University of Minho
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A Metric Equational System for Quantum Computation

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University of Minho, Braga, August 2024

Bruna Filipa Martins Salgado

Abstract

Noisy intermediate-scale quantum (NISQ) computers are expected to operate with severely limited hardware resources. Precisely controlling qubits in these systems comes at a high cost, is susceptible to errors, and faces scarcity challenges. Therefore, error analysis is indispensable for the design, optimization, and assessment of NISQ computing. Nevertheless, the analysis of errors in quantum programs poses a significant challenge. The overarching goal of the M.Sc. project is to provide a fully-fledged quantum programming language on which to study metric program equivalence in various scenarios, such as in quantum algorithmics and quantum information theory.

Keywords approximate equivalence, λ-calculus, metric equations

Resumo

Escrever aqui o resumo (pt)

Palavras-chave palavras, chave, aqui, separadas, por, vírgulas

Contents

List of Figures

List of Tables

Chapter 1

Introduction

1.1 Motivation and Context

Quantum computing dates back to 1982 when Nobel laureate Richard Feynman proposed the idea that constructing computers founded on the principles of quantum mechanics could efficiently simulate quantum systems of interest to physicists, whereas this seemed to be very difficult with classical computers [Feynman (1982)].

This paradigm holds immense promise, as evidenced by several compelling results in computational complexity theory [Shor (1994); Grover (1996)]. While hardware advancements have brought the scientific community closer to realizing this potential, the ultimate goal the ultimate goal is yet to be accomplished. A NISQ quantum computer equipped with 50-100 qubits may surpass the capabilities of current classical computers, yet the impact of quantum noise, such as decoherence in entangled states, imposes limitations on the size of quantum circuits that can be executed reliably [Preskill (2018)]. Unfortunately, general-purpose error correction techniques [Calderbank and Shor (1996); Gottesman (1997); Steane (1996)] consume a substantial number of qubits, making it difficult for NISQ devices to make use of them in the near term. For instance, the implementation of a single logical qubit may require between 10^3 and 10^4 physical qubits [Fowler et al. (2012)].

To reconcile quantum computation with NISQ computers, quantum compilers perform transformations for error mitigation [Wallman and Emerson (2016)] and noise-adaptive optimization [Murali et al. (2019)]. Additionally, current quantum computers only support a restricted, albeit universal, set of quantum operations. As a result, nonnative operations must be decomposed into sequences of native operations before execution [Harrow et al. (2002), Burgholzer and Wille (2020)]. In general, perfect computational universality is not sought, but only the ability to approximate any quantum algorithm, with a preference for minimizing the use of additional gates beyond the original requirements. The assessment of these compiler transformations necessitates a comparison of the error bounds between the source and compiled quantum programs. Furthermore, in quantum information theory, the concept of an $\epsilon -$ approximation channel is fundamental when studying quantum teleportation via noisy channels [Watrous (2018)]. This suggests the development of appropriate notions of approximate program equivalence, *in lieu* of the classical program equivalence and underlying theories that typically hinge on the idea that equivalence is binary, *i.e.* two programs are either equivalent or they are not [Winskel (1993)].

As previously noted, Shor's and Grover's algorithms have played a pivotal role in sparking heightened interest within the scientific community toward quantum computing research. On these bases, various endeavors to establish quantum programming languages have surfaced over the past 20 years. These include imperative languages such as Qiskit [Qiskit contributors (2023)] and Silq [Bichsel et al. (2020)], as well as functional languages such as Quipper [Green et al. (2013)] and Q# [Svore et al. (2018)]. On one hand, the design of quantum programming languages is strongly oriented towards implementing quantum algorithms. On the other hand, the definition of functional paradigmatic languages or functional calculi serves as a valuable tool for delving into theoretical aspects of quantum computing, particularly exploring the foundational basis of quantum computation [Zorzi (2016)]. Given the nature of this work, the focus will be on quantum languages designed with this latter aspect in mind.

QPL, a quantum language within the functional programming paradigm, marks a significant milestone in this context [Selinger (2004)]. This is a first-order functional language featuring a static type system based on the idea of classical control and quantum data.

Most of the current research on algorithms and programming languages assumes that addressing the challenge of noise during program execution will be resolved either by the hardware or through the implementation of fault-tolerant protocols designed independently of any specific application [Chong et al. (2017)]. As previously stated, this assumption is not realistic in the NISQ era. Nonetheless, there have been efforts to address the challenge of approximate program equivalence in the quantum setting. [Hung et al. (2019)] and [Tao et al. (2021)] reason about the issue of noise in a quantum while-language by developing a deductive system to determine how similar a quantum program is from its idealised, noise-free version. An alternative approach was explored in [Dahlqvist and Neves (2022)], using linear

λ-calculus as basis – *i.e* programs are written as linear λ-terms – which has deep connections to both logic and category theory [Girard et al. (1995), Benton (1994)]. Some positive results were achieved in this setting, but much remains to be done.

1.2 Goals

The notion of approximate equivalence for quantum programming explored in [Dahlqvist and Neves (2022)] does not take important operations into account. Specifically, the corresponding mathematical model does not include measurements, classical control flow, or discard operations. Also, the corresponding typing system is often times too strict and cannot properly handle multiple uses of the same resource, such as sampling exactly n-times from a distribution. The overarching goal of this M.Sc. project is to tackle the aforementioned limitations. A successful completion of this goal will provide a fully-fledged quantum programming language on which to study metric program equivalence in various scenarios. This includes not only quantum algorithmics – where, for example, the number of iterations in Grover's algorithm involves approximations – but also quantum information theory, where, for instance, quantum teleportation and the problem of the discrimination of quantum states have important roles [Nielsen and Chuang (2010)].

Chapter 2

Background

2.1 Affine Lambda Calculus

The Lambda-Calculus, developed by Church and Curry in the 1930s, serves as a formal language capturing the key attribute of higher-order functional languages, treating functions as first-class citizens, allowing them to be passed as arguments [Barendregt et al. (1984)]. Beyond its foundational aspects, this calculus incorporates extensions for modeling side effects, including probabilistic or non-deterministic behaviors and shared memory. Centered around the expression of higher-order functions, where functions can serve as inputs or outputs, it emerges as a potent computational tool. Higher-order functions form a pivotal abstraction in practical programming languages such as LISP, Scheme, ML, and Haskell.
In quantum information theory, the role of higher-order functions encompasses two fundamental aspects. The first involves the concept of entangled functions and how well-known quantum phenomena find natural descriptions through such functions. The second concerns the interplay between classical objects and quantum objects in a higher-order context. Quantum computation conventionally handles classical and quantum data, while the higher-order context introduces a third data type: functions. These functions fall into two categories - those ”quantum-like” (entangled, single-use) and those ”classical-like” (duplicable, reusable). Remarkably, this classification transcends input/output types, highlighting the coexistence of quantum-like functions operating on classical data and classical-like functions operating on quantum data. [Selinger et al. (2009)].

2.1.1 Syntax

The grammar and term formation rules of the affine lambda calculus, discussed in [Dahlqvist and Neves (2022)], are presented in this subsection.

The definition of the grammar for affine lambda calculus is as follows, where G represents a set of ground types.

$$\mathbb{A} ::= X \in G \mid \mathbb{I} \mid \mathbb{A} \otimes \mathbb{A} \mid \mathbb{A} \oplus \mathbb{A} \mid \mathbb{A} \multimap \mathbb{A} \tag{2.1}$$

Regarding the term formation rules, Σ corresponds to a class of sorted operation symbols $f : \mathbb{A}_1, \ldots, \mathbb{A}_n \to \mathbb{A}$, where $n \geq 1$. Typing contexts are represented as lists $x_1 : \mathbb{A}_1, \ldots, x_n : \mathbb{A}_n$ of typed variables, with each variable x_i (where $1 \leq i \leq n$) occurring at most once in $x_1, \ldots, x_n$. The typing contexts are denoted by greek letters Γ, Δ, and E. The concept of shuffling is employed to construct a linear typing system that ensures the admissibility of the exchange rule and enables unambiguous reference to judgment's denotations $[\![\Gamma \triangleright v : \mathbb{A}]\!]$. Shuffling is defined as a permutation of typed variables in a sequence of contexts, $\Gamma_1, \ldots, \Gamma_n$, preserving the relative order of variables within each Γ_i. For instance, if $\Gamma_1 = x : \mathbb{A}, y : \mathbb{B}$ and $\Gamma_2 = z : \mathbb{C}$, then $z : \mathbb{C}, x : \mathbb{A}, y : \mathbb{B}$ is a valid shuffle of Γ_1, Γ_2. On the other hand, $y : \mathbb{B}, x : \mathbb{A}, z : \mathbb{C}$ is not a shuffle because it alters the occurrence order of x and y in Γ_1. The set of shuffles in $\Gamma_1, \ldots, \Gamma_n$ is denoted as $\mathsf{Sf}(\Gamma_1, \ldots, \Gamma_n)$. The term formation rules of the linear lambda calculus are shown in Figure 1.

$$\frac{\Gamma_i \triangleright v_i : \mathbb{A}_i \quad f : \mathbb{A}_1, \ldots, \mathbb{A}_n \to \mathbb{A} \in \Sigma \quad E \in \mathsf{Sf}(\Gamma_1; \ldots; \Gamma_n)}{E \triangleright f(v_1, \ldots, v_n) : \mathbb{A}}\,(\mathrm{ax}) \qquad \frac{}{x : \mathbb{A} \triangleright x : \mathbb{A}}\,(\mathrm{hyp})$$

$$\frac{}{- \triangleright * : \mathbb{I}}\,(\mathbb{I}_i) \qquad \frac{\Gamma \triangleright v : \mathbb{A} \otimes \mathbb{B} \quad \Delta, x : \mathbb{A}, y : \mathbb{B} \triangleright w : \mathbb{C} \quad E \in \mathsf{Sf}(\Gamma; \Delta)}{E \triangleright \mathsf{pm}\ v\ \mathsf{to}\ x \otimes y.w : \mathbb{C}}\,(\otimes_e)$$

$$\frac{\Gamma \triangleright v : \mathbb{A} \quad \Delta \triangleright w : \mathbb{B} \quad E \in \mathsf{Sf}(\Gamma; \Delta)}{E \triangleright v \otimes w : \mathbb{A} \otimes \mathbb{B}}\,(\otimes_i) \qquad \frac{\Gamma \triangleright v : \mathbb{I} \quad \Delta \triangleright w : \mathbb{A} \quad E \in \mathsf{Sf}(\Gamma; \Delta)}{E \triangleright v\ \mathsf{to}\ *.w : \mathbb{A}}\,(\mathbb{I}_e)$$

$$\frac{\Gamma, x : \mathbb{A} \triangleright v : \mathbb{B}}{\Gamma \triangleright \lambda x : \mathbb{A}.v : \mathbb{A} \multimap \mathbb{B}}\,(\multimap_i) \quad \frac{\Gamma \triangleright v : \mathbb{A} \multimap \mathbb{B} \quad \Delta \triangleright w : \mathbb{A} \quad E \in \mathsf{Sf}(\Gamma; \Delta)}{E \triangleright vw : \mathbb{B}}\,(\multimap_e) \quad \frac{\Gamma \triangleright v : \mathbb{A}}{\Gamma \triangleright \mathsf{dis}(v) : \mathbb{I}}\,(\mathrm{dis})$$

Figure 1: Term formation rules of affine lambda calculus.

The no-cloning theorem states that it is impossible to duplicate a quantum bit [Wootters and Zurek (1982)]. This principle is upheld by the type system outlined in Figure 1, which does not allow the repeated use of a variable (seen as a quantum resource). Nevertheless, the linearity constraint is often deemed too restrictive, prompting research into relaxing it in various computational paradigms. In [Dahlqvist and Neves (2023)], the controlled use of a resource multiple times is explored within approximate program equivalence paradigms. Moreover, the grammar introduced allows the specification of how many times a resource can be used—a notion particularly relevant in quantum computation, especially within the NISQ era where resources are scarce.

Linear λ-calculus comes equipped with a class of equations, given in Figure 2, specifically equations-in-context $\Gamma \triangleright v = w : \mathbb{A}$.

Monoidal structure	Higher-order structure
$\text{pm } v \otimes w \text{ to } x \otimes y.\ u = u[v/x, w/y]$ $\text{pm } v \text{ to } x \otimes y.\ u[x \otimes y/z] = u[v/z]$ $* \text{ to } *.\ v = v$ $v \text{ to } *.\ w[*/z] = w[v/z]$	$(\lambda x : A.\ v)w = v[w/x]$ $\lambda x : A.(vx) = v$
Commuting conversions	
$u[v \text{ to } *.w/z] = v \text{ to } *.u[w/z]$ $u[\text{pm } v \text{ to } x \otimes y.\ w/z] = \text{pm } v \text{ to } x \otimes y.\ u[w/z]$	
Discard	
$v : \mathbb{I} = \text{dis}(x_1) \text{ to } * \ldots \text{dis}(x_{n-1}) \text{ to } * \ \text{dis}(x_n)$	

Figure 2: Equations-in-context for affine lambda calculus

2.1.2 Metric equational system

Metric equations [Mardare et al. (2016), Mardare et al. (2017)] are a strong candidate for reasoning about approximate program equivalence. These equations take the form of $t =_\epsilon s$, where ϵ is a non-negative rational representing the "maximum distance" between the two terms t and s. The metric equational system for linear lambda calculus is depicted in Figure 3 [Dahlqvist and Neves (2022)].

$$\frac{}{v =_0 v}\ (\text{refl}) \qquad \frac{v =_q w \quad w =_r u}{v =_{q+r} u}\ (\text{trans}) \qquad \frac{v =_q w \quad r \geq q}{v =_r w}\ (\text{weak})$$

$$\frac{\forall r < q.\ v =_r w}{v =_q w}\ (\text{arch}) \qquad \frac{\forall i \leq n.\ v =_{q_i} w}{v =_{\wedge q_i} w}\ (\text{join}) \qquad \frac{v =_q w \quad v' =_r w'}{v \otimes v' =_{q+r} w \otimes w'}$$

$$\frac{\forall i \leq n.\ v_i =_{q_i} w_i}{f(v_1, \ldots, v_n) =_{\Sigma q_i} f(w_1, \ldots, , w_n)} \qquad \frac{v =_q w \quad v' =_r w'}{v \text{ to } *.v' =_{q+r} w \text{ to } *.w'} \qquad \frac{v =_q w}{\lambda x : \mathbb{A}.\ v =_q \lambda x : \mathbb{A}.\ w}$$

$$\frac{v =_q w \quad v' =_r w'}{\text{pm } v \text{ to } x \otimes y.\ v' =_{q+r} \text{pm } w \text{ to } x \otimes y.\ w'} \qquad \frac{v =_q w \quad v' =_r w'}{vv' =_{q+r} ww'}$$

$$\frac{\Gamma \triangleright v =_q w : \mathbb{A} \quad \Delta \in \text{perm}(\Gamma)}{\Delta \triangleright v =_q w : \mathbb{A}} \qquad \frac{v =_q w \quad v' =_r w'}{v[v'/x] =_{q+r} w[w'/x]}$$

Figure 3: Metric equational system

In the quantum paradigm, a potential notion of approximate equivalence arises from the so-called diamond norm [Watrous (2018)], which induces a metric (roughly, a distance function) on the space of quantum programs (seen semantically as completely positive trace-preserving super-operators). This norm relies on another norm known as the trace norm. The $\|\|_1$ latter is defined by $\|A\|_1 = \text{Tr}\sqrt{A^\dagger A}$ for matrices $A \in \mathbb{C}^{n\times n}$. The trace norm induces a metric on the set of density matrices which is defined by $d(\rho, \sigma) = \|\rho - \sigma\|_1$. On the other hand, it is well known that the distance $d(vv^\dagger, uu^\dagger)$ between two quantum states v and u is their Euclidean distance in the Bloch sphere [Wallman and Emerson (2016); Nielsen and Chuang (2010)]. The Euclidean norm of a vector $u \in \mathbb{C}^n$ is defined as:

$$\|u\|_2 = \sqrt{\langle u, u\rangle} \tag{2.2}$$

The trace distance between two super-operators $E, E' : \mathbb{C}^{n\times n} \to \mathbb{C}^{m\times m}$, denoted as $T(E, E')$, is defined as follows:

$$T(E, E') = \max\{\|(E - E')A\|_1 \mid \|A\|_1 = 1\} \tag{2.3}$$

Unfortunately, this norm is not stable under tensoring [Watrous (2018)], and consequently, the diamond norm, which is based on the trace norm, is used instead. The diamond norm between two super-operators $E, E' : \mathbb{C}^{n\times n} \to \mathbb{C}^{m\times m}$ is defined as:

$$\|E - E'\|_\diamond = T(E \otimes I_n, E' \otimes I_n) \tag{2.4}$$

where I_n is the identity super-operator over the space $\mathbb{C}^{n\times n}$.

Consider an operator r : $(\mathbb{C}^n \to \mathbb{C}^m) \to (\mathbb{C}^{n\times n} \to \mathbb{C}^{m\times m})$ that sends an operator T to the mapping $A \mapsto TAT^\dagger$. The exact calculation of distances induced by $\|\|_\diamond$ tends to be quite complicated, but a useful property for calculating the distance between quantum channels in the image of r is provided [Watrous (2018)]: Consider two operators $T, S : n \to m$. There exists a unit vector $v \in \mathbb{C}^n$ such that,

$$\|r(T)(vv^\dagger) - r(S)(vv^\dagger)\|_1 = \|r(T) - r(S)\|_\diamond \tag{2.5}$$

The notion of a diamond norm is used in [Dahlqvist and Neves (2022)] which introduces a simple metric theory based on the idea of approximating a quantum operation. The authors argue that their deductive system allows to compute an approximate distance between two quantum programs easily as opposed to computing an exact distance "semantically" which tends to involve quite complex operators. Other works in this spirit include [Hung et al. (2019)] and [Tao et al. (2021)]. They reason about the issue of noise in a quantum while-language by developing a deductive system to determine how similar a quantum program is from its idealised, noise-free version. The former introduces the (Q,λ)-diamond norm which analyzes the output error given that the input quantum state satisfies some quantum predicate Q to degree λ. However, it does not specify any practical method for obtaining non-trivial quantum predicates. In fact, the methods used in [Hung et al. (2019)] cannot produce any post conditions other than $(I, 0)$ (*i.e.*, the identity matrix I to degree 0, analogous to a "true" predicate) for large quantum programs. The latter specifically addresses and delves into this aspect.

2.1.3 Interpretation

In order to define the interpretation of judgments $\Gamma \triangleright v : \mathbb{A}$, it is necessary to establish some notation first. Considering $v \in V$, $w \in W$, and $u \in U$ where V, W, U represent vector spaces, $\mathsf{sw}_{V,W} : V \otimes W \to W \otimes V$, denotes the swap operator, defined as $\mathsf{sw}_{V,W} = v \otimes w \mapsto w \otimes v$; $\rho_V : \mathbb{C} \otimes V \to V$ is the left unitor defined as $\rho_V = 1 \otimes v \mapsto v$; $\lambda_V : V \otimes \mathbb{C} \to V$ is the right unitor defined as $\lambda_V = v \otimes 1 \mapsto v$; $\alpha_{V,W,U} : V \otimes (W \otimes U) \to (V \otimes W) \otimes U$ is the left associator, defined as $\alpha_{V,W,U} = v \otimes (w \otimes u) \mapsto (v \otimes w) \otimes u$; and $!_V : V \to \mathbb{C}$ is the trace operation applied to a vector, defined as $!_V = v \to \mathrm{Tr}v$. Moreover, for all operators $f : V \otimes W \to U$, the operator $\overline{f} : V \to (W \multimap U)$ denotes the corresponding curried version, defined as $\overline{f}(v) = w \mapsto f(v, w)$. The subscripts in these operators will be omitted unless ambiguity arises.

For all ground types $X \in G$ the interpretation of $[\![X]\!]$ is postulated as a vector space V. Types are interpreted inductively using the unit $\mathbb{I}$, the tensor $\otimes$, and the linear map $\multimap$. Given a non-empty context $\Gamma = \Gamma', x : \mathbb{A}$, its interpretation is defined by $[\![\Gamma', x : \mathbb{A}]\!] = [\![\Gamma']\!] \otimes [\![\mathbb{A}]\!]$ if Γ' is non-empty and $[\![\Gamma', x : \mathbb{A}]\!] = [\![\mathbb{A}]\!]$ otherwise. The empty context $-$ is interpreted as $[\![-]\!] = \mathbb{I}$. Given $X_1, ..., X_n \in V$, the n-tensor $(\dots(X_1 \otimes X_2) \otimes \dots) \otimes X_n$ is denoted as $X_1 \otimes \dots \otimes X_n$, and similarly for operators.

"Housekeeping" operators are employed to handle interactions between context interpretation and the vectorial model. Given $\Gamma_1, \dots, \Gamma_n$, the operator that splits $[\![\Gamma_1, \dots, \Gamma_n]\!]$ into $[\![\Gamma_1]\!] \otimes \dots \otimes [\![\Gamma_n]\!]$ is denoted by $\mathsf{sp}_{\Gamma_1;\dots;\Gamma_n} : [\![\Gamma_1, \dots, \Gamma_n]\!] \to [\![\Gamma_1]\!] \otimes \dots \otimes [\![\Gamma_n]\!]$. On the other hand, $\mathsf{jn}_{\Gamma_1;\dots;\Gamma_n}$ denotes the inverse of $\mathsf{sp}_{\Gamma_1;\dots;\Gamma_n}$. Next, given $\Gamma, x : \mathbb{A}, y : \mathbb{B}, \Delta$, the operator permuting x and y is denoted by $\mathsf{exch}_{\Gamma,x:\mathbb{A},y:\mathbb{B},\Delta} : [\![\Gamma, x : \mathbb{A}, y : \mathbb{B}, \Delta]\!] \to [\![\Gamma, y : \mathbb{B}, x : \mathbb{A}, \Delta]\!]$. The shuffling operator $\mathsf{sh}_E : [\![E]\!] \to [\![\Gamma_1, \dots, \Gamma_n]\!]$ is defined as a suitable composition of exchange operators.

For every operation symbol $f : \mathbb{A}_1, \dots, \mathbb{A}_n \to \mathbb{A}$ we assume the existence of an operator $[\![f]\!] : [\![\mathbb{A}_1]\!] \otimes \dots \otimes [\![\mathbb{A}_n]\!] \to [\![\mathbb{A}]\!]$. The interpretation of judgments is defined by induction over derivations according to the rules in Figure 4 [Dahlqvist and Neves (2022)].

$$\frac{[\![\Gamma_i \triangleright v_i : \mathbb{A}_i]\!] = m_i \quad f : \mathbb{A}_1, \dots, \mathbb{A}_n \in \Sigma \quad E \in \mathsf{Sf}(\Gamma_1; \dots; \Gamma_n)}{[\![E \triangleright f(v_1, \dots, v_n) : \mathbb{A}]\!] = [\![f]\!] \cdot (m_1 \otimes \dots \otimes m_n) \cdot \mathsf{sp}_{\Gamma_1;\dots;\Gamma_n} \cdot \mathsf{sh}_E} \qquad \frac{}{[\![x : \mathbb{A} \triangleright x : \mathbb{A}]\!] = \mathsf{id}_{[\![\mathbb{A}]\!]}}$$

$$\frac{}{[\![- \triangleright * : \mathbb{I}]\!] = \mathsf{id}_{[\![\mathbb{I}]\!]}} \qquad \frac{[\![\Gamma \triangleright v : \mathbb{A} \otimes \mathbb{B}]\!] = m \quad [\![\Delta, x : \mathbb{A}, y : \mathbb{B} \triangleright w : \mathbb{C}]\!] = n \quad E \in \mathsf{Sf}(\Gamma; \Delta)}{[\![E \triangleright \mathsf{pm}\ v\ \mathsf{to}\ x \otimes y.w : \mathbb{C}]\!] = n \cdot \mathsf{jn}_{\Delta;\mathbb{A};\mathbb{B}} \cdot \alpha \cdot \mathsf{sw} \cdot (m \otimes \mathsf{id}) \cdot \mathsf{sp}_{\Gamma;\Delta} \cdot \mathsf{sh}_E}$$

$$\frac{[\![\Gamma \triangleright v : \mathbb{A}]\!] = m \quad [\![\Delta \triangleright w : \mathbb{B}]\!] = n \quad E \in \mathsf{Sf}(\Gamma; \Delta)}{[\![E \triangleright v \otimes w : \mathbb{A} \otimes \mathbb{B}]\!] = (m \otimes n) \cdot \mathsf{sp}_{\Gamma;\Delta} \cdot \mathsf{sh}_E}$$

$$\frac{[\![\Gamma \triangleright v : \mathbb{I}]\!] = m \quad [\![\Delta \triangleright w : \mathbb{A}]\!] = n \quad E \in \mathsf{Sf}(\Gamma; \Delta)}{[\![E \triangleright v\ \mathsf{to}\ *.w : \mathbb{A}]\!] = n \cdot \lambda \cdot (m \otimes \mathsf{id}) \cdot \mathsf{sp}_{\Gamma;\Delta} \cdot \mathsf{sh}_E} \qquad \frac{[\![\Gamma, x : \mathbb{A} \triangleright v : \mathbb{B}]\!] = m}{[\![\Gamma \triangleright \lambda x : \mathbb{A}.\, v : \mathbb{A} \multimap \mathbb{B}]\!] = \overline{m \cdot \mathsf{jn}_{\Gamma;\mathbb{A}}}}$$

$$\frac{[\![\Gamma \triangleright v : \mathbb{A} \multimap \mathbb{B}]\!] = m \quad [\![\Delta \triangleright w : \mathbb{A}]\!] = n \quad E \in \mathit{Sf}(\Gamma; \Delta)}{[\![E \triangleright vw : \mathbb{A}]\!] = \mathsf{app} \cdot (m \otimes n) \cdot \mathsf{sp}_{\Gamma;\Delta} \cdot \mathsf{sh}_E} \qquad \frac{[\![\Gamma \triangleright v : \mathbb{A}]\!] = f}{[\![\Gamma \triangleright \mathsf{dis}(v) : \mathbb{I}]\!] = !_{[\![\mathbb{A}]\!]} \cdot f}$$

Figure 4: Judgment interpretation

2.2 Graded Lambda Calculus

Intro

2.2.1 Syntax

Here, the following grammar of types is used.

$$\mathbb{A} ::= X \mid \mathbb{I} \mid \mathbb{A} \otimes \mathbb{A} \mid \mathbb{A} \oplus \mathbb{A} \mid \mathbb{A} \multimap \mathbb{A} \mid\, !_r\mathbb{A} \qquad\qquad X \in G, r \in \mathbb{N}$$

$$\frac{\Gamma_i \triangleright v_i :!_{r\cdot s_i}\mathbb{A}_i \quad x_1 :!_{s_1}\mathbb{A}_1, \ldots, x_n :!_{s_n}\mathbb{A}_n \triangleright u : \mathbb{A} \quad E \in \mathsf{Sf}(\Gamma_1; \ldots; \Gamma_n)}{E \triangleright \mathsf{pr}_{(r,[s_1,\ldots,s_n])} v_1, \ldots, v_n \;\mathsf{fr}\; x_1, \ldots, x_n.u :!_r\mathbb{A}}(!_i) \qquad \frac{\Gamma \triangleright v :!_1\mathbb{A}}{\Gamma \triangleright \mathsf{dr}\; v : \mathbb{A}}(!_e)$$

$$\frac{\Gamma \triangleright v :!_0\mathbb{A} \quad \Delta \triangleright u : \mathbb{B} \quad E \in \mathsf{Sf}(\Gamma, \Delta)}{E \triangleright v.u : \mathbb{B}}(!_0) \qquad \frac{\Gamma \triangleright v :!_{n+m}\mathbb{A} \quad \Delta, x :!_n\mathbb{A}, y :!_m\mathbb{B} \triangleright u : \mathbb{B} \quad E \in \mathsf{Sf}(\Gamma, \Delta)}{E \triangleright \mathsf{cp}_{(n,m)} v \;\mathsf{to}\; x, y.u : \mathbb{B}}(!_{n+m})$$

Figure 5: Term formation rules of graded lambda calculus.

