A Study of The Gibbs Phenomenon in Fourier Series and Wavelets

by

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B.A., Allameh Tabatabaie University, 1999

B.S., University of New Mexico, 2005

THESIS

Submitted in Partial Fulfillment of the

Requirements for the Degree of

Master of Science

Mathematics

The University of New Mexico

Albuquerque, New Mexico

August, 2008

Dedication

to Suzanne

Acknowledgments

First, I would like to thank my advisor, Professor Cristina Pereyra, for taking me on as her student, and her guidance and encouragement on this thesis.

I would like to thank Professors, Pedro Embid and John Hamm for being part of my thesis committee.

A very big thanks to my wife, Suzanne for all her support and encouragement during the course of writing this thesis.

I would like to thank my friends in Albuquerque, Alejandro and Iveth Acuna, Ronald Chen, Flor Espinoza, Oksana Guba, Jaime and Christina Hernandez, Daishu Komagata, Randy, Phyllis, and Kristin Lynn, and Henry Moncada. I could not ask for a more supportive group of friends.

Finally, I would like to thank my Mom, and my cousin Parisa for their love and support.

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ABSTRACT OF THESIS

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Abstract

In this thesis, we examine the Gibbs phenomenon in Fourier and wavelet expansions of functions with jump discontinuities. In short, the Gibbs phenomenon refers to the persistent overshoot or undershoot of the values of a partial sum expansion of a function near a jump discontinuity as compared to the values of the original function. First we introduce those elements of Fourier analysis needed and give an introduction to this thesis. Then we give a historical account of this phenomenon. Next, we examine the Gibbs phenomenon in the context of Fourier series. We calculate the size of the overshoot/undershoot for a simple function with a jump discontinuity at the origin and then prove the occurrence of the phenomenon at a general point of discontinuity for a class of functions. After that we give a method of removing the Gibbs phenomenon in the context of good kernels. Finally, we show the existence of the Gibbs phenomenon for certain class of wavelets.

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Chapter 1

Introduction

This thesis is a study of the so called Gibbs phenomenon in Fourier and wavelet approximations to functions. When we approximate a function with a jump discontinuity using its Fourier series an anomaly appears near the discontinuity. The values of the partial sums near the discontinuity overshoot or undershoot the function value. As we incorporate more and more terms in the partial sums, away from the discontinuity, their graphs begin to resemble that of the original function more. However, the blips near the discontinuity persist, and although they decrease in width they appear to remain at the same height. This lack of improvement in the approximations near the discontinuity manifested in the continual presence of the overshoot or undershoot is the Gibbs phenomenon.

To better understand this phenomenon, we introduce in the second chapter those mathematical tools necessary from Fourier analysis. We begin with the definitions of Fourier series, coefficients, and partial sums. Since the issue here is the lack of uniform convergence of partial Fourier sums at the points of discontinuity, we state a few well-known theorems concerning the convergence of Fourier series. Convolutions and their relation to summation methods involving some well-known kernels are introduced next. And finally we define the so called positive kernels and hint at their relevance to the Gibbs phenomenon.

In the first section of chapter three, we go back to the time when the existence of this phenomenon was first brought to the attention of the scientific community. We introduce the main players in the history of this phenomenon and discuss their contributions. Section two of chapter three is devoted to a careful analysis of the phenomenon for a simple function with a jump discontinuity at the origin. We start with a precise definition of the Gibbs phenomenon. Next, we prove that the size of the overshoot near the discontinuity is independent of the number of terms used in the Fourier partial sum and the size of the overshoot stays a fixed percentage of the size of the jump. Several graphs are included to illustrate the points.

In chapter four, we employ a different approach to show the existence of the Gibbs phenomenon for another simple function with a jump discontinuity at the origin. We, then use this result together with two lemmas and two theorems to show the existence of the phenomenon for a "general" function with a jump discontinuity at the origin. In the next section, we discuss jump discontinuities at a general point and the case of a finite number of discontinuities. In the last section, we use a theorem to explain the behavior of summation methods involving positive kernels observed in chapter two.

In chapter five, we give a brief introduction to wavelets and multiresolution analysis. Then, we define the Gibbs phenomenon for the wavelet approximations and prove a theorem which gives a condition for the existence of the phenomenon for a certain class of wavelets. This condition involves the integral of a kernel associated to the scaling function. Using this result, we show next that restrictions on the scaling function can be somewhat relaxed, hence proving the existence of the Gibbs phenomenon for a larger class of wavelets. In the next section, we show that the Shannon wavelets exhibit the Gibbs phenomenon. After that, we introduce the so called positive delta sequences and show that similar to positive kernels they result in the elimination of the Gibbs effect. We use this result to show the absence of the Gibbs phenomenon in the Haar wavelets. Finally, in chapter six we give a summary of our conclusions.

Chapter 2

Preliminaries

In this chapter we review some elements of Fourier analysis that are used in the subsequent chapters. The definitions and the theorems presented in this chapter can be found in [9], [15], [18], and [22]. The reader can also consult these references for the proofs of the theorems and a more in depth treatment of the ideas.

2.1 Fourier Coefficients, Series, and Partial Sums

Definition 2.1. Let $\mathbb{T} := \{z = e^{i\theta} | -\pi \leq \theta < \pi\}$. Suppose $f : \mathbb{T} \to \mathbb{C}$ is integrable and 2π periodic. Then we define the nth Fourier coefficient of f by

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-in\theta} d\theta.$$

Definition 2.2. Let f and $\hat{f}(n)$ be as in definition 2.1. Then the Fourier series of f is defined by

$$\sum_{n=-\infty}^{\infty} \hat{f}(n) e^{in\theta} = \sum_{n=0}^{\infty} [a_n \sin(n\theta) + b_n \cos(n\theta)],$$

where a_n and b_n are given by

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)d\theta, \tag{2.1.1}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)d\theta. \tag{2.1.2}$$

In general, if $f : [a, b) \to \mathbb{C}$ is L-periodic where $L = b-a$ then we define the L-Fourier coefficients of f and its Fourier series by

$$\hat{f}^L(n) = \frac{1}{L}\int_a^b f(\theta)e^{-2\pi in\theta/L}d\theta,$$

$$f(\theta) \sim \sum_{n=-\infty}^{\infty} \hat{f}^L(n)e^{2\pi in\theta/L}.$$

Definition 2.3. Let f be as in definition 2.1. Then the Nth partial Fourier sum of f is defined by

$$S_N f(\theta) = \sum_{n=-N}^{N} \hat{f}(n)e^{in\theta}.$$

The next two definitions tell us how to find the Fourier transform and the inverse Fourier transform on $L^1(\mathbb{R})$.

Definition 2.4. Let $f \in L^1(\mathbb{R})$ and $\xi \in \mathbb{R}$. Then we define the Fourier transform of f by

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-ix\xi}dx.$$

Definition 2.5. Let $g \in L^1(\mathbb{R})$ and $x \in \mathbb{R}$. Then the inverse Fourier transform of g is given by

$$\check{g}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\xi)e^{ix\xi}d\xi.$$

Remark 2.6. If $f \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ then one can verify that its Fourier transform is in $L^2(\mathbb{R})$. Furthermore, one can extend by continuity $\hat{f}$ to an isometry in $L^2(\mathbb{R})$. Then we can use the formulae in Definitions 2.4, and 2.5.

2.2 Trigonometric Series and Polynomials

Definition 2.7. By a trigonometric series of period L we mean a series of the form

$$\sum_{n=-\infty}^{\infty} c_n e^{2\pi in\theta/L}.$$

So the Fourier series are part of the class of trigonometric series.

Definition 2.8. A trigonometric polynomial is a trigonometric series of period L with finitely many terms. In other words,

$$\sum_{n=-N}^{N} c_n e^{2\pi in\theta/L}.$$

2.3 Some Convergence Results

There are some natural questions regarding the Fourier series of a function f as with any infinite series. For example, does the series converge, and if it does is the convergence pointwise or uniform? If the series converges at $x = \theta$ does it converge to $f(\theta)$ or some other value? Under what conditions does the Fourier series converge uniformly to f? The following theorems will provide answers to these questions.

Theorem 2.9. *Let $f \in C(\mathbb{T})$. Suppose that the series of Fourier coefficients of f converges absolutely. In other words, $\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty$. Then,*

$$\lim_{N\to\infty} S_N f \to f \quad \textit{uniformly on } \mathbb{T}$$

The following theorem shows how the smoothness of f is related to the rate of decay of $\hat{f}(n)$. The right rate of decay can result in the absolute convergence of the series of Fourier coefficients and by the previous theorem the uniform convergence of the Fourier series.

Theorem 2.10. *Let $f \in C^2(\mathbb{T})$. Then,*

$$\hat{f}(n) = \mathcal{O}\big(\frac{1}{|n|^2}\big) \quad \textit{as } n \to \infty,$$

in other words, there exists a constant $C > 0$ such that $\hat{f}(n) \leq \dfrac{c}{|n|^2}$.

In fact, the smoother f is the faster the Fourier coefficients decay. More precisely, if $f \in C^k(\mathbb{T})$ then

$$\hat{f}(n) = \mathcal{O}\big(\frac{1}{|n|^k}\big) \quad \text{as } n \to \infty,$$

in other words, there exists a constant $C > 0$ such that $\hat{f}(n) \leq \dfrac{c}{|n|^k}$.

The following result is a consequence of theorems 2.9, and 2.10.

Corollary 2.11. *Let $f \in C^2(\mathbb{T})$. Then,*

$$\lim_{N\to\infty} S_N f \to f \quad \textit{uniformly as } N \to \infty.$$

Actually, the smoothness condition can be relaxed to just one continuous derivative as the next theorem states.

Theorem 2.12. *Let $f \in C^1(\mathbb{T})$, and $\theta \in \mathbb{T}$. Then,*

$$\lim_{N\to\infty} S_N f(\theta) = f(\theta).$$

Moreover, the convergence is uniform on $\mathbb{T}$.

The next lemma, known as the Riemann-Lebesgue Lemma, shows that for the Fourier coefficients of f to decay to zero we do not necessarily need any number of continuous derivatives.

Lemma 2.13. *If $f \in C(\mathbb{T})$, then*

$$\hat{f}(n) \to 0 \quad \textit{as } |n| \to \infty.$$

In fact, using the density of continuous functions in the space of Lebesgue integrable functions on the circle, $L^1(\mathbb{T})$, one can prove the following version of the Riemann-Lebesgue Lemma.

