|v\| = 1\} = \sup\{\|(0, v)\| \mid \|v\| = 1\} = \sup\{\|0\| + \|v\| \mid \|v\| = 1\} \\
&= \sup\{0 + \|v\| \mid \|v\| = 1\} \quad \{\text{Positive definiteness}\} \\
&= \sup\{\|v\| \mid \|v\| = 1\} = 1
\end{aligned}
\tag{4.8}$$

Futhermore, given the submultiplicative property of the operator norm, for any super-operators P and Q , where $\|P\|_{\sigma} = 1$ the following holds:

Lemma 4.3.4. $\|PQ\|_{\sigma} \leq \|Q\|_{\sigma}, \quad \|P\|_{\sigma} = 1$

Using these properties it is possible to prove the validity of the metric equations for the injections. Demonstrating the correctness of the metric equations for the injections is equivalent to proving that for any non-negative rational q and super-operators v and w such that $d(v, w) \leq q$, where $d(v, w)$ represents the distance between v and w the following holds:

Theorem 4.3.1. $d(\mathbf{I}\mathbf{L}(v), \mathbf{I}\mathbf{L}(w)) \leq q$

Theorem 4.3.2. $d(\mathbf{I}\mathbf{R}(v), \mathbf{I}\mathbf{R}(w)) \leq q$

Proof. In the quantum paradigm, the distance between two super-operators E and E' corresponds to the diamond norm between E and E' . Therefore,

$$d(v, w) \leq q \Leftrightarrow \|v \otimes I - w \otimes I\|_{\sigma} \leq q \tag{4.9}$$

As a result, to prove that $d(\mathbf{I}\mathbf{L}(v), \mathbf{I}\mathbf{L}(w)) \leq q$, it suffices to show that:

$$\|\mathbf{I}\mathbf{L} \otimes I(v \otimes I) - \mathbf{I}\mathbf{L} \otimes I(w \otimes I)\|_{\sigma} \leq \|v \otimes I - w \otimes I\|_{\sigma} \tag{4.10}$$

$$\|\mathbf{I}\mathbf{R} \otimes I(v \otimes I) - \mathbf{I}\mathbf{R} \otimes I(w \otimes I)\|_{\sigma} \leq \|v \otimes I - w \otimes I\|_{\sigma} \tag{4.11}$$

Given that $\mathbf{I}\mathbf{L}$ and $\mathbf{I}\mathbf{R}$ possess a norm equal to 1, as established by [Lemmas 4.3.2](#) and [4.3.3](#) respectively, and considering the multiplicative property of the operator norm with respect to tensor products alongside the fact that the identity operator also exhibits a norm equal to 1, as demonstrated in [Lemma 4.3.1](#), it follows that both $\|\mathbf{I}\mathbf{L} \otimes I\|_{\sigma}$ and $\|\mathbf{I}\mathbf{R} \otimes I\|_{\sigma}$ are equal to

one 1. Hence, by [Lemma 4.3.4](#),

$$\|\mathbf{L} \otimes I(v \otimes I) - \mathbf{L} \otimes I(w \otimes I)\|_\sigma = \|\mathbf{L} \otimes I(v \otimes I - w \otimes I)\|_\sigma \leq \|v \otimes I - w \otimes I\|_\sigma \quad (4.12)$$

$$\|\mathbf{R} \otimes I(v \otimes I) - \mathbf{R} \otimes I(w \otimes I)\|_\sigma = \|\mathbf{R} \otimes I(v \otimes I - w \otimes I)\|_\sigma \leq \|v \otimes I - w \otimes I\|_\sigma \quad (4.13)$$

Now, regarding the metric equation for the **conditional statement**, before validating its correctness, it is necessary to prove a few intermediate results.

