University of Minho
School of Engineering

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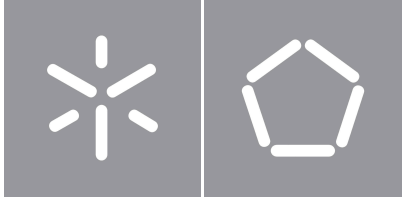
Metric λ-calculus with conditionals: from quantum to probabilistic and beyond

July 2025



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Metric λ-calculus with conditionals: from quantum to probabilistic and beyond

Master's Dissertation
Master in Physics Engineering

Work carried out under the supervision of
Renato Jorge Araújo Neves

July 2025

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University of Minho, Braga, July 2025

Bruna Filipa Martins Salgado

Abstract

Keywords approximate equivalence, λ-calculus, metric equations

Resumo

Escrever aqui o resumo (pt)

Palavras-chave palavras, chave, aqui, separadas, por, vírgulas

Contents

List of Figures

Acronyms

NISQ Noisy Intermediate-Scale Quantum 8

BNF Backus-Naur Form 13, 14

CPTP Completely Positive Trace-Preserving 107

OSR Operator Sum Representation 108

Notation

$FV(v)$ Set of free variables of a term v. 14

$v : \mathbb{A}$ Typed term. 15

Γ, Δ, E Typical names for typing contexts. 15

$\Gamma \triangleright v : \mathbb{A}$ Typing judgement. 15

$v[w/x]$ Substitution of a variable x for a term w in a term v. 19

$\Gamma \triangleright v = w : \mathbb{A}$ Equation-in-context. 23

$t =_\epsilon s$ Metric equation. 30

$\| \cdot \|$ Norm of an arbitrary vector. 76

$d(x, y)$ distance between vectors x and y. 76

$\| \cdot \|_{\mathbf{op}}$ Operator norm. 76

$\mathcal{B}(V, W)$ Vector space of all bounded linear operators from V to W. 76

$\mathcal{B}(V)$ Vector space of all bounded linear operators from V to itself. 76

$\otimes_{\mathbf{meas}}$ Product measure. 83

$\mathcal{M}\mathbb{R}$ Banach space of finite Borel measures on $\mathbb{R}$. 83

$\langle \cdot, \cdot \rangle$ Inner product. 92

$\overline{(-)}$ Complex conjugate operation. 92

$\mathcal{H}, \mathcal{K}$ Typical names for Hilbert spaces. 92

$(-)^\dagger$ Adjoint operation. 93

U Typical name for a unitary operator. 94

ρ Typical designation for a density matrix. 94

$(-)^{\otimes n}$ N-fold tensor product. 96

$\|\cdot\|_2$ Euclidean norm. 96

$\|\cdot\|_1$ Trace norm. 96

$\mathcal{H} \otimes_2 \mathcal{K}$ Hilbert space tensor product of $\mathcal{H}$ and $\mathcal{K}$. 97

$|\psi\rangle$ Quantum state. Also known as ket. 97

$\langle\psi|$ $|\psi\rangle^\dagger$. Also known as bra. 97

$\langle\psi|\phi\rangle$ Inner product between states $|\psi\rangle$ and $|\phi\rangle$. 97

$|\psi\rangle \otimes |\phi\rangle$ Tensor product of states $|\psi\rangle$ and $|\phi\rangle$. 99

$|\psi\rangle\,|\phi\rangle$ Tensor product of states $|\psi\rangle$ and $|\phi\rangle$. 99

$|\psi\phi\rangle$ Tensor product of states $|\psi\rangle$ and $|\phi\rangle$. 99

X Pauli operator σ_x. 101

Z Pauli operator σ_z. 101

H Hadamard gate 102

CNOT Controlled Not gate 102

$\|\cdot\|_\diamond$ Diamond norm. 108

$\mathcal{M}_n$ Vector space of complex $n \times n$ matrices. 114

Part I

Foundations

Chapter 1

Introduction

1.1 Motivation and Context

Some History

Hilbert's Optimistic Vision of Mathematics In September 1928, David Hilbert presented his vision for the foundations of mathematics at the International Congress of Mathematicians in Bologna. He believed it would be possible to place mathematics on an absolutely secure foundation. This would mean that no matter how difficult a mathematical problem might be, one would only need to "take up the pen, sit at the abacus, and calculate" [1]. The process would be entirely deterministic—requiring no intuition or creativity, only strict adherence to formal rules, like performing multiplication in decimal notation. Every problem would, in principle, be solvable by such mechanical procedures. Mathematics, would be both complete (able to answer every question) and consistent (free of contradictions).

Gödel's Incompleteness Theorems However, this vision was shattered by Kurt Gödel's Incompleteness Theorems (1931) [2], which showed that no set of mathematical rules powerful enough to handle basic arithmetic could ever be both complete (answering every question) and consistent (free of contradictions) at the same time.

Gödel's Completeness Theorems Interestingly, in his doctoral thesis, Gödel proved a foundational result in logic: first-order predicate logic—a formal system used to express statements involving quantifiers like "for all" and "there exists"—is *complete* [3]. It is important to note that the term "completeness" here differs from its use in Gödel's later Incompleteness Theorems. Before exploring this notion of completeness, it is helpful to introduce a few core concepts. *Syntax* refers to the formal symbols and inference rules used to construct well-formed statements, while *semantics* concerns the meaning assigned to these statements

through interpretations. A *model* of a first-order system is a mathematical structure in which the axioms (or rules) hold true under a given interpretation. With these notions in place, we can now turn to Gödel's result, known as the *Completeness Theorem*. This theorem states that if a statement holds in every possible model of a theory, then it can also be *syntactically* proven using the system's formal rules. In other words, completeness is the property that all universally valid statements are provable within the system. The converse also holds: any statement provable syntactically must hold true in all models. This is known as *soundness* [4].

Entscheidungsproblem We have just introduced two of the main pillars of this thesis — soundness and completeness. We now proceed to introduce a third, deeply tied to Hilbert's ambitious vision for mathematics. Recall that Hilbert not only sought a complete and consistent foundation for mathematics, but also believed in the possibility of an entirely *mechanical* method to resolve any mathematical problem—a process requiring no intuition or creative insight. In 1928, he formulated the Entscheidungsproblem (German for "decision problem"), which sought an *effective method* (also called a mechanical procedure or algorithm) to determine the truth or falsity of any mathematical statement [5]. A method or procedure is effective if:

1. it can be described by a finite number of exact instructions;
2. it produces the desired result after a finite number of steps (provided the instructions are followed without error);
3. it can, in principle, be carried out by a human using only paper and pencil;
4. it does not require any creativity or insight from the human.

The algorithms that children learn to perform basic arithmetic operations are examples of effective procedures.

Alonzo Church and the λ-calculus enter the scene Remarkably, it was Alonzo Church—using λ-*calculus*—who first addressed Hilbert's *Entscheidungsproblem*. This brings us to the third central theme of this dissertation: λ-*calculus*. In 1936, Alonzo Church published a solution to the *Entscheidungsproblem*, proving that no universal algorithmic method could decide the truth of all mathematical statements [6]. Today, this result is often referred to as emphChurch's Theorem. In the same work, he provided a mathematically precise notion

of an effective method. He proposed that a function is effectively computable a function is computable if and only if it can be written as a lambda term. This equivalence provided the first rigorous mathematical criterion for computability.

This calculus played an important role in functional programming, influencing the design of languages like LISP, Pascal, and GEDANKEN—many of which incorporate λ-calculus-inspired features, either explicitly or implicitly.Furthermore, λ-calculus may be employed to prove properties of programming language (such as: a well-formed program will not crash) and as a tool in the construction of compilers [7].

The idea that such important aspects of modern computer science emerged from foundational questions in mathematics is nothing short of extraordinary.

A note on Turing's work Around the same time, another researcher—unaware of Church's work—was independently addressing the same problem: Alan Turing, now widely regarded as the father of computer science. He introduced the concept of a universal machine, now known as the *Turing Machine*. While Turing's model is groundbreaking in its own right, it is largely orthogonal to this dissertation, as it is more closely associated with automata theory than with the syntax and semantics of programming languages.

λ-calculus

λ-calculus and functions The λ-calculus serves as a formal language that captures a key feature of higher-order functional languages: treating functions as "first-class citizens" that can be passed as arguments. Here functions are expressed as *formulae* of the form $\lambda x.\, f(x)$, with application denoted by juxtaposition. For example, the expression $f(2)$, where $f(x) = x + 1$, is written as $(\lambda x.\, x + 1)(2)$.

Typed-lambda calculus In this work, we use the typed λ-calculus, a variant where each term is "labeled" by a syntactic object called a *type*. Types serve as a mechanism for ensuring that programs are meaningful. In contrast, the untyped version allows, for example, a function to be applied to itself, as in $(\lambda x.\, x\, x)(\lambda x.\, x\, x)$, leading to non-termination. It also allows nonsensical operations, such as applying a boolean to a number or a function to a string.

λ-calculus and logic We previously mentioned that one of our main results pertains to soundness and completeness—a notion we deliberately introduced in the setting of (first-order) logic. In fact, the typed lambda calculus itself is equipped with an equational logic, i.e., a system of equations. These equations arise because the λ-calculus includes n-ary function

symbols, which may be accompanied by equality axioms specifying their intended properties. Moreover, the lambda calculus allows to establishe a correspondence between logical proofs and programs. This is known as the *Curry-Howard isomorphism* [8].

Semantics:λ-calculus and category theory In this work, we interpret programs as mathematical objects, particularly those arising in category theory. But why choose a categorical interpretation interpretation over other alternatives?

Consider an ancient Indian parable: six blind men encounter an elephant for the first time. Each man touches a different part of the animal—the side, tusk, trunk, leg, ear, or tail—and draws a conclusion based solely on that limited experience. One describes it as a spear (the tusk), another as a snake (the trunk), and another as a fan (the ear). Each is convinced of his own interpretation and dismisses the others as incorrect. None of them realises that they are each experiencing only a part of the same elephant, and that their individual descriptions are incomplete. In some versions of the story, the men stop arguing, begin listening to one another, and collaborate to form a more accurate understanding of the whole elephant.

Category theory plays a similar role in computer science. Each category embodies a distinct perspective —a "part of the elephant" —capturing a specific computational paradigms. Adopting a categorical approach, allows us to generalize our results across diverse computational paradigms.

However, there is a deeper reason for using category theory in this setting: it is intimately connected to the λ-calculus. First, it should be noted that λ-calculus is a type theory — and here lies the twist: categories themselves can be viewed as type theories. The objects may be regarded as types (of sorts), and the arrows as functions between those types. In this sense, a category may be thought of as a type theory stripped of its syntax. With this perspective in mind, in the 1970s, Joachim Lambek established a correspondence with cartesian closed categories and the λ-calculus [9]. That is, types correspond to objects, and programs correspond to arrows in such categories. This correspondence extends further to logic, under the so-called Curry–Howard–Lambek correspondence, where formulas correspond to types and proofs to arrows. Later, Lambek and Dana Scott independently observed that C-monoids (*i.e.*categories equipped with products, exponentials, and a single non-terminal object) correspond to the untyped λ-calculus [10].

Going quantitative

Beyond its foundational aspects, this calculus incorporates extensions for modeling side effects, including probabilistic or non-deterministic behaviors and shared memory. In this work, we are concerned with a version of λ-calculus that allows us to reason about approximate equivalence of programs, refered to as *metric λ-calculus*.

Program equivalence and its underlying theories traditionally rely on a binary notion of equivalence: two programs are either equivalent or not [11]. While this dichotomy is often sufficient for classical programming, it proves too coarse-grained for other computational paradigms. For instance, in various programming paradigms, interaction with the physical environment calls for notions of approximate program equivalence.

To adress this, [12, 13] incorporate a notion of approximate equivalence into the equational system of the affine λ-calculus by introducing, among other elements, *metric equations* [14, 15]. These are equations of the form $t =_\varepsilon s$, where ε is a non-negative real number representing the "maximum distance" between terms t and s. Here we start investigating the incorporation of a metric equation for the case statements (*i.e.* conditionals). Our motivation for it is highly practical: in trying to reason quantitatively about higher-order programs we often fell short when these involved conditionals.

Quantitative logics offer a way forward, extending beyond λ-calculus. They reflects a broader effort to move beyond rigid binary concepts, such as equality and bisimulation, and toward more flexible frameworks better suited to real-world computation.

Other works in the spirit of this dissertation include [14–17], which explore (generalized) metric universal algebras. A universal algebra, in simple terms, is a set equipped with any number of operations, further defined by axioms typically expressed as identities or equational laws. In a (generalized) metric universal algebra, these axioms are relaxed into (generalized) metric equations rather than strict equalities. In the higher-order setting, [18], following the framework introduced by Mardare [14], investigates the problem of defining quantitative algebras that are capable of interpreting terms in higher-order calculi.

Probabilistic Programming

Probabilistic programs are quite ubiquitous: they control autonomous systems, verify security protocols, and implement randomized algorithms for solving computationally intractable

problems. At their core, they aim to democratize probabilistic modeling by providing programmers with expressive, high-level abstractions for machine learning and statistical reasoning [19]. In this context, concerns such as the development of more eco-friendly programs and algorithms could greatly benefit from a quantitative approach.

Quantum Computation

In 1994, Peter Shor demonstrated that a quantum computer with sufficiently many qubits could pose a significant threat to the security of confidential data transmitted over the Internet [20]. This breakthrough spurred widespread interest in quantum computing. Nevertheless, **Noisy Intermediate-Scale Quantum** (**NISQ**) computers are expected to operate with severely limited hardware resources. Precisely controlling qubits in these systems comes at a high cost, is susceptible to errors, and faces scarcity challenges. Therefore, quantitative reasoning is indispensable for the design, optimization, and assessment of NISQ computing.

1.2 Contributions

Our contributions fall into three categories: syntactic, semantic, and concerned with the connection between the two.

Syntatic

We build on the work of [13] by introducing a metric equation for conditionals. Next, to illustrate the utility of the metric equation introduced, we present two simple examples: reasoning about approximate equivalence between boolean terms (*i.e.*terms of type $\mathbb{I} \oplus \mathbb{I}$) the extensionality of copairing.

Semantic

Returning to our earlier elephant parable, we study various "perspectives" by proving that the corresponding categories are indeed models:

- The category of metric spaces Met;
- Cocompletion of a Met-category C;

- Category Ban of Banach spaces and short maps;
- Selinger's Q, the category of quantum operationts (*i.e.*, completely positive, trace non-increasing superoperators) [21], and Cho's $(\mathsf{Wstar}_{\mathsf{CPSU}})^{\mathsf{op}}$, the opposite category of W*-algebras and normal, completely positive, subunital maps [22].

This demonstrates that our work is applicable across several domains. For the last two quantum models, we restrict our attention to the first-order fragment of the λ-calculus, noting that extensions to higher-order are possible using more more advances categorical tools, as in[13].

We investigate two paradigms in greater detail: probabilistic computation (via Ban) and quantum computation (via Q and $(\mathsf{Wstar}_{\mathsf{CPSU}})^{\mathsf{op}}$). In the probabilistic setting, we use a random walk to reason about approximate equivalence. In the quantum setting, we use three examples: quantum state discrimination, quantum teleportation, and quantum random walks.

Connection between syntax and semantics

We prove that the metric equation introduced its sound and complete. Soundness ensures that if a metric equation $t =_\varepsilon s$ can be derived in the calculus, then the distance between the interpretations of t and s is at most ε. Completeness guarantees that if ε is the maximal distance between the interpretations of two programs, then we can derive $t =_\varepsilon s$ in the calculus.

1.3 Document Structure

Chapter 2 introduces (metric) λ-calculus along with its categorical interpretation, accompanied by the necessary notions from category theory used throughout the thesis. One advantage of working with a metric equational system is the ability to reason syntactically about approximate equivalence. We leverage this idea in an interlude on booleans (i.e., terms of type $\mathbb{I} \oplus \mathbb{I}$) to illustrate the usefulness of the classical equational system. In Chapter 3, we introduce a metric equation for conditionals, prove its soundness and completeness, and present a few models in this setting, along with a few illustrative syntactic examples. Then, in Chapter 4 and Chapter 5, we focus on reasoning about higher-order probabilistic programs and first-order quantum programs, respectively, including the necessary background in each domain. The thesis concludes with directions for future work in Chapter 6.

Although this work uses knowlage across multiple areas, the author's engagement with them is mostly limited to the scope of this thesis.

Chapter 2

Metric Lambda Calculus

This chapter introduces the metric lambda calculus as presented in [13] drawing also from [23–25]. The metric lambda calculus integrates notions of approximation into the equational system of affine lambda calculus, a variant of lambda calculus that restricts each variable to being used at most once. This chapter offers a brief insight into lambda calculus and an overview of the syntax, metric equational system and interpretation of the metric lambda calculus. It also includes an exposition of the definitions and results from category theory used throughout the remainder of the thesis, based on [26–28]. We note that the vast majority of the examples used in this context are drawn form [26]. These categorical notions play a central role, as programs are interpreted within distributive symmetric monoidal (closed) categories. It is worth noting that we explicitly prove certain results concerning conditionals; although these results are well known, their proofs appear to be absent from the existing literature. Additionally, we illustrate the usefulness of the "traditional" equational system by utilizing it to demonstrate that the terms $\mathrm{inl}(*)$ and $\mathrm{inr}(*)$ possess certain properties that are characteristic of booleans in classical logic. For a more detailed study of lambda calculus theory, the reader is referred to [29].

2.1 The Lambda Calculus

The concept of a function takes a central role in the lambda calculus. But what exactly is a function? In most mathematics, the "functions as graph" paradigm is the most elegant and appropriate framework for understanding functions. Within this paradigm, each function f has a fixed domain X and a fixed codomain Y. The function f is then a subset of $X \times Y$ that satisfies the property that for each $x \in X$ there is a unique $y \in Y$ such that $(x, y) \in f$. Two functions f and g are equal if they yield the same output on each input, that is if $f(x) = g(x)$

for all $x \in X$. This perspective is known as the *extensional* view of functions, as it emphasizes that the only observable property of a function is how it maps inputs to outputs.

From a Computer Science perspective, this is not very useful. We are typically just as concerned with how a function computes its result as we are with what it produces. For instance, consider sorting: every correct sorting algorithm, from the simplest to the most sophisticated, produces the same output for a given input. Yet entire books and research papers are devoted to analyzing different sorting techniques. Clearly, something important is being overlooked. The casual use of the term "algorithm" in that context is revealing: a function should be represented not by its graph, but by the rule or process that describes how its result is computed. Such a rule is often expressed in the form of a formula, for example, $f(x) = x^2$. As with the mathematical paradigm, two functions are considered extensionally equal if they exhibit the same input-output behavior. However, this view also introduces the notion of *intensional* equality: two functions are intensionally equal if they are defined by (essentially) the same formula.

In the lambda calculus, functions are described explicitly as *formulae*. The function $f : x \mapsto f(x)$ is represented as $\lambda x.f(x)$. Applying a function to an argument is done by juxtaposing the two expressions. For instance consider the function $f : x \mapsto x + 1$, to compute $f(2)$ one writes $(\lambda x.x + 1)(2)$.

A major limitation of the notation appears to be that we can only define unary functions, that is, we can introduce only one argument at a time. However, this is not a true restriction. Suppose we have a binary function represented as an expression with formal arguments x and y, say $f(x, y)$. Then we can define a new function g as $g = \lambda y.\,(\lambda x.\,f(x, y))$. This function g is equivalent to the original binary function f, but it takes its arguments one at a time. This idea, known as *currying*, shows how functions of multiple arguments can be represented using only unary functions.

The expression of *higher-order functions*, functions whose inputs and/or outputs are themselves functions, in a simple manner is an essential feature of lambda calculus. For example, the composition operator $f, g \mapsto f \circ g$ is written as $\lambda f.\lambda g.\lambda x.f(g(x))$. Considering the functions $f : x \mapsto x^2$ and $g : x \mapsto x + 1$, to compute $(f \circ g)(2)$ one writes

$$(\lambda f.\lambda g.\lambda x.f(g(x)))(\lambda x.x^2)(\lambda x.x + 1)(2).$$

As mentioned above, within the "functions as rules" paradigm, is not always necessary to specify the domain and codomain of a function in advance. For instance, the identity func-

tion $f : x \mapsto x$, can have any set X as its domain and codomain, provided that the domain and codomain are the same. In this case, one says that f has type $X \to X$.

This flexibility regarding domains and codomains enables operations on functions that are not possible in ordinary mathematics. For instance, if $f = \lambda x.x$ is the identity function, then one has that $f(x) = x$ for any x. In particular, by substituting f for x, one obtains $f(f) = (\lambda x.x)(f) = f$. Note that the equation $f(f) = f$ is not valid in conventional mathematics, as it is not permissible, due to set-theoretic constraints, for a function to belong to its own domain.

Nevertheless this remarkable aspect of lambda calculus, this work focuses on a more restricted version of the lambda calculus, known as the simply-typed lambda calculus. Here, each expression is always assigned a type, which is very similar to the situation in mathematics. A function may only be applied to an argument if the argument's type aligns with the function's expected domain. Consequently, terms such as $f(f)$ are not allowed, even if f represents the identity function.

2.2 Syntax

2.2.1 Type system

As previously mentioned, this work focuses on the simply-typed lambda calculus, where each lambda term is assigned a *type*. Unlike sets, types are *syntactic* objects, meaning they can be discussed independently of their elements. One can conceptualize types as names or labels for set. The definition of the grammar of types for affine lambda calculus is as follows, where G represents a set of ground types, is given by the following **Backus-Naur Form** (**BNF**) [30].

$$\mathbb{A} ::= X \in G \mid \mathbb{I} \mid \mathbb{A} \otimes \mathbb{A} \mid \mathbb{A} \oplus \mathbb{A} \mid \mathbb{A} \multimap \mathbb{A} \tag{2.1}$$

Note that this is an inductive definition. Ground types are things such as booleans, integers, and so forth. The type $\mathbb{I}$ is the unit/empty type with only one element. The type $\mathbb{A} \otimes \mathbb{A}$ corresponds to the tensor of two types. The type $\mathbb{A} \oplus \mathbb{A}$ can be seen as the coproduct/sum of types. Finally, the type $\mathbb{A} \multimap \mathbb{B}$ is the type of linear maps one type to another.

2.2.2 (Raw)Terms

The expressions of the lambda calculus are called lambda terms. In the simply-typed lambda calculus, each lambda term is assigned a type. The terms without the specification of a type are called *raw typed lambda terms*. The grammar of *raw typed lambda terms* is given by the **BNF** below.

$$\begin{aligned} v, v_1, \ldots, v_n, w \quad ::= \quad & x \mid f(v_1, \ldots, v_n) \mid * \mid (\lambda x : \mathbb{A}.v) \mid (vw) \mid v \otimes w \mid \\ & \text{pm } v \text{ to } x \otimes y.w \mid v \text{ to } *.w \mid \text{dis}(v) \mid \text{inl}(v) \mid \text{inr}(v) \mid \\ & \text{case } v \, \{\text{inl}_{\mathbb{B}}(x) \Rightarrow w; \, \text{inr}_{\mathbb{A}}(y) \Rightarrow u\} \end{aligned}$$

Here x ranges over an infinite set of variables. $f \in \Sigma$, where Σ corresponds to a class of sorted operation symbols, and $f(v_1, \ldots, v_n)$ corresponds to the aplication of the function f to the arguments $v_1, \ldots, v_n$. The symbol $*$ is the unit element of the type $\mathbb{I}$. The term $(\lambda x : \mathbb{A}.v)$ is the lambda abstraction term, which represents a function that takes an argument of type $\mathbb{A}$ and returns the value of v. The term (vw) is the application term, which applies the function v to the argument w. The term $v \otimes w$ is the tensor product of v and w. The term pm v to $x \otimes y.w$ is the pattern-matching construct, which is used to deconstruct a tensor product into components x and y. The term v to $*.w$ is used to discard a variable v of the unit type. The term $\text{dis}(v)$ is the discard term, which is used to discard a term v. The terms $\text{inl}_{\mathbb{B}}(v)$ and $\text{inr}_{\mathbb{A}}(v)$ represent the left and right injections of v, respectively. Intuitively, the case statement executes w when v is a left injection, and u when v is a right injection, and a "mixture" of both otherwise.

Ver o que por antes do ::= porque v1,..., vn tb são termos

2.2.3 Free and Bound Variables

An occurrence of a variable x within a term of the form $\lambda x.v$ is referred to as *bound*. Similarly, the variables x and y in the term pm v to $x \otimes y.w$ are also bound. A variable occurrence that is not bound is said to be *free*. For example, in the term $\lambda x.xy$, the variable y is free, whereas the variable x is bound.

The set of free variables of a term v is denoted by $FV(v)$, and is defined inductively as follows:

$$FV(x) = \{x\}, \qquad FV(*) = \emptyset,$$

$$FV(f(v_1, \ldots, v_n)), = FV(v_1) \cup \ldots \cup FV(v_n) \qquad FV(\lambda x : \mathbb{A}.v) = FV(v) \backslash \{x\},$$

$$FV(vw) = FV(v) \cup FV(w), \qquad FV(v \otimes w) = FV(v) \cup FV(w),$$

$$FV(\mathsf{pm}\ v\ \mathsf{to}\ x \otimes y.w), = FV(v) \cup (FV(w) \backslash \{x, y\}) \qquad FV(\mathsf{dis}(v)) = FV(v),$$

$$FV(v\ \mathsf{to}\ *.w) = FV(v) \cup FV(w) \qquad FV(\mathsf{inl}_{\mathbb{B}}(v)) = FV(\mathsf{inr}_{\mathbb{A}}(v)) = FV(v).$$

$$FV(\mathsf{case}\ v\ \{\mathsf{inl}_{\mathbb{B}}(x) \Rightarrow w;\ \mathsf{inr}_{\mathbb{A}}(y) \Rightarrow u\}) = FV(v) \cup (FV(w) \backslash \{x\}) \cup (FV(u) \backslash \{y\})$$

2.2.4 Term formation rules

To prevent the formation of nonsensical terms within the context of lambda calculus, such as $(v \otimes w)(u)$, the *typing rules* are imposed.

A *typed* term is a pair consisting of a term and its corresponding type. The notation $v : \mathbb{A}$ denotes that the term v has type $\mathbb{A}$. Typing rules are formulated using *typing judgments*. A typing judgment is an expression of the form $x_1 : \mathbb{A}_1, \ldots, x_n : \mathbb{A}_n \triangleright v : \mathbb{A}$ (where $n \geq 1$), which asserts that the term v is a well-typed term of type $\mathbb{A}$ under the assumption that each variable variable x_i has type $\mathbb{A}_i$, for $1 \leq i \leq n$. The list $x_1 : \mathbb{A}_1, \ldots, x_n : \mathbb{A}_n$ of typed variables is called the *typing context* of the judgment, and it might be empty. Each variable x_i (where $1 \leq i \leq n$) must occur at most once in $x_1, \ldots, x_n$. The typing contexts are denoted by Greek letters Γ, Δ, E, and from now on, when referring to an abstract judgment, the notation $\Gamma \triangleright v : \mathbb{A}$ will be employed. The empty context is denoted by $-$. Note that in the affine lambda calculus, different contexts do not share variables. For example, if $\Gamma = x : \mathbb{A}, y : \mathbb{B}$ none of these variables can appear in any other context.

The concept of *shuffling* is employed to construct a linear typing system that ensures the admissibility of the exchange rule and enables unambiguous reference to judgment's denotation $[\![\Gamma \triangleright v : \mathbb{A}]\!]$. An admissible rule is not explicitly included in the formal definition of type theory, but its validity can be proven by demonstrating that whenever the premises can be derived, it is possible to construct a derivation of its conclusion. Shuffling is defined as a permutation of typed variables in a sequence of contexts, $\Gamma_1, \ldots, \Gamma_n$, preserving the relative order of variables within each Γ_i [31]. For instance, if $\Gamma_1 = x : \mathbb{A}, y : \mathbb{B}$ and $\Gamma_2 = z : \mathbb{D}$, then $z : \mathbb{D}, x : \mathbb{A}, y : \mathbb{B}$ is a valid shuffle of Γ_1, Γ_2. On the other hand, $y : \mathbb{B}, x : \mathbb{A}, z : \mathbb{D}$ is not a shuffle because it alters the occurrence order of x and y in Γ_1. The set of shuffles

in $\Gamma_1, \ldots, \Gamma_n$ is denoted as $\mathsf{Sf}(\Gamma_1, \ldots, \Gamma_n)$. A valid typing derivation is constructed using the inductive rules shown in Figure 1.

$$\frac{\Gamma_i \triangleright v_i : \mathbb{A}_i \quad f : \mathbb{A}_1, \ldots, \mathbb{A}_n \to \mathbb{A} \in \Sigma \quad E \in \mathsf{Sf}(\Gamma_1; \ldots; \Gamma_n)}{E \triangleright f(v_1, \ldots, v_n) : \mathbb{A}}\,(\mathsf{ax}) \qquad \frac{}{x : \mathbb{A} \triangleright x : \mathbb{A}}\,(\mathsf{hyp})$$

$$\frac{}{- \triangleright * : \mathbb{I}}\,(\mathbb{I}_i) \qquad \frac{\Gamma \triangleright v : \mathbb{A} \otimes \mathbb{B} \quad \Delta, x : \mathbb{A}, y : \mathbb{B} \triangleright w : \mathbb{D} \quad E \in \mathsf{Sf}(\Gamma; \Delta)}{E \triangleright \mathsf{pm}\ v\ \mathsf{to}\ x \otimes y.w : \mathbb{D}}\,(\otimes_e)$$

$$\frac{\Gamma \triangleright v : \mathbb{A} \quad \Delta \triangleright w : \mathbb{B} \quad E \in \mathsf{Sf}(\Gamma; \Delta)}{E \triangleright v \otimes w : \mathbb{A} \otimes \mathbb{B}}\,(\otimes_i) \qquad \frac{\Gamma \triangleright v : \mathbb{I} \quad \Delta \triangleright w : \mathbb{A} \quad E \in \mathsf{Sf}(\Gamma; \Delta)}{E \triangleright v\ \mathsf{to}\ *.w : \mathbb{A}}\,(\mathbb{I}_e)$$

$$\frac{\Gamma, x : \mathbb{A} \triangleright v : \mathbb{B}}{\Gamma \triangleright \lambda x : \mathbb{A}.v : \mathbb{A} \multimap \mathbb{B}}\,(\multimap_i) \qquad \frac{\Gamma \triangleright v : \mathbb{A} \multimap \mathbb{B} \quad \Delta \triangleright w : \mathbb{A} \quad E \in \mathsf{Sf}(\Gamma; \Delta)}{E \triangleright vw : \mathbb{B}}\,(\multimap_e)$$

$$\frac{\Gamma \triangleright v : \mathbb{A}}{\Gamma \triangleright \mathsf{dis}(v) : \mathbb{I}}\,(\mathsf{dis}) \qquad \frac{\Gamma \triangleright v : \mathbb{A}}{\Gamma \triangleright \mathrm{inl}_{\mathbb{B}}(v) : \mathbb{A} \oplus \mathbb{B}}\,(\mathsf{inl}) \qquad \frac{\Gamma \triangleright v : \mathbb{B}}{\Gamma \triangleright \mathrm{inr}_{\mathbb{A}}(v) : \mathbb{A} \oplus \mathbb{B}}\,(\mathsf{inr})$$

$$\frac{\Gamma \triangleright v : \mathbb{A} \oplus \mathbb{B} \qquad \Delta, x : \mathbb{A} \triangleright w : \mathbb{D} \qquad \Delta, y : \mathbb{B} \triangleright u : \mathbb{D} \qquad E \in \mathrm{Sf}(\Gamma; \Delta)}{E \triangleright \mathsf{case}\ v\ \{\mathrm{inl}_{\mathbb{B}}(x) \Rightarrow w;\ \mathrm{inr}_{\mathbb{A}}(y) \Rightarrow u\} : \mathbb{D}}\,(\mathsf{case})$$

Figure 1: Term formation rules of affine lambda calculus.

The rule (ax) states that if there is a function $f \in \Sigma$ that has type $\mathbb{A}_1, \ldots, \mathbb{A}_n \to \mathbb{A}$ and a set of variables $v_1, \ldots, v_n$ whose types match the type of the arguments of f, then if that function is applied to v_1, ..., v_n the respective result is of type $\mathbb{A}$. The rule (hyp) is a tautology: under the assumption that x has type $\mathbb{A}$, x has type $\mathbb{A}$. The rule $(\mathbb{I}_i)$ asserts that the unit element $*$ always has type $\mathbb{I}$. The rule $(\multimap_i)$ expresses that if v is a term of type $\mathbb{B}$ with a variable x of type $\mathbb{A}$, then $\lambda x : \mathbb{A}.v$ is a function of type $\mathbb{A} \multimap \mathbb{B}$. The rule $(\multimap_e)$ states that a function of type $\mathbb{A} \multimap \mathbb{B}$ can be applied to an argument of type $\mathbb{A}$ to produce a result of type $\mathbb{B}$. The rule $(\otimes_i)$ asserts that if there is a term v of type $\mathbb{A}$ and a term w of type $\mathbb{B}$, then the tensor of these terms is of type $\mathbb{A} \otimes \mathbb{B}$. The rule $(\otimes_e)$ expresses if there is a term w of type $\mathbb{D}$ with variables x and y of types $\mathbb{A}$ and $\mathbb{B}$, respectively, and a term v of type $\mathbb{A} \otimes \mathbb{B}$, then v can be deconstructed into $x \otimes y$. The rule $(\mathbb{I}_e)$ states that if there is a term w of type $\mathbb{A}$ and a term v of type $\mathbb{I}$, then v can be discarded, and only the term w remains. The rule (dis) asserts that a term v of type $\mathbb{A}$ can be discarded, resulting in a term of type $\mathbb{I}$. The rules $(\mathrm{inl}_{\mathbb{B}})$ (resp. $(\mathrm{inr}_{\mathbb{A}})$) states that if we inject a term v of type $\mathbb{A}$ (resp. $\mathbb{B}$) into $\mathbb{A} \oplus \mathbb{B}$ we obtain a term of type $\mathbb{A} \oplus \mathbb{B}$. Finally, the rule

(case) states that if there are two programs of type $\mathbb{D}$ to be executed depending on the value of a term v of type $\mathbb{A} \oplus \mathbb{B}$ (whose right and left components are assigned to variables x and y, respectively), then the resulting program also has type $\mathbb{D}$.
For a better understanding of the rules, a few straightforward programming examples are provided.

Example 2.2.1. For instance, the program that swaps the elements of a tensor product can be written as follows:

$$\textbf{SwapTensor} \triangleq x : \mathbb{A} \otimes \mathbb{B} \triangleright \mathsf{pm}\ x\ \mathsf{to}\ a \otimes b.b \otimes a : \mathbb{B} \otimes \mathbb{A}$$

Now, to prove that this program is well-typed one can write the following typing derivation:

$$\begin{array}{lll} 1 & x : \mathbb{A} \otimes \mathbb{B} \triangleright y : \mathbb{A} \otimes \mathbb{B} & (\mathsf{hyp}) \\ 2 & b : \mathbb{B} \triangleright b : \mathbb{B} & (\mathsf{hyp}) \\ 3 & a : \mathbb{A} \triangleright a : \mathbb{A} & (\mathsf{hyp}) \\ 4 & b : \mathbb{B}, a : \mathbb{A} \triangleright b \otimes a : \mathbb{B} \otimes \mathbb{A} & (2, 3, \otimes_i) \\ 5 & x : \mathbb{A}, y : \mathbb{B} \triangleright \mathsf{pm}\ x \otimes y\ \mathsf{to}\ a \otimes b.b \otimes a : \mathbb{B} \otimes \mathbb{A} & (1, 4, \otimes_e) \end{array}$$

Observe that in the notation of the third column, the numbers correspond to the premises utilized in the application of the rule.

Example 2.2.2. Another example is the function that recieves a tensor product and returns first element, discarding the second:

$$\textbf{Dis2nd} \triangleq - \triangleright \lambda x : \mathbb{A} \otimes \mathbb{A}.\mathsf{pm}\ x\ \mathsf{to}\ a \otimes b.\mathsf{dis}(b)\ \ \mathsf{to}\ *.a : \mathbb{A}$$

To prove that this program is well-typed one can write the following typing derivation:

$$\begin{array}{lll} 1 & b : \mathbb{A} \triangleright b : \mathbb{A} & (\mathsf{hyp}) \\ 2 & b : \mathbb{A} \triangleright \mathsf{dis}(b) : \mathbb{I} & (1, \mathsf{dis}) \\ 3 & a : \mathbb{A} \triangleright a : \mathbb{A} & (\mathsf{hyp}) \\ 4 & a : \mathbb{A}, b : \mathbb{A} \triangleright \mathsf{dis}(b)\ \ \mathsf{to}\ *.a & (2, 3, \mathbb{I}_e) \\ 5 & x : \mathbb{A} \otimes \mathbb{A} \triangleright x : \mathbb{A} \otimes \mathbb{A} & (\mathsf{hyp}) \\ 6 & x : \mathbb{A} \otimes \mathbb{A} \triangleright \mathsf{pm}\ x\ \mathsf{to}\ a \otimes b.\mathsf{dis}(b)\ \ \mathsf{to}\ *.a : \mathbb{A} & (4, 5, \otimes_e) \\ 7 & - \triangleright \lambda x : \mathbb{A} \otimes \mathbb{A}.\mathsf{pm}\ x\ \mathsf{to}\ a \otimes b.\mathsf{dis}(b)\ \ \mathsf{to}\ *.a : \mathbb{A} & (6, \multimap_i) \end{array}$$

Example 2.2.3. Consider an analogous program to the previous example that that recieves a tensor product and returns second element, discarding the first:

$$\textbf{Dis1st} \triangleq - \triangleright \lambda x : \mathbb{A} \otimes \mathbb{A}.\mathsf{pm}\ x\ \mathsf{to}\ a \otimes b.\mathsf{dis}(a)\ \ \mathsf{to}\ *.b : \mathbb{A} \otimes \mathbb{A} \multimap \mathbb{A}$$

Following a similar line of reasoning as in the previous example, it is straightforward to verify that this λ-term is also well-typed. Now, let us examine a program that receives both a co-product/sum and a tensor product, and applies either **Dis1st** or **Dis2nd** to the tensor product (or a mixture of both), depending on the value of the coproduct:

$$\textbf{Dis1stOR2nd} \triangleq - \triangleright \lambda x : \mathbb{B} \oplus \mathbb{B}.\ \lambda y : \mathbb{A} \otimes \mathbb{A}.\ \mathsf{case}\ x \begin{Bmatrix} \mathrm{inl}_{\mathbb{B}}(w) \Rightarrow \mathsf{dis}(w)\ \mathsf{to}\ *\ .\ \textbf{Dis1st}\ y; \\ \mathrm{inr}_{\mathbb{B}}(z) \Rightarrow \mathsf{dis}(z)\ \mathsf{to}\ *\ .\textbf{Dis2nd}\ y \end{Bmatrix}$$

To prove that this program is well-typed one can write the following typing derivation (the derivations regarding **Dis1st** and **Dis2nd** will be omitted):

$$\begin{array}{lll}
1 & y : \mathbb{A} \otimes \mathbb{A} \triangleright y : \mathbb{A} \otimes \mathbb{A} & (\mathsf{hyp}) \\
2 & y : \mathbb{A} \otimes \mathbb{A} \triangleright \textbf{Dis1st}\ y : \mathbb{A} & (\multimap_e) \\
3 & y : \mathbb{A} \otimes \mathbb{A} \triangleright \textbf{Dis2nd}\ y : \mathbb{A} & (\multimap_e) \\
4 & w : \mathbb{B} \triangleright w : \mathbb{B} & (\mathsf{hyp}) \\
5 & w : \mathbb{B} \triangleright \mathsf{dis}(w) : \mathbb{I} & (4, \mathsf{dis}) \\
6 & z : \mathbb{B} \triangleright z : \mathbb{B} & (\mathsf{hyp}) \\
7 & z : \mathbb{B} \triangleright \mathsf{dis}(z) : \mathbb{I} & (6, \mathsf{dis}) \\
8 & y : \mathbb{A} \otimes \mathbb{A}, w : \mathbb{B} \triangleright \mathsf{dis}(w)\ \mathsf{to}\ *\ .\ \textbf{Dis1st}\ y : \mathbb{A} & (2, 5, \mathbb{I}_e) \\
9 & y : \mathbb{A} \otimes \mathbb{A}, z : \mathbb{B} \triangleright \mathsf{dis}(z)\ \mathsf{to}\ *\ .\ \textbf{Dis2nd}\ y : \mathbb{A} & (3, 7, \mathbb{I}_e) \\
10 & x : \mathbb{B} \oplus \mathbb{B} \triangleright x : \mathbb{B} \oplus \mathbb{B} & (\mathsf{hyp}) \\
11 & x : \mathbb{B} \oplus \mathbb{B}, y : \mathbb{A} \otimes \mathbb{A} \triangleright\ \mathsf{case}\ x \begin{Bmatrix} \mathrm{inl}_{\mathbb{B}}(w) \Rightarrow \mathsf{dis}(w)\ \mathsf{to}\ *\ .\ \textbf{Dis1st}\ y; \\ \mathrm{inr}_{\mathbb{B}}(z) \Rightarrow \mathsf{dis}(z)\ \mathsf{to}\ *\ .\textbf{Dis2nd}\ y \end{Bmatrix} & (8, 9, 10, \mathsf{case}) \\
12 & x : \mathbb{B} \oplus \mathbb{B} \triangleright \lambda y : \mathbb{A} \otimes \mathbb{A}.\ \mathsf{case}\ x\ \{\ldots\} & (11, \multimap_i) \\
13 & - \triangleright \lambda x : \mathbb{B} \oplus \mathbb{B}.\ \lambda y : \mathbb{A} \otimes \mathbb{A}.\ \mathsf{case}\ x\ \{\ldots\} & (12, \multimap_i)
\end{array}$$

2.2.5 α-equivalence

A natural notion of equivalence definition stems from the fact that terms that differ only in the names of their bound variables represent the same program. For instance, the functions

$\lambda x : \mathbb{A}.x$ and $\lambda y : \mathbb{A}.y$ have the same input-output behavior, despite being represented by different lambda terms. This equivalence is called α-*equivalence*.

Definition 2.2.4. The α-equivalence is an equivalence relation on lambda terms that is used to rename bound variables. To rename a variable x as y in a term v, denoted by $v\{y/x\}$, is to replace all occurrences of x in v by y. Two terms v and w are α-equivalent, written $=_\alpha$, if one can be derived from the other by a series of changes of bound variables

Convention 2.2.5. Terms are considered up to α-equivalence from now on.

2.2.6 Substitution

The substitution of a variable x for a term w in a term v is denoted by $v[w/x]$. It is only permitted to replace free variables. For instance, $\lambda x.x\,[v/x]$ is $\lambda x.x$ and not $\lambda x.v$. Moreover, it is necessary to avoid the unintended binding of free variables. For example,

$$(\text{pm } x \otimes y \text{ to } a \otimes b.b \otimes a \otimes z)\,[z/\text{pm } c \otimes d \text{ to } e \otimes f.f \otimes e \otimes a]$$

is not the same as

$$\text{pm } x \otimes y \text{ to } a \otimes b.b \otimes a \otimes (\text{pm } c \otimes d \text{ to } e \otimes f.f \otimes e \otimes a).$$

Instead, the bounded variable a must be renamed before the substitution, and in this case, the proper substitution is

$$(\text{pm } x \otimes y \text{ to } t \otimes b.b \otimes t \otimes z)\,[z/\text{pm } c \otimes d \text{ to } e \otimes f.f \otimes e \otimes a]$$

which is equal to

$$\text{pm } x \otimes y \text{ to } t \otimes b.b \otimes t \otimes (\text{pm } c \otimes d \text{ to } e \otimes f.f \otimes e \otimes a).$$

Note that a simple way of ensuring these restrictions are satisfied is not allowing the variable x to occur in the context of w in $v[w/x]$. Since x is in the context of v, this is always the case in the affine lambda calculus.