Lemma 2.14. *If $f \in L^1(\mathbb{T})$, then*

$$\hat{f}(n) \to 0 \quad \textit{as } |n| \to \infty.$$

The following result of Du Bois-Reymond shows that continuity of f does not guarantee pointwise convergence of the Fourier series everywhere.

Theorem 2.15. *There exists a continuous function $f : \mathbb{T} \to \mathbb{C}$ such that*

$$\limsup_{N\to\infty} |S_N f(0)| = \infty.$$

Actually, the problem of pointwise convergence is not limited only to continuous functions. In 1926, Kolmogorov showed the following

Theorem 2.16. *There exist a Lebesgue-integrable function $f : \mathbb{T} \to \mathbb{C}$ such that*

$$\limsup_{N\to\infty} |S_N f(\theta)| = \infty \quad \textit{for all } \theta \in \mathbb{T}.$$

It was believed that sooner or later a continuous function with a similar property would be constructed. However, in 1966 Carleson proved the following:

Theorem 2.17. *If $f : \mathbb{T} \to \mathbb{C}$, $f \in L^2(\mathbb{T})$ then $S_N f(\theta)$ converges to $f(\theta)$ except possibly on a set of measure zero.*

As a corollary, we obtain the same result if f is continuous, or simply Riemann-integrable on $\mathbb{T}$.

2.4 Convolutions and their Properties

Definition 2.18. Let $f, g : \mathbb{T} \to \mathbb{C}$ be 2π periodic and integrable functions. Then their convolution $f * g$ on $\mathbb{T}$ is defined by

$$f * g(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)g(\theta - y)dy.$$

The following theorem shows some of the important properties of convolutions.

Theorem 2.19. *Let f , g , h : $\mathbb{T} \to \mathbb{C}$ be 2π periodic and integrable functions. Then*

(i) $f * g = g * f$

(ii) $(f * g) * h = f * (g * h)$

(ii) $f * (g + h) = (f * g) + (f * h)$

(iv) $(cf) * g = c(f * g) = f * (cg)$ *for all* $c \in \mathbb{C}$

(v) $\widehat{f * g}(n) = \hat{f}(n)\hat{g}(n)$

(vi) $f * g$ *is a continuous function*

In property (v), we can take the Fourier transform of $f*g$ since by Young's inequality we have

$$||f * g||_{L^1(\mathbb{T})} \leq ||f||_{L^1(\mathbb{T})}||g||_{L^1(\mathbb{T})},$$

which implies that $f * g$ is integrable.

2.5 The Dirichlet Kernel and its Properties

The Dirichlet kernel is a trigonometric polynomial of degree N defined by

$$D_N(\theta) = \sum_{n=-N}^{N} e^{in\theta},$$

which can also be written as

$$D_N(\theta) = 1 + 2\sum_{n=1}^{N} \cos(n\theta), \tag{2.5.1}$$

using the formula $2\cos(n\theta) = e^{i\theta} + e^{-i\theta}$.

The Dirichlet kernel can be expressed with a closed form formula as

$$D_N(\theta) = \frac{\sin((2N+1)\theta/2)}{\sin(\theta/2)}. \tag{2.5.2}$$

Some properties of the Dirichlet kernel are summarized in the following theorem.

Theorem 2.20. *Let $f : \mathbb{T} \to \mathbb{C}$ be integrable and let $\theta \in \mathbb{T}$. Then*

(i) $S_N f(\theta) = (f * D_N)(\theta)$

(ii) D_N *is an even function.*

(ii) $\frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(\theta)d\theta = 1.$

The graphs of the Dirichlet kernels for several values of N are depicted in figure 2.5.1 below.

2.6 The Fejér Kernel and the Cesàro Sum

The Fejér kernel is defined in terms of the Dirichlet kernels as

$$F_N f(\theta) = \frac{D_0(\theta) + D_1(\theta) + \ldots + D_{N-1}(\theta)}{N}. \tag{2.6.1}$$

The Fejér kernel is a positive kernel as seen in figure 2.6.1 below.

We can rewrite the Fejér kernel as the trigonometric polynomial

$$F_N f(\theta) = \sum_{n=-N}^{N} (\frac{N-|n|}{N})e^{in\theta}.$$

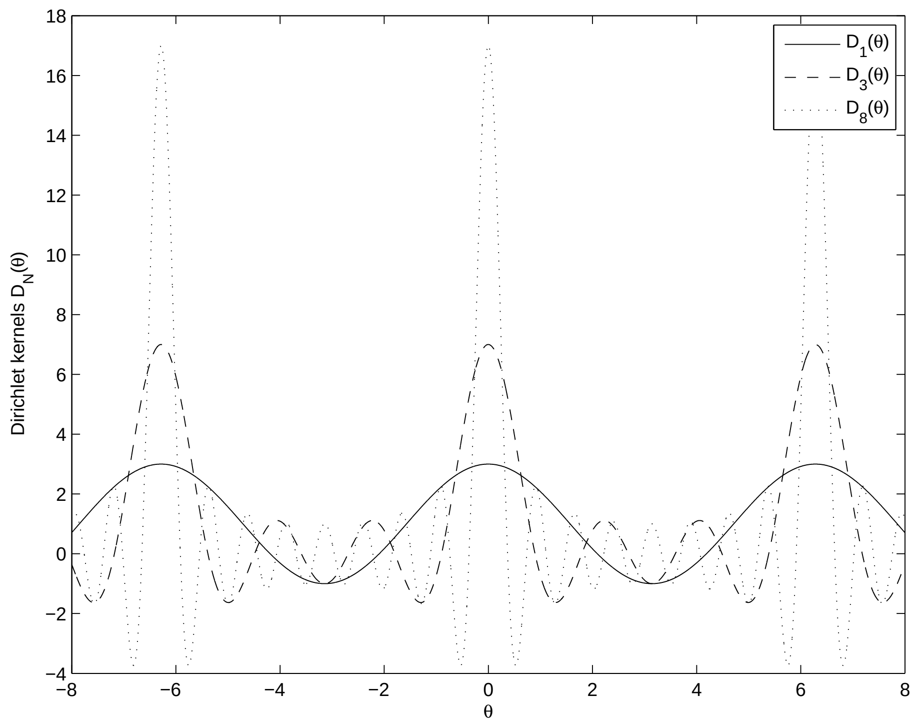

Figure 2.5.1: Graphs of the Dirichlet kernels

There is also the following closed form formula for the Fejér kernel

$$F_N f(\theta) = \frac{1}{N}\frac{\sin^2(N\theta/2)}{\sin^2(\theta/2)}.$$

The Cesàro sums, or Cesàro means, are defined by

$$\sigma_N f(\theta) = \frac{S_0 f(\theta) + S_1 f(\theta) + \ldots + S_{N-1} f(\theta)}{N}.$$

The Cesàro means can also be written as trigonometric polynomials by

$$\sigma_N f(\theta) = \sum_{k=-N}^{N} (\frac{N-|k|}{N}) a_k e^{in\theta},$$

where a_k is the kth Fourier coefficient of f. Now, Using $S_N f(\theta) = (f * D_N)(\theta)$ we obtain

$$\sigma_N f(\theta) = (f * F_N)(\theta).$$

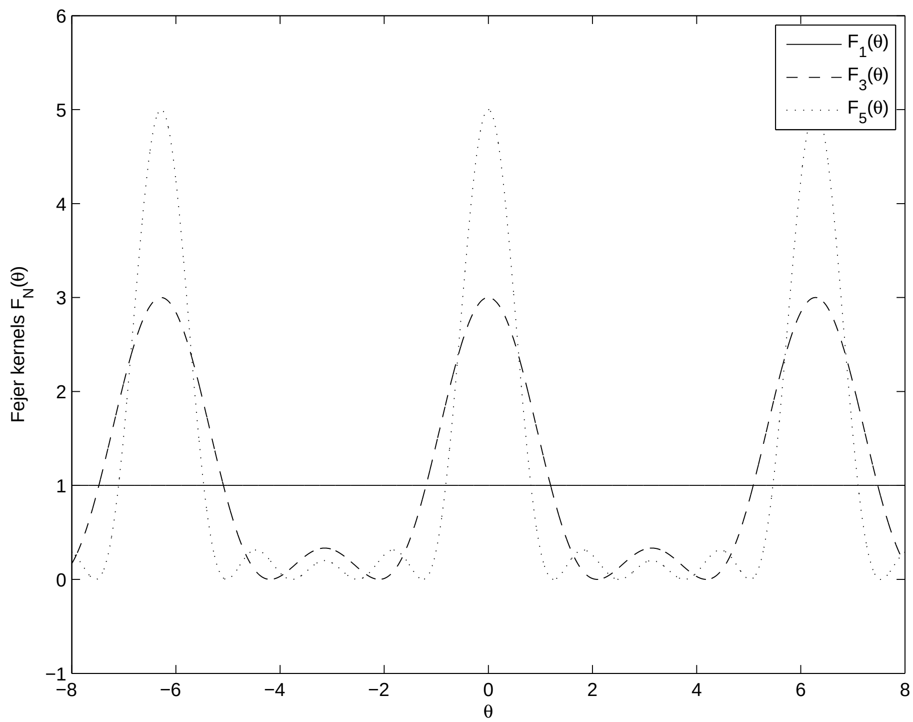

Figure 2.6.1: Graphs of the Fejér kernels

2.7 The Abel Means and the Poisson Kernel

The Poisson kernel $P_r(\theta)$ is defined by

$$P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta} = \frac{1 - r^2}{1 - 2r\cos\theta + r^2}, \qquad r \in [0, 1)$$

The rth Abel mean of the function $f(\theta)$ with Fourier series $\sum_{n=-\infty}^{\infty} a_n e^{in\theta}$ is defined by

$$A_r f(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} a_n e^{in\theta}.$$

Using convolution, the rth Abel mean can be written as

$$A_r f(\theta) = (f * P_r)(\theta).$$

Figure 2.7.1 shows the graphs of the Poisson kernels for several values of r.