The first step is to demonstrate that for any super-operators P and Q the following holds:

Lemma 4.3.5. $\|[P, Q]\|_\sigma \leq \max\{\|P\|_\sigma, \|Q\|_\sigma\}$

Proof. Employing the definition of the operator norm in [Equation 4.5](#), it follows that:

$$\begin{aligned} \sup\{\|[P, Q](v)\| \mid \|v\| = 1\} &\leq \max\{\sup\{\|P(w)\| \mid \|w\| = 1\}, \sup\{\|Q(u)\| \mid \|u\| = 1\}\} \\ &= \sup\{\|[P, Q](w + u)\| \mid \|w + u\| = 1\} \leq \max\{\sup\{\|P(w)\| \mid \|w\| = 1, \|Q(u)\| \mid \|u\| = 1\}\} \\ &= \sup\{\|P(w) + Q(u)\| \mid \|w + u\| = 1\} \leq \max\{\sup\{\|P(w)\| \mid \|w\| = 1, \|Q(u)\| \mid \|u\| = 1\}\} \\ &= \sup\{\|P(w) + Q(u)\| \mid \|w + u\| = 1\} \leq \sup\{\max\{\|P(w)\| \mid \|w\| = 1, \|Q(u)\| \mid \|u\| = 1\}\} \end{aligned} \quad (4.14)$$

Therefore, by the triangle inequality, proving the inequality in [Equation 4.15](#) suffices to establish [Lemma 4.3.5](#).

$$\sup\{\|P(w)\| + \|Q(u)\| \mid \|w + u\|_1 = 1\} \leq \sup\{\max\{\|P(w)\| \mid \|w\| = 1, \|Q(u)\| \mid \|u\| = 1\}\} \quad (4.15)$$

This can be rewritten as:

$$\|w + u\| = 1 \wedge \{\|P(w)\| + \|Q(u)\| \mid \|w + u\| = 1\} \leq \max\left\{\frac{1}{\|w\|}\|P(w)\|, \frac{1}{\|u\|}\|Q(u)\|\right\} \quad (4.16)$$

As a result,

$$\|w + u\| = 1 \wedge \sup\{\|P(w)\| + \|Q(u)\| \mid \|w + u\|_1\} \leq \max\left\{\left\|P\left(\frac{1}{\|w\|}w\right)\right\|, \left\|Q\left(\frac{1}{\|u\|}u\right)\right\|\right\}$$

(4.17)

This is equivalent to demonstrating that for $a + b = 1$,

$$x + y \leq \max \left\{ \frac{1}{a}x, \frac{1}{b}y \right\} \quad (4.18)$$

This is done by arguing by *reductio ad absurdum*, i.e., supposing otherwise leads to a contradiction:

$$\begin{aligned} x + y &> \max \left\{ \frac{1}{a}x, \frac{1}{b}y \right\} \\ \Rightarrow x + y &> \frac{1}{a}x \wedge x + y > \frac{1}{b}y \\ \Rightarrow a(x + y) &> x \wedge b(x + y) > y \\ \Rightarrow ax + ay &> x \wedge bx + by > y \\ \Rightarrow ax + ay &> x \wedge (1 - a)x + (1 - a)y > y \\ \Rightarrow ax + ay &> x \wedge x - ax + y - ay > y \\ \Rightarrow x &< ax + ay \wedge x > ax + ay \end{aligned} \quad (4.19)$$

Subsequently, it is imperative to prove that:

Lemma 4.3.6. $i = [\text{IL} \otimes I, \text{IR} \otimes I]$ is an isomorphism.

Proof. The proof is as follows:

For any vector spaces V , W , and U , $i : (V \otimes U) \oplus (W \otimes U) \rightarrow (V \oplus W) \otimes U$. If V has dimension m , W has dimension n , and U has dimension o , then the space $(V \otimes U) \oplus (W \otimes U)$ has dimension $mo + no = (m + n) \cdot o$. Similarly, the space $(V \oplus W) \otimes U$ has dimension $(m + n) \cdot o$. Hence, the spaces have the same dimension. Given that spaces with the same dimension are isomorphic [Hefferon (2006)], it follows that i is an isomorphism.