Definition 2.2.6. Given the typings judgments $\Gamma, x : \mathbb{A} \triangleright v : \mathbb{B}$ and $\Delta \triangleright w : \mathbb{A}$, the substitution $\Gamma, \Delta \triangleright v[w/x] : \mathbb{B}$ is defined below. The types of judgments are omitted as no ambiguity

arises.

$$\Gamma, \Delta \triangleright y[w/x] = \Gamma, \Delta \triangleright y,$$

$$\Delta \triangleright *[w/x] = \Delta \triangleright *,$$

$$\Gamma, \Delta \triangleright (\lambda y : \mathbb{B}.v)[w/x] = \Gamma, \Delta \triangleright \lambda y : \mathbb{B}.v[w/x],$$

$$(\mathsf{dis}(v))[w/x] = \mathsf{dis}(v[w/x]),$$

$$(\mathsf{inl}(v))[w/x] = \mathsf{inl}(v[w/x]),$$

$$(\mathsf{inr}(v))[w/x] = \mathsf{inr}(v[w/x]),$$

In the next three cases, $\Gamma, x : \mathbb{A} \in \mathsf{Sf}(\Gamma_1, \ldots, \Gamma_i, \ldots, \Gamma_n)$ and $\Gamma_i \triangleright v_i$

$$\Gamma, \Delta \triangleright (f(v_1, \ldots, v_n))[w/x] = \Gamma, \Delta \triangleright f(v_1[w/x], \ldots, v_n), \qquad (\text{if } x : \mathbb{A} \in \Gamma_1)$$

$$\Gamma, \Delta \triangleright (f(v_1, \ldots, v_i, \ldots, v_n))[w/x] = \Gamma, \Delta \triangleright f(v_1, \ldots, v_i[w/x], \ldots, v_n), \qquad (\text{if } x : \mathbb{A} \in T_i)$$

$$\Gamma, \Delta \triangleright (f(v_1, \ldots, v_n))[w/x] = \Gamma, \Delta \triangleright f(v_1, \ldots, v_n[w/x]), \qquad (\text{if } x : \mathbb{A} \in \Gamma_n)$$

In the next two cases, $\Gamma, x : \mathbb{A} \in \mathsf{Sf}(\Gamma_1, \Gamma_2), \Gamma_1 \triangleright v$, and $\Gamma_2 \triangleright u$

$$\Gamma, \Delta \triangleright (v\,u)[w/x] = \Gamma, \Delta \triangleright (v[w/x]\,u), \qquad (\text{if } x : \mathbb{A} \in \Gamma_1)$$

$$\Gamma, \Delta \triangleright (v\,u)[w/x] = \Gamma, \Delta \triangleright (v\,u[w/x]), \qquad (\text{if } x : \mathbb{A} \in \Gamma_2)$$

In the next two cases, $\Gamma, x : \mathbb{A} \in \mathsf{Sf}(\Gamma_1, \Gamma_2), \Gamma_1 \triangleright v$, and $\Gamma_2 \triangleright u$

$$\Gamma, \Delta \triangleright (v \otimes u)[w/x] = \Gamma, \Delta \triangleright v[w/x] \otimes u, \qquad (\text{if } x : \mathbb{A} \in \Gamma_1)$$

$$\Gamma, \Delta \triangleright (v \otimes u)[w/x] = \Gamma, \Delta \triangleright v \otimes u[w/x], \qquad (\text{if } x : \mathbb{A} \in \Gamma_2)$$

In the next two cases, $\Gamma, x : \mathbb{A} \in \mathsf{Sf}(\Gamma_1, \Gamma_2), \Gamma_1 \triangleright v$, and $\Gamma_2, y : \mathbb{D}, z : \mathbb{E} \triangleright u$

$$\Gamma, \Delta \triangleright (\mathsf{pm}\ v\ \mathsf{to}\ y \otimes z.u)[w/x] = \Gamma, \Delta \triangleright \mathsf{pm}\ v[w/x]\ \mathsf{to}\ y \otimes z.u, \qquad (\text{if } x : \mathbb{A} \in \Gamma_1)$$

$$\Gamma, \Delta \triangleright (\mathsf{pm}\ v\ \mathsf{to}\ y \otimes z.u)[w/x] = \Gamma, \Delta \triangleright \mathsf{pm}\ v\ \mathsf{to}\ y \otimes z.u[w/x], \qquad (\text{if } x : \mathbb{A} \in \Gamma_2)$$

In the next two cases, $\Gamma, x : \mathbb{A} \in \mathsf{Sf}(\Gamma_1, \Gamma_2), \Gamma_1 \triangleright v$, and $\Gamma_2 \triangleright u$

$$\Gamma, \Delta \triangleright (v\ \mathsf{to}\ *.u)[w/x] = \Gamma, \Delta \triangleright v[w/x]\ \mathsf{to}\ *.u \qquad (\text{if } x : \mathbb{A} \in \Gamma_1),$$

$$\Gamma, \Delta \triangleright (v\ \mathsf{to}\ *.u)[w/x] = \Gamma, \Delta \triangleright v\ \mathsf{to}\ *.u[w/x] \qquad (\text{if } x : \mathbb{A} \in \Gamma_2).$$

In the next two cases, $\Gamma, x : \mathbb{A} \in \mathsf{Sf}(\Gamma_1, \Gamma_2), \Gamma_1 \triangleright v$, $\Gamma_2, y : \mathbb{D} \triangleright a$, and

$$\Gamma_2, z : \mathbb{E} \triangleright b$$

$$\mathsf{case}\ v \left\{ \begin{array}{l} \mathrm{inl}_{\mathbb{I}}(y) \Rightarrow a; \\ \mathrm{inr}_{\mathbb{I}}(z) \Rightarrow b \end{array} \right\} [w/x] = \mathsf{case}\ v[w/x] \left\{ \begin{array}{l} \mathrm{inl}_{\mathbb{I}}(y) \Rightarrow a; \\ \mathrm{inr}_{\mathbb{I}}(z) \Rightarrow b \end{array} \right\} \qquad (\text{if } x : \mathbb{A} \in \Gamma_1)$$

$$\mathsf{case}\ v \begin{Bmatrix} \mathrm{inl}_{\mathbb{E}}(y) \Rightarrow a; \\ \mathrm{inr}_{\mathbb{D}}(z) \Rightarrow b \end{Bmatrix} [w/x] = \mathsf{case}\ v \begin{Bmatrix} \mathrm{inl}_{\mathbb{E}}(y) \Rightarrow a[w/x]; \\ \mathrm{inr}_{\mathbb{D}}(z) \Rightarrow b[w/x] \end{Bmatrix} \qquad (\ \mathsf{if}\ x : \mathbb{A} \in \Gamma_2)$$

The sequential substitutions $M[M_i/x_i] \dots [M_n/x_n]$ are writen as $M[M_i/x_i, \dots, M_n/x_n]$.

2.2.7 Properties

The calculus defined in Figure 1 possesses several desirable properties, which are listed below. Before proceeding, it is necessary to introduce some auxiliary notation. Given a context Γ, $te(\Gamma)$ denotes context Γ with all types erased. The expression $\Gamma \simeq_\pi \Gamma'$ denotes that the contexts Γ is a permutation of context Γ'. This notation also applies to non-repetitive lists of untyped variables $te(\Gamma)$. Additionally, a judgment $\Gamma \triangleright v : \mathbb{A}$ will often be abbreviated into $\Gamma \triangleright v$ or even just v when no ambiguities arise.

Theorem 2.2.7. *The lambda calculus defined by the rules of Figure 1 has the following properties:*

1. *for all judgements $\Gamma \triangleright v$ and $\Gamma' \triangleright v$, te(Γ) $\simeq_\pi$ te(Γ');*
2. *additionally if $\Gamma \triangleright v : \mathbb{A}, \Gamma' \triangleright v : \mathbb{A}'$, and $\Gamma \simeq_\pi \Gamma'$, then $\mathbb{A}$ must be equal to $\mathbb{A}'$;*
3. *all judgements $\Gamma \triangleright v : \mathbb{A}$ have a unique derivation.*

Proof. Since these properties are established in [32, Theorem 2.3] for the lambda calculus without conditionals, it suffices to consider the cases involving conditionals.
It follows in all three cases from induction over the length of derivation trees.
Let us focus first on Property (1). The case of the rules concerning injections is direct. As for rule (case) take two contexts E and E' for the same conditional. According to this rule we obtain contexts Γ, Γ', Δ, Δ' such that $E \in \mathrm{Sf}(\Gamma; \Delta)$ and $E' \in \mathrm{Sf}(\Gamma'; \Delta')$. It follows from induction that $\Gamma \simeq_\pi \Gamma'$ and $\Delta \simeq_\pi \Delta'$, and the proof is then obtained from the sequence of equivalences,

$$\begin{aligned} \mathsf{te}(E) &\simeq_\pi \mathsf{te}(\Gamma, \Delta) \\ &\simeq_\pi \mathsf{te}(\Gamma', \Delta') \\ &\simeq_\pi \mathsf{te}(E') \end{aligned}$$

Concerning Property (2), the case of the rules concerning injections is direct and the case of rule (case) is a corollary of Property (1). Finally let us consider Property (3). Again the case concerning injections is direct and we thus focus only on rule (case). According to this rule we obtain contexts Γ, Γ', Δ, Δ' such that $E \in \mathrm{Sf}(\Gamma;\Delta)$ and $E \in \mathrm{Sf}(\Gamma';\Delta')$. By an appeal to Property (1) we also obtain $\Gamma \simeq_\pi \Gamma'$ and $\Delta \simeq_\pi \Delta'$, and thus since shuffling preserves relative orders we obtain $\Gamma = \Gamma'$ and $\Delta = \Delta'$. The proof then follows by induction. □

Lemma 2.2.8 (Exchange and Substitution)**.** *For every judgement* $\Gamma, x : \mathbb{A}, y : \mathbb{B}, \Delta \triangleright v : \mathbb{D}$ *the judgement* $\Gamma, y : \mathbb{B}, x : \mathbb{A}, \Delta \triangleright v : \mathbb{D}$ *is derivable. Not only this, given judgements* $\Gamma, x : \mathbb{A} \triangleright v : \mathbb{B}$ *and* $\Delta \triangleright w : \mathbb{A}$ *the judgement* $\Gamma, \Delta \triangleright v[w/x] : \mathbb{B}$ *is also derivable.*

Proof. Once again, these properties are established in [13, Theorem 2.1] for the lambda calculus without conditionals, so it suffices to consider the cases involving conditionals.
We start with the exchange property which follows by induction over the length of derivation trees. The rules that involve injections are direct. The rule (case) calls for case distinction, more specifically we distinguish between the cases in which both variables ($x : \mathbb{A}$ and $y : \mathbb{B}$) are in Γ, both are in Δ, and otherwise. The first two cases follow straightforwardly by induction and the definition of a shuffle. For the third case consider a judgement $E_1, x : \mathbb{A}, y : \mathbb{B}, E_2 \triangleright \mathsf{case}\, v\, \{\mathrm{inl}_{\mathbb{F}}(a) \Rightarrow w;\ \mathrm{inr}_{\mathbb{E}}(b) \Rightarrow u\} : \mathbb{D}$, and assume with no loss of generality that Γ is of the form $\Gamma_1, x : \mathbb{A}, \Gamma_2$ and Δ of the form $\Delta_1, y : \mathbb{B}, \Delta_2$. The proof now follows directly from the implication,

$$\begin{aligned} & E_1, x : \mathbb{A}, y : \mathbb{B}, E_2 \in \mathrm{Sf}(\Gamma_1, x : \mathbb{A}, \Gamma_2;\ \Delta_1, y : \mathbb{B}, \Delta_2) \\ \Longrightarrow & E_1, y : \mathbb{B}, x : \mathbb{A}, E_2 \in \mathrm{Sf}(\Gamma_1, x : \mathbb{A}, \Gamma_2;\ \Delta_1, y : \mathbb{B}, \Delta_2) \end{aligned}$$

(which holds by the definition of a shuffle).
Finally we now focus on the substitution rule which also follows by induction over the length of derivation trees. Again the cases involving the injections are direct, and we thus only detail the proof of rule (case). Consider then judgements $E, x : \mathbb{A} \triangleright \mathsf{case}\, v\, \{\mathrm{inl}_{\mathbb{D}}(a) \Rightarrow w;\ \mathrm{inr}_{\mathbb{E}}(b) \Rightarrow u\} : \mathbb{B}$ and $Z \triangleright t : \mathbb{A}$ with $E \in \mathrm{Sf}(\Gamma;\Delta)$. According to the definition of a shuffle either Γ is of the form $\Gamma_1, x : \mathbb{A}$ or Δ is of the form $\Delta_1, x : \mathbb{A}$. The first case follows directly and the second case is a corollary of the exchange rule. □

2.2.8 Equations-in-context

The simply typed lambda calculus is a formal language that captures operations like the application of a function to an argument and the elimination of variables. To express these operations there is a set of rules known as reduction rules. These rules fall into two primary categories: the β-*reductions*, which perform operations and enforce the implicit meaning of the term, and η-*reductions*, which simplify terms by exploiting the extensionality of functions. There is also a secondary class of reductions known as *commuting conversions*, which serve to disambiguate terms that, while equivalent, have different representations. As a result, affine λ-calculus comes equipped with the so-called equations-in-context $\Gamma \triangleright v = w : \mathbb{A}$, depicted in Figure 2.

$$
\begin{array}{lll}
(\beta) & \Gamma, \Delta \triangleright (\lambda x : \mathbb{A}.\, v)\, w = v[w/x] : \mathbb{B} & (\eta) \quad \Gamma \triangleright \lambda x : \mathbb{A}.(v\, x) = v : \mathbb{A} \multimap \mathbb{B} \\
(\beta_{\mathbb{I}_e}) & \Gamma \triangleright * \text{ to } * \,.\, v = v : \mathbb{A} & (\eta_{\mathbb{I}_e}) \quad \Delta, \Gamma \triangleright v \text{ to } * \,.\, w[*/z] = w[v/z] : \mathbb{A} \\
(\beta_{\otimes_e}) & E, \Gamma, \Delta \triangleright \mathsf{pm}\ v \otimes w \text{ to } x \otimes y.\, u = u[v/x, w/y] : \mathbb{A} & \\
(\eta_{\otimes_e}) & \Delta, \Gamma \triangleright \mathsf{pm}\ v \text{ to } x \otimes y.\, u[x \otimes y/z] = u[v/z] : \mathbb{A} & \\
(c_{\mathbb{I}_e}) & \Delta, \Gamma, E \triangleright u[v \text{ to } * \,.w/z] = v \text{ to } * \,.u[w/z] : \mathbb{A} & \\
(c_{\otimes_e}) & \Delta, \Gamma, E \triangleright u[\mathsf{pm}\ v \text{ to } x \otimes y.\, w/z] = \mathsf{pm}\ v \text{ to } x \otimes y.\, u[w/z] : \mathbb{A} & \\
(\eta_{\mathsf{dis}}) & x_1 : \mathbb{A}_1, \ldots, x_n : \mathbb{A}_n \triangleright v = \mathsf{dis}(x_1) \text{ to } * \ldots . \mathsf{dis}(x_{n-1}) \text{ to } * \ \mathsf{dis}(x_n) : \mathbb{I} & \\
(\beta^{inl}_{case}) & \mathsf{case}\ \mathrm{inl}_{\mathbb{B}}(v)\ \{\mathrm{inl}_{\mathbb{B}}(x) \Rightarrow w;\ \mathrm{inr}_{\mathbb{A}}(y) \Rightarrow u\} = w[v/x] & \\
(\beta^{inr}_{case}) & \mathsf{case}\ \mathrm{inl}_{\mathbb{B}}(v)\ \{\mathrm{inl}_{\mathbb{B}}(x) \Rightarrow w;\ \mathrm{inr}_{\mathbb{A}}(y) \Rightarrow u\} = u[v/y] & \\
(\eta_{case}) & \mathsf{case}\ v\ \{\mathsf{inl}_{\mathbb{B}}(y) \Rightarrow w[\mathsf{inl}_{\mathbb{B}}(y)/x];\ \mathsf{inr}_{\mathbb{A}}(z) \Rightarrow w[\mathsf{inr}_{\mathbb{A}}(z)/x]\} = w[v/x] &
\end{array}
$$

Figure 2: Equations-in-context for affine lambda calculus

It is evident that, for example, equation (β) enforces the meaning of $(\lambda x : \mathbb{A}.\, v)w$, which is interpreted as "v with w in place of x". The equation (η), on the other hand, is a simplification rule that states that a function that applies another function v to an argument x can be simplified to the function v itself. The remaining β e η equations follow similar reasoning. The commuting conversion $(c_{\mathbb{I}_e})$ expresses that substituting a variable z by a term that maps a term v to the unit element $*$ in a term w is equivalent to mapping a term v to the unit element $*$ and then replacing z by w. The other commuting conversion has a similar interpretation.

Example 2.2.9. For instance, consider the λ-term

$$- \triangleright \left(\lambda z : \mathbb{I} \otimes \mathbb{A}.\, \mathsf{pm}\ z \text{ to } * \otimes y.\, y\right)(v \otimes w) : \mathbb{A}$$

Applying the β reduction, we have:

$$- \triangleright \left(\lambda z : \mathbb{I} \otimes \mathbb{A}.\ \text{pm}\ z\ \text{to}\ * \otimes y.\ y\right)(v \otimes w) = \text{pm}\ v \otimes w\ \text{to}\ * \otimes y.\ y : \mathbb{A}.$$

Next, applying the $\beta_{\otimes_e}$ reduction, it follows:

$$\text{pm}\ v \otimes w\ \text{to}\ * \otimes y.\ y : \mathbb{A} = w : \mathbb{A}.$$

Não sei se este é o lugar mais indicado para estas definições, mas tb não estou a ver outro dado que as categorias só vêm a seguir. Ou então passar isto para a interpretação.

Definition 2.2.10. Let S be a set. A *relation* on S is a subset $R \subseteq S \times S$. An ordered pair $(s_1, s_2) \in R$ means that s_1 is related to s_2.

Definition 2.2.11. A relation on a set S is an *equivalence relation* if it is

- reflexive: for all $x \in S$, $x \sim x$,
- symmetric: for all $x, y \in S$, if $x \sim y$ then $y \sim x$, and
- transitive: for all $x, y, z \in S$, if $x \sim y$ and $y \sim z$, then $x \sim z$.

We denote such a relation by $\sim \subseteq S \times S$, and write $x \sim y$ to mean that $(x, y) \in \sim$.

Definition 2.2.12. Given an equivalence relation on a set S, we can describe disjoint subsets of S called *equivalence classes*. If $s \in S$, then the *equivalence class* of s is the set of all elements related to it:

$$[s] = \{r \in S \mid r \sim s\}.$$

That is, $[s]$ is the set of all elements that are considered "the same" as s under the relation $\sim$. For a given set S and an equivalence relation $\sim$ on S, we define the *quotient set*, denoted $S/\sim$, whose elements are all the equivalence classes of elements in S. There is an obvious *quotient function* from S to $S/\sim$ that maps each element $s \in S$ to its equivalence class $[s]$.

Definition 2.2.13. Consider a pair (G, Σ), where G is a class of ground types and Σ is a class of sorted operation symbols. A λ-*theory* is a triple $((G, \Sigma), Ax)$, where Ax is a class of equations-in-context over λ-terms constructed from (G, Σ). The elements of Ax are called the *axioms* of the theory.

Let $Th(Ax)$ denote the smallest class containing Ax, the equations presented in Figure 2, and closed under exchange and substitution (Lemma 2.2.8). The elements of $Th(Ax)$ are called the *theorems* of the theory.
For instance recall Example 2.2.9:

$$- \triangleright \left(\lambda z : \mathbb{I} \otimes \mathbb{A}.\ \mathsf{pm}\ z\ \mathsf{to} * \otimes y.\ y\right)(v \otimes w) : \mathbb{A} = w : \mathbb{A}$$

is a theorem.

2.2.9 Interlude: Booleans

Booleans can be represented as the type $\mathbb{I} \oplus \mathbb{I}$, in which case True $= \mathrm{inl}(*)$, False $= \mathrm{inr}(*)$ [25]. We will use the equations in Figure 2 to demonstrate that they possess certain properties that are characteristic of booleans in classical logic. The remaining properties can be proven through similar reasoning.
Given variables $v : \mathbb{I} \oplus \mathbb{I}$ and $w : \mathbb{I} \oplus \mathbb{I}$, their conjunction and disjunction correspond to the following programs:

$$\textbf{Conjunction} \triangleq v : \mathbb{I} \oplus \mathbb{I}, w : \mathbb{I} \oplus \mathbb{I} \triangleright \mathsf{case}\ v \begin{Bmatrix} \mathrm{inl}_{\mathbb{I}}(x) \Rightarrow x\ \mathsf{to} * .\ w; \\ \mathrm{inr}_{\mathbb{I}}(y) \Rightarrow\ \mathsf{to} * .\ \mathsf{dis}(w)\ \mathsf{to} * .\mathrm{inr}_{\mathbb{I}}(*) \end{Bmatrix}$$

$$\textbf{Disjunction} \triangleq v : \mathbb{I} \oplus \mathbb{I}, w : \mathbb{I} \oplus \mathbb{I} \triangleright \mathsf{case}\ v \begin{Bmatrix} \mathrm{inl}_{\mathbb{I}}(x) \Rightarrow\ \mathsf{to} * .\ \mathsf{dis}(w)\ \mathsf{to} * .\mathrm{inl}_{\mathbb{I}}(*); \\ \mathrm{inr}_{\mathbb{I}}(y) \Rightarrow y\ \mathsf{to} * .\ w \end{Bmatrix}$$

Moreover, negation can be expressed by the following program:

$$\textbf{Negation} \triangleq v : \mathbb{I} \oplus \mathbb{I} \triangleright \mathsf{case}\ v \begin{Bmatrix} \mathrm{inl}_{\mathbb{I}}(x) \Rightarrow \mathrm{inr}_{\mathbb{I}}(x); \\ \mathrm{inr}_{\mathbb{I}}(y) \Rightarrow \mathrm{inl}_{\mathbb{I}}(y) \end{Bmatrix}$$

Lemma 2.2.14. $\mathrm{inl}(*)$ *acts as the neutral element for conjunction, whereas* $\mathrm{inr}(*)$ *serves as the absorbing element,* i.e.

$$\textbf{\textit{Conjunction}}\ [\mathrm{inl}(*)/v, w'/w] = w' \quad \textit{and} \quad \textbf{\textit{Conjunction}}\ [\mathrm{inr}(*)/v, w'/w] = \mathrm{inr}(*)$$

.

Proof. These properties follow from the equations β^{inl}_{case}, β^{inr}_{case}, $\beta_{\mathbb{I}_e}$, and η_{dis}.

$$
\begin{aligned}
&\textbf{Conjunction}\ [\mathrm{inl}(*)/v, w'/w] \\
&= \mathsf{case}\ \mathrm{inl}(*)\ \{\mathrm{inl}_{\mathbb{I}}(x) \Rightarrow x\ \mathsf{to}\ *\ .\ w';\ \mathrm{inr}_{\mathbb{I}}(y) \Rightarrow y\ \mathsf{to}\ *\ .\ \mathsf{dis}(w')\ \mathsf{to}\ *\ .\mathrm{inr}_{\mathbb{I}}(*)\} \\
&= *\ \mathsf{to}\ *\ .\ w' && (\beta^{inl}_{case}) \\
&= *\ \mathsf{to}\ *\ .\ w' && (\beta_{\mathbb{I}_e}) \\
&= w'
\end{aligned}
$$

For the second equality to be well-defined, we observe that the context of w' must be empty. Moreover, we remark on the following:

$$
\begin{aligned}
&x : \mathbb{I} \triangleright x\ \mathsf{to}\ *\ .\ \mathsf{dis}(w) = x\ \mathsf{to}\ *\ .\ x : \mathbb{I} && (\eta_{\mathsf{dis}}) \\
&\quad \Longrightarrow\ - \triangleright x\ \mathsf{to}\ *\ .\ \mathsf{dis}(w)[*/x] = x\ \mathsf{to}\ *\ .\ x[*/x] : \mathbb{I} \\
&\quad \Longrightarrow\ - \triangleright *\ \mathsf{to}\ *\ .\ \mathsf{dis}(w) = *\ \mathsf{to}\ *\ .\ * : \mathbb{I} \\
&\quad \Longrightarrow\ - \triangleright \mathsf{dis}(w) = * : \mathbb{I} && (\beta_{\mathbb{I}_e})
\end{aligned}
$$

With the equation above in hand, we have:

$$
\begin{aligned}
&\textbf{Conjunction}\ [\mathrm{inr}(*)/v, w'/w] \\
&= \mathsf{case}\ \mathrm{inr}(*)\ \{\mathrm{inl}_{\mathbb{I}}(x) \Rightarrow x\ \mathsf{to}\ *\ .\ w';\ \mathrm{inr}_{\mathbb{I}}(y) \Rightarrow y\ \mathsf{to}\ *\ .\ \mathsf{dis}(w')\ \mathsf{to}\ *\ .\mathrm{inr}_{\mathbb{I}}(*)\} \\
&= *\ y\ \mathsf{to}\ *\ .\ \mathsf{dis}(w')\ \mathsf{to}\ *\ .\mathrm{inr}_{\mathbb{I}}(*) && (\beta^{inr}_{case}) \\
&= \mathsf{dis}(w')\ \mathsf{to}\ *\ .\mathrm{inr}_{\mathbb{I}}(*) && (\beta_{\mathbb{I}_e}) \\
&= *\ \mathsf{to}\ *\ .\mathrm{inr}_{\mathbb{I}}(*) && (\eta_{\mathsf{dis}}\ \text{and}\ \beta_{\mathbb{I}_e}) \\
&= \mathrm{inr}_{\mathbb{I}}(*)
\end{aligned}
$$

□

Note that the idempotency property of conjunction (for $\mathrm{inl}(*)$ and $\mathrm{inr}(*)$) follows directly from the equalities above.

Lemma 2.2.15. *The conjunction of two terms is comutative ,* i.e.*,*

$$\textbf{\textit{Conjunction}}\ [v'/v, w'/w] = \textbf{\textit{Conjunction}}\ [w'/v, v'/w]$$

Proof. This property follows from the equations η_{case}, $c_{\mathbb{I}_e}$, $c_{\mathbb{I}_e}$, $\eta_{\mathbb{I}_e}$, and η_{dis}, and α-equivalence.

Conjunction $[v'/v, w'/w]$

$$= \mathsf{case}\ v\ \{\mathrm{inl}_{\mathbb{I}}(x) \Rightarrow x\ \mathsf{to} * .\ w;\ \mathrm{inr}_{\mathbb{I}}(y) \Rightarrow y\ \mathsf{to} * .\ \mathsf{dis}(w)\ \mathsf{to} * .\mathrm{inr}_{\mathbb{I}}(*)\}[w'/w]$$

$$= \mathsf{case}\ w' \left\{ \begin{array}{l} \mathrm{inl}_{\mathbb{I}}(a) \Rightarrow\ \mathsf{case}\ v' \left\{ \begin{array}{l} \mathrm{inl}_{\mathbb{I}}(x) \Rightarrow x\ \mathsf{to} * .\ \mathrm{inl}_{\mathbb{I}}(a); \\ \mathrm{inr}_{\mathbb{I}}(y) \Rightarrow y\ \mathsf{to} * .\ \mathsf{dis}(\mathrm{inl}_{\mathbb{I}}(a))\ \mathsf{to} * .\mathrm{inr}_{\mathbb{I}}(*) \end{array} \right\} \\ \mathrm{inr}_{\mathbb{I}}(b) \Rightarrow\ \mathsf{case}\ v' \left\{ \begin{array}{l} \mathrm{inl}_{\mathbb{I}}(x) \Rightarrow x\ \mathsf{to} * .\ \mathrm{inr}_{\mathbb{I}}(b); \\ \mathrm{inr}_{\mathbb{I}}(y) \Rightarrow y\ \mathsf{to} * .\ \mathsf{dis}(\mathrm{inr}_{\mathbb{I}}(b))\ \mathsf{to} * .\mathrm{inr}_{\mathbb{I}}(*) \end{array} \right\} \end{array} \right\} \quad (\eta_{\mathsf{case}})$$

$$= \mathsf{case}\ w' \left\{ \begin{array}{l} \mathrm{inl}_{\mathbb{I}}(a) \Rightarrow\ \mathsf{case}\ v' \left\{ \begin{array}{l} \mathrm{inl}_{\mathbb{I}}(x) \Rightarrow x\ \mathsf{to} * .\ \mathrm{inl}_{\mathbb{I}}(a); \\ \mathrm{inr}_{\mathbb{I}}(y) \Rightarrow y\ \mathsf{to} * .\ a\ \mathsf{to} * .\mathrm{inr}_{\mathbb{I}}(*) \end{array} \right\} \\ \mathrm{inr}_{\mathbb{I}}(b) \Rightarrow\ \mathsf{case}\ v' \left\{ \begin{array}{l} \mathrm{inl}_{\mathbb{I}}(x) \Rightarrow x\ \mathsf{to} * .\ \mathrm{inr}_{\mathbb{I}}(b); \\ \mathrm{inr}_{\mathbb{I}}(y) \Rightarrow y\ \mathsf{to} * .\ b\ \mathsf{to} * .\mathrm{inr}_{\mathbb{I}}(*) \end{array} \right\} \end{array} \right\} \quad (\eta_{\mathsf{dis}})$$

$$= \mathsf{case}\ w' \left\{ \begin{array}{l} \mathrm{inl}_{\mathbb{I}}(a) \Rightarrow\ \mathsf{case}\ v' \left\{ \begin{array}{l} \mathrm{inl}_{\mathbb{I}}(x) \Rightarrow x\ \mathsf{to} * .\ a\ \mathsf{to} * .\mathrm{inl}_{\mathbb{I}}(*)\ ; \\ \mathrm{inr}_{\mathbb{I}}(y) \Rightarrow y\ \mathsf{to} * .\ a\ \mathsf{to} * .\mathrm{inl}_{\mathbb{I}}(*) \end{array} \right\} \\ \mathrm{inr}_{\mathbb{I}}(b) \Rightarrow\ \mathsf{case}\ v' \left\{ \begin{array}{l} \mathrm{inl}_{\mathbb{I}}(x) \Rightarrow x\ \mathsf{to} * .\ b\ \mathsf{to} * .\ \mathrm{inr}_{\mathbb{I}}(*); \\ \mathrm{inr}_{\mathbb{I}}(y) \Rightarrow y\ \mathsf{to} * .\ b\ \mathsf{to} * .\mathrm{inr}_{\mathbb{I}}(*) \end{array} \right\} \end{array} \right\} \quad (\eta_{\mathbb{I}_e})$$

$$= \mathsf{case}\ w' \left\{ \begin{array}{l} \mathrm{inl}_{\mathbb{I}}(a) \Rightarrow\ \mathsf{case}\ v' \left\{ \begin{array}{l} \mathrm{inl}_{\mathbb{I}}(x) \Rightarrow a\ \mathsf{to} * .\ \mathrm{inl}_{\mathbb{I}}(*)[x\ \mathsf{to} * .\ * /*]; \\ \mathrm{inr}_{\mathbb{I}}(y) \Rightarrow a\ \mathsf{to} * .\ \mathrm{inl}_{\mathbb{I}}(*)[y\ \mathsf{to} * .\ * /*] \end{array} \right\} \\ \mathrm{inr}_{\mathbb{I}}(b) \Rightarrow\ \mathsf{case}\ v' \left\{ \begin{array}{l} \mathrm{inl}_{\mathbb{I}}(x) \Rightarrow b\ \mathsf{to} * .\ \mathrm{inl}_{\mathbb{I}}(*)[x\ \mathsf{to} * .\ * /*]; \\ \mathrm{inr}_{\mathbb{I}}(y) \Rightarrow b\ \mathsf{to} * .\ \mathrm{inl}_{\mathbb{I}}(*)[y\ \mathsf{to} * .\ * /*] \end{array} \right\} \end{array} \right\} \quad (c_{\mathbb{I}_e})$$

$$= \mathsf{case}\ w' \left\{ \begin{array}{l} \mathrm{inl}_{\mathbb{I}}(a) \Rightarrow\ \mathsf{case}\ v' \left\{ \begin{array}{l} \mathrm{inl}_{\mathbb{I}}(x) \Rightarrow a\ \mathsf{to} * .\ \mathrm{inl}_{\mathbb{I}}(x); \\ \mathrm{inr}_{\mathbb{I}}(y) \Rightarrow a\ \mathsf{to} * .\ \mathrm{inr}_{\mathbb{I}}(y) \end{array} \right\} \\ \mathrm{inr}_{\mathbb{I}}(b) \Rightarrow\ \mathsf{case}\ v' \left\{ \begin{array}{l} \mathrm{inl}_{\mathbb{I}}(x) \Rightarrow b\ \mathsf{to} * .\ x\ \mathsf{to} * .\ \mathrm{inr}_{\mathbb{I}}(*); \\ \mathrm{inr}_{\mathbb{I}}(y) \Rightarrow b\ \mathsf{to} * .\ y\ \mathsf{to} * .\mathrm{inr}_{\mathbb{I}}(*) \end{array} \right\} \end{array} \right\} \quad (\eta_{\mathbb{I}_e})$$

$$= \mathsf{case}\ w' \left\{ \begin{array}{l} \mathrm{inl}_{\mathbb{I}}(a) \Rightarrow\ \mathsf{case}\ v' \left\{ \begin{array}{l} \mathrm{inl}_{\mathbb{I}}(x) \Rightarrow a\ \mathsf{to} * .\ \mathrm{inl}_{\mathbb{I}}(x); \\ \mathrm{inr}_{\mathbb{I}}(y) \Rightarrow a\ \mathsf{to} * .\mathrm{inr}_{\mathbb{I}}(y) \end{array} \right\} \\ \mathrm{inr}_{\mathbb{I}}(b) \Rightarrow\ \mathsf{case}\ v' \left\{ \begin{array}{l} \mathrm{inl}_{\mathbb{I}}(x) \Rightarrow b\ \mathsf{to} * .\ \mathsf{dis}(\mathrm{inr}_{\mathbb{I}}(x))\ \mathsf{to} * .\ \mathrm{inr}_{\mathbb{I}}(*); \\ \mathrm{inr}_{\mathbb{I}}(y) \Rightarrow b\ \mathsf{to} * .\ \mathsf{dis}(\mathrm{inr}_{\mathbb{I}}(y))\ \mathsf{to} * .\mathrm{inr}_{\mathbb{I}}(*) \end{array} \right\} \end{array} \right\} \quad (\eta_{\mathsf{dis}})$$

$$= \mathsf{case}\ w'\ \{\mathrm{inl}_{\mathbb{I}}(a) \Rightarrow a\ \mathsf{to} * .\ v';\ \mathrm{inr}_{\mathbb{I}}(b) \Rightarrow b\ \mathsf{to} * .\ \mathsf{dis}(v')\ \mathsf{to} * .\mathrm{inr}_{\mathbb{I}}(*)\} \quad (\eta_{case})$$

$=$ **Conjunction** $[w'/v, v'/w]$

□

To address the double negation property and De Morgan's laws, we begin by establishing a key equality known as the *syntactic fusion law*.

Lemma 2.2.16 (*Syntactic fusion law*)**.** *The following equality holds:*

$$v\,[(\textit{case}\ a\,\{\mathrm{inl}_{\mathbb{B}}(x) \Rightarrow w;\ \mathrm{inr}_{\mathbb{A}}(y) \Rightarrow u\})\,/z] = \textit{case}\ a\,\{\mathrm{inl}_{\mathbb{B}}(x) \Rightarrow v[w/z];\ \mathrm{inr}_{\mathbb{A}}(y) \Rightarrow v[u/z]\}.$$

Proof. This equality follows from α-equivalence and the equations η_{case}, $\beta^{\mathsf{inl}}_{\mathsf{case}}$, and $\beta^{\mathsf{inr}}_{\mathsf{case}}$.

$$
\begin{aligned}
& v\,[(\mathsf{case}\ a\,\{\mathrm{inl}_{\mathbb{B}}(x) \Rightarrow w;\ \mathrm{inr}_{\mathbb{A}}(y) \Rightarrow u\})\,/z] \\
&= v\,[(\mathsf{case}\ a\,\{\mathrm{inl}_{\mathbb{B}}(x') \Rightarrow w[x'/x];\ \mathrm{inr}_{\mathbb{A}}(y') \Rightarrow u[y'/y]\})\,/z] && (\alpha) \\
&= \mathsf{case}\ a \left\{ \begin{array}{l} \mathrm{inl}_{\mathbb{I}}(x) \Rightarrow v \left[\mathsf{case}\ \mathrm{inl}_{\mathbb{I}}(x) \left\{ \begin{array}{l} \mathrm{inl}_{\mathbb{I}}(x') \Rightarrow w[x'/x]; \\ \mathrm{inr}_{\mathbb{I}}(y') \Rightarrow u[y'/y] \end{array} \right\} /z \right]; \\ \mathrm{inr}_{\mathbb{I}}(y) \Rightarrow v \left[\mathsf{case}\ \mathrm{inr}_{\mathbb{I}}(y) \left\{ \begin{array}{l} \mathrm{inl}_{\mathbb{I}}(x') \Rightarrow w[x'/x]; \\ \mathrm{inr}_{\mathbb{I}}(y') \Rightarrow u[y'/y] \end{array} \right\} /z \right] \end{array} \right\} && (\eta_{\mathsf{case}}) \\
&= \mathsf{case}\ a\,\{\mathrm{inl}_{\mathbb{B}}(x) \Rightarrow v[w[x'/x][x/x']/z];\ \mathrm{inr}_{\mathbb{A}}(y) \Rightarrow v[u[y'/y][y/y']/z]\} && (\beta^{\mathsf{inl}}_{\mathsf{case}}\ \text{and}\ \beta^{\mathsf{inr}}_{\mathsf{case}}) \\
&= \mathsf{case}\ a\,\{\mathrm{inl}_{\mathbb{B}}(x) \Rightarrow v[w/z];\ \mathrm{inr}_{\mathbb{A}}(y) \Rightarrow v[u/z]\}
\end{aligned}
$$

□

Lemma 2.2.17. *The double negation of a term w is equivalent to that term,* i.e.*,*

$$\textbf{\textit{Negation}}[\textbf{\textit{Negation}}\,[w/v]/v] = w.$$

Proof. This property follows from the syntactic fusion law and equations β^{inl}_{case}, β^{inr}_{case}, and η_{case}.

$$
\begin{aligned}
& \textbf{Negation}[\textbf{Negation}\,[w/v]/v] \\
&\triangleq \textbf{Negation}[\mathsf{case}\ w\,\{\mathrm{inl}_{\mathbb{I}}(x) \Rightarrow \mathrm{inr}_{\mathbb{I}}(x);\ \mathrm{inr}_{\mathbb{I}}(y) \Rightarrow \mathrm{inl}_{\mathbb{I}}(y)\}/v] \\
&= \mathsf{case}\ w \left\{ \begin{array}{l} \mathrm{inl}_{\mathbb{I}}(x) \Rightarrow \textbf{Negation}[\mathrm{inr}_{\mathbb{I}}(x)/v]; \\ \mathrm{inr}_{\mathbb{I}}(y) \Rightarrow \textbf{Negation}[\mathrm{inl}_{\mathbb{I}}(y)/v] \end{array} \right\} && (\text{syntactic fusion law}) \\
&= \mathsf{case}\ w\,\{\mathrm{inl}_{\mathbb{I}}(x) \Rightarrow \mathrm{inr}_{\mathbb{I}}(x);\ \mathrm{inl}_{\mathbb{I}}(x) \Rightarrow \mathrm{inr}_{\mathbb{I}}(y)\} && (\beta^{inl}_{case}\ \text{and}\ \beta^{inr}_{case}) \\
&= w && (\eta_{case})
\end{aligned}
$$

□

Lemma 2.2.18. *De Morgan's laws hold for terms* $\Gamma \triangleright v' : \mathbb{I} \oplus \mathbb{I}$ *and* $\Delta \triangleright w : \mathbb{I} \oplus \mathbb{I}$, i.e.,

Disjunction$[$***Negation*** $[v'/v]/v,$ ***Negation*** $[w'/v]/w] =$ ***Negation***$[$***Conjunction***$[v'/v, w'/w]/v]$

and

Conjunction$[$***Negation*** $[v'/v]/v,$ ***Negation*** $[w'/v]/w] =$ ***Negation***$[$***Disjunction***$[v'/v, w'/w]/v]$.

Proof. Both De Morgan's laws follow from the equations $\eta_{case}, \beta^{inl}_{case}, \beta^{inr}_{case}, c_{\mathbb{I}_e}, \eta_{\mathsf{dis}}$, the syntatic fusion law (SFL), and α-equivalence.

$$
\begin{aligned}
&\textbf{Disjunction}[\textbf{Negation}\ [v'/v]/v, \textbf{Negation}\ [w'/v]/w] \\
&\triangleq \mathsf{case}\ \ \mathsf{case}\, v \left\{ \begin{array}{l} \mathrm{inl}_{\mathbb{I}}(a) \Rightarrow \mathrm{inr}_{\mathbb{I}}(a); \\ \mathrm{inl}_{\mathbb{I}}(b) \Rightarrow \mathrm{inl}_{\mathbb{I}}(b) \end{array} \right\} \left\{ \begin{array}{l} \mathrm{inl}_{\mathbb{I}}(x) \Rightarrow \ldots; \\ \mathrm{inl}_{\mathbb{I}}(y) \Rightarrow \ldots \end{array} \right\} [v/v'] \\
&= \mathsf{case}\, v' \left\{ \begin{array}{l} \mathrm{inl}_{\mathbb{I}}(c) \Rightarrow \mathsf{case}\ \ \mathsf{case}\, \mathrm{inl}_{\mathbb{I}}(c) \left\{ \begin{array}{l} \mathrm{inl}_{\mathbb{I}}(a) \Rightarrow \mathrm{inr}_{\mathbb{I}}(a); \\ \mathrm{inr}_{\mathbb{I}}(b) \Rightarrow \mathrm{inl}_{\mathbb{I}}(b) \end{array} \right\} \ldots; \\ \mathrm{inl}_{\mathbb{I}}(d) \Rightarrow \mathsf{case}\ \ \mathsf{case}\, \mathrm{inr}_{\mathbb{I}}(d) \left\{ \begin{array}{l} \mathrm{inl}_{\mathbb{I}}(a) \Rightarrow \mathrm{inr}_{\mathbb{I}}(a); \\ \mathrm{inr}_{\mathbb{I}}(b) \Rightarrow \mathrm{inl}_{\mathbb{I}}(b) \end{array} \right\} \ldots \end{array} \right\} && (\eta_{case}) \\
&= \mathsf{case}\, v' \left\{ \begin{array}{l} \mathrm{inl}_{\mathbb{I}}(c) \Rightarrow \mathsf{case}\, \mathrm{inr}_{\mathbb{I}}(c) \left\{ \begin{array}{l} \mathrm{inl}_{\mathbb{I}}(x) \Rightarrow \ldots; \\ \mathrm{inr}_{\mathbb{I}}(y) \Rightarrow \ldots \end{array} \right\}; \\ \mathrm{inl}_{\mathbb{I}}(d) \Rightarrow \mathsf{case}\, \mathrm{inl}_{\mathbb{I}}(d) \left\{ \begin{array}{l} \mathrm{inl}_{\mathbb{I}}(x) \Rightarrow \ldots; \\ \mathrm{inr}_{\mathbb{I}}(y) \Rightarrow \ldots \end{array} \right\} \end{array} \right\} && (\beta^{inl}_{case} \text{ and } \beta^{inr}_{case}) \\
&= \mathsf{case}\, v' \left\{ \begin{array}{l} \mathrm{inl}_{\mathbb{I}}(c) \Rightarrow c\ \mathsf{to}\ * \,.\, \textbf{Negation}[w'/v]; \\ \mathrm{inl}_{\mathbb{I}}(d) \Rightarrow d\ \mathsf{to}\ * \,.\, \mathsf{dis} \left(\mathsf{case}\, w' \left\{ \begin{array}{l} \mathrm{inl}_{\mathbb{I}}(a) \Rightarrow \mathrm{inr}_{\mathbb{I}}(a); \\ \mathrm{inr}_{\mathbb{I}}(b) \Rightarrow \mathrm{inl}_{\mathbb{I}}(b) \end{array} \right\} \right) \ldots \end{array} \right\} && (\beta^{inl}_{case} \text{ and } \beta^{inr}_{case}) \\
&= \mathsf{case}\, v' \left\{ \begin{array}{l} \mathrm{inl}_{\mathbb{I}}(c) \Rightarrow c\ \mathsf{to}\ * \,.\, \textbf{Negation}[w'/v]; \\ \mathrm{inl}_{\mathbb{I}}(d) \Rightarrow d\ \mathsf{to}\ * \,.\, \mathsf{dis}\,(w')\ \ \mathsf{to}\ * \,.\, \mathrm{inl}_{\mathbb{I}}(*) \end{array} \right\} && (\eta_{\mathsf{dis}}) \\
&= \mathsf{case}\, v' \left\{ \begin{array}{l} \mathrm{inl}_{\mathbb{I}}(c) \Rightarrow c\ \mathsf{to}\ * \,.\, \textbf{Negation}[w'/v]; \\ \mathrm{inl}_{\mathbb{I}}(d) \Rightarrow d\ \mathsf{to}\ * \,.\, \mathsf{dis}\,(w')\ \ \mathsf{to}\ * \,.\, \mathrm{inr}_{\mathbb{I}}(*) \left\{ \begin{array}{l} \mathrm{inl}_{\mathbb{I}}(a) \Rightarrow \mathrm{inr}_{\mathbb{I}}(a); \\ \mathrm{inr}_{\mathbb{I}}(b) \Rightarrow \mathrm{inl}_{\mathbb{I}}(b) \end{array} \right\} \end{array} \right\} && (\beta^{inr}_{case})
\end{aligned}
$$

$$= \text{case } v' \left\{ \begin{array}{l} \text{inl}_{\mathbb{I}}(c) \Rightarrow c \text{ to } * \,.\, \textbf{Negation}[w'/v]; \\ \text{inl}_{\mathbb{I}}(d) \Rightarrow d \text{ to } * \,.\, \text{dis}\,(w') \text{ to } * \,.\textbf{Negation}[\text{inr}_{\mathbb{I}}(*)/v] \end{array} \right\}$$

$$= \text{case } v' \left\{ \begin{array}{l} \text{inl}_{\mathbb{I}}(c) \Rightarrow \textbf{Negation}[c \text{ to } * \,.\, w'/v]; \\ \text{inl}_{\mathbb{I}}(d) \Rightarrow \textbf{Negation}[d \text{ to } * \,.\, \text{dis}\,(w') \text{ to } * \,.\text{inr}_{\mathbb{I}}(*)/v] \end{array} \right\} \qquad (c_{\mathbb{I}_e})$$

$$= \textbf{Negation}\,[\text{case } v' \,\{\text{inl}_{\mathbb{I}}(c) \Rightarrow c \text{ to } * \,.\, w;\ \text{inr}_{\mathbb{I}}(d) \Rightarrow d \text{ to } * \,.\, \text{dis}(w) \text{ to } * \,.\text{inr}_{\mathbb{I}}(*)\}/v] \qquad (\text{SFL})$$

$$= \textbf{Negation}[\textbf{Conjunction}[v'/v, w', w]/v] \qquad (\alpha)$$

□

2.2.10 Metric equational system

Metric equations [14, 15] are a strong candidate for reasoning about approximate program equivalence. These equations take the form of $t =_\epsilon s$, where ϵ is a non-negative rational representing the "maximum distance" between the two terms t and s. The metric equational system for affine lambda calculus is depicted in Figure 3. Strictly speaking, the equations $\Gamma \triangleright v = w : A$ in Figure 2, which in this setting abbreviate $\Gamma \triangleright w =_0 v : A$, are also part of the metric equational system.

$$\frac{}{v =_0 v}\ (\text{refl}) \qquad \frac{v =_q w \quad w =_r u}{v =_{q+r} u}\ (\text{trans}) \qquad \frac{v =_q w \quad r \geq q}{v =_r w}\ (\text{weak})$$

$$\frac{\forall r > q.\ v =_r w}{v =_q w}\ (\text{arch}) \qquad \frac{\forall i \leq n.\ v =_{q_i} w}{v =_{\wedge q_i} w}\ (\text{join}) \qquad \frac{v =_q w}{w =_q v}\ (\text{sym})$$

$$\frac{v =_q w \quad v' =_r w'}{v \otimes v' =_{q+r} w \otimes w'} \qquad \frac{\forall i \leq n.\ v_i =_{q_i} w_i}{f(v_1, ..., v_n) =_{\Sigma q_i} f(w_1, ..., , w_n)} \qquad \frac{v =_q w}{\lambda x : \mathbb{A}.\ v =_q \lambda x : \mathbb{A}.\ w}$$

$$\frac{v =_q w \quad v' =_r w'}{\text{pm } v \text{ to } x \otimes y.\ v' =_{q+r} \text{pm } w \text{ to } x \otimes y.\ w'} \qquad \frac{v =_q w}{\text{dis}(v) =_q \text{dis}(w)} \qquad \frac{v =_q w \quad v' =_r w'}{v v' =_{q+r} w w'}$$

$$\frac{\Gamma \triangleright v =_q w : \mathbb{A} \quad \Delta \in \text{perm}(\Gamma)}{\Delta \triangleright v =_q w : \mathbb{A}} \qquad \frac{v =_q w \quad v' =_r w'}{v \text{ to } * \,.v' =_{q+r} w \text{ to } * \,.w'} \qquad \frac{v =_q w \quad v' =_r w'}{v[v'/x] =_{q+r} w[w'/x]}$$

Figure 3: Metric equational system

Here, $\text{perm}(\Gamma)$ denotes the set of possible permutions of context Γ. The rules (refl), (trans),

and (sym) generalize the properties of reflexivity, transitivity, and symmetry of equality. Rule (weak) asserts that if two terms are at a maximum distance q from each other, then they are also separated by any $r \geq q$. Rule (arch) states that if $v =_r w$ for all approximations r of q, then it necessarily follows that $v =_q w$. The rule (join) expresses that if several maximum distances between two terms are known, the actual maximum distance between them is the minimum of these distances. In particular, it is always the case that $v =_\infty w$. The rule that follows conveys that if the maximum distance between two terms v and w is q, and the maximum distance between terms v' and w' is r, then the maximum distance between the tensor products $v \otimes v'$ and $w \otimes w'$ is $q + r$. The remaining rules follow similar reasoning.