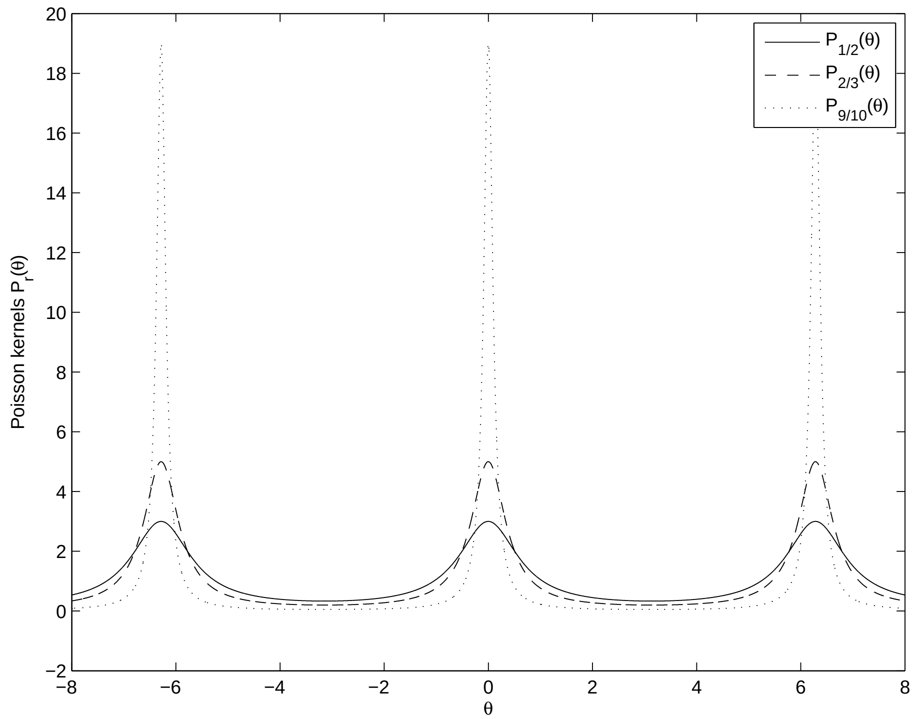

Figure 2.7.1: Graphs of the Poisson kernels

2.8 Good Kernels

Definition 2.21. Let $\{K_n(\theta)\}_{n=1}^{\infty}$ be a sequence of functions such that $K_n : \mathbb{T} \to \mathbb{R}$. Then we call this sequence, a sequence, or family, of *good kernels* if the following properties are satisfied:

(i) For all $n \in \mathbb{N}$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(\theta)d\theta = 1. \tag{2.8.1}$$

(ii) There exists $M > 0$ such that for all $n \in \mathbb{N}$,

$$\int_{-\pi}^{\pi} |K_n(\theta)| d\theta \leq M. \tag{2.8.2}$$

(iii) For every $\delta > 0$,

$$\int_{\delta \leq |\theta| \leq \pi} |K_n(\theta)| d\theta \to 0 \quad \text{as} \quad n \to \infty. \tag{2.8.3}$$

The Dirichlet kernel D_n fails to satisfy the second property. In fact,

$$\int_{-\pi}^{\pi} |D_n(\theta)| d\theta \geq c \log N, \quad c > 0.$$

Thus, D_n is not a good kernel. On the other hand, the Fejér kernel F_n and the Poisson kernel P_r are good kernels.

A family of good kernels is also called an approximation to the identity due to the following theorem.

Theorem 2.22. *If $\{K_n\}_{n=1}^{\infty}$ is a family of good kernels, $K_n : \mathbb{T} \to \mathbb{R}$, and $f : \mathbb{T} \to \mathbb{C}$ is integrable then*

$$\lim_{n \to \infty} (f * K_n)(\theta) = f(\theta)$$

whenever f is continuous at θ. Moreover, if f is continuous everywhere then $f * K_n \to f$ uniformly.

As a consequence of Theorem 2.22 we have the following theorem of Fejér about the Cesàro summability of a function f.

Theorem 2.23. *Let $f : \mathbb{T} \to \mathbb{C}$ be an integrable function. Then,*

$$\lim_{N \to \infty} \sigma_N f(\theta) = f(\theta)$$

at all points of continuity of f. Furthermore, the convergence is uniform if f is continuous on $\mathbb{T}$.

A similar result holds if we replace the Cesàro means with the Abel means.

Definition 2.24. We call a family of good kernels $\{K_n\}_{n=1}^{\infty}$ *positive kernels* if we replace the conditions (ii), and (iii) in Definition 2.21 by the following:
(ii′) For all $n \in \mathbb{N}$,

$$K_n(\theta) \geq 0. \tag{2.8.4}$$

(iii′) For every $\delta > 0$,

$$\max_{\delta \leq |\theta| \leq \pi} |K_n(\theta)| \to 0. \tag{2.8.5}$$

As implied in the definition, positive kernels are good kernels since conditions (i) and (ii′) imply condition (ii) and condition (iii′) implies condition (iii). The Fejér kernel F_n and the Poisson kernel P_r are examples of positive kernels.

2.9 Retrieving a Discontinuous Function Using Cesàro Means

In this section we examine a function f with a jump discontinuity at 0, comparing the behavior of its partial Fourier sums with that of its partial Cesàro sums near the origin.

Define the *step* function f by

$$f(\theta) = \begin{cases} -1, & -\pi < \theta < 0; \\ 1, & 0 \leq \theta < \pi. \end{cases} \tag{2.9.1}$$

Figure 2.9.1 shows the behavior of $S_N f(x)$ and $\sigma_N f(x)$ on the interval $[-1, 1]$. Notice the oscillations of $S_N f(x)$ near the discontinuity at $x = 0$. These oscillations,

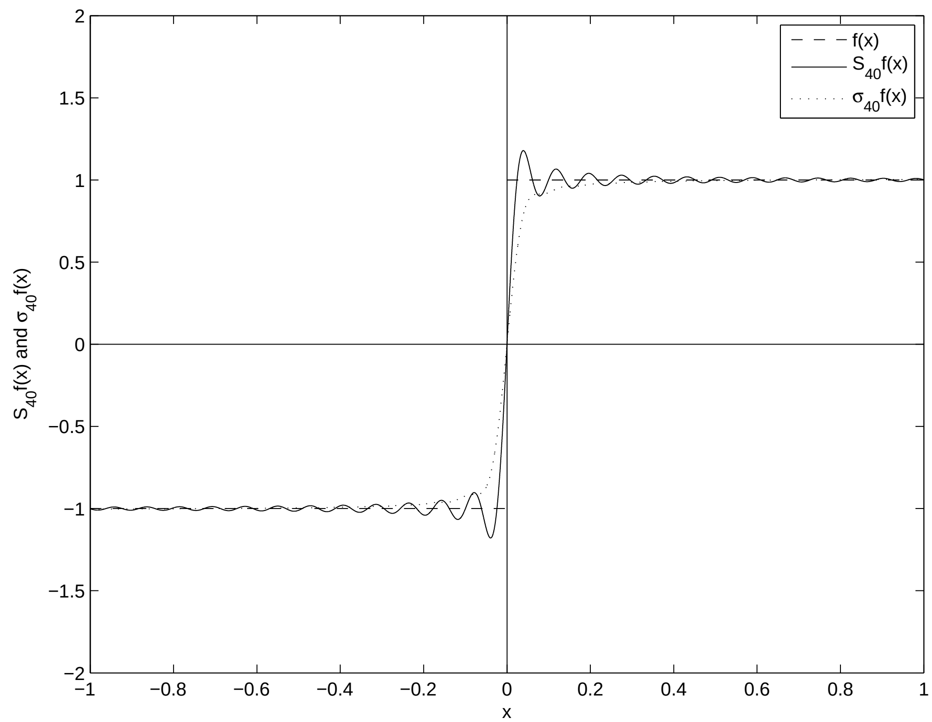

Figure 2.9.1: Graphs of the step function and its 40th partial Fourier and Cesàro sums

which persist no matter how large N is, are the manifestation of the Gibbs phenomenon. We will see an analysis of this phenomenon in Chapter 2. Also, notice the absence of these oscillations near 0 on the graph of $\sigma_N f(x)$. In fact, near the origin we see that

$$\min f(x) \leq \sigma_N f(x) \leq \max f(x).$$

As we will see in Chapter 4, this is no mere coincidence.

Remark 2.25. We can see from figure 2.5.1 that

$$\lim_{N\to\infty} S_N f(x) = \frac{f(0^+) + f(0^-)}{2} = 0, \tag{2.9.2}$$

$$\lim_{N\to\infty} \sigma_N f(x) = \frac{f(0^+) + f(0^-)}{2} = 0. \tag{2.9.3}$$

The limit in (2.9.2) is a consequence of the following theorem of Dirichlet.

Theorem 2.26. *Let $f : \mathbb{T} \to \mathbb{C}$ be continuous except at finitely many points with a continuous bounded derivative. Then*

$$\lim_{N\to\infty} S_N f(\theta) = \frac{f(\theta^+) + f(\theta^-)}{2} \quad \textit{for all } \theta \in \mathbb{T},$$

where,

$$f(\theta^+) = \lim_{\substack{h\to 0\\ h>0}} f(\theta + h) \quad \textit{and} \quad f(\theta^-) = \lim_{\substack{h\to 0\\ h>0}} f(\theta - h).$$

If f is continuous at θ then $S_N f(\theta) \to f(\theta)$ as $N \to \infty$.

The following theorem of Jordan shows that the same results follow if f is of bounded variation.

Theorem 2.27. *Suppose f is a function on $\mathbb{T}$ with bounded variation. Then*

$$\lim_{N\to\infty} S_N f(\theta) = \frac{f(\theta^+) + f(\theta^-)}{2} \quad \textit{for all } \theta \in \mathbb{T},$$

where $f(\theta^+)$ and $f(\theta^-)$ are as in Theorem 2.26. At all points of continuity $S_N f(\theta) \to f(\theta)$, and on closed intervals of continuity the convergence is uniform.

Results similar to Theorem 2.26 hold for the Abel and Cesàro means of f, thus explaining (2.9.3).

Chapter 3

A Historical Account

In this chapter we give a brief historical background to the Gibbs phenomenon introducing the main contributors. We also do a careful analysis of the calculation of the overshoot/undershoot of the Fourier partial sums near a discontinuity for a particular function.

3.1 Wilbraham, Michelson, and Gibbs

In 1898 American scientists Albert Michelson and Samuel Stratton built a mechanical machine, called a harmonic analyzer that was capable of computing partial Fourier sums up to 80 terms. See figure 3.1.1.

In the same year they published a description of their machine together with some graphs produced by the harmonic analyzer [13]. One of their graphs was that of the partial Fourier sum of the function, known as the sawtooth function, given by

$$f(x) = \frac{1}{2}x, \qquad -\pi < x < \pi, \qquad f(x + 2\pi) = f(x).$$

Figure 3.1.1: The harmonic analyzer built by Michelson and Stratton

Figures 3.1.2 and 3.1.3 show the graphs of this function and its partial Fourier sums.