Next, it is necessary to demonstrate that for any operators P and Q , the identity operator I , and an isomorphism $i = [\text{IL} \otimes I, \text{IR} \otimes I]$ the following holds:

Lemma 4.3.7. $([P, Q] \otimes I) \cdot i = [P \otimes I, Q \otimes I]$

Which is equivalent to showing that for any vector spaces V , W , U , and Z and super-operators $P : V \rightarrow Z$, $Q : W \rightarrow Z$, and $I : U \rightarrow U$, the following diagram holds:

$$\begin{array}{ccc}
V \otimes U \oplus W \otimes U & \xrightarrow{i} & (V \oplus W) \otimes U \\
\downarrow [P \otimes I, Q \otimes I] & \nearrow [P, Q] \otimes I & \\
Z \otimes U & &
\end{array}$$

Proof. The proof is straightforward:

$$\begin{aligned}
& ([P, Q] \otimes I) \cdot [I_L \otimes I, I_R \otimes I] \\
&= [([P, Q] \otimes I) \cdot (I_L \otimes I), ([P, Q] \otimes I) \cdot (I_R \otimes I)] \\
&= [P \otimes I, Q \otimes I]
\end{aligned} \tag{4.20}$$

Furhtermore, it is imperative to show that the following relation holds:

Lemma 4.3.8. $[P \otimes I, Q \otimes I] \cdot i^{-1} = [P, Q] \otimes I$

Demonstrating this is equivalent to establishing that for any vector spaces V, W, U , and Z , and super-operators $P : V \rightarrow Z, Q : W \rightarrow Z$, and $I : U \rightarrow U$, the following diagram commutes:

$$\begin{array}{ccc}
V \otimes U \oplus W \otimes U & \xleftarrow{i^{-1}} & (V \oplus W) \otimes U \\
\downarrow [P \otimes I, Q \otimes I] & \nearrow [P, Q] \otimes I & \\
Z \otimes U & &
\end{array}$$

Proof. The proof is as follows:

$$\begin{aligned}
& ([P, Q] \otimes I) \cdot i = [P \otimes I, Q \otimes I] && \{\text{Lemma 4.3.7}\} \\
& \Leftrightarrow ([P, Q] \otimes I) \cdot i \cdot i^{-1} = [P \otimes I, Q \otimes I] \cdot i^{-1} && (4.21) \\
& \Leftrightarrow ([P, Q] \otimes I) = [P \otimes I, Q \otimes I] \cdot i^{-1} && \{\text{Lemma 4.3.6}\}
\end{aligned}$$

With [Lemma 4.3.7](#) and [Lemma 4.3.8](#), it has been proved that the diagram below is valid:

$$\begin{array}{ccc}
V \otimes U \oplus W \otimes U & \begin{array}{c} \xrightarrow{i} \\ \xleftarrow{i^{-1}} \end{array} & (V \oplus W) \otimes U \\
\downarrow [P \otimes I, Q \otimes I] & \nearrow [P, Q] \otimes I & \\
Z \otimes U & &
\end{array}$$

Now, it is possible to prove that i has a norm equal to 1.

Lemma 4.3.9. $\|i\|_\sigma = 1$

Proof. Employing the definition of the operator norm as defined in [Equation 4.5](#), it follows that:

$$\begin{aligned}
& \|[\mathbf{L} \otimes I, \mathbf{R} \otimes I]\| \\
&= \max\{\|[\mathbf{L} \otimes I, \mathbf{R} \otimes I](A)\| \mid \|A\| = 1\} \\
&= \max\left\{\left\|[\mathbf{L} \otimes I, \mathbf{R} \otimes I] \left(\sum_i v_i \otimes u_i, \sum_i w_i \otimes u_i\right)\right\| \mid \left\|\left(\sum_i v_i \otimes u_i, \sum_i w_i \otimes u_i\right)\right\| = 1\right\} \\
&= \max\left\{\left\|\mathbf{L} \otimes I \left(\sum_i v_i \otimes u_i\right) + \mathbf{R} \otimes I \left(\sum_i w_i \otimes u_i\right)\right\| \mid \left\|\left(\sum_i v_i \otimes u_i, \sum_i w_i \otimes u_i\right)\right\| = 1\right\} \\
&= \max\left\{\left\|\mathbf{L} \left(\sum_i v_i\right) \otimes I \left(\sum_i u_i\right) + \mathbf{R} \left(\sum_i w_i\right) \otimes I \left(\sum_i u_i\right)\right\| \mid \left\|\left(\sum_i v_i \otimes u_i, \sum_i w_i \otimes u_i\right)\right\| = 1\right\} \\
&= \max\left\{\left\|\left(\sum_i v_i, 0\right) \otimes \sum_i u_i + \left(0, \sum_i w_i\right) \otimes \sum_i u_i\right\| \mid \left\|\left(\sum_i v_i \otimes u_i, \sum_i w_i \otimes u_i\right)\right\| = 1\right\} \\
&= \max\left\{\left\|\left(\sum_i v_i \otimes u_i, 0\right) + \left(0, \sum_i w_i \otimes u_i\right)\right\| \mid \left\|\left(\sum_i v_i \otimes u_i, \sum_i w_i \otimes u_i\right)\right\| = 1\right\} \\
&= \max\left\{\left\|\left(\sum_i v_i \otimes u_i, \sum_i w_i \otimes u_i\right)\right\| \mid \left\|\left(\sum_i v_i \otimes u_i, \sum_i w_i \otimes u_i\right)\right\| = 1\right\} \\
&= 1
\end{aligned} \tag{4.22}$$

Finally, it is possible to demonstrate that i^{-1} has a norm less or equal to 1

Lemma 4.3.10. $\|i^{-1}\|_\sigma \leq 1$

Proof. Given that i is an isomorphism, it follows that

$$\begin{aligned}
& \|i \cdot i^{-1}\|_\sigma = 1 \\
& \leq \|i\|_\sigma \cdot \|i^{-1}\|_\sigma = 1 \quad \{\text{Norm submultiplicative with respect to compositions}\} \\
& \Leftrightarrow 1 \cdot \|i^{-1}\|_\sigma = 1 \quad \{\text{Lemma 4.3.10}\} \\
& \Leftrightarrow \|i^{-1}\|_\sigma = 1
\end{aligned} \tag{4.23}$$

Now, it is finally possible to adress the proof of the metric equation for the conditional statement. Considering the the semantics of the "case" rule in [Figure 7](#), proving that the "case" rule in [Figure 8](#) is valid is equivalent to demonstrating that for any super-operators P and Q and their respective erroneous versions P' and Q' , denoting the distance between super-operators A and B as $d(A, B)$, the following holds:

Theorem 4.3.3. $d([P, Q], [P', Q']) \leq \max\{d(P, P'), d(Q, Q')\}$

Proof. In the quantum paradigm, the distance between two super-operators corresponds to the diamond norm between the two super-operators. Hence, denoting $[L \otimes I, R \otimes I]$ by i it follows that:

$$\begin{aligned}
d([P, Q], [P', Q']) &= \|[P, Q] \otimes I - [P', Q'] \otimes I\|_\sigma \\
&= \|[P \otimes I, Q \otimes I] \cdot i^{-1} - [P' \otimes I, Q' \otimes I] \cdot i^{-1}\|_\sigma && \{\text{Lemma 4.3.8}\} \\
&= \|[P - P' \otimes I, Q - Q' \otimes I] \cdot i^{-1}\|_\sigma \\
&\leq \|[P - P' \otimes I, Q - Q' \otimes I]\| \|i^{-1}\|_\sigma && \{\text{Norm submultiplicative with respect to compositions}\} \\
&\leq \|[(P - P') \otimes I, (Q - Q') \otimes I]\|_\sigma && \{\text{Lemma 4.3.10}\} \\
& && (4.24)
\end{aligned}$$

and

$$\begin{aligned}
&\max\{d(P, P'), d(Q, Q')\} \\
&= \max\{\|P \otimes I - P' \otimes I\|_\sigma, \|Q \otimes I - Q' \otimes I\|_\sigma\} && (4.25) \\
&= \max\{\|(P - P') \otimes I\|_\sigma, \|(Q - Q') \otimes I\|_\sigma\}
\end{aligned}$$