2.3 Quantum Computing Preliminaries

This section presents background on quantum information and quantum computation [Nielsen and Chuang (2010)].

The basic unit of information in quantum computation is a quantum bit or qubit [Perdrix (2008)]. The state of a single qubit is described by a normalized vector of the 2-dimensional Hilbert space $\mathbb{C}^2$. When global phases are ignored we can represent a quantum state $|\psi\rangle \in \mathbb{C}^2$ in the form,

$$\cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle \tag{2.6}$$

which corresponds to a point in the unit sphere where θ marks the latitude (*i.e.* the polar angle) and ϕ marks the longitude (*i.e.* the azimuthal angle). This representation is traditionally called the Bloch sphere representation. A point in the latter representation corresponds to the vector in $\mathbb{R}^3$ defined by $(\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)$ and often called Bloch vector.

An n-qubit state can be represented by a unit vector in 2^n-dimensional Hilbert space $\mathbb{C}^{2^n}$. An n-qubit mixed state can be represented by a density operator $\mathbb{C}^{2^n} \to \mathbb{C}^{2^n}$, whose matrix representation is $\rho = \Sigma_i\, p_i|\phi_i\rangle\langle\phi_i|$. A density operator encodes uncertainty about the current state of the quantum system at hand. For example, a mixed state with half probability of $|0\rangle$ and $|1\rangle$ can be represented by $\frac{|0\rangle\langle 0|+|1\rangle\langle 1|}{2} = I/2$, where I is the identity matrix. One usually denotes density matrices by the greek letters ρ, σ, and so forth. The set of density operators is denoted by $\mathcal{D}_n \subseteq \mathbb{C}^{2^{n\times n}}$.

Measurements extract classical information from quantum states. If a measurement M_m is performed on a state ρ, the outcome m is observed with probability $p_m = \text{Tr}(M_m \rho M_m^\dagger)$ for each m. Moreover, after a measurement yielding outcome m, the state collapses to $M_m \rho M_m^\dagger / p_m$.

Operations on quantum systems can be described using unitary operators. An operator, U, is unitary if its Hermitian conjugate is its own inverse, *i.e.*, $U^\dagger U = UU^\dagger = I$. For a pure state $|\psi\rangle$, a unitary operator U describes an evolution from $|\psi\rangle$ to $U|\psi\rangle$. Similarly, for a density operator ρ, the corresponding evolution is $\rho \mapsto U\rho U^\dagger$. For example, the bit flip gate $X = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ maps $|0\rangle$ to $|1\rangle$ and $|1\rangle$ to $|0\rangle$. On the other hand, the Hadamard gate $H = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$ maps $|0\rangle$ to $\frac{|0\rangle+|1\rangle}{\sqrt{2}}$ (denoted as $|+\rangle$) and $|1\rangle$ to $\frac{|0\rangle-|1\rangle}{\sqrt{2}}$ (denoted as $|-\rangle$). There are also multi-qubit gates, such as $CNOT$, which leaves the states $|00\rangle$ and $|01\rangle$ unchanged, and maps $|10\rangle$ and $|11\rangle$ to each other.

More broadly, the evolution of a quantum system can be defined by a super-operator E, which is a completely-positive and trace-preserving linear map from $\mathcal{D}(n)$ to $\mathcal{D}(m)$. A super-operator E is called positive if it sends positive matrices to positive matrices, *i.e.* $A \geq 0 \Rightarrow EA \geq 0$. A super-operator is said to be completely positive if, for any positive integer k and any k-dimensional Hilbert space $\mathbb{C}^{2^k}$, the super-operator $E \otimes I_{\mathbb{C}^{2^k}}$ is a positive map on $\mathcal{D}(n \times k)$. Finally, a super-operator E is called trace-preserving if $\text{Tr}\, EA = \text{Tr} A$ [Watrous (2018)]. Completely-positive, trace-preserving super-operators are traditionally called quantum channels.

For every super-operator $E : \mathcal{D}(n) \to \mathcal{D}(m)$, there exists a set of Kraus operators $\{\epsilon_k\}_k$ such that $E(\rho) = \Sigma_k\, \epsilon_k\, \rho\, \epsilon_k^\dagger$ for any input $\rho \in \mathcal{D}(n)$. Note that the set of Kraus operators is finite if the Hilbert space is finite-dimensional. The Kraus form of E is written as $E = \Sigma_k\, \epsilon_k \circ \epsilon_k^\dagger$.

A matrix $A \in \mathbb{C}^{n\times n}$ is Hermitian if $A = A^\dagger$. A matrix $A \in \mathbb{C}^{n\times n}$ is said to be normal if $AA^\dagger = A^\dagger A$. Clearly every Hermitian matrix is normal. Note also that for every matrix $A \in C^{n\times n}$ the matrix $A^\dagger A$ is Hermitian. Next, it is well-known that by appealing to the spectral theorem [NC16], every normal matrix $A \in \mathbb{C}^{n\times n}$ can be expressed as a linear combination $\sum_i \lambda_i b_i b_i^\dagger$ where the set $\{b_i, \ldots, bn\}$ is an orthonormal basis of $\mathbb{C}^n$. Using this last result we can extend any function $f : \mathbb{C} \to \mathbb{C}$, to normal matrices via,

$$f(A) = \sum_i f(\lambda_i) b_i b_i^\dagger \tag{2.7}$$

... The Bloch vector is given by

$$r_\mu = \mathrm{Tr}(\rho\sigma_\mu) \tag{2.8}$$

add trace, partial trace, reduced density matrix, and respective Bloch Vector, put the last paragragh in the right place and rewrite it

2.4 Quantum Lambda Calculus

Adicionar os operadores CPTP que vou usar

In the case of quantum lambda calculus, which combines classical and quantum features, it is natural to consider two distinct basic data types: a type *bit* of classical bits and a type *qbit* of quantum bits. The interpretation of these types is defined as $[\![bit]\!] = \mathbb{C} \oplus \mathbb{C}$ and $[\![qbit]\!] = \mathbb{C}^{2 \cdot 2}$. The type $\mathbb{I}$ is interpreted as $[\![\mathbb{I}]\!] = \mathbb{C}$.

The following operations are considered: *new* $0 : \mathbb{I} \multimap$ *bit*, *new* $1 : \mathbb{I} \multimap$ *bit*, q : *bit* $\multimap$ *qbit*, *meas* : *qbit* $\to$ *bit*, and *U* : *qbit*, ..., *qbit* $\to$ *qbit*$^{\otimes n}$. Their correspondent judgment interpretation is shown in Figure 6.

$$[\![\mathit{new}\,0]\!] : \mathbb{C} \multimap [\![\mathit{bit}]\!] \qquad 1 \mapsto (1,0)$$

$$[\![\mathit{new}\,1]\!] : \mathbb{C} \multimap [\![\mathit{bit}]\!] \qquad 1 \mapsto (0,1)$$

$$[\![q]\!] : [\![\mathit{bit}]\!] \multimap [\![\mathit{qbit}]\!] \qquad (a,b) \mapsto \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$$

$$[\![\mathit{meas}]\!] : [\![\mathit{qbit}]\!] \to [\![\mathit{bit}]\!] \qquad \rho \mapsto (\mathrm{Tr}(M_0 \rho M_0^\dagger), \mathrm{Tr}(M_1 \rho M_1^\dagger))$$

$$[\![U]\!] : [\![\mathit{qbit}]\!]^{\otimes n} \to [\![\mathit{qbit}]\!]^{\otimes n} \qquad \rho \mapsto U \rho U^\dagger$$

Figure 6: Judgment interpretation of the operations in quantum lambda calculus.

Chapter 3

Conditionals

3.1 Measurements

In order to established that the theory introduced is valid in quantum programming, it is necessary to build a model. The model can be seen as a category where the morphisms are the CPTP super-operators (quantum channels). The algebric structure of this model is given by the vector spaces. Any completely-positive and trace-preserving map has a diamond norm equal to one [Watrous (2018)]. Since the measurement operation is completely positive and trace-preserving, its diamond norm is equal to one. This is a desirable property, as it ensures that the measurement operation does not increase the distance between states, and as a consequence, composition of programs remains valid.

3.1.1 Example: Deutsch's Algorithm

In 1985, David Deutsch presented an algorithm that determines whether a function f is constant for a single-bit input (*i.e.*, either equal to 1 for all x or equal to 0 for all x) or balanced (*i.e.*, equal to 1 for half of the values of x and equal to 0 for the other half) [Deutsch (1985)]. Classically, to determine which case holds requires running f twice. Quantumly, it suffices to run f once. The Deutsch-Jozsa Algorithm is a simple example of a quantum algorithm that outperforms its classical counterpart. The algorithm is based on the concept of a quantum oracle, which is a black box that implements a unitary transformation U_f such that $U_f |x\rangle |y\rangle = |x\rangle |y \oplus f(x)\rangle$, where $\oplus$ denotes addition modulo 2. The quantum circuit implementing Deutsch's algorithm is presented in Figure 7.

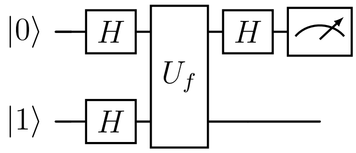

Figure 7: Quantum circuit implementing Deutsch's algorithm

Using lambda calculus, the Deutsch-Jozsa Algorithm can be expressed as:

$$\text{Deutsch} : \ (\mathit{qbit} \otimes \mathit{qbit} \multimap \mathit{qbit} \otimes \mathit{qbit}) \multimap \mathit{bit} \otimes \mathit{qbit}$$

$$\begin{aligned}\text{Deutsch} = U_f : \mathit{qbit} \otimes \mathit{qbit} \multimap \mathit{qbit} \otimes \mathit{qbit} \triangleright \\ \text{pm } U_f(H(q\,(\mathit{new}\,0\,(*))), (H(q\,(\mathit{new}\,1\,(*)))) \ \mathit{to}\ q_1 \otimes q_2 \,.\, \mathit{meas}(H(q_1)) \otimes q_2\end{aligned}$$

Regarding the interpretation of the Deutsch Algorithm, one has that:

$$\begin{aligned} & |0\rangle \otimes |1\rangle \\ \xmapsto{H \otimes H} \quad & \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |-\rangle \end{aligned} \tag{3.1}$$

With respecto to quantum oracle U_f, it is possible to show that:

$$\begin{aligned} & |x\rangle \otimes |-\rangle = |x\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = \frac{1}{\sqrt{2}}(|x\rangle \otimes |0\rangle - |x\rangle \otimes |1\rangle)) \\ \xmapsto{U_f} \ & \frac{1}{\sqrt{2}}(|x\rangle \otimes |0 \oplus f(x)\rangle - |x\rangle \otimes |1 \oplus f(x)\rangle) && \{\text{Defn. of } U_f\} \\ = \ & \frac{1}{\sqrt{2}}(|x\rangle\,|f(x)\rangle - |x\rangle\,|\neg f(x)\rangle) && \{0 \oplus x = x, 1 \oplus x = \neg x\} \\ = \ & \frac{1}{\sqrt{2}}(|x\rangle \otimes (|f(x)\rangle - |\neg f(x)\rangle)) \end{aligned} \tag{3.2}$$

Proceding by case distinction:

$$\frac{1}{\sqrt{2}}(|x\rangle \otimes (|f(x)\rangle - |\neg f(x)\rangle) = \begin{cases} |x\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) & \text{if } f(x) = 0 \\ |x\rangle \otimes \frac{1}{\sqrt{2}}(|1\rangle - |0\rangle) & \text{if } f(x) = 1 \end{cases} \tag{3.3}$$

And conclude that

$$|x\rangle \otimes \frac{1}{\sqrt{2}}(|f(x)\rangle - |\neg f(x)\rangle) = (-1)^{f(x)}\,|x\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = (-1)^{f(x)}\,|x\rangle \otimes |-\rangle \tag{3.4}$$

Returning to the interpretation of the Deutsch Algorithm, one has that:

$$
\begin{aligned}
& \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |-\rangle \\
\xmapsto{U_f} \quad & \frac{1}{\sqrt{2}}(U_f |0\rangle \otimes |-\rangle + U_f |1\rangle \otimes |-\rangle)) \\
= \quad & \frac{1}{\sqrt{2}}((-1)^{f(0)} |0\rangle \otimes |-\rangle + (-1)^{f(1)} |1\rangle \otimes |-\rangle) \\
= \quad & \begin{cases} (\pm 1) |+\rangle \otimes |-\rangle & \text{if } f(0) = f(1) \\ (\pm 1) |-\rangle \otimes |-\rangle & \text{if } f(0) \neq f(1) \end{cases} \\
\xmapsto{H \otimes I} \quad & \begin{cases} (\pm 1) |0\rangle \otimes |-\rangle & \text{if } f(0) = f(1) \\ (\pm 1) |1\rangle \otimes |-\rangle & \text{if } f(0) \neq f(1) \end{cases}
\end{aligned}
\tag{3.5}
$$

Attending to the interpretation of quantum states, concerning the measurement of the first qubit, one has that:

$$
\begin{aligned}
& \begin{cases} |0\rangle \langle 0| \otimes |-\rangle \langle -| & \text{if } f(0) = f(1) \\ |1\rangle \langle 1| \otimes |-\rangle \langle -| & \text{if } f(0) \neq f(1) \end{cases} \\
\xmapsto{\textit{meas} \otimes I} \quad & \begin{cases} (|-\rangle \langle -| \,, 0) & \text{if } f(0) = f(1) \\ (0, |-\rangle \langle -|) & \text{if } f(0) \neq f(1) \end{cases}
\end{aligned}
\tag{3.6}
$$

A measurement error is characterized by reading a ”1” as a ”0” or vice versa. Furthermore, it’s important to note that measurement errors do not impact all states uniformly [Tannu and Qureshi (2019)]. Consequently, there is a discrepancy in how frequently the state ”1” is incorrectly read as ”0” compared to how often the state ”0” is measured as ”1” or vice versa. For example, considering there is a 10% chance of measuring a ”0” as a ”1” and a 30% chance of measuring a ”1” as a ”0”, the resulting state after measurement is:

$$
\begin{cases} (0.9 \, |-\rangle \langle -| \,, 0.1 \, |-\rangle \langle -|) & \text{if } f(0) = f(1) \\ (0.3 \, |-\rangle \langle -| \,, 0.7 \, |-\rangle \langle -|) & \text{if } f(0) \neq f(1) \end{cases}
\tag{3.7}
$$

The norm of a tuple is defined as the sum of the norms of its components, *i.e.*, for any operators v and w:

$$
\|(v, w)\| = \|v\| + \|w\|
\tag{3.8}
$$

As a result, the discrepancy between the ideal and actual measurement results corresponds to:

$$
\begin{aligned}
&\begin{cases} \|(\left|-\right\rangle\left\langle-\right|, 0) - (0.9\left|-\right\rangle\left\langle-\right|, 0.1\left|-\right\rangle\left\langle-\right|)\|_{\diamond} & \text{if } f(0) = f(1) \\ \|(0, \left|-\right\rangle\left\langle-\right|) - (0.3\left|-\right\rangle\left\langle-\right|, 0.7\left|-\right\rangle\left\langle-\right|)\|_{\diamond} & \text{if } f(0) \neq f(1) \end{cases} \\
= &\begin{cases} \|(0.1\left|-\right\rangle\left\langle-\right|, -0.1\left|-\right\rangle\left\langle-\right|)\|_{\diamond} & \text{if } f(0) = f(1) \\ \|(-0.3\left|-\right\rangle\left\langle-\right|, 0.3\left|-\right\rangle\left\langle-\right|)\|_{\diamond} & \text{if } f(0) \neq f(1) \end{cases} \\
= &\begin{cases} \|0.1\left|-\right\rangle\left\langle-\right|\|_{\diamond} + \|-0.1\left|-\right\rangle\left\langle-\right|\|_{\diamond} & \text{if } f(0) = f(1) \\ \|-0.3\left|-\right\rangle\left\langle-\right|\|_{\diamond} + \|-0.3\left|-\right\rangle\left\langle-\right|\|_{\diamond} & \text{if } f(0) \neq f(1) \end{cases}
\end{aligned} \tag{3.9}
$$

Employing Equation 2.8, it is easily concluded that the Bloch vector of the state $\left|-\right\rangle\left\langle-\right|$ is $(-1, 0, 0)$. Consequently, the discrepancy between the ideal and actual measurement results is:

$$
\begin{aligned}
&\begin{cases} \|(-0.1, 0, 0)\|_2 + \|(0.1, 0, 0)\left|-\right\rangle\left\langle-\right|\|_2 & \text{if } f(0) = f(1) \\ \|(0.3, 0, 0)\left|-\right\rangle\left\langle-\right|\|_2 + \|(-0.3, 0, 0)\left|-\right\rangle\left\langle-\right|\|_2 & \text{if } f(0) \neq f(1) \end{cases} \\
= &\begin{cases} \sqrt{(-0.1)^2 + 0^2 + 0^2} + \sqrt{(0.1)^2 + 0^2 + 0^2} & \text{if } f(0) = f(1) \\ \sqrt{(0.3)^2 + 0^2 + 0^2} + \sqrt{(-0.3)^2 + 0^2 + 0^2} & \text{if } f(0) \neq f(1) \end{cases} \\
= &\begin{cases} 2\sqrt{0.01} & \text{if } f(0) = f(1) \\ 2\sqrt{0.09} & \text{if } f(0) \neq f(1) \end{cases}
\end{aligned} \tag{3.10}
$$

Via the metric deductive system in Figure 3, it is easily verified that for an arbitrary error ϵ:

$$
\begin{aligned}
&U_f : \mathit{qbit} \otimes \mathit{qbit} \multimap \mathit{qbit} \otimes \mathit{qbit} \triangleright \\
&\text{pm } U_f(H(q\,(\mathit{new}\, 0\,(*))), (H(q\,(\mathit{new}\, 1\,(*)))) \;\mathit{to}\; q_1 \otimes q_2 \,.\, \mathit{meas}(H(q_1)) \otimes q_2 \\
=_{\epsilon}& \\
&U_f : \mathit{qbit} \otimes \mathit{qbit} \multimap \mathit{qbit} \otimes \mathit{qbit} \triangleright \\
&\text{pm } U_f(H(q\,(\mathit{new}\, 0\,(*))), (H(q\,(\mathit{new}\, 1\,(*)))) \;\mathit{to}\; q_1 \otimes q_2 \,.\, \mathit{meas}^{\epsilon}(H(q_1)) \otimes q_2
\end{aligned}
$$

Therefore, Deutsch $=_{\epsilon}$ Deutsch$^{\epsilon}$, and consequently, for scenario under consideration, if f is a constant function, Deutsch $= 2\sqrt{0.01}$Deutsch$^{0.1,0.3}$; otherwise, Deutsch $=_{2\sqrt{0.09}}$ Deutsch$^{0.1,0.3}$.

3.2 Conditionals

The notion of approximate equivalence for quantum programming explored in [Dahlqvist and Neves (2022)] does not encompass classical control flow. As a result, preliminary work based on [Crole (1993); Selinger (2013)] has been undertaken to address the integration of conditionals.

3.2.1 Syntax

The term formation rules for conditionals are depicted in Figure 8.

$$\dfrac{\Gamma \triangleright v : \mathbb{A}}{\Gamma \triangleright \mathsf{inl}_{\mathbb{B}}(v) : \mathbb{A} \oplus \mathbb{B}}\,(\mathsf{inl}) \qquad \dfrac{\Gamma \triangleright v : \mathbb{B}}{\Gamma \triangleright \mathsf{inr}_{\mathbb{A}}(v) : \mathbb{A} \oplus \mathbb{B}}\,(\mathsf{inr})$$

$$\dfrac{\Gamma \triangleright v : \mathbb{A} \oplus \mathbb{B} \quad \Delta, x : \mathbb{A} \triangleright w : \mathbb{C} \quad \Delta, y : \mathbb{B} \triangleright u : \mathbb{C} \quad E \in \mathsf{Sf}(\Gamma; \Delta)}{E \triangleright \ \mathsf{case}\ v\ \{\mathsf{inl}_{\mathbb{B}}(x) \Rightarrow w; \mathsf{inr}_{\mathbb{A}}(y) \Rightarrow u\} : \mathbb{C}}\,(\mathsf{case})$$

Figure 8: Term formation rules for conditionals

Properties

The rules presented in Figure 8 are subject the properties in Theorem 3.2.1.

Theorem 3.2.1. *Lambda calculus with conditionals has the following properties:*

1. *for all judgements $\Gamma \triangleright v$ and $\Gamma' \triangleright v$, te(Γ) $\simeq_\pi$ te(Γ');*
2. *additionally if $\Gamma \triangleright v : \mathbb{A}, \Gamma' \triangleright v : \mathbb{A}'$, and $\Gamma \simeq_\pi \Gamma'$, then $\mathbb{A}$ must be equal to $\mathbb{A}'$;*
3. *all judgements $\Gamma \triangleright v : \mathbb{A}$ have a unique derivation.*

Proof Regarding the first property, considering the case statement in Figure 8 and

$$\dfrac{\Gamma' \triangleright v : \mathbb{A} \oplus \mathbb{B} \quad \Delta', x : \mathbb{A} \triangleright w : \mathbb{C} \quad \Delta', y : \mathbb{B} \triangleright u : \mathbb{C} \quad E' \in \mathsf{Sf}(\Gamma'; \Delta')}{E' \triangleright \ \mathsf{case}\ v\ \{\mathsf{inl}_{\mathbb{B}}(x) \Rightarrow w; \mathsf{inr}_{\mathbb{A}}(y) \Rightarrow u\} : \mathbb{C}}$$

we want to prove that te$(E) \simeq_\pi$ te(E'). By induction hypothesis, te$(\Gamma) \simeq_\pi$ te(Γ), te$(\Delta, x) \simeq_\pi$ te(Δ', x) and te$(\Delta, y) \simeq_\pi$ te(Δ', y). This implies that te$(\Delta) \simeq_\pi$ te(Δ'). Since, $E \in \mathsf{Sf}(\Gamma; \Delta)$ and $E' \in \mathsf{Sf}(\Gamma'; \Delta')$, one has that te$(E) \simeq_\pi$ te(Γ, Δ) and te$(E') \simeq_\pi$ te(Γ', Δ'). Consequently, te(E) $\simeq_\pi$ te(E').

With respect to the second property, considering the case statement in Figure 8 and

$$\frac{\Gamma \rhd v : \mathbb{A} \oplus \mathbb{B} \quad \Delta', x : \mathbb{A} \rhd w : \mathbb{C}' \quad \Delta', y : \mathbb{B} \rhd u : \mathbb{C}' \quad E' \in \mathsf{Sf}(\Gamma'; \Delta')}{E' \rhd \mathsf{case}\, v \,\{\mathsf{inl}_{\mathbb{B}}(x) \Rightarrow w; \mathsf{inr}_{\mathbb{A}}(y) \Rightarrow u\} : \mathbb{C}'}$$

we want to prove that if $E \rhd \mathsf{case}\, v \,\{\mathsf{inl}(x) \Rightarrow w; \mathsf{inr}(y) \Rightarrow u\} : \mathbb{C}; \Gamma' \rhd \mathsf{case}\, v \,\{\mathsf{inl}(x) \Rightarrow w; \mathsf{inr}(y) \Rightarrow u\} : \mathbb{C}'$, and $E \simeq_\pi E'$, then $\mathbb{C}$ must be equal to $\mathbb{C}'$. Assuming, that $E \simeq_\pi E'$ and knowing that $E \in \mathsf{Sf}(\Gamma; \Delta)$ and $E' \in \mathsf{Sf}(\Gamma'; \Delta')$, one has that

$$\begin{aligned}
& x : \mathbb{A} \in \Delta \\
\Longrightarrow\ & x : \mathbb{A} \in E && \{E \in \mathsf{Sf}(\Gamma; \Delta)\} \\
\Longrightarrow\ & x : \mathbb{A} \in E' && \{E \simeq_\pi E'\} \\
\Longrightarrow\ & x : \mathbb{A} \in \Delta' && \{\text{All terms are well typed and contexts do not share variables}\}
\end{aligned}$$

This proves that $\Delta \simeq_\pi \Delta'$. Therefore, by induction hypothesis on the premises of the conditional statement, one has that $\mathbb{C}$ must be equal to $\mathbb{C}'$.

Finally, concerning the third property it is necessary to demonstrate that the case statement in Figure 8 has a unique derivation. This means proving that the premises in Figure 8 and in

$$\frac{\Gamma' \rhd v : \mathbb{A} \oplus \mathbb{B} \quad \Delta', x : \mathbb{A} \rhd w : \mathbb{C}' \quad \Delta', y : \mathbb{B} \rhd u : \mathbb{C} \quad E' \in \mathsf{Sf}(\Gamma'; \Delta')}{E \rhd \mathsf{case}\, v \,\{\mathsf{inl}_{\mathbb{B}}(x) \Rightarrow w; \mathsf{inr}_{\mathbb{A}}(y) \Rightarrow u\} : \mathbb{C}}$$

are equal, more concretely that $\Gamma = \Gamma'$ and $\Delta = \Delta'$.

$$
\begin{aligned}
& x : \mathbb{A} \in \Gamma \\
\Longrightarrow\ & x : \mathbb{A} \in E \wedge te(x : \mathbb{A}) \in \Gamma' && \{E \in \mathsf{Sf}(\Gamma; \Delta), te(\Gamma) \simeq_\pi te(\Gamma')\} \\
\Longrightarrow\ & x : \mathbb{A} \in E \wedge x : \mathbb{A} \in \Gamma' && \{E \in \mathsf{Sf}(\Gamma; \Delta), E \in \mathsf{Sf}(\Gamma'; \Delta')\}
\end{aligned}
$$

This last implication is related with the fact that in E, there can only exist one variable designated by x. Given that a shuffle preserves the relative order of the variables in each context, it follows that $\Gamma = \Gamma'$. The same reasoning can be applied to Δ and Δ', which concludes the proof.

$\square$

Lemma 3.2.1. *(Exchange and Substitution) For every judgement $\Gamma, x : \mathbb{A}, y : \mathbb{B}, \Delta \rhd v : \mathbb{C}$ it is possible to derive $\Gamma, y : \mathbb{B}, x : \mathbb{A}, \Delta \rhd v : \mathbb{C}$. For all judgements $\Gamma, x : \mathbb{A} \rhd v : \mathbb{B}$ and $\Delta \rhd w : \mathbb{A}$ it is possible to derive $\Gamma, \Delta \rhd v[w/x] : \mathbb{B}$.*

Proof Regarding the exchange property, for the case statement it is necessary to prove that for every judgment $\Gamma, x : \mathbb{A}, y : \mathbb{B}, \Delta \rhd \mathsf{case}\ v\ \{\mathsf{inl}_{\mathbb{E}}\ (a) \Rightarrow w; \mathsf{inr}_{\mathbb{D}}\ (b) \Rightarrow u\} : \mathbb{C}$, it is possible to derive $\Gamma, y : \mathbb{B}, x : \mathbb{A}, \Delta \rhd \mathsf{case}\ v\ \{\mathsf{inl}_{\mathbb{E}}(a) \Rightarrow w; \mathsf{inr}_{\mathbb{D}}(b) \Rightarrow u\} : \mathbb{C}$. It is necessary to consider three scenarios:

1. $x : \mathbb{A}, y : \mathbb{B}$ are variables in the context of v;
2. $x : \mathbb{A}, y : \mathbb{B}$ are variables in the context of w and u;
3. $x : \mathbb{A}$ is a variable in the context of v and $y : \mathbb{B}$ is a variable in the context of w and u.