Example 2.2.19. To ilustrate the usefulness of these equations, consider the program P that recieves a tensor product, swaps its elements and then applies a function $f : \mathbb{A} \to \mathbb{D} \in \Sigma$ to the new second element of the tensor pair:

$$P = x : \mathbb{A}, y : \mathbb{B} \triangleright \mathsf{pm}\ x \otimes y\ \mathsf{to}\ a \otimes b.b \otimes f(a) : \mathbb{B} \otimes \mathbb{D}$$

Let f^ϵ be an erroneous implementation of f. The program above is thus rewritten as:

$$P^\epsilon = x : \mathbb{A}, y : \mathbb{B} \triangleright \mathsf{pm}\ x \otimes y\ \mathsf{to}\ a \otimes b.b \otimes f(a)^\epsilon : \mathbb{B} \otimes \mathbb{D}$$

Consider we can postulate the axiom $f^\epsilon(a) =_\epsilon f(a)$. Then, it is possible to show that $P^\epsilon =_\epsilon P$ using our metric equational system. The prove is as follows. The types and contexts are omitted for brevity as no ambiguity arises.

1	$f(a)^\epsilon =_\epsilon f(a)$	
2	$b =_0 b$	(refl)
3	$b \otimes f(a)^\epsilon =_\epsilon b \otimes f(a)$	$(1, 2, \otimes_i)$
4	$x \otimes y =_0 x \otimes y$	(refl)
5	$\mathsf{pm}\ x \otimes y\ \mathsf{to}\ a \otimes b.b \otimes f(a)^\epsilon =_\epsilon \mathsf{pm}\ x \otimes y\ \mathsf{to}\ a \otimes b.b \otimes f(a)$	$(3, 4, \otimes_e)$

Note that without a metric equation for conditionals, we cannot, for instance, reason about approximate equivalence in programs such as **Conjunction**, discussed in the previous subsubsection. For example, it would be interesting to know whether, given that $v' =_\varepsilon \mathrm{inl}_{\mathbb{I}}(*)$, it follows that

$$\textbf{Conjunction}[v/v', w/w'] =_\varepsilon \textbf{Conjunction}[\mathrm{inl}_{\mathbb{I}}(*)/v', w/w'].$$

Definition 2.2.20. Consider a tuple (G, Σ), where G is a class of ground types and Σ is a class of sorted operation symbols of the form $f : A_1, \ldots, A_n \to A$ with $n \geq 1$. A *metric* λ-*theory* is a tuple $((G, \Sigma), Ax)$, where Ax is a class of *metric equations-in-context* over λ-terms constructed from (G, Σ).

The elements of Ax are called the *axioms* of the theory. Let $Th(Ax)$ denote the smallest class that contains Ax and is closed under the rules presented in Figure 2 (i.e., the classical equational system) and Figure 3. The elements of $Th(Ax)$ are called the *theorems* of the theory.
For instance, in Example 2.2.19, **Conjunction**$[v/v', w/w'] =_\varepsilon$ **Conjunction**$[\mathrm{inl}_{\mathbb{I}}(*)/v', w/w']$ is a theorem.

2.3 Category theory

Category theory originated as an effort to connect and unify two distinct areas of mathematics. The goal was to study and classify specific geometric structures—such as topological spaces, manifolds, and bundles—by associating them with corresponding algebraic structures like groups, rings, and abelian groups. It became clear that a language was needed to connect geometric and algebraic objects—one not explicitly tailored to geometry or algebra. Only a language of such generality could allow meaningful discussion across both fields. This is the birth of category theory. Described as ”a language about nothing, and therefore about everything,” category theory provides a highly general way of discussing mathematical concepts. It was invented by Samuel Eilenberg and Saunders MacLane [33]. They organized various mathematical structures into categories called geometric others algebraic. To connect these categories, they defined functors, which map objects and morphisms from one category to another, much like functions do. They further introduced natural transformations, which provide a way to compare functors, translating the results of one functor into those of another. [26]

2.3.1 Categories

Definition 2.3.1. A *category* C consists of

- a collection of objects $A, B, C, \ldots$, denoted $|\mathsf{C}|$ or $\mathrm{Obj}(\mathsf{C})$;

- a collection of morphisms $f, g, \ldots$, usually denoted $\mathsf{C}(A, B)$, $\mathrm{Hom}_{\mathsf{C}}(A, B)$, or $\mathrm{Hom}(A, B)$ if there is no ambiguity.

The collection for morphisms has the following structure:

- Each morphism has a specified domain and codomain; the notation $f : A \to B$ indicates that f is a morphism from object A to object B.
- Every object A has an identity morphism $\mathrm{id}_A : A \to A$.
- For any pair of morphisms $f : A \to B$ and $g : B \to C$, where the codomain of f matches the domain of g, there exists a composite morphism $g \circ f : X \to Z$. We will also write $g \circ f$ as $f \cdot g$ or simply fg.

The composition is required to satisfy the two following laws: if $f : A \to B, g : B \to C,$ and $h : C \to D$ are morphisms, then

- $f \circ \mathrm{id}_a = f = \mathrm{id}_b \circ f$;
- $(f \circ g) \circ h = f \circ (g \circ h)$.

Example 2.3.2. Set is the category whose objects are sets and whose morphisms are functions between them. Given a function $f : A \to B$, it assigns to each element $a \in A$ a unique image $f(a) \in B$. For any two functions $f : A \to B$ and $g : B \to C$, their composition is defined by

$$(g \circ f)(a) = g(f(a)) \quad \text{for all } a \in A.$$

This composition is *associative*. That is, for any further function $h : C \to D$, we have

$$(h \circ g) \circ f = h \circ (g \circ f),$$

since for every $a \in A$,

$$((h \circ g) \circ f)(a) = h(g(f(a))) = (h \circ (g \circ f))(a).$$

Moreover, for every set A, there exists an *identity function*

$$\mathrm{id}_A : A \to A, \quad \text{defined by } \mathrm{id}_A(a) = a,$$

which satisfies the unit laws for composition:

$$f \circ \mathrm{id}_A = f \quad \text{and} \quad \mathrm{id}_B \circ f = f$$

for any function $f : A \to B$.
Therefore, $\mathbf{Set}$, with sets as objects and functions as morphisms, satisfies the axioms of a category.

Another common type of example consists of categories of sets equipped with additional structure, along with functions that preserve that structure.

Definition 2.3.3. A *partially ordered set* or *partial order* is a set A equipped with a binary relation $\leq_A$ satisfying the following properties for all $a, b, c \in A$:

- Reflexivity: $a \leq_A a$;
- Transitivity: If $a \leq_A b$ and $b \leq_A c$, then $a \leq_A c$;
- Antisymmetry: If $a \leq_A b$ and $b \leq_A a$, then $a = b$.

Example 2.3.4. The set of real numbers $\mathbb{R}$, equipped with the usual ordering $\leq$, forms a poset. Moreover, it is *linearly ordered* (or *totally ordered*), since for any $x, y \in \mathbb{R}$, either $x \leq y$ or $y \leq x$ holds.

Definition 2.3.5. Given two partial orders $(A, \leq_A)$ and $(B, \leq_B)$, a function $m : A \to B$ is called a *monotone map* (or *order-preserving map*) if for all $a, a' \in A$,

$$a \leq_A a' \quad \Rightarrow \quad m(a) \leq_B m(a').$$

Example 2.3.6. PO is the category of all partial orders and all monotone maps. First, for any poset A, the identity function $\mathrm{id}_A : A \to A$ is monotone. Indeed, for all $a \in A$,

$$a \leq_A a \quad \Rightarrow \quad \mathrm{id}_A(a) \leq_A \mathrm{id}_A(a).$$

Next, given monotone maps $f : A \to B$ and $g : B \to C$, their composition $g \circ f : A \to C$ is also monotone. For all $a, a' \in A$, if $a \leq_A a'$, then

$$f(a) \leq_B f(a') \quad \text{and} \quad g(f(a)) \leq_C g(f(a')),$$

so it follows that

$$(g \circ f)(a) \leq_C (g \circ f)(a').$$

Example 2.3.7. Each partially ordered set naturally defines a category. Let $(P, \leq)$ be a poset. We define a category $\mathsf{B}(\mathsf{P}, \leq)$, often denoted simply by $\mathsf{B}(\mathsf{P})$ or even P, where the objects are the elements of P, and there is a unique morphism $p \to q$ if and only if $p \leq q$. The reflexivity of the order $\leq$ ensures the existence of identity morphisms, while transitivity guarantees that morphisms compose appropriately. Moreover, since there is at most one morphism between any two objects, composition is trivially associative.

Example 2.3.8. CVect is the category of finite complex vector spaces and linear mappings.

Example 2.3.9. Given a functional programming language L, we can define a category CompFunc. In this category, the objects represent the data types of the language L, and the morphisms correspond to computable functions or programs. A function is considered *computable* if a computer program is capable of executing that function.
The composition of morphisms is defined as follows: given two morphisms $f\colon X \to Y$ and $g\colon Y \to Z$, the composition $g \circ f\colon X \to Z$ is defined by applying g to the output of f. This composition is often written as $f; g$.
Additionally, the identity morphism $\mathrm{id}_X\colon X \to X$ represents the "identity program," which returns its input without making any changes (i.e., it "does nothing").

Definition 2.3.10. A morphism $f : A \to B$ in a category C be a category is called an *isomorphism* if there exists a morphism $f^{-1} : B \to A$ such that

$$f^{-1} \circ f = \mathrm{id}_A \quad \text{and} \quad f \circ g = \mathrm{id}_B.$$

In this case, f^{-1} is called the *inverse* of f, and it is unique. If such an isomorphism exists, we say that A and B are *isomorphic*, written $A \cong B$.

One of the central ideas in category theory is *duality*. Simply put, for a given definition of a structure, there is often a corresponding dual concept obtained by reversing the directions of all the arrows.

Definition 2.3.11. Let C be a category. The *opposite category*, denoted C^{op}, is defined as follows:

- The objects of C^{op} are the same as those of C.
- For any pair of objects A, B, the hom-set in C^{op} is defined by

$$\mathit{Hom}_{\mathsf{C}^{\mathrm{op}}}(A, B) = \mathit{Hom}_{\mathsf{C}}(B, A),$$

that is, each morphism $f : A \to B$ in C^{op} corresponds to a morphism $f : B \to A$ in C.

- Composition in C^{op} is defined using the composition in C, but in reverse order. That is, if

$$A \xrightarrow{f} B \xrightarrow{g} C$$

are morphisms in C^{op}, corresponding to morphisms

$$C \xrightarrow{g} B \xrightarrow{f} A$$

in C, then the composition in C^{op} is defined by

$$g \circ f := f \circ_{\mathsf{C}} g.$$

Thus, C^{op} reverses the direction of morphisms and composition while retaining the same collection of objects.

Definition 2.3.12. A subcategory D of a category C is a category such that

- All the objects of D are objects of C, and all the morphisms of D are morphisms of C (that is, $\mathsf{D}_0 \subseteq \mathsf{C}_0$ and $\mathsf{D}_1 \subseteq \mathsf{C}_1$).
- The domain and codomain of any morphism in D are the same as in C (in other words, the domain and codomain maps for D are the restrictions of those for C).
- It follows that for any objects A and B in D, we have $\mathrm{Hom}_{\mathsf{D}}(A, B) \subseteq \mathrm{Hom}_{\mathsf{C}}(A, B)$. If A is an object of D, then its identity morphism id_A in C also belongs to D.
- If $f : A \to B$ and $g : B \to C$ are morphisms in D, then the composite $g \circ f$, as defined in C, is also in D, and coincides with the composite in D.

Example 2.3.13. The category FinSet, whose objects are finite sets and whose morphisms are functions between them, forms a subcategory of the category Set.

Definition 2.3.14. A category is called *small* if both its collection of objects and its collection of morphisms form sets. A category is called *locally small* if, for every pair of objects, the corresponding hom-set is a set.

2.3.2 Produts and coproducts

A category frequently possesses more intricate structure than a mere collection of objects and their morphisms. The existence of particular relationships among certain objects and morphisms can can make some objects have important properties.
It should be noted that a diagram is said to commute if, for every pair of objects X and Y in the diagram, all directed paths from X to Y yield equal morphisms.

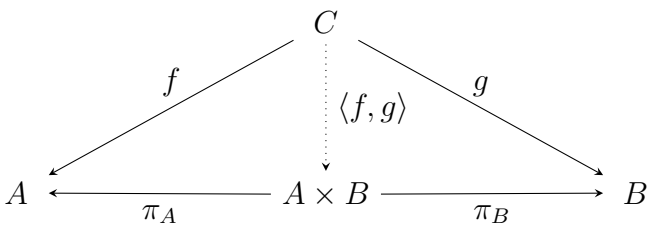
Definition 2.3.15. An object 0 in a category C is called an *initial object* if for every object $A \in$ C, there exists a unique morphism $f : 0 \to A$.

Definition 2.3.16. An object 1 in a category C is called a *terminal object* if for every object $A \in$ C, there exists a unique morphism $f : A \to 1$.

Example 2.3.17. In the category Set, the empty set $\emptyset$ is an initial object, since for any set S, there exists a unique function$f : \emptyset \to S$. This function is unique because there are no elements in $\emptyset$ to map.
Any singleton set, such as $\{*\}$ or $\{a\}$, is a terminal object in this category. For any set S, there exists a unique function $f : S \to \{*\}$, which maps every element of S to the sole element of the singleton set

Example 2.3.18. Let $(P, \leq)$ be a partial order and P be its associated category. Here, the initial object is the *bottom element*—an element that is less than or equal to every other element in P. The terminal object in P is the *top element*—an element that is greater than or equal to every other element in P.

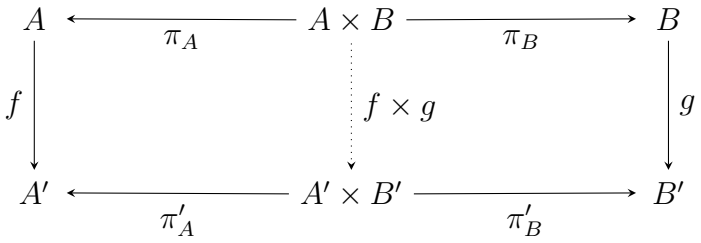
Definition 2.3.19 (*Product*)**.** Consider a category C. We say that it has (binary) products if for any objects A and B in C there also exists an object $A \times B$ C with morphisms $\pi_A : A \times B \to A$ and $\pi_B : A \times B \to B$ that satisfy a certain universal property: specifically for every two morphisms $f : C \to A$ and $g : C \to B$ there exists a *unique* morphism $\langle f, g\rangle : C \to A \times B$ called *pairing* that makes the diagram below commute.



Definition 2.3.20. Let $A \times B$ be a product of objects A and B, and let $A' \times B'$ be a product of objects A' and A' in a category C. Suppose we are given morphisms $f : A \to A'$ and $g : B \to B'$.
Then there exists a unique morphism

$$f \times g : a \times b \to a' \times b'$$

such that the following diagram commutes.



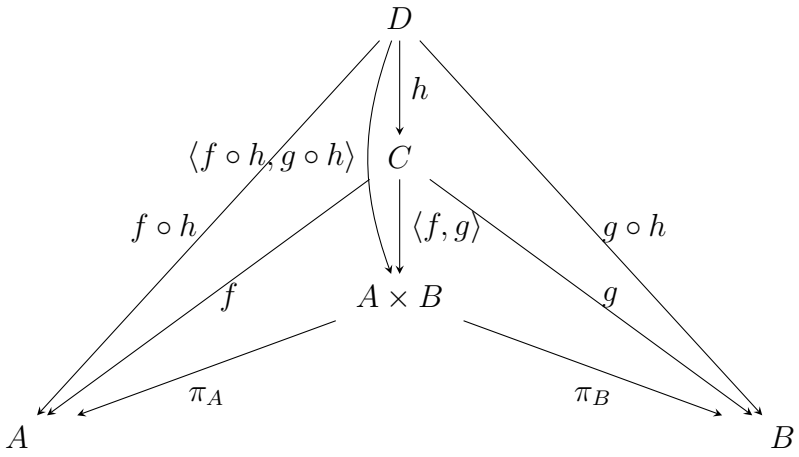
This induced morphism $f \times g$ is called the *product of the morphisms* f and g, and it is given explicitly by

$$f \times g = \langle f \circ \pi_A,\, g \circ \pi_B \rangle.$$

Theorem 2.3.21. *Let $A \times B$ be the product of objects A and B in a category* C. *For any object C and morphisms $f : C \to A$ and $g : C \to B$ are morphisms, it holds that:*

$$\langle f \circ h,\, g \circ h \rangle = \langle f, g \rangle \circ h.$$

Proof. The universal property of the product induces a unique morphism $\langle f, g \rangle : C \to A \times B$ such that $\pi_A \circ \langle f, g \rangle = f \quad$ and $\quad \pi_B \circ \langle f, g \rangle = g$.
Now, let $h : D \to C$ be another morphism. Then the compositions $f \circ h : D \to A$ and $g \circ h : D \to B$ also induce a unique morphism $\langle f \circ h,\, g \circ h \rangle : D \to A \times B$ by the universal property of the product. As a result, by the universal property of the produt the following diagram commutes.



□

Example 2.3.22. In the category Set, the product of two sets A and B is given by their Cartesian product, denoted

$$A \times B = \{(a, b) \mid a \in A,\ b \in B\}.$$

The projection maps are defined by

$$\pi_A(a, b) = a \quad \text{and} \quad \pi_B(a, b) = b.$$

Given a set C and morphisms $f : C \to A$ and $g : C \to B$, their pairing is the map

$$\langle f, g\rangle(c) = (f(c), g(c)).$$

Example 2.3.23. Let $(P, \leq)$ be a partial order and P be its associated category. Consider a product of elements $p \times q \in P$. Then, by definition, there must exist projections satisfying

$$p \times q \leq p \quad \text{and} \quad p \times q \leq q.$$

Furthermore, for any element $x \in P$, if

$$x \leq p \quad \text{and} \quad x \leq q,$$

then it follows that

$$x \leq p \times q.$$

This operation $p \times q$ corresponds to what is commonly known as the *greatest lower bound* or *meet*, and is typically denoted by $p \wedge q$.

Maybe tirar este exemplo

Example 2.3.24. In the category CVect, the product of two vector spaces V and W corresponds to their direct sum, denoted by $V \oplus W$. The projection maps are the linear maps

$$\pi_V : V \oplus W \to V, \quad \pi_V(v, w) = v,$$

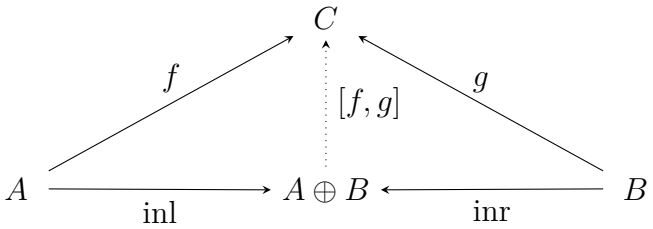
$$\pi_W : V \oplus W \to W, \quad \pi_W(v, w) = w.$$

Given any vector space U and linear maps $f : U \to V$ and $g : U \to W$, the unique map $\langle f, g \rangle : U \to V \oplus W$ is defined by

$$\langle f, g \rangle(u) = (f(u), g(u)).$$

The *coproduct* is the dual of the *product*—it is obtained by reversing all the morphisms in the definition of a product. Consequently, a product in a category C corresponds to a coproduct in the opposite category C^{op}.

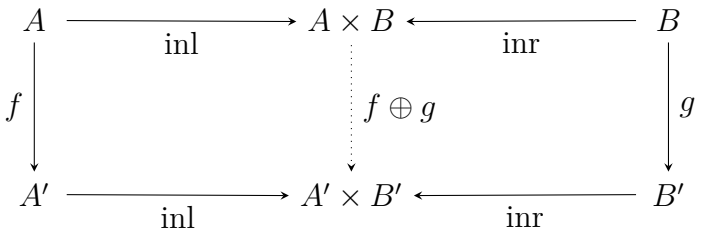
Definition 2.3.25. Consider a category C. We say that it has (binary) coproducts if for any objects A and B in C there also exists an object $A \oplus B$ in C with morphisms $\mathrm{inl} : A \to A \oplus B$ and $\mathrm{inr} : B \to A \oplus B$ that satisfy a certain universal property: specifically for every two morphisms $f : A \to C$ and $g : B \to C$ there exists a *unique* morphism $[f, g] : A \oplus B \to C$ known as *co-pairing* that makes the diagram below commute.



Definition 2.3.26. Let $A \oplus B$ be a coproduct of objects A and B, and let $A' \times B'$ be a coproduct of objects A' and A' in a category C. Suppose we are given morphisms $f : A \to A'$ and $g : B \to B'$.
Then there exists a unique morphism

$$f \times g : a \times b \to a' \times b'$$

such that the following diagram commutes.



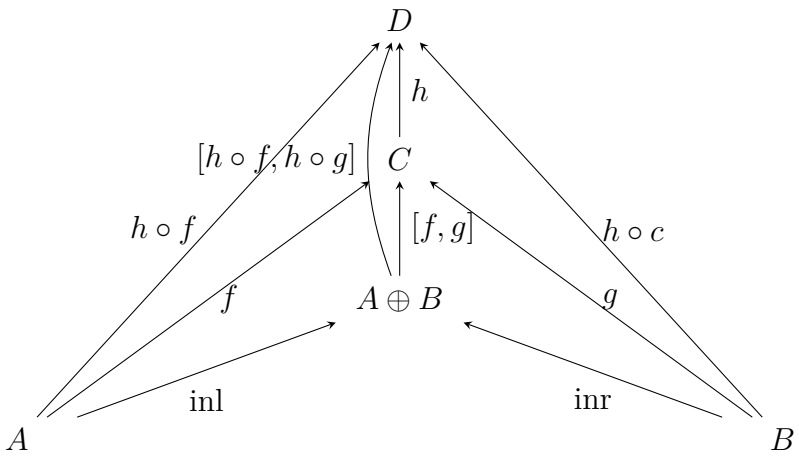
This induced morphism $f \oplus g$ is called the *coproduct of the morphisms* f and g, and it is given explicitly by

$$f \oplus g = [\text{inl} \circ f,\ \text{inr} \circ g].$$

Theorem 2.3.27. *Let $A \oplus B$ be the product of objects A and B in a category* C*. For any object C and morphisms $f : A \to C$ and $g : B \to C$ are morphisms, it holds that:*

$$[h \circ f,\ h \circ g] = h \circ [f, g].$$

Proof. By the universal property of the coprodut the following diagram commutes.



□

Proposition 2.3.28. *[34, Proposition 3.12] Coproducts are unique up to isomorphism. Alternatively, this can be formulated as follows: let $(C, \text{inl} : A \to C, \text{inr} : B \to C)$ and $(C', \text{inl}' : A \to C', \text{inr}' : B \to C')$ be two coproducts of objects A and B in a category. Then there exists a unique isomorphism $\varphi : C \to C'$ such that*

$$\varphi \cdot \text{inl} = \text{inl}' \quad \textit{and} \quad \varphi \cdot \text{inr} = \text{inr}'.$$

Example 2.3.29. In the category Set, the coproduct $A \oplus B$ of two sets is their disjoint union, which can be constructed as

$$A \oplus B = \{(a, 1) \mid a \in A\} \cup \{(b, 2) \mid b \in B\}.$$

The canonical coproduct injections are defined by

$$i_1(a) = (a, 1), \quad i_2(b) = (b, 2).$$

Given any set C and functions $f : A \to C$ and $g : B \to C$, the copairing $[f, g] : A \oplus B \to C$ is defined by

$$[f, g](x, \delta) = \begin{cases} f(x) & \text{if } \delta = 1, \\ g(x) & \text{if } \delta = 2. \end{cases}$$

Example 2.3.30. Let $(P, \leq)$ be a partial order and P be its associated category. Consider a coproduct of elements $p \oplus q \in P$. Then, by definition, there must exist injections satisfying

$$p \leq p + q \quad \text{and} \quad q \leq p + q.$$

Furthermore, for any element $z \in P$, if

$$p \leq z \quad \text{and} \quad q \leq z,$$

then it follows that

$$p + q \leq z.$$

This operation $p + q$ corresponds to what is commonly known as the *least upper bound* or *join*, and is typically denoted by $p \vee q$.

Maybe tirar este exemplo

Example 2.3.31. In CVect, the coproduct coincides with the product. In such cases, this structure is called a *biproduct*. In both CVect. the injection maps are the linear maps

$$\text{inl} : V \to V \oplus W, \quad \text{inl}(v) = (v, 0),$$

$$\text{inr} : W \to V \oplus W, \quad \text{inr}(w) = (0, w).$$

Given any vector space U and linear maps $f : V \to U$ and $g : W \to U$, the unique map $\langle f, g \rangle : V \oplus W \to U$ is defined by

$$[f, g](v, w) = f(v) + g(w).$$

2.3.3 Functors

Although categories are interesting on their own, the real strength of category theory lies in understanding how categories relate to one another. Just as functions express relationships between sets, functors play a similar role for categories. A functor maps each object in one category to an object in another category, and it does the same for morphisms, preserving the structure of the relationships.

Definition 2.3.32. Let C and D be two categories. A *functor* $F : \mathsf{C} \to \mathsf{D}$ consists of a mapping that assigns to each object A in C an object FA in D, and to each morphism $f \in \mathrm{Hom}_{\mathsf{C}}(A, B)$ a morphism $Ff \in \mathrm{Hom}_{\mathsf{D}}(FA, FB)$, in such a way that the following two conditions are satisfied for all objects A, B, C in C and all morphisms $f \in \mathrm{Hom}_{\mathsf{C}}(A, B)$ and $g \in \mathrm{Hom}_{\mathsf{C}}(B, C)$:

$$F(\mathrm{id}_A) = \mathrm{id}_{FA}, \qquad F(g \circ f) = F(g) \circ F(f).$$

A functor $F : \mathsf{C} \to \mathsf{D}$ is said to be *full* if, for all objects A and B in C, the induced map

$$F_{A,B} : \mathrm{Hom}_{\mathsf{C}}(A, B) \longrightarrow \mathrm{Hom}_{\mathsf{D}}(FA, FB), \quad f \mapsto Ff,$$

is surjective. The functor is called *faithful* if each $F_{A,B}$ is injective, and *fully faithful* if each $F_{A,B}$ is bijective.
A *full embedding* is a functor that is fully faithful and, in addition, injective on objects.

Example 2.3.33. Let C be a category. Then there exists an *identity functor* $\mathrm{id}_{\mathsf{C}} : \mathsf{C} \to \mathsf{C}$, which is defined on objects by $\mathrm{id}_{\mathsf{C}}(A) = A$ for every object A in C, and analogously on morphisms, that is, $\mathrm{id}_{\mathsf{C}}(f) = f$ for every morphism f in C.

Example 2.3.34. Consider the natural numbers $\mathbb{N}$ as partial order category. There is a functor $(-) + 5 : \mathbb{N} \to \mathbb{N}$ that maps each object $m \in \mathbb{N}$ to $m + 5$. This defines a functor because it preserves morphisms: if $m \leq m'$, then $m + 5 \leq m' + 5$. Moreover, the identity morphisms are preserved, since $(\mathrm{id}_m) + 5 = \mathrm{id}_{m+5}$.

Example 2.3.35. Consider the set of real numbers $\mathbb{R}$ and the set of integers $\mathbb{Z}$, each regarded as a partial order category. In this context, there exists a functor $\mathrm{Floor} : \mathbb{R} \to \mathbb{Z}$ that assigns to each real number $r \in \mathbb{R}$ the greatest integer less than or equal to r, denoted $\lfloor r \rfloor$. For instance, $\lfloor 6.2 \rfloor = 6$ and $\lfloor -1.66 \rfloor = -2$.
Similarly, there exists a *ceiling functor* $\mathrm{Ceil} : \mathbb{R} \to \mathbb{Z}$ that maps each real number r to the least integer greater than or equal to r, denoted $\lceil r \rceil$.

Definition 2.3.36. Given categories C, D, and E, a *bifunctor* $F : \mathsf{C} \times \mathsf{D} \to \mathsf{E}$ is simply a functor from the product category $\mathsf{C} \times \mathsf{D}$ to E. In particular, F is a rule that assigns:

- to every object $A \in \mathsf{C}$ and $B \in \mathsf{D}$, an object $F(A, B) \in \mathsf{E}$;
- to every morphism $f : A \to A'$ in C and $g : B \to B'$ in D, a morphism $F(f, g) : F(A, B) \to F(A', B') \in \mathsf{E}$.

These assignments must satisfy the following two requirements:

- Respect for composition: For morphisms $f : A \to A'$, $f' : A' \to A''$ in C and $g : B \to B'$, $g' : B' \to B''$ in D, it should hold that

  $$F(f' \circ f,\, g' \circ g) = F(f', g') \circ F(f, g),$$

  where the $\circ$ on the right-hand side is composition in E.
- Respect for identities: For all objects $A \in \mathsf{C}$ and $B \in \mathsf{D}$, it should hold that

  $$F(\mathrm{id}_A, \mathrm{id}_B) = \mathrm{id}_{F(A,B)},$$

  where id_A and id_B are the identity morphisms in C and D, respectively, and $\mathrm{id}_{F(A,B)}$ is the identity morphism in E.

Many times, rather than writing the name of the bifunctor before the input, like $F(A, B)$, we write the bifunctor as an operation between the inputs, for example, $a \,\square\, b$. If we use this notation, the condition

$$F(f' \circ f,\, g' \circ g) = F(f', g') \circ F(f, g)$$

becomes

$$(f' \circ f) \,\square\, (g' \circ g) = (f' \,\square\, g') \circ (f \,\square\, g).$$

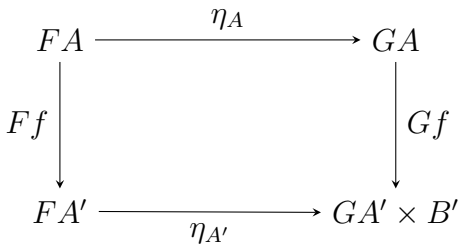
2.3.4 Natural Tranformations

A *natural transformation* is a morphism between functors. It provides a way of relating two functors that have the same domain and codomain. Intuitively, if we consider two functors $F, G : \mathsf{C} \to \mathsf{D}$ as different ways of assigning images of the category C into the category D, then a natural transformation $\eta : F \Rightarrow G$ is a coherent way of transforming the image of F into the image of G.

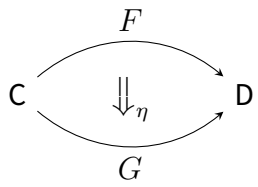
Definition 2.3.37. Let C and D be categories, and let $F, G : \mathsf{C} \to \mathsf{D}$ be functors. A *natural transformation* $\eta : F \Rightarrow G$ is a family of morphisms in D,

$$(\eta_A : FA \to GA)_{A \in \mathrm{Ob}(\mathsf{C})},$$

indexed by the objects of C, such that for every morphism $f : A \to A'$ in C, the following diagram commutes.



Given a natural transformation $\eta : F \Rightarrow G$, the morphism $\eta_A : F(A) \to G(A)$ in D is called the *component* of η at A. A natural transformation $\eta : F \Rightarrow G$ is represented diagrammatically as



Example 2.3.38. For every functor $F : \mathsf{C} \to \mathsf{D}$, there exists a natural transformation

$$\iota_F : F \Rightarrow F$$

known as *identity natural transformation*, such that for each object $A \in \mathsf{C}$, each component of ι_F is the identity morphism:

$$(\iota_F)_A = \mathrm{id}_{F(A)} : F(A) \to F(A).$$

Example 2.3.39. The *list functor*

$$\mathrm{List} : \mathsf{Set} \to \mathsf{Set}$$

assigns to each set S the set of all finite sequences (or lists) of its elements.
For instance, if $S = \{a, b, c\}$, then

$$\mathrm{List}(S) = \{\varepsilon, a, b, c, aa, ab, ac, ba, \ldots, abc, cba, \ldots\},$$

where ε denotes the empty list.
Given a function $f : S \to T$, where $T = \{1, 2\}$, the functor maps it to $\mathrm{List}(f) : \mathrm{List}(S) \to \mathrm{List}(T)$, which applies f to each element of a list. For example, if

$$f(a) = 2, \quad f(b) = 1, \quad f(c) = 2,$$

then $\mathrm{List}(f)(aabccba) = 2212212$.
There exists a natural transformation

$$\mathrm{Reverse} : \mathrm{List} \Rightarrow \mathrm{List},$$

whose component at a set S, $\mathrm{Reverse}_S$, maps each list to its reversal. For example:

$$\mathrm{Reverse}_S(accbab) = babcca.$$

Definition 2.3.40. A natural transformation $\eta : F \Rightarrow G$ between functors $F, G : \mathsf{C} \to \mathsf{D}$ is called a *natural isomorphism* if, for every object $A \in \mathsf{C}$, $\eta_A : F(A) \to G(A)$ is an isomorphism in D.

2.3.5 Equivalence of Categories

In category theory, the concept of isomorphism between categories can be quite strict - is is not often it arises in a non-trivial manner. So, we weaken the notion of isomorphism and come to the concept of an *equivalence of categories*.
If $F : \mathsf{C} \to \mathsf{D}$ is an isomorphism of categories, then for every object $B \in \mathsf{D}$, there exists a *unique* object $A \in \mathsf{C}$ such that $F(A) = B$. This expresses the idea that C and D are structurally identical.
An *equivalence of categories* relaxes this requirement. For every object $B \in \mathsf{D}$, there exists an object $A \in \mathsf{C}$ such that $F(A)$ is not necessarily equal to B, but is *isomorphic* to B.

Definition 2.3.41. Categories C and D are said to be *equivalent* if there exist functors $F : \mathsf{C} \to \mathsf{D}$ and $G : \mathsf{D} \to \mathsf{C}$ such that $G \circ F \cong \mathrm{id}_{\mathsf{C}}$ and $F \circ G \cong \mathrm{id}_{\mathsf{D}}$. The functors F and G are called *quasi-inverses*, and we write $\mathsf{C} \simeq \mathsf{D}$. This means that for every $A \in \mathsf{C}$, there is a $B \in \mathsf{D}$ with $G(B) \cong A$, and for every $B \in \mathsf{D}$, there is an $A \in \mathsf{C}$ with $F(A) \cong B$.

Example 2.3.42. One of the simplest examples of an equivalence of categories is the relationship between the one-object category $\mathbf{1}$ and the category $\mathbf{2}_I$, which has two objects and

a single isomorphism between them. We can visualize this as:

$$* \quad \simeq \quad a \xrightarrow{\cong} b$$

More precisely, there is a unique functor $! : \mathbf{2}_I \to \mathbf{1}$, and a functor $L : \mathbf{1} \to \mathbf{2}_I$ defined by $L(*) = a$. Clearly, the composition $! \circ L$ is equal to $\mathrm{id}_{\mathbf{1}}$, and $L \circ ! \cong \mathrm{id}_{\mathbf{2}_I}$, since both objects a and b in $\mathbf{2}_I$ are isomorphic. Thus, $\mathbf{1} \simeq \mathbf{2}_I$.

2.3.6 Adjoints

If we further weaken the notion of an equivalence of categories, we we arrive at the concept of an *adjunction*. A central idea in category theory is that *the weaker the assumptions we impose, the more mathematical phenomena we can capture*. This principle explains why adjunctions are among category theory's most important structures.

Definition 2.3.43. An adjunction between categories C and D is given by a quadruple (L, R, η, ϵ), where $L : \mathsf{C} \to \mathsf{D}$ and $R : \mathsf{D} \to \mathsf{C}$ are functors, and $\eta : \mathrm{id}_{\mathsf{C}} \Rightarrow RL$ and $\epsilon : LR \Rightarrow \mathrm{id}_{\mathsf{D}}$ are natural transformations such that

$$(\eta R) \circ (R\epsilon) = \mathrm{id}_R \quad \text{and} \quad (L\eta) \circ (\epsilon L) = \mathrm{id}_L.$$

One says that R is right adjoint to L, or that L is left adjoint to R, and one calls η the *unit* and ϵ the *counit* of the adjunction. Such an adjunction is denoted by $L \dashv R$, where the turn of the symbol $\dashv$ always points to the left adjoint.

An adjunction between categories can be characterized in multiple equivalent ways. It follows one of the alternative formulations.

Given categories C and D, a pair of functors $L : \mathsf{C} \to \mathsf{D}$ and $R : \mathsf{D} \to \mathsf{C}$ form an *adjunction* $L \dashv R$ if there exists a natural isomorphism:

$$\mathrm{Hom}_{\mathsf{D}}(L(A), B) \xrightarrow{\Phi_{A,B}} \mathrm{Hom}_{\mathsf{C}}(A, R(B)).$$

Example 2.3.44. Consider the set of real numbers $\mathbb{R}$ and the set of integers $\mathbb{Z}$, each viewed as partial order categories. There is an inclusion functor $\mathrm{inc} : \mathbb{Z} \hookrightarrow \mathbb{R}$ which simply maps each integer to itself. This inclusion has a left adjoint $L : \mathbb{R} \to \mathbb{Z}$.

To determine this left adjoint L, we use the definition of an adjunction: for all $N \in \mathbb{Z}$ and $R \in \mathbb{R}$, we have a natural isomorphism:

$$\mathrm{Hom}_{\mathbb{Z}}(L(R), N) \cong \mathrm{Hom}_{\mathbb{R}}(R, \mathrm{inc}(N)).$$

Since both $\mathbb{Z}$ and $\mathbb{R}$ are partial orders, the hom-sets contain at most one morphism. Hence, this isomorphism reduces to the logical equivalence:

$$L(R) \leq N \text{ if and only if } R \leq \mathrm{inc}(N) = N.$$

Take $R = 7.27$ as an example. Then the inequality $R \leq N$ holds precisely when N is an integer greater than or equal to 7.27. That is:

$$7.27 \not\leq 5, \quad 7.27 \not\leq 6, \quad 7.27 \not\leq 7, \quad 7.27 \leq 8, \quad 7.27 \leq 9, \quad \dots$$

By the condition above, we must then have:

$$L(7.27) \not\leq 5, \quad L(7.27) \not\leq 6, \quad L(7.27) \not\leq 7, \quad L(7.27) \leq 8, \quad L(7.27) \leq 9, \quad \dots$$

From this, we conclude that $L(7.27) = 8$. In general, $L(R)$ is the least integer greater than or equal to R, *i.e.*, the ceiling function:

$$L(r) = \lceil r \rceil.$$

Thus, the inclusion functor inc has $\lceil\ \rceil$ as a left adjoint, *i.e.*, $\lceil\ \rceil \dashv \mathrm{inc}$. The unit of this adjunction is the natural transformation $\eta : \mathrm{id}_{\mathbb{R}} \Rightarrow \mathrm{inc} \circ \lceil\ \rceil$, which expresses the inequality $r \leq \lceil r \rceil$ for all $r \in \mathbb{R}$. The counit of the adjunction is the identity, since for any integer n, it holds that $\lceil N \rceil = N$.

Definition 2.3.45. Let $F : \mathsf{C} \to \mathsf{D}$ and $G : \mathsf{D} \to \mathsf{E}$ be functors. It is said that G *preserves coproducts* if whenever L is a coproduct of F, then $G(L)$ is a coproduct of $G \circ F$.

Theorem 2.3.46. *Left adjoints preserve coproducts.*

2.3.7 Monoidal categories

Definition 2.3.47. A **monoid** is a triple $(M, \cdot, u)$, where M is a set equipped with a binary operation $\cdot : M \times M \to M$ and a distinguished element $u \in M$ called the *unit*, satisfying the following axioms for all $x, y, z \in M$:

$$\text{(Associativity)} \qquad x \cdot (y \cdot z) = (x \cdot y) \cdot z,$$

$$\text{(Unit laws)} \qquad u \cdot x = x = x \cdot u.$$

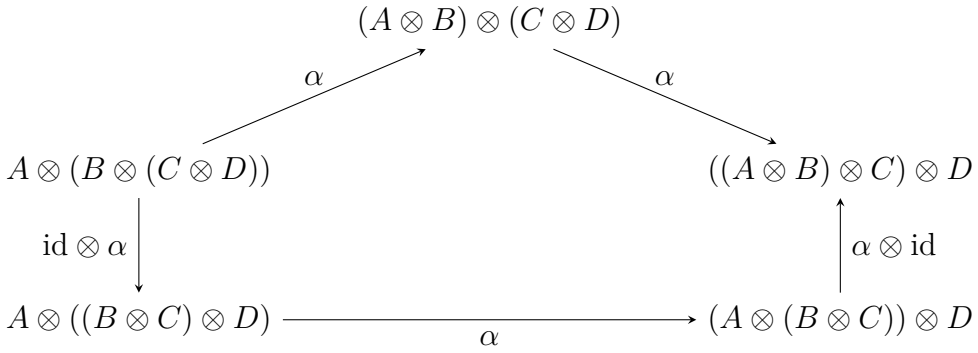
Monoidal categories are named so because they are categories equipped with an additional structure that resembles the structure of monoids.