Noticing the oscillations near the jump discontinuities, Michelson tried to fine tune his machine as he associated the appearance of these blips to mechanical defect. However, his efforts were fruitless and the oscillations persisted. Eventually, hand calculations confirmed the existence of them. Michelson's conclusion, which he asserted in a 1898 letter to *Nature* [12], was that the series

$$\sin x + \frac{1}{2}\sin 2x + \ldots + \frac{1}{n}\sin nx + \ldots \tag{3.1.1}$$

did not converge to $y = \frac{1}{2}x$ on the interval $-\pi < x < \pi$, contrary to what he believed the textbooks were claiming. A. E. H. Love, a British mathematician, responded to Michelson's letter by pointing out Michelson's lack of knowledge on the concept of uniform convergence [10]. However, he did not address Michelson's problem of

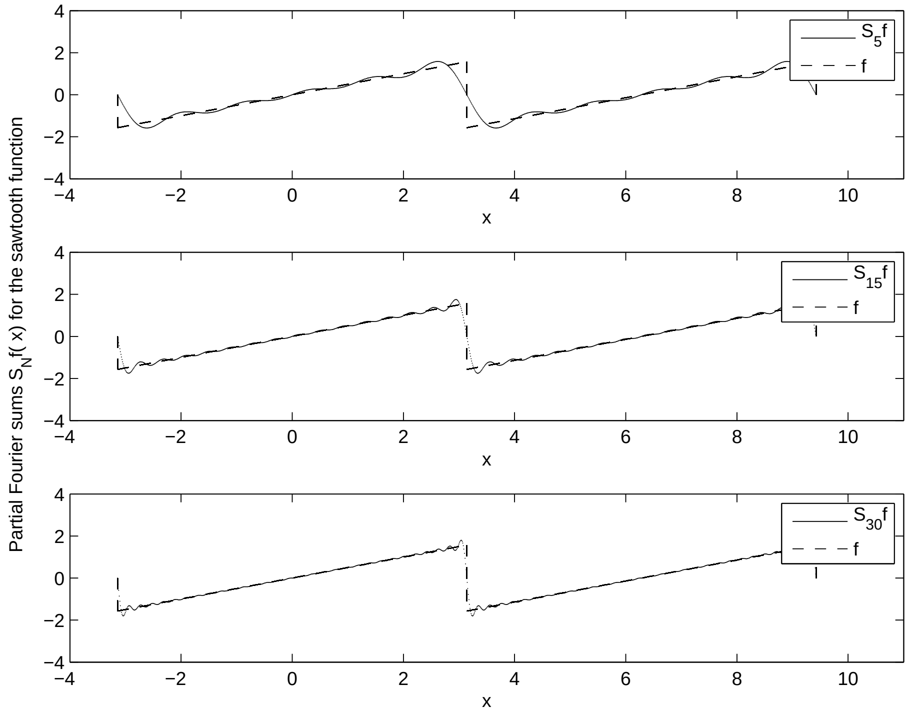

Figure 3.1.2: Graphs of the sawtooth function and its partial Fourier sums

retrieving a discontinuous function from its Fourier coefficients. He just suggested a book for reading to Michelson, and any other physicist troubled by nonuniform convergence [6]. Eventually, in two letters to *Nature* [4], [5] J. W. Gibbs explained that the limit of graphs is not necessarily the same as the graph of the limit. In other words, the series (3.1.1) converges pointwise to the function $f(x) = \frac{1}{2}x$, but this does not imply that the graphs of the partial sums $S_N f(x)$ need to start looking like the graph of f as $N \to \infty$ [9]. Furthermore, Gibbs gave the correct size of the overshoot and the undershoot of $S_N f(x)$ near the point of discontinuity. However, Gibbs did not provide any proofs or calculations and it was not until 1906 when Maxime Bôcher gave a rigorous analysis of this phenomenon [1]. He also named this phenomenon after Gibbs as he believed Gibbs to be the first person noticing it. However, 50 years before Gibbs' letter appeared in *Nature* Henry Wilbraham,

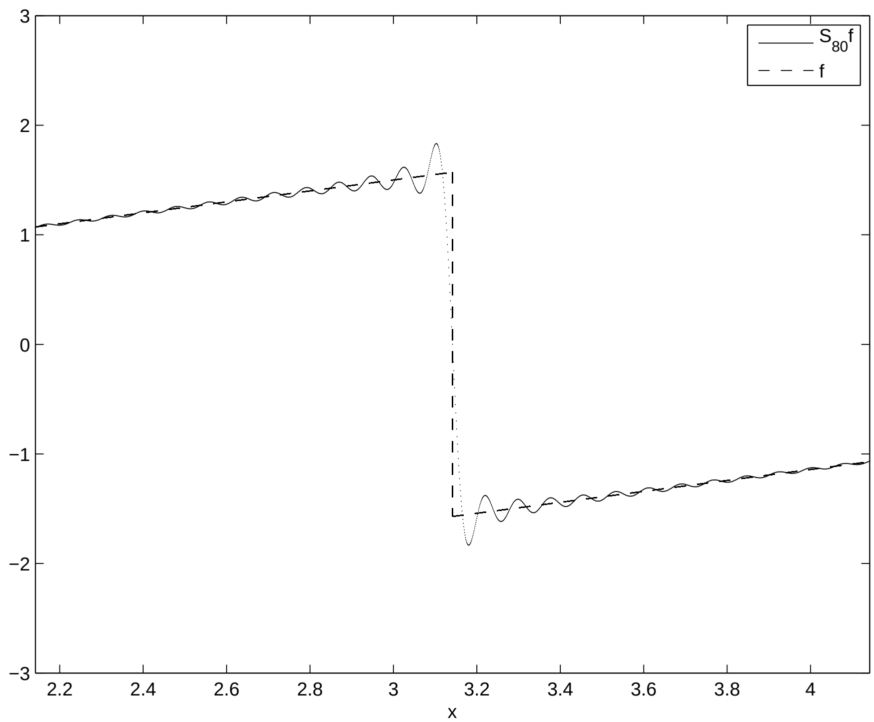

Figure 3.1.3: Graphs of the sawtooth function and its 80th partial Fourier sum

an English mathematician, had observed this phenomenon, but his paper [21] went unnoticed by Gibbs and Bôcher. He did not get recognition for his work until 1925 [2]. Wilbraham's paper is in response to F. Newman's claim [14] that the values of the partial Fourier sums of the function

$$f(x) = \begin{cases} \frac{\pi}{4} & \text{if} \quad |x| < \frac{\pi}{2}, \\ -\frac{\pi}{4} & \text{if} \quad \frac{\pi}{2} < |x| < \pi, \end{cases}$$

depicted in figure 3.1.4, are "contained between the limits $\pm\frac{1}{4}\pi$".

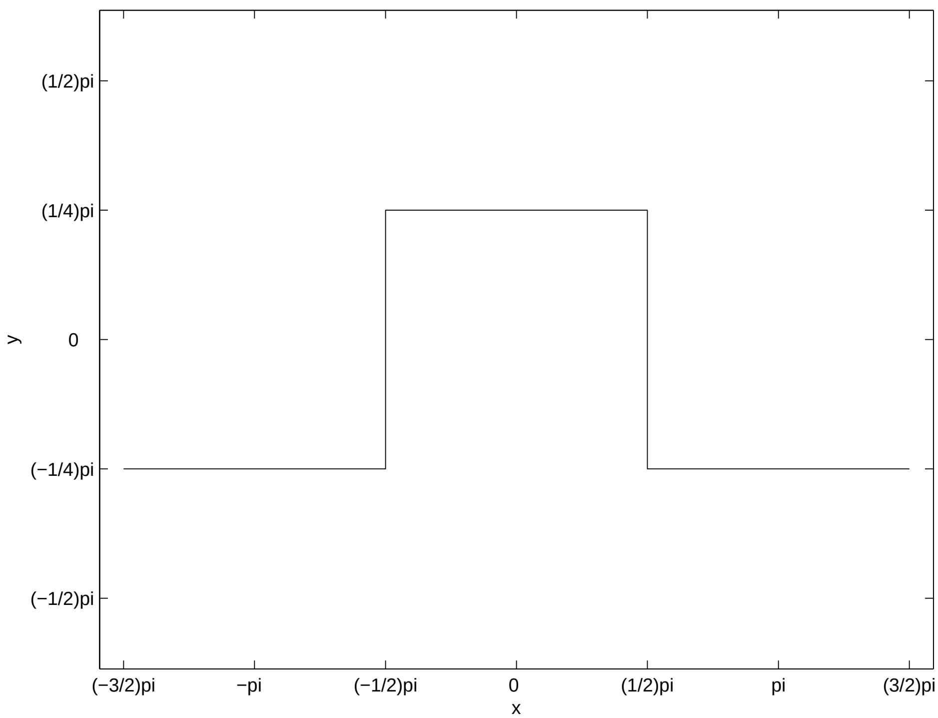

Figure 3.1.4: Graph of the square wave function

3.2 A Careful Analysis of the Square Wave Function

Here we do an analysis similar to that of Wilbraham's for a variation of the square wave function[7]. Let us first define the Gibbs phenomenon in a more precise way.

Definition 3.1. Suppose a function $f(x)$ has a jump discontinuity at $x = a$, i.e, $f(a^+) = \lim_{x\to a^+} f(x) < \infty$, $f(a^-) = \lim_{x\to a^-} f(x) < \infty$, and $f(a^+) \neq f(a^-)$. We say that the Fourier partial sum approximation of f exhibits the Gibbs phenomenon at the right hand side of $x = a$ if there is sequence $x_n > a$ converging to a such that $\lim_{n\to\infty} S_n f(x_n) > f(a^+)$ if $f(a^+) > f(a^-)$, or $\lim_{n\to\infty} S_n f(x_n) < f(a^+)$ if $f(a^+) < f(a^-)$. Similarly, we say that the Fourier partial sum approximation of f

exhibits the Gibbs phenomenon at the left hand side of $x = a$ if there is sequence $x_n < a$ converging to a such that $\lim_{n\to\infty} S_n f(x_n) < f(a^-)$ if $f(a^+) > f(a^-)$, or $\lim_{n\to\infty} S_n f(x_n) > f(a^-)$ if $f(a^+) < f(a^-)$.

As we see from this definition, it is not the mere existence of the overshoot/undershoot of the partial Fourier sums that constitutes the Gibbs phenomenon, but rather the persistence of this overshoot/undershoot, i.e the lack of improvement in the approximations $S_n f(x_n)$.