Finally, by [Lemma 4.3.5](#), it can be deduced that $d([P, Q], [P', Q']) \leq \max\{d(P, P'), d(Q, Q')\}$, which concludes the proof of theorem [Theorem 4.3.3](#).

An alternative method to establish [Theorem 4.3.3](#) is now presented. *Proof.* The proof is as

follows:

$$\begin{aligned}
& d([P, Q], [P', Q']) \\
&= \|[P, Q] \otimes I - [P', Q'] \otimes I\|_\sigma \\
&= \|([P, Q] - [P', Q']) \otimes I\|_\sigma \\
&= \|[P - P', Q - Q'] \otimes I\|_\sigma \\
&= \|[P - P', Q - Q']\|_\sigma \|I\|_\sigma \quad \{\text{Norm multiplicative with respect to tensor products}\} \\
&= \|[P - P', Q - Q']\|_\sigma \quad \{\text{Lemma 4.3.1}\} \\
& \quad (4.26)
\end{aligned}$$

Moreover,

$$\begin{aligned}
& \max\{d(P, P'), d(Q, Q')\} \\
&= \max\{\|P \otimes I - P' \otimes I\|_\sigma, \|Q \otimes I - Q' \otimes I\|_\sigma\} \\
&= \max\{\|(P - P') \otimes I\|_\sigma, \|(Q - Q') \otimes I\|_\sigma\} \\
&= \max\{\|(P - P')\|_\sigma \|I\|_\sigma, \|(Q - Q')\|_\sigma \|I\|_\sigma\} \quad \{\text{Norm multiplicative with respect to tensor products}\} \\
&= \max\{\|(P - P')\|_\sigma, \|(Q - Q')\|_\sigma\} \quad \{\text{Lemma 4.3.1}\} \\
& \quad (4.27)
\end{aligned}$$

Therefore, by [Lemma 4.3.5](#), it can be deduced that $d([P, Q], [P', Q']) \leq \max\{d(P, P'), d(Q, Q')\}$, which concludes the proof of theorem [Theorem 4.3.3](#).

Next, one has to prove that for any super-operators P and Q and their respective erroneous versions P' and Q' , the following holds:

Lemma 4.3.11. $\|P \cdot Q - P' \cdot Q'\|_\sigma \leq \|P - P'\|_\sigma + \|Q - Q'\|_\sigma$

4.3.2 Quantum Teleportation

The circuit corresponding to the implementation of the quantum teleportation protocol is depicted in [Figure 9](#).

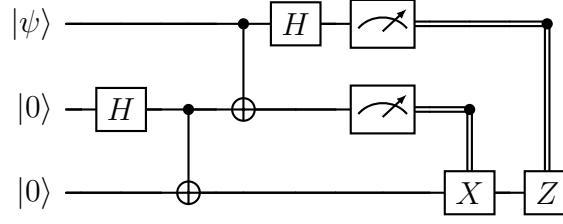


Figure 9: Quantum Teleportation Protocol

When formalizing the quantum teleportation protocol within the lambda calculus framework, each part of the protocol is instantiated as a distinct function. This entails the definition of three specific functions:

EPR : $\mathbb{I} \multimap (\text{qbit} \otimes \text{qbit})$

BellMeasure : $\text{qbit} \otimes \text{qbit} \multimap \text{bit} \otimes \text{bit}$