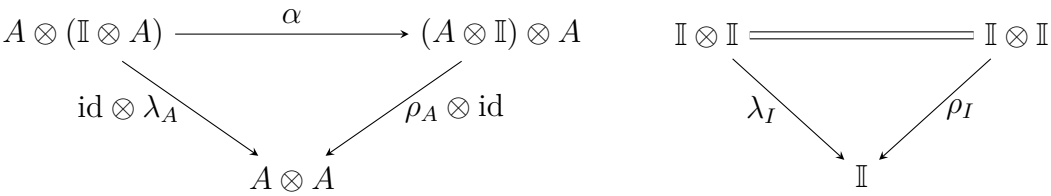
Definition 2.3.48. A *monoidal category* consists of a category C equipped with a bifunctor $\otimes : \mathsf{C} \times \mathsf{C} \to \mathsf{C}$ called *tensor product* and a distinguished object $I \in \mathsf{C}$, called *unit* together with natural isomorphisms

$$\alpha_{A,B,C} : A \otimes (B \otimes C) \to (A \otimes B) \otimes C,$$

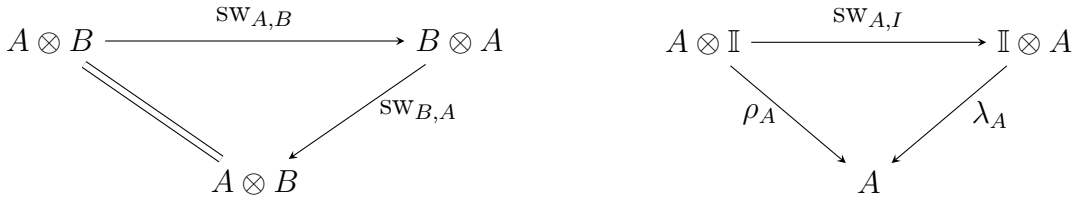
$$\lambda_A : \mathbb{I} \otimes A \to A, \quad \rho_A : A \otimes \mathbb{I} \to A,$$

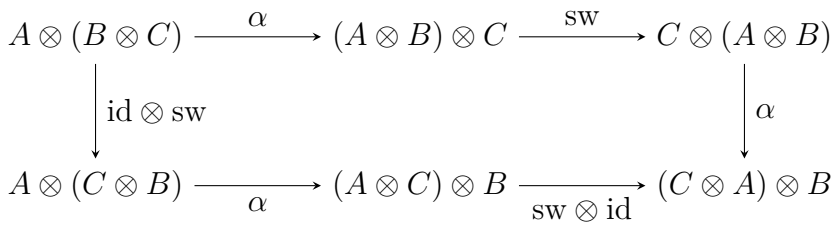
known as *reassociator*, *left unitor*,and *right unitor*, respectively.
Moreover, these natural isomorphisms are required to make the following coherence diagrams commute.





Definition 2.3.49. A monoidal category is said to be *symmetric* when it is equipped with a natural isomorphism $\mathrm{SW} : A \otimes B \to B \otimes A$ known as *braiding* such that the following diagrams commute.





Definition 2.3.50. A monoidal category C is said to be *closed* if for each object A in C the functor $- \otimes A$ has a right adjoint, denoted by $A \multimap -$.

Definition 2.3.51. A monoidal category C with coproducts is called *distributive* if for every object A in C the functor $- \otimes A$ preserves coproducts. Explicitly this means that the morphism,

$$[\mathrm{inl} \otimes \mathrm{id}, \mathrm{inr} \otimes \mathrm{id}] : B \otimes A \oplus C \otimes A \to (B \oplus C) \otimes A$$

is actually an isomorphism. We will denote the respective inverse by dist. Note that if C is monoidal closed then it is automatically distributive as left adjoints preserve all colimits.

Definition 2.3.52. The *terminal map to the unit object* (of a monoidal category) is the unique mapping form each object to the monoidal unit and is represented by $!$.

Example 2.3.53. Examples of monoidal closed categories with coproducts include Set and CVect. In Set, the tensor product is the cartesian product, the monoidal unit is the singleton set, the coproduct is the disjoint union, and the internal hom consists of all functions between sets. For CVect, the tensor product is the standard tensor product of complex vector spaces, the field of complex number $\mathbb{C}$, the coproduct is the direct sum, and the internal hom is the space of complex linear maps.

Theorem 2.3.54 (*Coherence Theorem for Symmetric Monoidal Categories*)**.** *Any diagram in a symmetric monoidal category constructed only from associators α, unitors λ, ρ, the symmetry sw, and inverses and their composition and tensor product necessarily commutes.*

2.4 Interpretation

Sets, sub, exchange, soundness and completeness, conds, diagrama a explicar

Up to this point, we have discussed λ-calculus in abstract terms: we explored which programs can be written, but we have not yet assigned them any meaning. This process—assigning

meaning to syntactic expressions—is known as the *interpretation* or *semantics* of the language. In fact, the word "semantics" comes from the Greek word for "meaning".
There are different kinds of semantics, in particular, *denotational semantics* interprets terms as mathematical objects. This is done by defining a function that maps syntactic entities (such as types and terms) to semantic entities (such as sets and elements). This mapping is called the *interpretation function*, typically denoted by $[\![-]\!]$. Thus, given a term v, we write $[\![v]\!]$ to denote its meaning under a specific interpretation.
Naturally, this raises important questions: what guarantees do we have that the interpretation of terms respects the classical equations of the calculus? This leads us to the notions of *soundness* and *completeness*.
With respect to a given class of interpretations:

- *Soundness* is the property

  $$M = N \;\Rightarrow\; [\![M]\!] = [\![N]\!] \quad \text{for all interpretations in the class.}$$

  That is, if two terms are provably equal, then they are interpreted as equal.

- *Completeness* is the property

  $$[\![M]\!] = [\![N]\!] \;\Rightarrow\; M = N \quad \text{for all interpretations in the class.}$$

  That is, if two terms are interpreted as equal, then they are provably equal.

Soundness ensures that our equations are *correct*—all derivable equations are semantically valid. Completeness ensures that our equations are *sufficient*—we can derive all semantically valid equations.
We note that, in the case of the metric equations, the underlying idea is similar, although soundness and completeness are defined differently.

In order to define the interpretation of judgments $\Gamma \triangleright v : \mathbb{A}$, it is necessary to establish some notation first. Let C be a symmetric monoidal category and A, B and C be objects of this category. For all morphisms $f : A \otimes B \to C$, the morphism $\overline{f} : A \to (B \multimap C)$ denotes the corresponding curried version. Moreover, there is the application morphism, app $: (A \multimap B) \otimes A \to B$.
For all ground types $X \in G$ the interpretation of $[\![X]\!]$ is postulated to be an object of C. Types are interpreted inductively using the unit $\mathbb{I}$, the tensor $\otimes$, the coproduct $\oplus$, and the

linear map $\multimap$. Given a non-empty context $\Gamma = \Gamma', x : \mathbb{A}$, its interpretation is defined by $[\![\Gamma', x : \mathbb{A}]\!] = [\![\Gamma']\!] \otimes [\![\mathbb{A}]\!]$ if Γ' is non-empty and $[\![\Gamma', x : \mathbb{A}]\!] = [\![\mathbb{A}]\!]$ otherwise. The empty context $-$ is interpreted as $[\![-]\!] = \mathbb{I}$. Given $A_1, ..., A_n \in \mathsf{C}$, the n-tensor $(\ldots(A_1 \otimes A_2) \otimes \ldots) \otimes A_n$ is denoted as $X_1 \otimes \ldots \otimes A_n$, and similarly for morphisms.

2.4.1 Classical Semantics

"Housekeeping" morphisms are employed to handle interactions between context interpretation and the vectorial model. Given $\Gamma_1, \ldots, \Gamma_n$, the morphism that splits $[\![\Gamma_1, \ldots, \Gamma_n]\!]$ into $[\![\Gamma_1]\!] \otimes \ldots \otimes [\![\Gamma_n]\!]$ is denoted by $\mathsf{sp}_{\Gamma_1;\ldots;\Gamma_n} : [\![\Gamma_1, \ldots, \Gamma_n]\!] \to [\![\Gamma_1]\!] \otimes \ldots \otimes [\![\Gamma_n]\!]$. For $n = 1$, $\mathsf{sp}_{\Gamma_1} = \mathrm{id}$. Let Γ_1 and Γ_2 be two contexts, $\mathsf{sp}_{\Gamma_1,\Gamma_2} \to \Gamma_1 \otimes \Gamma_2$ is defined as:

$$\mathsf{sp}_{-;\Gamma_2} = \lambda^{-1} \qquad \mathsf{sp}_{\Gamma_1;-} = \rho^{-1} \qquad \mathsf{sp}_{\Gamma_1;x:\mathbb{A}} = \mathrm{id} \qquad \mathsf{sp}_{\Gamma_1;\Delta,x:\mathbb{A}} = \alpha \cdot (\mathsf{sp}_{\Gamma_1;\Delta} \otimes \mathrm{id})$$

For $n > 2$, $\mathsf{sp}_{\Gamma_1;\ldots;\Gamma_n}$ is is defined recursively based on the previous definition, using induction on n:

$$\mathsf{sp}_{\Gamma_1;\ldots;\Gamma_n} = (\mathsf{sp}_{\Gamma_1;\ldots;\Gamma_{n-1}} \otimes \mathrm{id}) \cdot \mathsf{sp}_{\Gamma_1,\ldots,\Gamma_{n-1};\Gamma_n}$$

On the other hand, $\mathsf{jn}_{\Gamma_1;\ldots;\Gamma_n}$ denotes the inverse of $\mathsf{sp}_{\Gamma_1;\ldots;\Gamma_n}$. Next, given $\Gamma, x : \mathbb{A}, y : \mathbb{B}, \Delta$, the morphism permuting x and y is denoted by $\mathsf{exch}_{\Gamma,x:\mathbb{A},y:\mathbb{B},\Delta} : [\![\Gamma, \underline{x : \mathbb{A}, y : \mathbb{B}}, \Delta]\!] \to [\![\Gamma, y : \mathbb{B}, x : \mathbb{A}, \Delta]\!]$ and defined as:

$$\mathsf{exch}_{\Gamma,\underline{x:\mathbb{A},y:\mathbb{B}},\Delta} = \mathsf{jn}_{\Gamma;y:\mathbb{B},x:\mathbb{A};\Delta} \cdot (\mathrm{id} \otimes \mathsf{sw} \otimes \mathrm{id}) \cdot \mathsf{sp}_{\Gamma;x:\mathbb{A},y:\mathbb{B};\Delta}$$

The shuffling morphism $\mathsf{sh}_E : [\![E]\!] \to [\![\Gamma_1, \ldots, \Gamma_n]\!]$ is defined as a suitable composition of exchange morphisms.

For every operation symbol $f : \mathbb{A}_1, \ldots, \mathbb{A}_n \to \mathbb{A}$ it is assumed the existence of an morphism $[\![f]\!] : [\![\mathbb{A}_1]\!] \otimes \ldots \otimes [\![\mathbb{A}_n]\!] \to [\![\mathbb{A}]\!]$. The interpretation of judgments is defined by induction over derivations according to the rules in Figure 4.

$$\frac{[\![\Gamma_i \triangleright v_i : \mathbb{A}_i]\!] = m_i \quad f : \mathbb{A}_1, \ldots, \mathbb{A}_n \to \mathbb{A} \in \Sigma \quad E \in \mathsf{Sf}(\Gamma_1; \ldots; \Gamma_n)}{[\![E \triangleright f(v_1, \ldots, v_n) : \mathbb{A}]\!] = [\![f]\!] \cdot (m_1 \otimes \ldots \otimes m_n) \cdot \mathsf{sp}_{\Gamma_1;\ldots;\Gamma_n} \cdot \mathsf{sh}_E} \qquad \frac{}{[\![x : \mathbb{A} \triangleright x : \mathbb{A}]\!] = \mathrm{id}_{[\![\mathbb{A}]\!]}}$$

$$\frac{}{[\![- \triangleright * : \mathbb{I}]\!] = \mathrm{id}_{[\![\mathbb{I}]\!]}} \qquad \frac{[\![\Gamma \triangleright v : \mathbb{A} \otimes \mathbb{B}]\!] = m \quad [\![\Delta, x : \mathbb{A}, y : \mathbb{B} \triangleright w : \mathbb{D}]\!] = n \quad E \in \mathsf{Sf}(\Gamma; \Delta)}{[\![E \triangleright \mathsf{pm}\ v\ \mathsf{to}\ x \otimes y.w : \mathbb{D}]\!] = n \cdot \mathsf{jn}_{\Delta;\mathbb{A};\mathbb{B}} \cdot \alpha \cdot \mathsf{sw} \cdot (m \otimes \mathrm{id}) \cdot \mathsf{sp}_{\Gamma;\Delta} \cdot \mathsf{sh}_E}$$

$$\frac{[\![\Gamma \triangleright v : \mathbb{A}]\!] = m \quad [\![\Delta \triangleright w : \mathbb{B}]\!] = n \quad E \in \mathsf{Sf}(\Gamma; \Delta)}{[\![E \triangleright v \otimes w : \mathbb{A} \otimes \mathbb{B}]\!] = (m \otimes n) \cdot \mathsf{sp}_{\Gamma;\Delta} \cdot \mathsf{sh}_E}$$

$$\frac{[\![\Gamma \triangleright v : \mathbb{I}]\!] = m \quad [\![\Delta \triangleright w : \mathbb{A}]\!] = n \quad E \in \mathsf{Sf}(\Gamma; \Delta)}{[\![E \triangleright v\ \mathsf{to}\ *.w : \mathbb{A}]\!] = n \cdot \lambda \cdot (m \otimes \mathrm{id}) \cdot \mathsf{sp}_{\Gamma;\Delta} \cdot \mathsf{sh}_E} \qquad \frac{[\![\Gamma, x : \mathbb{A} \triangleright v : \mathbb{B}]\!] = m}{[\![\Gamma \triangleright \lambda x : \mathbb{A}.\, v : \mathbb{A} \multimap \mathbb{B}]\!] = \overline{m \cdot \mathsf{jn}_{\Gamma;\mathbb{A}}}}$$

$$\frac{[\![\Gamma \triangleright v : \mathbb{A} \multimap \mathbb{B}]\!] = m \quad [\![\Delta \triangleright w : \mathbb{A}]\!] = n \quad E \in Sf(\Gamma; \Delta)}{[\![E \triangleright vw : \mathbb{B}]\!] = \mathsf{app} \cdot (m \otimes n) \cdot \mathsf{sp}_{\Gamma;\Delta} \cdot \mathsf{sh}_E} \qquad \frac{[\![\Gamma \triangleright v : \mathbb{A}]\!] = f}{[\![\Gamma \triangleright \mathsf{dis}(v) : \mathbb{I}]\!] = !\cdot f}$$

$$\frac{[\![\Gamma \triangleright v : \mathbb{A}]\!] = m}{[\![\Gamma \triangleright \mathsf{inl}_{\mathbb{B}}(v) : \mathbb{A} \oplus \mathbb{B}]\!] = \mathrm{inl} \cdot m} \qquad \frac{[\![\Gamma \triangleright v : \mathbb{B}]\!] = m}{[\![\Gamma \triangleright \mathsf{inr}_{\mathbb{A}}(v) : \mathbb{A} \oplus \mathbb{B}]\!] = \mathrm{inr} \cdot m}$$

$$\frac{[\![\Gamma \triangleright v : \mathbb{A} \oplus \mathbb{B}]\!] = b \quad [\![\Delta, x : \mathbb{A} \triangleright w : \mathbb{D}]\!] = p \quad [\![\Delta, y : \mathbb{B} \triangleright u : \mathbb{D}]\!] = q \quad E \in \mathrm{Sf}(\Gamma; \Delta)}{\begin{array}{l}[\![E \triangleright \mathsf{case}\ v\ \{\mathsf{inl}_{\mathbb{B}}(x) \Rightarrow w;\ \mathsf{inr}_{\mathbb{A}}(y) \Rightarrow u\} : \mathbb{D}]\!] = [p, q] \cdot (\mathsf{jn}_{\Delta;\mathbb{A}} \cdot \mathrm{sw} \oplus \mathsf{jn}_{\Delta;\mathbb{B}} \cdot \mathrm{sw}) \cdot \mathrm{dist} \cdot \\ \qquad (b \otimes \mathsf{id}) \cdot \mathsf{sp}_{\Gamma;\Delta} \cdot \mathsf{sh}_E\end{array}}$$

Figure 4: Judgment interpretation

The following diagrams are useful for a clearer understanding of the interpretation of judgements given in Figure 4.

$[\![\mathsf{ax}]\!]$:
$$[\![E]\!] \xrightarrow{\mathsf{sh}_E} [\![\Gamma_1, \ldots, \Gamma_n]\!] \xrightarrow{\mathsf{sp}_{\Gamma;\Delta}} [\![\Gamma_1]\!] \otimes \ldots \otimes [\![\Gamma_n]\!] \xrightarrow{m_1 \otimes \ldots \otimes m_n} [\![\mathbb{A}_1]\!] \otimes \ldots \otimes [\![\mathbb{A}_n]\!] \xrightarrow{[\![f]\!]} [\![\mathbb{A}]\!]$$

$[\![\mathsf{hyp}]\!]$:
$$[\![\mathbb{A}]\!] \xrightarrow{\mathrm{id}_{[\![\mathbb{A}]\!]}} [\![\mathbb{A}]\!]$$

$[\![\mathbb{I}_i]\!]$:
$$[\![\mathbb{I}]\!] \xrightarrow{\mathrm{id}_{[\![\mathbb{I}]\!]}} [\![\mathbb{I}]\!]$$

$[\![\otimes_e]\!]$:
$$[\![E]\!] \xrightarrow{\mathsf{sh}_E} [\![\Gamma, \Delta]\!] \xrightarrow{\mathsf{sp}_{\Gamma;\Delta}} [\![\Gamma]\!] \otimes [\![\Delta]\!] \xrightarrow{m \otimes \mathrm{id}} ([\![\mathbb{A}]\!] \otimes [\![\mathbb{B}]\!]) \otimes [\![\Delta]\!] \xrightarrow{\mathsf{sw}} [\![\Delta]\!] \otimes ([\![\mathbb{A}]\!] \otimes [\![\mathbb{B}]\!]) \xrightarrow{\alpha} ([\![\Delta]\!] \otimes [\![\mathbb{A}]\!]) \otimes [\![\mathbb{B}]\!] \xrightarrow{\mathsf{jn}_{\Delta;\mathbb{A};\mathbb{B}}} [\![\Delta, \mathbb{A}, \mathbb{B}]\!] \xrightarrow{n} [\![\mathbb{D}]\!]$$

$[\![\otimes_i]\!]$:
$$[\![E]\!] \xrightarrow{\mathsf{sh}_E} [\![\Gamma, \Delta]\!] \xrightarrow{\mathsf{sp}_{\Gamma;\Delta}} [\![\Gamma]\!] \otimes [\![\Delta]\!] \xrightarrow{m \otimes n} [\![\mathbb{A}]\!] \otimes [\![\mathbb{B}]\!]$$

$[\![\mathbb{I}_e]\!]$:
$$[\![E]\!] \xrightarrow{\mathsf{sh}_E} [\![\Gamma, \Delta]\!] \xrightarrow{\mathsf{sp}_{\Gamma;\Delta}} [\![\Gamma]\!] \otimes [\![\Delta]\!] \xrightarrow{m \otimes \mathrm{id}} [\![\mathbb{I}]\!] \otimes [\![\Delta]\!] \xrightarrow{\lambda} [\![\Delta]\!] \xrightarrow{n} [\![\mathbb{A}]\!]$$

$$
\begin{aligned}
[\![\multimap_i]\!] :&\quad [\![\Gamma]\!] \otimes [\![\mathbb{A}]\!] \xrightarrow{\overline{m \cdot \mathsf{jn}_{\Gamma;\mathbb{A}}}} [\![\mathbb{A}]\!] \multimap [\![\mathbb{B}]\!] \qquad ([\![\Gamma]\!] \otimes [\![\mathbb{A}]\!] \xrightarrow{\mathsf{jn}_{\Gamma;\mathbb{A}}} [\![\Gamma, \mathbb{A}]\!] \xrightarrow{m} [\![\mathbb{B}]\!])\\
[\![\multimap_e]\!] :&\quad [\![E]\!] \xrightarrow{\mathsf{sh}_E} [\![\Gamma, \Delta]\!] \xrightarrow{\mathsf{sp}_{\Gamma;\Delta}} [\![\Gamma]\!] \otimes [\![\Delta]\!] \xrightarrow{m \otimes n} [\![\mathbb{A}]\!] \multimap [\![\mathbb{B}]\!] \otimes [\![\mathbb{A}]\!] \xrightarrow{\mathsf{app}} [\![\mathbb{B}]\!]\\
[\![\mathsf{dis}]\!] :&\quad [\![\Gamma]\!] \xrightarrow{f} [\![\mathbb{A}]\!] \xrightarrow{!} [\![\mathbb{I}]\!]\\
[\![\mathsf{inl}]\!] :&\quad [\![\Gamma]\!] \xrightarrow{m} [\![\mathbb{A}]\!] \xrightarrow{\mathrm{inl}} [\![\mathbb{A} \oplus \mathbb{B}]\!]\\
[\![\mathsf{inr}]\!] :&\quad [\![\Gamma]\!] \xrightarrow{m} [\![\mathbb{B}]\!] \xrightarrow{\mathrm{inr}} [\![\mathbb{A} \oplus \mathbb{B}]\!]\\
[\![\mathsf{case}]\!] :&\quad [\![E]\!] \xrightarrow{\mathsf{sh}_E} [\![\Gamma, \Delta]\!] \xrightarrow{\mathsf{sp}_{\Gamma;\Delta}} [\![\Gamma]\!] \otimes [\![\Delta]\!] \xrightarrow{b \otimes \mathrm{id}} ([\![\mathbb{A}]\!] \oplus [\![\mathbb{B}]\!]) \otimes [\![\Delta]\!]\\
&\qquad \xrightarrow{\mathrm{dist}} ([\![\mathbb{A}]\!] \otimes [\![\Delta]\!]) \oplus ([\![\mathbb{B}]\!] \otimes [\![\Delta]\!])\\
&\qquad \xrightarrow{\mathsf{jn}_{\Delta;\mathbb{A}} \cdot \mathrm{sw} \oplus \mathsf{jn}_{\Delta;\mathbb{B}} \cdot \mathrm{sw}} ([\![\Delta, \mathbb{A}]\!]) \oplus ([\![\Delta, \mathbb{B}]\!]) \xrightarrow{[p,q]} [\![\mathbb{D}]\!]
\end{aligned}
$$

Regarding the interpretation of the exhange and substitution properties, we have the following lemma.

Lemma 2.4.1. *For any judgements* $\Gamma, x : \mathbb{A}, y : \mathbb{B}, \Delta \triangleright v : \mathbb{D}$*,* $\Gamma, x : \mathbb{A} \triangleright v : \mathbb{B}$*, and* $\Delta \triangleright w : \mathbb{A}$*, the following holds:*

$$
\begin{aligned}
[\![\Gamma, x : \mathbb{A}, y : \mathbb{B}, \Delta \triangleright v : \mathbb{D}]\!] &= [\![\Gamma, y : \mathbb{B}, x : \mathbb{A}, \Delta \triangleright v : \mathbb{D}]\!] \cdot \mathit{exch}_{\Gamma, \underline{x:\mathbb{A}, y:\mathbb{B}}, \Delta}\\
[\![\Gamma, \Delta \triangleright v[w/x] : \mathbb{B}]\!] &= [\![\Gamma, x : \mathbb{A} \triangleright v : \mathbb{B}]\!] \cdot \mathit{jn}_{\Gamma;\mathbb{A}} \cdot (\mathrm{id} \otimes [\![\Delta \triangleright w : \mathbb{A}]\!]) \cdot \mathit{sp}_{\Gamma;\Delta}
\end{aligned}
$$

Proof. This lemma is proved in [13, Lemma 2.2] for the lambda calculus without conditionals, so we only need to address the conditional cases.

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□

Theorem 2.4.2. *The equations presented in Figure 2 are sound with respect to judgement interpretation. More specifically, if* $\Gamma \triangleright v = w : \mathbb{A}$ *is one of the equations in Figure 2, then* $[\![\Gamma \triangleright v : A]\!] = [\![\Gamma \triangleright w : \mathbb{A}]\!]$.

Proof. Given the theorem is already proven in [13, Theorem 2.3] for the lambda calculus without conditionals, it suffices to consider the cases involving conditionals.

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□

Definition 2.4.3 (Models of linear λ-theories)**.** Consider a linear λ-theory $((G, \Sigma), Ax)$ and a symmetric monoidal closed category with coproducts C. Suppose that for each $X \in G$,

we have an interpretation $[\![X]\!]$, which is an object of C, and analogously for the operation symbols in Σ. This interpretation structure is a *model* of the theory if all axioms in Ax are satisfied by the interpretation.

Theorem 2.4.4 (Completeness)**.** *Consider a linear λ-theory $\mathscr{T}$. Then an equation $\Gamma \triangleright v = w : \mathbb{A}$ is a theorem of $\mathscr{T}$ if and only if it is satisfied by all models of the theory.*

Proof. This theorem is proved in [13, Lemma 2.6] for the lambda calculus without conditionals, so we only need to address the conditional cases.

Passar para aqui as provas qd elas estiverem vistas

□

Example 2.4.5. We now illustrate how the programs presented in Examples 2.2.1, 2.2.2, and 2.2.3, with slight modifications, are interpreted in Set. To this effect, we consider a type N representing set of natural numbers, along with a family of operations $\{n : \mathbb{I} \to N \mid n \in \mathbb{N}\}$, each mapping the monoidal unit to a corresponding natural number n. In Set have $[\![\mathbb{I}]\!] = \{*\}$, and define $[\![N]\!] = \mathbb{N}$, and $[\![n]\!] = \{*\} \to \mathbb{N}, * \mapsto n$. Consider the following λ-term:

$$x : N \otimes N \triangleright \mathsf{pm}\ x\ \mathsf{to}\ a \otimes b.b \otimes a\,[1(*) \otimes 2(*)/x] : N \otimes N$$

Attending to Figure 4 this program is interpreted as follows:

$$\begin{aligned}
&[\![\mathsf{pm}\ x\ \mathsf{to}\ a \otimes b.b \otimes a\,[1(*) \otimes 2(*)/x]]\!] \\
&= [\![\mathsf{pm}\ x\ \mathsf{to}\ a \otimes b.b \otimes a]\!] \cdot \mathrm{jn} \cdot (\mathrm{id} \otimes [\![1(*) \otimes 2(*)]\!]) \cdot \mathrm{sp} && \text{(Lemma 2.4.1)} \\
&= [\![b \otimes a]\!] \cdot \mathrm{jn} \cdot \alpha \cdot \mathrm{sw} \cdot ([\![x]\!] \otimes \mathrm{id}) \cdot \mathrm{sp} \cdot \mathrm{sh} \cdot \lambda \cdot (\mathrm{id} \otimes ([\![1(*)]\!] \otimes [\![2(*)]\!] && (\otimes_e, \otimes_i) \\
&\quad \cdot \mathrm{sp} \cdot \mathrm{sh})) \cdot \lambda^{-1} \\
&= [\![b]\!] \otimes [\![a]\!] \cdot \mathrm{sp} \cdot \mathrm{sh} \cdot (\lambda \otimes \mathrm{id}) \cdot \alpha \cdot \mathrm{sw} \cdot \rho^{-1} \cdot \lambda \cdot (\mathrm{id} \otimes (1 \cdot [\![*]\!] \cdot \mathrm{sp} \cdot \mathrm{sh} \otimes && (\otimes_i, \mathsf{hyp}, \\
&\quad 2 \cdot [\![*]\!] \cdot \mathrm{sp} \cdot \mathrm{sh} \cdot \lambda^{-1})) \cdot \lambda^{-1} && \mathsf{ax}) \\
&= \mathrm{sw} \cdot \lambda \cdot (\mathrm{id} \otimes (1 \otimes 2 \cdot \lambda^{-1})) \cdot \lambda^{-1} && (\mathsf{hyp}, \mathbb{I}_i, \mathsf{coh}) \\
&= 2 \otimes 1 \cdot \rho^{-1} && (\mathsf{nat}, \mathsf{coh})
\end{aligned}$$

Next, consider the λ-terms below.

$$\mathbf{Dis2nd} \triangleq - \triangleright \lambda x : N \otimes N.\mathsf{pm}\ x\ \mathsf{to}\ a \otimes b.\mathsf{dis}(b)\ \ \mathsf{to}\ \ * .a : N \otimes N \multimap N$$

$$\mathbf{Dis2nd}\ (1(*) \otimes 2(*))$$

In this case, we will "cheat" slightly by first using the equations in Figure 2 to simplify the term. We note that here we will not explicitly illustrate the interpretation of the case statement, because in Set, any term of type $\mathbb{A} \otimes \mathbb{A}$, for any type $\mathbb{A}$, corresponds to an injection ($\mathsf{inl}(v)$ or $\mathsf{inr}(v)$). As a result, we can "escape" any case statement by applying the equations β^{inl}_{case} and β^{inr}_{case}.

Applying the β equation we have

$$\begin{aligned}\textbf{Dis2nd}\ (1(*) \otimes 2(*)) &= \textbf{Dis2nd}\ [1(*) \otimes 2(*)/x] \\ &= \mathsf{pm}\ 1(*) \otimes 2(*)\ \mathsf{to}\ a \otimes b.\mathsf{dis}(b)\ \ \mathsf{to}\ \ *.a\end{aligned}$$

The resulting program is interpreted as follows:

$$\begin{aligned}&[\![\mathsf{pm}\ 1(*) \otimes 2(*)\ \mathsf{to}\ a \otimes b.\mathsf{dis}(b)\ \ \mathsf{to}\ \ *.a]\!] \\ &= [\![\mathsf{dis}(b)\ \ \mathsf{to}\ \ *.a]\!] \cdot \mathrm{jn} \cdot \alpha \cdot \mathrm{sw} \cdot ([\![1(*) \otimes 2(*)]\!] \otimes \mathrm{id}) \cdot \mathrm{sp} \cdot \mathrm{sh} && (\otimes_e) \\ &= [\![a]\!] \cdot \lambda \cdot ([\![\mathsf{dis}(b)]\!] \otimes \mathrm{id}) \cdot \mathrm{sp} \cdot \mathrm{sh} \cdot (\lambda \otimes \mathrm{id}) \cdot \alpha \cdot \mathrm{sw} \cdot (([\![1(*)]\!] \otimes [\![2(*)]\!] && (\mathbb{I}_e, \otimes_i) \\ &\quad \cdot \mathrm{sp} \cdot \mathrm{sh}) \otimes \mathrm{id}) \cdot \lambda^{-1} \\ &= \lambda \cdot (! \cdot [\![b]\!] \otimes \mathrm{id}) \cdot \mathrm{sw} \cdot (\lambda \otimes \mathrm{id}) \cdot \alpha \cdot \mathrm{sw} \cdot ((1 \cdot [\![*]\!] \cdot \mathrm{sp} \cdot \mathrm{sh}\ \otimes 2 \cdot [\![*]\!] \cdot \mathrm{sp} && (\mathsf{hyp}, \mathsf{dis}, \mathsf{ax}) \\ &\quad \cdot \mathrm{sh} \cdot \lambda^{-1}) \otimes \mathrm{id}) \cdot \lambda^{-1} \\ &= \lambda \cdot (! \otimes \mathrm{id}) \cdot \mathrm{sw} \cdot \rho \cdot ((1\ \otimes 2 \cdot \lambda^{-1}) \otimes \mathrm{id}) \cdot \lambda^{-1} && (\mathsf{hyp}, \mathbb{I}_i, \mathsf{coh}) \\ &= \lambda \cdot (! \otimes \mathrm{id}) \cdot 2\ \otimes 1 \cdot \rho^{-1} && (\mathsf{nat}, \mathsf{coh})\end{aligned}$$

The diagram below illustrates the resulting interpretation.

$$\{*\} \xrightarrow{\rho^{-1}} \{(*,*)\} \xrightarrow{2 \otimes 1} \{(2,1)\} \xrightarrow{(!\otimes \mathrm{id})} \{(*,1)\} \xrightarrow{\lambda} \{1\}$$

2.4.2 Semantics of metric equations

We will now turn our attention to the semantics of the metric equations. First, we recall the definition of a metric space.

Definition 2.4.6. A *metric space* is a pair (X, d) where X is a set and $d : X \times X \to [0, \infty]$ is a function known as *distance* satisfying:

1. $0 \leq d(x, y)$, with equality if and only if $x = y$,
2. $d(x, y) = d(y, x)$,

3. $d(x,z) \leq d(x,y) + d(y,z)$ for all $x, y, z \in X$.

Here, we equip each hom-set $\mathsf{C}(A,B)$ of a category C with a metric $d_{A,B}$, and impose that both composition and precomposition are non-expansive. That is, for all morphisms $f, f_1, f_2 \in \mathsf{C}(A,B)$ and any $g, g_1, g_2 \in \mathsf{C}(B,C)$, the following inequalities holds:

$$d_{A,C}(g \circ f_1, g \circ f_2) \leq d_{A,B}(f_1, f_2) \qquad d_{A,C}(g_1 \circ f, g_2 \circ f) \leq d_{B,C}(g_1, g_2).$$

Note that, given the triangle inequality, we have:

$$d_{A,C}(g_1 \circ f_1, g_2 \circ f_2) \leq d_{A,C}(g_1 \circ f_1, g_1 \circ f_2) + d_{A,C}(g_1 \circ f_2, g_2 \circ f_2) \leq d_{A,B}(f_1, f_2) + d_{B,C}(g_1, g_2).$$

This is known as *enriching* the category C *over metric spaces*.
In this context, *soundness* and *completeness* concepts are extended to emcompass not only the classical equations but also the metric equations. Note that the $=$ in classical equations can be written as $=_0$. As a result, in this setting we define

- *Soundness* is the property

  $$M =_\varepsilon N \;\Rightarrow\; d([\![N]\!], [\![M]\!]) \leq \varepsilon \quad \text{for all interpretations in the class.}$$

  That is, if two terms are provably at a maximum distance $_\varepsilon$, so are there respective interpretations

- *Completeness* is the property

  $$d([\![N]\!], [\![M]\!]) \leq \varepsilon \;\Rightarrow\; M =_\varepsilon N \quad \text{for all interpretations in the class.}$$

  That is, if $_\varepsilon$ is the maximum distance between the interpretation of two programs, then they are provably at a maximum distance $_\varepsilon$.

In this context, the notions of *soundness* and *completeness* are extended to encompass not only classical equations but also *metric equations*. Note that classical equations, written as $M = N$, can be viewed as a special case of metric equations where the distance is zero, *i.e.*, $M =_0 N$. We define:

- *Soundness* is the property

  $$M =_\varepsilon N \;\Rightarrow\; d\big([\![M]\!], [\![N]\!]\big) \leq \varepsilon \quad \text{for all interpretations in the class.}$$

  That is, if two terms are provably within a distance ε, then so are their interpretations.

- *Completeness* is the property

$$d\big([\![M]\!],[\![N]\!]\big) \leq \varepsilon \;\Rightarrow\; M =_\varepsilon N \quad \text{for all interpretations in the class.}$$

That is, if the interpretations of two terms are within a distance ε, then the terms are provably at most a distance ε apart.

Esta parte acho que vai ficar um pouco dúbia, melhor confirmar com prof. Renato.

O que fazer com a def de adjunção -> theorem 4.2 artigo

Definition 2.4.7. Met denotes the category whose objects are metric spaces and whose morphisms are non-expansive maps, *i.e.*, functions that do not increase the distance between points. More precisely, for two metric spaces (X, d_X) and (Y, d_Y), a morphism $f : (X, d_X) \to (Y, d_Y)$ is a function $f : X \to Y$ such that

$$d_Y(f(x), f(y)) \leq d_X(x, y) \quad \text{for all } x, y \in X.$$

Definition 2.4.8. A Met-*category* C is a category in which each hom-set $\mathsf{C}(A, B)$ is equipped with a metric $d_{A,B}$, such that for all morphisms $f_1, f_2 \in \mathsf{C}(A, B)$ and $g_1, g_2 \in \mathsf{C}(B, C)$, the following inequality holds:

$$d_{A,C}(g_1 \circ f_1, g_2 \circ f_2) \leq d_{A,B}(f_1, f_2) + d_{B,C}(g_1, g_2).$$

A *symmetric monoidal* Met-*category* C is a category that is both symmetric monoidal and a Met-category, such that for all morphisms $f_1, f_2 \in \mathsf{C}(A, B)$ and $g_1, g_2 \in \mathsf{C}(C, D)$, we have:

$$d_{A\otimes C,B\otimes D}(f_1 \otimes g_1, f_2 \otimes g_2) \leq d_{A,B}(f_1, f_2) + d_{C,D}(g_1, g_2).$$

A *symmetric monoidal closed* Met-*category* C is a category that is both symmetric monoidal closed and a symmetric monoidal Met-category, such that for all objects A, B, and C, there exists a Met-isomorphism

$$\mathsf{C}(B \otimes A, C) \cong \mathsf{C}(C, A \multimap B)$$

natural in B and C.

Theorem 2.4.9. *[13, Theorem 4.2] Let* C *be a category that is both monoidal closed and a symmetric monoidal* Met- *category, and suppose that for all objects* $X \in \mathsf{C}$*, the functor* $X \multimap - : \mathsf{C} \to \mathsf{C}$ *is non-expansive. Then* C *is a* symmetric monoidal closed Met-category.

Theorem 2.4.10 (Soundness)**.** *[13, Theorem 3.14] The rules in Figures 2 and 3 are sound for a symmetric monoidal closed* Met*-category* C *over metric spaces. Specifically, if* $\Gamma \rhd v =_q w : \mathbb{A}$ *results from the rules in Figures 2 and 3 then* $q \geq d([\![\Gamma \rhd v : \mathbb{A}]\!], [\![\Gamma \rhd w : \mathbb{A}]\!])$.

Definition 2.4.11. Consider a metric λ-theory $((G, \Sigma), Ax)$ and a symmetric monoidal closed Met-category C. Suppose that for each $X \in G$ we have an interpretation $[\![X]\!]$ as a C-object and analogously for the operation symbols. This interpretation structure is a model of the theory if all axioms in Ax are satisfied by the interpretation.

For two types $\mathbb{A}$ and $\mathbb{B}$ of a metric λ-theory $\mathscr{T}$, consider the class $\texttt{Values}(\mathbb{A}, \mathbb{B})$ of values v such that $x : \mathbb{A} \rhd v : \mathbb{B}$. We equip $\texttt{Values}(\mathbb{A}, \mathbb{B})$ with the function $d : \texttt{Values}(\mathbb{A}, \mathbb{B}) \times \texttt{Values}(\mathbb{A}, \mathbb{B}) \to \mathbb{Q}_0^+$ defined by,

$$d(v, w) = \inf \{q \mid v =_q w \text{ is a theorem of } \mathscr{T}\}.$$

Given that the equations $\Gamma \rhd v = w : \mathbb{A}$ are abbreviations of $\Gamma \rhd v =_0 w : \mathbb{A}$, $\texttt{Values}((\mathbb{A}, \mathbb{B}), d)$ is a pseudometric category, *i.e.*, it allows distinct terms to have distance zero. Consequently, we quotient this category by the relation $\sim$ (identifying elements at distance zero) to obtain a separated category, denoted by $(\texttt{Values}(\mathbb{A}, \mathbb{B}), d)/\!\sim$, which is a Met-category.
We say that a theory $\mathscr{T}$ is *varietal* if $(\texttt{Values}(\mathbb{A}, \mathbb{B}), d)/\!\sim$ is a small Met-category for all types $\mathbb{A}$ and $\mathbb{B}$. For the remainder of this work, we restrict our attention to varietal theories and locally small categories.

Theorem 2.4.12 (Completeness)**.** *[13, Theorem 3.16] Consider a varietal metric* λ*-theory. A metric equation in context* $\Gamma \rhd v =_q w : \mathbb{A}$ *is a theorem if and only if it holds in all models of the theory.*

Chapter 3

Metric λ-calculus with conditionals

As noted in Section 2.2.9, a metric equation for the conditional statement would be extremely helpful for reasoning about approximate equivalence within the λ-calculus with conditionals. In this chapter, we address this gap by introducing such an equation and proving both its soundness and completeness. We present the category of metric spaces as an example of a setting suitable for reasoning about approximate equivalence in this context. Furthermore, we prove if C is a Met-category, then its coproduct cocompletion admits a metric making it a Met-enriched category with coproducts. Finally, we illustrate the usefulness of the introduced equation through two small examples: reasoning about approximate equivalence between boolean terms (i.e., terms of type $\mathbb{I} \oplus \mathbb{I}$) and the extensionality of copairing. We conclude the chapter with a brief discussion on the suitability of the chosen metric equation.

3.1 Syntax

The metric equation for conditionals is presented in Figure 5.

$$\frac{v =_q v' \qquad w =_r w' \qquad u =_s u'}{\mathsf{case}\, v\, \{\mathsf{inl}(x) \Rightarrow w;\ \mathsf{inr}(y) \Rightarrow u\} =_{q+\sup\{r,s\}} \mathsf{case}\, v'\, \{\mathsf{inl}(x) \Rightarrow w';\ \mathsf{inr}(y) \Rightarrow u'\}}$$

Figure 5: Metric equation for condicionals

Intuitively, this equation states that the maximum distance between the two conditional statements is determined the:

- The larger of the maximum distances between the corresponding branches, *i.e.*, the program pairs (w, w') and (u, u'); and

- The maximum distance between the terms v and v' on which their execution depends.

Recall Definition 2.2.20. Now, we extend $Th(Ax)$ to denote the smallest class that contains Ax and is closed under the rules presented in Figure 2, Figure 3 and Figure 5.

3.2 Interpretation

In this subsection, we extend the concepts introduced in Section 2.4.2 to include the interpretation of the new metric equation.

Definition 3.2.1. A *symmetric monoidal closed* Met-*category with (binary) coproducts* C is a category that is both symmetric monoidal closed ith (binary) coproducts and a symmetric monoidal closed Met-category, such that, for all morphisms $f_1, f_2 \in \mathsf{C}(A, C)$ and $g_1, g_2 \in \mathsf{C}(B, C)$, we have:

$$d_{A\oplus B,C}([f_1, g_1], [f_2, g_2]) \leq \sup\{d_{A,C}(f_1, f_2), d_{B,C}(g_1, g_2)\}.$$

Definition 3.2.2. Consider a metric λ-theory $((G, \Sigma), Ax)$ and a symmetric monoidal closed Met-category with coproducts C. Suppose that for each $X \in G$ we have an interpretation $[\![X]\!]$ as a C-object and analogously for the operation symbols. This interpretation structure is a model of the theory if all axioms in Ax are satisfied by the interpretation.

3.3 Lattice Theory Preliminaries

In this section, we introduce a few concepts from lattice theory that will be useful for the completeness proof and for the broader discussion of our results.

Definition 3.3.1. A *lattice* is a partial order in which every finite subset has both a meet and a join. A *complete lattice* is a partial order in which every subset, finite or infinite, has a meet and a join.

Definition 3.3.2. *Quantales* are complete lattices equipped with an associative composition, typically denoted $\otimes$, that preserves suprema in both arguments.

Example 3.3.3. In this work, we consider only the standard metric quantale, whose underlying complete lattice is the interval $[0, \infty]$ ordered by $\geq$, with associative composition given by addition. Note that, since the order is reversed in this quantale, suprema in general lattices correspond to infima here, and vice versa.

Lemma 3.3.4 (Way-below relation for the metric quantale)**.** *[13] Let L denote the complete lattice $[0,\infty]$ ordered by $\geq$. For all $x, y \in L$ and for every subset $X \subseteq L$, if $y > x$, then whenever $x \geq \inf X$, there exists a* finite *subset $A \subseteq X$ such that $y \geq \inf A$.*
This property characterizes the way-below relation *in the metric quantale. The lattice L is* continuous*,* i.e.*for every $x \in L$,*

$$x = \inf\{y \mid y \in L \text{ and } y > x\}.$$

Definition 3.3.5. A subset D of a lattice L is called *directed* if it is nonempty and every finite subset of D has an upper bound in D. A partially order is said to be *directed complete* if every directed subset has a supremum. A directed complete poset is commonly referred to as a *dcpo*.

Definition 3.3.6. A semilattice L is called *meet continuous* if it is directed complete, i.e., a dcpo, and satisfies the condition

$$x \wedge \sup D = \sup\{x \wedge d \mid d \in D\} \tag{MC}$$

for all $x \in L$ and all directed subsets $D \subseteq L$.