Recall that the step function f is defined by

$$f(x) = \begin{cases} -1, & -\pi < x < 0; \\ \ \ 1, & \ \ 0 \le x < \pi. \end{cases} \tag{3.2.1}$$

Then the nth Fourier coefficient of f can be calculated as follows:

$$\begin{aligned}
\hat{f}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{0} f(x)e^{-inx}dx + \frac{1}{2\pi}\int_{0}^{\pi} f(x)e^{-inx}dx \\
&= -\frac{1}{2\pi}\int_{-\pi}^{0} e^{-inx}dx + \frac{1}{2\pi}\int_{0}^{\pi} e^{-inx}dx \\
&= -\frac{1}{2\pi}\frac{e^{-inx}}{-in}\Big|_{-\pi}^{0} + \frac{1}{2\pi}\frac{e^{-inx}}{-in}\Big|_{0}^{\pi} \\
&= -\frac{1}{2\pi}\left(\frac{1}{-in} - \frac{e^{n\pi i}}{-in}\right) + \frac{1}{2\pi}\left(\frac{e^{-n\pi i}}{-in} - \frac{1}{-in}\right) \\
&= \frac{1}{2\pi}\left(\frac{1-(-1)^n}{in}\right) + \frac{1}{2\pi}\left(\frac{1-(-1)^n}{in}\right) \\
&= \frac{1}{n\pi i}\big(1-(-1)^n\big) \\
&= \begin{cases} \dfrac{2}{n\pi i} & \text{if } n \text{ odd,} \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

So, the Fourier series of f is:

$$\begin{aligned}
\sum_{n\,\mathrm{odd}} \hat{f}(n)e^{inx} &= \frac{2}{\pi}\sum_{n\,\mathrm{odd}} \frac{e^{inx}}{in} \\
&= \frac{2}{\pi}\sum_{n\,\mathrm{odd}} \frac{\cos(nx)+i\sin(nx)}{in} \\
&= \frac{2}{\pi}\sum_{n\,\mathrm{odd}} \frac{\sin(nx)-i\cos(nx)}{n} \\
&= \frac{2}{\pi}\sum_{\substack{n=1\\ n\,\mathrm{odd}}}^{\infty} \frac{\sin(nx)-i\cos(nx)}{n} + \frac{\sin(-nx)-i\cos(-nx)}{-n} \\
&= \frac{4}{\pi}\sum_{\substack{n=1\\ n\,\mathrm{odd}}}^{\infty} \frac{\sin(nx)}{n} \\
&= \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\sin((2n-1)x)}{2n-1}.
\end{aligned}$$

So the $(2N-1)$th partial Fourier sum of f is given by

$$S_{2N-1}f(x) = \frac{4}{\pi}\sum_{n=1}^{N} \frac{\sin((2n-1)x)}{2n-1}. \tag{3.2.2}$$

We can rewrite expression (3.2.2), using the fact that

$$\int_0^x \cos((2n-1)t)dt = \frac{\sin((2n-1)x)}{2n-1}$$

as follows

$$\begin{aligned}
S_{2N-1}f(x) &= \frac{4}{\pi}\sum_{n=1}^{N}\int_0^x \cos((2n-1)t)dt \\
&= \frac{4}{\pi}\int_0^x \left(\sum_{n=1}^{N}\cos((2n-1)t)\right)dt. \qquad (3.2.3)
\end{aligned}$$

Now, using the trigonometric identity $\sin u \cos v = \frac{1}{2}[\sin(u+v) + \sin(u-v)]$ we have:

$$\begin{aligned}
2\sin t \sum_{n=1}^{N} \cos((2n-1)t) &= \sum_{n=1}^{N} 2\sin t \cos((2n-1)t) \\
&= \sum_{n=1}^{N} (\sin(t + (2n-1)t) + \sin(t - (2n-1)t)) \\
&= \sum_{n=1}^{N} (\sin(2nt) + \sin((2-2n)t)) \\
&= \sum_{n=1}^{N} (\sin(2nt) - \sin((2n-2)t)) \\
&= (\sin(2t) - \sin(0)) + (\sin(4t) - \sin(2t)) + ... \\
&\quad + (\sin(2Nt) - \sin((2N-2)t)) \\
&= \sin(2Nt).
\end{aligned}$$

Therefore,

$$\sum_{n=1}^{N} \cos((2n-1)t) = \frac{\sin(2Nt)}{2\sin t}.$$

Plugging this into (3.2.3) we obtain:

$$\begin{aligned}
S_{2N-1}f(x) &= \frac{4}{\pi} \int_0^x \frac{\sin(2Nt)}{2\sin t} dt \\
&= \frac{2}{\pi} \int_0^x \frac{\sin(2Nt)}{\sin t} dt. \qquad (3.2.4)
\end{aligned}$$

Now, we find the first extremum of this function to the right of $x = 0$.

$$\frac{d}{dx} S_{2N-1}f(x) = \frac{2}{\pi} \frac{\sin((2Nx)}{\sin x} = 0.$$

Therefore,

$$\sin(2Nx) = 0 \quad \Rightarrow \quad x = \frac{k\pi}{2N},\ k \in \mathbb{Z}.$$

For $k = 1$, $x = \dfrac{\pi}{2N}$, and

$$\frac{d^2}{dx^2} S_{2N-1}f\left(\frac{\pi}{2N}\right) = -\frac{2N}{\sin(\frac{\pi}{2N})} < 0.$$

So, $S_{2N-1}f(x)$ has its first maximum to the right of $x = 0$ at $x = \dfrac{\pi}{2N}$ and

$$S_{2N-1}f\left(\frac{\pi}{2N}\right) = \frac{2}{\pi}\int_0^{\frac{\pi}{2N}} \frac{\sin(2Nt)}{\sin t}dt.$$

For N large, $0 < t < \dfrac{\pi}{2N}$ is small, hence $\sin t \sim t$, therefore we obtain:

$$\begin{aligned} S_{2N-1}f\left(\frac{\pi}{2N}\right) &\sim \frac{2}{\pi}\int_0^{\frac{\pi}{2N}} \frac{\sin(2Nt)}{t}dt \\ &= \frac{2}{\pi}\int_0^{\pi} \frac{\sin(u)}{u}du = \frac{2}{\pi}\mathrm{Si}(\pi), \end{aligned}$$

where $\mathrm{Si}(\pi) = \dfrac{\pi}{2}(1.17898)$. This value can be found using the Taylor expansion of $\sin x$ as follows:

We can write $\sin x$ as

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

Then

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ldots$$

Therefore

$$\begin{aligned} \frac{2}{\pi}\int_0^{\pi} \frac{\sin x}{x}dx &= \frac{2}{\pi}\int_0^{\pi} \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!}\right) + \ldots \\ &= \frac{2}{\pi}\left(\pi - \frac{\pi^3}{18} + \frac{\pi^5}{600} - \frac{\pi^7}{35280}\right) + \ldots \\ &= 2 - \frac{\pi^2}{9} + \frac{\pi^4}{300} - \frac{\pi^6}{17640} + \ldots \\ &= 2 - 1.11 + 0.33 - 0.04 + \ldots \\ &\approx 1.18 \end{aligned} \tag{3.2.5}$$

Thus

$$S_{2N-1}f\left(\frac{\pi}{2N}\right) \sim \frac{2}{\pi}\frac{\pi}{2}(1.17898) = 1.17898.$$

The magnitude of the jump at $x = 0$ is $|f(0^+) - f(0^-)| = 2$. The overshoot at $x = \dfrac{\pi}{2N}$, for N large, is 0.17898 which is approximately 9% of the magnitude of the jump.

Because of Wilbraham's contributions, which predate those of Gibbs', some authors prefer to use the term Gibbs-Wilbraham phenomenon [2].

In the next chapter, we will see that the Gibbs phenomenon occurs not just for the step function but for a class of functions with jump discontinuities at general points.

Chapter 4

The Gibbs Phenomenon in Fourier Series

In this chapter we show the existence of the Gibbs phenomenon at $x = 0$ for another simple function [20]. Then we generalize this result for a certain class of functions with jump discontinuities at the origin [19], [20]. We then illustrate the case of a jump discontinuity at a general point. In the last section, we discuss some summation methods used in removing the Gibbs effect [20], [22].

4.1 A Simple Function with a Jump Discontinuity at 0

Consider the *ramp* function

$$f(x) = \frac{\pi}{2}\operatorname{sgn}(x) - \frac{x}{2}, \qquad -\pi < x < \pi, \tag{4.1.1}$$

where

$$\operatorname{sgn}(x) = \begin{cases} -1 & \text{if} \quad x < 0; \\ 1 & \text{if} \quad 0 \leq x. \end{cases}$$

As seen in the figure 4.1.1, f has a jump discontinuity at $x = 0$. We will examine

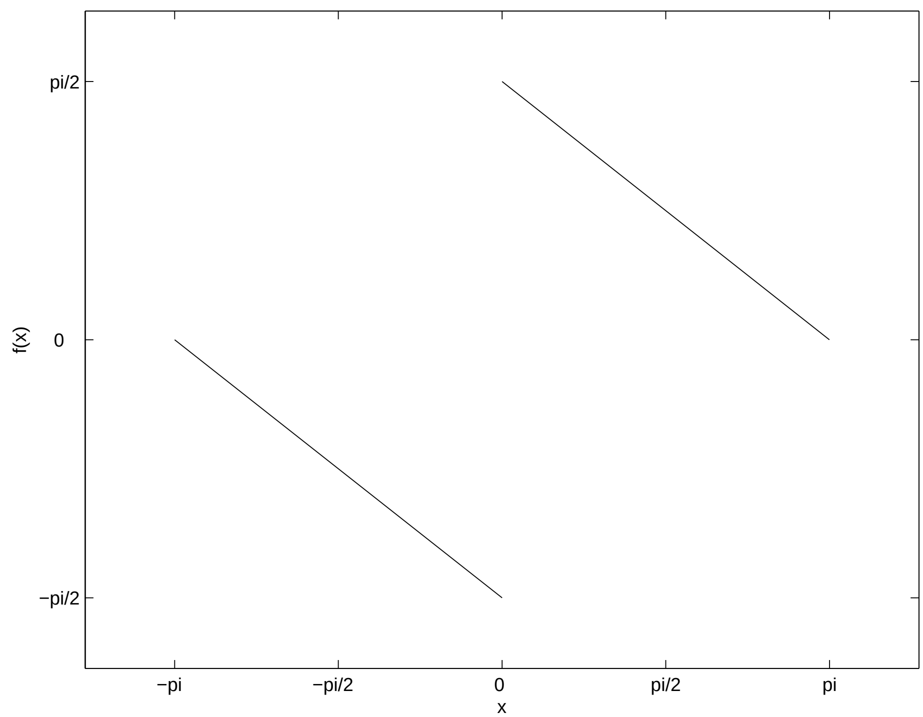

Figure 4.1.1: Graph of the ramp function

$S_N f(x)$ when $x \to 0^+$. First, we calculate the nth Fourier coefficient of f for $n \neq 0$ as follows