With respect to the first case, by induction hypothesis and applying the case rule, one has that:

$$
\dfrac{\dfrac{\Gamma_1, x : \mathbb{A}, y : \mathbb{B}, \Gamma_2 \rhd v : \mathbb{D} \oplus \mathbb{E} \quad \begin{array}{c}\Sigma, a : \mathbb{D} \rhd w : \mathbb{C} \\ \Sigma, b : \mathbb{E} \rhd u : \mathbb{C}\end{array} \quad \Gamma, y : \mathbb{B}, x : \mathbb{A}; \Delta \in \mathsf{Sf}(\Gamma_1, x : \mathbb{A}, y : \mathbb{B}, \Gamma_2; \Sigma)}{\Gamma_1, y : \mathbb{B}, x : \mathbb{A}, \Gamma_2 \rhd v : \mathbb{D} \oplus \mathbb{E} \quad \begin{array}{c}\Sigma, a : \mathbb{D} \rhd w : \mathbb{C} \\ \Sigma, b : \mathbb{E} \rhd u : \mathbb{C}\end{array} \quad \Gamma, y : \mathbb{B}, x : \mathbb{A}; \Delta \in \mathsf{Sf}(\Gamma_1, y : \mathbb{B}, x : \mathbb{A}, \Gamma_2; \Sigma)}}{\Gamma, y : \mathbb{B}, x : \mathbb{A}, \Delta \rhd \mathsf{case}\ v\ \{\mathsf{inl}_{\mathbb{E}}(a) \Rightarrow w; \mathsf{inr}_{\mathbb{D}}(b) \Rightarrow u\} : \mathbb{C}}
$$

Next, for the second case, by induction hypothesis and applying the case rule, one has that:

$$\dfrac{\dfrac{\Sigma \triangleright v : \mathbb{D} \oplus \mathbb{E} \quad \begin{array}{c}\Delta_1, x : \mathbb{A}, y : \mathbb{B}, \Delta_2, a : \mathbb{D} \triangleright w : \mathbb{C} \\ \Delta_1, x : \mathbb{A}, y : \mathbb{B}, \Delta_2, b : \mathbb{E} \triangleright u : \mathbb{C}\end{array} \quad \Gamma, x : \mathbb{A}, y : \mathbb{B}; \Delta \in \mathsf{Sf}(\Sigma; \Delta_1, x : \mathbb{A}, y : \mathbb{B}, \Delta_2)}{\Sigma \triangleright v : \mathbb{D} \oplus \mathbb{E} \quad \begin{array}{c}\Delta_1, y : \mathbb{B}, x : \mathbb{A}, \Delta_2, a : \mathbb{D} \triangleright w : \mathbb{C} \\ \Delta_1, y : \mathbb{B}, x : \mathbb{A}, \Delta_2, b : \mathbb{E} \triangleright u : \mathbb{C}\end{array} \quad \Gamma, y : \mathbb{B}, x : \mathbb{A}; \Delta \in \mathsf{Sf}(\Sigma; \Delta_1, y : \mathbb{B}, x : \mathbb{A}, \Delta_2)}}{\Gamma, y : \mathbb{B}, x : \mathbb{A}, \Delta \triangleright \text{ case } v \, \{\mathsf{inl}_{\mathbb{E}}(a) \Rightarrow w; \mathsf{inr}_{\mathbb{D}}(b) \Rightarrow u\} : \mathbb{C}}$$

Finally, for the third case, considering the premises

$$\Gamma_1, x : \mathbb{A}, \Gamma_2 \triangleright v : \mathbb{D} \oplus \mathbb{E} \quad \begin{array}{c}\Delta_1, y : \mathbb{B}, \Delta_2, a : \mathbb{D} \triangleright w : \mathbb{C} \\ \Delta_1, y : \mathbb{B}, \Delta_2, b : \mathbb{E} \triangleright u : \mathbb{C}\end{array} \quad \begin{array}{l}\Gamma, x : \mathbb{A}, y : \mathbb{B}, \Delta \in \\ \mathsf{Sf}(\Gamma_1, x : \mathbb{A}, \Gamma_2; \Delta_2, y : \mathbb{B}, \Delta_1)\end{array}$$

and attending to the definition of shuffle, a possibility for $\Gamma, y : \mathbb{B}, x : \mathbb{A}, \Delta \in \mathsf{Sf}(\Gamma_1, x : \mathbb{A}, \Gamma_2; \Delta_2, y : \mathbb{B}, \Delta_1)$, given that in exchanging these variables the relative order of the variables in $\Gamma_1, x : \mathbb{A}, \Gamma_2$ and $\Delta_2, y : \mathbb{B}, \Delta_1$ is preserved. As a result, it is possible to derive $\Gamma, y : \mathbb{B}, x : \mathbb{A}, \Delta \triangleright$ case $v \, \{\mathsf{inl}_{\mathbb{E}}(a) \Rightarrow w; \mathsf{inr}_{\mathbb{D}}(b) \Rightarrow u\} : \mathbb{C}$.

With respect to the substitution property, for the case statement it is necessary to prove that for all judgements $E, z : \mathbb{D} \triangleright$ case $v \, \{\mathsf{inl}_{\mathbb{A}}(x) \Rightarrow w; \mathsf{inr}_{\mathbb{B}}(y) \Rightarrow u\} : \mathbb{C}$ and $\Sigma \triangleright a : \mathbb{D}$ it is possible to derive $E, \Sigma \triangleright$ case $v \, \{\mathsf{inl}_{\mathbb{A}}(x) \Rightarrow w; \mathsf{inr}_{\mathbb{B}}(y) \Rightarrow u\}[a/z] : \mathbb{C}$. In this case, it is necessary to consider two scenarios:

1. $z : \mathbb{D}$ is a variable in the context of v;
2. $z : \mathbb{D}$ is a variable in the context of w and u.

Regarding the first case, by induction and applying the case rule, one has that:

$$\dfrac{\dfrac{\dfrac{\Gamma, z : \mathbb{D} \triangleright v : \mathbb{A} \oplus \mathbb{B} \quad \Delta, x : \mathbb{A} \triangleright w : \mathbb{C} \quad \Delta, y : \mathbb{B} \triangleright u : \mathbb{C} \quad E, z : \mathbb{D} \in \mathsf{Sf}(\Gamma, z : \mathbb{D}; \Delta) \quad \Sigma \triangleright a : \mathbb{D}}{\Gamma, \Sigma \triangleright v[a/z] : \mathbb{A} \oplus \mathbb{B} \quad \Delta, x : \mathbb{A} \triangleright w : \mathbb{C} \quad \Delta, y : \mathbb{B} \triangleright u : \mathbb{C} \quad E, \Sigma \in \mathsf{Sf}(\Gamma, \Sigma; \Delta)}}{E, \Sigma \triangleright \text{ case } v \, \{\mathsf{inl}_{\mathbb{B}}(x) \Rightarrow w[a/z]; \mathsf{inr}_{\mathbb{A}}(y) \Rightarrow u[a/z]\} : \mathbb{C}}}{E, \Sigma \triangleright \text{ case } v \, \{\mathsf{inl}_{\mathbb{B}}(x) \Rightarrow w; \mathsf{inr}_{\mathbb{A}}(y) \Rightarrow u\}[a/z] : \mathbb{C}}$$

The second case is similar to the first one, applying the exchange property, then the induction,followed by the exchange property once more, and finally the case rule, one has that

$$\dfrac{\dfrac{\dfrac{\Gamma \triangleright v : \mathbb{A} \oplus \mathbb{B} \quad \begin{array}{c}\Delta, z : \mathbb{D}, y : \mathbb{B}, y : \mathbb{B} \triangleright u : \mathbb{C} \\ \Delta, z : \mathbb{D}, x : \mathbb{A} \triangleright w : \mathbb{C}\end{array} \quad E, z : \mathbb{D} \in \mathsf{Sf}(\Gamma; \Delta, z : \mathbb{D}) \quad \Sigma \triangleright a : \mathbb{D}}{\Gamma \triangleright v : \mathbb{A} \oplus \mathbb{B} \quad \begin{array}{c}\Delta, y : \mathbb{B}, z : \mathbb{D} \triangleright u : \mathbb{C} \\ \Delta, x : \mathbb{A}, z : \mathbb{D} \triangleright w : \mathbb{C}\end{array} \quad E, z : \mathbb{D} \in \mathsf{Sf}(\Gamma; \Delta, z : \mathbb{D}) \quad \Sigma \triangleright a : \mathbb{D}}}{\dfrac{\Gamma \triangleright v : \mathbb{A} \oplus \mathbb{B} \quad \Delta, x : \mathbb{A}, \Sigma \triangleright w[a/z] : \mathbb{C} \quad \Delta, y : \mathbb{B}, \Sigma \triangleright u[a/z] : \mathbb{C} \quad E, \Sigma \in \mathsf{Sf}(\Gamma; \Delta, \Sigma)}{\dfrac{\Gamma \triangleright v : \mathbb{A} \oplus \mathbb{B} \quad \Delta, \Sigma, x : \mathbb{A} \triangleright w[a/z] : \mathbb{C} \quad \Delta, \Sigma, y : \mathbb{B} \triangleright u[a/z] : \mathbb{C} \quad E, \Sigma \in \mathsf{Sf}(\Gamma; \Delta, \Sigma)}{E, \Sigma \triangleright \mathsf{case}\ v[a/z]\ \{\mathsf{inl}_{\mathbb{B}}(x) \Rightarrow w; \mathsf{inr}_{\mathbb{A}}(y) \Rightarrow u\} : \mathbb{C}}}}}{E, \Sigma \triangleright \mathsf{case}\ v\ \{\mathsf{inl}_{\mathbb{B}}(x) \Rightarrow w; \mathsf{inr}_{\mathbb{A}}(y) \Rightarrow u\}[a/z] : \mathbb{C}}$$

3.2.2 Interpretation

Considering $v \in V$, $w \in W$, and $u \in U$ where V, W, U represent vector spaces, $\mathsf{IL}_V : V \to V \oplus W$, denotes the left injection operator, defined as $\mathsf{IL}_V = v \mapsto (v, 0)$; $\mathsf{IR}_V : V \to W \oplus V$, denotes the right injection operator, defined as $\mathsf{IR}_V = v \mapsto (0, v)$; and $\mathsf{dist}_{V,W,U} : V \otimes (W \oplus U) \to (V \otimes W) \oplus (V \otimes U)$, denotes the distributive property of the tensor product over the direct sum, defined as $\mathsf{dist}_{V,W,U} = v \otimes (w, u) \mapsto (v \otimes w, v \otimes u)$. The subscripts in these operators will be omitted unless ambiguity arises. Moreover, the operation either corresponds to:

$$\begin{array}{l} \dfrac{\begin{array}{c} V \to U \\ W \to U \end{array}}{[T, S] : V \oplus W \to U} \\[2ex] [T, S] = (v, w) \mapsto T(v) + S(w) \end{array} \tag{3.11}$$

The interpretation of conditionals is illustrated in Figure 9.

$$\frac{[\![\Gamma \triangleright v : \mathbb{A}]\!] = m}{[\![\Gamma \triangleright \mathsf{inl}(v) : \mathbb{A} \oplus \mathbb{B}]\!] = \mathsf{l_L} \cdot m} \qquad \frac{[\![\Gamma \triangleright v : \mathbb{B}]\!] = m}{[\![\Gamma \triangleright \mathsf{inr}(v) : \mathbb{A} \oplus \mathbb{B}]\!] = \mathsf{l_R} \cdot m}$$

$$\frac{[\![\Gamma \triangleright v : \mathbb{A} \oplus \mathbb{B}]\!] = b \quad [\![\Delta, x : \mathbb{A} \triangleright w : \mathbb{C}]\!] = p \quad [\![\Delta, y : \mathbb{B} \triangleright u : \mathbb{C}]\!] = q \quad E \in \mathsf{Sf}(\Gamma; \Delta)}{[\![E \triangleright \; \mathsf{case}\, v \, \{\mathsf{inl}(x) \Rightarrow w; \, \mathsf{inr}(y) \Rightarrow u\} : \mathbb{C}]\!] = [p \cdot \mathsf{jn}_{\Delta;\mathbb{A}}, q \cdot \mathsf{jn}_{\Delta;\mathbb{B}}] \cdot \mathsf{dist} \cdot \mathsf{sw} \cdot (b \otimes \mathsf{id}) \cdot \mathsf{sp}_{\Gamma;\Delta} \cdot \mathsf{sh}_E} \tag{3.12}$$

Figure 9: Judgment interpretation for conditionals

Proof In order to validate the judgment interpretation for conditionals, it is necessary to demonstrate its correctness.

For the booleans:

$$\begin{gathered} [\![\Gamma]\!] \xrightarrow{m} [\![\mathbb{A}]\!] \xrightarrow{\mathsf{l_L}} [\![\mathbb{A} \oplus \mathbb{B}]\!] \\ [\![\Gamma]\!] \xrightarrow{m} [\![\mathbb{B}]\!] \xrightarrow{\mathsf{l_R}} [\![\mathbb{A} \oplus \mathbb{B}]\!] \end{gathered} \tag{3.13}$$

Now, for the conditional statement:

$$\begin{aligned} &[\![E]\!] \xrightarrow{\mathsf{sh}_E} [\![\Gamma, \Delta]\!] \xrightarrow{\mathsf{sp}_{\Gamma;\Delta}} [\![\Gamma]\!] \otimes [\![\Delta]\!] \xrightarrow{b \otimes \mathsf{id}} ([\![\mathbb{A}]\!] \oplus [\![\mathbb{B}]\!]) \otimes [\![\Delta]\!] \xrightarrow{\mathsf{sw}} [\![\Delta]\!] \otimes ([\![\mathbb{A}]\!] \oplus [\![\mathbb{B}]\!]) \\ &\xrightarrow{\mathsf{dist}} ([\![\Delta]\!] \otimes [\![\mathbb{A}]\!]) \oplus ([\![\Delta]\!] \otimes [\![\mathbb{B}]\!]) \xrightarrow{[p \cdot \mathsf{jn}_{\Delta;\mathbb{A}}, q \cdot \mathsf{jn}_{\Delta;\mathbb{B}}]} [\![\mathbb{C}]\!] \end{aligned} \tag{3.14}$$

Next, it is necessary to demonstrate that the interpretation of exchange and substitution holds for injections and the case statement.

Lemma 3.2.2. *(Exchange) For all judgements* $\Gamma, x : \mathbb{A}, y : \mathbb{B}, \Delta \triangleright v : \mathbb{C}$*, the following equation holds:* $[\![\Gamma, x : \mathbb{A}, y : \mathbb{B}, \Delta \triangleright v : \mathbb{C}]\!] = [\![\Gamma, y : \mathbb{B}, x : \mathbb{A}, \Delta \triangleright v : \mathbb{C}]\!] \cdot \mathit{exch}_{\Gamma,\underline{\mathbb{A},\mathbb{B}},\Delta}$

Proof It is necessary to consider three scenarios:

1. $x : \mathbb{A}, y : \mathbb{B}$ are variables in the context of v;
2. $x : \mathbb{A}, y : \mathbb{B}$ are variables in the context of w and u;
3. $x : \mathbb{A}$ is a variable in the context of v and $y : \mathbb{B}$ is a variable in the context of w and u.

For the first case,

$$
\begin{aligned}
&[\![\Gamma, x : \mathbb{A}, y : \mathbb{B}, \Delta \rhd \textsf{case } \{\textsf{inl}_{\mathbb{E}}(a) \Rightarrow w; \textsf{inr}_{\mathbb{E}}(b) \Rightarrow u\} : \mathbb{C}]\!] \\
=\;& \big[[\![\Sigma, a : \mathbb{D} \rhd w]\!] \cdot \textsf{jn}_{\Sigma;\mathbb{D}}, [\![\Sigma, b : \mathbb{E} \rhd u]\!] \cdot \textsf{jn}_{\Sigma;\mathbb{E}}\big] \cdot \textsf{dist} \cdot \textsf{sw} \\
&\cdot ([\![\Gamma_1, x : \mathbb{A}, y : \mathbb{B}, \Gamma_2 \rhd v : \mathbb{D} \oplus \mathbb{E}]\!] \otimes \textsf{id}) \cdot \textsf{sp}_{\Gamma_1,\mathbb{A},\mathbb{B},\Gamma_2;\Sigma} \cdot \textsf{sh}_{\Gamma,\mathbb{A},\mathbb{B},\Delta} \\
=\;& \big[[\![\Sigma, a : \mathbb{D} \rhd w]\!] \cdot \textsf{jn}_{\Sigma;\mathbb{D}}, [\![\Sigma, b : \mathbb{E} \rhd u]\!] \cdot \textsf{jn}_{\Sigma;\mathbb{E}}\big] \cdot \textsf{dist} \cdot \textsf{sw} \\
&\cdot ([\![\Gamma_1, y : \mathbb{B}, x : \mathbb{A}, \Gamma_2 \rhd v : \mathbb{C}]\!] \cdot \textsf{exch}_{\Gamma_1,\underline{\mathbb{A},\mathbb{B}},\Gamma_2} \otimes \textsf{id}) \cdot \textsf{sp}_{\Gamma_1,\mathbb{A},\mathbb{B},\Gamma_2;\Sigma} \cdot \textsf{sh}_{\Gamma,\mathbb{A},\mathbb{B},\Delta} \\
=\;& \big[[\![\Sigma, a : \mathbb{D} \rhd w]\!] \cdot \textsf{jn}_{\Sigma;\mathbb{D}}, [\![\Sigma, b : \mathbb{E} \rhd u]\!] \cdot \textsf{jn}_{\Sigma;\mathbb{E}}\big] \cdot \textsf{dist} \cdot \textsf{sw} \\
&\cdot ([\![\Gamma_1, y : \mathbb{B}, x : \mathbb{A}, \Gamma_2 \rhd v : \mathbb{D} \oplus \mathbb{E}]\!] \otimes \textsf{id}) \cdot \textsf{sp}_{\Gamma_1,\mathbb{B},\mathbb{A},\Gamma_2;\Sigma} \cdot \textsf{exch}_{\Gamma_1,\underline{\mathbb{A},\mathbb{B}},\Gamma_2,\Sigma} \cdot \textsf{jn}_{\Gamma_1,\mathbb{A},\mathbb{B},\Gamma_2;\Sigma} \\
&\cdot \textsf{sp}_{\Gamma_1,\mathbb{A},\mathbb{B},\Gamma_2;\Sigma} \cdot \textsf{sh}_{\Gamma,\mathbb{A},\mathbb{B},\Delta} \\
=\;& \big[[\![\Sigma, a : \mathbb{D} \rhd w]\!] \cdot \textsf{jn}_{\Sigma;\mathbb{D}}, [\![\Sigma, b : \mathbb{E} \rhd u]\!] \cdot \textsf{jn}_{\Sigma;\mathbb{E}}\big] \cdot \textsf{dist} \cdot \textsf{sw} \\
&\cdot ([\![\Gamma_1, y : \mathbb{B}, x : \mathbb{A}, \Gamma_2 \rhd v : \mathbb{D} \oplus \mathbb{E}]\!] \otimes \textsf{id}) \cdot \textsf{sp}_{\Gamma_1,\mathbb{B},\mathbb{A},\Gamma_2;\Sigma} \cdot \textsf{exch}_{\Gamma,\underline{\mathbb{A},\mathbb{B}},\Delta} \cdot \textsf{sh}_{\Gamma,\mathbb{A},\mathbb{B},\Delta} \\
=\;& \big[[\![\Sigma, a : \mathbb{D} \rhd w]\!] \cdot \textsf{jn}_{\Sigma;\mathbb{D}}, [\![\Sigma, b : \mathbb{E} \rhd u]\!] \cdot \textsf{jn}_{\Sigma;\mathbb{E}}\big] \cdot \textsf{dist} \cdot \textsf{sw} \\
&\cdot ([\![\Gamma_1, y : \mathbb{B}, x : \mathbb{A}, \Gamma_2 \rhd v : \mathbb{D} \oplus \mathbb{E}]\!] \otimes \textsf{id}) \cdot \textsf{sp}_{\Gamma_1,\mathbb{B},\mathbb{A},\Gamma_2;\Sigma} \cdot \textsf{sh}_{\Gamma,\mathbb{B},\mathbb{A},\Delta} \cdot \textsf{exch}_{\Gamma,\underline{\mathbb{A},\mathbb{B}},\Delta} \\
=\;& [\![\Gamma, y : \mathbb{B}, x : \mathbb{A}, \Delta \rhd \textsf{case } \{\textsf{inl}_{\mathbb{E}}(a) \Rightarrow w; \textsf{inr}_{\mathbb{E}}(b) \Rightarrow u\} : \mathbb{C}]\!] \cdot \textsf{exch}_{\Gamma,\underline{\mathbb{A},\mathbb{B}},\Delta}
\end{aligned}
$$

Now, for the second case,

$$
\begin{aligned}
&\llbracket \Gamma, x : \mathbb{A}, y : \mathbb{B}, \Delta \triangleright \textsf{case } \{\textsf{inl}_{\mathbb{E}}(a) \Rightarrow w; \textsf{inr}_{\mathbb{E}}(b) \Rightarrow u\} : \mathbb{C} \rrbracket \\
&= \big[\llbracket \Delta_1, x : \mathbb{A}, y : \mathbb{B}, \Delta_2, a : \mathbb{D} \triangleright w \rrbracket \cdot \textsf{jn}_{\Delta_1,\mathbb{A},\mathbb{B},\Delta_2,\mathbb{D}}, \llbracket \Delta_1, x : \mathbb{A}, y : \mathbb{B}, \Delta_2, b : \mathbb{E} \triangleright u \rrbracket \cdot \textsf{jn}_{\Delta_1,\mathbb{A},\mathbb{B},\Delta_2,\mathbb{D}}\big] \\
&\quad \cdot \textsf{dist} \cdot \textsf{sw} \cdot (\llbracket \Sigma \triangleright v : \mathbb{D} \oplus \mathbb{E} \rrbracket \otimes \textsf{id}) \cdot \textsf{sp}_{\Sigma;\Delta_1,\mathbb{A},\mathbb{B},\Delta_2} \cdot \textsf{sh}_{\Gamma;\mathbb{A},\mathbb{B},\Delta} \\
&= [\llbracket \Delta_1, y : \mathbb{B}, x : \mathbb{A}, \Delta_2, a : \mathbb{D} \triangleright w \rrbracket \cdot \textsf{exch}_{\Delta_1,\underline{\mathbb{A},\mathbb{B}},\Delta_2,\mathbb{D}} \cdot \textsf{jn}_{\Delta_1,\mathbb{A},\mathbb{B},\Delta_2,\mathbb{D}}, \\
&\quad \llbracket \Delta_1, y : \mathbb{B}, x : \mathbb{A}, \Delta_2, b : \mathbb{E} \triangleright w \rrbracket \cdot \textsf{exch}_{\Delta_1,\underline{\mathbb{A},\mathbb{B}},\Delta_2,\mathbb{E}} \cdot \textsf{jn}_{\Delta_1,\mathbb{A},\mathbb{B},\Delta_2,\mathbb{E}}] \cdot \textsf{dist} \cdot \textsf{sw} \\
&\quad \cdot (\llbracket \Sigma \triangleright v : \mathbb{D} \oplus \mathbb{E} \rrbracket \otimes \textsf{id}) \cdot \textsf{sp}_{\Sigma;\Delta_1,\mathbb{A},\mathbb{B},\Delta_2} \cdot \textsf{sh}_{\Gamma,\mathbb{A},\mathbb{B},\Delta} \\
&= [\llbracket \Delta_1, y : \mathbb{B}, x : \mathbb{A}, \Delta_2, a : \mathbb{D} \triangleright w \rrbracket \cdot \textsf{jn}_{\Delta_1,\mathbb{B},\mathbb{A},\Delta_2,\mathbb{D}}, \llbracket \Delta_1, y : \mathbb{B}, x : \mathbb{A}, \Delta_2, b : \mathbb{E} \triangleright w \rrbracket \cdot \textsf{jn}_{\Delta_1,\mathbb{B},\mathbb{A},\Delta_2,\mathbb{E}}] \\
&\quad \cdot \textsf{dist} \cdot \textsf{sp}_{\Delta_1,\mathbb{B},\mathbb{A},\Delta_2;\mathbb{D}\oplus\mathbb{E}} \cdot \textsf{exch}_{\Delta_1,\underline{\mathbb{A},\mathbb{B}},\Delta_2,\mathbb{D}\oplus\mathbb{E}} \cdot \textsf{jn}_{\Delta_1,\mathbb{A},\mathbb{B},\Delta_2;\mathbb{D}\oplus\mathbb{E}} \cdot \textsf{sw} \cdot (\llbracket \Sigma \triangleright v : \mathbb{D} \oplus \mathbb{E} \rrbracket \otimes \textsf{id}) \\
&\quad \cdot \textsf{sp}_{\Sigma;\Delta_1,\mathbb{A},\mathbb{B},\Delta_2} \cdot \textsf{sh}_{\Gamma,\mathbb{A},\mathbb{B},\Delta} \\
&= [\llbracket \Delta_1, y : \mathbb{B}, x : \mathbb{A}, \Delta_2, a : \mathbb{D} \triangleright w \rrbracket \cdot \textsf{jn}_{\Delta_1,\mathbb{B},\mathbb{A},\Delta_2,\mathbb{D}}, \llbracket \Delta_1, y : \mathbb{B}, x : \mathbb{A}, \Delta_2, b : \mathbb{E} \triangleright w \rrbracket \cdot \textsf{jn}_{\Delta_1,\mathbb{B},\mathbb{A},\Delta_2,\mathbb{E}}] \\
&\quad \cdot \textsf{dist} \cdot \textsf{sw} \cdot (\llbracket \Sigma \triangleright v : \mathbb{D} \oplus \mathbb{E} \rrbracket \otimes \textsf{id}) \cdot \textsf{sp}_{\Gamma;\Delta_1,\mathbb{B},\mathbb{A},\Delta_2} \cdot \textsf{exch}_{\Gamma,\Delta_1,\underline{\mathbb{A},\mathbb{B}},\Delta_2} \cdot \textsf{jn}_{\Gamma;\Delta_1,\mathbb{A},\mathbb{B},\Delta_2} \\
&\quad \cdot \textsf{sp}_{\Sigma;\Delta_1,\mathbb{A},\mathbb{B},\Delta_2} \cdot \textsf{sh}_{\Gamma,\mathbb{A},\mathbb{B},\Delta} \\
&= [\llbracket \Delta_1, y : \mathbb{B}, x : \mathbb{A}, \Delta_2, a : \mathbb{D} \triangleright w \rrbracket \cdot \textsf{jn}_{\Delta_1,\mathbb{B},\mathbb{A},\Delta_2,\mathbb{D}}, \llbracket \Delta_1, y : \mathbb{B}, x : \mathbb{A}, \Delta_2, b : \mathbb{E} \triangleright w \rrbracket \cdot \textsf{jn}_{\Delta_1,\mathbb{B},\mathbb{A},\Delta_2,\mathbb{E}}] \\
&\quad \cdot \textsf{dist} \cdot \textsf{sw} \cdot (\llbracket \Sigma \triangleright v : \mathbb{D} \oplus \mathbb{E} \rrbracket \otimes \textsf{id}) \cdot \textsf{sp}_{\Sigma;\Delta_1,\mathbb{B},\mathbb{A},\Delta_2} \cdot \textsf{sh}_{\Gamma,\mathbb{B},\mathbb{A},\Delta} \cdot \textsf{exch}_{\Gamma,\underline{\mathbb{A},\mathbb{B}},\Delta} \\
&= \llbracket \Gamma, y : \mathbb{B}, x : \mathbb{A}, \Delta \triangleright \textsf{case } \{\textsf{inl}_{\mathbb{E}}(a) \Rightarrow w; \textsf{inr}_{\mathbb{E}}(b) \Rightarrow u\} : \mathbb{C} \rrbracket \cdot \textsf{exch}_{\Gamma,\underline{\mathbb{A},\mathbb{B}},\Delta}
\end{aligned}
$$

Finally, for the third case, note that, the shuffle operator is permutation of typed variables that preserves the relative order of the variables in both contexts, and, as a result, $\Gamma, \mathbb{B}, \mathbb{A}, \Delta \in \textsf{Sf}(\Gamma_1, x : \mathbb{A}, \Gamma_2; \Delta_2, y : \mathbb{B}, \Delta_1)$. Thus, the proof is as follows:

$$
\begin{aligned}
&\llbracket \Gamma, x : \mathbb{A}, y : \mathbb{B}, \Delta \triangleright \textsf{case } \{\textsf{inl}_{\mathbb{E}}(a) \Rightarrow w; \textsf{inr}_{\mathbb{E}}(b) \Rightarrow u\} : \mathbb{C} \rrbracket \\
&= \big[\llbracket \Delta_2, y : \mathbb{B}, \Delta_1, a : \mathbb{D} \triangleright w \rrbracket \cdot \textsf{jn}_{\Delta_2,\mathbb{B},\Delta_1;\mathbb{D}}, \llbracket \Delta_2, y : \mathbb{B}, \Delta_1, b : \mathbb{E} \triangleright u \rrbracket \cdot \textsf{jn}_{\Delta_2,\mathbb{B},\Delta_1;\mathbb{E}}\big] \cdot \textsf{dist} \cdot \textsf{sw} \\
&\quad \cdot (\llbracket \Delta_2, y : \mathbb{B}, \Delta_1 \triangleright v : \mathbb{D} \oplus \mathbb{E} \rrbracket \otimes \textsf{id}) \cdot \textsf{sp}_{\Gamma_1,x:\mathbb{A},\Gamma_2;\Delta_2,y:\mathbb{B},\Delta_1} \cdot \textsf{sh}_{\Gamma,\mathbb{A},\mathbb{B},\Delta} \\
&= \big[\llbracket \Delta_2, y : \mathbb{B}, \Delta_1, a : \mathbb{D} \triangleright w \rrbracket \cdot \textsf{jn}_{\Delta_2,\mathbb{B},\Delta_1;\mathbb{D}}, \llbracket \Delta_2, y : \mathbb{B}, \Delta_1, b : \mathbb{E} \triangleright u \rrbracket \cdot \textsf{jn}_{\Delta_2,\mathbb{B},\Delta_1;\mathbb{E}}\big] \cdot \textsf{dist} \cdot \textsf{sw} \\
&\quad \cdot (\llbracket \Delta_2, y : \mathbb{B}, \Delta_1 \triangleright v : \mathbb{D} \oplus \mathbb{E} \rrbracket \otimes \textsf{id}) \cdot \textsf{sp}_{\Gamma_1,x:\mathbb{A},\Gamma_2;\Delta_2,y:\mathbb{B},\Delta_1} \cdot \textsf{sh}_{\Gamma,\mathbb{A},\mathbb{B},\Delta} \cdot \textsf{exch}_{\Gamma,\underline{\mathbb{A},\mathbb{B}},\Delta} \\
&= \llbracket \Gamma, y : \mathbb{B}, x : \mathbb{A}, \Delta \triangleright \textsf{case } \{\textsf{inl}_{\mathbb{E}}(a) \Rightarrow w; \textsf{inr}_{\mathbb{E}}(b) \Rightarrow u\} : \mathbb{C} \rrbracket \cdot \textsf{exch}_{\Gamma,\underline{\mathbb{A},\mathbb{B}},\Delta}
\end{aligned}
$$

Lemma 3.2.3. *(Substitution) For all judgements $\Gamma, x : \mathbb{A} \triangleright v : \mathbb{B}$ and $\Delta \triangleright w : \mathbb{A}$ the following equation holds: $\llbracket \Gamma, \Delta \triangleright v[w/x] : \mathbb{B} \rrbracket = \llbracket \Gamma, x : \mathbb{A} \triangleright v : \mathbb{B} \rrbracket \cdot jn_{\Gamma;\mathbb{A}} \cdot (id \otimes \llbracket \Delta \triangleright w : \mathbb{A} \rrbracket) \cdot sp_{\Gamma;\Delta}$*

In this case, it is necessary to consider two scenarios:

1. $z : \mathbb{D}$ is a variable in the context of v;

2. $z : \mathbb{D}$ is a variable in the context of w and u.