Correction : $\text{qbit} \otimes \text{bit} \otimes \text{bit} \multimap \text{qbit}$

Considering the unitary operations $H : \text{qbit} \rightarrow \text{qbit}$, $X : \text{qbit} \rightarrow \text{qbit}$, $Z : \text{qbit} \rightarrow \text{qbit}$, $I : \text{qbit} \rightarrow \text{qbit}$, and $CNOT : \text{qbit}, \text{qbit} \rightarrow \text{qbit} \otimes \text{qbit}$, these functions are defined as follows:

EPR = $\lambda . \triangleright CNOT(H(q(new\ 0(*))), (q(new\ 0(*))))$

BellMeasure = $q : \text{qbit}, q_2 : \text{qbit} \triangleright (\text{pm } CNOT(q_1, q_2) \text{ to } x \otimes y. \text{meas}(H(x)) \otimes \text{meas}(y))$

Correction = $q : \text{qbit}, x : \text{bit}, y : \text{bit} \triangleright \text{cond } x \{ \text{inl}(x_0) \Rightarrow (\text{cond } y \{ \text{inl}(y_0) \Rightarrow Z(X(q));$
 $\text{inr}(y_1) \Rightarrow Z(q) \}) \}$
 $\text{inr}(x_1) \Rightarrow (\text{cond } y \{ \text{inl}(y_0) \Rightarrow X(q);$
 $\text{inr}(y_1) \Rightarrow I(q) \}) \}$

4.3.3 Illustration: Noisy Quantum Teleportation

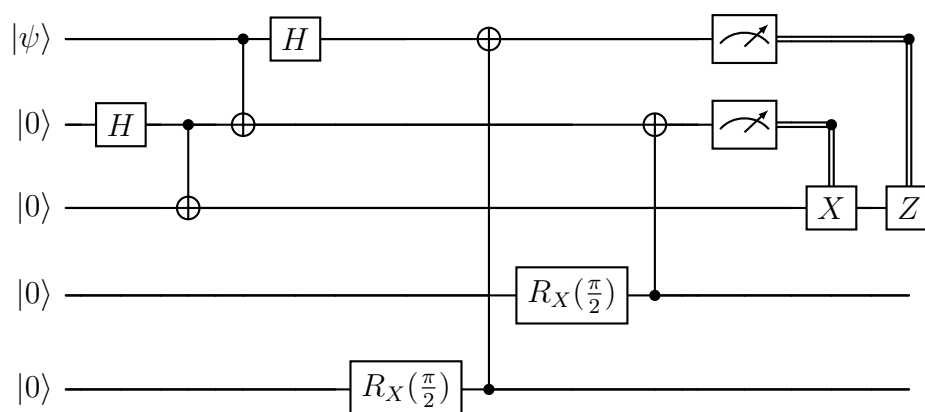


Figure 10: Quantum Teleportation Protocol with bit flip with a 50% probability.

Chapter 5

Applications

Application of main result (examples and case studies)

5.1 Introduction

5.2 Summary

Chapter 6

Conclusions and future work

Conclusions and future work.

6.1 Conclusions

6.2 Prospect for future work

Chapter 7

Planned Schedule

7.1 Activities

Task	Oct	Nov	Dec	Jan	Feb	Mar	Apr	May	Jun	Jul
Background and SOA	•	•	•							
PDR preparation		•	•	•						
Contribution				•	•	•	•	•	•	
Writing up							•	•	•	•

Table 1: Activities Plan

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Part III

Appendices

Appendix A

Support work

Auxiliary results which are not main-stream.

Appendix B

Details of results

Details of results whose length would compromise readability of main text.

Appendix C

Listings

Should this be the case.

Appendix D

Tooling

(Should this be the case)

Anyone using [L^AT_EX](#) should consider having a look at [TUG](#) , the [T_EX Users Group](#) .

Place here information about funding, FCT project, etc. in which the work is framed. Leave empty otherwise.