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx.$$

We break this integral into two parts to obtain

$$
\begin{aligned}
\hat{f}(n) &= \frac{1}{2\pi}\int_{-\pi}^{0} f(x)e^{-inx}dx + \frac{1}{2\pi}\int_{0}^{\pi} f(x)e^{-inx}dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{0} (-\frac{\pi}{2} - \frac{x}{2})e^{-inx}dx + \frac{1}{2\pi}\int_{0}^{\pi} (\frac{\pi}{2} - \frac{x}{2})e^{-inx}dx \\
&= -\frac{1}{4}\int_{-\pi}^{0} e^{-inx}dx - \frac{1}{2\pi}\int_{-\pi}^{0} \frac{x}{2}e^{-inx}dx + \frac{1}{4}\int_{0}^{\pi} e^{-inx}dx - \frac{1}{2\pi}\int_{0}^{\pi} \frac{x}{2}e^{-inx}dx \\
&= I_1 + I_2 + I_3 + I_4.
\end{aligned}
$$

From the first and the third integrals, using a change of variables on the first integral, we obtain

$$
\begin{aligned}
I_1 + I_3 &= -\frac{1}{4}\int_{0}^{\pi} e^{inx}dx + \frac{1}{4}\int_{0}^{\pi} e^{-inx}dx \\
&= -\frac{i}{2}\int_{0}^{\pi} \frac{e^{inx} - e^{-inx}}{2i}dx \\
&= -\frac{i}{2}\int_{0}^{\pi} \sin(nx)dx \\
&= \frac{i\cos(nx)}{2n}\bigg|_{0}^{\pi} \\
&= \frac{i\cos(n\pi)}{2n} - \frac{i}{2n}.
\end{aligned}
$$

From the second and the fourth integrals, using a change of variables on the second integral, we obtain

$$
\begin{aligned}
I_2 + I_4 &= \frac{1}{2\pi}\int_{0}^{\pi} \frac{x}{2}e^{inx}dx - \frac{1}{2\pi}\int_{0}^{\pi} \frac{x}{2}e^{-inx}dx \\
&= \frac{i}{2\pi}\int_{0}^{\pi} x\left(\frac{e^{inx} - e^{-inx}}{2i}\right)dx \\
&= \frac{i}{2\pi}\int_{0}^{\pi} x\sin(nx)dx.
\end{aligned}
$$

Integration by parts will yield

$$\begin{aligned}
I_2 + I_4 &= \frac{i}{2\pi}\left(- x\frac{\cos(nx)}{n}\bigg|_0^\pi + \int_0^\pi \frac{\cos(nx)}{n} dx \right) \\
&= \frac{i}{2\pi}\left(- \pi\frac{\cos(n\pi)}{n} + \frac{\sin(nx)}{n^2}\bigg|_0^\pi \right) \\
&= \frac{i}{2\pi}\left(- \pi\frac{\cos(n\pi)}{n} + 0 \right) \\
&= \frac{i\cos(n\pi)}{2n}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\hat{f}(n) &= I_1 + I_2 + I_3 + I_4 \\
&= \frac{i\cos(n\pi)}{2n} - \frac{i}{2n} - \frac{i\cos(n\pi)}{2n} \\
&= -\frac{i}{2n}, \quad \text{for } n \neq 0.
\end{aligned} \tag{4.1.2}$$

When $n = 0$ we have

$$\hat{f}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = 0.$$

So the Fourier series of f is given by:

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{inx} &= \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} -\frac{i}{2n}(\cos(nx) + i\sin(nx)) \\
&= \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{\sin(nx) - i\cos(nx)}{2n}
\end{aligned}$$

Now, we look at the sums over positive and negative integers n separately as follows

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{inx} &= \sum_{n>0} \frac{\sin(nx)}{2n} + \sum_{n<0} \frac{\sin(nx)}{2n} - \sum_{n>0} \frac{i\cos(nx)}{2n} - \sum_{n<0} \frac{i\cos(nx)}{2n} \\
&= \sum_{n>0} \frac{\sin(nx)}{2n} + \sum_{\substack{-m \\ m>0}} \frac{\sin(-mx)}{-2m} - \sum_{n>0} \frac{i\cos(nx)}{2n} - \sum_{\substack{-m \\ m>0}} \frac{i\cos(-mx)}{-2m} \\
&= \sum_{n>0} \frac{\sin(nx)}{2n} + \sum_{n>0} \frac{\sin(nx)}{2n} - \sum_{n>0} \frac{i\cos(nx)}{2n} + \sum_{n>0} \frac{i\cos(nx)}{2n} \\
&= \sum_{n=1}^{\infty} \frac{\sin(nx)}{n} \\
&= \sum_{n=1}^{\infty} \int_0^x \cos(nt)dt \\
&= \lim_{n\to\infty} \int_0^x \left(\frac{1}{2} + \sum_{k=1}^{n} \cos(kt)\right) dt - \frac{x}{2} \\
&= \lim_{n\to\infty} \left(\frac{1}{2}\int_0^x D_n(t)dt\right) - \frac{x}{2},
\end{aligned}$$

where D_n is the Dirichlet kernel given by (2.5.1). Using the closed form formula for D_n given in (2.5.2), and the identity $\sin(u+v) = \sin u \cos v + \cos u \sin v$, we can rewrite the integral in (2.8.3) as follows:

$$\begin{aligned}
\frac{1}{2}\int_0^x D_n(t)dt &= \int_0^x \left(\frac{\sin(nt)\cos(\frac{t}{2})}{2\sin(\frac{t}{2})} + \frac{\cos(nt)}{2}\right) dt \\
&= \int_0^x \frac{\sin(nt)}{t} dt + \int_0^x \sin(nt)\left(\frac{\cos(\frac{t}{2})}{2\sin(\frac{t}{2})} - \frac{1}{t}\right) dt + \int_0^x \frac{\cos(nt)}{2} dt.
\end{aligned}$$

Thus, we can write $S_n f(x)$ as

$$\begin{aligned}
S_n f(x) &= \int_0^x \frac{\sin(nt)}{t} dt + \int_0^x \sin(nt)\left(\frac{\cos(\frac{t}{2})}{2\sin(\frac{t}{2})} - \frac{1}{t}\right) dt + \int_0^x \frac{\cos(nt)}{2} dt - \frac{x}{2} \\
&= I_1 + I_2 + I_3 - \frac{x}{2}.
\end{aligned}$$

We Define the sequence $\{x_n\}_{n=1}^{\infty}$ by $x_n = \dfrac{\pi}{n}$ and examine $S_n f(x_n)$ as $n \to \infty$. With this choice $I_2(x_n)$ becomes

$$\begin{aligned} I_2(x_n) &= \int_0^{x_n} \sin(nt)\left(\frac{\cos(\frac{t}{2})}{2\sin(\frac{t}{2})} - \frac{1}{t}\right)dt \\ &= \int_0^{\pi} \left[\left(\frac{\cos(\frac{t}{2})}{2\sin(\frac{t}{2})} - \frac{1}{t}\right)\chi_{[0,x_n]}(t)\right]\sin(nt)dt \\ &= \int_0^{\pi} g_n(t)\sin(nt)dt, \end{aligned}$$

where the sequence of functions $\{g_n\}$ is given by,

$$g_n(t) = \left(\frac{\cos(\frac{t}{2})}{2\sin(\frac{t}{2})} - \frac{1}{t}\right)\chi_{[0,x_n]}.$$

Now, $g_n(t)$ is continuous on the interval $[-\pi, \pi]$ except the singularities at zero, and at x_n. However, near zero $\lim_{t\to 0} g_n(t) = 0$ by twice applying L'Hospital's Rule. Thus, we define the continuous extension at 0 of g_n by

$$G_n(t) = \begin{cases} g_n(t) & \text{if} \quad -\pi \le t \le \pi,\ t \neq 0; \\ 0 & \text{if} \quad t = 0. \end{cases}$$

Notice that $G_n(t)$ is still discontinuous at x_n. But when $n = 1$ G_1 is continuous on $[-\pi, \pi]$, hence it attains its maximum. Therefore, we have

$$\max_{t\in[0,\pi/2]} G_n(t) \le \max_{t\in[0,\pi]} G_1(t) =: M.$$

So we obtain

$$\begin{aligned} |I_2(x_n)| &\le \int_{-\pi}^{\pi} |G_n(t)||\sin(nt)|dt \\ &\le \max_{t\in[0,\pi]} G_1(t) \int_0^{x_n} dt \\ &= Mx_n \end{aligned}$$

Thus $I_2(x_n)$ converges to zero as $n \to \infty$ since $x_n \to 0$. Similarly,

$$\begin{aligned} I_3(x_n) &= \int_0^{x_n} \frac{\cos(nt)}{2} dt \\ &= \int_{-\pi}^{\pi} \frac{1}{2} \chi_{[0,x_n]}(t) \cos(nt) dt \\ &= \int_{-\pi}^{\pi} h_n(t) \cos(nt) dt, \end{aligned}$$

where $h_n(t) = \frac{1}{2}\chi_{[0,x_n]}(t)$. Now, as $n \to \infty$ $I_3(x_n) \to 0$ since $\cos(nt)$ is bounded and the support of $h_n(t)$ shrinks to zero. Therefore, near the origin we have

$$\begin{aligned} \lim_{n\to\infty} S_n f(x_n) &= \left(\lim_{n\to\infty} \int_0^{x_n} \frac{\sin(nt)}{t} dt - \frac{x_n}{2} \right) + \mathcal{O}(1) \\ &= \left(\lim_{n\to\infty} \int_0^{x_n} \frac{\sin(nt)}{t} dt \right) + \mathcal{O}(1). \end{aligned}$$

By the change of variable $u = nt$, we obtain

$$\lim_{n\to\infty} S_n f(x_n) = \lim_{n\to\infty} \int_0^{n\pi/n} \frac{\sin(u)}{u/n} du/n = \int_0^{\pi} \frac{\sin(u)}{u} du = \mathrm{Si}(\pi), \tag{4.1.3}$$

where, as calculated in (3.2.5),

$$\mathrm{Si}(\pi) = \int_0^{\pi} \frac{\sin u}{u} = \frac{\pi}{2}(1.17898).$$

However, for fixed $x > 0$, we have

$$\int_0^{nx} \frac{\sin u}{u} du \quad \overrightarrow{n \to \infty} \quad \int_0^{\infty} \frac{\sin u}{u} du = \frac{\pi}{2}.$$

As seen in (4.1.3), the value of $S_n f(x_n)$ is independent of n near 0 and it will always be greater than that of $f(x)$ no matter how many terms we use in the Fourier partial sum. Taking $x_n = -\dfrac{\pi}{n}$ and a similar argument will show the undershoot of the values of the Fourier partial sums on the left hand side of $x = 0$.