For the first case,

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$$
\begin{aligned}
&[\![E, \Sigma \triangleright \text{case } \{\mathsf{inl}_{\mathbb{B}}(x) \Rightarrow w; \mathsf{inr}_{\mathbb{A}}(y) \Rightarrow u\}[a/z] : \mathbb{C}]\!] \\
&= \left[[\![\Delta, x : \mathbb{A} \triangleright w : \mathbb{C}]\!] \cdot \mathsf{jn}_{\Delta;\mathbb{A}}, [\![\Delta, y : \mathbb{B} \triangleright u]\!] \cdot \mathsf{jn}_{\Delta;\mathbb{B}}\right] \cdot \mathsf{dist} \cdot \mathsf{sw} \cdot ([\![\Gamma, \Sigma \triangleright v[a/z] : \mathbb{A} \oplus \mathbb{B}]\!] \otimes \mathsf{id}) \\
&\quad \cdot \mathsf{sp}_{\Gamma,\Sigma;\Delta} \cdot \mathsf{sh}_{E,\Sigma} \\
&= \left[[\![\Delta, x : \mathbb{A} \triangleright w : \mathbb{C}]\!] \cdot \mathsf{jn}_{\Delta;\mathbb{A}}, [\![\Delta, y : \mathbb{B} \triangleright u]\!] \cdot \mathsf{jn}_{\Delta;\mathbb{B}}\right] \cdot \mathsf{dist} \cdot \mathsf{sw} \cdot (([\![\Gamma, z : \mathbb{D} \triangleright v : \mathbb{A} \oplus \mathbb{B}]\!] \cdot \mathsf{jn}_{\Gamma;\mathbb{D}} \\
&\quad \cdot (\mathsf{id} \otimes [\![\Sigma \triangleright a : \mathbb{D}]\!]) \cdot \mathsf{sp}_{\Gamma;\Sigma}) \otimes \mathsf{id}) \cdot \mathsf{sp}_{\Gamma,\Sigma;\Delta} \cdot \mathsf{sh}_{E,\Sigma} \\
&= \left[[\![\Delta, x : \mathbb{A} \triangleright w : \mathbb{C}]\!] \cdot \mathsf{jn}_{\Delta;\mathbb{A}}, [\![\Delta, y : \mathbb{B} \triangleright u]\!] \cdot \mathsf{jn}_{\Delta;\mathbb{B}}\right] \cdot \mathsf{dist} \cdot \mathsf{sw} \cdot ([\![\Gamma, z : \mathbb{D} \triangleright v : \mathbb{A} \oplus \mathbb{B}]\!] \otimes \mathsf{id}) \\
&\quad \cdot \mathsf{sp}_{\Gamma,\mathbb{D};\Delta} \cdot \mathsf{jn}_{\Gamma;\mathbb{D};\Delta} \cdot (\mathsf{id} \otimes [\![\Sigma \triangleright a : \mathbb{D}]\!] \otimes \mathsf{id}) \cdot \mathsf{sp}_{\Gamma;\Sigma;\Delta} \cdot \mathsf{sh}_{E,\Sigma} \\
&= \left[[\![\Delta, x : \mathbb{A} \triangleright w : \mathbb{C}]\!] \cdot \mathsf{jn}_{\Delta;\mathbb{A}}, [\![\Delta, y : \mathbb{B} \triangleright u]\!] \cdot \mathsf{jn}_{\Delta;\mathbb{B}}\right] \cdot \mathsf{dist} \cdot \mathsf{sw} \cdot ([\![\Gamma, z : \mathbb{D} \triangleright v : \mathbb{A} \oplus \mathbb{B}]\!] \otimes \mathsf{id}) \\
&\quad \ldots \\
&= \left[[\![\Delta, x : \mathbb{A} \triangleright w : \mathbb{C}]\!] \cdot \mathsf{jn}_{\Delta;\mathbb{A}}, [\![\Delta, y : \mathbb{B} \triangleright u]\!] \cdot \mathsf{jn}_{\Delta;\mathbb{B}}\right] \cdot \mathsf{dist} \cdot \mathsf{sw} \cdot ([\![\Gamma, z : \mathbb{D} \triangleright v : \mathbb{A} \oplus \mathbb{B}]\!] \otimes \mathsf{id}) \\
&\quad \cdot \mathsf{sp}_{\Gamma,\mathbb{D};\Delta} \cdot \mathsf{sh}_{E,\mathbb{D}} \cdot \mathsf{jn}_{E;\mathbb{D}} \cdot (\mathsf{id} \otimes [\![\Sigma \triangleright a : \mathbb{D}]\!]) \cdot \mathsf{sp}_{E;\Sigma} \\
&= [\![E, z : \mathbb{D} \triangleright \text{case } \{\mathsf{inl}_{\mathbb{B}}(x) \Rightarrow w; \mathsf{inr}_{\mathbb{A}}(y) \Rightarrow u\} : \mathbb{C}]\!] \cdot \mathsf{jn}_{E;\mathbb{D}} \cdot (\mathsf{id} \otimes [\![\Sigma \triangleright a : \mathbb{D}]\!]) \cdot \mathsf{sp}_{E;\Sigma}
\end{aligned}
$$

For the second case,

$$
\begin{aligned}
&[\![E, \Sigma \triangleright \mathsf{case}\ \{\mathsf{inl}_{\mathbb{B}}(x) \Rightarrow w; \mathsf{inr}_{\mathbb{A}}(y) \Rightarrow u\}[a/z] : \mathbb{C}]\!] \\
= &\ [[\![\Delta, \Sigma, x : \mathbb{A} \triangleright w[a/z] : \mathbb{C}]\!] \cdot \mathsf{jn}_{\Delta,\Sigma,\mathbb{A}}, [\![\Delta, \Sigma, y : \mathbb{B} \triangleright u[a/z] : \mathbb{C}]\!] \cdot \mathsf{jn}_{\Delta,\Sigma,\mathbb{B}}] \cdot \mathsf{dist} \cdot \mathsf{sw} \\
&\cdot ([\![\Gamma \triangleright v : \mathbb{A} \oplus \mathbb{B}]\!] \otimes \mathsf{id}) \cdot \mathsf{sp}_{\Gamma;\Sigma,\Delta} \cdot \mathsf{sh}_{E,\Sigma} \\
= &[[\![\Delta, x : \mathbb{A}, \Sigma \triangleright w[a/z] : \mathbb{C}]\!] \cdot \mathsf{exch}_{\Delta,\Sigma,\mathbb{A}} \cdot \mathsf{jn}_{\Delta,\Sigma,\mathbb{A}}, [\![\Delta, y : \mathbb{B}, \Sigma \triangleright u[a/z] : \mathbb{C}]\!] \cdot \mathsf{exch}_{\Delta,\Sigma,\mathbb{B}} \\
&\cdot \mathsf{jn}_{\Delta,\Sigma,\mathbb{B}}] \cdot \mathsf{dist} \cdot \mathsf{sw} \cdot ([\![\Gamma \triangleright v : \mathbb{A} \oplus \mathbb{B}]\!] \otimes \mathsf{id}) \cdot \mathsf{sp}_{\Gamma;\Sigma,\Delta} \cdot \mathsf{sh}_{E,\Sigma} \\
= &\ [[\![\Delta, x : \mathbb{A}, z : \mathbb{D} \triangleright w : \mathbb{C}]\!] \cdot \mathsf{jn}_{\Delta,\mathbb{A};\mathbb{D}} \cdot (\mathsf{id} \otimes [\![\Sigma \triangleright a : \mathbb{D}]\!]) \cdot \mathsf{sp}_{\Delta,\mathbb{A};\Sigma} \cdot \mathsf{exch}_{\Delta,\Sigma,\mathbb{A}} \cdot \mathsf{jn}_{\Delta,\Sigma,\mathbb{A}}, \\
&[\![\Delta, y : \mathbb{B}, z : \mathbb{D} \triangleright u : \mathbb{C}]\!] \cdot \mathsf{jn}_{\Delta,\mathbb{A};\mathbb{D}} \cdot (\mathsf{id} \otimes [\![\Sigma \triangleright a : \mathbb{D}]\!]) \cdot \mathsf{sp}_{\Delta,\mathbb{B};\Sigma} \cdot \mathsf{exch}_{\Delta,\Sigma,\mathbb{B}}] \cdot \mathsf{jn}_{\Delta,\Sigma,\mathbb{B}} \cdot \mathsf{dist} \\
&\cdot \mathsf{sw} \cdot ([\![\Gamma \triangleright v : \mathbb{A} \oplus \mathbb{B}]\!] \otimes \mathsf{id}) \cdot \mathsf{sp}_{\Gamma;\Sigma,\Delta} \cdot \mathsf{sh}_{E,\Sigma} \\
= &\ [[\![\Delta, x : \mathbb{A}, z : \mathbb{D} \triangleright w : \mathbb{C}]\!] \cdot \mathsf{exch}_{\Delta,\mathbb{D},\mathbb{A}} \cdot \mathsf{jn}_{\Delta,\mathbb{D},\mathbb{A}}, [\![\Delta, y : \mathbb{B}, z : \mathbb{D} \triangleright u : \mathbb{C}]\!] \cdot \mathsf{exch}_{\Delta,\mathbb{D},\mathbb{B}} \cdot \mathsf{jn}_{\Delta,\mathbb{D},\mathbb{B}}] \\
&\cdot \mathsf{dist} \cdot \mathsf{sp}_{\Delta,\mathbb{D};\mathbb{A}\oplus\mathbb{B}} \cdot \mathsf{jn}_{\Delta;\mathbb{D};\mathbb{A}\oplus\mathbb{B}} \cdot (\mathsf{id} \otimes [\![\Sigma \triangleright a : \mathbb{D}]\!] \otimes \mathsf{id}) \cdot \mathsf{sp}_{\Delta;\Sigma,\mathbb{A}\oplus\mathbb{B}} \cdot \mathsf{sw} \\
&\cdot ([\![\Gamma \triangleright v : \mathbb{A} \oplus \mathbb{B}]\!] \otimes \mathsf{id}) \cdot \mathsf{sp}_{\Gamma;\Sigma,\Delta} \cdot \mathsf{sh}_{E,\Sigma} \\
= &\ [[\![\Delta, x : \mathbb{A}, z : \mathbb{D} \triangleright w : \mathbb{C}]\!] \cdot \mathsf{exch}_{\Delta,\mathbb{D},\mathbb{A}} \cdot \mathsf{jn}_{\Delta,\mathbb{D},\mathbb{A}}, [\![\Delta, y : \mathbb{B}, z : \mathbb{D} \triangleright u : \mathbb{C}]\!] \cdot \mathsf{exch}_{\Delta,\mathbb{D},\mathbb{B}} \cdot \mathsf{jn}_{\Delta,\mathbb{D},\mathbb{B}}] \\
&\cdot \mathsf{dist} \cdot \mathsf{sw} \cdot ([\![\Gamma \triangleright v : \mathbb{A} \oplus \mathbb{B}]\!] \otimes \mathsf{id}) \cdot \mathsf{sp}_{\Gamma;\Delta,\mathbb{D}} \cdot \mathsf{jn}_{\Gamma;\Delta;\mathbb{D}} \cdot (\mathsf{id} \otimes [\![\Sigma \triangleright a : \mathbb{D}]\!] \otimes \mathsf{id}) \\
&\cdot \mathsf{sp}_{\Gamma;\Sigma;\Delta} \cdot \mathsf{sh}_{E,\Sigma} \\
= &\ [[\![\Delta, x : \mathbb{A}, z : \mathbb{D} \triangleright w : \mathbb{C}]\!] \cdot \mathsf{exch}_{\Delta,\mathbb{D},\mathbb{A}} \cdot \mathsf{jn}_{\Delta,\mathbb{D},\mathbb{A}}, [\![\Delta, y : \mathbb{B}, z : \mathbb{D} \triangleright u : \mathbb{C}]\!] \cdot \mathsf{exch}_{\Delta,\mathbb{D},\mathbb{B}} \cdot \mathsf{jn}_{\Delta,\mathbb{D},\mathbb{B}}] \\
&\cdot \mathsf{dist} \cdot \mathsf{sw} \cdot ([\![\Gamma \triangleright v : \mathbb{A} \oplus \mathbb{B}]\!] \otimes \mathsf{id}) \cdot \mathsf{sp}_{\Gamma;\Delta,\mathbb{D}} \cdot \mathsf{sh}_{E,\mathbb{D}} \cdot \mathsf{jn}_{E;\mathbb{D}} \cdot (\mathsf{id} \otimes [\![\Sigma \triangleright a : \mathbb{D}]\!]) \cdot \mathsf{sp}_{E;\Sigma} \\
= &\ [[\![\Delta, z : \mathbb{D}, x : \mathbb{A} \triangleright w : \mathbb{C}]\!] \cdot \mathsf{jn}_{\Delta,\mathbb{D},\mathbb{A}}, [\![\Delta, z : \mathbb{D}, y : \mathbb{B} \triangleright u : \mathbb{C}]\!] \cdot \mathsf{jn}_{\Delta,\mathbb{D},\mathbb{B}}] \cdot \mathsf{dist} \cdot \mathsf{sw} \\
&\cdot ([\![\Gamma \triangleright v : \mathbb{A} \oplus \mathbb{B}]\!] \otimes \mathsf{id}) \cdot \mathsf{sp}_{\Gamma;\Delta,\mathbb{D}} \cdot \mathsf{sh}_{E,\mathbb{D}} \cdot \mathsf{jn}_{E;\mathbb{D}} \cdot (\mathsf{id} \otimes [\![\Sigma \triangleright a : \mathbb{D}]\!]) \cdot \mathsf{sp}_{E;\Sigma} \\
= &[\![E, z : \mathbb{D} \triangleright \mathsf{case}\ \{\mathsf{inl}_{\mathbb{B}}(x) \Rightarrow w; \mathsf{inr}_{\mathbb{A}}(y) \Rightarrow u\}[a/z] : \mathbb{C}]\!] \cdot \mathsf{jn}_{E;\mathbb{D}} \cdot (\mathsf{id} \otimes [\![\Sigma \triangleright a : \mathbb{D}]\!]) \cdot \mathsf{sp}_{E;\Sigma}
\end{aligned}
$$

3.2.3 β and η Reductions

Fazer intro a o que isto é

In this subsection it will be shown that the following equations hold for the model considered.

$$
\begin{aligned}
\beta^{inl}_{case} &: \mathsf{case}\ \mathsf{inl}_{\mathbb{B}}(v)\ \{\mathsf{inl}_{\mathbb{B}}(x) \Rightarrow w; \mathsf{inr}_{\mathbb{A}}(y) \Rightarrow u\} : \mathbb{C} = w[v/x] : \mathbb{C} \\
\beta^{inr}_{case} &: \mathsf{case}\ \mathsf{inr}_{\mathbb{A}}(v)\ \{\mathsf{inl}_{\mathbb{B}}(x) \Rightarrow w; \mathsf{inr}_{\mathbb{A}}(y) \Rightarrow u\} : \mathbb{C} = w[v/y] : \mathbb{C} \\
\eta_{case} &: \mathsf{case}\ (v)\ \{\mathsf{inl}_{\mathbb{B}}(y) \Rightarrow w[\mathsf{inl}_{\mathbb{B}}(y)/x]; \mathsf{inr}_{\mathbb{A}}(z) \Rightarrow w[\mathsf{inr}_{\mathbb{A}}(z)/x]\} : \mathbb{C} = w[v/x] : \mathbb{C}
\end{aligned}
$$

Proof It is necessary to demonstrate that

$$\llbracket\, \mathsf{case}\ \mathsf{inl}_{\mathbb{B}}(v)\ \{\mathsf{inl}_{\mathbb{B}}(x) \Rightarrow w;\ \mathsf{inr}_{\mathbb{A}}(y) \Rightarrow u\} : \mathbb{C}\rrbracket = \llbracket w[v/x] : \mathbb{C}\rrbracket$$

$$\llbracket\, \mathsf{case}\ \mathsf{inr}_{\mathbb{A}}(v)\ \{\mathsf{inl}_{\mathbb{B}}(x) \Rightarrow w;\ \mathsf{inr}_{\mathbb{A}}(y) \Rightarrow u\} : \mathbb{C}\rrbracket = \llbracket w[v/x] : \mathbb{C}\rrbracket$$

$$\llbracket\, \mathsf{case}\ (v)\ \{\mathsf{inl}_{\mathbb{B}}(y) \Rightarrow w[\mathsf{inl}_{\mathbb{B}}(y)/x];\ \mathsf{inr}_{\mathbb{A}}(z) \Rightarrow w[\mathsf{inr}_{\mathbb{A}}(z)/x]\} : \mathbb{C}\rrbracket = \llbracket w[v/x] : \mathbb{C}\rrbracket$$

Regarding the first equation,

$$\begin{aligned}
&\llbracket\, \mathsf{case}\ \mathsf{inl}_{\mathbb{B}}(v)\ \{\mathsf{inl}_{\mathbb{B}}(x) \Rightarrow w;\ \mathsf{inr}_{\mathbb{A}}(y) \Rightarrow u\} : \mathbb{C}\rrbracket \\
=&[\llbracket \Delta, x : \mathbb{A} \rhd w : \mathbb{C}\rrbracket \cdot \mathsf{jn}_{\Delta;\mathbb{A}}, \llbracket \Delta, y : \mathbb{B} \rhd u : \mathbb{C}\rrbracket \cdot \mathsf{jn}_{\Delta;\mathbb{B}}] \cdot \mathsf{dist} \cdot \mathsf{sw} \cdot (\llbracket \Gamma \rhd \mathsf{inl}_{\mathbb{B}}(v) : \mathbb{A} \oplus \mathbb{B}\rrbracket \\
&\otimes \mathsf{id}) \cdot \mathsf{sp}_{\Gamma;\Delta} \cdot \mathsf{sh}_E \\
=&[\llbracket \Delta, x : \mathbb{A} \rhd w : \mathbb{C}\rrbracket \cdot \mathsf{jn}_{\Delta;\mathbb{A}}, \llbracket \Delta, y : \mathbb{B} \rhd u : \mathbb{C}\rrbracket \cdot \mathsf{jn}_{\Delta;\mathbb{B}}] \cdot \mathsf{dist} \cdot \mathsf{sw} \cdot (\mathsf{IL} \cdot \llbracket \Gamma \rhd v : \mathbb{A}\rrbracket \otimes \mathsf{id}) \\
&\cdot \mathsf{sp}_{\Gamma;\Delta} \cdot \mathsf{sh}_E \\
=&[\llbracket \Delta, x : \mathbb{A} \rhd w : \mathbb{C}\rrbracket \cdot \mathsf{jn}_{\Delta;\mathbb{A}}, \llbracket \Delta, y : \mathbb{B} \rhd u : \mathbb{C}\rrbracket \cdot \mathsf{jn}_{\Delta;\mathbb{B}}] \cdot \mathsf{dist} \cdot \mathsf{sw} \cdot (\mathsf{IL} \otimes \mathsf{id}) \cdot (\llbracket \Gamma \rhd v : \mathbb{A}\rrbracket \\
&\otimes \mathsf{id}) \cdot \mathsf{sp}_{\Gamma;\Delta} \cdot \mathsf{sh}_E \\
=&[\llbracket \Delta, x : \mathbb{A} \rhd w : \mathbb{C}\rrbracket \cdot \mathsf{jn}_{\Delta;\mathbb{A}}, \llbracket \Delta, y : \mathbb{B} \rhd u : \mathbb{C}\rrbracket \cdot \mathsf{jn}_{\Delta;\mathbb{B}}] \cdot \mathsf{dist} \cdot (\mathsf{id} \otimes \mathsf{IL}) \cdot \mathsf{sw} \cdot (\llbracket \Gamma \rhd v : \mathbb{A}\rrbracket \\
&\otimes \mathsf{id}) \cdot \mathsf{sp}_{\Gamma;\Delta} \cdot \mathsf{sh}_E
\end{aligned}$$

Given that $[\mathsf{id} \otimes \mathsf{IL}, \mathsf{id} \otimes \mathsf{IR}] \cdot \mathsf{IL} = \mathsf{id} \otimes \mathsf{IL}$, it follows that the following diagram commutes.

$$\begin{tikzcd}
X \otimes Y \arrow[rr, "\mathsf{id} \otimes \mathsf{IL}"] \arrow[d, "\mathsf{IL}"'] & & X \otimes (Y \oplus Y) \arrow[dll, bend right, "\mathsf{dist}"'] \\
X \otimes Y \oplus X \otimes Z \arrow[urr, bend right, "{[\mathsf{id} \otimes \mathsf{IL}, \mathsf{id} \otimes \mathsf{IR}]}"'] & &
\end{tikzcd}$$

And as a result, $\mathsf{dist} \cdot (\mathsf{id} \otimes \mathsf{IL}) = \mathsf{IL}$. Therefore,

$$\begin{aligned}
&[\llbracket \Delta, x : \mathbb{A} \rhd w : \mathbb{C}\rrbracket \cdot \mathsf{jn}_{\Delta;\mathbb{A}}, \llbracket \Delta, y : \mathbb{B} \rhd u : \mathbb{C}\rrbracket \cdot \mathsf{jn}_{\Delta;\mathbb{B}}] \cdot \mathsf{dist} \cdot (\mathsf{id} \otimes \mathsf{IL}) \cdot \mathsf{sw} \cdot (\llbracket \Gamma \rhd v : \mathbb{A}\rrbracket \\
&\otimes \mathsf{id}) \cdot \mathsf{sp}_{\Gamma;\Delta} \cdot \mathsf{sh}_E \\
=&[\llbracket \Delta, x : \mathbb{A} \rhd w : \mathbb{C}\rrbracket \cdot \mathsf{jn}_{\Delta;\mathbb{A}}, \llbracket \Delta, y : \mathbb{B} \rhd u : \mathbb{C}\rrbracket \cdot \mathsf{jn}_{\Delta;\mathbb{B}}] \cdot \mathsf{IL} \cdot \mathsf{sw} \cdot (\llbracket \Gamma \rhd v : \mathbb{A}\rrbracket \otimes \mathsf{id}) \cdot \mathsf{sp}_{\Gamma;\Delta} \\
&\cdot \mathsf{sh}_E \\
=&\llbracket \Delta, x : \mathbb{A} \rhd w : \mathbb{C}\rrbracket \cdot \mathsf{jn}_{\Delta;\mathbb{A}} \cdot \mathsf{sw} \cdot (\llbracket \Gamma \rhd v : \mathbb{A}\rrbracket \otimes \mathsf{id}) \cdot \mathsf{sp}_{\Gamma;\Delta} \cdot \mathsf{sh}_E \\
=&\llbracket \Delta, x : \mathbb{A} \rhd w : \mathbb{C}\rrbracket \cdot \mathsf{jn}_{\Delta;\mathbb{A}} \cdot (\mathsf{id} \otimes \llbracket \Gamma \rhd v : \mathbb{A}\rrbracket) \cdot \mathsf{sp}_{\Delta;\Gamma} \cdot \mathsf{sh}_E \\
=&\llbracket w[v/x] : \mathbb{C}\rrbracket
\end{aligned}$$

The proof for the second equation is analogous to the first one.