Lemma 3.3.7. *[35, Proposition I-1.8] Every continuous lattice is meet continuous.*

Lemma 3.3.8. *[36, Lemma 2.23] Let P be a lattice, let $S, T \subseteq P$ and assume that $\bigvee S$, $\bigvee T$, $\bigwedge S$ and $\bigwedge T$ exist in P. Then*

$$\bigvee(S \cup T) = \left(\bigvee S\right) \vee \left(\bigvee T\right) \quad \text{and} \quad \bigwedge(S \cup T) = \left(\bigwedge S\right) \wedge \left(\bigwedge T\right).$$

3.4 Soundeness and Completeness

The proofs in this section are based on the proofs of Theorem 2.4.10 and Theorem 2.4.12 in [13].

Theorem 3.4.1. *The rules in Figures 2, 3 and 5 are sound for a symmetric monoidal closed* Met*-category with coproducts* C*. Specifically, if $\Gamma \rhd v =_q w : \mathbb{A}$ results from the rules in Figures 2, 3 and 5 then $q \geq d(\llbracket \Gamma \rhd v : \mathbb{A} \rrbracket, \llbracket \Gamma \rhd w : \mathbb{A} \rrbracket)$.*

Proof. We follow the same strategy as [13]. This proof uses induction over the depth of proof trees that arise from the metric deductive system. The general strategy for each inference rule is to use the definition of a symmetric monoidal closed Met-category with coproducts.

More concretely, first, consider the equations on Figure 2 which abreviate equations $\Gamma \triangleright w =_0 v : \mathbb{A}$. By Theorem 2.4.2, these equations are sound for symmetric monoidal closed categories with binary coproducts, *i.e.*if $v = w$, then $[\![v]\!] = [\![w]\!]$ in C. Then by the definition of a Met-Category (Definition 2.4.8) we obtain $d([\![v]\!], [\![w]\!]) = d([\![w]\!], [\![v]\!]) = 0$.
The rules in Figure 3 follow from the definition of a symmetric monoidal closed Met-category. Finally, regarding the rule in Figure 5, it follows from the fact that C is symmetric monoidal closed Met-category with binary coproducts. We reason as follows:

$$
\begin{aligned}
&d([\![\,\mathsf{case}\; v\,\{\mathsf{inl}_{\mathbb{B}}(x) \Rightarrow w;\; \mathsf{inr}_{\mathbb{A}}(y) \Rightarrow u\}]\!], [\![\,\mathsf{case}\; v'\,\{\mathsf{inl}_{\mathbb{B}}(x) \Rightarrow w';\; \mathsf{inr}_{\mathbb{A}}(y) \Rightarrow u'\}]\!]) \\
&= d([[\![w]\!], [\![u]\!]] \cdot (\mathsf{jn}_{\Delta;\mathbb{A}} \cdot \mathrm{sw} \oplus \mathsf{jn}_{\Delta;\mathbb{B}} \cdot \mathrm{sw}) \cdot \mathrm{dist} \cdot ([\![v]\!] \otimes \mathsf{id}) \cdot \mathsf{sp}_{\Gamma;\Delta} \cdot \mathsf{sh}_E, \\
&\qquad [[\![w']\!], [\![u']\!]] \cdot (\mathsf{jn}_{\Delta;\mathbb{A}} \cdot \mathrm{sw} \oplus \mathsf{jn}_{\Delta;\mathbb{B}} \cdot \mathrm{sw}) \cdot \mathrm{dist} \cdot ([\![v']\!] \otimes \mathsf{id}) \cdot \mathsf{sp}_{\Gamma;\Delta} \cdot \mathsf{sh}_E) \\
&\leq d([[\![w]\!], [\![u]\!]] \cdot (\mathsf{jn}_{\Delta;\mathbb{A}} \cdot \mathrm{sw} \oplus \mathsf{jn}_{\Delta;\mathbb{B}} \cdot \mathrm{sw}) \cdot \mathrm{dist} \cdot ([\![v]\!] \otimes \mathsf{id}), [[\![w']\!], [\![u']\!]] \cdot (\mathsf{jn}_{\Delta;\mathbb{A}} \cdot \mathrm{sw} \oplus \\
&\qquad \mathsf{jn}_{\Delta;\mathbb{B}} \cdot \mathrm{sw}) \cdot \mathrm{dist} \cdot ([\![v']\!] \otimes \mathsf{id})) \\
&\leq d([\![v]\!] \otimes \mathsf{id}, [\![v']\!] \otimes \mathsf{id}) + d([[\![w]\!], [\![u]\!]] \cdot (\mathsf{jn}_{\Delta;\mathbb{A}} \cdot \mathrm{sw} \oplus \mathsf{jn}_{\Delta;\mathbb{B}} \cdot \mathrm{sw}) \cdot \mathrm{dist}, [[\![w']\!], [\![u']\!]] \cdot \\
&\qquad (\mathsf{jn}_{\Delta;\mathbb{A}} \cdot \mathrm{sw} \oplus \mathsf{jn}_{\Delta;\mathbb{B}} \cdot \mathrm{sw}) \cdot \mathrm{dist}) \\
&\leq q + d([[\![w]\!], [\![u]\!]] \cdot (\mathsf{jn}_{\Delta;\mathbb{A}} \cdot \mathrm{sw} \oplus \mathsf{jn}_{\Delta;\mathbb{B}} \cdot \mathrm{sw}) \cdot \mathrm{dist}, [[\![w']\!], [\![u']\!]] \cdot (\mathsf{jn}_{\Delta;\mathbb{A}} \cdot \mathrm{sw} \oplus \mathsf{jn}_{\Delta;\mathbb{B}} \cdot \mathrm{sw}) \\
&\qquad \cdot \mathrm{dist}) \\
&\leq q + d([[\![w]\!], [\![u]\!]], [[\![w']\!], [\![u']\!]]) \\
&\leq q + \sup(d([\![w]\!], [\![w']\!]), d([\![u]\!], [\![u']\!])) \\
&\leq q + \sup(r, s)
\end{aligned}
$$

The second step follows from the fact that $\mathsf{sp}_{\Gamma;\Delta} \cdot \mathsf{sh}_E$ is a morphism in C and that C is a Met-category. The third and fifth step follow from an analogous reasoning. The fourth step follows from the premises of the rule in question and the fact that C is a symmetric monoidal Met-category. The sixth step follows from the fact that C is a symmetric monoidal Met-category with binary coproducts. Finally, the last step follows from the premise of the rule in question. □

Next, we extend $\mathtt{Values}((\mathbb{A}, \mathbb{B}), d)$, by incorporating the new theorems and quotient this category into a Met-category $(\mathtt{Values}(\mathbb{A}, \mathbb{B}), d)/\!\sim$.

Theorem 3.4.2 (Completeness)**.** *Consider a varietal metric λ-theory. A metric equation in context $\Gamma \triangleright v =_q w : \mathbb{A}$ is a theorem if and only if it holds in all models of the theory.*

Proof. Here, we focus only on the coproduct, as the remaining cases are proven in [13]. Completeness arises from constructing the syntactic category $\mathrm{Syn}(\mathscr{T})$ of the underlying theory $\mathscr{T}$ and then show that provability of $\Gamma \rhd v =_q w : \mathbb{A}$ in $\mathscr{T}$ is equivalent to $a([\![v]\!], [\![w]\!]) \geq q$ in the category $\mathrm{Syn}(\mathscr{T})$. We use the category $(\mathtt{Values}(\mathbb{A}, \mathbb{B}), d)/\!\sim$ to this end. Note that the quotienting process identifies all terms $x : \mathbb{A} \rhd v : \mathbb{B}$ and $x : \mathbb{A} \rhd w : \mathbb{B}$ such that $v =_0 w$ and $w =_0 v$. This relation includes the equations-in-context from Figure 2. First, we remark that all sets of the form $\{q \mid v =_q w\}$ are directed: they are non-empty, since by equation (join) we always have at least $v =_\infty w$, and again by (join), every finite subset of $\{q \mid v =_q w\}$ has a lower bound in the set. This will be useful for applying Lemma 3.3.7.

Next, we need to prove that the copairing is well defined in $(\mathtt{Values}(\mathbb{A}, \mathbb{B}), d)/\!\sim$, *i.e.*, for any a, a', b, b' if $a \sim a'$ and $b \sim b'$, then $[a, b] \sim [a', b']$.

This is equivalent to demonstrating the following implication:

$$\begin{cases} \inf\{q \mid v =_q w\} \leq 0 \\ \inf\{r \mid v =_r w\} \leq 0 \end{cases} \implies \inf \left\{ q \;\middle|\; \begin{array}{l} \mathsf{case}\ z\ \{\mathsf{inl}(x) \Rightarrow v;\ \mathsf{inr}(y) \Rightarrow w\} =_q \\ \mathsf{case}\ z\ \{\mathsf{inl}(x) \Rightarrow v';\ \mathsf{inr}(y) \Rightarrow w'\} \end{array} \right\} \leq 0.$$

Let L be a lattice, and let $D, F \subseteq L$ be directed sets. Observe the following:

$$\begin{aligned} &\sup\{\inf D, \inf F\} \\ &= \inf\{\sup\{\inf D, f\} \mid f \in F\} && \text{(Lemma 3.3.7)} \\ &= \inf\{\inf\{\sup\{\inf d, f\} \mid d \in D\} \mid f \in F\} && \text{(Lemma 3.3.7)} \\ &= \inf\{\sup\{d, f\} \mid d \in D, f \in F\} && \text{(Lemma 3.3.8)} \end{aligned} \tag{3.1}$$

With the equality above, we have

$$\begin{aligned} &\begin{cases} \inf\{q \mid v =_q w\} \leq 0 \\ \inf\{r \mid v =_r w\} \leq 0 \end{cases} \\ &\implies \sup\{\inf\{q \mid v =_q w\}, \inf\{r \mid v =_r w\}\} \leq 0 \\ &\implies \inf\{\sup\{q, r\} \mid v =_q w, v =_r w\} \leq 0 && \text{(Equation 3.1)} \\ &\implies \left\{ q \;\middle|\; \begin{array}{l} \mathsf{case}\ z\ \{\mathsf{inl}(x) \Rightarrow v;\ \mathsf{inr}(y) \Rightarrow w\} =_q \\ \mathsf{case}\ z\ \{\mathsf{inl}(x) \Rightarrow v';\ \mathsf{inr}(y) \Rightarrow w'\} \end{array} \right\} \leq 0 \end{aligned}$$

Thus, we obtain a category $\mathrm{Syn}(\mathscr{T})$ whose objects are the types of the language and whose hom-sets are the underlying sets of the Met-category $(\mathbf{Values}(\mathbb{A}, \mathbb{B}), d)/\!\sim$.

The next step is to show that $\mathrm{Syn}(\mathscr{T})$ is a symmetric monoidal Met-category with binary coproducts. To this effect, with repect to the case statement, we reason as follows:

$$
\begin{aligned}
&\sup\{d([v],[w]), d([v'],[w'])\} \\
&= \sup\{d(v,w), d(v',w')\} \\
&= \sup\{\inf\{q \mid v =_q v'\}, \inf\{r \mid w =_r w'\}\} \\
&= \inf\{\sup\{q,r\} \mid v =_q v', w =_r w'\} \\
&\geq \inf\{q \mid \mathsf{case}\ z\ \{\mathsf{inl}(x) \Rightarrow v;\ \mathsf{inr}(y) \Rightarrow w\} =_q \ \mathsf{case}\ z\ \{\mathsf{inl}(x) \Rightarrow v';\ \mathsf{inr}(y) \Rightarrow w'\}\} \\
&= d(\mathsf{case}\ z\ \{\mathsf{inl}(x) \Rightarrow v;\ \mathsf{inr}(y) \Rightarrow w\}, \mathsf{case}\ z\ \{\mathsf{inl}(x) \Rightarrow v';\ \mathsf{inr}(y) \Rightarrow w'\}) \\
&= d([\mathsf{case}\ z\ \{\mathsf{inl}(x) \Rightarrow v;\ \mathsf{inr}(y) \Rightarrow w\}], [\mathsf{case}\ z\ \{\mathsf{inl}(x) \Rightarrow v';\ \mathsf{inr}(y) \Rightarrow w'\}]) \\
&= d([[v],[v']],[[w],[w']])
\end{aligned}
$$

The third step follows from Equation 3.1, and the fourth step follows from the fact that for any sets A and B, if $A \subseteq B$, then $\inf\{A\} \geq \inf\{B\}$.

The final step is to show that if an equation $\Gamma \triangleright v =_q v' : \mathbb{A}$ with $q \in [0,\infty]$ is satisfied by $\mathsf{Syn}(\mathscr{T})$ then it is a theorem of the linear metric λ-theory. By assumption, $d([v],[v']) = d(v,v') = \inf\{r \mid v =_r v'\} \leq q$. It follows from the definition of the way-below relation that for all $x \in [0,\infty]$ with $x > q$ there exists a finite set $A \subseteq \{r \mid v =_r v'\}$ such that $x \geq \inf A$. Then by an application of rule (join) we obtain $v =_{\inf A} v'$, and consequently, rule (weak) provides $v =_x v'$ for all $x > q$. Finally, by an application of rule (arch) we deduce that $v =_q v'$ is part of the theory. □

3.5 A small example: Met

Proposition 3.5.1. *The category* Met *is a symmetric monoidal closed with binary coproducts* Met*-category*

Proof. By [13, Example 3.8], the category Met is a symmetric monoidal closed Met-category. As a result, it suffices to show that for all morphisms $f_1, f_2 \in \mathsf{Met}(A,C)$ and $g_1, g_2 \in \mathsf{Met}(B,C)$, the following inequality holds:

$$d_{A\oplus B,C}([f_1,g_1],[f_2,g_2]) \leq \sup\{d_{A,C}(f_1,f_2), d_{B,C}(g_1,g_2)\}.$$

In this category, the coproduct is defined as in Set. The distance function d on the coproduct

$A \oplus B$ is given by:

$$\begin{cases} d_{A\oplus B}(\mathrm{inl}(a_1), \mathrm{inl}(x_2)) = d_A(a_1, a_2) \\ d_{A\oplus B}(\mathrm{inr}(b_1), \mathrm{inr}(b_2)) = d_B(b_1, b_2) \\ d_{A\oplus B}(\mathrm{inl}(a), \mathrm{inr}(b)) = d_{A\oplus B}(\mathrm{inr}(b), \mathrm{inl}(a)) = \infty \end{cases}$$

The co-pairing is defined as in Set. The inequality we aim to prove follows directly from the fact that, given two morphisms $f, g \in \mathsf{Met}(A, B)$ the distance between them is defined as

$$d_B(\mathrm{inl}(x_1), \mathrm{inl}(x_2)) = \sup\{d_A(fa, ga) \mid a \in A\},$$

together with Lemma 3.3.8. For $f_1, f_2 \in \mathsf{Met}(A, C)$ and $g_1, g_2 \in \mathsf{Met}(B, C)$, we calculate:

$$\begin{aligned} & d_{A\oplus B,C}([f_1, g_1], [f_2, g_2]) \\ & = \sup\{d_{A\oplus B}([f_1, g_1](a, b), [f_2, g_2](a, b)) \mid (a, b) \in A \oplus B\} \\ & = \sup\{\{d_{A\oplus B}([f_1, g_1](\mathrm{inl}(a)), [f_2, g_2](\mathrm{inl}(a))) \mid a \in A\} \\ & \quad \cup \{d_{A\oplus B}([f_1, g_1](\mathrm{inr}(b)), [f_2, g_2](\mathrm{inr}(b))) \mid b \in B\}\} \\ & = \sup\{\{d_A(f_1(a), f_2(a)) \mid a \in A\} \cup \{d_B(g_1(b), g_2(b)) \mid b \in B\}\} \\ & = \sup\{\sup\{d_A(f_1(a), f_2(a)) \mid a \in A\}, \sup\{d_B(g_1(b), g_2(b)) \mid b \in B\}\} \\ & = \sup\{d_{A,C}(f_1, f_2), d_{B,C}(g_1, g_2)\}. \end{aligned}$$

□

3.6 Coproduct cocompletion

Porque é que queriamos fazer corpoduct completion de uma categoria C de q não tem productos? Os productos nesta cat não respeitariam as condições de C

We note that more powerful machinery, based on advanced categorical structures such as presheaves, can be employed to address the fact that the coproduct cocompletion of a category C forms a Met-category. However, since categories are used here as a tool rather than the main focus of this thesis, the elegance of the proof is not a primary concern.

Up until how we have only discussed binary products/coproducts. However, we can also define *ternary products* $A_1 \times A_2 \times A_3$ with an analogous universal property. That is, there exist three projection morphisms $\pi_i : A_1 \times A_2 \times A_3 \to A_i$ for $i = 1, 2, 3$, and for any object

B and morphisms $f_i : B \to A_i$, there exists a unique morphism $\langle f_1, f_2, f_3\rangle :\to A_1 \times A_2 \times A_3$ such that $\pi_i \cdot \langle f_1, f_2, f_3\rangle = f_i$ for each $i = 1, 2, 3$. Such a condition can be formulated for any number of factors and if a category has binary products, then it has all finite products with two or more factors. Any object A is the unary product of A with itself one time. Observe also that a terminal object is a "nullary" product, that is, a product of no objects: given no objects, there exists an object $\mathbf{1}$ with no maps, and for any other object A with no maps, there exists a unique arrow $! : A \to \mathbf{1}$ making no additional diagrams commute. One can also define the product of a family of objects $(C_i)_{i\in I}$ indexed by a set I, as an object $\prod_{i\in I} C_i$ together with a family of projection morphisms

$$\pi_i : \prod_{j\in I} C_j \to C_i \quad \text{for each } i \in I,$$

such that for every object A and every family of morphisms $(f_i : A \to C_i)_{i\in I}$, there exists a unique morphism $u : A \to \prod_{i\in I} C_i$ such that $\pi_i \cdot u = f_i \quad$ for all $i \in I$.
Reversing all arrows in the definitions above yields the notion of n-*ary coproducts* and the *coproduct* of a family of objects $(C_i)_{i\in I}$.

Definition 3.6.1. A C is said to have *all small products* if every small set of objects in C has a product.

The idea behind the coproduct completion of a category C is to create a new category where all small coproducts exist by formally adding them to C in the simplest way possible.

Definition 3.6.2. A *(free) coproduct completion* of a category C, denoted C^+, is the category whose objects are families $(A_i)_{i\in I}$ of objects of C, where I is a set; an arrow $(A_i)_{i\in I} \to (B_j)_{j\in J}$ consists of a pair $(f, (\phi_i)_{i\in I})$, where $f : I \to J$ is a function between the index sets, and $(\phi_i)_{i\in I}$ is a family of morphisms $\phi_i : X_i \to Y_{f(i)}$ in C. Given morphisms $(f, (\phi)_i) : (A_i)_{i\in I} \to (B_j)_{j\in J}$ and $(g, (\psi)_i) : (Y_j)_{j\in J} \to (Z_k)_{k\in K}$, their composition is defined as the morphism, given by the pair $(g \cdot f, (\theta_i)_{i\in I})$, where $\theta_i := \psi_{f(i)} \circ \varphi_i : X_i \to Z_{g(f(i))}$. From now on, unless ambiguity arises, we will omit the indexing function and use letters Φ, Ψ, ξ to refer to families of morphisms.

É para colocar a parte monoidal?

If C is a Met-category, one may define a metric on the morphims of its coproduct cocompletion C^+ as follows:

$$d(\Phi, \Psi) = \sup\left\{d'(\phi_i, \psi_i) \mid i \in I\right\}, \text{ where } d'(\phi_i, \psi_i) = \begin{cases} \infty, & \text{if } f(i) \neq g(i), \\ d(\phi_i, \psi_i), & \text{otherwise.} \end{cases}$$

Proposition 3.6.3. *The coproduct completion of a* Met*-category* C *is a* Met*-category.*

Proof. This follows from the fact that C is a Met-category. This follows from the fact that C is a Met-category. For all objects A, B in C^+, and for any morphisms $\Phi, \Psi, \Psi' \in \mathsf{C}^+(A, B)$, we have:

$$d(\Phi \cdot \Psi, \Phi \cdot \Psi') = \sup\{d'(\phi_{f(i)} \cdot \psi_i, \phi_{g(i)} \cdot \psi') \mid i \in I\}$$

First, suppose $f(i) = g(i)$ for all $i \in I$,

$$\begin{aligned} &\sup\{d'(\phi_{f(i)} \cdot \psi_i, \phi_{g(i)} \cdot \psi') \mid i \in I\} \\ &= \sup\{d(\phi_{f(i)} \cdot \psi_i, \phi_{f(i)} \cdot \psi_i') \mid i \in I\} \\ &\leq \sup\{d(\psi_i, \psi_i') \mid i \in I\} && (\mathsf{C} \text{ is a Met-category}) \\ &= d(\Psi, \Psi') \end{aligned}$$

Next, assume $f(i) \neq g(i)$ for any $i \in I$, is is straightforward that

$$\sup\{d'(\phi_{f(i)} \cdot \psi_i, \phi_{g(i)} \cdot \psi')\} = \infty = \sup\{d'(\psi_i, \psi_i')\} = d(\Psi, \Psi')$$

The proof for precomposition follows similar reasoning. □

Proposition 3.6.4. *The coproduct completion of a* Met*-category* C *is a* Met*-category with binary coproducts.*

Proof. This follows from Proposition 3.6.3 and the definition of the metric in C^+. For any objects A, B, C in C^+ and morphisms $\Phi, \Phi' \in \mathsf{C}^+(A, C)$ and $\Psi, \Psi' \in \mathsf{C}^+(B, C)$, we may regard the copairings

$$[(f, (\phi_i)_{i \in I'}), (g, (\psi_i)_{i \in I''})] \quad \text{and} \quad [(f', (\phi_i')_{i \in I'}), (g', (\psi_i')_{i \in I''})]$$

as morphisms $(h, (\xi_i)_{i \in I})$ and $(h', (\xi_i')_{i \in I})$ from $A \oplus B$ to C, where $I = I' \cup I''$. For instance, the indexing function h underlying $(\xi_i)_{i \in I}$ is defined as

$$h(i) = \begin{cases} f(i) & \text{if } i \in I', \\ g(i) & \text{if } i \in I''. \end{cases}$$

Then, we have:

$$
\begin{aligned}
& d([\Phi, \Psi], [\Phi', \Psi']) \\
&= d(\xi, \xi') = \sup\{d'(\xi_i, \xi_i') \mid i \in I' \cup I''\} \\
&= \sup\{\{d'(\phi_i, \phi_i') \mid i \in I'\} \cup \{d'(\psi_i, \psi_i') \mid i \in I''\}\} \\
&= \sup\{\sup\{d'(\phi_i, \phi_i') \mid i \in I'\}, \sup\{d'(\psi_i, \psi_i') \mid i \in I''\}\} \qquad \text{(Lemma 3.3.8)} \\
&= \sup\{d(\Phi, \Phi'), d(\Psi, \Psi')\}
\end{aligned}
$$

□

3.7 Interlude:Booleans - Part 2

Having proven that the metric equation for conditionals is both sound and complete, we can now use it to explore the booleans introduced in the previous chapter in subsection 2.2.9. For instance, given axioms

$$v' =_{\epsilon_1} a \quad \text{and} \quad w' =_{\epsilon_2} b$$

we can postulate

$$\textbf{Conjunction}[v'/v, w'/w] =_{\epsilon_1+\epsilon_2} \textbf{Conjunction}[a/v, b/w].$$

First note that using the equation η_{dis}, we have that $\Gamma \rhd \mathsf{dis}(w) =_0 \mathsf{dis}(b)$. Then,using our metric deductive system we calculate

$$
\begin{aligned}
& \mathsf{case}\ v' \left\{ \mathrm{inl}_{\mathbb{I}}(x) \Rightarrow x\ \mathsf{to}\ * .\ w; \mathrm{inl}_{\mathbb{I}}(y) \Rightarrow y\ \mathsf{to}\ * .\ \mathsf{dis}(w)\ \mathsf{to}\ * .\ \mathrm{inr}_{\mathbb{I}}(*) \right\} \\
=_{\epsilon_1+\epsilon_2}\ & \mathsf{case}\ a \left\{ \mathrm{inl}_{\mathbb{I}}(x) \Rightarrow x\ \mathsf{to}\ * .\ b; \mathrm{inl}_{\mathbb{I}}(y) \Rightarrow y\ \mathsf{to}\ * .\ \mathsf{dis}(b)\ \mathsf{to}\ * .\ \mathrm{inr}_{\mathbb{I}}(*) \right\}.
\end{aligned}
$$

Applying similar reasoning, is straightforward to see that we have a similar equation for the **Disjunction** program and that given an axiom $v' =_\epsilon a$, we can postulate $\textbf{Negation}[v'/v] =_\epsilon \textbf{Negation}[a/v]$.

Now, consider the axiom

$$v' =_\epsilon \mathrm{inl}_{\mathbb{I}}(*)$$

Considering Lemma 2.2.14, we have

$$\textbf{Conjunction}[v'/v, w'/w] =_\epsilon \textbf{Conjunction}[\mathrm{inl}_{\mathbb{I}}(*)'/v, w'/w] = w'$$

3.8 Extensionality of the copairing

A λ-abstraction that receives inputs of a disjunctive type is determined by what it does to inputs "from the left and from the right", *i.e.*,

$$\begin{cases} (\lambda x.t)\,\mathsf{inl}(y) = (\lambda x.s)\,\mathsf{inl}(y) \\ (\lambda x.t)\,\mathsf{inr}(z) = (\lambda x.s)\,\mathsf{inr}(z) \end{cases} \implies \lambda x.t = \lambda x.s$$

The proof is as follows:

Using the β equation we have

$$\begin{cases} (\lambda x.t)\,\mathsf{inl}(y) = (\lambda x.s)\,\mathsf{inl}(y) \\ (\lambda x.t)\,\mathsf{inr}(z) = (\lambda x.s)\,\mathsf{inr}(z) \end{cases} = \begin{cases} t\,[\mathsf{inl}(y)/x] = s\,[\mathsf{inl}(y)/x] \\ t\,[\mathsf{inr}(z)/x] = s\,[\mathsf{inr}(z]/x] \end{cases} \tag{3.2}$$

Next, considering the equations above, η_{case}, and the α-equivalence we reason as follows:

$$\begin{aligned} t =& t[x'/x] = \mathsf{case}\ x' \left\{ \mathrm{inl}_{\mathbb{I}}(y) \Rightarrow t\,[\mathsf{inl}(y)/x]; \mathrm{inl}_{\mathbb{I}}(z) \Rightarrow t\,[\mathsf{inr}(z)/x] \right\} && (\alpha, \eta_{case}) \\ &= \mathsf{case}\ x' \left\{ \mathrm{inl}_{\mathbb{I}}(y) \Rightarrow s\,[\mathsf{inl}(y)/x]; \mathrm{inl}_{\mathbb{I}}(z) \Rightarrow s\,[\mathsf{inr}(z)/x] \right\} = s && (\text{Equation 3.2}, \eta_{case}, \alpha) \end{aligned}$$

Finally, it is straightforward that $t = s \implies \lambda x.t = \lambda x.s$.

Now assume that

$$\begin{cases} (\lambda x.t)\,\mathsf{inl}(y) =_{\epsilon_1} (\lambda x.s)\,\mathsf{inl}(y) \\ (\lambda x.t)\,\mathsf{inr}(z) =_{\epsilon_2} (\lambda x.s)\,\mathsf{inr}(z) \end{cases} \implies \lambda x.t = \lambda x.s$$

Following the same reasoning as before and applying the metric equation for condicionals we obtain:

$$\begin{aligned} t =& \mathsf{case}\ x' \left\{ \mathrm{inl}_{\mathbb{I}}(y) \Rightarrow t\,[\mathsf{inl}(y)/x]; \mathrm{inl}_{\mathbb{I}}(z) \Rightarrow t\,[\mathsf{inr}(z)/x] \right\} && (\alpha, \eta_{case}) \\ =_{\max\{\epsilon_1,\epsilon_2\}}& \mathsf{case}\ x' \left\{ \mathrm{inl}_{\mathbb{I}}(y) \Rightarrow s\,[\mathsf{inl}(y)/x]; \mathrm{inl}_{\mathbb{I}}(z) \Rightarrow s\,[\mathsf{inr}(z)/x] \right\} = s && (\text{Figure 5}, \eta_{case}, \alpha) \end{aligned}$$

As a result, given our metric deductive system we obtain $\lambda x.t =_{\max\{\epsilon_1,\epsilon_2\}} \lambda x.s$.

3.9 Some remarks on the metric equation chosen

Falta exemplo de quantale em que não yemos distributividade

One might wonder why, in the definition of the metric, we chose to use the expression $q + \sup\{r, s\}$ rather than $\sup\{q + r, q + s\}$, given that these two expressions are, in fact, equal in the metric quantale. More generally, this equality corresponds to the identity

$$q \otimes \inf\{r, s\} = \inf\{q \otimes r, q \otimes s\},$$

for $r, s \in L$, where L denotes the underlying lattice of the quantale.

However, this identity does *not* hold in arbitrary quantales **[CONTRAEXAMPLE NEEDED]**. Moreover, as becomes clear in the soundness proof, it is the expression $q \otimes \sup\{r, s\}$ that arises naturally.

Part II

Applications

Chapter 4

Probabilistic Programming

Computer science and probability theory have shared a fruitful relationship since the early days [37]. Over the years, probabilistic algorithms have emerged as powerful tools, across diverse domains—from machine learning [38] and robotics [39] to computational linguistics [40]. These algorithms play a pivotal role in modern cryptography, particularly in public-key systems [41], and tackle computationally intractable problems [42].

The growing influence of probabilistic methods has also spurred the development of probabilistic programming languages, both concrete and abstract. Early examples include higher-order probabilistic languages like Church [43], while more recent innovations, such as Anglican [44], continue to expand the expressive power and practicality of probabilistic programming.

In this setting, Crubillé and Dal Lago introduced the notion of a *context distance* in [45, 46], as a metric analogue of Morris' context equivalence. In Morris' framework, two programs are said to be *context equivalent* if their observable behavior—that is, what an external observer can measure during execution—is identical in any context. This distance was first developed for an affine λ-calculus and later extended to a more general setting that, for instance, allows copying.

In [13], the authors reason about approximate equivalence using the *operator norm* which induces a metric (roughly, a distance function) on the space of probabilistic programs (which are interpreted as short maps between Banach spaces).

This chapter begins with an introduction to Banach spaces based on [47–49]. We then show that Ban, the category of Banach spaces and short maps, is a model in this setting. The next section introduces the fundamentals of measure theory, a key component in defining the semantics of probabilistic programs, drawing inspiration primarily from [48, 50, 51]. Finally, we study approximate equivalence in the context of a random walk on the real line.

4.1 Banach spaces

4.1.1 Normed spaces

Definition 4.1.1. A *norm* $\|\cdot\|$ is a function that associates an element of a vector space V with a non-negative real number, such that the following properties hold:

1. Positive definiteness: $\|v\| \geq 0$ for all $v \in V$, with $\|v\| = 0$ if and only if $v = 0$;
2. Positive scalability: $\|av\| = |a|\|v\|$ for all $v \in V$ scalar a;
3. The triangle inequality: $\|v + w\| \leq \|v\| + \|w\|$ for all $v, w \in V$.

Definition 4.1.2. A vector space together with a norm is called a *normed vector space*.

Every normed space may be regarded as a metric space, in which the distance $d(x, y)$ between vectors x and y is $\|x - y\|$. The relevant properties of $d(x, y)$ are

1. $d(x, y) \leq 0$ for all x and y, with equality holding iff $x = y$,
2. $d(x, y) = d(y, x)$ for all x and y,
3. $d(x, z) < d(x, y) + d(y, z)$ for all x, y, z.

Definition 4.1.3. Let V and W be normed vector spaces, and let $T : V \to W$ be a linear operator. The *operator norm*, denoted by $\|\cdot\|_{\text{op}}$, is defined as

$$\|T\|_{\text{op}} = \sup\{\|T(v)\| : v \in V,\ \|v\| = 1\}.$$

If $\|T\|_{\text{op}} < \infty$, we say that T is a *bounded operator*; otherwise, if $\|T\|_{\text{op}} = \infty$, we say that T is *unbounded*. We denote by $\mathcal{B}(V, W)$ the vector space of all bounded linear operators from V to W, and we write $\mathcal{B}(V)$ for $\mathcal{B}(V, V)$.

Lemma 4.1.4. *[48, Lemma 6.4] Let $T : V \to W$ be a bounded linear operator between normed spaces. Then the following statements hold:*

1. *For every $v \in V$, we have $\|T(v)\| \leq \|T\|_{op} \cdot \|x\|$.*
2. *The operator T is continuous if and only if it is bounded.*

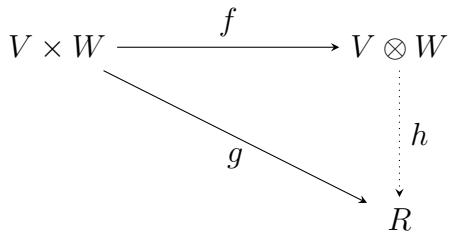
Definition 4.1.5. Let V and W be normed vector spaces, and let $T : V \to W$ be a linear operator. T is called a *short map* if $\|T\|_{\text{op}} \leq 1$.

4.1.2 Banach spaces

Definition 4.1.6. (*Cauchy sequence*) Suppose d is a metric on a set X. A sequence $\{x_n\} \subset X$ is called a *Cauchy sequence* if, for every $\varepsilon > 0$, there exists an integer $N \in \mathbb{N}$ such that $d(x_m, x_n) < \varepsilon$ for all $m, n > N$. The metric d is said to be *complete* if every Cauchy sequence in X converges to a point in X

Definition 4.1.7. A *Banach space* is a normed vector space that is complete with respect to the metric induced by its norm. In other words, every Cauchy sequence in the space must converge to a limit within the space.

Definition 4.1.8 (Algebraic Tensor Product)**.** The tensor product of vector spaces V and W is a vector space $V \otimes W$ together with a bilinear function (*i.e.*linear in both variables) $f : V \times W \to V \otimes W$ such that for every bilinear function $g : V \times W \to R$, there exists a unique linear function $h : V \otimes W \to R$ such that $g = h \circ f$.



The function f usually remains anonymous and is written as $(a, b) \mapsto a \otimes b$.
It follows that arbitrary elements of $V \otimes W$ take the form

$$\sum_{i=1}^{n} a_i (v_i \otimes w_i)$$

for $a_i \in \mathbb{C}$, $v_i \in V$, and $w_i \in W$.
The tensor product also extends to linear maps. If $f_1 : V_1 \to W_1 \quad$ and $\quad f_2 : V_2 \to W_2$ are linear maps, then there is a unique linear map $f_1 \otimes f_2 : V_1 \otimes V_2 \to W_1 \otimes W_2$ that satisfies

$$(f_1 \otimes f_2)(v_1 \otimes v_2) = f_1(v_1) \otimes f_2(v_2)$$

for all $v_1 \in V_1$, $v_2 \in V_2$.

Definition 4.1.9. Let V and W be Banach spaces. Let u be any element of $V \otimes W$. The *projective norm*, denoted $\|\cdot\|_\pi$, is defined by:

$$\|u\|_\pi = \inf \left\{ \sum_{i=1}^{n} \|v_i\| \, \|w_i\| \;\middle|\; u = \sum_{i=1}^{n} v_i \otimes w_i \right\}.$$

Definition 4.1.10. Let V and W be Banach spaces. The *projective tensor product* of V and W, denoted $V \widehat{\otimes}_\pi W$, is the completion of the algebraic tensor product $V \otimes W$ with respect to the projective norm $\|\cdot\|_\pi$.

4.2 Category Ban

Definition 4.2.1. The category Ban is the category of Banach spaces and short maps. It has a (symmetric) monoidal structure where the tensor product is the projective tensor $\widehat{\otimes}_\pi$. The (binary) coproduct of two Banach spaces V, W is given by their direct sum equipped with the norm $\|(v, w)\| = v + w$.

Lemma 4.2.2. *Let V, W and U be Banach spaces. Let $T : V \to U$ and $S : W \to U$ be short maps. Then, it holds that*

$$\|[T, S]\|_{op} \leq \sup\{\|T\|_{op}, \|S\|_{op}\}$$

Proof. We calculate,

$$\begin{aligned}
\|[T, S]\|_{\mathsf{op}} &= \mathsf{sup}\{\|[T, S](v_0)\| \mid \|(v_0)\| = 1\} \\
&= \mathsf{sup}\{\|[T, S](v, w)\| \mid \|(v, w)\| = 1\} \\
&= \sup\{\|T(v) + S(w)\| \mid \|v\| + \|w\| = 1\} \\
&\leq \sup\{\|T(v)\| + \|S(w)\| \mid \|v\| + \|w\| = 1\} \\
&= \sup\{\|v\| \cdot T(v/\|v\|) + \|w\| \cdot S(w/\|w\|) \mid \|v\| + \|w\| = 1\} \\
&= \sup\{\sup\{\|T(v)\| \mid \|(v)\| = 1\}, \sup\{\|S(w)\| \mid \|(w)\| = 1\}\} \\
&= \sup\{\|T\|_{\mathsf{op}}, \|S\|_{\mathsf{op}}\}
\end{aligned}$$

□

Theorem 4.2.3. *[13, Theorem 4.3] The category* Ban *is a symmetric monoidal* Met*-category.*

Theorem 4.2.4. *The category* Ban *is a symmetric monoidal* Met*-category with binary coproducts.*

Proof. Considering the definition of a symmetric monoidal Met-category with binary coproducts, this follows directly from Theorem 4.2.3 and Lemma 4.2.2. □

4.3 Measure theory

Probabilistic computation involves running programs incorporating randomness, leading to output behaviors characterized by probability distributions rather than deterministic outcomes. To effectively understand and analyze these programs, it is essential to have a solid foundation in reasoning about probability distributions. That is where measure theory comes into play.

In this work, we only consider finitemeasures; hence, the term "measure" implicitly refers to a finite measure unless stated otherwise.

4.3.1 What is measure theory?

Throughout history, mathematicians sought to extend the ideas of length, area, and volume. The most effective way to generalize these concepts is through the idea of a measure. Abstractly, a measure is a function defined on sets with additive properties mirroring length, area, and volume.

We begin with a simple example inspired by [52] to develop an intuition for the concepts of measure and measure space. Imagine an open field S covered in snow after a storm. Suppose we wish to measure the amount of snow accumulated in as many field regions as possible. Assume we have accurate tools for measuring snow over standard geometric shapes like triangles, rectangles, and circles. We can approximate irregularly shaped regions using combinations of these standard shapes and then apply a limiting process to assign a consistent measure to such regions. Let $\mathcal{B}$ denote the collection of subsets of S that are deemed *measurable*, let $\lambda(A)$ represent the amount of snow in each $A \in \mathcal{B}$, and let A^c denote the complement of a set A.

For this framework to make sense, it is reasonable to require that $\mathcal{B}$ and $\lambda(\cdot)$ satisfy the following properties:

Properties of B:

1. If $A \in B$, then the complement $A^c \in \mathcal{B}$. (*i.e.*, if we can measure the snow on a set A, and we know the total amount on S, then we can determine the snow on the remaining part A^c.)

2. If $A_1, A_2 \in \mathcal{B}$, then $A_1 \cup A_2 \in \mathcal{B}$. (*i.e.*, if we can measure the snow on two regions A_1 and A_2, we should also be able to measure it on their union.)

3. If $\{A_n\}_{n\geq 1} \subset \mathcal{B}$ is an increasing sequence, *i.e.*, $A_n \subset A_{n+1}$ for all n, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{B}$. (*i.e.*, if each set in a increasing sequence of regions is measurable, then their limit—the union—should also be measurable.)

4. The collection $\mathcal{B}$ contains a base class C of simple, well-behaved sets (e.g., triangles, rectangles, circles) for which measurement is initially defined.

Properties of $\lambda(\cdot)$:

1. $\lambda(A) \geq 0$ for all $A \in \mathcal{B}$ (*i.e.*, the amount of snow on any set must be nonnegative).

2. If $A_1, A_2 \in \mathcal{B}$ and $A_1 \cap A_2 = \emptyset$, then $\lambda(A_1 \cup A_2) = \lambda(A_1) + \lambda(A_2)$ (*i.e.*, The total amount of snow over two non-overlapping regions is just the sum of the snow in each region. This characteristic of λ is known as *finite additivity*.)

3. If $\{A_n\}_{n\geq 1} \subset \mathcal{B}$ is an increasing sequence, *i.e.*, $A_n \subset A_{n+1}$ then $\lambda(\lim_{n\to\infty} A_n) = \lambda\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \lambda(A_n)$. (*i.e.*, if a set can be approximated by an increasing sequence of measurable sets $\{A_n\}_{n\geq 1}$, then $\lambda(A) = \lim_{n\to\infty} \lambda(A_n)$. This is known as *monotone continuity from below*.)

4.3.2 Measurable spaces and measures

Remarkably, these intuitive conditions give rise to a profoundly versatile and far-reaching theoretical framework. The requirements imposed on $\mathcal{B}$ and λ can equivalently be stated as follows:

Properties of $\mathcal{B}$:

1. $\emptyset \in \mathcal{B}$.

2. $A \in \mathcal{B} \implies A^c \in \mathcal{B}$.

3. $A_1, A_2, \cdots \in \mathcal{B} \implies \bigcup_i A_i \in \mathcal{B}$ (this is known as *closure under countable unions*).

Properties of λ:

1. $\lambda(\cdot) \geq 0$ and $\lambda(\emptyset) = 0$.

2. If $\{A_n\}_{n\geq 1} \subset \mathcal{B}$ is a sequence of pairwise disjoint sets (i.e., $A_i \cap A_j = \emptyset$ for $i \neq j$), then $\lambda\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \lambda(A_n)$ (this is known as *countable additivity*).

A collection $\mathcal{B}$ of subsets of S satisfying the above conditions for B is designated a *σ-algebra*. Similarly, a set function λ defined on a σ-algebra $\mathcal{B}$ that fulfills the above properties for λ qualifies as a *measure*.

Definition 4.3.1. A *σ-algebra* $\mathcal{B}$ on a set S is a collection of subsets of S that includes the empty set, is closed under complementation with respect to S, and is closed under countable unions.

Definition 4.3.2. The *Borel σ-algebra* of a metric space space is the smallest family of sets that includes the closed sets and is closed under countable intersections and countable unions. Elements of the Borel σ-algebra are known as *Borel sets*.

Definition 4.3.3. The pair $(S, \mathcal{B})$ where $\mathcal{B}$ is a σ-algebra constitutes a *measurable space*. The elements of $\mathcal{B}$ are called the *measurable sets* of the space.

Definition 4.3.4. A *measure* on the measurable space $(S, \mathcal{B})$ is a function $\mu : \mathcal{B} \to \mathbb{R}$ that is countably additive and satisfies $\mu(\emptyset) = 0$. A measure on $(S, \mathcal{B})$ is called a *probability measure* if $\mu(S) = 1$.

In the context of probability theory, such measures are often referred to as *distributions*. In what follows, we will use the terms measure and distribution interchangeably.
One of the most important measures is the Lebesgue measure on the real line (*i.e.*, the length in $\mathbb{R}$), and its generalizations to $\mathbb{R}^n$. It is characterized as the unique measure on the Borel sets, whose value on every interval is its length. That is, for any interval $(a, b) \subset \mathbb{R}$, $\lambda((a, b)) = b - a$. The Lebesgue measure is *translation-invariant*, meaning that shifting a set does not change its measure.
For any $s \in S$, the *Dirac measure* (also known as the *Dirac delta* or *point mass* at s) is the probability measure defined by

$$\delta_s(B) = \begin{cases} 1, & \text{if } s \in B, \\ 0, & \text{if } s \notin B. \end{cases} \qquad \text{for all } B \in \mathcal{B}$$

A measure is called *discrete* if represented as a countable weighted sum of Dirac measures. In particular, a *convex combination* of Dirac measures yields a discrete probability measure. These are of the form $\sum_i a_i \delta_i$, where $a_i \geq 0$, and the weights satisfy $\sum_i a_i = 1$.
On the other hand, a measure μ on a measurable space $(S, \mathcal{B})$ is called *continuous* if it assigns zero measure to all singleton sets, i.e., $\mu(\{s\}) = 0$ for every $s \in S$. An example of a continuous measure is the *Lebesgue measure* on $\mathbb{R}^n$ (for $n \in \mathbb{N}$).

Definition 4.3.5. Let $(S, \mathcal{B}_S)$ and $(T, \mathcal{B}_T)$ be measurable spaces. Given a function $f : (S, \mathcal{B}_S) \to (T, \mathcal{B}_T)$ and a measure μ on $\mathcal{B}_S$, the *pushforward measure* $f_*(\mu)$ on $\mathcal{B}_T$ is defined by:

$$f_*(\mu)(B) = \mu(f^{-1}(B)), \quad B \in \mathcal{B}_T.$$

Definition 4.3.6. Let $(S, \mathcal{B}_S)$ and $(T, \mathcal{B}_T)$ be measurable spaces. A function $f : S \to T$ is said to be *measurable* if for every measurable subset $B \in \mathcal{B}_T$, the preimage $f^{-1}(B) \in \mathcal{B}_S$.