4.2 Generalization to other Functions with a Jump Discontinuity at 0

In this section, we use the result of the previous section to show the existence of the Gibbs phenomenon for a general function with a discontinuity at the origin. We will use the following definitions and theorems [19], [20].

Definition 4.1. Let $f(x)$ be a 2π periodic function whose restriction to $(-\pi, \pi)$ is in $\mathbf{L}^1(-\pi, \pi)$. Then f satisfies a Lipschitz condition of order $\alpha > 0$ at $x_\circ$ if there exists a constant C such that $|f(x) - f(x_\circ)| \leq C|x - x_\circ|^\alpha$ in a neighborhood of $x_\circ$. If the condition holds $\forall\ x_\circ$ with the same constant C then f is said to satisfy a uniform Lipschitz condition.

In case $\alpha > 1$ the condition will imply that $f'(x) = 0$ which means that f is constant. Therefore, we assume that $\alpha \leq 1$.

Definition 4.2. We define $f(x_\circ^+)$ and $f(x_\circ^-)$ as follows

$$f(x_\circ^+) = \lim_{x \to x_\circ^+} f(x),$$
$$f(x_\circ^-) = \lim_{x \to x_\circ^-} f(x).$$

Definition 4.3. If $f(x_\circ^+)$ (respectively $f(x_\circ^-)$) exists, and the condition in definition 4.1 holds for $x > x_\circ$, (respectively $x < x_\circ$), then f is said to satisfy a right (respectively left) hand Lipschitz condition.

First, we need the following version of the Riemann-Lebesgue lemma. For a proof, see [20], page 74.

Lemma 4.4. *If $f \in \mathrm{L}^1(-\pi, \pi)$ then $|a_n| + |b_n| \to 0$ as $n \to \infty$, where a_n and b_n are the Fourier coefficients of f as defined in* (2.1.1) *and* (2.1.2).

We also need the following uniform version of the Riemann-Lebesgue lemma.

Lemma 4.5. *Suppose* $f \in \mathrm{L}^1[-\pi, \pi]$ *is a periodic function and* $h \in \mathrm{C}^1[\alpha, \beta]$ *such that* $[\alpha, \beta] \in [-\pi, \pi]$. *Then*

$$\int_{\alpha}^{\beta} f(x-u)h(u)\sin(\lambda u)du \to 0 \quad as \quad \lambda \to \infty$$

uniformly in x.

Proof. By the density of $C^1[-\pi, \pi]$ in $L^1[-\pi, \pi]$ we can find $g \in C^1[-\pi, \pi]$ that is close to f in the L^1-norm. In other words,

$$||f - g||_{L^1} = \int_{-\pi}^{\pi} |f(x) - g(x)|dx < \varepsilon.$$

Now, we define I to be the following integral

$$I = \int_{\alpha}^{\beta} g(x-u)h(u)\sin(\lambda u)du.$$

Using integration by parts we obtain

$$I = -g(x-u)h(u)\frac{\cos(\lambda u)}{\lambda}\bigg|_{\alpha}^{\beta} + \int_{\alpha}^{\beta} \frac{d}{du}(g(x-u)h(u))\frac{\cos(\lambda u)}{\lambda}du.$$

now, $I \to 0$ uniformly as $\lambda \to \infty$ since $g(x-u)h(u)$ and $\dfrac{d}{du}(g(x-u)h(u))$ are uniformly bounded. Then,

$$\begin{aligned}
&\left|\int_{\alpha}^{\beta} f(x-u)h(u)\sin(\lambda u)du\right| \\
\leq &\left|\int_{\alpha}^{\beta} (f(x-u) - g(x-u))h(u)\sin(\lambda u)du\right| + |I| \\
\leq &\max_{\alpha \leq u \leq \beta} |h(u)| \int_{\alpha}^{\beta} |(f(x-u) - g(x-u))|du + |I| \\
\leq &\max_{\alpha \leq u \leq \beta} |h(u)|\varepsilon + |I|.
\end{aligned}$$

Since ε is arbitrary and I goes to 0 as $\lambda \to \infty$ the desired result follows. $\square$

Now, using the following theorem we show the existence of the Gibbs phenomenon for a "general" function with a jump discontinuity at 0.

Theorem 4.6. *Let f be a function satisfying a left and right hand Lipschitz condition at $x_\circ$. Then*

$$S_n f(x_\circ) \to \frac{f(x_\circ^+) + f(x_\circ^-)}{2}$$

as $n \to \infty$. Moreover, if f satisfies a uniform two sided Lipschitz condition in a neighborhood of $x_\circ$, then $S_n f \to f$ uniformly in a neighborhood of $x_\circ$ as $n \to \infty$.

Proof. First we assume that $x_\circ \neq \pm\pi$. From parts (i) and (ii) of Theorem 2.20 we know that,

$$S_n f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-u) D_n(u) du,$$

and

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} D_n(u) = 1$$

Using these facts we have

$$S_n f(x) - f(x) = \int_{-\pi}^{\pi} \big[f(x-u) - f(x)\big] D_n(u) du. \tag{4.2.1}$$

Let

$$\bar{f}(x_\circ) = \frac{f(x_\circ^+) + f(x_\circ^-)}{2}. \tag{4.2.2}$$

Then by (4.2.1) we have

$$S_n f(x_\circ) - \bar{f}(x_\circ) = \int_{-\pi}^{\pi} \big(f(x_\circ - u) - \bar{f}(x_\circ)\big) D_n(u) du.$$

Now, using (4.2.2) we obtain

$$\begin{aligned}
S_n f(x_\circ) - \bar{f}(x_\circ) &= \int_{-\pi}^{\pi} \left(f(x_\circ - u) - \frac{f(x_\circ^+)}{2} - \frac{f(x_\circ^-)}{2} \right) D_n(u)du \\
&= \int_0^{\pi} f(x_\circ - u) D_n(u)du - \int_0^{\pi} \frac{f(x_\circ^-)}{2} D_n(u)du \\
&\quad - \int_0^{\pi} \frac{f(x_\circ^+)}{2} D_n(u)du + \int_{-\pi}^{0} f(x_\circ - u) D_n(u)du \\
&\quad - \int_{-\pi}^{0} \frac{f(x_\circ^+)}{2} D_n(u)du - \int_{-\pi}^{0} \frac{f(x_\circ^-)}{2} D_n(u)du \\
&= \int_0^{\pi} \big(f(x_\circ - u) - f(x_\circ^-)\big) D_n(u)du + \int_0^{\pi} \frac{f(x_\circ^-)}{2} D_n(u)du \\
&\quad - \int_0^{\pi} \frac{f(x_\circ^+)}{2} D_n(u)du + \int_{-\pi}^{0} \big(f(x_\circ - u) - f(x_\circ^+)\big) D_n(u) \\
&\quad + \int_{-\pi}^{0} \frac{f(x_\circ^+)}{2} D_n(u)du - \int_{-\pi}^{0} \frac{f(x_\circ^-)}{2} D_n(u)du \\
&= I_1 + I_2 + I_3 + I_4 + I_5 + I_6.
\end{aligned}$$

Now, using the change of variables $y = -u$ and the fact that the Dirichlet kernel $D_n(u)$ is an even function we rewrite the fifth and the sixth integrals as follows

$$\begin{aligned}
I_5 + I_6 &= \int_{\pi}^{0} \frac{f(x_\circ^+)}{2} D_n(-y)(-dy) - \int_{\pi}^{0} \frac{f(x_\circ^-)}{2} D_n(-y)(-dy) \\
&= -\int_{\pi}^{0} \frac{f(x_\circ^+)}{2} D_n(y)(dy) + \int_{\pi}^{0} \frac{f(x_\circ^-)}{2} D_n(y)(dy) \\
&= \int_0^{\pi} \frac{f(x_\circ^+)}{2} D_n(y)(dy) - \int_0^{\pi} \frac{f(x_\circ^-)}{2} D_n(y)(dy) \\
&= -(I_2 + I_3).
\end{aligned}$$

Therefore, $I_2 + I_3 + I_5 + I_6 = 0$. Thus,

$$\begin{aligned}
S_n f(x_\circ) - \bar{f}(x_\circ) &= I_1 + I_4 \\
&= \int_{-\pi}^{0} \big(f(x_\circ - u) - f(x_\circ^+)\big) D_n(u)du \\
&\quad + \int_0^{\pi} \big(f(x_\circ - u) - f(x_\circ^-)\big) D_n(u)du.
\end{aligned}$$

Multiplying and dividing by u, we obtain

$$
\begin{aligned}
&= \int_{-\pi}^{-\delta} \left(\frac{f(x_\circ - u) - f(x_\circ^+)}{u} \right) u D_n(u) du \\
&\quad + \int_{-\delta}^{0} \left(\frac{f(x_\circ - u) - f(x_\circ^+)}{u} \right) u D_n(u) du \\
&\quad + \int_{0}^{\delta} \left(\frac{f(x_\circ - u) - f(x_\circ^-)}{u} \right) u D_n(u) du \\
&\quad + \int_{\delta}^{\pi} \left(\frac{f(x_\circ - u) - f(x_\circ^-)}{u} \right) u D_n(u) du \\
&= J_1 + J_2 + J_3 + J_4.
\end{aligned}
$$