Taking into account that $[\mathsf{id} \otimes \mathsf{IL}, \mathsf{id} \otimes \mathsf{IR}] \cdot \mathsf{IR} = \mathsf{id} \otimes \mathsf{IR}$, it follows that the following diagram commutes.

$$\begin{tikzcd}
X \otimes Y \arrow[r, "\mathsf{id} \otimes \mathsf{IR}"] \arrow[d, "\mathsf{IR}"'] & X \otimes (Z \oplus Y) \arrow[dl, bend right, "\mathsf{dist}"'] \\
X \otimes Z \oplus X \otimes Y \arrow[ur, bend right, "{[\mathsf{id} \otimes \mathsf{IL}, \mathsf{id} \otimes \mathsf{IR}]}"'] &
\end{tikzcd}$$

Consequentlty, $\mathsf{dist} \cdot (\mathsf{id} \otimes \mathsf{IR}) = \mathsf{IR}$. Thus,

$$\begin{aligned}
&[\![\text{ case } \mathsf{inr}_{\mathbb{A}}(v) \{\mathsf{inl}_{\mathbb{B}}(x) \Rightarrow w; \mathsf{inr}_{\mathbb{A}}(y) \Rightarrow u\} : \mathbb{C}]\!] \\
=&[[\![\Delta, x : \mathbb{A} \triangleright w : \mathbb{C}]\!] \cdot \mathsf{jn}_{\Delta;\mathbb{A}}, [\![\Delta, y : \mathbb{B} \triangleright u : \mathbb{C}]\!] \cdot \mathsf{jn}_{\Delta;\mathbb{B}}] \cdot \mathsf{dist} \cdot (\mathsf{id} \otimes \mathsf{IR}) \cdot \mathsf{sw} \cdot ([\![\Gamma \triangleright v : \mathbb{B}]\!] \\
&\otimes \mathsf{id}) \cdot \mathsf{sp}_{\Gamma;\Delta} \cdot \mathsf{sh}_E \\
=&[[\![\Delta, x : \mathbb{A} \triangleright w : \mathbb{C}]\!] \cdot \mathsf{jn}_{\Delta;\mathbb{A}}, [\![\Delta, y : \mathbb{B} \triangleright u : \mathbb{C}]\!] \cdot \mathsf{jn}_{\Delta;\mathbb{B}}] \cdot \mathsf{IR} \cdot \mathsf{sw} \cdot ([\![\Gamma \triangleright v : \mathbb{B}]\!] \otimes \mathsf{id}) \cdot \mathsf{sp}_{\Gamma;\Delta} \\
&\cdot \mathsf{sh}_E \\
=&[\![\Delta, y : \mathbb{B} \triangleright w : \mathbb{C}]\!] \cdot \mathsf{jn}_{\Delta;\mathbb{B}} \cdot (\mathsf{id} \otimes [\![\Gamma \triangleright v : \mathbb{B}]\!]) \cdot \mathsf{sp}_{\Delta;\Gamma} \cdot \mathsf{sh}_E \\
=&[\![w[v/y] : \mathbb{C}]\!]
\end{aligned}$$

With respect to the third equation,

$$\begin{aligned}
&[\![\text{ case } (v) \{\mathsf{inl}_{\mathbb{B}}(y) \Rightarrow w[\mathsf{inl}_{\mathbb{B}}(y)/x]; \mathsf{inr}_{\mathbb{A}}(z) \Rightarrow w[\mathsf{inr}_{\mathbb{A}}(z)/x]\} : \mathbb{C}]\!] \\
=&[[\![\Delta, y : \mathbb{B} \triangleright w[\mathsf{inl}_{\mathbb{B}}(y)/x] : \mathbb{C}]\!] \cdot \mathsf{jn}_{\Delta;\mathbb{A}}, [\![\Delta, z : \mathbb{A} \triangleright w[\mathsf{inr}_{\mathbb{A}}(z)/x] : \mathbb{C}]\!] \cdot \mathsf{jn}_{\Delta;\mathbb{B}}] \cdot \mathsf{dist} \cdot \mathsf{sw} \\
&\cdot ([\![\Gamma \triangleright v : \mathbb{A} \oplus \mathbb{B}]\!] \otimes \mathsf{id}) \cdot \mathsf{sp}_{\Gamma;\Delta} \cdot \mathsf{sh}_E \\
=&[[\![\Delta, x : \mathbb{A} \oplus \mathbb{B} \triangleright w : \mathbb{C}]\!] \cdot \mathsf{jn}_{\Delta;\mathbb{A}\oplus\mathbb{B}} \cdot (\mathsf{id} \otimes [\![y : \mathbb{A} \triangleright \mathsf{inl}_{\mathbb{B}}(y) : \mathbb{A} \oplus \mathbb{B}]\!]) \cdot \mathsf{sp}_{\Delta;\mathbb{A}} \cdot \mathsf{jn}_{\Delta;\mathbb{A}}, \\
&[\![\Delta, x : \mathbb{A} \oplus \mathbb{B} \triangleright w : \mathbb{C}]\!] \cdot \mathsf{jn}_{\Delta;\mathbb{A}\oplus\mathbb{B}} \cdot (\mathsf{id} \otimes [\![z : \mathbb{B} \triangleright \mathsf{inr}_{\mathbb{A}}(z) : \mathbb{A} \oplus \mathbb{B}]\!]) \cdot \mathsf{sp}_{\Delta;\mathbb{B}} \cdot \mathsf{jn}_{\Delta;\mathbb{B}}] \\
&\cdot \mathsf{dist} \cdot \mathsf{sw} \cdot ([\![\Gamma \triangleright v : \mathbb{A} \oplus \mathbb{B}]\!] \otimes \mathsf{id}) \cdot \mathsf{sp}_{\Gamma;\Delta} \cdot \mathsf{sh}_E \\
=&[[\![\Delta, x : \mathbb{A} \oplus \mathbb{B} \triangleright w : \mathbb{C}]\!] \cdot \mathsf{jn}_{\Delta;\mathbb{A}\oplus\mathbb{B}} \cdot (\mathsf{id} \otimes \mathsf{IL} \cdot [\![y : \mathbb{A} \triangleright y : \mathbb{A}]\!]), [\![\Delta, x : \mathbb{A} \oplus \mathbb{B} \triangleright w : \mathbb{C}]\!] \\
&[\![\Delta, x : \mathbb{A} \oplus \mathbb{B} \triangleright w : \mathbb{C}]\!] \cdot \mathsf{jn}_{\Delta;\mathbb{A}\oplus\mathbb{B}} \cdot (\mathsf{id} \otimes \mathsf{IR} \cdot [\![z : \mathbb{B} \triangleright z : \mathbb{B}]\!])] \cdot \mathsf{dist} \cdot \mathsf{sw} \cdot ([\![\Gamma \triangleright v : \mathbb{A} \oplus \mathbb{B}]\!]) \\
&\otimes \mathsf{id} \cdot \mathsf{sp}_{\Gamma;\Delta} \cdot \mathsf{sh}_E \\
=&[[\![\Delta, x : \mathbb{A} \oplus \mathbb{B} \triangleright w : \mathbb{C}]\!] \cdot \mathsf{jn}_{\Delta;\mathbb{A}\oplus\mathbb{B}} \cdot (\mathsf{id} \otimes \mathsf{IL} \cdot \mathsf{id}_{[\![\mathbb{A}]\!]}), [\![\Delta, x : \mathbb{A} \oplus \mathbb{B} \triangleright w : \mathbb{C}]\!] \cdot \mathsf{jn}_{\Delta;\mathbb{A}\oplus\mathbb{B}} \\
&\cdot (\mathsf{id} \otimes \mathsf{IR} \cdot \mathsf{id}_{[\![\mathbb{B}]\!]})] \cdot \mathsf{dist} \cdot \mathsf{sw} \cdot ([\![\Gamma \triangleright v : \mathbb{A} \oplus \mathbb{B}]\!] \otimes \mathsf{id}) \cdot \mathsf{sp}_{\Gamma;\Delta} \cdot \mathsf{sh}_E \\
=&[\![\Delta, x : \mathbb{A} \oplus \mathbb{B} \triangleright w : \mathbb{C}]\!] \cdot \mathsf{jn}_{\Delta;\mathbb{A}\oplus\mathbb{B}} \cdot (\mathsf{id} \otimes [\mathsf{IL} \cdot \mathsf{id}_{[\![\mathbb{A}]\!]}, \mathsf{IR} \cdot \mathsf{id}_{[\![\mathbb{B}]\!]}]) \cdot \mathsf{dist} \cdot \mathsf{sw} \cdot ([\![\Gamma \triangleright v : \mathbb{A} \oplus \mathbb{B}]\!] \\
&\otimes \mathsf{id}) \cdot \mathsf{sp}_{\Gamma;\Delta} \cdot \mathsf{sh}_E
\end{aligned}$$

Considering that $[\mathsf{IL} \cdot \mathsf{id}, \mathsf{IR} \cdot \mathsf{id}] = \mathsf{id} + \mathsf{id} = \mathsf{id}$, it follows that the following diagram commutes.

$$\begin{tikzcd}
X \otimes (Y \oplus Y) \arrow[r, shift left, "{[\mathsf{id} \otimes \mathsf{IL}, \mathsf{id} \otimes \mathsf{IR}]}"] \arrow[d, "\mathsf{id} \otimes \mathsf{id}"'] & X \otimes Y \oplus X \otimes Z \arrow[l, shift left, "\mathsf{dist}"] \arrow[dl, "{\mathsf{id} \otimes [\mathsf{IL} \cdot \mathsf{id}, \mathsf{IR} \cdot \mathsf{id}]}"] \\
X \otimes (Y \oplus Y) &
\end{tikzcd}$$

And as a result, one has that $\mathsf{id} \otimes [\mathsf{IL} \cdot \mathsf{id}_{[\![\mathbb{A}]\!]}, \mathsf{IR} \cdot \mathsf{id}_{[\![\mathbb{B}]\!]}] \cdot \mathsf{dist} = \mathsf{id} \otimes \mathsf{id}$. Therefore,

$$\begin{aligned}
&[\![\Delta, x : \mathbb{A} \oplus \mathbb{B} \triangleright w : \mathbb{C}]\!] \cdot \mathsf{jn}_{\Delta;\mathbb{A}\oplus\mathbb{B}} \cdot (\mathsf{id} \otimes [\mathsf{IL} \cdot \mathsf{id}_{[\![\mathbb{A}]\!]}, \mathsf{IR} \cdot \mathsf{id}_{[\![\mathbb{B}]\!]}]) \cdot \mathsf{dist} \cdot \mathsf{sw} \cdot ([\![\Gamma \triangleright v : \mathbb{A} \oplus \mathbb{B}]\!] \\
&\otimes \mathsf{id}) \cdot \mathsf{sp}_{\Gamma;\Delta} \cdot \mathsf{sh}_E \\
=&[\![\Delta, x : \mathbb{A} \oplus \mathbb{B} \triangleright w : \mathbb{C}]\!] \cdot \mathsf{jn}_{\Delta;\mathbb{A}\oplus\mathbb{B}} \cdot (\mathsf{id} \otimes \mathsf{id}) \cdot \mathsf{sw} \cdot ([\![\Gamma \triangleright v : \mathbb{A} \oplus \mathbb{B}]\!] \otimes \mathsf{id}) \cdot \mathsf{sp}_{\Gamma;\Delta} \cdot \mathsf{sh}_E \\
=&[\![\Delta, x : \mathbb{A} \oplus \mathbb{B} \triangleright w : \mathbb{C}]\!] \cdot \mathsf{jn}_{\Delta;\mathbb{A}\oplus\mathbb{B}} \cdot \mathsf{sw} \cdot ([\![\Gamma \triangleright v : \mathbb{A} \oplus \mathbb{B}]\!] \otimes \mathsf{id}) \cdot \mathsf{sp}_{\Gamma;\Delta} \cdot \mathsf{sh}_E \\
=&[\![w[v/x] : \mathbb{C}]\!]
\end{aligned}$$

3.2.4 Metric equations

The metric equations for conditionals are presented in Figure 10. Note that the first two equations are redundant.

$$\frac{v =_q w}{\mathsf{inl}(v) =_q \mathsf{inl}(w)} \qquad \frac{v =_q w}{\mathsf{inr}(v) =_q \mathsf{inr}(w)}$$

$$\frac{v =_q v' \quad w =_r w' \quad u =_s u'}{\mathsf{case}\ v\ \{\mathsf{inl}(x) \Rightarrow w; \mathsf{inr}(y) \Rightarrow u\} =_{q+\mathsf{max}(r,s)} \mathsf{case}\ v'\ \{\mathsf{inl}(x) \Rightarrow w'; \mathsf{inr}(y) \Rightarrow u'\}}$$

Figure 10: Metric equational system for condicionals

Proof In order to validate the metric equational system for conditionals, it is necessary to demonstrate its correctness.

The diamond norm is a particular instance of the operator norm. The operator norm [Guide (2006)] for a super-operator E is defined as:

$$\|E\|_\sigma = \mathsf{sup}\{\|E(v)\| \mid \|v\| = 1\} \tag{3.15}$$

For the **injections**:

Firstly, it is necessary to prove that the identity operator I has a norm equal to 1.

Lemma 3.2.4. $\|I\|_\sigma = 1$

Proof. Using the definition of operator norm in Equation 3.15, it follows that:

$$\|I\|_\sigma = \text{sup}\{\|I(v)\| \mid \|v\| = 1\} = \text{sup}\{\|v\| \mid \|v\| = 1\} = 1 \tag{3.16}$$

Thereafter, it is imperative to show that the injection operators IL and IR are have a norm equal to 1.

Lemma 3.2.5. $\|\text{IL}\|_\sigma = 1$

Lemma 3.2.6. $\|\text{IR}\|_\sigma = 1$

Proof. Employing the definition of operator norm as defined in Equation 3.15, it ensues that:

$$\begin{aligned}
\|\text{IL}\|_\sigma &= \text{sup}\{\|\text{IL}(v)\| \mid \|v\| = 1\} = \text{sup}\{\|(v,0)\| \mid \|v\| = 1\} = \text{sup}\{\|v\| + \|0\| \mid \|v\| = 1\} \\
&= \text{sup}\{\|v\| \ + 0 \mid \|v\| = 1\} && \{\text{Positive definiteness}\} \\
&= \text{sup}\{\|v\| \mid \|v\| = 1\} = 1
\end{aligned} \tag{3.17}$$

The proof for Lemma 3.2.6 is analogous to the proof for Lemma 3.2.5.

$$\begin{aligned}
\|\text{IR}\|_\sigma &= \text{sup}\{\|\text{IR}(v)\| \mid \|v\| = 1\} = \text{sup}\{\|(0,v)\| \mid \|v\| = 1\} = \text{sup}\{\|0\| + \|v\| \mid \|v\| = 1\} \\
&= \text{sup}\{0 + \|v\| \ \mid \|v\| = 1\} && \{\text{Positive definiteness}\} \\
&= \text{sup}\{\|v\| \mid \|v\| = 1\} = 1
\end{aligned} \tag{3.18}$$

Futhermore, given the submultiplicative property of the operator norm, for any super-operators P and Q,where $\|P\|_\sigma = 1$ the following holds:

Lemma 3.2.7. $\|PQ\|_\sigma \leq \|Q\|_\sigma, \quad \|P\|_\sigma = 1$

Using these properties it is possible to prove the validity of the metric equations for the injections. Demonstrating the correctness of the metric equations for the injections is equivalent to proving that for any non-negative rational q and super-operators v and w such that $d(v,w) \leq q$, where $d(v,w)$ represents the distance between v and w the following holds:

Theorem 3.2.2. $d(\text{IL}(v), \text{IL}(w)) \leq q$

Theorem 3.2.3. $d(\text{IR}(v), \text{IR}(w)) \leq q$

Proof. In the quantum paradigm, the distance between two super-operators E and E' corresponds to the diamond norm between E and E'. Therefore,

$$d(v, w) \leq q \Leftrightarrow \|v \otimes I - w \otimes I\|_\sigma \leq q \tag{3.19}$$

As a result, to prove that $d(\text{IL}(v), \text{IL}(w)) \leq q$, it suffices to show that:

$$\|\text{IL} \otimes I(v \otimes I) - \text{IL} \otimes I(w \otimes I)\|_\sigma \leq \|v \otimes I - w \otimes I\|_\sigma \tag{3.20}$$

$$\|\text{IR} \otimes I(v \otimes I) - \text{IR} \otimes I(w \otimes I)\|_\sigma \leq \|v \otimes I - w \otimes I\|_\sigma \tag{3.21}$$

Given that IL and IR possess a norm equal to 1, as established by Lemmas 3.2.5 and 3.2.6 respectively, and considering the multiplicative property of the operator norm with respect to tensor products alongside the fact that the identity operator also exhibits a norm equal to 1, as demonstrated in Lemma 3.2.4, it follows that both $\|\text{IL} \otimes I\|_\sigma$ and $\|\text{IR} \otimes I\|_\sigma$ are equal to one 1. Hence, by Lemma 3.2.7,

$$\|\text{IL} \otimes I(v \otimes I) - \text{IL} \otimes I(w \otimes I)\|_\sigma = \|\text{IL} \otimes I(v \otimes I - w \otimes I)\|_\sigma \leq \|v \otimes I - w \otimes I\|_\sigma \tag{3.22}$$

$$\|\text{IR} \otimes I(v \otimes I) - \text{IR} \otimes I(w \otimes I)\|_\sigma = \|\text{IR} \otimes I(v \otimes I - w \otimes I)\|_\sigma \leq \|v \otimes I - w \otimes I\|_\sigma \tag{3.23}$$

Now, regarding the metric equation for the **conditional statement**, before validating its correctness, it is necessary to prove a few intermediate results.
The first step is to demonstrate that for any super-operators P and Q the following holds:

Lemma 3.2.8. $\|[P, Q]\|_\sigma \leq \max\{\|P\|_\sigma, \|Q\|_\sigma\}$

Proof. Employing the definition of the operator norm in Equation 3.15, it follows that:

$$\text{sup}\{\|[P, Q](v)\| \mid \|v\| = 1\} \leq \text{max}\{\text{sup}\{\|P(w)\| \mid \|w\| = 1\}, \text{sup}\{\|Q(u)\| \mid \|u\| = 1\}\}$$
$$= \text{sup}\{\|[P, Q](w, u)\| \mid \|w\| + \|u\| = 1\} \leq \text{max}\{\text{sup}\{\|P(w)\| \mid \|w\| = 1, \|Q(u)\| \mid \|u\| = 1\}\}$$
$$= \text{sup}\{\|P(w) + Q(u)\| \mid \|w\| + \|u\| = 1\| = 1\} \leq \text{max}\{\text{sup}\{\|P(w)\| \mid \|w\| = 1, \|Q(u)\| \mid \|u\| = 1\}\}$$
$$= \text{sup}\{\|P(w) + Q(u)\| \mid \|w\| + \|u\| = 1\} \leq \text{sup}\{\text{max}\{\|P(w)\| \mid \|w\| = 1, \|Q(u)\| \mid \|u\| = 1\}\}$$

(3.24)

Therefore, by the triangle inequality, proving the inequality in Equation 3.25 suffices to establish Lemma 3.2.8.

$$\sup\{\|P(w)\| + \|Q(u)\| \mid \|w\| + \|u\| = 1\} \leq \sup\{\max\{\|P(w)\| \mid \|w\| = 1, \|Q(u)\| \mid \|u\| = 1\}\} \tag{3.25}$$

This can be rewritten as:

$$\|w\| + \|u\| = 1 \wedge \sup\{\|P(w)\| + \|Q(u)\| \} \leq \max\left\{\frac{1}{\|w\|}\|P(w)\|, \frac{1}{\|u\|}\|Q(u)\|\right\} \tag{3.26}$$

As a result,

$$\|w\| + \|u\| = 1 \wedge \sup\{\|P(w)\| + \|Q(u)\|\} \leq \max\left\{\left\|P\left(\frac{1}{\|w\|}w\right)\right\|, \left\|Q\left(\frac{1}{\|u\|}u\right)\right\|\right\} \tag{3.27}$$

This is equivalent to demonstrating that for $a + b = 1$,

$$x + y \leq \max\left\{\frac{1}{a}x, \frac{1}{b}y\right\} \tag{3.28}$$

This is done by arguing by *reductio ad absurdum*, *i.e.*, supposing otherwise leads to a contradiction:

$$\begin{aligned}
& x + y > \max\left\{\frac{1}{a}x, \frac{1}{b}y\right\} \\
& \Rightarrow x + y > \frac{1}{a}x \wedge x + y > \frac{1}{b}y \\
& \Rightarrow a(x + y) > x \wedge b(x + y) > y \\
& \Rightarrow ax + ay > x \wedge bx + by > y \\
& \Rightarrow ax + ay > x \wedge (1 - a)x + (1 - a)y > y \\
& \Rightarrow ax + ay > x \wedge x - ax + y - ay > y \\
& \Rightarrow x < ax + ay \wedge x > ax + ay
\end{aligned} \tag{3.29}$$

Subsequently, it is imperative to prove that:

Lemma 3.2.9. *$i = [\mathsf{IL} \otimes I, \mathsf{IR} \otimes I]$ is an isomorphism.*

Proof. The proof is as follows:

For any vector spaces V, W, and U, $i : (V \otimes U) \oplus (W \otimes U) \to (V \oplus W) \otimes U$. If V has dimension m, W has dimension n, and U has dimension o, then the space $(V \otimes U) \oplus (W \otimes U)$ has dimension $mo + no = (m+n) \cdot o$. Similarly, the space $(V \oplus W) \otimes U$ has dimension $(m+n) \cdot o$. Hence, the spaces have the same dimension. Given that spaces with the same dimension are isomorphic [Hefferon (2006)], it follows that i is an isomorphism.

Next, it is necessary to demonstrate that for any operators P and Q, the identity operator I, and an isomorphism $i = [\mathsf{IL} \otimes I, \mathsf{IR} \otimes I]$ the following holds:

Lemma 3.2.10. $([P, Q] \otimes I) \cdot i = [P \otimes I, Q \otimes I]$

Which is equivalent to showing that for any vector spaces V, W, U, and Z and super-operators $P : V \to Z$, $Q : W \to Z$, and $I : U \to U$, the following diagram holds:

$$\begin{tikzcd}
V \otimes U \oplus W \otimes U \arrow[r, "i"] \arrow[d, "{[P \otimes I, Q \otimes I]}"'] & (V \oplus W) \otimes U \arrow[dl, "{[P, Q] \otimes I}"] \\
Z \otimes U &
\end{tikzcd}$$

Proof. The proof is straightforward:

$$\begin{aligned}
&([P, Q] \otimes I) \cdot [\mathsf{IL} \otimes I, \mathsf{IR} \otimes I] \\
&= [([P, Q] \otimes I) \cdot (\mathsf{IL} \otimes I), ([P, Q] \otimes I) \cdot (\mathsf{IR} \otimes I)] \qquad (3.30) \\
&= [P \otimes I, Q \otimes I]
\end{aligned}$$

Furhtermore, it is imperative to show that the following relation holds:

Lemma 3.2.11. $[P \otimes I, Q \otimes I] \cdot i^{-1} = [P, Q] \otimes I$

Demonstrating this is equivalent to establishing that for any vector spaces V, W, U, and Z, and super-operators $P : V \to Z$, $Q : W \to Z$, and $I : U \to U$, the following diagram commutes:

$$\begin{tikzcd}
V \otimes U \oplus W \otimes U \arrow[d, "{[P \otimes I, Q \otimes I]}"'] & (V \oplus W) \otimes U \arrow[l, "i^{-1}"'] \arrow[dl, "{[P, Q] \otimes I}"] \\
Z \otimes U &
\end{tikzcd}$$

Proof. The proof is as follows:

$$
\begin{aligned}
&([P,Q]\otimes I)\cdot i = [P\otimes I, Q\otimes I] && \{\text{Lemma 3.2.10}\} \\
\Leftrightarrow\ &([P,Q]\otimes I)\cdot i\cdot i^{-1} = [P\otimes I, Q\otimes I]\cdot i^{-1} && \\
\Leftrightarrow\ &([P,Q]\otimes I) = [P\otimes I, Q\otimes I]\cdot i^{-1} && \{\text{Lemma 3.2.9}\}
\end{aligned}
\tag{3.31}
$$

With Lemma 3.2.10 and Lemma 3.2.11, it has been proved that the diagram below is valid:

$$
\begin{tikzcd}
V\otimes U \oplus W\otimes U \arrow[r, bend left, "i"] \arrow[d, "{[P\otimes I, Q\otimes I]}"'] & (V\oplus W)\otimes U \arrow[l, bend left, "i^{-1}"] \arrow[dl, "{[P,Q]\otimes I}"] \\
Z\otimes U &
\end{tikzcd}
$$

Now, it is possivel to prove that i has a norm equal to 1.

Lemma 3.2.12. $\|i\|_\sigma \geq 1$

Proof. Considering the vector $(v\otimes u, 0)$ with $\|(v\otimes u, 0)\| = 1$, and attending the multiplicative property of the operator norm with respect to tensor products, along with the definition of the norm of a tuple as in Equation 3.8, it holds that $\|v\| = 1$ and $\|u\| = 1$. Therefore, using this same property and definition, it is possible to demonstrate that the following holds:

$$
\|[\mathsf{IL}\otimes I, \mathsf{IR}\otimes I](v\otimes u, 0)\| = \|(v,0)\otimes u\| = (\|v\| + \|0\|)\|u\| = \|v\|\|u\| = 1 \tag{3.32}
$$

Given the definition of the operator norm as presented in Equation 3.15, it follows that:

$$
\|[\mathsf{IL}\otimes I, \mathsf{IR}\otimes I]\|_\sigma = \mathsf{sup}\{\|[\mathsf{IL}\otimes I, \mathsf{IR}\otimes I](a)\| \mid \|a\| = 1\} \tag{3.33}
$$

From this, it can be deduced that $\|i\|_\sigma \geq 1$.

Subsequently, it is possible to demontrate that i^{-1} has a norm greater than or equal to 1,

Lemma 3.2.13. $\|i^{-1}\|_\sigma \leq 1$

Proof. Given that i is an isomophism, it follows that

$$
\begin{aligned}
&\|i\cdot i^{-1}\|_\sigma = 1 && \\
\leq\ &\|i\|_\sigma \cdot \|i^{-1}\|_\sigma = 1 && \{\text{Norm submultiplicative with respect to compositions}\} \\
\leq\ &1\cdot \|i^{-1}\|_\sigma = 1 && \{\text{Lemma 3.2.13}\} \\
\Leftrightarrow\ &\|i^{-1}\|_\sigma = 1 &&
\end{aligned}
$$

(3.34)

Next, one has to prove that for any super-operators P and Q and their respective erroneous versions P' and Q', the following holds:

Lemma 3.2.14. $\|P \cdot Q \otimes I - P' \cdot Q' \otimes I\|_\sigma \leq \|(P - P') \otimes I\|_\sigma + \|(Q - Q') \otimes I\|_\sigma$

Proof. Applying the triangle inequality, he submultiplicative property of the operator norm with respect to compositions, and given that a positive and trace-preserving operator map, E, has norm $\|E \otimes I\|_\sigma = 1$ (Watrous (2018)), it follows that:

$$\begin{aligned}
&\|P \cdot Q \otimes I - P' \cdot Q' \otimes I\|_\sigma \\
&= \|P \cdot Q \otimes I - P \cdot Q' \otimes I + P \cdot Q' \otimes I - P' \cdot Q' \otimes I\|_\sigma \\
&\leq \|P \cdot Q \otimes I - P \cdot Q' \otimes I\|_\sigma + \|P \cdot Q' \otimes I - P' \cdot Q' \otimes I\|_\sigma \\
&\leq \|P\|_\sigma \|Q \otimes I - Q' \otimes I\|_\sigma + \|P \otimes I - P' \otimes I\|_\sigma \|Q'\|_\sigma \\
&= \|P\|_\sigma \|(Q - Q') \otimes I\|_\sigma + \|(P - P') \otimes I\|_\sigma \|Q'\|_\sigma \\
&= \|(P - P') \otimes I\|_\sigma + \|(Q - Q') \otimes I\|_\sigma
\end{aligned} \tag{3.35}$$

Finally, considering the the semantics the conditional statement in Figure 9, demonstrating the conditional statement rule in Figure 10 includes proving that for any super-operators P, Q, P' and Q', denoting the distance between super-operators A and B as $d(A, B)$, the following holds:

Lemma 3.2.15. *d*$([P, Q], [P', Q']) \leq$ *max*$\{$*d*$(P, P'),$ *d*$(Q, Q')\}$

Proof. In the quantum paradigm, the distance between two super-operators corresponds to the diamond norm between the two super-operators. Hence, denoting $[\text{IL} \otimes I, \text{IR} \otimes I]$ by i it follows that:

$$\begin{aligned}
&\mathrm{d}([P, Q], [P', Q']) \\
&= \|[P, Q] \otimes I - [P', Q'] \otimes I\|_\sigma \\
&= \|[P \otimes I, Q \otimes I] \cdot i^{-1} - [P' \otimes I, Q' \otimes I] \cdot i^{-1}\|_\sigma && \{\text{Lemma 3.2.11}\} \\
&= \|[P - P' \otimes I, Q - Q' \otimes I] \cdot i^{-1}\|_\sigma \\
&\leq \|[P - P' \otimes I, Q - Q' \otimes I]\| \|i^{-1}\| \|_\sigma && \{\text{Norm submultiplicative with respect to compositions}\} \\
&\leq \|[(P - P') \otimes I, (Q - Q') \otimes I]\|_\sigma && \{\text{Lemma 3.2.13}\}
\end{aligned}$$

$$(3.36)$$

and

$$\begin{aligned}
&\max\{\mathsf{d}(P,P'),\mathsf{d}(Q,Q')\}\\
=&\max\{\|P\otimes I-P'\otimes I\|_\sigma,\|Q\otimes I-Q'\otimes I\|_\sigma\}\\
=&\max\{\|(P-P')\otimes I\|_\sigma,\|(Q-Q')\otimes I\|_\sigma\}
\end{aligned} \tag{3.37}$$

Finally, by Lemma 3.2.8, it can be deduced that $\mathsf{d}([P,Q],[P',Q'])\leq \max\{\mathsf{d}(P,P'),\mathsf{d}(Q,Q')\}$, which concludes the proof of theorem Lemma 3.2.15.