Definition 4.3.7. The *Lebesgue integral* generalizes the familiar Riemann integral. Consider a measurable space $(S, \mathcal{B})$ and a bounded measurable function $f \colon S \to \mathbb{R}$, with upper and lower bounds M and m, respectively. The *Lebesgue integral* of f with respect to a measure $\mu \colon \mathcal{B} \to \mathbb{R}$, denoted $\int f \, d\mu$, is defined as the limit of finite weighted sums of the form:

$$\sum_{i=0}^{n} f(s_i)\mu(B_i),$$

where $\{B_0, \ldots, B_n\}$ forms a measurable partition of S, and within each B_i, the variation of f does not exceed $(M - m)/n$. Here, $s_i \in B_i$ for each i, and the limit is taken over increasingly refined partitions.

In the case of a finite discrete space $n = \{1, 2, \ldots, n\}$, the Lebesgue integral simplifies to a weighted sum:

$$\int f \, d\mu = \sum_{i=1}^{n} f(i)\mu(i).$$

Given two measurable spaces $(S_1, \mathcal{B}_1)$ and $(S_2, \mathcal{B}_2)$, their product is the measurable space $(S_1 \times S_2, \mathcal{B}_1 \otimes \mathcal{B}_2)$, where $S_1 \times S_2$ is the cartesian product and $\mathcal{B}_1 \otimes \mathcal{B}_2$ is the σ-algebra generated by all measurable rectangles $B_1 \times B_2$ with $B_1 \in \mathcal{B}_1$ and $B_2 \in \mathcal{B}_2$:

$$\mathcal{B}_1 \otimes_{\text{meas}} \mathcal{B}_2 \coloneqq \sigma\left(\{B_1 \times B_2 \mid B_1 \in \mathcal{B}_1,\ B_2 \in \mathcal{B}_2\}\right).$$

Measures on this product space are called *joint distributions*, and are uniquely determined by their values on measurable rectangles due to the inductive structure of the product σ-algebra. *Product measures* are a significant class of joint distributions and are defined from measures

Definition 4.3.8. Let $(S_1, \mathcal{B}_1)$ and $(S_2, \mathcal{B}_2)$ be measurable spaces, and let μ_1 and μ_2 be measures on these spaces, respectively. The *product measure* $\mu_1 \otimes_{\text{meas}} \mu_2$ is defined on measurable rectangles by

$$(\mu_1 \otimes_{\text{meas}} \mu_2)(B_1 \times B_2) = \mu_1(B_1)\mu_2(B_2).$$

This definition extends uniquely to a joint distribution $\mu_1 \otimes \mu_2 : \mathcal{B}_1 \otimes \mathcal{B}_2 \to \mathbb{R}$, and reflects the notion of probabilistic independence: sampling from $\mu_1 \otimes \mu_2$ is equivalent to independently sampling from μ_1 and μ_2.

Definition 4.3.9. Let $(S, \mathcal{B}_S)$ and $(T, \mathcal{B}_T)$ be measurable spaces. A *Markov kernel* is a mapping $P : S \times \mathcal{B}_T \to [0, 1]$ satisfying the following two properties:

1. For each fixed $s \in S$, the function $P(s, \cdot) : \mathcal{B}_T \to [0, 1]$ is a probability measure on $(T, \mathcal{B}_T)$.
2. For each fixed $A \in \mathcal{B}_T$, the function $P(\cdot, A) : S \to [0, 1]$ is a measurable function on $(S, \mathcal{B}_S)$.

By a slight abuse of terminology, we will also refer to a mapping $P' : S \to (\mathcal{B}_T \to [0, 1])$, $P'(s) = \mu$, as a Markov kernel, when there exists a Markov kernel $P : S \times \mathcal{B}_T \to [0, 1]$, $P(s, A) = \mu(A)$ for all $s \in S$ and $A \in \mathcal{B}_T$.

Definition 4.3.10. Given a measure μ on S and a Markov kernel $P : S \times \mathcal{B}_T \to [0, 1]$, we can define the *pushforward* of μ under the Markov kernel P as:

$$P_*(\mu)(B) = \int_S P(s, B)\, \mu(ds), \quad \forall B \in \mathcal{B}_T.$$

Any measurable function $f : (S, \mathcal{B}_S) \to (T, \mathcal{B}_T)$ induces a simple Markov kernel $s \mapsto \delta_{f(s)}$. As a result, the usual definition of the pushforward measure (Definition 4.3.5) becomes a special case of the general formulation defined above.

4.3.3 Spaces of Measures

The set of all finite measures on a measurable space $(S, \mathcal{B})$ will be denoted by $\mathcal{M}(S, \mathcal{B})$, or simply $\mathcal{M}S$ when the σ-algebra $\mathcal{B}$ is clear from context. In particular, $\mathcal{M}\mathbb{R}$ denotes the Banach space of finite Borel measures on $\mathbb{R}$.

$\mathcal{M}S$ forms a real vector space, where addition and scalar multiplication are defined pointwise. Specifically, for $B \in \mathcal{B}$, $\mu, \nu \in \mathcal{M}S$, and $\alpha \in \mathbb{R}$, the operations are given by

$$(\mu + \nu)(B) = \mu(B) + \nu(B), \quad (a\mu)(B) = \alpha\mu(B).$$

$\mathcal{M}S$ is also a normed space equipped with the *total variation norm*.

Definition 4.3.11. A *partition* of a set $B \in \mathcal{B}$ is any finite collection $\{B_1, \ldots, B_n\}$ of pairwise disjoint subsets of B satisfying $\bigcup_{i=1}^{n} B_i = B$. For a measure μ, the *total variation norm* is defined as

$$\|\mu\| := \sup \left\{ \sum_{i=1}^{n} |\mu(B_i)| : \{B_1, \ldots, B_n\} \text{ is a finite measurable partition of } S \right\}.$$

For positive measures, this reduces to $\mu(S)$, and for probability measures, the norm is 1. The total variation norm turns $\mathcal{M}S$ into a Banach space, meaning it is complete under this norm. The following alternative definition is useful to compute the total variation norm between measures.

Theorem 4.3.12. *[51, Theorem 3.1.1] Let μ measure on a measurable space $(S, \mathcal{B})$. Then, there exist disjoint sets $S^-, S^+ \in \mathcal{A}$ such that $S^- \cup S^+ = S$ and for all $B \in \mathcal{B}$, one has*

$$\mu(B \cap S^-) \leq 0 \quad \textit{and} \quad \mu(B \cap S^+) \geq 0.$$

Corollary 4.3.13. *[51, Corollary 3.1.2] Attending to the theorem above, define:*

$$\mu^+(B) := \mu(B \cap S^+), \quad \mu^-(B) := -\mu(B \cap S^-).$$

Then μ^+ and μ^- are nonnegative measures, and we have:

$$\mu = \mu^+ - \mu^-.$$

The measures μ^+ and μ^- have the following properties:

$$\mu^+(B) = \sup\{\mu(B_i) : B_i \subset B,\ B_i \in \mathcal{B}\},$$

$$\mu^-(B) = \sup\{-\mu(B_i) : B_i \subset B,\ B_i \in \mathcal{B}\},$$

for all $B \in \mathcal{B}$.

Definition 4.3.14. Let μ^+ and μ^- be defined as in the corollary above. Then, For a measure μ, the *total variation norm* is defined as

$$\|\mu\| = \mu^+(S) + \mu^-(S)$$

4.4 Example

We proceed by presenting a metric λ-theory on which to reason about random walks, as previously discussed and briefly illustrating of the synergy between syntax and semantics that our framework provides. Our (only) ground type will be $\texttt{real}$ to represent measures over real numbers, *i.e.* we set $[\![\texttt{real}]\!]$ to be the space of measures over the real line, $\mathcal{M}(\mathbb{R})$. Concerning operations we take a pre-determined set of coin toss functions $\mathit{CoinToss}_p : \mathbb{I} \to \mathbb{I} \oplus \mathbb{I}$ whose interpretation takes the form $[\![\mathit{CoinToss}_p]\!] : \mathbb{R} \to \mathbb{R} \oplus \mathbb{R},\ 1 \mapsto (p, 1-p)$. We also take a pre-determined set of measures

$$m = \{\texttt{unif}(a,b) : \mathbb{I} \to \texttt{real}\} \cup \{\texttt{delta}_{p_1,\ldots,p_n,x_1,\ldots,x_n} : \mathbb{I} \to \texttt{real}\}$$

which are interpreted as

$$[\![\texttt{unif}(a,b)]\!](1) = \texttt{unif}(a,b) \quad \text{and} \quad [\![\texttt{delta}_{p_1,\ldots,p_n,x_1,\ldots,x_n}]\!](1) = \sum_{i=1}^{n} p_i \cdot \delta_{x_i},$$

for all $a, b, p_1, \ldots, p_n, x_1, \ldots, x_n \in \mathbb{Q}$, such that $\sum_i p_i = 1$. Here $\texttt{unif}(0,1) \in \mathcal{M}(\mathbb{R})$ is the uniform distribution on the interval $[a, b]$. Note that we are slightly abusing notation by using $\texttt{unif}(a,b)$ both as syntactic and semantic objects. We consider yet addition $+ : \texttt{real}, \texttt{real} \to \texttt{real}$ whose interpretation is given by $\mu \otimes_{\mathsf{meas}} \nu \mapsto +_*(\mu \otimes \nu)$. Finally, we consider a pre-determined set of jumps $j : \mathbb{I} \to (\texttt{real} \multimap \texttt{real})$ interpreted as

$$[\![j]\!](1) = \mu \mapsto +_*(\mu \otimes [\![m_0]\!](*)),$$

where $m_0 \in m$.

Example 4.4.1 (Random walk)**.** In general terms, a *random walk* on $\mathbb{R}$ is a stochastic process in which a particle—the walker—starts at an initial position and repeatedly "tosses a coin" (possibly biased) to decide whether to move left or right.
Given jumps jl, jr we can describe a single step of the random walk a follows,

$$\textbf{step} = - \triangleright \mathsf{case}\ \mathit{CoinToss}_p(*)\{\mathrm{inl}(x) \Rightarrow jl(x); \mathrm{inr}(y) \Rightarrow jr(y)\} : \texttt{real} \multimap \texttt{real}$$

Given an argument r representing the walker's current position, the program **step** performs a random jump: with probability p, the jump is sampled from the underlying distribution of jl; with probability $1-p$, it is sampled from the distribution of jr.

Now, consider the λ-term **apply-n** $= \lambda f_1, \ldots, f_n, r.\ f_1(f_2(\ldots(f_n(r))))$ which operationally speaking sequences n terms given as input. The term that follows represents a n-step walk.

$$\textbf{rwalk} = \textbf{apply-n}\ \ \textbf{step} \ldots \textbf{step}\ (\mathtt{delta}_{1,0}) : \mathtt{real} \multimap \mathtt{real}$$

Recall the interpretation of the jump operations. It is straightforward to prove that the following axiom is sound for each of them:

$$j(*) =_0 \lambda x.\ +(x, m_0(*)).$$

Now, consider we set the interpretations of $[\![jl]\!], [\![jr]\!] : \mathbb{R} \to (\mathcal{M}(\mathbb{R}) \multimap \mathcal{M}(\mathbb{R}))$ to be:

$$[\![jl]\!](1) = \mu \mapsto +_*(\mu \otimes \mathtt{unif}(0,-1)) \qquad [\![jr]\!](1) = \mu \mapsto +_*(\mu \otimes \mathtt{unif}(1,0))$$

Operationally jl corresponds to a jump to the left with magnitude between 0 and 1, and analogously for jr. Suppose we have another jump $[\![jr^\delta]\!] : \mathbb{R} \to (\mathcal{M}(\mathbb{R}) \multimap \mathcal{M}(\mathbb{R}))$ whose interpretation is that of jr except for the fact that $\mathtt{unif}(0,1)$ is replaced by $\mathtt{unif}(0,1+\delta)$. What will be the effect on the random walk when replacing jr by jr^δ? Observe that one can put an upper bound between $jr(*)$ and $jr^\delta(*)$ via the previous axioms and an upper bound between the terms $\mathtt{unif}(0,1)(*)$ and $\mathtt{unif}(0,1+\delta)(*)$. The latter upper bound is obtained *semantically* by computing the norm $\|\mathtt{unif}(0,1) - \mathtt{unif}(0,1+\delta)\|$. First, attending to Definition 4.3.14, we have,

$$\begin{aligned}
&\|\mathtt{unif}(0,1) - \mathtt{unif}(0,1+\delta)\| \\
&= (\mathtt{unif}(0,1) - \mathtt{unif}(0,1+\delta))^+(\mathbb{R}) + (\mathtt{unif}(0,1) - \mathtt{unif}(0,1+\delta))^-(\mathbb{R})
\end{aligned}$$

and proceed by computing the left-hand side of the addition,

$$\begin{aligned}
&(\mathtt{unif}(0,1) - \mathtt{unif}(0,1+\delta))^+(\mathbb{R}) \\
&= \sup\{\mathtt{unif}(0,1)(U) - \mathtt{unif}(0,1+\delta)(U) \mid U \subseteq \mathbb{R}\} \\
&= \sup\{\mathtt{unif}(0,1)(U \cap [0,1]) - \mathtt{unif}(0,1+\delta)(U \cap [0,1]) \\
&\quad - \mathtt{unif}(0,1+\delta)(U \cap (1,1+\delta]) \mid U \subseteq \mathbb{R}\} \\
&= \sup\left\{\left(1 - \frac{1}{1+\delta}\right)\mathtt{unif}(0,1)(U \cap [0,1]) - \mathtt{unif}(0,1+\delta)(U \cap (1,1+\delta]) \mid U \subseteq \mathbb{R}\right\} \\
&= 1 - \frac{1}{1+\delta}
\end{aligned}$$

It follows from an analogous reasoning the right-hand side of the addition will be $\frac{\delta}{1+\delta}$ and therefore the norm will be $2 \cdot \frac{\delta}{1+\delta}$.

Additionally, suppose $\mathit{CoinToss}_p$ is replaced by $\mathit{CoinToss}_q$. We calculate:

$$\begin{aligned}\|[\![\mathit{CoinToss}_p(*)]\!] - [\![\mathit{CoinToss}_q(*)]\!]\| &= \|(p, 1-p) - (q, 1-q)\| \\ &= \|(p-q, q-p)\| = 2|p-q|\end{aligned}$$

Then as our final step we proceed *syntactically* via our metric deductive system, as follows.

$$\begin{aligned}&\text{case } \mathit{CoinToss}_p(*) \text{ of } \mathrm{inl}(x) \Rightarrow jl(x); \mathrm{inr}(y) \Rightarrow jr(y) \\ =_0\ &\text{case } \mathit{CoinToss}_p \text{ of } \mathrm{inl}(x) \Rightarrow jl(y); \mathrm{inr}(y) \Rightarrow x \text{ to } * \,.\, jr(*) \\ =_{2\cdot\left(|p-q|+\frac{\delta}{1+\delta}\right)}\ &\text{case } \mathit{CoinToss}_q \text{ of } \mathrm{inl}(x) \Rightarrow jl(y); \mathrm{inr}(y) \Rightarrow x \text{ to } * \,.\, jr^{\delta}(*) \\ =_0\ &\text{case } \mathit{CoinToss}_q \text{ of } \mathrm{inl}(x) \Rightarrow jl(x); \mathrm{inr}(y) \Rightarrow jr^{\delta}\end{aligned}$$

Thus if **rwalk** is the random walk that involves jump jl and $\mathit{CoinToss}_p$ and **rwalk'** the random walk that involves jump jl^{δ} and $\mathit{CoinToss}_q$ we deduce from the framework the metric equation,

$$\textbf{rwalk} =_{2n\cdot\left(|p-q|+\frac{\delta}{1+\delta}\right)} \textbf{rwalk'}$$

which will converge to 0 as δ and $|p-q|$ tends to 0.

The same reasoning applies to alternative interpretations of jl and jr. For example, consider:

$$[\![jl]\!](1) = \mu \mapsto +_*\left(\mu \otimes \sum_{i=1}^{n} p_i \cdot \delta_{-x_i}\right) \qquad [\![jr]\!](1) = \mu \mapsto +_*\left(\mu \otimes \sum_{i=1}^{n} p_i \cdot \delta_{x_i}\right).$$

Operationally, this means that jl performs a jump to the left, landing at position $-x_i$ with probability p_i, and jl behaves analogously, jumping to x_i with the same probabilities. Now, consider another jump jl^{q_i} whose interpretation corresponds to that of jl xcept for the fact that $\sum_{i=1}^{n} p_i \cdot \delta_{-x_i}$ is replaced with $\sum_{i=1}^{n} q_i \cdot \delta_{-x_i}$. Given Definition 4.3.11, we compute,

$$\begin{aligned}&\left\|\sum_i p_i\delta_{-x_i} - \sum_i q_i\delta_{-x_i}\right\| \\ &= \sup\left\{\sum_{i=1}^{n}\left|\left(\sum_i p_i\delta_{-x_i} - \sum_i q_i\delta_{-x_i}\right)(B_i)\right| \;\middle|\; B_i \in \mathbb{R}, B_i \cap B_j = \emptyset i, \neq j, n \in \mathbb{N}\right\} \\ &= \sum_i |p_i - q_i|\end{aligned}$$

Here, we use the inequality

$$\left|\sum_{i=1}^{n} a_i\right| \leq \sum_{i=1}^{n} |a_i|, \quad \text{for all } n \in \mathbb{N}.$$

Applying the same reasoning as above, if **rwalk** is the random walk that involves jump jl and *CoinToss*$_p$ and **rwalk'** the random walk that involves jump jl^{q_i} and *CoinToss*$_q$, we obtain

$$\textbf{rwalk} =_{n\cdot\left(2|p-q|+\sum_i |p_i-q_i|\right)} \textbf{rwalk'}.$$

Chapter 5

Quantum computation

Quantum computing dates back to 1982 when Nobel laureate Richard Feynman proposed the idea of constructing computers based on quantum mechanics principles to efficiently simulate quantum phenomena [53].

The field has since evolved into a multidisciplinary research area that combines quantum mechanics, computer science, and information theory. Quantum information theory, in particular, is based on the idea that if there are new physics laws, there should be new ways to process and transmit information. In classical information theory, all systems (computers, communication channels, etc.) are fundamentally equivalent, meaning they adhere to consistent scaling laws. These laws, therefore, govern the ultimate limits of such systems. For instance, if the time required to solve a particular problem, such as the factorization of a large number, increases exponentially with the size of the problem, this scaling behavior remains true irrespective of the computational power available. Such a problem, growing exponentially with the size of the object, is known as a "difficult problem". However, as demonstrated by Peter Shor, the use of a quantum computer with a sufficient number of quantum bits (qubits) could significantly accelerate the factorization of large numbers [20]. This advancement poses a significant threat to the security of confidential data transmitted over the Internet, as the RSA algorithm is based on the computational difficulty of factorizing large numbers. This result underscores the promise of the quantum computing paradigm.

Quantum computing and the need for quantitative reasoning While hardware advancements have brought the scientific community closer to realizing the transformative potential of quantum computing, the ultimate goal is yet to be accomplished. A NISQ computer equipped with 50-100 qubits may surpass the capabilities of current classical computers, yet the impact of quantum noise, such as decoherence in entangled states, imposes limitations

on the size of quantum circuits that can be executed reliably [54]. Unfortunately, general-purpose error correction techniques [55–57] consume a substantial number of qubits, making it difficult for NISQ devices to make use of them in the near term. For instance, the implementation of a single logical qubit may require between 10^3 and 10^4 physical qubits [58]. As a result, it is unreasonable to expect that the idealized quantum algorithm will run perfectly on a quantum device, instead only a mere approximation will be observed.

To reconcile quantum computation with NISQ computers, quantum compilers perform transformations for error mitigation [59] and noise-adaptive optimization [60]. Additionally, current quantum computers only support a restricted, albeit universal, set of quantum operations. As a result, nonnative operations must be decomposed into sequences of native operations before execution [61, 62]. The assessment of these compiler transformations necessitates a comparison of the error bounds between the source and compiled quantum programs, which calls for the development of appropriate notions of approximate program equivalence.

As previously noted, Shor's algorithm has played a pivotal role in sparking heightened interest within the scientific community toward quantum computing research. Several quantum programming languages have surfaced over the past 25 years [63, 64]. Among them, we remark on Selinger's and Valerion's work. In 2004, Selinger introduced a first-order functional language for quantum computation, QPL, along with its denotational semantics [21]. Building on this, Selinger and Valiron later developed a higher-order functional language for quantum computation—commonly referred to as a quantum lambda calculus. They first presented a version with classical control and its operational semantics in [65]. This was followed by a denotational semantics for a fragment of the language in [66]. In subsequent work, they extended the quantum lambda calculus to include recursion and infinite types, along with its operational semantics [67]. Later, they proposed an alternative approach to its denotational semantics [68].

These works adopt the Schrödinger picture, where quantum programs are interpreted as maps between quantum states (*i.e.*, density operators). In contrast, [22, 69] consider the Heisenberg picture, in which programs are modeled as maps between observables (*i.e.*, self-adjoint operators). Particularly, [22] presents a model based on von Neumann algebras, which can be viewed as an infinite-dimensional extension of [21]. Moreover, [22] presents a model of Selinger and Valiron's quantum lambda calculus [65, 67, 70], also based on von

Neumann algebras, and proves the model's adequacy.

However, most of the current research on algorithms and programming languages assumes that addressing the challenge of noise during program execution will be resolved either by the hardware or through the implementation of fault-tolerant protocols designed independently of any specific application [71]. As previously stated, this assumption is not realistic in the NISQ era. Nonetheless, there have been efforts to address the challenge of approximate program equivalence in the quantum setting.

[72] and [73] reason about the issue of noise in a quantum while-language by developing a deductive system to determine how similar a quantum program is from its idealised, noise-free version. The former introduces the (Q,λ)-diamond norm which analyzes the output error given that the input quantum state satisfies some quantum predicate Q to degree λ. However, it does not specify any practical method for obtaining non-trivial quantum predicates. In fact, the methods used in [72] cannot produce any post conditions other than $(I,0)$ (*i.e.*, the identity matrix I to degree 0, analogous to a "true" predicate) for large quantum programs. The latter specifically addresses and delves into this aspect.

An alternative approach was explored in [13], using linear λ-calculus as basis. A notion of approximate equivalence is then integrated in the calculus via the so-called diamond norm, which induces a metric on the space of quantum programs (seen semantically as completely positive trace-preserving super-operators) [74].

The first two sections of this chapter present the mathematical and quantum computing preliminaries necessary for understanding the theory of quantum computation. The introduction to quantum computing draws primarily from [74, 75], while the mathematical foundations are also based on [76–78]. The next section introduces core concepts from functional analysis that are essential for understanding W*-algebras, based on [47, 48]. This is followed by a section presenting the fundamentals of W*-algebras, drawing on [79–81]. We then show that both Selinger's category Q of quantum operations (i.e., completely positive, trace-nonincreasing super-operators) [21], and Cho's category $(\mathsf{Wstar}_{\mathsf{CPSU}})^{\mathsf{op}}$, the opposite category of W*-algebras with normal, completely positive, subunital maps [22], are first-order models. Finally, the last section provides a few illustrative examples.

5.1 Hilbert Spaces

It is impossible to present the theory of quantum computation without introducing some concepts of theory of Hilbert spaces and operators. This section provides a brief overview of the aspects of Hilbert spaces that are most pertinent to the study of quantum computation.

In this section and the one that follows, vector spaces are assumed to be finite-dimensional, unless otherwise stated.

5.1.1 Inner product

A linearidade é no segundo ou no primeiro argumentos, é que uns livros usam no primeiro e outros no segundo?

Definition 5.1.1. An *inner product* $\langle \cdot, \cdot \rangle$ on a vector space V is a function from a mapping $V \times V$ to the field of scalars, $\langle \cdot, \cdot \rangle : V \times V \to \mathcal{F}$, that satisfies the following properties for all $v, w, w_1, \ldots, w_n \in V$ and $a_1, \ldots, a_n \in \mathcal{F}$

1. Linearity in the second argument,
$$\left\langle v, \sum_{i=1}^{n} a_i w_i \right\rangle = \sum_{i=1}^{n} a_i \langle v, w_i \rangle.$$
2. $\langle v, w \rangle = \overline{\langle w, v \rangle}$, where $\overline{(-)}$ is the complex conjugate operation.
3. $\langle v, w \rangle \geq 0$ with equality if and only if $v = 0$.

For instance, the inner product $\langle v, w \rangle$ of two vectors $v = (a_1, \ldots, a_n), w = (b_1, \ldots, b_n) \in \mathbb{C}^n$ is defined as

$$\langle v, w \rangle = \sum_i \overline{a}_i b_i.$$

Every inner product space is a normed space, where the norm of a vector $v \in V$ is defined as $\|v\| = \sqrt{\langle v, v \rangle}$.

Definition 5.1.2. A *Hilbert space* $\mathcal{H}$ is an inner product space.

The letters $\mathcal{H}, \mathcal{K}$ will often be used to refer to Hilbert spaces.

5.1.2 Trace

Definition 5.1.3. Let $\mathcal{H}$ be an Hilbert space and $T \in \mathcal{B}(\mathcal{H})$ a positive operator (Definition 5.1.7). The trace of T is defined as

$$\mathrm{Tr}(T) := \sum_i \langle Tv_i, v_i \rangle \in [0, \infty],$$

where $\{v_i\}$ is an orthonormal basis for $\mathcal{H}$.

The trace is *linear*, $\mathrm{Tr}(A + B) = \mathrm{Tr}(A) + \mathrm{Tr}(B)$, $\mathrm{Tr}(a \cdot A) = a \cdot \mathrm{Tr}(A)$, where $A, B \in \mathcal{B}(\mathcal{H})$, and a is a complex number.

The trace of a square matrix can alternatively be defined as follows.

Definition 5.1.4. The trace of a square matrix $A \in \mathbb{C}^{n \times n}$ defined to be the sum of its diagonal elements,

$$\mathrm{Tr}(A) = \sum_i A_{ii}.$$

By means of the trace, one defines the inner product of two operators $A, B \in \mathbb{C}^{m \times n}$ as follows

$$\langle A, B \rangle = \mathrm{Tr}(A^\dagger B),$$

where $(-)^\dagger$ denotes the adjoint operation.

5.1.3 Important classes of operators

In a finite-dimensional Hilbert space $\mathcal{H}$ every lin ear mapping is continuous, hence a bounded operator. For an n-dimensional Hilbert space $\mathcal{H}$, we can identify $\mathcal{B}(\mathcal{H})$ with the space $\mathbb{C}^{n \times n}$ of $n \times n$ complex matrices known as square matrices. As a result,linear operators mapping a Hilbert space to itself are known as *square operators*.

The following classes of operators are of particular interest in quantum information theory.

Definition 5.1.5. *Normal operators.* A square operator $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is *normal* if $AA^\dagger = A^\dagger A$.

Definition 5.1.6. *Hermitian operators.* A square operator $A \in \mathcal{B}(\mathcal{H})$ is *hermitian* if $A = A^\dagger$. Every Hermitian operator is a normal operator.

Definition 5.1.7. *Positive (semidefinite) operators*. A square operator $A \in \mathcal{B}(\mathcal{H})$ is *positive*, denoted $A \geq 0$, if $\langle v, Av \rangle \geq 0$ for all $v \in \mathcal{B}(\mathcal{H})$. Positive matrices are hermitian, and consequently, by the spectral decomposition, have diagonal representation $A = \sum_i \lambda_i v_i v_i^\dagger$, with non-negative eigenvalues λ_i.

Definition 5.1.8. *Unitary operators.* A square operator $U \in \mathcal{B}(\mathcal{H})$ is *unitary* if $U^\dagger U = UU^\dagger = \mathrm{id}$. The letter U will often be used to refer to unitary operators.

Geometrically, unitary operators are important because they preserve inner products between vectors, $\langle Uv, Uw\rangle = \langle v, w\rangle$ for any two vectors v and w.

Definition 5.1.9. *Density operator.* Positive trace class operators of trace 1 are called *density operators*. By convention, such operators are denoted by the lowercase Greek letter ρ, often accompanied by subscripts.

Definition 5.1.10. *Isometries*. An operator $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is as isometry if $\|Av\| = \|v\|$ for all elements all elements $v \in \mathcal{H}$.

Definition 5.1.11. *Projectors*. A positive operator $P \in \mathcal{B}(\mathcal{H})$ is a projector if $P^2 = P$.

5.1.4 Spectral theorem

Theorem 5.1.12. *[74, Corollary 1.4] Let $\mathcal{H}$ be a Hilbert space. Every normal operator $A \in \mathcal{L}(\mathcal{H})$ can be expressed as a linear combination $\sum_{i=1}^{n} \lambda_i b_i b_i^\dagger$ where the set $\{b_1, \ldots, bn\}$ is an orthonormal basis on $\mathcal{H}$.*

Using this last result any function $f : \mathbb{C} \to \mathbb{C}$, can be extended to normal operators via,

$$f(A) = \sum_i f(\lambda_i) b_i b_i^\dagger \tag{5.1}$$

where $A = \sum_i \lambda_i b_i b_i^\dagger$ is the spectral decomposition of A.

5.1.5 Riez Representation Theorem

Definition 5.1.13 (Riesz Representation Theorem)**.** Let $\mathcal{H}$ be a Hilbert space, and let φ be a linear functional on V. Then there exists a unique vector $v_\varphi \in V$ such that

$$\varphi(w) = \langle w, v_\varphi\rangle \quad \text{for all } w \in V.$$

We call v_φ the *Riesz vector* for φ and denote it by v_φ.

Colocar reparo das notas

5.1.6 Tensor Products and Direct Sums of Hilbert Spaces

Definition 5.1.14. The *direct sum* of two finite-dimensinal Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, denoted $\mathcal{H} \oplus \mathcal{K}$, is the space of all pairs (v, w) where $v \in \mathcal{H}$ and $w \in \mathcal{K}$.

The inner product in $\mathcal{H} \oplus \mathcal{K}$ is defined as follows:

$$\langle (v_1, w_1), (v_2, w_2) \rangle = \langle v_1, v_2 \rangle + \langle w_1, w_2 \rangle.$$

Definition 5.1.15. Consider two finite dimensinal Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ with respective basis $v = (v_1, \ldots, v_n)$ and $w = (w_1, \ldots, w_m)$. Then $\mathcal{H} \otimes \mathcal{K}$ is an mn dimensional vector space and $v \otimes w$ corresponds to the vector

$$(v_1 w_1, \ldots, v_1 w_m, \ldots, v_n w_1, \ldots, v_n w_m),$$

which is a basis for $\mathcal{H} \otimes \mathcal{K}$. The tensor product of two elements $v = \sum_i a_i v_i$ and w = $\sum_j b_j w_j$ is:

$$v \otimes w = \sum_{i,j} a_i b_j \cdot v_i \otimes w_j.$$

The inner product in $V \otimes W$ is defined as follows

$$\langle v_1 \otimes w_1, v_2 \otimes w_2 \rangle = \langle v_1, v_2 \rangle \langle w_1, w_2 \rangle.$$

This is definition is known as the *Kronecker product*.

Definition 5.1.16. Consider two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$. The tensor product of two operators $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ is an operator $A \otimes B \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ defined by the equation

$$(A \otimes B)(v \otimes w) = Av \otimes Bw.$$

The definition of $A \otimes B$ is extended to all elements of $\mathcal{H} \otimes \mathcal{K}$ in the natural way to ensure linearity of the tensor product operator. That is,

$$(A \otimes B) \left(\sum_i a_i \, v_i \otimes w_i \right) = \sum_i a_i \, (Av_i \otimes Bw_i).$$

Suppose A is an $n \times n$ matrix, and B is a $m \times m$ matrix. Then we have the matrix representation:

$$A \otimes B \coloneqq \begin{bmatrix} A_{11}B & A_{12}B & \cdots & A_{1n}B \\ A_{21}B & A_{22}B & \cdots & A_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1}B & A_{n2}B & \cdots & A_{nn}B \end{bmatrix}$$

In this representation, each block $A_{ij}B$ is a $p \times q$ submatrix whose entries are proportional to B, with overall proportionality constant A_{ij}.
For example the tensor product of the matrices $A = \left(\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right)$ and $A = \mathrm{id}$ is

$$A \otimes B = \begin{bmatrix} 1 \cdot \mathrm{id} & 2 \cdot \mathrm{id} \\ 3 \cdot \mathrm{id} & 0 \cdot \mathrm{id} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 4 & 0 \\ 0 & 3 & 0 & 4 \end{bmatrix}. \tag{5.2}$$

The notation $(-)^{\otimes n}$ will be used to denote the tensor product of a vector space, vector, or operator with itself n times.

5.1.7 Useful norms

In this section we only consider finite dimensional Hilbert spaces.

Definition 5.1.17. The *euclidean norm*, $\| \cdot \|_2$, of a vector $v \in \mathcal{H}$ is defined as:

$$\|v\|_2 = \sqrt{\langle v, v \rangle}.$$

Definition 5.1.18. The *trace norm*, $\| \cdot \|_1$, of a matrix $A \in \mathcal{B}(\mathcal{H})$ is defined as:

$$\|A\|_1 = \mathrm{Tr}\sqrt{A^\dagger A}$$

This norm is also known as the Schatten 1-norm. The trace norm induces a metric on the set of density matrices which is defined by $d(\rho, \sigma) = \|\rho - \sigma\|_1$.

5.1.8 Infinite-dimensional Hilbert Spaces

In this subsection we lift the restriction to finite-dimensional Hilbert spaces.
Definition 5.1.1 extends naturally to infinite-dimensional vector spaces, as stated, and the same applies to Definition 5.1.3.

Definition 5.1.19. A *Hilbert space* $\mathcal{H}$ is an inner product space that is complete with respect to the norm induced by the inner product.

Definition 5.1.20. A Hilbert space $\mathcal{H}$ is called *separable* if it has a countable orthonormal basis.

Definition 5.1.21. Let $\mathcal{H}$ be an Hilbert space. An operator $A \in \mathcal{B}(\mathcal{H})$ is *trace class* if $\mathrm{Tr}(|A|) < \infty$, where $|A| = \left(\overline{A}A\right)^{1/2}$. We denote by $\mathcal{T}(\mathcal{H})$ the set of trace class operators on $\mathcal{H}$.

If $\mathcal{H}$ is infinite-dimensional, the set $\mathcal{T}(\mathcal{H})$ forms a proper subset of $\mathcal{B}(\mathcal{H})$. In the finite-dimensional case, however, the two spaces coincide and can be identified with one another.

Definition 5.1.22. Let $\mathcal{H}$and $\mathcal{K}$ be Hilbert spaces. We denote by $\mathcal{H} \otimes_2 \mathcal{K}$ the Hilbert space tensor product, obtained by completing $\mathcal{H} \otimes \mathcal{K}$ equipped with the standard inner product

$$\langle w_1 \otimes v_1,\ w_2 \otimes v_2 \rangle = \langle w_1, v_2 \rangle \cdot \langle w_1, v_2 \rangle.$$

5.2 Quantum Computing Preliminaries

The basic unit of information in quantum computation is a *quantum bit* or *qubit* [82]. Just as classical bit, which can be in one of two states, 0 or 1, a qubit also has a state. Qubits are represented using *Dirac notation*, where the ket symbol $|\psi\rangle$ is used to denote a quantum state ψ. The corresponding bra symbol $\langle\psi|$ is used to denote the conjugate transpose of the state ψ. In this setting, the inner product of two states $|\psi\rangle$ and $|\phi\rangle$ is denoted $\langle\psi|\phi\rangle$ and is the same as $\langle\psi|\,|\phi\rangle$.

Definition 5.2.1. Each isolated physical system is associated with a Hilbert space, known as the system's *state space*. The system's state is fully characterized by a *state vector*, which is a unit vector within this state space.

5.2.1 The 2-Dimensional Hilbert Space

Definition 5.2.2. The *state* of a single qubit is described by a normalized vector in the 2-dimensional Hilbert space $\mathbb{C}^2$. The states

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

correspond to the classical states 0 and 1, respectively. These states, known as the *computational basis* states, form an orthonormal basis for this vector space.

Definition 5.2.3. Unlike classical bits, a qubit is not restricted to the basis states $|0\rangle$ and $|1\rangle$. It can exist in a linear combination of these states, known as a *superposition*. A general qubit

state can be written as

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle\,,$$

where $\alpha, \beta \in \mathbb{C}$ are called *amplitudes*, and must satisfy the normalization condition $|\alpha|^2 + |\beta|^2 = 1$. The values $|\alpha|^2$ and $|\beta|^2$ represent the probabilities of measuring the qubit in the states $|0\rangle$ and $|1\rangle$, respectively.

Definition 5.2.4. Any normalized qubit state $|\psi\rangle$ can be written (up to a global phase) as

$$|\psi\rangle = e^{i\gamma}\left(\cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle\right),$$

where $\theta, \phi, \gamma \in \mathbb{R}$. The global phase factor $e^{i\gamma}$ has no observable effect on the outcome of measurements and is often disregarded. Thus, the state is usually represented as:

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle\,. \tag{5.3}$$

The above parametrization defines a point on the unit sphere in $\mathbb{R}^3$, known as the *Bloch sphere*. The angles θ and ϕ represent the polar and azimuthal angles, respectively. Each pure qubit state corresponds to a point on this sphere, with associated *Bloch vector* given by $(\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)$.

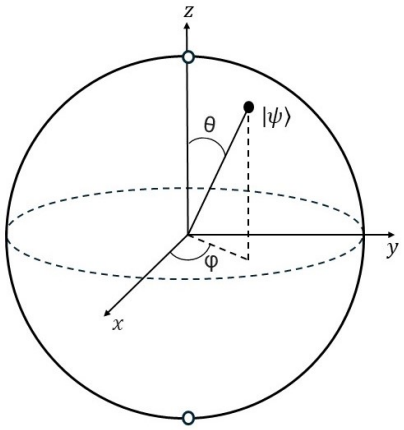


Figure 6: Bloch sphere representation of a qubit

The distance between two quantum states $|\psi\rangle$ and $|\psi'\rangle$ is their Euclidean distance in the Bloch sphere [59, 75].

There are infinite points in the Bloch sphere, which might suggest the possibility of encoding an infinite amount of information in the infinite binary expansion of the angle θ. However,

when a qubit is measured, it collapses to one of the basis states, so only one bit of information can be extracted from a qubit. To accurately determine the amplitudes α and β, an infinite number of identical qubit copies would need to be measured. Nevertheless, it is still conceptually valid to think of these amplitudes as "hidden information". One could say that quantum computation is the art of manipulating this hidden information using phenomena such as interference and superposition to perform tasks that would be impossible or inefficient with classical computers.

5.2.2 Multi-qubit States

Definition 5.2.5. The state space of a composite physical system is the *tensor product* of the state spaces of the component physical systems. As a result, an n-*qubit state* can be represented by a unit vector in 2^n-dimensional Hilbert space, $\mathbb{C}^{2^n}$. The notations $|\psi\rangle \otimes |\phi\rangle$, $|\psi\rangle\,|\phi\rangle$, and $|\psi\phi\rangle$ are used to denote the tensor product of two states $|\psi\rangle$ and $|\phi\rangle$. As for any complex vector, $|\psi\rangle^{\otimes n}$ denotes the n-fold tensor product of state $|\psi\rangle$ with itself. The computational basis states of an n-qubit system are of the form $|x_1 \ldots x_n\rangle$ and so a quantum state of such a system is specified by 2^n amplitudes. For instance, a two-qubit state can be written as

$$|\psi\rangle = \alpha_{00}\,|00\rangle + \alpha_{01}\,|01\rangle + \alpha_{10}\,|10\rangle + \alpha_{11}\,|11\rangle\,.$$

It should be noted that unfortunately, no simple generalization of the Bloch sphere known for multiple qubits.

Entanglement

Definition 5.2.6. An interesting aspect of multi-qubit states is the phenomenon of *entanglement*. This term means strong intrinsic correlations between two (or more) particles when the quantum state of each of them cannot be described independently of the state of the other, i.e. cannot be written as a product of states of the individual qubits. Measuring one qubit of the entangled pair affects the state of the other qubit. This must happen even if the particles are far apart.

In order to better understand this concept, consider the follow *Bell state* or *EPR pair*:

$$\left|\Phi^+\right\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).$$

Upon measuring the first qubit, there are two possible outcomes: 0 with probability $1/2$ and 1 with probability $1/2$. If the first qubit is measured to be 0, the second qubit will also be 0 with probability 1. If the first qubit is measured to be 1, the second qubit will also be 1 with probability 1. Therefore, the measurement outcomes are correlated.

These correlations prompted Einstein, Podolsky, and Rosen to publish a paper [83] questioning the completeness of quantum mechanics in 1935. The EPR paradox presented a dilemma: the existence of entanglement (i.e., correlations that persist regardless of distance) versus local realism and hidden variables. Einstein argued that if two objects, which have interacted in the past but are now separated, exhibit perfect correlation, they must possess a set of properties determined before their separation. These properties would persist in each object, dictating the outcomes of measurements on both sides. Einstein believed that the strong correlations predicted by quantum mechanics necessitate the existence of additional properties not accounted for by the quantum formalism that determine the measurement results. Therefore, he argued that quantum mechanics might require supplementation, as it may not represent a complete or ultimate description of reality.

In 1964, John Bell made a remarkable discovery: the measurement correlations in the Bell state are stronger than those that could ever occur between classical systems [84]. He explored the idea that each entangled particle might possess hidden properties — unaccounted for by quantum mechanics—that determine the measurement outcomes. Then, through mathematical reasoning, Bell demonstrated that the correlations predicted by any local hidden variable theory cannot exceed a specific level. There is an upper limit of correlations fixed by what today is called the “Bell inequalities”. He found that quantum theory sometimes predicts correlations that exceed this limit. Consequently, an experiment could settle the debate by testing whether or not correlations surpass the bounds he had found following Einstein’s position.

In 1982, Alain Aspect conducted an experiment that confirmed the violation of the Bell inequalities [85]. In this experiment, polarizers were placed more than twelve meters apart. This meant that the correlation obtained could not be explained by the fact that the particles carry within them unmeasured properties. Moreover, it proved that the outcome of the measurement is not determined until the moment of measurement. There seemed to be an instantaneous exchange between two particles at the time of measurement when they were twelve meters apart.

Sixteen years later, Nicolas Gisin [86] and Anton Zeilinger [87] conducted similar experiments, demonstrating that entanglement persists over distances of several kilometers. More recently, [88] extended these tests using entangled photon pairs sent from a satellite to verify Bell's inequalities over a distance of one thousand kilometers, further confirming that, regardless of the distance, entangled particles behave as an indivisible, inseparable whole. The connection between them is so profound that it appears to challenge the principles of relativity. This phenomenon is known as *quantum nonlocality*.

5.2.3 Unitary operators

Pauli Matrices

Definition 5.2.7. The Pauli matrices are a set of three 2×2 hermitian matrices that are defined as follows:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The eigenvectors and eigenvalues of the Pauli matrices are as follows:

$$\sigma_x \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \sigma_y \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} 1 \\ i \end{pmatrix}, \qquad \sigma_z \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$\sigma_x \begin{pmatrix} 1 \\ -1 \end{pmatrix} = - \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad \sigma_y \begin{pmatrix} 1 \\ -i \end{pmatrix} = - \begin{pmatrix} 1 \\ -i \end{pmatrix}, \qquad \sigma_z \begin{pmatrix} 0 \\ 1 \end{pmatrix} = - \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The normalized eigenvectors of σ_x are $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$, and normalized eigenvectors of σ_y are $|+i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i\,|1\rangle)$ and $|-i\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i\,|1\rangle)$. The eigenvectors of σ_z are $|0\rangle$ and $|1\rangle$. These eigenvectors correspond to the $\hat{x}, \hat{y}$ and $\hat{z}$ axes of the Bloch sphere in Figure 6, respectively.

When matrices σ_x, σ_y or σ_z are applied to a state on the Bloch sphere, they rotate the state by π radians around the $\hat{x}, \hat{y}$ or $\hat{z}$ axis, respectively. For example, the action of σ_x on the state $|0\rangle$ is to rotate it to $|1\rangle$, and vice versa. Note that for the eigenstates of these matrices with eigenvalue -1, this still applies if considering a global phase of $-1 = e^{i\pi}$, given that two quantum states $|\psi\rangle$ and $e^{i\phi}\,|\psi\rangle$ are indistinguishable by any quantum measurement.

Matrices σ_x and σ_z will also be referred to as X and Z, respectively.

Unitary operators

Definition 5.2.8. *Closed systems*, i.e., systems that do not interact with other systems evolve according to unitary operators. In quantum computation, these unitary operators are also known as *gates*. For a state $|\psi\rangle$, a *unitary operator* U describes an evolution from $|\psi\rangle$ to $U|\psi\rangle$.