An alternative method to establish Theorem 3.2.4 is now presented.

Proof. The proof is as follows:

$$\begin{aligned}
&\mathsf{d}([P,Q],[P',Q'])\\
&=\|[P,Q]\otimes I-[P',Q']\otimes I\|_\sigma\\
&=\|([P,Q]-[P',Q'])\otimes I\|_\sigma\\
&=\|[P-P',Q-Q']\otimes I\|_\sigma\\
&=\|[P-P',Q-Q']\|_\sigma\|I\|_\sigma && \{\text{Norm multiplicative with respect to tensor products}\}\\
&=\|[P-P',Q-Q']\|_\sigma && \{\text{Lemma 3.2.4}\}
\end{aligned} \tag{3.38}$$

Moreover,

$$\begin{aligned}
&\max\{\mathsf{d}(P,P'),\mathsf{d}(Q,Q')\}\\
=&\max\{\|P\otimes I-P'\otimes I\|_\sigma,\|Q\otimes I-Q'\otimes I\|_\sigma\}\\
=&\max\{\|(P-P')\otimes I\|_\sigma,\|(Q-Q')\otimes I\|_\sigma\}\\
=&\max\{\|(P-P')\|_\sigma\|I\|_\sigma,\|(Q-Q')\|_\sigma\|I\|_\sigma\} && \{\text{Norm multiplicative with respect to tensor products}\}\\
=&\max\{\|(P-P')\|_\sigma,\|(Q-Q')\|_\sigma\} && \{\text{Lemma 3.2.4}\}
\end{aligned} \tag{3.39}$$

Therefore, by Lemma 3.2.8, it can be deduced that $\mathsf{d}([P,Q],[P',Q'])\leq \max\{\mathsf{d}(P,P'),\mathsf{d}(Q,Q')\}$, which concludes the proof of theorem Lemma 3.2.15.

Now, it is finally possible to adress the proof of the metric equation for the conditional statement as a whole. Considering the the semantics of the conditional statement in Figure 9, the rule for the conditional statement in Figure 10 is valid is equivalent to demonstrating that the distance between the evalution of a boolen B followed by the execution of a program P or a program Q and the evalution of a boolean B' followed by the execution of a program P' or a program Q' is less or equal to the distance between the evaluation of the boolean B and the evaluation of the boolean B' plus the maximum distance between the execution of the programs P and P' and the execution of the programs Q and Q', *ergo*, that for any booleand B and B' super-operators P, Q, P' and Q', the following holds:

Theorem 3.2.4. $d(B \cdot [P, Q], B' \cdot [P', Q']) \leq d(B, B') + max\{d(P, P'), d(Q, Q')\}$

Proof. Considering that in the quantum paradigm, the distance between two super-operators corresponds to the diamond norm between the two super-operators, it follows that:

$$
\begin{aligned}
& \mathrm{d}(B \cdot [P, Q], B' \cdot [P', Q']) \\
&= \|B \cdot [P, Q] \otimes I - B' \cdot [P', Q'] \otimes I\|_\sigma \\
&\leq \|(B - B') \otimes I\|_\sigma + \|([P, Q] - [P', Q']) \otimes I\|_\sigma && \{\text{Lemma 3.2.14}\} \\
&= d(B, B') + \|[P, Q] \otimes I - [P', Q'] \otimes I\|_\sigma \\
&= \mathrm{d}(B, B') + \mathrm{d}([P, Q], [P', Q']) \\
&= d(B, B') + \max\{\mathrm{d}(P, P'), \mathrm{d}(Q, Q')\} && \{\text{Lemma 3.2.15}\}
\end{aligned} \tag{3.40}
$$

3.2.5 Quantum Teleportation

[Bennett et al. (1993)] introduced the concept of quantum teleportation, which is a protocol that allows the transfer of unknown quantum states between distant parties. The quantum teleportation protocol is a fundamental building block for quantum communication, quantum computation, and quantum networks, its applications ranging from secure quantum communication to distributed quantum computing [Briegel et al. (1998); Gottesman and Chuang (1999); Kimble (2008)].

The circuit corresponding to the implementation of the quantum teleportation protocol is depicted in Figure 11.

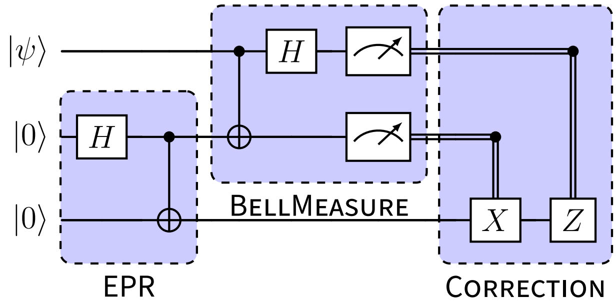

Figure 11: Quantum Teleportation Protocol

When formalizing the quantum teleportation protocol within the lambda calculus framework, each part of the protocol is instantiated as a distinct function. This entails the definition of three specific functions:

$$\textbf{EPR}: \ \mathbb{I} \multimap (qbit \otimes qbit)$$
$$\textbf{BellMeasure}: \ qbit \otimes qbit \multimap bit \otimes bit$$
$$\textbf{Correction}: \ qbit \otimes bit \otimes bit \multimap qbit$$

The only part that is not self- explanatory is EPR, an acronym derived from a famous article written in 1935 by Albert Einstein, Boris Podolsky, and Nathan Rosen , where these authors questioned the completeness of Quantum Mechanics [Einstein et al. (1935)].
Considering the unitary operations $H : qbit \to qbit$, $X : qbit \to qbit$, $Z : qbit \to qbit$, $I : qbit \to qbit$, and $CNOT : qbit, qbit \to qbit \otimes qbit$, these functions are defined as follows:

$$\textbf{EPR} = - \triangleright CNOT\,(H\,(q\,(new\,0\,(*))), (q\,(new\,0\,(*))))$$
$$\textbf{BellMeasure} = q_1 : qbit, q_2 : qbit \triangleright (\text{pm } CNOT(q_1, q_2) \text{ to } x \otimes y.\, meas(H(x)) \otimes meas(y))$$
$$\begin{aligned}\textbf{Correction} = q : qbit, x : \text{bit}, y : \text{bit} \triangleright \text{ case } x\, \{&\text{inl}(x_0) \Rightarrow (\text{cond}\, y\, \{\text{inl}(y_0) \Rightarrow I(q); \\ &\qquad \text{inr}(y_1) \Rightarrow X(q))\}; \\ &\text{inr}(x_1) \Rightarrow (\text{cond}\, y\, \{\text{inl}(y_0) \Rightarrow Z(q); \\ &\qquad \text{inr}(y_1) \Rightarrow Z(X(q))\})\}\end{aligned}$$

Designating the qubit to be teleported as q_0, one can conceptualize the teleportation procedure as follows:

$$q_0 : qbit \triangleright \text{pm } \mathbf{EPR}(*) \text{ to } q_1 \otimes q_2.$$
$$\text{pm } \mathbf{BellMeasure}(q_0, q_1) \text{ to } c_0 \otimes c_1.$$
$$\text{pm } \mathbf{Correction}(q_2, c_0, c_1) \text{ to } q.\, q$$

Regarding the interpretation of the quantum teleportation protocol, considering $\rho = |\phi\rangle\langle\phi|$ as the state of the system before measurement, $|\phi\rangle$ is calculated as follows, where $|\psi\rangle$ is the state of the qubit to be teleported:

$$\begin{aligned}
& |\psi\rangle \otimes |0\rangle \otimes |0\rangle = (\alpha|0\rangle + \beta|1\rangle) \otimes |0\rangle \otimes |0\rangle \\
\xmapsto{I\otimes H\otimes I} \quad & (\alpha|0\rangle + \beta|1\rangle) \otimes \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle) \\
\xmapsto{I\otimes CNOT} \quad & (\alpha|0\rangle + \beta|1\rangle) \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle) \\
\xmapsto{CNOT\otimes I} \quad & \frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|011\rangle + \beta|110\rangle + \beta|101\rangle) \\
\xmapsto{H\otimes I\otimes I} \quad & \frac{1}{2}(\alpha|000\rangle + \alpha|001\rangle + \alpha|011\rangle + \alpha|111\rangle + \beta|010\rangle - \beta|110\rangle + \beta|101\rangle - \beta|001\rangle) \\
= \quad & \frac{1}{2}(|00\rangle \otimes (\alpha|0\rangle + \beta|1\rangle) + |01\rangle \otimes (\alpha|1\rangle + \beta|0\rangle) + |10\rangle \otimes (\alpha|0\rangle - \beta|1\rangle) \\
& + |11\rangle \otimes (\alpha|1\rangle - \beta|0\rangle)) \\
= \quad & |00\rangle \otimes |\psi\rangle + |01\rangle \otimes X|\psi\rangle + |10\rangle \otimes Z|\psi\rangle + |11\rangle \otimes XZ|\psi\rangle = |\phi\rangle
\end{aligned} \tag{3.41}$$

Regarding the remaining steps of the protocol,

$$\begin{aligned}
|\phi\rangle\langle\phi| = \quad & \frac{1}{4}(|00\rangle\langle 00| \otimes |\psi\rangle\langle\psi| + |00\rangle\langle 01| \otimes |\psi\rangle\langle\psi|X + |00\rangle\langle 10| \otimes |\psi\rangle\langle\psi|Z \\
& + |00\rangle\langle 11| \otimes |\psi\rangle\langle\psi|ZX + X|01\rangle\langle 00| \otimes |\psi\rangle\langle\psi| + |01\rangle\langle 01| \otimes X|\psi\rangle\langle\psi|X \\
& + |01\rangle\langle 10| \otimes X|\psi\rangle\langle\psi|Z + |01\rangle\langle 11| \otimes X|\psi\rangle\langle\psi|ZX + |10\rangle\langle 00| \otimes Z|\psi\rangle\langle\psi| \\
& + |10\rangle\langle 01| \otimes Z|\psi\rangle\langle\psi|X + |10\rangle\langle 10| \otimes Z|\psi\rangle\langle\psi|Z + |10\rangle\langle 11| \otimes Z|\psi\rangle\langle\psi|ZX \\
& + |00\rangle\langle 11| \otimes |\psi\rangle\langle\psi|ZX + |01\rangle\langle 11| \otimes X|\psi\rangle\langle\psi|ZX + |10\rangle\langle 11| \otimes Z|\psi\rangle\langle\psi|ZX \\
& + |11\rangle\langle 11| \otimes ZX|\psi\rangle\langle\psi|ZX) \\
\xmapsto{\text{meas}\otimes\text{meas}\otimes I} \quad & \Big(\Big(\frac{1}{4}|\psi\rangle\langle\psi|, \frac{1}{4}X|\psi\rangle\langle\psi|X\Big), \Big(\frac{1}{4}Z|\psi\rangle\langle\psi|Z, \frac{1}{4}XZ|\psi\rangle\langle\psi|ZX\Big)\Big)
\end{aligned} \tag{3.42}$$

With respect to the final step of the protocol, attending to the interpretation of the conditional statement (Figure 9), the state of the system after the application of the correction function is given by:

$$\begin{aligned}&\frac{1}{4}|\psi\rangle\langle\psi| + \frac{1}{4}XX|\psi\rangle\langle\psi|XX + \frac{1}{4}ZZ|\psi\rangle\langle\psi|ZZ + \frac{1}{4}ZXXZ|\psi\rangle\langle\psi|ZXXZ \\ =&\frac{1}{4}\left(|\psi\rangle\langle\psi| + |\psi\rangle\langle\psi| + |\psi\rangle\langle\psi| + |\psi\rangle\langle\psi|\right) = |\psi\rangle\langle\psi|\end{aligned} \tag{3.43}$$

3.2.6 Ilustration: Noisy Quantum Teleportation

Noisy Quantum Teleportation: Decoherence

Realistic quantum systems are never isolated, but are immersed in the surrounding environment and interact continuously with it [Schlosshauer (2005)]. Decoherence can be seen as the consequence of that 'openness' of quantum systems to their environments . To study decoherence in a quantum channel within the presented metric deductive system, one can consider the application of a dephasing channel in the quantum teleportation protocol with a certain probability p.

The Kraus operators of the dephasing channel with probability p are expressed as:

$$D_0 = \frac{\sqrt{2-p}}{\sqrt{2}}I, D_1 = \frac{\sqrt{p}}{\sqrt{2}}Z \tag{3.44}$$

Considering a density operator $\rho = |\alpha|^2|0\rangle\langle 0| + \alpha\beta^\dagger|0\rangle\langle 1| + \alpha^\dagger\beta|1\rangle\langle 0| + |\beta|^2|1\rangle\langle 1|$, using these Kraus operators, it is possible to easily verify that after applying the dephasing channel with probability p, the resulting operator ρ' is given by:

$$\rho' = A_0\rho A_0^\dagger + A_1\rho A_1^\dagger = |\alpha|^2|0\rangle\langle 0| + (1-p)\alpha\beta^\dagger|0\rangle\langle 1| + (1-p)\alpha^\dagger\beta|1\rangle\langle 0| + |\beta|^2|1\rangle\langle 1| \tag{3.45}$$

This shows that the dephasing channel with probability p preserves the diagonal elements of the density matrix while attenuating the off-diagonal elements by a factor of $(1-p)$.

The circuit representing the introduction of decoherence after EPR is illustrated in Figure 12.

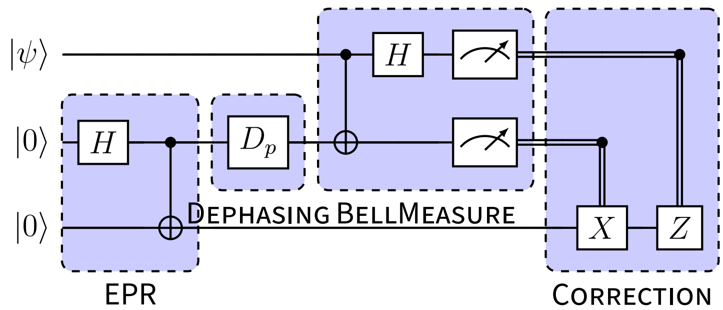

Figure 12: Quantum Teleportation Protocol: Dephasing with probability p after EPR pair creation.

In this case, to facilate the analysis, the quantum teleportion protocol is devided in four parts: EPR, BellMeasure, Identity and Correction. This entails the definition of an aditional function and respective version subjected to decoherence with probability p:

$$\begin{aligned}
\textbf{Identity} &: \; \textit{qbit} \multimap \textit{qbit} \\
\textbf{Identity}^p &: \; \textit{qbit} \multimap \textit{qbit}
\end{aligned}$$

Considering the unitary operation I : *qbit* $\rightarrow$ *qbit*, and the operation D_p : *qbit* $\rightarrow$ *qbit* the ideal version of this function, **Identity**, and its respective version subjected to decoherence with probability p, **Identity**^{p}, are defined as follows:

$$\textbf{Identity} = q : \textit{qbit} \triangleright I(q) : \textit{qbit} \tag{3.46}$$

$$\textbf{Identity}^p = q : \textit{qbit} \triangleright D_p(q) : \textit{qbit} \tag{3.47}$$

Designating the qubit to be teleported as q_0, one can conceptualize the teleportation procedure as follows:

$$\begin{aligned}
&\text{pm } \textbf{EPR}(*) \text{ to } q_1 \otimes q_2. \\
&\text{pm } \textbf{Identity}(q_1) \text{ to } id_q_1. \\
&\text{pm } \textbf{BellMeasure}(q_0, id_q_1) \text{ to } c_0 \otimes c_1. \\
&\text{pm } \textbf{Correction}(q_2, c_0, c_1) \text{ to } q.\, q
\end{aligned}$$

To evaluate the disparity between the ideal implementation of the quantum teleportation protocol and its realization subjected to decoherence, the initial step involves computing the distance between the density operators of the ideal and noisy implementations of the EPR state, denoted as ρ and ρ', respectively.

$$\begin{aligned}
& |0\rangle\langle 0| \otimes |0\rangle\langle 0| \\
\xmapsto{\text{EPR}} \;& \frac{1}{2}(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|) = \rho \\
\xmapsto{D(p)\otimes I} \;& \frac{1}{2}(|00\rangle\langle 00| + (1-p)\,|00\rangle\langle 11| + (1-p)|11\rangle\langle 00| + |11\rangle\langle 11|) = \rho'
\end{aligned} \tag{3.48}$$

The distance between the r-image of the mapping $1 \mapsto \frac{1}{2}(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|)$ and the mapping $1 \mapsto \frac{1}{2}(|00\rangle\langle 00| + (1-p)\,|00\rangle\langle 11| + (1-p)|11\rangle\langle 00| + |11\rangle\langle 11|)$ is given by: $f(p) = \|\frac{p}{2}(|00\rangle\langle 11| + |11\rangle\langle 00|)\|_1$. Therefore, attending to Equation 2.5, $\|\rho -$

$\rho'(p)\|_\diamond = f(p)$.

$$\begin{array}{rll} f(p) = & \left\| \frac{p}{2} \left(|00\rangle \langle 11| + |11\rangle \langle 00| \right) \right\|_1 & \\ = & \text{Tr} \left(\sqrt{\frac{p^2}{4}(|00\rangle \langle 11| + |11\rangle \langle 00|)(|00\rangle \langle 11| + |11\rangle \langle 00|)^\dagger} \right) & \{\|\|_1 \text{ defn. for matrices}\} \\ = & \text{Tr} \left(\sqrt{\frac{p^2}{4}(|00\rangle \langle 00| + |11\rangle \langle 11|)} \right) & \\ = & \text{Tr} \left(\frac{p}{2}(|00\rangle \langle 00| + |11\rangle \langle 11|) \right) & \{\text{Equation 2.7}\} \\ = & \frac{p}{2} + \frac{p}{2} = p & \end{array} \tag{3.49}$$

Therefore, the distance between the ideal and noisy implementations of the EPR state is given by $\|\rho - \rho'(p)\|_\diamond = p$.

Next, via the metric deductive system in Figure 3, it is easily verified that for an error p,

$$q : \textit{qbit} \triangleright I(q) =_p q : \textit{qbit} \triangleright D_p(q) : \textit{qbit} \tag{3.50}$$

Therefore **Identity** $=_p$ **Identity**^{p} and finally, considering the entirety of the quantum teleportation protocol denoted as **QTP**, it follows that **QTP** $=_p$ **QTP**^{p}. This final metric equation indicates that by bounding the error associated with the application of decoherence with a specified probability p to the initial qubit before measurement, it becomes feasible to limit the overall error of the entire quantum teleportation protocol. Moreover, it is interesting to observe that the error associated with the application of decoherence with a certain probability p in one of the qubits corresponds exactly to that probability p.

Noisy Quantum Teleportation: Amplitude Damping

Next, the amplitude-damping channel is considered as a source of noise in the quantum teleportation protocol. Similarly to the dephasing channel, the amplitude damping channel serves as a model illustrating the dissipation of energy between a qubit and its environment. An example of this type of noise is found in the spontaneous emission of a photon by a two-level atom into an electromagnetic field environment with either a finite or infinite number of modes at zero temperature [Salles et al. (2008); Wang et al. (2011)].

The amplitude damping channel with probability γ is described by the Kraus operators:

$$A_0 = |0\rangle \langle 0| + \sqrt{1-\gamma}\, |1\rangle \langle 1| \,, A_1 = \sqrt{\gamma}\, |0\rangle \langle 1| \tag{3.51}$$

Applying these Kraus operators to the density operator $\rho = |\alpha|^2|0\rangle\langle 0|+\alpha\beta^\dagger|0\rangle\langle 1|+\alpha^\dagger\beta|1\rangle\langle 0|+|\beta|^2|1\rangle\langle 1|$, the the resulting operator ρ' is given by:

$$\begin{aligned}\rho' &= A_0\rho A_0^\dagger + A_1\rho A_1^\dagger \\ &= (|\alpha|^2+\gamma|\beta|^2)\,|0\rangle\langle 0| + \sqrt{1-\gamma}\,\alpha\beta^\dagger\,|0\rangle\langle 1| + \sqrt{1-\gamma}\,\alpha^\dagger\beta\,|1\rangle\langle 0| + (1-\gamma)|\beta|^2\,|1\rangle\langle 1|\end{aligned} \tag{3.52}$$

It is possible to observe that as γ increases, while the $|1\rangle\langle 1|$ component, alongside the non-diagonal elements, are attenuated, the $|0\rangle\langle 0|$ element is amplified.
The circuit representing the introduction of amplitude damping after the correction step is presented in Figure 13.

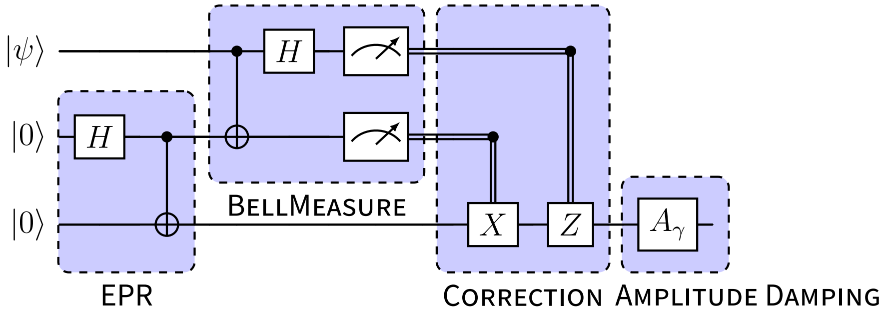

Figure 13: Quantum Teleportation Protocol: Amplitude Dampling with probability γ after Correction.

Once again, a fourth part of the teleportation protocol, the **Identity**, is considered to facilitate the error analysis. In this case, it is neccessay to define the erroneous version of **Identity**, **Identity**$^{A(\gamma)}$:

$$\textbf{Identity}^{A(\gamma)} : \textit{qbit} \multimap \textit{qbit} \tag{3.53}$$

Considering the operation $A_\gamma : \textit{qbit} \to \textit{qbit}$ the respective version of **Identity** subjected to amplitude damping with probability γ, **Identity**$^{A(\gamma)}$, is defined as follows:

$$\textbf{Identity}^{A(\gamma)} = q : \textit{qbit} \triangleright A_\gamma(q) : \textit{qbit} \tag{3.54}$$

Designating the qubit to be teleported as q_0, one can conceptualize the teleportation procedure as follows:

pm **EPR**$(*)$ to $q_1 \otimes q_2$.
pm **BellMeasure**(q_0, q_1) to $c_0 \otimes c_1$.
pm **Correction**(q_2, c_0, c_1) to q. **Identity**(q)

The first step to evaluate the distance between the ideal quantum teleportation protocol and the one subjected to amplitude damping with probability γ is to compute the distance between the density operators of the ideal and noisy implementations of the teleported qubit, denoted as ρ and ρ', respectively.

As shown in Equation 3.43, the state of the teleported qubit is $\rho = |\psi\rangle\langle\psi|$. Given Equation 3.52, the state of the teleported qubit after amplitude damping with probability γ is $(|\alpha|^2+\gamma|\beta|^2)|0\rangle\langle 0| + \sqrt{1-\gamma}\alpha\beta^\dagger|0\rangle\langle 1| + \sqrt{1-\gamma}\alpha^\dagger\beta|1\rangle\langle 0| + (1-\gamma)|\beta|^2|1\rangle\langle 1|$, which is denoted as ρ'.

As a result,

$$\begin{aligned}\rho-\rho' =&|\alpha|^2|0\rangle\langle 0| + \alpha\beta^\dagger|0\rangle\langle 1| + \alpha^\dagger\beta|1\rangle\langle 0| + |\beta|^2|1\rangle\langle 1| - \big((|\alpha|^2+\gamma|\beta|^2)|0\rangle\langle 0| \\ &+ \sqrt{1-\gamma}\alpha\beta^\dagger|0\rangle\langle 1| + \sqrt{1-\gamma}\alpha^\dagger\beta|1\rangle\langle 0| + (1-\gamma)|\beta|^2|1\rangle\langle 1|\big) \\ =&\gamma|\beta|^2|0\rangle\langle 0| + (1-\sqrt{1-\gamma})(\alpha\beta^\dagger|0\rangle\langle 1| + \alpha^\dagger\beta|1\rangle\langle 0|) - \gamma|\beta|^2|1\rangle\langle 1|\end{aligned} \tag{3.55}$$

Employing Equation 2.8, the components of the Bloch vector of the state $\rho-\rho'$ are as follows:

$$\begin{aligned}r_x =&\mathrm{Tr}\left[\begin{pmatrix}\gamma|\beta|^2 & (1-\sqrt{1-\gamma})\alpha\beta^\dagger \\ (1-\sqrt{1-\gamma})\alpha^\dagger\beta & -\gamma|\beta|^2\end{pmatrix}\cdot\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right] \\ =&\mathrm{Tr}\left[\begin{pmatrix}(1-\sqrt{1-\gamma})\alpha\beta^\dagger & \gamma|\beta|^2 \\ -\gamma|\beta|^2 & (1-\sqrt{1-\gamma})\alpha^\dagger\beta\end{pmatrix}\right] = (1-\sqrt{1-\gamma})(\alpha\beta^\dagger+\alpha^\dagger\beta) \\ r_y =&\mathrm{Tr}\left[\begin{pmatrix}\gamma|\beta|^2 & (1-\sqrt{1-\gamma})\alpha\beta^\dagger \\ (1-\sqrt{1-\gamma})\alpha^\dagger\beta & -\gamma|\beta|^2\end{pmatrix}\cdot\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\right] \\ =&\mathrm{Tr}\left[\begin{pmatrix}i(1-\sqrt{1-\gamma})\alpha\beta^\dagger & -i\gamma|\beta|^2 \\ i\gamma|\beta|^2 & -i(1-\sqrt{1-\gamma})\alpha^\dagger\beta\end{pmatrix}\right] = i(1-\sqrt{1-\gamma})(\alpha\beta^\dagger-\alpha^\dagger\beta) \\ r_z =&\mathrm{Tr}\left[\begin{pmatrix}\gamma|\beta|^2 & (1-\sqrt{1-\gamma})\alpha\beta^\dagger \\ (1-\sqrt{1-\gamma})\alpha^\dagger\beta & -\gamma|\beta|^2\end{pmatrix}\cdot\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right] \\ =&\mathrm{Tr}\left[\begin{pmatrix}\gamma|\beta|^2 & -(1-\sqrt{1-\gamma})\alpha\beta^\dagger \\ (1-\sqrt{1-\gamma})\alpha^\dagger\beta & \gamma|\beta|^2\end{pmatrix}\right] = \gamma|\beta|^2+\gamma|\beta|^2 = 2\gamma|\beta|^2\end{aligned} \tag{3.56}$$

Consequently, and knowing that the distance between two vectors corresponds to their Euclidean distance, it follows that the distance between the ideal and noisy implementations of the teleported qubit corresponds to:

$$
\begin{aligned}
&\|\rho - \rho'\|_\diamond \\
&= \left\| \left((1-\sqrt{1-\gamma})(\alpha\beta^\dagger + \alpha^\dagger\beta), i(1-\sqrt{1-\gamma})(\alpha\beta^\dagger - \alpha^\dagger\beta), 2\gamma|\beta|^2 \right) \right\|_2 && \{\text{Equation 3.56}\} \\
&= \sqrt{\left((1-\sqrt{1-\gamma})(\alpha\beta^\dagger + \alpha^\dagger\beta)\right)^2 + \left(i(1-\sqrt{1-\gamma})(\alpha\beta^\dagger - \alpha^\dagger\beta)\right)^2 + (2\gamma|\beta|^2)^2} && \{\text{Equation 2.2}\} \\
&= \sqrt{\left((1-\sqrt{1-\gamma})(\alpha\beta^\dagger + \alpha^\dagger\beta)\right)^2 - \left((1-\sqrt{1-\gamma})(\alpha\beta^\dagger - \alpha^\dagger\beta)\right)^2 + (2\gamma|\beta|^2)^2} \\
&= \sqrt{4 \cdot (1-\sqrt{1-\gamma})^2|\alpha|^2|\beta|^2 + 4\gamma^2|\beta|^4} \\
&= 2 \cdot \sqrt{(1-\sqrt{1-\gamma})^2|\alpha|^2|\beta|^2 + \gamma^2|\beta|^4}
\end{aligned}
\tag{3.57}
$$

Note that, as expected when $\gamma \to 0$ or $\beta \to 0$,$\|\rho - \rho'\|_\diamond \to 0$, and when $\gamma \to 1$, $\|\rho - \rho'\|_\diamond \to 2\left(\sqrt{|\alpha|^2|\beta|^2 + \gamma^2|\beta|^4}\right)$.