Example 5.2.9. Pauli matrices are examples of unitary operators. The X and Z gates are often referred to as the *not* and *phase flip* gates, respectively. Other important unitary operators include *Hadamard gate*, denoted H, which maps $|0\rangle$ to $|+\rangle$ and $|1\rangle$ to $|-\rangle$, and the *phase-shift gate*, denoted P, which leaves $|0\rangle$ unaltered applies a phase shift of θ to the state $|1\rangle$:

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad P = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}.$$

When the Pauli matrices are exponentiated, they result in three valuable classes of unitary matrices, corresponding to the rotation operators around the $\hat{x}$, $\hat{y}$, and $\hat{z}$ axes, which are defined as follows:

$$R_x(\theta) = e^{-i\theta\sigma_x/2} = \cos\left(\frac{\theta}{2}\right)\mathrm{id} - i\sin\left(\frac{\theta}{2}\right)\sigma_x = \begin{pmatrix} \cos(\frac{\theta}{2}) & -i\sin(\frac{\theta}{2}) \\ -i\sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{pmatrix},$$

$$R_y(\theta) = e^{-i\theta\sigma_y/2} = \cos\left(\frac{\theta}{2}\right)\mathrm{id} - i\sin\left(\frac{\theta}{2}\right)\sigma_y = \begin{pmatrix} \cos(\frac{\theta}{2}) & -\sin(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{pmatrix},$$

$$R_z(\theta) = e^{-i\theta\sigma_z/2} = \cos\left(\frac{\theta}{2}\right)\mathrm{id} - i\sin\left(\frac{\theta}{2}\right)\sigma_z = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}.$$

Theorem 5.2.10. *[75] Suppose U is a unitary operation on a single qubit. Then there exist real numbers α, β, γ and δ such that*

$$U = e^{i\alpha}R_z(\beta)R_y(\gamma)R_z(\delta).$$

Example 5.2.11. There are also multi-qubit gates, such as the *controlled-not* gate, denoted ***CNOT***, which leaves the states $|00\rangle$ and $|01\rangle$ unchanged, and maps $|10\rangle$ and $|11\rangle$ to each other:

$$\mathit{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

In this case, as the state of the first qubit determines if the X gate is applied to the second qubit, the first qubit is called the *control qubit* and the second qubit the *target qubit*.
There is an "extension" of the controlled-not gate, the controlled-U gate, where U is a unitary gate acting on a single qubit. This gate applies the gate U to the target qubit if the control qubit is in state $|1\rangle$ and does nothing otherwise. It is defined as:

$$CU(|0\rangle \otimes |\psi\rangle) = |0\rangle \otimes |\psi\rangle$$
$$CU(|1\rangle \otimes |\psi\rangle) = |1\rangle \otimes U|\psi\rangle .$$

It should be noted that no completely closed systems exist in the universe. Nevertheless, for many systems, the approximation of a closed system is valid.

5.2.4 Measurements

There are times when it necessary to observe the system to extract information. This interection leaves the system no longer closed and, consequently, the evolution of the system is no longer unitary.

Definition 5.2.12. The act of measuring a qubit is represented by a set of operators called *measurement operators*, denoted $\{M_m\}$. These operators act on the state space of the system being measured. The index m refers possible measurement outcomes. These measurement operators must satisfy the completeness equation $\sum_m M_m^\dagger M_m = \mathrm{id}$, which ensures that the probabilities of all possible outcomes sum to 1. If a measurement M_m is performed on a state $|\psi\rangle$ the outcome m is observed with probability $p_m = \langle\psi| M_m^\dagger M_m |\psi\rangle$ for each m. Moreover, after a measurement yielding outcome m, the state collapses to

$$|\psi'\rangle = \frac{M_m|\psi\rangle}{\sqrt{p_m}}.$$

Definition 5.2.13. A measurement is called a *projective measurement* if its measurement operators are projectors.

Example 5.2.14. In the case of the computational basis, the measurement operators are the projectors onto the basis states $|0\rangle$ and $|1\rangle$ denoted by $M_0 = |0\rangle\langle 0|$ and $M_1 = |1\rangle\langle 1|$, respectively. Considering an arbitrary state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, the probabilities of measuring 0 and 1 are $p_0 = \langle\psi| M_0 M_0^\dagger |\psi\rangle = \langle\psi| M_0 |\psi\rangle = |\alpha|^2$, and $p_1 = \langle\psi| M_1 M_1^\dagger |\psi\rangle = \langle\psi| M_1 |\psi\rangle =$

$|\beta|^2$, respectively. Consequently the states after measurement are

$$|\psi'\rangle = \frac{M_0 |\psi\rangle}{|\alpha|} = \frac{\alpha}{|\alpha|} |0\rangle = |0\rangle \text{ (with } p = p_0) \quad \text{and}$$
$$|\psi''\rangle = \frac{M_1 |\psi\rangle}{|\beta|} = \frac{\beta}{|\beta|} |1\rangle = |1\rangle \text{ (with } p = p_1)$$

From now on, unless stated otherwise, any reference to measurement should be understood as pertaining to the computational basis.

As previously mentioned, any states $|\psi\rangle$ and $e^{i\phi} |\psi\rangle$ are indistinguishable by any quantum measurement. Consider a measurement operator M_m, the probabilities of obtaining outcome m are $\langle\psi| M_m^\dagger M_m |\psi\rangle$ and $\langle\psi| e^{-i\theta} M_m^\dagger M_m e^{i\theta} |\psi\rangle = \langle\psi| M_m^\dagger M_m |\psi\rangle$. For this reason, it is said that these states are equal from an observational point of view.

5.2.5 Density operators

Until now the state vector formalism was used. However there is an alternative formulation using density operators. The density operator is often known as the *density matrix*, the two terms will be used interchangeably.

Definition 5.2.15. A quantum state $|\psi\rangle$ is said to be a *pure state* if it is completely known, *i.e.*if it can be written as a ket. In this case, the state can be written in the density operator formalism as $\rho = |\psi\rangle \langle\psi|$.

Definition 5.2.16. A state that is a probabilistic mixture of pure states is designated a *mixed state*. A mixed state $\sum_i \alpha_i |\psi_i\rangle$ can be represented by a density operator $\rho = \sum_i |\alpha_i|^2 |\psi_i\rangle \langle\psi_i|$. Note that $|\alpha_i|^2$ is the probability of the system being in state $|\psi_i\rangle$.

Definition 5.2.17 (Unitary Evolution of a Density Operator)**.** When a unitary operator U is applied to a quantum state described by a density matrix ρ, the resulting state is given by $\rho' = U\rho U^\dagger$.

Definition 5.2.18 (Measurement of a Density Operator)**.** Given a collection of measurement operators $\{M_m\}$, the probability of obtaining outcome m when measuring a state ρ is $p_m = \text{Tr}(M_m \rho M_m^\dagger)$. After observing outcome m, the post-measurement state collapses to:

$$\rho' = \frac{M_m \rho M_m^\dagger}{\text{Tr}(M_m \rho M_m^\dagger)}.$$

Definition 5.2.19. In subsection 5.2.1 it was shown how to determine the cartesian coordenates of a pure state in the Bloch sphere from the state vector. For an arbitrary 2×2 density matrix, the following holds

$$\rho = \frac{1}{2}(\mathrm{id} + r_x\sigma_x + r_y\sigma_y + r_z\sigma_z), \tag{5.4}$$

where $r = (r_x, r_y, r_z)$ is a real three-dimensional vector such that $\|r\|_2 \leq 1$. This vector is known as the *Bloch vector for the state* ρ. Since ρ is Hermitian. r_x, r_y and r_z are always real. The inverse map of Equation 5.4 is

$$r_\mu = \mathrm{Tr}(\rho\sigma_\mu) \tag{5.5}$$

Note that given that the trace is linear and matrix multiplication distributes over matrix addition, the cartesian coordinates of an operator consisting of the sum or subtraction of density operators can also be determined by Equation 5.5.

Reduced density operator

Density operators are particularly well-suited for describing individual subsystems of a composite quantum system. This type of description is provided by the *reduced density operator*

Definition 5.2.20. Let $|a_1\rangle, |a_2\rangle$ be vectors in the Hilbert space of system A, and $|b_1\rangle, |b_2\rangle$ vectors in the Hilbert space of system B. The *partial trace over* B is defined by:

$$\begin{aligned}
\mathrm{Tr}_B(|a_1\rangle\langle a_2| \otimes |b_1\rangle\langle b_2|) &= |a_1\rangle\langle a_2|\,\mathrm{Tr}(|b_1\rangle\langle b_2|) \\
&= |a_1\rangle\langle a_2| \sum_\mu \langle\mu|b_1\rangle\langle b_2|\mu\rangle \\
&= |a_1\rangle\langle a_2| \sum_\mu \langle\mu|b_1\rangle\langle b_2|\mu\rangle \\
&= |a_1\rangle\langle a_2| \sum_\mu \langle b_2|\mu\rangle\langle\mu|b_1\rangle \\
&= |a_1\rangle\langle a_2|\langle b_2|b_1\rangle
\end{aligned}$$

where $\{|\mu\rangle\}$ is an orthonormal basis spanning the state space of B. In certain situations it is more advantageous to consider the reduced density operator for subsystem A defined as:

$$\rho_A = \mathrm{Tr}_B(\rho_{AB}) = \sum_\mu \langle\mu|\rho_{AB}|\mu\rangle,$$

where $\{|\mu\rangle\}$, span the state space of B and act only in the state space of B.

Definition 5.2.21. Given physical systems A and B whose composite system is given by the density operator ρ_{AB}, the *reduced density operator* for subsystem A is

$$\rho_A = \mathsf{Tr}_B(\rho_{AB}).$$

Note that, by linearly, the partial trace of $\rho_{AB} = \sum_{ijkl} p_{ijkl} |a_i\rangle \langle a_j| \otimes |b_k\rangle \langle b_l|$ is

$$\begin{aligned}\mathsf{Tr}_B(\rho_{AB}) &= \sum_{ijkl} p_{ijkl} \mathsf{Tr}_B \left(|a_i\rangle \langle a_j| \otimes |b_k\rangle \langle b_l| \right) \\ &= \sum_{ijkl} p_{ijkl} |a_i\rangle \langle a_j| \sum_{\mu} \langle b_l|\mu\rangle \langle \mu|b_k\rangle = \sum_{ijkl} p_{ijkl} |a_i\rangle \langle a_j| \langle b_l|b_k\rangle\end{aligned}$$

Similarly, the *reduced density operator for subsystem* B is $\rho_B = \mathsf{Tr}_A(\rho_{AB})$.

5.2.6 Quantum Channels

Thus far, only two types of quantum operations have been discussed: unitary operators, which describe the evolution of a closed quantum system, and measurements, which describe the act of observing a quantum system. Now, a new type of quantum operation that accounts for the more realistic notion of interaction between a quantum system and an environment will be introduced. Nonetheless, it is necessary to first introduce a few key concepts.

Definition 5.2.22. Operators that map operators to other opertors are known as *super-operators*.

Definition 5.2.23. A super-operator $\Phi : \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{K})$ is called *positive* if it sends positive operators to positive operators, *i.e.* $A \geq 0 \Rightarrow QA \geq 0$.

Definition 5.2.24. The tensor product of two super-operators $\Phi : \mathcal{T}(\mathcal{H}_1) \to \mathcal{T}(\mathcal{K}_1)$ and $\Psi : \mathcal{T}(\mathcal{H}_2) \to \mathcal{T}(\mathcal{K}_2)$ is an operator $\Phi \otimes \Psi : \mathcal{T}(\mathcal{H}_1) \otimes \mathcal{T}(\mathcal{K}_1) \to \mathcal{T}(\mathcal{H}_2) \otimes \mathcal{T}(\mathcal{K}_2)$ defined by the equation:

$$(P \otimes Q)(A \otimes B) = P(A) \otimes Q(B).$$

Definition 5.2.25. A linear mapping $\Phi : \mathcal{T}(\mathcal{H}_1) \to \mathcal{T}(\mathcal{H}_2)$ is *completely positive* if the mapping $\Phi \otimes \mathrm{id}$ on $\mathcal{T}(\mathcal{H}_A \otimes \mathcal{K}) \to \mathcal{T}(\mathcal{T}(\mathcal{H}_2) \otimes \mathcal{K})$ is positive for any Hilbert space $\mathcal{K}$.

Definition 5.2.26. A super-operator Φ is called *trace-preserving* (resp. *trace-nonincreasing*) if $\mathsf{Tr}\left(\Phi A\right) = \mathsf{Tr}(A)$ (resp. ($\mathsf{Tr}\left(\Phi A\right) \leq \mathsf{Tr}(A)$)).

Since density matrices are positive, any physically allowed transformation must be represented by a positive operator. Nonetheless, this is not sufficient on its own: since one can always extend the space $\mathbb{C}^{n\times n}$ to $\mathbb{C}^{n\times n} \otimes \mathbb{C}^{m\times m}$ by adjoining a new quantum system, any physically allowed transformation must be completely positive. Finally, since the trace of a density matrix is always 1, any physically allowed transformation must be trace-preserving. A **Completely Positive Trace-Preserving** (**CPTP**) operator is traditionally called a *quantum channel*.

It is sometimes convenient to relax the trace-preserving condition to a trace-non-increasing one, resulting in what is known as a *quantum operation*.

Kraus operator sum representation

Assume that there is a quantum system S of interest which is a subsystem of a larger system which also includes an environment E. These systems have a joint unitary evolution described by a unitary operator U acting on the composite system, $U(\rho_{SE}) = U\rho_{SE}U^\dagger$.

Given that density matrices are positve operators, the density operator of the environment ρ_E initially can be written as

$$\rho_E = \sum_i p_i \left|i\right\rangle \left\langle i\right|$$

where $\left|i\right\rangle$ form an orthonormal basis for the state space of E and p_i are positive.

The state of the subsystem S after the unitary evolution corresponds to the partial trace of the joint state over the environment,

$$\begin{aligned}\rho'_S =& \mathsf{Tr}_E(U\rho_{SE}U^\dagger)\\ =& \sum_\mu \left\langle\mu\right| U\rho_{SE}U^\dagger \left|\mu\right\rangle\end{aligned}$$

where $\{\left|\mu\right\rangle\}$ span the state space of E.

Considering that initially both systems are completely decoupled, the initial state of the system can be written as $\rho_{SE} = \rho_S \otimes \rho_E$. Thus,

$$\begin{aligned}\rho'_S &= \sum_\mu \left\langle\mu\right| U\rho_S \otimes \sum_i p_i \left|i\right\rangle \left\langle i\right| U^\dagger \left|\mu\right\rangle\\ &= \sum_{\mu i} \sqrt{p_i}\left\langle\mu\right| U \left|i\right\rangle \rho_S \sqrt{p_i}\left\langle i\right| U^\dagger \left|\mu\right\rangle\\ &= \sum_{\mu i} \mathsf{K}_{\mu i}\rho_S\mathsf{K}_{\mu i}^\dagger\end{aligned}$$

where the set of operators $\{\mathsf{K}_{\mu i}\}$ is designated *Kraus operators* and $\mathsf{K}_{\mu i} = \sqrt{p_i}\,\langle\mu|\,U\,|i\rangle$. Note that $\{|\mu\rangle\}$ and $\{|i\rangle\}$, act only in the state space of E.

Definition 5.2.27. The equation $\rho'_S = \sum_{\mu i} \mathsf{K}_{\mu i}\rho_S\mathsf{K}^\dagger_{\mu i}$ is called an ***Operator Sum Representation (OSR)***. An OSR can be thought of as a quantum channel that maps ρ_S to $\sum_{\mu i} \mathsf{K}_{\mu i}\rho_S\mathsf{K}^\dagger_{\mu i}$, given this map is CPTP [74, 89].

Non-selective measurements

In the previously presented formalism to represent all the possible outcomes of a measurement, described by a set of operators $\{M_m\}$, on a state ρ, it would be necessary to write that state ρ collapse to state $\rho_m = \frac{M_m^\dagger \rho M_m}{\mathsf{Tr}(M_m \rho M_m^\dagger)}$ with probability $p_m = \mathsf{Tr}(M_m \rho M_m^\dagger)$, for each possible outcome m. Although the selective description above is useful conceptually, it is often impractical for calculations. Instead, one uses *non-selective measurements*, in which the possible outcomes are not explicitly stated.

Definition 5.2.28. A *non-selective measurement* is a quantum measurement in which the post-measurement state of the system is then given by the weighted sum over all possible outcomes:

$$\rho = \sum_m p(m)\rho_m = \sum_m M_m \rho M_m^\dagger.$$

This last equality corresponds to an Kraus operator sum representation, where the set of Kraus operators is $\{M_m\}$.

5.2.7 Norms on quantum channels

Definition 5.2.29. The trace norm of a super-operator $Q : \Phi : \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{K})$ is defined as:

$$\|Q\|_1 = \max\{\|Q\,A\|_1 \mid \|A\|_1 = 1\},$$

where $A \in \mathcal{T}(\mathcal{H})$

Unfortunately, this norm is not stable under tensoring , given that the inequation $\|\Phi \otimes I_{\mathcal{T}(\mathcal{H})}\|_1 \geq \|\Phi\|_1$ does not hold [74]. As a result, the diamond norm, which is based on the trace norm, is used instead in the context of quantum channels.

Definition 5.2.30. Given a super-operator $\Phi : \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{K})$, the diamond norm, $\|\cdot\|_\diamond$, is defined as:

$$\|\Phi\|_\diamond = \|\Phi \otimes \mathrm{id}_{\mathcal{T}(\mathcal{H})}\|_1$$

Since the diamond norm is generally difficult to compute, we will rely on the following properties:

Theorem 5.2.31. *Let $\Phi \in \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{K})$ be a positive map. Then it holds that*

$$\|\Phi\|_1 = \max\{\mathrm{Tr}\left(\Phi(vv^*)\right) \mid \|v\|_2 = 1\}.$$

Proposition 5.2.32. *For all maps $\Phi \in \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{K})$ and $\Psi \in \mathcal{T}(\mathcal{K}) \to \mathcal{T}(\mathcal{L})$, it holds that*

$$\|\Psi\Phi\|_\diamond \leq \|\Psi\|_\diamond \|\Phi\|_\diamond$$

Theorem 5.2.33. *Let $\Phi \in \mathcal{T}(\mathcal{H}_1) \to \mathcal{T}(\mathcal{K}_1)$ and $\Psi \in \mathcal{T}(\mathcal{H}_2) \to \mathcal{T}(\mathcal{K}_2)$ be maps. Then it holds that*

$$\|\Phi \otimes \Psi\|_\diamond = \|\Phi\|_\diamond \|\Phi\|_\diamond$$

Theorem 5.2.34. *[74, Theorem 3.55] Let $n \leq m$, let $V_0, V_1 : \mathcal{T}(\mathcal{H}, \mathcal{K})$ be isometries, and define CPTP operators $\Phi_0, \Phi_1 : \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{K})$ as*

$$\phi_0(\rho) = V_0 \rho V_0^\dagger \quad \textit{and} \quad \phi_1(\rho) = V_1 \rho V_1^\dagger$$

for all $\rho \in \mathcal{T}(\mathcal{H})$. There exists a unit vector $u \in \mathcal{H}$ such that

$$\left\|\phi_0(uu^\dagger) - \phi_1(uu^\dagger)\right\|_1 = \|\phi_0 - \phi_1\|_\diamond .$$

Theorem 5.2.35. *[74, Theorem 3.56] Let $\Phi : \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{K})$ be a quantum channel, let $\varepsilon \in [0, 2]$, and suppose that*

$$\|\phi(\rho) - \rho\|_1 \leq \varepsilon$$

for every density operator $\rho \in \mathcal{T}(\mathcal{H})$. It holds that

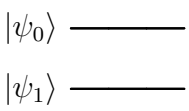
$$\|\phi - \mathrm{id}_{\mathcal{T}(\mathcal{H})}\|_1 \leq \sqrt{2\varepsilon}.$$

5.2.8 Quantum circuits

As quantum computation remains in its early stages of development, programming is primarily based on the use of *quantum circuits*.

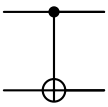
Definition 5.2.36. A *quantum circuit* consists of wires and quantum gates, which serve to transmit and manipulate quantum information. Each wire corresponds to a qubit, while the gates represent operations that can be applied to these qubits.

In this subsection the notation for the quantum gates used in this work will be introduced. Wires in parallel represent the tensor product of the respective qubits. For instance, $\psi_0 \otimes \psi_1$ corresponds to

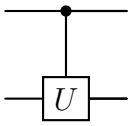


The single bit gates presented in Section 5.2.3 are represented as a box with the symbom of the gate inside. For example, the Hadamard gate is represented as

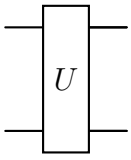
The controlled-not gate, which is a two-qubit gate, is represented as



Similarly, the controlled-U gate, where U is an unitary single-qubit gate, is represented as



An arbitrary unitary operator acting on n qubits is represented as a box acting on n wires. For instance, the operator U acting on two qubits is represented as



CPTP maps are depicted as boxes containing the corresponding map symbols.
The measurement operation is representes by a “meter” symbol. Given that output of a measurement is a classical bit, the wire representing the output of a measurement is a classical wire, representes by a double line.

5.2.9 No-cloning theorem

The no-cloning theorem states that it is impossible to duplicate an unknown quantum bit [90]. In this subsection, an elementary proof of this theorem will be presented.
Suppose that there exists a unitary operator U that recieves a qubit $|\psi\rangle$ and some standard pure state $|s\rangle$ as input and oututs the state $|\psi\rangle \otimes |\psi\rangle$. The action of U can be written as

$$U\left(|\psi\rangle \otimes |s\rangle\right) = |\psi\rangle \otimes |\psi\rangle$$

Consider the aplication of U to two pure states $|\psi_0\rangle$ and $|\psi_1\rangle$,

$$\begin{aligned} U\left(|\psi_0\rangle \otimes |s\rangle\right) &= |\psi_0\rangle \otimes |\psi_0\rangle \\ U\left(|\psi_1\rangle \otimes |s\rangle\right) &= |\psi_1\rangle \otimes |\psi_1\rangle . \end{aligned}$$

Given that unitary operators preserve inner products, the following equality should hold:

$$\langle\psi_0|\psi_1\rangle = \left(\langle\psi_0|\psi_1\rangle\right)^2$$

This equation is only satisfied if $\langle\psi_0|\psi_1\rangle = 0$ or $\langle\psi_0|\psi_1\rangle = 1$. The first case implies that $|\psi_0\rangle$ and $|\psi_1\rangle$ are orthogonal, and the second case implies that they are in the same state. Therefore, it is only possible to clone orthogonal states. These are the states perfectly distinguishable by measurement and thus are equivalent to copying classical information. For instance, it is impossible to the clone qubits $\psi_0 = |0\rangle$ and $\psi_1 = |-\rangle$, since they are not orthogonal.
It should be noted that this principle is upheld by the type system outlined in Figure 1, which does not allow the repeated use of a variable (seen as a quantum resource).

5.3 Functional Analysis

Passar primeios conceitos para a parte de probabilistic programming por causa dos Borel sets lá?

In this section, we are no longer restricted to finite dimensional vector spaces; the term "vector space" now also encompasses infinite-dimensional ones.
Topology is the abstract mathematical study of concepts like convergence and approximation, generalizing familiar notions from calculus and analysis. Note that, for instance, in a metric space, a sequence $\{x_n\}$ with $n \in \mathbb{N}$ converges to a point x if the distance $d(x_n, x)$ tends to zero; that is, for every $\varepsilon > 0$, there exists n_0 such that $d(x_n, x) < \varepsilon$ for all $n \geq n_0$.

However, metric spaces are not sufficient to describe all types of convergence. An example is the pointwise convergence of all real-valued functions on the interval $[0, 1]$. In fact, there is no metric on the space of all real functions on the interval $[0, 1]$ for which one can define a distance function $d(f_n, f)$ such that $d(f_n, f) \to 0$ if and only if $f_n(x) \to f(x)$ for every $x \in [0, 1]$. A foundational idea in topology is that of a *neighborhood*—a collection of points considered "sufficiently close" to a given point. From this arises the concept of *open sets*, which are sets that serve as neighborhoods for all their points. The collection of all such open sets defines a *topology*, and a set equipped with a topology becomes a *topological space*. This framework introduces some subtleties: for example, traditional sequences are often inadequate for capturing convergence, requiring the more general notion of *nets*, which are indexed over broader structures than the natural numbers.

Definition 5.3.1. A *topology* τ on a set X is a collection of subsets of X satisfying the following properties:

1. $\varnothing \in \tau$ and $X \in \tau$.
2. τ is closed under finite intersections: if $U_1, U_2, \ldots, U_n \in \tau$, then $\bigcap_{i=1}^{n} U_i \in \tau$.
3. τ is closed under arbitrary unions: if $\{U_\alpha\}_{\alpha \in A} \subseteq \tau$, then $\bigcup_{\alpha \in A} U_\alpha \in \tau$.

A nonempty set S equipped with a topology τ is called a *topological space*, and is denoted by (S, τ) (or simply X when no ambiguity arises). A member of τ is called an *open set* in S. The complement of an open set is a *closed set*.
A set S can have many different topologies. The family of all topologies on S is partially ordered by set inclusion. If $\tau_1 \subset \tau_2$, that is, if every τ_1-open set is also τ_2-open, then we say that τ_1 is *weaker* or *coarser* than τ_2, and that τ_2 is *stronger* or *finer* than τ_1.

Example 5.3.2. Standard examples of topologies are presented below:

1. *Trivial (or indiscrete) topology:* On a set S, the trivial topology consists only of the sets $\varnothing$ and X. These are also the only closed sets.
2. *Discrete topology:* The discrete topology on a set S consists of all possible subsets of X. In this topology, every set is both open and closed.
3. *Standard topology on* $\mathbb{R}$*:* The metric $d(x, y) = |x - y|$ on $\mathbb{R}$ induces a topology where open sets are unions of open intervals. This is known as the standard topology on $\mathbb{R}$.

Definition 5.3.3. The *norm topology* induced by a norm $\|\cdot\|$ is the metrizable topology generated by the metric $d(x, y) = \|x - y\|$.

Definition 5.3.4. A *topological vector space* is a vector space V equipped with a linear τ such that:

1. every singleton $\{v\} \subset V$ is a closed set, and
2. the vector space operations (addition and scalar multiplication) are continuous with respect to τ. That is, the addition map $(x, y) \mapsto x + y$, from the Cartesian product $V \times V$ into V, is continuous, and the scalar multiplication map $(r, x) \mapsto rx$, from $\mathcal{F} \times V$ into V, is also continuous.

Definition 5.3.5. Let V be a vector space. Linear maps from V to its scalar field are called *linear functionals*. The set of all continuous linear functionals on V forms a vector space, called the *(topological) dual space* of V, and is denoted by V^*. It is common to designate elements of the dual space V^* by v^*.

Theorem 5.3.6. *[47, Theorem 4.3] Let V be a normed vector space. For each $v^* \in V^*$, define its norm by*

$$\|v^*\| := \sup\{|v^*(v)| : \|v\| = 1\}.$$

This defines a norm on V^* under which V^* is a Banach space. Moreover, for every $v \in V$, we have

$$\|v\| = \sup\{|v^*(v)| : \|v^*\| = 1\}.$$

As a consequence, the map $v^* \mapsto v^*(v)$ defines a bounded linear functional on V^*, and its norm equals $\|v\|$.

Definition 5.3.7. Let V be a vector space. The *weak*-topology* on the dual space V^* is the coarsest topology that makes all evaluation maps

$$v^* \mapsto v^*(v)$$

continuous for every $v \in V$.

Proposition 5.3.8. *[91, Proposition 2.3.10] Let $f : V \to W$ be a bounded linear map. Then its dual $f^* : V^* \to W^*$, defined by*

$$f^*(\varphi) = \varphi \circ f,$$

is also a bounded linear map.

5.4 W*-Algebras

5.5 Categories for (first-order) quantum computation

Given that the categories that "naturally arise" in quantum computation are first-order, in this section we restrict the λ-calculus to the first-order setting. Consequently, our models will be distributive symmetric monoidal Met-categories, that is, we do not require a Met-isomorphism

$$\mathsf{C}(B \otimes A, C) \cong \mathsf{C}(C, A \multimap B).$$

We begin by presenting the category CPTP of completely positive trace-preserving maps, which was shown in [13] to form a symmetric monoidal Met-category. In this section $\mathcal{M}_n$ will denote the vector space of complex $n \times n$ matrices.

Definition 5.5.1. The category CPTP is the category whose objects are natural numbers $n \geq 1$ and whose morphisms $n \to m$ are quantum channels $\mathcal{M}_n \to \mathcal{M}_m$.

A natural candidate for interpreting quantum programs with coproducts is the category CPTP$^+$, obtained via the coproduct cocompletion of CPTP. For CPTP$^+$ to serve as a suitable model for quantum computation, its morphisms should be able to express the measurement operation, *i.e.*, an operation mapping a density matrix $\rho \in \mathcal{M}_2$ to a classical bit $b \in \mathbb{C} \oplus \mathbb{C}$. However, this is not the case. Recall that in CPTP$^+$, a morphism consists of a pair $(f, (\phi_i)_{i \in I})$, where: $f : I \to J$ is a function between index sets, and $(\phi_i)_{i \in I}$ is a family of CPTP-morphisms $\phi_i : A_i \to B_{f(i)}$. Because f is a function, the category does not support morphisms of the form $(f, \phi_i : A \to C_i)_{i \in I}$ when I has more than one element. As a result, we do not have products, and consequently, measurements.
Given this negative result, we will shift our focus to a category that is known to possess both products (and therefore supports measures) and coproducts: Selinger's QPL model, the category Q [21].

Definition 5.5.2. The category Q is defined as the the finite biproduct completion of CP (which extends CPTP to include all completely positive maps), further restricted to trace-nonincreasing morphisms.

- An object is a signature $\sigma = n_1, \ldots, n_s$. We denote these signatures by the Greek letters σ, τ and μ.

- Given signatures $\sigma = n_1, \ldots, n_s$ and $\tau = m_1, \ldots, m_t$, a morphism $\Phi \in \sigma \to \tau$ is a matrix

$$\begin{pmatrix} \Phi_{11} & \ldots & \Phi_{s1}, \\ \vdots & \ddots & \vdots \\ \Phi_{1t} & \ldots & \Phi_{ts} \end{pmatrix}$$

of arrows $\Phi_{ij} : \mathcal{M}_{n_i} \to \mathcal{M}_{m_j}$ in CP which is trace-nonincreasing, *i.e.*, the following condition holds:

$$\sum_j \sum_i \mathsf{Tr}\left(\Phi_{ij}(A_i)\right) \leq \sum_i \mathsf{Tr}\left(A_i\right)$$

for all positive $A_i \in \mathcal{M}_{n_i}$.

More concretely, ij-component of Φ is given by the function $\Phi_{ij} = \pi_j \cdot \Phi \cdot \mathrm{in}_i : \mathcal{M}_{n_i} \to \mathcal{M}_{n_j}$, where in_i is the injection of $\mathcal{M}_{n_i}$ into the input space of Φ and π_j is the projection onto the j-th component.

Trace-non increasing maps are used here instead of strictly trace-preserving maps to account for possible non-termination in programs, particularly since the language QPL includes while loops.

Every signature σ, is associated with a complex vector space $\mathcal{M}_\sigma \coloneqq \mathcal{M}_{n_1} \oplus \cdots \oplus \mathcal{M}_{n_s}$

This space consists of matrix vectors

$$A = \begin{pmatrix} A_1 \\ \vdots \\ A_n \end{pmatrix}$$

where the signature σ specifies both the number of matrices, s, and their respective dimensions, $n_i \times n_i$. The elements of $\mathcal{M}_\sigma$ are represented uppercase letters such as A, B, etc. We also establish that $\mathbb{C}^\sigma \coloneqq \mathbb{C}^{n_1} \oplus \cdots \oplus \mathbb{C}^{n_s}$.

Note that attending to the definition of trace, for $A = \begin{pmatrix} A_1 & \ldots & A_n \end{pmatrix}^T$, we have $\mathsf{Tr}(A) = \sum_{i=1}^n \mathsf{Tr}(A_i)$.

Definition 5.5.3. *Product and Coproduct* The biproduct is given by the direct sum $\oplus$. Consequently, $\sigma \oplus \sigma'$ represents the concatenation of signatures. The co-pairing map $[\Phi, \Psi] : \sigma \oplus \sigma' \to \tau$ is defined as $[\Phi, \Psi](A, B) = \Phi(A) + \Psi(B)$, and the pairing map $\langle \Phi, \Psi \rangle : \sigma \to \tau \oplus \tau'$ is given by $\langle \Phi, \Psi \rangle(A) = (\Phi(A), \Psi(A))$.

Definition 5.5.4. *Tensor Product* For signatures $\sigma = n_1, \ldots, n_s$ and $\tau = m_1, \ldots, m_t$, the tensor product of σ and τ is defined as $\sigma \otimes \tau = n_1 m_1, \ldots, n_1 m_t, \ldots, n_s m_1, ..., n_s m_t$. The

morphism part of the tensor product follows the definition in the category of vector spaces. If $\Phi : \sigma \to \tau$ and $\Psi : \sigma' \to \tau'$, then their tensor product $\Psi \otimes \Phi : \sigma \otimes \sigma' \to \tau \otimes \tau'$ is defined on a basis element $A \otimes B$ by

$$(\Phi \otimes \Psi)(A \otimes B) = \Phi(A) \otimes \Psi(B),$$

and extends to arbitrary elements by linearity. More concretely, the tensor product $\Phi \otimes \Psi$ is the Kronecker product of their matrices representation, *i.e.*,

$$\Phi \otimes \Psi = \begin{pmatrix} \Phi_{11} \otimes \Psi_{11} & \dots & \Phi_{11} \otimes \Psi_{s'1} & \dots & \Phi_{s1} \otimes \Psi_{11} & \dots & \Phi_{s1} \otimes \Psi_{s'1} \\ \vdots & & & & & & \vdots \\ \Phi_{1t} \otimes \Psi_{1t'} & \dots & \Phi_{1t} \otimes \Psi_{s't'} & \dots & \Phi_{st} \otimes \Psi_{1t'} & \dots & \Phi_{st} \otimes \Psi_{s't'} \end{pmatrix}$$

Moreover, dist is an identity map:

$$(\sigma \oplus \sigma') \otimes \tau = (\sigma \otimes \tau) \oplus (\sigma' \otimes \tau)$$

The category Q is a distributive symmetric monoidal category with (finite) biproducts. However, this category is not closed [92].

We begin by applying the insights from Section 3.6 and considering the following metric on the pairing:

$$d\big(\langle \phi, \psi \rangle, \langle \phi', \psi' \rangle\big) \coloneqq \sup\big\{d(\phi, \phi'),\ d(\psi, \psi')\big\}.$$

For this to be a suitable metric, for Q-maps $\phi_1, \phi_2, \phi_1', \phi_2', \psi_1, \psi_2$ the following should hold:

$$d([\psi_1, \psi_2] \cdot \langle \phi_1, \phi_2 \rangle, [\psi_1, \psi_2] \cdot \langle \phi_1', \phi_2' \rangle) \leq d(\langle \phi_1, \phi_2 \rangle, \langle \phi_1', \phi_2' \rangle)$$

Considering $\phi_1 = \phi_2$, $\phi_1' = \phi_2'$, $\phi_1 \neq \phi_1'$ and $\psi_1 = \psi_2 = \mathrm{id}$, we have

$$\begin{aligned} & d([\psi_1, \psi_2] \cdot \langle \phi_1, \phi_2 \rangle, [\psi_1, \psi_2] \cdot \langle \phi_1', \phi_2' \rangle) \\ &= d(\psi_1 \cdot \phi_1 + \psi_2 \cdot \phi_2, \psi_1 \cdot \phi_1' + \psi_2 \cdot \phi_2') \\ &= d(2\phi_1, 2\phi_1') \end{aligned}$$

Futhermore, if the metric is induced by a norm, *i.e.* $d(\psi_1, \psi_2) = \|\psi_1 - \psi_2\|$, it follows that

$$\begin{aligned} & d(2\phi_1, 2\phi_1') \\ &= \|2\phi_1 - 2\phi_1'\| \\ &= 2 \cdot \|\phi_1 - \phi_1'\| \\ &= 2 \cdot d(\phi_1, \phi_1') > d(\phi_1, \phi_1') = \sup\{d(\phi_1, \phi_1'), d(\phi_2, \phi_2')\} = d(\langle \phi_1, \phi_2 \rangle, \langle \phi_1', \phi_2' \rangle) \end{aligned}$$

Thus, this metric is not a suitable candidate for making Q a Met-category.

Now let us consider the metric below

$$d\big(\langle\phi,\psi\rangle,\langle\phi',\psi'\rangle\big) \coloneqq d(\phi,\phi') + d(\psi,\psi').$$

For Q-maps $\phi, \phi', \phi_1, \phi_2$ the following should hold:

$$d(\langle\phi_1,\phi_2\rangle\cdot\phi, \langle\phi_1,\phi_2\rangle\cdot\phi') \leq d(\phi,\phi')$$

Considering $\phi_1 = \phi_2 = \mathrm{id}$ and $\phi \neq \phi'$, we have

$$\begin{aligned}
&d(\langle\phi_1,\phi_2\rangle\cdot\phi, \langle\phi_1,\phi_2\rangle\cdot\phi')\\
&= d(\langle\phi_1\cdot\phi, \phi_2\cdot\phi\rangle\langle\phi_1\cdot\phi', \phi_2\cdot\phi'\rangle)\\
&= d(\langle\phi,\phi\rangle,\langle\phi',\phi'\rangle)\\
&= d(\phi,\phi') + d(\phi,\phi')\\
&= 2\cdot d(\phi,\phi') > d(\phi,\phi')
\end{aligned}$$

Next, we will consider a metric between the other two, *i.e.*,

$$\sup\{d(\phi,\phi'), d(\psi,\psi')\} \leq d(\langle\phi,\psi\rangle,\langle\phi',\psi'\rangle) \leq d(\phi,\phi') + d(\psi,\psi').$$

In order to verify if this metric is valid, we must ensure that for Q-maps $\phi_1, \phi_2, \phi_1', \phi_2', \psi_1, \psi_2$ the following holds:

$$d([\psi_1,\psi_2]\cdot\langle\phi_1,\phi_2\rangle, [\psi_1,\psi_2]\cdot\langle\phi_1',\phi_2'\rangle) \geq \sup\{d(\phi_1,\phi_1'), d(\phi_2,\phi_2')\}.$$

Assuming that $\phi_2 = \phi_2'$, the metric is induced by a submultiplicative norm with respect to composition, and $\|\phi_1\| < 1$, we have

$$\begin{aligned}
&d([\psi_1,\psi_2]\cdot\langle\phi_1,\phi_2\rangle, [\psi_1,\psi_2]\cdot\langle\phi_1',\phi_2'\rangle)\\
&= d(\psi_1\cdot\phi_1 + \psi_2\cdot\phi_2, \psi_1\cdot\phi_1' + \psi_2\cdot\phi_2')\\
&= \|\psi_1(\phi_1 - \phi_1')\|\\
&\leq \|\psi_1\|\|\phi_1 - \phi_1'\|\\
&= \|\phi_1 - \phi_1'\|\\
&= d(\phi_1,\phi_1').
\end{aligned}$$

Consequently this norm is not fit either.

Given that our attempts to consider a simpler metric were not fruitful, we now turn our attention to the diamond norm (Definition 5.2.30). First, note the following inclusions, given signatures σ and τ:

$$\mathcal{M}_{n_1} \oplus \cdots \oplus \mathcal{M}_{n_s} \overset{\text{inc}}{\hookrightarrow} \mathcal{T}(\mathbb{C}_{n_1} \oplus \cdots \oplus \mathbb{C}_{n_s}), \quad \text{and} \quad \mathcal{M}_{\sigma\otimes\tau} \overset{\text{inc}}{\hookrightarrow} \mathcal{T}(\mathbb{C}_\sigma \otimes \mathbb{C}_\tau),$$

where the second inclusion arises from the fact that $\mathrm{dist} = \mathrm{id}$, and thus $\mathbb{C}_\sigma \otimes \mathbb{C}_\tau = \mathbb{C}_{\sigma\otimes\tau}$. Let us denote by $\mathcal{LS}(\mathcal{H}, \mathcal{K})$ the vector space of linear maps from $\mathcal{T}(\mathcal{H})$ to $\mathcal{T}(\mathcal{K})$. Given signatures $\sigma, \sigma', \tau, \tau'$ and super-operators $\Phi\colon \sigma \to \tau$, $\Psi\colon \sigma' \to \tau'$, we have:

$$\Phi : \mathcal{M}_\sigma \to \mathcal{M}_\tau \in \mathcal{LS}(\mathbb{C}_\sigma, \mathbb{C}_\tau) \quad \text{and} \quad \Phi\otimes\Psi : \mathcal{M}_{\sigma\otimes\sigma'} \to \mathcal{M}_{\tau\otimes\tau'} \in \mathcal{LS}(\mathbb{C}_\sigma\otimes\mathbb{C}_{\sigma'}, \mathbb{C}_\tau\otimes\mathbb{C}_{\tau'})$$

With these inclusions established, it is now clear that we can use the diamond norm on Q's morphisms.

Proposition 5.5.5. *For all $\Phi, \Phi' \in \mathsf{Q}(\sigma, \mu)$ and $\Psi, \Psi' \in \mathsf{Q}(\tau, \mu)$, it holds that*

$$\|[\Phi - \Phi', \Psi - \Psi']\|_\diamond \leq \sup\{\Phi - \Phi', \Psi - \Psi'\}.$$

Proof. Attending to the definition of the diamond norm, the fact that dist is an identity and Lemma 4.2.2 we compute:

$$\begin{aligned}
&\|[\Phi, \Psi]\|_{\diamond\,\mathsf{gen}} \\
&= \left\|[\Phi, \Psi] \otimes \mathrm{id}_{\mathcal{M}_{\sigma\oplus\tau}}\right\|_1 \\
&= \|[\Phi \otimes \mathrm{id}_{\mathcal{M}_\sigma}, \Psi \otimes \mathrm{id}_{\mathcal{M}_\tau}]\|_1 && (\mathrm{dist} = \mathrm{id}) \\
&\leq \sup\{\|\Phi \otimes \mathrm{id}_{\mathcal{M}_\sigma}\|_1, \|\Psi \otimes \mathrm{id}_{\mathcal{M}_\tau}\|_1\} && (\text{Lemma 4.2.2}) \\
&= \sup\left\{\|\Phi\|_\diamond, \|\Psi\|_\diamond\right\}
\end{aligned}$$

The inequality in Proposition 5.5.5 follows from the linearity of the co-pairing map. □

Corollary 5.5.6. *Let $\sigma : n_1, \ldots, n_s$ and $\tau : m_1, \ldots, m_t$ be signatures. Let $\Phi : \sigma \to \tau$ be a completely positive trace-nonincreasing super-operator.*

Proof. Given that Φ is a completely positive trace-nonincreasing super-operator, if follows

that $\Phi \otimes \mathrm{id}_{\mathcal{M}_\sigma}$ is a positive trace-nonincreasing super-operator. Let $\Psi = \Phi \otimes \mathrm{id}$, it holds that,

$$
\begin{aligned}
&\|\Phi\|_\diamond = \|\Psi\|_1 \\
&= \max\left\{\mathsf{Tr}\left(\Psi(uu^\dagger)\right) \mid \|u\|_2 = 1\right\} && \text{(Theorem 5.2.31)} \\
&= \max\left\{\sum_i \sum_j \mathsf{Tr}\left(\Psi_{ij}(u_i u_i^\dagger)\right) \;\middle|\; \left\|\left(u_1, \ldots, u_s^2\right)^T\right\|_2 = 1\right\} \\
&\leq \max\left\{\sum_i \mathsf{Tr}\left(u_i u_i^\dagger\right) \;\middle|\; \sqrt{\sum_i \|u_i\|_2^2} = 1\right\} && (\Psi \text{ is trace-nonincreasing}) \\
&= 1
\end{aligned}
$$

□

Proposition 5.5.7. *The category* Q *a symmetric monoidal* Met*-category.*

Proof. Here we follow the same reasoning as [13, Proof of Proposition 4.1].

First, we establish that Q is Met-enriched. By unpacking the relevant definitions, this reduces to proving the following: for all Q-morphisms $\Phi, \Phi' : \sigma \to \tau$ and $\Psi, \Psi' : \tau \to \mu$ the inequation $\|\Phi - \Phi'\|_\diamond + \|\Psi - \Psi'\|_\diamond \geq \|\Psi\Phi - \Psi'\Phi'\|_\diamond$ holds. We proceed as follows:

$$
\begin{aligned}
&\|\Phi - \Phi'\|_\diamond + \|\Psi - \Psi'\|_\diamond \\
&\geq \|(\Phi - \Phi')\Psi\|_\diamond + \|\Phi'(\Psi - \Psi')\|_\diamond && \text{(Proposition 5.2.32, and Corollary 5.5.6)} \\
&= \|\Phi\Psi - \Phi'\Psi\|_\diamond + \|\Phi'\Psi - \Phi'\Psi'\|_\diamond \\
&\geq \|\Phi\Psi - \Phi'\Psi + \Phi'\Psi - \Phi'\Psi'\|_\diamond && \{\text{Triangle inequality}\} \\
&= \|\Psi\Phi - \Psi'\Phi'\|_\diamond .
\end{aligned}
$$

Next, to prove that $\|\Phi - \Phi'\|_\diamond + \|\Psi - \Psi'\|_\diamond \geq \|\Psi \otimes \Phi - \Psi' \otimes \Phi'\|_\diamond$, we calculate

$$
\begin{aligned}
&\|\Psi - \Psi'\| + \|\Phi - \Phi'\| \\
&\geq \|\mathrm{id} \otimes (\Psi - \Psi')\|_\diamond + \|\mathrm{id} \otimes (\Phi - \Phi')\|_\diamond && \text{(Theorem 5.2.33 and Corollary 5.5.6)} \\
&= \|\mathrm{id} \otimes \Psi - \mathrm{id} \otimes \Psi'\|_\diamond + \|\mathrm{id} \otimes \Phi - \mathrm{id} \otimes \Phi'\|_\diamond \\
&\geq \|(\mathrm{id} \otimes \Psi) \cdot (\Phi \otimes \mathrm{id})\|_\diamond - \|(\mathrm{id} \otimes \Psi') \cdot (\Phi' \otimes \mathrm{id})\|_\diamond && \text{(Q is aMet-category)} \\
&= \|\Psi \otimes \Phi - \Psi' \otimes \Phi'\|_\diamond
\end{aligned}
$$

□

Theorem 5.5.8. Q *is a distributive symmetric monoidal* Met*-category.*

Proof. It follows directly from Proposition 5.5.5 and Proposition 5.5.7. □

5.6 Examples

We now illustrate the use of (first-order) λ-calculus with conditionals for describing quantum programs. To this effect, we first consider a type $\mathtt{qbit}$ of qubits, the basic unit of information in quantum computation. We then regard $\mathbb{I} \oplus \mathbb{I}$ to be the type of bits. Next we propound the following basic quantum operations: the conversion of a bit into a qubit, $q : \mathbb{I} \oplus \mathbb{I} \to \mathtt{qbit}$, the measurement of a qubit, *meas* $: \mathtt{qbit} \to \mathbb{I} \oplus \mathbb{I}$, and pre-determined sets of operations on n-qubits, *U* $: \mathtt{qbit}, \ldots, \mathtt{qbit} \to \mathtt{qbit}^{\otimes n}$ and *CPTP* $: \mathtt{qbit}, \ldots, \mathtt{qbit} \to \mathtt{qbit}^{\otimes n}$. The former includes unitary operations, as the Hadamard gate $H : \mathtt{qbit} \to \mathtt{qbit}$, the not-gate $X : \mathtt{qbit} \to \mathtt{qbit}$, and the cnot-gate $CNOT : \mathtt{qbit}, \mathtt{qbit} \to \mathtt{qbit}^{\otimes 2}$, and the latterr set included operations such as the dephasing with probability p, $D_p : \mathtt{qbit}, \mathtt{qbit} \to \mathtt{qbit}^{\otimes 2}$. We consider as well a pre-determined set of quantum states $|\psi\rangle : \mathbb{I} \to \mathtt{qbit}$.

Q forms a model of the metric λ-theory for quantum computation via the following interpretation: $[\![\mathbb{I}]\!] = \mathbb{C} \ni 1$, $[\![\mathtt{qbit}]\!] = \mathbb{C}^2$, $[\![q]\!]((a,b)) = \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$, for $\psi \in \{0,1\}$ we define $[\![|\psi\rangle]\!](1) = |\psi\rangle\langle\psi|$, $[\![\mathit{meas}]\!](\rho) = (\mathsf{Tr}(M_0 \rho M_0^\dagger), \mathsf{Tr}(M_1 \rho M_1^\dagger))$, for unitary operations U we define $[\![U]\!] = U\rho U^\dagger$. For completely positive trace-preserving operators *CPTP*, defined as $\mathit{CPTP}(\rho) = \sum_i K_i \rho K_i^\dagger$, we define $[\![\mathit{CPTP}]\!] = \mathit{CPTP}(\rho)$.

Let us now apply this machinery to two well-known problems in quantum computation and quantum information.

5.6.1 Quantum state discrimination

Example 5.6.1 (Coin-Toss)**.** In the quantum setting, tossing a "fair" coin can be described as preparing a qubit in a superposition of two states, $|0\rangle$ and $|1\rangle$, representing 'heads' and 'tails', each with an equal probability of 0.5 and then measuring it. This is achieved by simply applying a Hadamard gate to the initial state $|0\rangle$, followed by a measurement. More generally, tossing a coin (whether "fair" or "unfair") can be described as preparing a qubit in a superposition of $|0\rangle$ and $|1\rangle$, with probabilities p and $1-p$, respectively, and then measuring it. Considering $p = \cos(\theta/2)^2$ and the quantum gate $R_{y,\theta} : \mathtt{qbit} \to \mathtt{qbit}$, representing a single-qubit rotation by an angle θ around the y-axis, this process is described by the following λ-term:

$$\textbf{CoinToss} = - \triangleright \mathit{meas}(R_{y,\theta}(|0\rangle)) : \mathbb{I} \oplus \mathbb{I}$$

When running a quantum program on a real quantum computer, the quantum circuits are mapped to the hardware's native quantum gates during compilation. For instance consider 2020 IBM's native quantum gate set U_1, U_2, U_3, CX where

$$
\begin{aligned}
U_1(\lambda) &= \begin{pmatrix} 1 & 0 \\ 0 & e^{i\lambda} \end{pmatrix} \\
U_2(\phi, \lambda) &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -e^{i\lambda} \\ e^{i\phi} & e^{i(\phi+\lambda)} \end{pmatrix} \\
U_3(\theta, \phi, \lambda) &= \begin{pmatrix} \cos(\theta/2) & -e^{i\lambda}\sin(\theta/2) \\ e^{i\phi}\sin(\theta/2) & e^{i(\phi+\lambda)}\cos(\theta/2) \end{pmatrix}
\end{aligned}
$$

Here, $R_{y,\theta}$, can be expressed as $U_3(\theta, 0, 0)$. We now examine how the coin toss outcome is affected when the U_3 gate is faulty, particularly when its parameter θ is perturbed by an error ϵ. In this case, the implemented gate becomes $U_3(\theta + \epsilon, \phi, \lambda)$, *i.e.*, $R_{y,\theta+\epsilon}$. First, we compute the action of the unitary operator $U_3(\theta, \phi, \lambda)$ on an arbitrary quantum state $|\psi\rangle$.

$$
\begin{aligned}
U_3(\theta, \phi, \lambda)\,|\psi\rangle &= U_3(\theta, \phi, \lambda)\left(\cos(\alpha/2)\,|0\rangle + e^{i\beta}\sin(\alpha/2)\,|1\rangle\right) \\
&= \left(\cos(\alpha/2)\cos(\theta/2) - e^{i(\lambda+\beta)}\sin(\alpha/2)\sin(\theta/2)\right)|0\rangle \\
&+ \left(e^{i\phi}\cos(\alpha/2)\sin(\theta/2) + e^{i(\beta+\lambda+\phi)}\sin(\alpha/2)\cos(\theta/2)\right)|1\rangle
\end{aligned}
$$

Designating $U_3(\theta, \phi, \lambda)\,|\psi\rangle = a\,|0\rangle + b\,|1\rangle$, one has

$$
\begin{aligned}
aa^* &= |\cos(\alpha/2)\cos(\theta/2) - e^{i(\lambda+\beta)}\sin(\alpha/2)\sin(\theta/2)|^2 \\
&= \cos^2(\alpha/2)\cos^2(\theta/2) - 2\cos(\beta+\lambda)\cos(\alpha/2)\cos(\theta/2)\sin(\alpha/2)\sin(\theta/2) \\
&\quad + \sin(\alpha/2)^2\sin^2(\theta/2) \\
&= \cos^2(\alpha/2)\cos^2(\theta/2) + \sin^2(\alpha/2)\sin^2(\theta/2) - 1/2\cos(\beta+\lambda)\sin(\alpha)\sin(\theta) \\
&= \cos^2((\theta+\alpha)/2) + (1 - \cos(\beta+\lambda))\sin(\alpha)\sin(\theta) - 1 \\
a^*b &= \left(\cos(\alpha/2)\cos(\theta/2) - e^{-i(\lambda+\beta)}\sin(\alpha/2)\sin(\theta/2)\right)\left(e^{i\phi}\cos(\alpha/2)\sin(\theta/2)\right. \\
&\quad \left. + e^{i(\beta+\lambda+\phi)}\sin(\alpha/2)\cos(\theta/2)\right) \\
&= (1/2)\left(e^{i\phi}\cos^2(\alpha/2)\sin(\theta) + e^{i(\beta+\lambda+\phi)}\sin(\alpha)\cos^2(\theta/2) - e^{-i(\beta+\lambda-\phi)}\sin(\alpha)\sin^2(\theta/2)\right. \\
&\quad \left. - e^{i\phi}\sin^2(\alpha/2)\cos(\theta/2)\right) \\
&= (1/2)\left(e^{i\phi}\cos(\alpha)\sin(\theta) + \sin(\alpha)\left(e^{i(\beta+\lambda+\phi)}\cos^2(\theta/2) - e^{-i(\beta+\lambda-\phi)}\sin^2(\theta/2)\right)\right)
\end{aligned}
$$

Then, we calculate the vector Bloch of $U_3(\theta,\phi,\lambda)\,|\psi\rangle$,

$$
\begin{aligned}
x &= 2\mathsf{Im}\,(a^*b) = \cos(\phi)\cos(\alpha)\sin(\theta) + \sin(\alpha)\big(\cos\left(\beta+\lambda+\phi\right)\cos^2(\theta/2)\\
&\quad - \cos\left(\beta+\lambda-\phi\right)\sin^2(\theta/2)\big)\\
y &= 2\mathsf{Re}\,(a^*b) = \sin(\phi)\cos(\alpha)\sin(\theta) + \sin(\alpha)\big(\sin\left(\beta+\lambda+\phi\right)\cos^2(\theta/2)\\
&\quad + \sin\left(\beta+\lambda-\phi\right)\sin^2(\theta/2)\big)\\
z &= 2aa^* - 1 = 2\cos^2((\theta+\alpha)/2) + (1-\cos(\beta+\lambda))\sin(\alpha)\sin(\theta) - 1
\end{aligned}
$$

As a result, we have,

$$
\begin{aligned}
&\left\|U_3(\theta,0,0)\,|\psi\rangle\,\langle\psi|\,U_3(\theta,0,0)^\dagger - U_3(\theta+\epsilon,0,0)\,|\psi\rangle\,\langle\psi|\,U_3(\theta+\epsilon,0,0)^\dagger\right\|_1\\
&= \|(\cos(\alpha)(\sin(\theta)-\sin(\theta+\epsilon)) + \sin(\alpha)\cos\beta\,(\cos(\theta)-\cos(\theta+\epsilon))\,,0,\\
&\qquad 2(\cos^2((\theta+\alpha)/2) - \cos^2((\theta+\epsilon+\alpha)/2)) + \cos(\beta)\sin(\alpha)(\sin(\theta)-\sin(\theta+\epsilon)))\|_2\\
&\leq \|(\cos(\alpha)\epsilon + \sin(\alpha)\cos(\beta)\epsilon, 0, 2\epsilon + \sin(\alpha)\cos(\beta)\epsilon)\|_2\\
&= \sqrt{\epsilon^2(\cos^2(\alpha) + 2\sin^2(\alpha)\cos^2(\beta) + \sin(2\alpha)\cos(\beta) + 2\sin(\alpha)\cos(\beta) + 4)}\\
&\leq \epsilon\sqrt{(1+\sin^2(\alpha) + \sin(2\alpha) + 2\sin(\alpha) + 4)}\\
&\leq \epsilon\sqrt{(1+4+4)} = 3\epsilon
\end{aligned}
$$

The first inequality arises from both functions $\cos$ and $\sin$ being Lipschitz continuous. Attending to Theorem 5.2.34, it follows that

$$\|R_{y,\theta} - R_{y,\theta+\epsilon}\|_\diamondsuit \leq 3\epsilon$$

Using our metric deductive system, we can easily conclude that **CoinToss** $=_{3\epsilon}$ **CoinToss**$^\epsilon$, where **CoinToss**$^\epsilon$ is the the judgement that results from replacing $R_{y,\theta}$ by $U^{R_{y,\theta+\epsilon}}$.

Example 5.6.2 (Quantum state discrimination)**.** Quantum state discrimination is a pivotal challenge in quantum theory of communications [74, 93] and quantum cryptography [94]. While orthogonal states can be perfectly distinguished, the same does not apply to nonorthogonal states. In fact, even when the set of possible nonorthogonal states is known, determining the optimal discrimination strategy is considered a nontrivial problem.
The problem of quantum state discrimination can be naturally introduced through its connection with quantum communication. Consider two parties, Alice and Bob, who want to communicate with each other using a quantum channel. Alice chooses a state from a known

set $\{|\psi_i\rangle\}$, each occurring with a known probability p_i and sends it to Bob through the channel. Bob, who knows both the set of possible states and their associated probabilities, performs a suitable measurement to determine which state Alice sent. This scenario defines the quantum state discrimination problem: how to optimally distinguish between a known set of quantum states, each prepared with a known prior probability p_i.

When distinguishing between two pure states, the optimal measurement known as the *Helstrom measurement* is given by a projective measurement [93]. When operating within the computational basis, a projective measurement can be understood as the application of a unitary operator followed by a subsequent measurement in the computational basis. Thus the optimal measurement can be interpreted as a unitary transformation applied to the quantum state, followed by a measurement in the computational basis.

We will now show how to describe this discrimination task in λ-calculus. Consider two pure states $|\psi_0\rangle$ and $|\psi_1\rangle$, prepared *a priori* with probabilities p_0 and $p_1 = 1 - p_0$, respectively. Consider as well an operation $U : \texttt{qbit} \to \texttt{qbit}$ which corresponds to the basis-change associated with the optimal measurement. The relevant λ-terms are then:

$$\textbf{StatePreparation} = b : \mathbb{I} \oplus \mathbb{I} \triangleright \text{case}\, b \,\{\text{inl}_{\mathbb{B}}(x) \Rightarrow |\psi_0\rangle \,; \text{inr}_{\mathbb{A}}(y) \Rightarrow |\psi_1\rangle\} : \texttt{qbit}$$

$$\textbf{HMeasure} = x : \texttt{qbit} \triangleright \mathit{meas}(U(x)) : \mathbb{I} \oplus \mathbb{I}$$

$$\textbf{Discrimination} = - \triangleright \textbf{HMeasure}[\textbf{StatePreparation}[\textbf{CoinToss}(*)/b]/x] : \mathbb{I} \oplus \mathbb{I}$$

An arbitrary single qubit unitary $U \in \mathbb{C}^{2\times 2}$ may be written

$$U = e^{i\alpha} R_{z,\beta} R_{y,\gamma} R_{z,\delta},$$

for appropriate choices of angles α, β, γ and δ.

As in the previous example, we assume the hardware's native gate set consists of $\{U_1, U_2, U_3, \text{CNOT}\}$, and the quantum circuit is compiled into these gates. As previously noted, the $R_y(\theta)$ gate can be implemented as $U_3(\theta, 0, 0)$. Similarly, the $R_z(\lambda)$ gate is equivalent to $U_1(\lambda)$ up to a global phase factor $e^{-i\lambda/2}$, consequently, it can be directly implemented using this gate. We will also consider that the gates U_1 and U_3 are affected by errors ϵ_1 and ϵ_2, respectively. More precisely, we will consider erroneous implementations of this gates $U_1(\lambda+\epsilon_1)$ and $U_3(\theta+\epsilon_2, \phi, \lambda)$. From the previous example, we know that the error in the R_y gate is bounded by $3\epsilon_2$. Consider the single-qubit state

$$|\psi\rangle = \cos\left(\frac{\alpha}{2}\right)|0\rangle + e^{i\beta}\sin\left(\frac{\alpha}{2}\right)|1\rangle\,.$$

Applying the $U_1(\lambda)$ gate yields

$$U_1(\lambda)\left|\psi\right\rangle = \cos\left(\frac{\alpha}{2}\right)\left|0\right\rangle + e^{i(\beta+\lambda)}\sin\left(\frac{\alpha}{2}\right)\left|1\right\rangle.$$

The corresponding Bloch vector is then

$$(\cos(\beta+\lambda)\sin\alpha,\ \sin(\beta+\lambda)\sin\alpha,\ \cos\alpha).$$

Consequently, applying the same reasoning as in the previous example, it follows that

$$\begin{aligned}
&\left\|U_1(\lambda)\left|\psi\right\rangle\left\langle\psi\right|U_1(\lambda)^\dagger - U_1(\lambda+\epsilon)\left|\psi\right\rangle\left\langle\psi\right|U_1(\lambda+\epsilon)^\dagger\right\|_1\\
&= \left\|\left(\sin(\alpha)\left(\cos(\beta+\lambda)-\cos(\beta+\lambda+\epsilon)\right), \sin(\alpha)\left(\sin(\beta+\lambda)-\sin(\beta+\lambda+\epsilon)\right), 0\right)\right\|_2\\
&\leq \left\|\left(\cos(\beta+\lambda)-\cos(\beta+\lambda+\epsilon), \sin(\beta+\lambda)-\sin(\beta+\lambda+\epsilon), 0\right)\right\|_2\\
&\leq \left\|(\epsilon,\epsilon,0)\right\|_2\\
&= \sqrt{2}\epsilon
\end{aligned}$$

Attending to Theorem 5.2.34, it follows that

$$\left\|U_1(\lambda) - U_1(\lambda+\epsilon)\right\|_\diamond \leq \sqrt{2}\epsilon$$

As a result, considering the λ-term **HMeasure** and the erroneous implementation of U described above, denoted $U^{\epsilon_1,\epsilon_2}$, using our deductive metric system, we have $U =_{\sqrt{2}\epsilon_1+3\epsilon_2} U^{\epsilon_1,\epsilon_2}$. Therefore, we deduce **HMeasure** $=_{\sqrt{2}\epsilon_1+3\epsilon_2}$ **HMeasure**$^{\epsilon_1,\epsilon_2}$, where **HMeasure**$^{\epsilon_1,\epsilon_2}$ is the the judgement that results from replacing U by $U^{\epsilon_1,\epsilon_2}$. Moreover, considering the erroneous implementation of the R_y gate also afecting the **CoinToss** term, as discussed in the previous example, we deduce that

$$\textbf{Discrimination} =_{\sqrt{2}\epsilon_1+6\epsilon_2} \textbf{Discrimination}^{\epsilon_1,\epsilon_2},$$

where **Discrimination**$^{\epsilon_1,\epsilon_2}$ denotes the judgement that results from replacing **HMeasure** by **HMeasure**$^{\epsilon_1,\epsilon_2}$ and **CoinToss** by **CoinToss**$^{\epsilon_2}$.

5.6.2 Quantum teleportation protocol

Example 5.6.3 (Quantum teleportation)**.** [95] introduced the concept of quantum teleportation, a protocol that allows the transfer of unknown quantum states between distant parties.

The quantum teleportation protocol is a fundamental building block of quantum communication, quantum computation, and quantum networks, its applications ranging from secure quantum communication to distributed quantum computing [96–98].

Conceptually it can be described as follows: Alice and Bob share an entangled pair of qubits, specifically in a Bell state. Alice keeps the first qubit and Bob the second. Moreover, Alice has a qubit in an unknown state $|\psi\rangle$ that she wants to send to Bob. Alice entangles her qubit and the first qubit in the Bell state, and then measures both. The result of this measurement is two classical bits that Alice then sends to Bob though a classical channel. Based on the measurement results, Bob applies a correction to his qubit so it matches the initial state $|\psi\rangle$. The circuit corresponding to the implementation of quantum teleportation is depicted in Figure 7.

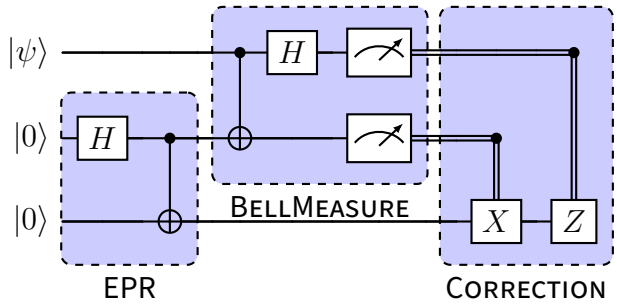


Figure 7: Quantum Teleportation Protocol

We first describe each of the rectangles filled in blue separately, and using standard quantum gate operations, namely $H : \texttt{qbit} \to \texttt{qbit}$, $X : \texttt{qbit} \to \texttt{qbit}$, $Z : \texttt{qbit} \to \texttt{qbit}$, and $\mathit{CNOT} : \texttt{qbit}, \texttt{qbit} \to \texttt{qbit} \otimes \texttt{qbit}$:

EPR $= \mathit{CNOT}(H\,|0\rangle\,, |0\rangle) : \texttt{qbit} \otimes \texttt{qbit}$

BellMeasure $= q_1 : \texttt{qbit}, q_2 : \texttt{qbit} \triangleright \mathsf{pm}\ \mathit{CNOT}(q_1, q_2)\ \mathsf{to}\ x \otimes y.$

$$\mathit{meas}(H(x)) \otimes \mathit{meas}(y) : (\mathbb{I} \oplus \mathbb{I}) \otimes (\mathbb{I} \oplus \mathbb{I})$$

Correction $= q : \texttt{qbit}, x : \mathbb{I} \oplus \mathbb{I}, y : \mathbb{I} \oplus \mathbb{I} \triangleright$

$$\mathsf{case}\ x \left\{ \begin{array}{l} \mathrm{inl}(x_0) \Rightarrow x_0\ \mathsf{to}\ *.\mathsf{case}\ y \left\{ \begin{array}{l} \mathrm{inl}(y_0) \Rightarrow y_0\ \mathsf{to}\ *.q; \\ \mathrm{inr}(y_1) \Rightarrow y_1\ \mathsf{to}\ *.X \end{array} \right\}; \\ \\ \mathrm{inr}(x_1) \Rightarrow x_1\ \mathsf{to}\ *.\mathsf{case}\ y \left\{ \begin{array}{l} \mathrm{inl}(y_0) \Rightarrow y_0\ \mathsf{to}\ *.Z(q); \\ \mathrm{inr}(y_1) \Rightarrow y_1\ \mathsf{to}\ *.Z(X(q)) \end{array} \right\} \end{array} \right\} : \texttt{qbit}$$

Designating the qubit to be teleported as qb_0, one then describes the teleportation procedure

in λ-calculus as follows:

$$\begin{aligned}\textbf{QTP} = qb_0 : \texttt{qbit} \triangleright \text{pm } \textbf{EPR} \text{ to } qb_1 \otimes qb_2.\\ \text{pm } \textbf{BellMeasure } [qb_0/q_1, qb_1/q_2] \text{ to } c_0 \otimes c_1.\\ \textbf{Correction } [qb_2/q, c_0/x, c_1/y] : \texttt{qbit}\end{aligned}$$

Following the approach of previous examples, we analyze erroneous implementations of the gates U_1 and U_3 within the hardware's native gate set. Additionally, we consider the action of both dephasing and amplitude damping channels. Furthermore, we account for an adversarial agent that applies a bit-flip operation immediately prior to measurement with probability $p = 0.5$.

Here, we consider imperfect implementations of the gates U_1 and U_3, given by $U_1(\lambda)$ and $U_3(\theta, \phi + \epsilon_2, \lambda + \epsilon_3)$, respectively. Recall from the previous example that we established the bound $\|U_1(\lambda) - U_1(\lambda + \epsilon)\|_\diamond \leq \sqrt{2}\epsilon$. The Hadamard gate, H, is the composition $U_3(\pi/2, 0, 0) \cdot U_1(\pi)$. We calculate,

$$\begin{aligned}&\left\|U_3(\pi/2,0,0)\left|\psi\right\rangle\left\langle\psi\right|U_3(\pi/2,0,0)^\dagger - U_3(\pi/2,\epsilon_2,\epsilon_3)\left|\psi\right\rangle\left\langle\psi\right|U_3(\pi/2,\epsilon_2,\epsilon_3)^\dagger\right\|_1\\ &= \|(\cos(\alpha)(1-\cos\epsilon_2) + \sin(\alpha)(1/2(\cos(\beta+\epsilon_2+\epsilon_3) - \cos(\beta-\epsilon_2+\epsilon_3))),\\ &\quad \cos(\alpha)(1-\sin\epsilon_2) + (1/2)\sin(\alpha)(\sin(\beta) - \sin(\beta+\epsilon_2+\epsilon_3)\\ &\quad + \sin(\beta) - \sin(\beta-\epsilon_2+\epsilon_3)), (\cos(\beta+\epsilon_3) - \cos(\beta))\sin(\alpha))\|_2\\ &\leq ||(\cos(\alpha)\epsilon_2 + (1/2)\sin(\alpha)(\epsilon_2+\epsilon_3), \cos(\alpha)\epsilon_2 + (1/2)\sin(\alpha)(\epsilon_2+\epsilon_3+|\epsilon_3-\epsilon_2|),\\ &\quad \sin(\alpha)\epsilon_3)||_2\\ &< \|(\epsilon_2 + (1/2)(\epsilon_2+\epsilon_3), \epsilon_2 + (1/2)(\epsilon_2+\epsilon_3+|\epsilon_3-\epsilon_2|), \epsilon_3)\|_2\\ &\leq 3\epsilon_2 + 2\epsilon_3 + |\epsilon_2 - \epsilon_3|\end{aligned}$$

Attending to Theorem 5.2.34, it follows that

$$\|U_3(\pi/2,0,0) - U_3(\pi/2,\epsilon_2,\epsilon_3)\|_\diamond \leq 3\epsilon_2 + 2\epsilon_3 + |\epsilon_2 - \epsilon_3|$$

As a result, denoting the imperfect implementation of the Hadamard gate as $H^{\epsilon_1,\epsilon_2,\epsilon_3}$, we have

$$H =_{\sqrt{2}\epsilon_1 + 3\epsilon_2 + 2\epsilon_3 + |\epsilon_2 - \epsilon_3|} H^{\epsilon_1,\epsilon_2,\epsilon_3}. \tag{5.6}$$

The gate X can be implemented as $U_3(\pi, \psi, 0)$. We compute,

$$\begin{aligned}
&\left\| U_3(\pi,0,\pi)\left|\psi\right\rangle\left\langle\psi\right| U_3(\pi,0,\pi)^\dagger - U_3(\pi,\epsilon_2,\pi+\epsilon_3)\left|\psi\right\rangle\left\langle\psi\right| U_3(\pi,\epsilon_2,\pi+\epsilon_3)^\dagger \right\|_1 \\
&= \|(\sin(\alpha)(\cos(\beta)+\cos(\beta+\epsilon_3-\epsilon_2)), \sin(\alpha)(\sin(\beta+\epsilon_3-\epsilon_2)-\sin(\beta)), 0)\|_2 \\
&\leq \|(|\epsilon_3-\epsilon_2|, |\epsilon_3-\epsilon_2|, 0)\|_2 \\
&= \sqrt{2}|\epsilon_3-\epsilon_2|
\end{aligned}$$

Considering Theorem 5.2.34, it holds that

$$\|U_3(\pi,0,\pi) - U_3(\pi,\epsilon_2,\pi+\epsilon_3)\|_\diamondsuit \leq \sqrt{2}|\epsilon_3-\epsilon_2|$$

As a result, denoting the erroneous implementation of the X gate as $X^{\epsilon_2,\epsilon_3}$, we have

$$X =_{\sqrt{2}|\epsilon_3-\epsilon_2|} X^{\epsilon_2,\epsilon_3}. \tag{5.7}$$

Finally, the gate Z corresponds to $U_1(\pi)$, therefore, denoting the erroneous implementation of the X gate as X^{ϵ_1}, we postulate the following axiom

$$Z =_{\sqrt{2}\epsilon_1} Z^{\epsilon_1}. \tag{5.8}$$

Designating the **Correction** block with the imperfect implementations of X and Z by **Correction**$^{\epsilon_1,\epsilon_2,\epsilon_3}$, in light of the axioms in equations (5.7) and (5.7) and our metric deductive system we have that

$$\textbf{Correction} =_{\sqrt{2}(\epsilon_1+|\epsilon_3-\epsilon_2|)} \textbf{Correction}^{\epsilon_1,\epsilon_2,\epsilon_3}. \tag{5.9}$$

Dephasing channel

Realistic quantum systems are never isolated, but are immersed in the surrounding environment and interact continuously with it. Decoherence can be seen as the consequence of that 'openness' of quantum systems to their environments . To study decoherence in a quantum channel within the presented metric deductive system, one can consider the application of a dephasing channel in the quantum teleportation protocol with a certain probability p.

The Kraus operators of the dephasing channel with probability p are expressed as:

$$D_0 = \frac{\sqrt{2-p}}{\sqrt{2}} I, D_1 = \frac{\sqrt{p}}{\sqrt{2}} Z$$

Considering a density operator $\rho = |\alpha|^2 \left|0\right\rangle\left\langle 0\right| + \alpha\beta^* \left|0\right\rangle\left\langle 1\right| + \alpha^*\beta \left|1\right\rangle\left\langle 0\right| + |\beta|^2 \left|1\right\rangle\left\langle 1\right|$, using these Kraus operators, it is possible to easily verify that after applying the dephasing channel with probability p, the resulting operator ρ' is given by:

$$\rho' = D_p(\rho) = D_0\rho D_0^\dagger + D_1\rho D_1^\dagger = |\alpha|^2 \left|0\right\rangle\left\langle 0\right| + (1-p)\alpha\beta^* \left|0\right\rangle\left\langle 1\right| + (1-p)\alpha^*\beta \left|1\right\rangle\left\langle 0\right| + |\beta|^2 \left|1\right\rangle\left\langle 1\right|$$

This shows that the dephasing channel with probability p preserves the diagonal elements of the density matrix while attenuating the off-diagonal elements by a factor of $(1-p)$.
In this scenario (and in subsequent ones), we will add identity gates to the ideal program to simplify the calculations. Thus, attending to the definition of trace norm for matrices and Equation 5.1, we have:

$$
\begin{aligned}
&\|\mathrm{id}(\rho) - D_p(\rho)\|_1 \\
&\|\alpha\beta^* |0\rangle\langle 1| + \alpha^*\beta |1\rangle\langle 0| - (1-p)\alpha\beta^* |0\rangle\langle 1| - (1-p)\alpha^*\beta |1\rangle\langle 0|\|_1 \\
&= p \cdot \|\alpha\beta^* |0\rangle\langle 1| + \alpha^*\beta |1\rangle\langle 0|\|_1 \\
&= p \cdot \mathsf{Tr}\left(\sqrt{(\alpha\beta^* |0\rangle\langle 1| + \alpha^*\beta |1\rangle\langle 0|)^2}\right) \\
&= p \cdot \mathsf{Tr}\left(\sqrt{|\alpha|^2|\beta|^2(|0\rangle\langle 0| + |1\rangle\langle 1|)}\right) \\
&= 2 \cdot p \cdot |\alpha||\beta| \\
&\leq p
\end{aligned}
$$

The last step arises from the fact that the expression is maximized when $|\alpha| = |\beta| = 1/\sqrt{2}$. Considering Theorem 5.2.35, it holds that

$$\|\mathrm{id} - D_p\|_\diamond \leq \sqrt{2p}$$

Consequently, we can postulate the following axiom:

$$\mathrm{id} =_{\sqrt{2p}} D_p. \tag{5.10}$$

If a dephasing channel acts on the first qubit of the EPR state, we are interested in reasoning about the following judgements:

$$\mathbf{EPR} = (\mathrm{id} \otimes \mathrm{id})(\mathit{CNOT}(H|0\rangle, |0\rangle)) : \mathtt{qbit} \otimes \mathtt{qbit}$$

$$\mathbf{EPR}^{\epsilon_1,\epsilon_2,\epsilon_3,p} = (D_p \otimes \mathrm{id})(\mathit{CNOT}(H^{\epsilon_1,\epsilon_2,\epsilon_3}|0\rangle, |0\rangle)) : \mathtt{qbit} \otimes \mathtt{qbit}$$

Given axioms in equations (5.6) and (5.10), using our metric deductive system, we infer that

$$\mathbf{EPR} =_{\sqrt{2}\epsilon_1+3\epsilon_2+2\epsilon_3+|\epsilon_2-\epsilon_3|+\sqrt{2p}} \mathbf{EPR}^{\epsilon_1,\epsilon_2,\epsilon_3,p} \tag{5.11}$$

Amplitude Dephasing channel

Next, the amplitude-damping channel is considered as a source of noise in the quantum teleportation protocol. Similarly to the dephasing channel, the amplitude damping channel

serves as a model illustrating the dissipation of energy between a qubit and its environment. An example of this type of noise is found in the spontaneous emission of a photon by a two-level atom into an electromagnetic field environment with either a finite or infinite number of modes at zero temperature [99, 100].

The amplitude damping channel with probability γ is described by the Kraus operators:

$$A_0 = |0\rangle\langle 0| + \sqrt{1-\gamma}\,|1\rangle\langle 1|\,, A_1 = \sqrt{\gamma}\,|0\rangle\langle 1|$$

Applying these Kraus operators an arbitray density operator $\rho = |\alpha|^2\,|0\rangle\,|0\rangle + \alpha\beta^*\,|0\rangle\,|1\rangle + \alpha^*\beta\,|1\rangle\,|0\rangle + |\beta|^2\,|1\rangle\,|1\rangle$, we obtain the state ρ' as follows:

$$\begin{aligned}\rho' &= A_\gamma(\rho) = A_0\rho A_0^\dagger + A_1\rho A_1^\dagger \\ &= (|\alpha|^2 + \gamma|\beta|^2)\,|0\rangle\langle 0| + \sqrt{1-\gamma}\alpha\beta^*\,|0\rangle\langle 1| + \sqrt{1-\gamma}\alpha^*\beta\,|1\rangle\langle 0| + (1-\gamma)|\beta|^2\,|1\rangle\langle 1|\end{aligned}$$

Once again, we will add identity gates to the ideal program to simplify the calculations, as a result it is necessary to compute the trace norm of the diference between the identity applied to the density operator $\rho = |\psi\rangle\langle\psi|$ and the amplitude damping channel applied to ρ. First, attending to the definition of trace norm for matrices and Equation 5.1, we calculate,

$$\begin{aligned}&\|\mathrm{id}(\rho) - A_\gamma(\rho)\|_1 \\ &= \Big\| |\alpha|^2\,|0\rangle\,|0\rangle + \alpha\beta^*\,|0\rangle\,|1\rangle + \alpha^*\beta\,|1\rangle\,|0\rangle + |\beta|^2\,|1\rangle\,|1\rangle - \big((|\alpha|^2 + \gamma|\beta|^2)\,|0\rangle\langle 0| \\ &\quad + \sqrt{1-\gamma}\alpha\beta^*\,|0\rangle\langle 1| + \sqrt{1-\gamma}\alpha^*\beta\,|1\rangle\langle 0| + (1-\gamma)|\beta|^2\,|1\rangle\langle 1|\,\big)\Big\|_1 \\ &= \Big\| \gamma|\beta|^2\,|0\rangle\langle 0| + (1-\sqrt{1-\gamma})(\alpha\beta^*\,|0\rangle\langle 1| + \alpha^*\beta\,|1\rangle\langle 0|) - \gamma|\beta|^2\,|1\rangle\langle 1|\Big\|_1 \\ &= \mathrm{Tr}\left(\sqrt{\Big(\gamma|\beta|^2\,|0\rangle\langle 0| + (1-\sqrt{1-\gamma})(\alpha\beta^*\,|0\rangle\langle 1| + \alpha^*\beta\,|1\rangle\langle 0|) - \gamma|\beta|^2\,|1\rangle\langle 1|\Big)^2}\right) \\ &= \mathrm{Tr}\left(\sqrt{\Big((1-\sqrt{1-\gamma})^2|\alpha|^2|\beta|^2 + \gamma^2|\beta|^4\Big)\,(|0\rangle\langle 0| + |1\rangle\langle 1|)}\right) \\ &= 2\cdot\sqrt{(1-\sqrt{1-\gamma})^2|\alpha|^2|\beta|^2 + \gamma^2|\beta|^4} \\ &\leq 2\gamma\end{aligned}$$

This final step follows because the expression attains its maximum when $|\alpha| = 0$ and $|\beta| = 0$. Attending to Theorem 5.2.35, it holds that

$$\|\mathrm{id} - A_\gamma\|_\diamond \leq 2\sqrt{\gamma}$$

As a result, we can postulate the following axiom:

$$\mathrm{id} =_{2\sqrt{\gamma}} A_\gamma. \tag{5.12}$$

When an amplitude damping channel acts on the final qubit following the Correction block, we define two new lambda terms consisting of the ideal operation **Id** and its erroneous counterpart $\mathbf{Id}^\gamma$.

$$\mathbf{Id} = qb : \texttt{qbit} \triangleright \mathrm{id}(qb) \tag{5.13}$$

$$\mathbf{Id}^\gamma = A_\gamma(q) : \texttt{qbit} \triangleright A_\gamma(qb) \tag{5.14}$$

Consequently the ideal version of teleportation protocol is now defined as follows

$$\begin{aligned}\mathbf{QTP} = qb_0 : \texttt{qbit} \triangleright\ & \text{pm } \mathbf{EPR} \text{ to } qb_1 \otimes qb_2. \\ & \text{pm } \mathbf{BellMeasure}\,[qb_0/q_1, qb_1/q_2] \text{ to } c_0 \otimes c_1. \\ & \mathbf{Id}\,[\mathbf{Correction}/qb]\,[qb_2/q, c_0/x, c_1/y] : \texttt{qbit}\end{aligned}$$

Considering the axiom in equation (5.12) and our metric deductive system, it holds that

$$\mathbf{Id} =_{2\sqrt{\gamma}} \mathbf{Id}^\gamma$$

Malicious attack

Finally, consider a malicious attack on the quantum teleportation protocol in the form of a bit-flip occurring with a 50% probability before measurement. More generally, one can define an operation T that applies a unitary operation U to the state given as input with 50% probability. Operation T can be defined as follows:

$$T : \ \texttt{qbit} \to \texttt{qbit}$$

$$T = q : \texttt{qbit} \triangleright \text{pm}\, CU(R_{x,\frac{\pi}{2}}(|0\rangle), q) \text{ to } newq \otimes qb.\, \mathit{disc}(newq)$$

Here, CU denotes the controlled operation that applies U to the second qubit when the first qubit is in the state $|1\rangle\langle 1|$, and leaves it unchanged when the first qubit is in the state $|0\rangle\langle 0|$. The operator $R_{x,\frac{\pi}{2}}$ represents a rotation by $\frac{\pi}{2}$ around the x-axis of the Bloch sphere.
This operation is depicted in Figure 8.

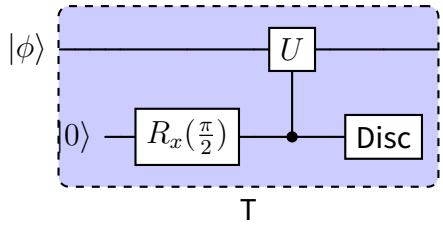


Figure 8: T operation

First, let us verify the result of applying operation T to a quantum state $\rho = |\psi\rangle\langle\psi|$:

$$
\begin{aligned}
& |\psi\rangle\langle\psi| \\
\xmapsto{\mathrm{id}\otimes[\![|0\rangle]\!]} \quad & |\psi\rangle\langle\psi| \otimes |0\rangle\langle 0| \\
\xmapsto{\mathrm{id}\otimes\left[\!\left[R_{x,\frac{\pi}{2}}\right]\!\right]} \quad & |\psi\rangle\langle\psi| \otimes \frac{1}{2}\left(|0\rangle\langle 0| - i\,|0\rangle\langle 1| + i\,|1\rangle\langle 0| + |1\rangle\langle 1|\right) \\
& = \frac{1}{2}\left(|\psi\rangle\langle\psi|\,|0\rangle\langle 0| - i\,|\psi\rangle\langle\psi|\,|0\rangle\langle 1| + i\,|\psi\rangle\langle\psi|\,|1\rangle\langle 0| + |\psi\rangle\langle\psi|\,|1\rangle\langle 1|\right) \\
\xmapsto{[\![\mathsf{CU}]\!]} \quad & \frac{1}{2}\left(|\psi\rangle\langle\psi|\,|0\rangle\langle 0| - i\,|\psi\rangle\langle\psi|\,|0\rangle\langle 1|\,U^\dagger + iU\,|\psi\rangle\langle\psi|\,|1\rangle\langle 0| + U\,|\psi\rangle\langle\psi|\,|1\rangle\langle 1|\,U^\dagger\right) \\
\xmapsto{\mathrm{id}\otimes\mathsf{Tr}} \quad & \frac{1}{2}\left(|\psi\rangle\langle\psi| + U\,|\psi\rangle\langle\psi|\,U^\dagger\right)
\end{aligned}
$$

Considering X as U, we compute

$$
\begin{aligned}
& \|\mathrm{id}(\rho) - T(\rho)\|_1 \\
& = \Big|\Big|(1/2)\Big(\left(|\alpha|^2 - |\beta|^2\right)|0\rangle\langle 0| + \left(\alpha\beta^* - \alpha^*\beta\right)|0\rangle\langle 1| + \left(\alpha^*\beta - \alpha\beta^*\right)|0\rangle\langle 1| \\
& \quad + \left(|\beta|^2 - |\alpha|^2\right)|1\rangle\langle 1|\Big)\Big|\Big|_1 \\
& = (1/2)\mathsf{Tr}\Big(\sqrt{\left(\left(|\alpha|^2 - |\beta|^2\right)|0\rangle\langle 0| + \left(\alpha\beta^* - \alpha^*\beta\right)|0\rangle\langle 1| + \left(\alpha^*\beta - \alpha\beta^*\right)|0\rangle\langle 1|\right.} \\
& \quad \overline{\left.+\left(|\beta|^2 - |\alpha|^2\right)|1\rangle\langle 1|\right)^2}\Big) \\
& = (1/2)\mathsf{Tr}\left(\sqrt{\left(\left(|\alpha|^2 - |\beta|^2\right)^2 + \left(\alpha\beta^* - \alpha^*\beta\right)\left(\alpha^*\beta - \alpha\beta^*\right)\right)\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right)}\right) \\
& = (1/2)\mathsf{Tr}\left(\sqrt{\left(\left(|\alpha|^2 - |\beta|^2\right)^2 + 2|\alpha|^2|\beta|^2 - (\alpha^*)^2\beta^2 - (\beta^*)^2\alpha^2\right)\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right)}\right) \\
& = (1/2)\mathsf{Tr}\left(\sqrt{\left(\left(|\alpha|^4 + |\beta|^4 - 2\mathsf{Re}(\alpha\beta)\right)\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right)\right)}\right) \\
& = \sqrt{|\alpha|^4 + |\beta|^4 - 2\mathsf{Re}(\alpha\beta)} \\
& \leq 1
\end{aligned}
$$

This last step holds because the expression is maximized when the imaginary part of α or β is 1.

Considering Theorem 5.2.35, it holds that

$$\|\mathrm{id} - T\|_{\diamond} \leq \sqrt{2}$$

Consequently, we postulate the following axiom:

$$\mathrm{id} =_{\sqrt{2}} T. \tag{5.15}$$

In this case we reason about the following λ-terms:

$$\textbf{BellMeasure} = q_1 : \mathtt{qbit}, q_2 : \mathtt{qbit} \triangleright \mathsf{pm}\ \mathit{CNOT}(q_1, q_2)\ \mathsf{to}\ x \otimes y.\ \mathit{meas}(\mathit{H}(x)) \otimes \mathit{meas}(y) : (\mathbb{I} \oplus \mathbb{I}) \otimes (\mathbb{I} \oplus \mathbb{I})$$

$$\textbf{BellMeasure}^{\epsilon_1,\epsilon_2,\epsilon_3,T} = q_1 : \mathtt{qbit}, q_2 : \mathtt{qbit} \triangleright \mathsf{pm}\ \mathit{CNOT}(q_1, q_2)\ \mathsf{to}\ x \otimes y.\ \mathit{meas}(T(\mathit{H}^{\epsilon_1,\epsilon_2,\epsilon_3}(x))) \otimes \mathit{meas}(T(y)) : (\mathbb{I} \oplus \mathbb{I}) \otimes (\mathbb{I} \oplus \mathbb{I})$$

Attending to the axioms in equations (5.6) and (5.15), via our metric deductive system, we infer that

$$\textbf{BellMeasure} =_{2\sqrt{2}+\sqrt{2}\epsilon_1+3\epsilon_2+2\epsilon_3+|\epsilon_2-\epsilon_3|} \textbf{BellMeasure}^{\epsilon_1,\epsilon_2,\epsilon_3,T} \tag{5.16}$$

Lastly, designating the judgment **QTP** with the erroneous implementations of **EPR**, **BellMeasure**, **Correction**, **Id**, by $\textbf{QTP}^{\epsilon_1,\epsilon_2,\epsilon_3,T,p,\gamma}$, given equations (5.11), (5.16), (5.8), and (5.13), using our deductive metric system, it follows that

$$\textbf{QTP} =_{3\sqrt{2}\epsilon_1+6\epsilon_2+4\epsilon_3+(2+\sqrt{2})|\epsilon_2-\epsilon_3|+2\sqrt{2}+\sqrt{2p}+2\sqrt{\gamma}} \textbf{QTP}^{\epsilon_1,\epsilon_2,\epsilon_3,T,p,\gamma}$$

Chapter 6

Future work

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