From this result, it follows that **Identity** $=_{2\cdot\sqrt{(1-\sqrt{1-\gamma})^2|\alpha|^2|\beta|^2+\gamma^2|\beta|^4}}$ **Identity**$^{A(\gamma)}$. Thus, **QTP** $=_{2\cdot\sqrt{(1-\sqrt{1-\gamma})^2|\alpha|^2|\beta|^2+\gamma^2|\beta|^4}}$ **QTP**$^{A(\gamma)}$.

Noisy Quantum Teleportation: An imperfect implementation of the Hadamard gate

Now, it will be considered an imperfect implementation of a Hadamard gate, denoted as H^ϵ. Therefore, a new operation is added H^ϵ : *qbit* $\to$ *qbit* and it is postulated as an axiom that q : *qbit* $\triangleright H =_\epsilon H^\epsilon$: *qbit*. In this example, considering the Hadamard gate as the composition $R_y(\frac{\pi}{2}) \cdot P(\pi)$, H^ϵ is regarded as the composition $R_y(\frac{\pi}{2}) \cdot P(\pi + \delta)$. This imperfect implementation deviates from a precise rotation of π radians along the z-axis, rotating by $\pi + \delta$ radians instead. This type of imperfection is inevitable during the implementation of quantum gates. The circuit representing the introduction of an erroneous Hadamard gate is presented in Figure 14.

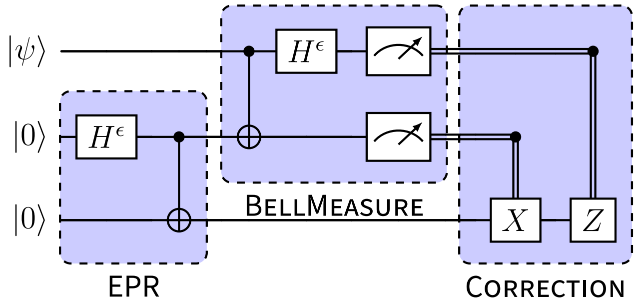

Figure 14: Quantum Teleportation Protocol: Erroneous implementation of the Hadamard gate. H^ϵ is regarded as the composition $R_y(\frac{\pi}{2}) \cdot P(\pi + \epsilon)$.

As usual, the initial step consists of evaluating the distance between the density operators of the ideal and noisy implementations of the Hadamard gate within each block. With respect to the EPR block, as presented in Equation 3.41 the ideal state of the EPR pair is $\frac{1}{2}(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|)$. Regarding, the imperfect Hadamard gate one has that:

$$\begin{aligned}
& |0\rangle \otimes |0\rangle \\
\xmapsto{H^\epsilon \otimes I} \quad & R_y\left(\frac{\pi}{2}\right) \cdot P(\pi+\epsilon)\,|0\rangle \otimes |0\rangle = R_y\left(\frac{\pi}{2}\right)|0\rangle \otimes |0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle) \\
\xmapsto{CNOT} \quad & \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = |\Phi\rangle
\end{aligned} \tag{3.58}$$

Therefore, the state of the EPR pair with an imperfect Hadamard gate is $|\Phi\rangle\langle\Phi| = \frac{1}{2}(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|)$. Hence, the imperfect Hadamard gate does not affect the state of the EPR pair and, as a result, the distance between the ideal and noisy implementations of the EPR pair is zero, **EPR** $=_0$ **EPR**$^{H(\epsilon)}$.

Next, it is necessary to repeat this exercise regarding the BellMeasure block. As shown in Equation 3.42, the ideal state of the BellMeasure block is
$\rho = \left(\left(\frac{1}{4}|\psi\rangle\langle\psi|, \frac{1}{4}X|\psi\rangle\langle\psi|X\right), \left(\frac{1}{4}Z|\psi\rangle\langle\psi|Z, \frac{1}{4}XZ|\psi\rangle\langle\psi|ZX\right)\right)$. Regarding the imperfect Hadamard gate, knowing that:

$$\begin{aligned}
& \alpha|0\rangle + \beta|1\rangle \\
\xmapsto{H^\epsilon} \quad & R_y\left(\frac{\pi}{2}\right) \cdot P(\pi+\epsilon)(\alpha|0\rangle + \beta|1\rangle) = R_y\left(\frac{\pi}{2}\right) \cdot \left(\alpha|0\rangle + e^{i(\pi+\epsilon)}\beta|1\rangle\right) \\
= \quad & R_y\left(\frac{\pi}{2}\right) \cdot \left(\alpha|0\rangle - e^{i\epsilon}\beta|1\rangle\right) = \frac{1}{\sqrt{2}}\left((\alpha + e^{i\epsilon}\beta)|0\rangle + (\alpha - e^{\epsilon}\beta)|1\rangle\right)
\end{aligned} \tag{3.59}$$

It follows, that:

$$
\begin{aligned}
& |\psi\rangle \otimes |0\rangle \otimes |0\rangle \\
\xmapsto{\text{EPR}} \quad & \frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle) \\
\xmapsto{CNOT \otimes I} \quad & \frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|011\rangle + \beta|110\rangle + \beta|101\rangle) \\
\xmapsto{H^{\epsilon} \otimes I \otimes I} \quad & \frac{1}{2}\left(\alpha(|000\rangle + |100\rangle + |011\rangle + |111\rangle) + \beta e^{i\epsilon}(|010\rangle - |110\rangle + |001\rangle - |101\rangle)\right) \\
& = \frac{1}{2}\left(\alpha(|000\rangle + |100\rangle + |011\rangle + |111\rangle) + \beta e^{i\epsilon}(|010\rangle - |110\rangle + |001\rangle - |101\rangle)\right) \\
& = \frac{1}{2}(|00\rangle \otimes (\alpha|0\rangle + \beta e^{i\epsilon}|1\rangle) + |01\rangle \otimes (\alpha|1\rangle + e^{i\epsilon}\beta|0\rangle) + |10\rangle \otimes (\alpha|0\rangle - e^{i\epsilon}\beta|1\rangle)) \\
& + |11\rangle \otimes (\alpha|1\rangle - e^{i\epsilon}\beta|0\rangle)) \\
& = |00\rangle \otimes P(\epsilon)|\psi\rangle + |01\rangle \otimes XP(\epsilon)|\psi\rangle + |10\rangle \otimes ZP(\epsilon)|\psi\rangle + |11\rangle \otimes XZP(\epsilon)|\psi\rangle \\
& = |\phi'\rangle
\end{aligned}
\tag{3.60}
$$

Finally, measuring the first two qubits:

$$
\begin{aligned}
|\phi'\rangle\langle\phi'| \xmapsto{\text{meas} \otimes \text{meas} \otimes I} & \Big(\Big(\frac{1}{4}P(\epsilon)|\psi\rangle\langle\psi|P^{\dagger}(\epsilon), \frac{1}{4}XP(\epsilon)|\psi\rangle\langle\psi|XP^{\dagger}(\epsilon)\Big), \\
& \Big(\frac{1}{4}ZP(\epsilon)|\psi\rangle\langle\psi|P^{\dagger}(\epsilon)Z, \frac{1}{4}XZP(\epsilon)|\psi\rangle\langle\psi|P^{\dagger}(\epsilon)ZX\Big)\Big) = \rho'
\end{aligned}
\tag{3.61}
$$

Given that,

$$
\begin{aligned}
|\psi\rangle\langle\psi| - P(\epsilon)|\psi\rangle\langle\psi|P^{\dagger}(\epsilon) = & |\alpha|^2|0\rangle\langle 0| + \alpha\beta^{\dagger}|0\rangle\langle 1| + \alpha^{\dagger}\beta|1\rangle\langle 0| + |\beta|^2|1\rangle\langle 1| - \\
& (|\alpha|^2|0\rangle\langle 0| + e^{-i\epsilon}\alpha\beta^{\dagger}|0\rangle\langle 1| + e^{i\epsilon}\alpha^{\dagger}\beta|1\rangle\langle 0| + |\beta|^2|1\rangle\langle 1|) \\
= & (1 - e^{-i\epsilon})\alpha\beta^{\dagger}|0\rangle\langle 1| + (1 - e^{i\epsilon})\alpha^{\dagger}\beta|1\rangle\langle 0|
\end{aligned}
\tag{3.62}
$$

$$
\begin{aligned}
X|\psi\rangle\langle\psi|X - XP(\epsilon)|\psi\rangle\langle\psi|XP^{\dagger}(\epsilon) = & |\alpha|^2|1\rangle\langle 1| + \alpha\beta^{\dagger}|1\rangle\langle 0| + \alpha^{\dagger}\beta|0\rangle\langle 1| + |\beta|^2|0\rangle\langle 0| - \\
& (|\alpha|^2|1\rangle\langle 1| + e^{-i\epsilon}\alpha\beta^{\dagger}|1\rangle\langle 0| + e^{i\epsilon}\alpha^{\dagger}\beta|0\rangle\langle 1| + |\beta|^2|0\rangle\langle 0|) \\
= & (1 - e^{-i\epsilon})\alpha\beta^{\dagger}|1\rangle\langle 0| + (1 - e^{i\epsilon})\alpha^{\dagger}\beta|0\rangle\langle 1|
\end{aligned}
\tag{3.63}
$$

$$
\begin{aligned}
Z|\psi\rangle\langle\psi|Z - ZP(\epsilon)|\psi\rangle\langle\psi|P^{\dagger}(\epsilon)Z = & |\alpha|^2|0\rangle\langle 0| - \alpha\beta^{\dagger}|0\rangle\langle 1| - \alpha^{\dagger}\beta|1\rangle\langle 0| + |\beta|^2|1\rangle\langle 1| - \\
& (|\alpha|^2|0\rangle\langle 0| - e^{-i\epsilon}\alpha\beta^{\dagger}|0\rangle\langle 1| - e^{i\epsilon}\alpha^{\dagger}\beta|1\rangle\langle 0| + |\beta|^2|1\rangle\langle 1|) \\
= & (e^{-i\epsilon} - 1)\alpha\beta^{\dagger}|0\rangle\langle 1| + (e^{i\epsilon} - 1)\alpha^{\dagger}\beta|1\rangle\langle 0|
\end{aligned}
$$

$$ \tag{3.64} $$

$$
\begin{aligned}
XZ\,|\psi\rangle\,\langle\psi|\,ZX - XZP(\epsilon)|\psi\rangle\langle\psi|P^{\dagger}(\epsilon)ZX =& |\alpha|^2|1\rangle\langle 1| - \alpha\beta^{\dagger}|1\rangle\langle 0| - \alpha^{\dagger}\beta|0\rangle\langle 1| + |\beta|^2|0\rangle\langle 0| - \\
& (|\alpha|^2|1\rangle\langle 1| - e^{-i\epsilon}\alpha\beta^{\dagger}|1\rangle\langle 0| - e^{i\epsilon}\alpha^{\dagger}\beta|0\rangle\langle 1| + |\beta|^2|0\rangle\langle 0|) \\
=& (e^{-i\epsilon} - 1)\alpha\beta^{\dagger}|1\rangle\langle 0| + (e^{i\epsilon} - 1)\alpha^{\dagger}\beta|0\rangle\langle 1|
\end{aligned}
\tag{3.65}
$$

Consequently,

$$
\begin{aligned}
\rho - \rho' =& \Big(\Big(\frac{1}{4}|\psi\rangle\langle\psi| - \frac{1}{4}P(\epsilon)|\psi\rangle\langle\psi|P^{\dagger}(\epsilon), \frac{1}{4}XP(\epsilon)|\psi\rangle\langle\psi|XP^{\dagger}(\epsilon) - \frac{1}{4}XP(\epsilon)|\psi\rangle\langle\psi|XP^{\dagger}(\epsilon)\Big), \\
& \Big(\frac{1}{4}ZP(\epsilon)|\psi\rangle\langle\psi|P^{\dagger}(\epsilon)Z - \frac{1}{4}ZP(\epsilon)|\psi\rangle\langle\psi|P^{\dagger}(\epsilon)Z, \\
& \frac{1}{4}P(\epsilon)XZ|\psi\rangle\langle\psi|P^{\dagger}(\epsilon)ZX - \frac{1}{4}XZP(\epsilon)|\psi\rangle\langle\psi|P^{\dagger}(\epsilon)ZX\Big)\Big) \\
=& \Big(\Big(\frac{1}{4}(1 - e^{-i\epsilon})\alpha\beta^{\dagger}|0\rangle\langle 1| + \frac{1}{4}(1 - e^{i\epsilon})\alpha^{\dagger}\beta|1\rangle\langle 0|, \frac{1}{4}(1 - e^{-i\epsilon})\alpha\beta^{\dagger}|1\rangle\langle 0| + \frac{1}{4}(1 - e^{i\epsilon})\alpha^{\dagger}\beta|0\rangle\langle 1|\Big), \\
& \Big(\frac{1}{4}(e^{-i\epsilon} - 1)\alpha\beta^{\dagger}|0\rangle\langle 1| + \frac{1}{4}(e^{i\epsilon} - 1)\alpha^{\dagger}\beta|1\rangle\langle 0|, \frac{1}{4}(e^{-i\epsilon} - 1)\alpha\beta^{\dagger}|1\rangle\langle 0| + \frac{1}{4}(e^{i\epsilon} - 1)\alpha^{\dagger}\beta|0\rangle\langle 1||\Big)\Big) \\
=& \left(\left(\frac{1}{4}\sigma, \frac{1}{4}\sigma'\right), \left(\frac{1}{4}\sigma'', \frac{1}{4}\sigma'''\right)\right)
\end{aligned}
\tag{3.66}
$$

Employing Equation 2.8, the components of the Bloch vector of each state $\sigma, \sigma', \sigma'', \sigma'''$ are as follows:

$$
\begin{aligned}
r_x =&\mathsf{Tr}\left[\begin{pmatrix} 0 & (1-e^{-i\epsilon})\alpha\beta^\dagger \\ (1-e^{i\epsilon})\alpha^\dagger\beta & 0 \end{pmatrix}\cdot\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] \\
=&\mathsf{Tr}\left[\begin{pmatrix} (1-e^{-i\epsilon})\alpha\beta^\dagger & 0 \\ 0 & (1-e^{i\epsilon})\alpha^\dagger\beta \end{pmatrix}\right] = (1-e^{-i\epsilon})\alpha\beta^\dagger + (1-e^{i\epsilon})\alpha^\dagger\beta \\
r_y =&\mathsf{Tr}\left[\begin{pmatrix} 0 & (1-e^{-i\epsilon})\alpha\beta^\dagger \\ (1-e^{i\epsilon})\alpha^\dagger\beta & 0 \end{pmatrix}\cdot\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right] \\
=&\mathsf{Tr}\left[\begin{pmatrix} i(1-e^{-i\epsilon})\alpha\beta^\dagger & 0 \\ 0 & -i(1-e^{i\epsilon})\alpha^\dagger\beta \end{pmatrix}\right] = i(1-e^{-i\epsilon})\alpha\beta^\dagger - i(1-e^{i\epsilon})\alpha^\dagger\beta \\
r_z =&\mathsf{Tr}\left[\begin{pmatrix} 0 & (1-e^{-i\epsilon})\alpha\beta^\dagger \\ (1-e^{i\epsilon})\alpha^\dagger\beta & 0 \end{pmatrix}\cdot\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] \\
=&\mathsf{Tr}\left[\begin{pmatrix} 0 & (1-e^{-i\epsilon})\alpha\beta^\dagger \\ -(1-e^{i\epsilon})\alpha^\dagger\beta & 0 \end{pmatrix}\right] = 0
\end{aligned}
\tag{3.67}
$$

$$
\begin{aligned}
r_x =&\mathsf{Tr}\left[\begin{pmatrix} 0 & (1-e^{i\epsilon})\alpha^\dagger\beta \\ (1-e^{-i\epsilon})\alpha\beta^\dagger & 0 \end{pmatrix}\cdot\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] \\
=&\mathsf{Tr}\left[\begin{pmatrix} (1-e^{i\epsilon})\alpha^\dagger\beta & 0 \\ 0 & (1-e^{-i\epsilon})\alpha\beta^\dagger \end{pmatrix}\right] = (1-e^{i\epsilon})\alpha^\dagger\beta + (1-e^{-i\epsilon})\alpha\beta^\dagger \\
r_y =&\mathsf{Tr}\left[\begin{pmatrix} 0 & (1-e^{i\epsilon})\alpha^\dagger\beta \\ (1-e^{-i\epsilon})\alpha\beta^\dagger & 0 \end{pmatrix}\cdot\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right] \\
=&\mathsf{Tr}\left[\begin{pmatrix} i(1-e^{i\epsilon})\alpha^\dagger\beta & 0 \\ 0 & -i(1-e^{-i\epsilon})\alpha\beta^\dagger \end{pmatrix}\right] = i(1-e^{i\epsilon})\alpha^\dagger\beta - i(1-e^{-i\epsilon})\alpha\beta^\dagger \\
r_z =&\mathsf{Tr}\left[\begin{pmatrix} 0 & (1-e^{i\epsilon})\alpha^\dagger\beta \\ (1-e^{-i\epsilon})\alpha\beta^\dagger & 0 \end{pmatrix}\cdot\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] \\
=&\mathsf{Tr}\left[\begin{pmatrix} 0 & (1-e^{i\epsilon})\alpha^\dagger\beta \\ -(1-e^{-i\epsilon})\alpha\beta^\dagger & 0 \end{pmatrix}\right] = 0
\end{aligned}
\tag{3.68}
$$

$$
\begin{aligned}
r_x =& \mathrm{Tr}\left[\begin{pmatrix} 0 & (e^{-i\epsilon}-1)\alpha\beta^\dagger \\ (e^{i\epsilon}-1)\alpha^\dagger\beta & 0 \end{pmatrix}\cdot\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] \\
=& \mathrm{Tr}\left[\begin{pmatrix} (e^{-i\epsilon}-1)\alpha\beta^\dagger & 0 \\ 0 & (e^{i\epsilon}-1)\alpha^\dagger\beta \end{pmatrix}\right] = (e^{-i\epsilon}-1)\alpha\beta^\dagger + (e^{i\epsilon}-1)\alpha^\dagger\beta \\
r_y =& \mathrm{Tr}\left[\begin{pmatrix} 0 & (e^{-i\epsilon}-1)\alpha\beta^\dagger \\ (e^{i\epsilon}-1)\alpha^\dagger\beta & 0 \end{pmatrix}\cdot\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right] \\
=& \mathrm{Tr}\left[\begin{pmatrix} i(e^{-i\epsilon}-1)\alpha\beta^\dagger & 0 \\ 0 & -i(e^{i\epsilon}-1)\alpha^\dagger\beta \end{pmatrix}\right] = i(e^{-i\epsilon}-1)\alpha\beta^\dagger - i(e^{i\epsilon}-1)\alpha^\dagger\beta \\
r_z =& \mathrm{Tr}\left[\begin{pmatrix} 0 & (e^{-i\epsilon}-1)\alpha\beta^\dagger \\ (e^{i\epsilon}-1)\alpha^\dagger\beta & 0 \end{pmatrix}\cdot\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] \\
=& \mathrm{Tr}\left[\begin{pmatrix} 0 & (e^{-i\epsilon}-1)\alpha\beta^\dagger \\ -(e^{i\epsilon}-1)\alpha^\dagger\beta & 0 \end{pmatrix}\right] = 0
\end{aligned}
\tag{3.69}
$$

$$
\begin{aligned}
r_x =& \mathrm{Tr}\left[\begin{pmatrix} 0 & (e^{i\epsilon}-1)\alpha^\dagger\beta \\ (e^{-i\epsilon}-1)\alpha\beta^\dagger & 0 \end{pmatrix}\cdot\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] \\
=& \mathrm{Tr}\left[\begin{pmatrix} (e^{i\epsilon}-1)\alpha^\dagger\beta & 0 \\ 0 & (e^{-i\epsilon}-1)\alpha\beta^\dagger \end{pmatrix}\right] = (e^{i\epsilon}-1)\alpha^\dagger\beta + (e^{-i\epsilon}-1)\alpha\beta^\dagger \\
r_y =& \mathrm{Tr}\left[\begin{pmatrix} 0 & (e^{i\epsilon}-1)\alpha^\dagger\beta \\ (e^{-i\epsilon}-1)\alpha\beta^\dagger & 0 \end{pmatrix}\cdot\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right] \\
=& \mathrm{Tr}\left[\begin{pmatrix} i(e^{i\epsilon}-1)\alpha^\dagger\beta & 0 \\ 0 & -i(e^{-i\epsilon}-1)\alpha\beta^\dagger \end{pmatrix}\right] = i(e^{i\epsilon}-1)\alpha^\dagger\beta - i(e^{-i\epsilon}-1)\alpha\beta^\dagger \\
r_z =& \mathrm{Tr}\left[\begin{pmatrix} 0 & (e^{i\epsilon}-1)\alpha^\dagger\beta \\ (e^{-i\epsilon}-1)\alpha\beta^\dagger & 0 \end{pmatrix}\cdot\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] \\
=& \mathrm{Tr}\left[\begin{pmatrix} 0 & (e^{i\epsilon}-1)\alpha^\dagger\beta \\ -(e^{-i\epsilon}-1)\alpha\beta^\dagger & 0 \end{pmatrix}\right] = 0
\end{aligned}
\tag{3.70}
$$

As a result, and given that the distance between two vectors corresponds to their Euclidean distance, it follows that the distance between the ideal and noisy implementations of the

Hadamard gate in the BellMeasure block corresponds to:

$$
\begin{aligned}
\|\rho - \rho'\|_\diamond =& \left\| \left(\left(\frac{1}{4}\sigma, \frac{1}{4}\sigma' \right), \left(\frac{1}{4}\sigma'', \frac{1}{4}\sigma''' \right) \right) \right\|_\diamond \\
=& \left\| \frac{1}{4}\sigma \right\|_\diamond + \left\| \frac{1}{4}\sigma' \right\|_\diamond + \left\| \frac{1}{4}\sigma'' \right\|_\diamond + \left\| \frac{1}{4}\sigma''' \right\|_\diamond \qquad \{\text{Equation 3.8}\} \\
=& \left\| \frac{1}{4}((1-e^{-i\epsilon})\alpha\beta^\dagger + (1-e^{i\epsilon})\alpha^\dagger\beta, i(1-e^{-i\epsilon})\alpha\beta^\dagger - i(1-e^{i\epsilon})\alpha^\dagger\beta) \right\|_2 + \\
& \left\| \frac{1}{4}((1-e^{i\epsilon})\alpha^\dagger\beta + (1-e^{-i\epsilon})\alpha\beta^\dagger, i(1-e^{i\epsilon})\alpha^\dagger\beta - i(1-e^{-i\epsilon})\alpha\beta^\dagger) \right\|_2 + \\
& \left\| \frac{1}{4}((e^{-i\epsilon}-1)\alpha\beta^\dagger + (e^{i\epsilon}-1)\alpha^\dagger\beta, i(e^{-i\epsilon}-1)\alpha\beta^\dagger - i(e^{i\epsilon}-1)\alpha^\dagger\beta) \right\|_2 + \\
& \left\| \frac{1}{4}((e^{i\epsilon}-1)\alpha^\dagger\beta + (e^{-i\epsilon}-1)\alpha\beta^\dagger, i(e^{i\epsilon}-1)\alpha^\dagger\beta - i(e^{-i\epsilon}-1)\alpha\beta^\dagger) \right\|_2
\end{aligned}
\tag{3.71}
$$

Applying Equation 2.2 to each term, it follows that:

$$
\begin{aligned}
\|\rho - \rho'\|_\diamond =& \frac{1}{4}\sqrt{((1-e^{-i\epsilon})\alpha\beta^\dagger + (1-e^{i\epsilon})\alpha^\dagger\beta)^2 + (i((1-e^{-i\epsilon})\alpha\beta^\dagger - (1-e^{i\epsilon})\alpha^\dagger\beta))^2} \\
&+ \frac{1}{4}\sqrt{((1-e^{i\epsilon})\alpha^\dagger\beta + (1-e^{-i\epsilon})\alpha\beta^\dagger)^2 + (i((1-e^{i\epsilon})\alpha^\dagger\beta - (1-e^{-i\epsilon})\alpha\beta^\dagger))^2} \\
&+ \frac{1}{4}\sqrt{((e^{-i\epsilon}-1)\alpha\beta^\dagger + (e^{i\epsilon}-1)\alpha^\dagger\beta)^2 + (i((e^{-i\epsilon}-1)\alpha\beta^\dagger - (e^{i\epsilon}-1)\alpha^\dagger\beta))^2} \\
&+ \frac{1}{4}\sqrt{((e^{i\epsilon}-1)\alpha^\dagger\beta + (e^{-i\epsilon}-1)\alpha\beta^\dagger)^2 + (i((e^{i\epsilon}-1)\alpha^\dagger\beta - (e^{-i\epsilon}-1)\alpha\beta^\dagger))^2} \\
=& \frac{1}{4}\sqrt{((1-e^{-i\epsilon})\alpha\beta^\dagger + (1-e^{i\epsilon})\alpha^\dagger\beta)^2 + (i((1-e^{-i\epsilon})\alpha\beta^\dagger - (1-e^{i\epsilon})\alpha^\dagger\beta))^2} \\
&+ \frac{1}{4}\sqrt{((1-e^{-i\epsilon})\alpha\beta^\dagger + (1-e^{i\epsilon})\alpha^\dagger\beta)^2 + (-(i((1-e^{-i\epsilon})\alpha\beta^\dagger - (1-e^{i\epsilon})\alpha^\dagger\beta)))^2} \\
&+ \frac{1}{4}\sqrt{(-((1-e^{-i\epsilon})\alpha\beta^\dagger + (1-e^{i\epsilon})\alpha^\dagger\beta))^2 + (i((1-e^{-i\epsilon})\alpha\beta^\dagger - (1-e^{i\epsilon})\alpha^\dagger\beta))^2} \\
&+ \frac{1}{4}\sqrt{(-((1-e^{-i\epsilon})\alpha\beta^\dagger + (1-e^{i\epsilon})\alpha^\dagger\beta))^2 + (-(i((1-e^{-i\epsilon})\alpha\beta^\dagger - (1-e^{i\epsilon})\alpha^\dagger\beta)))^2} \\
&= \frac{1}{4}\sqrt{((1-e^{-i\epsilon})\alpha\beta^\dagger + (1-e^{i\epsilon})\alpha^\dagger\beta)^2 + (i((1-e^{-i\epsilon})\alpha\beta^\dagger - (1-e^{i\epsilon})\alpha^\dagger\beta))^2} \\
&= \sqrt{4(1-e^{-i\epsilon})(1-e^{i\epsilon})|\alpha|^2|\beta|^2} = 2\sqrt{(1-e^{i\epsilon}-e^{-i\epsilon}+1)|\alpha|^2|\beta|^2} \\
&= 2\sqrt{2(1-\cos(\epsilon))|\alpha|^2|\beta|^2} = 2\sqrt{2}\sqrt{(1-\cos(\epsilon))|\alpha|^2|\beta|^2}
\end{aligned}
\tag{3.72}
$$

It is possible to observe that when $\epsilon = 0$, the distance between the ideal and noisy implementations of the Hadamard gate in the BellMeasure block is zero, which is consistent with the fact that the ideal and noisy implementations are the same. The same goes for $\epsilon = \pi$,

$\alpha = 0$ and $\beta = 0$ given that only the non-diagonal components of the density matrix are affected by an erroneous phase gate.

Given this result it is possible to conclude that **BellMeasure** $=_{2\sqrt{2}\sqrt{(1-\cos(\epsilon))|\alpha|^2|\beta|^2}}$ **BellMeasure**$^{H(\epsilon)}$. Hence, **QTP** $=_{0+2\sqrt{2}\sqrt{(1-\cos(\epsilon))|\alpha|^2|\beta|^2}}$ **QTP**$^{H(\epsilon)}$, *i.e.*, **QTP** $=_{2\sqrt{2}\sqrt{(1-\cos(\epsilon))|\alpha|^2|\beta|^2}}$ **QTP**$^{H(\epsilon)}$.

3.3 Discard Operation

The discard operation was defined as the trace, and therefore is also completely positive and trace-preserving.

3.3.1 Example: Proving an equivalence using the discard equation-in-context

This subsection aims to ilustrate how to prove that

$$- \triangleright \text{disc}\left(q\left(\textit{new}0\left(*\right)\right)\right) \text{to} * . q\left(\textit{new}0\left(*\right)\right) : \textit{qbit} = - \triangleright q\left(\textit{new}0\left(*\right)\right) : \textit{qbit} \tag{3.73}$$

using the discard equation-in-context.

The discard equation in the bottom line in Figure 2 states that all judgements $\Gamma \triangleright v : \mathbb{I}$ (with $\Gamma = x_1 : \mathbb{A}_1, ..., x_n : \mathbb{A}_n$) carry no different information than that of just discarding all variables available in context Γ. Therefore considering an empty context $\Gamma = -$, the discard equation states that:

$$- \triangleright v : \mathbb{I} = - \triangleright * : \mathbb{I} \tag{3.74}$$

Given that as established in Figure 1, dis$(v) : \mathbb{I}$, it follows that:

$$- \triangleright \text{disc}\left(q\left(\textit{new}0\left(*\right)\right)\right) \text{to} * . q\left(\textit{new}0\left(*\right)\right) : \textit{qbit} = - \triangleright * \text{to} * . q\left(\textit{new}0\left(*\right)\right) : \textit{qbit} \tag{3.75}$$

Subsequently applying the rule $*$ to $* . v = v$ in Figure 2, it holds that

$$\begin{aligned} - \triangleright \text{disc}\left(q\left(\textit{new}0\left(*\right)\right)\right) \text{to} * . q\left(\textit{new}0\left(*\right)\right) : \textit{qbit} &= - \triangleright * \text{ to} * . q\left(\textit{new}0\left(*\right)\right) : \textit{qbit} \\ &= - \triangleright q\left(\textit{new}0\left(*\right)\right) : \textit{qbit} \end{aligned} \tag{3.76}$$

3.3.2 Ilustration: A malicious attack on the quantum teleportation protocol

Now, consider a malicious attack on the quantum teleportation protocol in the form of a bit-flip occurring with a 50% probability before measurement. More generally, one can define an operation T that applies a unitary operation U to the state given as input with 50% probability. Operation T can be defined as follows:

$$T: \textit{qbit}, ..., \textit{qbit} \multimap \textit{qbit}^{\otimes n}$$
$$T = q_1 : \textit{qbit}, ..., q_n : \textit{qbit} \triangleright \text{pm}\, CU(R_X^{\frac{\pi}{2}}(q(\textit{new}\,0(*))), q_1, ..., q_n) \text{ to } newq \otimes q.\,\textit{disc}(newq)$$

This operation is depicted in Figure 15.

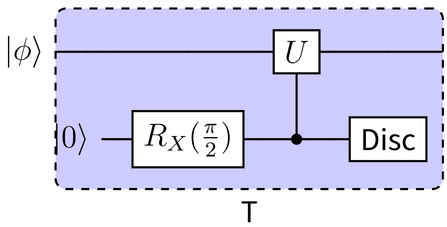

Figure 15: T operation

Regarding the calculations, applying operation T to the state $|\psi\rangle$, one has that:

$$\begin{aligned}
& |\phi\rangle\langle\phi| \\
\xmapsto{I\otimes q(\textit{new}\,0(*))} \quad & |\phi\rangle\langle\phi| \otimes |0\rangle\langle 0| \\
\xmapsto{I\otimes R_X(\frac{\pi}{2})} \quad & |\phi\rangle\langle\phi| \otimes \frac{1}{2}(|0\rangle\langle 0| - i|0\rangle\langle 1| + i|1\rangle\langle 0| + |1\rangle\langle 1|) \\
& = \frac{1}{2}(|\phi\rangle\langle\phi| |0\rangle\langle 0| - i|\phi\rangle\langle\phi| |0\rangle\langle 1| + i|\phi\rangle\langle\phi| |1\rangle\langle 0| + |\phi\rangle\langle\phi| |1\rangle\langle 1|) \\
\xmapsto{\text{CU}} \quad & \frac{1}{2}\left(|\phi\rangle\langle\phi| |0\rangle\langle 0| - i|\phi\rangle\langle\phi| |0\rangle\langle 1| U^\dagger + iU|\phi\rangle\langle\phi| |1\rangle\langle 0| + U|\phi\rangle\langle\phi| |1\rangle\langle 1| U^\dagger\right) \\
\xmapsto{I\otimes \text{Disc}} \quad & \frac{1}{2}\left(|\phi\rangle\langle\phi| + U|\phi\rangle\langle\phi| U^\dagger\right)
\end{aligned} \tag{3.77}$$

Revisiting the example at hand, the circuit that represents the quantum teleportation protocol with a 50% probability of occurring a bit flip prior to measurement is depicted in Figure 16.

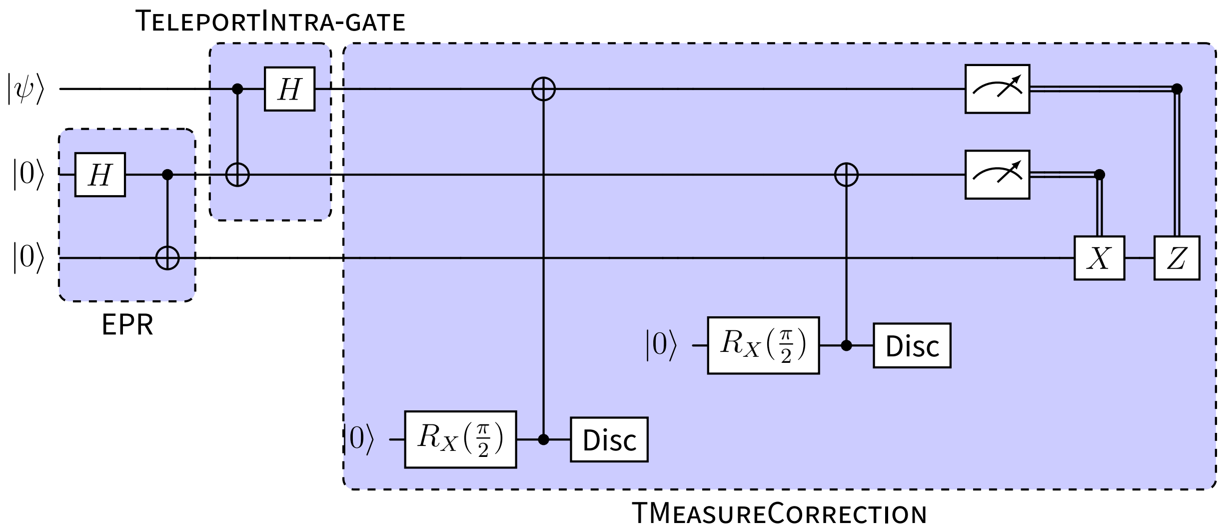

Figure 16: Quantum Teleportation Protocol: Bit flip with 50% probability before measurement.

In this case, the quantum teleportation protocol is divided into three parts: **EPR**, **TeleportIntra-gate** and **TMeasureCorrection**. As a result, it is necessary to define the new functions (note that the function **EPR** is the same as the one defined in subsection 3.2.5):

BellMeasure : $\quad$ *qbit* $\otimes$ *qbit* $\multimap$ *qbit* $\otimes$ *qbit*

TeleportIntra-gate : $\quad$ *qbit* $\otimes$ *qbit* $\otimes$ *qbit* $\multimap$ *qbit* $\otimes$ *qbit* $\otimes$ *qbit*

TMeasureCorrection : $\quad$ *qbit* $\otimes$ *qbit* $\otimes$ *qbit* $\multimap$ *qbit*

Considering the operation $T_{X\otimes I\otimes I}$ as the operation T with the unitary U represented by $X \otimes I \otimes I$, and similarly, $T_{I\otimes X\otimes I}$ as T with U denoted by $I \otimes U \otimes I$, these funtions can be defined as follows:

TeleportIntra-gate $= q_1 : \mathit{qbit}, q_2 : \mathit{qbit} \triangleright (\mathsf{pm}\ \mathit{CNOT}(q_1, q_2)\ \mathsf{to}\ x \otimes y.\, \mathit{H}(x) \otimes y)$

$$\begin{aligned}
\textbf{TMeasureCorrection} = {} & q_1 : \mathit{qbit}, q_2 : \mathit{qbit}, q_3 : \mathit{qbit} \triangleright \mathsf{pm}\ T_{X\otimes I\otimes I}(q_1, q_2, q_3)\ \mathsf{to}\ a \otimes b \otimes c. \\
& \mathsf{pm}\ T_{I\otimes X\otimes I}(a, b, c)\ \mathsf{to}\ d \otimes e \otimes q. \\
& \mathsf{pm}\ \mathit{meas}(d) \otimes \mathit{meas}(e)\ \mathsf{to}\ x \otimes y. \\
& \mathsf{case}\ \ x\ \{\mathsf{inl}(x_0) \Rightarrow (\mathsf{cond}\ y\ \{\mathsf{inl}(y_0) \Rightarrow \mathit{I}(q);\ \ \mathsf{inr}(y_1) \Rightarrow\ \mathit{X}(q))\}; \\
& \qquad \mathsf{inr}(x_1) \Rightarrow (\mathsf{cond}\ y\ \{\mathsf{inl}(y_0) \Rightarrow \mathit{Z}(q); \mathsf{inr}(y_1) \Rightarrow \mathit{Z}(\mathit{X}(q))\})\}
\end{aligned}$$

Designating the qubit to be teleported as q_0, one can conceptualize he quantum teleportation protocol with a 50% probability of occurring a bit flip prior to measurement as follows:

$$\begin{aligned}
&\text{pm } \textbf{EPR}(*) \text{ to } q_1 \otimes q_2.\\
&\text{pm } \textbf{TeleportIntra-gate}(q_0, q_1) \text{ to } tiq_0 \otimes tiq_1.\\
&\text{pm } \textbf{TMeasureCorrection}(tiq_0, tiq_1, q_2) \text{ to } q.\, q
\end{aligned}$$

Per Equation 3.43, the state of the system post-teleportation protocol corresponds to $|\psi\rangle\langle\psi|$ in the absence of a malicious attack, denoted as ρ.

Regarding the first two parts of the teleportation protocol, given Equation 3.41, one has that:

$$\begin{aligned}
&|\psi\rangle\langle\psi| \otimes |0\rangle\langle 0| \otimes |0\rangle\langle 0|\\
\xmapsto{\text{EPR}}\ &\frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle)\\
\xmapsto{\text{TeleportIntra-gate}}\ &|00\rangle \otimes |\psi\rangle + |01\rangle \otimes X|\psi\rangle + |10\rangle \otimes Z|\psi\rangle + |11\rangle \otimes XZ|\psi\rangle = |\phi\rangle
\end{aligned} \tag{3.78}$$

Consequently, the state of the system post-teleportation protocol corresponds to $|\phi\rangle\langle\phi|$.
With respect to **TMeasureCorrection**, considering that,

$$\begin{aligned}
&|\phi\rangle = |00\rangle \otimes |\psi\rangle + |01\rangle \otimes X|\psi\rangle + |10\rangle \otimes Z|\psi\rangle + |11\rangle \otimes XZ|\psi\rangle\\
\xmapsto{X\otimes I\otimes I}\ &|10\rangle \otimes |\psi\rangle + |11\rangle \otimes X|\psi\rangle + |00\rangle \otimes Z|\psi\rangle + |01\rangle \otimes XZ|\psi\rangle\\
&= |00\rangle \otimes Z|\psi\rangle + |01\rangle \otimes XZ|\psi\rangle + |10\rangle \otimes |\psi\rangle + |11\rangle \otimes X|\psi\rangle = |\phi'\rangle
\end{aligned} \tag{3.79}$$

And,

$$\begin{aligned}
&|\phi\rangle = |00\rangle \otimes |\psi\rangle + |01\rangle \otimes X|\psi\rangle + |10\rangle \otimes Z|\psi\rangle + |11\rangle \otimes XZ|\psi\rangle\\
\xmapsto{I\otimes X\otimes I}\ &|01\rangle \otimes |\psi\rangle + |00\rangle \otimes X|\psi\rangle + |11\rangle \otimes Z|\psi\rangle + |10\rangle \otimes XZ|\psi\rangle\\
&= |00\rangle \otimes X|\psi\rangle + |01\rangle \otimes |\psi\rangle + |10\rangle \otimes XZ|\psi\rangle + |11\rangle \otimes Z|\psi\rangle = |\phi''\rangle
\end{aligned} \tag{3.80}$$

And finally,

$$\begin{aligned}
&|\phi'\rangle = |00\rangle \otimes Z|\psi\rangle + |01\rangle \otimes XZ|\psi\rangle + |10\rangle \otimes |\psi\rangle + |11\rangle \otimes X|\psi\rangle\\
\xmapsto{I\otimes X\otimes I}\ &|01\rangle \otimes Z|\psi\rangle + |00\rangle \otimes XZ|\psi\rangle + |11\rangle \otimes |\psi\rangle + |10\rangle \otimes X|\psi\rangle\\
&= |00\rangle \otimes XZ|\psi\rangle + |01\rangle \otimes Z|\psi\rangle + |10\rangle \otimes X|\psi\rangle + |11\rangle \otimes |\psi\rangle = |\phi'''\rangle
\end{aligned} \tag{3.81}$$

It follows that,

$$
\begin{aligned}
& |\phi\rangle \langle\phi| \\
\xmapsto{T_{X\otimes I\otimes I}} \quad & \frac{1}{2}\left(|\phi\rangle\langle\phi| + |\phi'\rangle\langle\phi'|\right) && \{\text{Equation 3.77}\} \\
\xmapsto{T_{I\otimes X\otimes I}} \quad & \frac{1}{4}\left(|\phi\rangle\langle\phi| + |\phi'\rangle\langle\phi'| + |\phi''\rangle\langle\phi''| + |\phi'''\rangle\langle\phi'''|\right) && \{\text{Equation 3.77}\} \\
\xmapsto{\text{meas}\otimes\text{meas}\otimes I} \quad & \frac{1}{4}\bigg(\Big(\Big(\frac{1}{4}|\psi\rangle\langle\psi|, \frac{1}{4}X|\psi\rangle\langle\psi|X\Big), \Big(\frac{1}{4}Z|\psi\rangle\langle\psi|Z, \frac{1}{4}XZ|\psi\rangle\langle\psi|ZX\Big)\Big) \\
& + \Big(\Big(\frac{1}{4}Z|\psi\rangle\langle\psi|Z, \frac{1}{4}XZ|\psi\rangle\langle\psi|XZ\Big), \Big(\frac{1}{4}|\psi\rangle\langle\psi|, \frac{1}{4}X|\psi\rangle\langle\psi|X\Big)\Big) \\
& + \Big(\Big(\frac{1}{4}X|\psi\rangle\langle\psi|X, \frac{1}{4}|\psi\rangle\langle\psi|\Big), \Big(\frac{1}{4}XZ|\psi\rangle\langle\psi|ZX, \frac{1}{4}Z|\psi\rangle\langle\psi|Z\Big)\Big) \\
& + \Big(\Big(\frac{1}{4}XZ|\psi\rangle\langle\psi|ZX, \frac{1}{4}Z|\psi\rangle\langle\psi|Z\Big), \Big(\frac{1}{4}X|\psi\rangle\langle\psi|X, \frac{1}{4}|\psi\rangle\langle\psi|\Big)\Big)\bigg)
\end{aligned}
\tag{3.82}
$$

Next, regarding the condicional statements, applying correction to $\Big(\big(\frac{1}{4}|\psi\rangle\langle\psi|, \frac{1}{4}X|\psi\rangle\langle\psi|X\big), \big(\frac{1}{4}Z|\psi\rangle\langle\psi|Z,$ results in the state $|\psi\rangle$ (Equation 3.43). Moreover, with respect to $\Big(\big(\frac{1}{4}|\psi\rangle\langle\psi|, \frac{1}{4}X|\psi\rangle\langle\psi|X\big), \big(\frac{1}{4}Z|\psi\rangle\langle\psi|Z, \frac{1}{4}X$ one has that applying the condicional statements:

$$
\begin{aligned}
& \frac{1}{4}Z|\psi\rangle\langle\psi|Z + \frac{1}{4}XXZ|\psi\rangle\langle\psi|ZXX + \frac{1}{4}Z|\psi\rangle\langle\psi|Z + \frac{1}{4}ZXX|\psi\rangle\langle\psi|XXZ \\
= & \frac{1}{4}Z|\psi\rangle\langle\psi|Z + \frac{1}{4}Z|\psi\rangle\langle\psi|Z + \frac{1}{4}Z|\psi\rangle\langle\psi|Z + \frac{1}{4}Z|\psi\rangle\langle\psi|Z = Z|\psi\rangle\langle\psi|Z
\end{aligned}
\tag{3.83}
$$

Furthermore, applying correction to $\Big(\big(\frac{1}{4}X|\psi\rangle\langle\psi|X, \frac{1}{4}|\psi\rangle\langle\psi|\big), \big(\frac{1}{4}XZ|\psi\rangle\langle\psi|ZX, \frac{1}{4}Z|\psi\rangle\langle\psi|Z\big)\Big)$ results in

$$
\begin{aligned}
& \frac{1}{4}X|\psi\rangle\langle\psi|X + \frac{1}{4}X|\psi\rangle\langle\psi|X + \frac{1}{4}ZXZ|\psi\rangle\langle\psi|ZXZ + \frac{1}{4}ZXZ|\psi\rangle\langle\psi|ZXZ \\
= & \frac{1}{4}X|\psi\rangle\langle\psi|X + \frac{1}{4}X|\psi\rangle\langle\psi|X + \frac{1}{4}X|\psi\rangle\langle\psi|X + \frac{1}{4}X|\psi\rangle\langle\psi|X = X|\psi\rangle\langle\psi|X
\end{aligned}
\tag{3.84}
$$

And, at last, regarding $\Big(\big(\frac{1}{4}XZ|\psi\rangle\langle\psi|ZX, \frac{1}{4}Z|\psi\rangle\langle\psi|Z\big), \big(\frac{1}{4}X|\psi\rangle\langle\psi|X, \frac{1}{4}|\psi\rangle\langle\psi|\big)\Big)$,

$$
\begin{aligned}
& \frac{1}{4}XZ|\psi\rangle\langle\psi|ZX + \frac{1}{4}XZ|\psi\rangle\langle\psi|ZX + \frac{1}{4}ZX|\psi\rangle\langle\psi|XZ + \frac{1}{4}ZX|\psi\rangle\langle\psi|XZ \\
= & \frac{1}{4}XZ|\psi\rangle\langle\psi|ZX + \frac{1}{4}XZ|\psi\rangle\langle\psi|ZX + \frac{1}{4}XZ|\psi\rangle\langle\psi|ZX + \frac{1}{4}XZ|\psi\rangle\langle\psi|ZX \\
= & ZX|\psi\rangle\langle\psi|XZ
\end{aligned}
\tag{3.85}
$$

Consequentlty, applying the conditional statements to the state obtained in Equation 3.82, it follows that,

$$
\begin{aligned}
&\frac{1}{4}\left(\left|\psi\right\rangle\left\langle\psi\right| + Z\left|\psi\right\rangle\left\langle\psi\right|Z + X\left|\psi\right\rangle\left\langle\psi\right|X + ZX\left|\psi\right\rangle\left\langle\psi\right|XZ\right)\\
=&\frac{1}{4}(|\alpha|^2|0\rangle\langle 0| + \alpha\beta^\dagger|0\rangle\langle 1| + \alpha^\dagger\beta|1\rangle\langle 0| + |\beta|^2|1\rangle\langle 1|\\
&+|\alpha|^2\left|0\right\rangle\left\langle 0\right| - \alpha\beta^\dagger\left|0\right\rangle\left\langle 1\right| - \alpha^\dagger\beta\left|1\right\rangle\left\langle 0\right| + |\beta|^2\left|1\right\rangle\left\langle 1\right|\\
&+|\beta|^2\left|0\right\rangle\left\langle 0\right| + \alpha^\dagger\beta\left|0\right\rangle\left\langle 1\right| + \alpha\beta^\dagger\left|1\right\rangle\left\langle 0\right| + |\alpha|^2\left|1\right\rangle\left\langle 1\right|\\
&+|\beta|^2\left|0\right\rangle\left\langle 0\right| - \alpha^\dagger\beta\left|0\right\rangle\left\langle 1\right| - \alpha\beta^\dagger\left|1\right\rangle\left\langle 0\right| + |\alpha|^2\left|1\right\rangle\left\langle 1\right|)\\
=&\frac{|\alpha|^2+|\beta|^2}{2}\left|0\right\rangle\left\langle 0\right| + \frac{|\alpha|^2+|\beta|^2}{2}\left|1\right\rangle\left\langle 1\right| = \rho'
\end{aligned}
\tag{3.86}
$$

Therefore, $\rho - \rho'$ corresponds to:

$$
\begin{aligned}
\rho - \rho' =&|\alpha|^2\left|0\right\rangle\left\langle 0\right| + \alpha\beta^\dagger\left|0\right\rangle\left\langle 1\right| + \alpha^\dagger\beta\left|1\right\rangle\left\langle 0\right| + |\beta|^2\left|1\right\rangle\left\langle 1\right|\\
&-\left(\frac{|\alpha|^2+|\beta|^2}{2}\left|0\right\rangle\left\langle 0\right| + \frac{|\alpha|^2+|\beta|^2}{2}\left|1\right\rangle\left\langle 1\right|\right)\\
=&\frac{|\alpha|^2-|\beta|^2}{2}\left|0\right\rangle\left\langle 0\right| + \alpha\beta^\dagger\left|0\right\rangle\left\langle 1\right| + \alpha^\dagger\beta\left|1\right\rangle\left\langle 0\right| + \frac{|\beta|^2-|\alpha|^2}{2}\left|1\right\rangle\left\langle 1\right|
\end{aligned}
\tag{3.87}
$$

Employing Equation 2.8, the components of the Bloch vector of the state $\rho - \rho'$ are as follows:

$$
\begin{aligned}
r_x =&\text{Tr}\left[\begin{pmatrix}\frac{|\alpha|^2-|\beta|^2}{2} & \alpha\beta^\dagger\\ \alpha^\dagger\beta & \frac{|\beta|^2-|\alpha|^2}{2}\end{pmatrix}\cdot\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right] = \text{Tr}\left[\begin{pmatrix}\alpha\beta^\dagger & \frac{|\alpha|^2-|\beta|^2}{2}\\ \frac{|\beta|^2-|\alpha|^2}{2} & \alpha^\dagger\beta\end{pmatrix}\right] = \alpha\beta^\dagger + \alpha^\dagger\beta\\
r_y =&\text{Tr}\left[\begin{pmatrix}\frac{|\alpha|^2-|\beta|^2}{2} & \alpha\beta^\dagger\\ \alpha^\dagger\beta & \frac{|\beta|^2-|\alpha|^2}{2}\end{pmatrix}\cdot\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\right] = \text{Tr}\left[\begin{pmatrix}i\alpha\beta^\dagger & \frac{|\alpha|^2-|\beta|^2}{2}\\ \frac{|\beta|^2-|\alpha|^2}{2} & -i\alpha^\dagger\beta\end{pmatrix}\right] = i(\alpha\beta^\dagger - \alpha^\dagger\beta)\\
r_z =&\text{Tr}\left[\begin{pmatrix}\frac{|\alpha|^2-|\beta|^2}{2} & \alpha\beta^\dagger\\ \alpha^\dagger\beta & \frac{|\beta|^2-|\alpha|^2}{2}\end{pmatrix}\cdot\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right] = \text{Tr}\left[\begin{pmatrix}\frac{|\alpha|^2-|\beta|^2}{2} & -\alpha\beta^\dagger\\ \alpha^\dagger\beta & -\frac{|\beta|^2-|\alpha|^2}{2}\end{pmatrix}\right] = |\alpha|^2 - |\beta|^2
\end{aligned}
\tag{3.88}
$$

Considering that the distance between two vectors corresponds to their Euclidean distance, it follows that the distance between the ideal state and its version subjected to the malicious attack is given by:

$$
\begin{aligned}
&\|\rho - \rho'\|_\diamond \\
&= \left\|\left(\alpha\beta^\dagger + \alpha^\dagger\beta, i(\alpha\beta^\dagger - \alpha^\dagger\beta), |\alpha|^2 - |\beta|^2\right)\right\|_2 && \{\text{Equation 3.88}\} \\
&= \sqrt{(\alpha\beta^\dagger + \alpha^\dagger\beta)^2 + (i(\alpha\beta^\dagger - \alpha^\dagger\beta))^2 + (|\alpha|^2 - |\beta|^2)^2} && \{\text{Equation 2.2}\} \\
&= \sqrt{(\alpha\beta^\dagger + \alpha^\dagger\beta)^2 + -(\alpha\beta^\dagger - \alpha^\dagger\beta)^2 + (|\alpha|^2 - |\beta|^2)^2} \\
&= \sqrt{4\alpha\beta^\dagger\alpha^\dagger\beta + |\alpha|^4 - 2|\alpha|^2|\beta|^2 + |\beta|^4} = \sqrt{4|\alpha|^2|\beta|^2 + |\alpha|^4 - 2|\alpha|^2|\beta|^2 + |\beta|^4} \\
&= \sqrt{|\alpha|^4 + 2|\alpha|^2|\beta|^2 + |\beta|^4} = \sqrt{(|\alpha|^2 + |\beta|^2)^2} = |\alpha|^2 + |\beta|^2 = 1
\end{aligned}
\tag{3.89}
$$

Chapter 4

Graded modalities

4.1 Conditionals

4.2 Quantum and relation with symmetric subspaces and construction of graded modalities

4.3 Quantum State Discrimination

Chapter 5

Conclusions and future work

Conclusions and future work.

5.1 Conclusions

5.2 Prospect for future work

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