We can write J_1 as

$$
\begin{aligned}
J_1 &= \int_{-\pi}^{-\delta} \big(f(x_\circ - u) - f(x_\circ^+) \big) \frac{\sin(n + \frac{1}{2})u}{\sin(\frac{u}{2})} du \\
&= \int_{-\pi}^{-\delta} \big(f(x_\circ - u) - f(x_\circ^+) \big) \frac{1}{\sin(\frac{u}{2})} \big(\sin(nu) \cos(\frac{u}{2}) + \cos(nu) \sin(\frac{u}{2}) \big) du \\
&= \int_{-\pi}^{-\delta} \left(f(x_\circ - u) - f(x_\circ^+) \frac{\cos(\frac{u}{2})}{\sin(\frac{u}{2})} \right) \sin(nu) du \\
&\quad + \int_{-\pi}^{-\delta} \big(f(x_\circ - u) - f(x_\circ^+) \big) \cos(nu) du \\
&= \int_{-\pi}^{\pi} \left(f(x_\circ - u) - f(x_\circ^+) \frac{\cos(\frac{u}{2})}{\sin(\frac{u}{2})} \chi_{[-\pi,-\delta]}(u) \right) \sin(nu) du \\
&\quad + \int_{-\pi}^{-\delta} \big(f(x_\circ - u) - f(x_\circ^+) \chi_{[-\pi,-\delta]}(u) \big) \cos(nu) du \\
&= \int_{-\pi}^{\pi} g(x_\circ, u) \sin(nu) du + \int_{-\pi}^{\pi} h(x_\circ, u) \cos(nu) du \\
&= A + B,
\end{aligned}
$$

where,

$$
\begin{aligned}
g(x_\circ, u) &:= \Big(f(x_\circ - u) - f(x_\circ^+) \Big) \frac{\cos(\frac{u}{2})}{\sin(\frac{u}{2})} \chi_{[-\pi,-\delta]}(u) \\
h(x_\circ, u) &:= \Big(f(x_\circ - u) - f(x_\circ^+) \Big) \chi_{[-\pi,-\delta]}(u).
\end{aligned}
$$

Since $f \in \mathrm{L}^1(-\pi, \pi)$ and $\sin(\frac{u}{2})$ is bounded away from zero in $[-\pi, -\delta]$ both g and h are in $\mathrm{L}^1(-\pi, \pi)$. Also, notice that the integrals A and B are the Fourier coefficients for g and h. Therefore, by Lemma 4.4, $A \to 0$ and $B \to 0$ as $n \to \infty$. Thus, $J_1 \to 0$ as $n \to \infty$. A similar argument shows that $J_4 \to 0$ as $n \to \infty$. As for the other two integrals, we argue as follows

$$\begin{aligned} |J_2| &= \left| \int_{-\delta}^{0} \left(\frac{f(x_\circ - u) - f(x_\circ^+)}{u} \right) \frac{u}{\sin(\frac{u}{2})} \sin\big((n + \frac{1}{2})u\big) du \right| \\ &\leq \int_{-\delta}^{0} \left| \left(\frac{f(x_\circ - u) - f(x_\circ^+)}{u} \right) \frac{u}{\sin(\frac{u}{2})} \sin\big((n + \frac{1}{2})u\big) \right| du. \end{aligned}$$

By the Lipschitz condition on $f(x)$ we have

$$|f(x) - f(x_\circ)| \leq C_1 |x - x_\circ|^\alpha, \quad \text{if } |x - x_\circ| < \delta.$$

Therefore,

$$|f(x_\circ - u) - f(x_\circ)| \leq C_1 |x_\circ - u - x_\circ|^\alpha = C_1 |u|^\alpha.$$

Also, $\left|\sin\big((n + \frac{1}{2})u\big)\right| \leq 1$, and

$$\lim_{u \to 0} \frac{\frac{u}{2}}{\sin(\frac{u}{2})} = 1,$$

So if $|u| < \delta$ then

$$\left| \frac{u}{\sin(\frac{u}{2})} \right| \leq C_3$$

Therefore,

$$|J_2| \leq C_3 \int_{-\delta}^{0} C_1 |u|^{\alpha - 1} du \leq C \int_{-\delta}^{0} |u|^{\alpha - 1} du.$$

Similarly,

$$\begin{aligned} |J_3| &= \left| \int_{0}^{\delta} \left(\frac{f(x_\circ - u) - f(x_\circ^-)}{u} \right) \frac{u}{\sin(\frac{u}{2})} \sin\big((n + \frac{1}{2})u\big) du \right| \\ &\leq C_3 \int_{0}^{\delta} C_2 |u|^{\alpha - 1} du \leq C \int_{-\delta}^{0} |u|^{\alpha - 1} du, \end{aligned}$$

where $C = \max\{C_3C_1, C_3C_2\}$. Thus,

$$\begin{aligned}
|J_2| + |J_3| &\leq C \int_{-\delta}^{0} |u|^{\alpha-1} du + C \int_{0}^{\delta} |u|^{\alpha-1} du \\
&\leq \frac{C}{2} \int_{-\delta}^{0} |u|^{\alpha-1} du + \frac{C}{2} \int_{0}^{\delta} |u|^{\alpha-1} du \\
&= \frac{C}{2} \int_{-\delta}^{\delta} |u|^{\alpha-1} du \\
&= 2\frac{C}{2} \int_{0}^{\delta} |u|^{\alpha-1} du \\
&= C \int_{0}^{\delta} |u|^{\alpha-1} du \\
&= C \frac{|u|^{\alpha}}{\alpha}\Big|_{0}^{\delta} \\
&= \frac{C}{\alpha}\delta^{\alpha}.
\end{aligned}$$

Now, given $\varepsilon > 0$, we choose δ and N such that $\frac{C}{\alpha}\delta^{\alpha} < \frac{\varepsilon}{2}$ and $|J_1| + |J_4| < \frac{\varepsilon}{2}$ for all $n \geq N$. Then, these choices imply that $|S_n f(x_\circ) - \bar{f}(x_\circ)| < \varepsilon$.

For the case $x_\circ = \pm\pi$, a translation by π and a similar argument shows that $S_n(x - \pi) \to \bar{f}(x - \pi)$ at $x = 0$.

The case of the two sided Lipschitz condition is proved by replacing $f(x_\circ^+)$ and $f(x_\circ^-)$ by $f(x_\circ)$ in the integrals J_1 and J_4. □

Lemma 4.5 and the following theorem will give us uniform convergence.

Theorem 4.7. *Suppose that f satisfies a uniform Lipschitz condition of order $0 < \alpha \leq 1$ in (a, b). Then $S_n f \to f$ uniformly in any interior subinterval $[c, d] \subset (a, b)$.*

Proof. Let $\delta < \min(c - a, b - d)$. By the proof of Theorem 4.6 we have

$$|J_2| + |J_3| \leq \frac{C}{\alpha}\delta^{\alpha}.$$

Now, we write J_1 as

$$\begin{aligned} J_1 &= \int_{-\pi}^{-\delta} f(x_\circ - u)\frac{1}{\sin(u/2)}\sin((n+\frac{1}{2})u)du \\ &- \int_{-\pi}^{-\delta} f(x_\circ^+)\frac{1}{\sin(u/2)}\sin((n+\frac{1}{2})u)du. \end{aligned}$$

Since $\frac{1}{\sin(u/2)} \in C^1[-\pi, -\delta]$ we can apply Lemma 4.5 with $\alpha = -\pi$, $\beta = -\delta$, and $\lambda = n + \frac{1}{2}$ to the first integral to obtain

$$\frac{1}{\pi}\int_{-\pi}^{-\delta} f(x_\circ - u)\frac{1}{\sin(u/2)}\sin(\lambda u)du \to 0 \quad \text{uniformly as } \lambda \to \infty.$$

The second integral also converges to 0 uniformly. Thus, $J_1 \to 0$ uniformly as $n \to \infty$. A similar argument works for J_4 and the required result follows. □

Now, using the above convergence results, to show that a "general" function g exhibits the Gibbs phenomenon at 0 it is enough to show the existence of the phenomenon for the function defined in (4.1.1), which we have already done. To elaborate, suppose $g(x)$ is piecewise smooth with a jump at 0 such that

$$g(0^+) = \lim_{x\to 0^+} g(x) \neq \pm\infty \quad \text{and} \quad g(0^-) = \lim_{x\to 0^-} g(x) \neq \pm\infty.$$

Then to remove the discontinuity at 0 we define a new function $h(x)$ as follows

$$h(x) = g(x) - \left(\frac{g(0^+) - g(0^-)}{\pi}\right) f(x),$$

where f is the ramp function defined by (4.1.1). Now, as $x \to 0^+$ we obtain

$$\begin{aligned} \lim_{x\to 0^+} h(x) &= \lim_{x\to 0^+} g(x) - \left(\frac{g(0^+) - g(0^-)}{\pi}\right) \lim_{x\to 0^+} f(x) \\ &= g(x_\circ^+) - \left(\frac{g(0^+) - g(0^-)}{\pi}\right)\frac{\pi}{2} \\ &= \frac{g(0^+) + g(0^-)}{2}, \end{aligned}$$

and as $x \to 0^-$ we obtain

$$\begin{aligned}\lim_{x\to 0^-} h(x) &= \lim_{x\to 0^-} g(x) - \left(\frac{g(0^+) - g(0^-)}{\pi}\right) \lim_{x\to 0^-} f(x) \\ &= g(0^-) - \left(\frac{g(0^+) - g(0^-)}{\pi}\right)\left(-\frac{\pi}{2}\right) \\ &= \frac{g(0^+) + g(0^-)}{2}.\end{aligned}$$

Now, define $h(0)$ as

$$h(0) := \frac{g(0^+) + g(0^-)}{2}.$$

Then $h(x)$ is continuous at 0 and it satisfies the hypothesis of Theorem 4.6. Therefore, $S_n h$ converges at 0. In fact, it converges uniformly in a neighborhood of 0 thus, if either of the functions f or g shows the Gibbs phenomenon at 0 then so does the other. Since we already showed that f exhibits the Gibbs phenomenon near 0, so does g.

4.3 Jump Discontinuity at a General Point

Suppose that the function g has a jump discontinuity at $x = x_\circ$ and is piecewise smooth everywhere else. Define $h(x)$ by

$$h(x) = \begin{cases} g(x) - \left(\dfrac{g(x_\circ^+) - g(x_\circ^-)}{\pi}\right) f(x - x_\circ) & \text{if} \quad x \neq x_\circ, \\ \dfrac{g(x_\circ^+) + g(x_\circ^-)}{2} & \text{if} \quad x = x_\circ. \end{cases}$$

Then,

$$\lim_{x\to x_\circ^+} h(x) = \lim_{x\to x_\circ^-} h(x) = h(x_\circ) = \frac{g(x_\circ^+) + g(x_\circ^-)}{2}.$$

Thus, h is continuous at $x = x_\circ$ and therefore by Theorem 4.6, $S_n h$ converges uniformly in a neighborhood of $x_\circ$. Hence the function $g(x)$ displays the Gibbs phenomenon at $x = x_\circ$ since $f(x - x_\circ)$ does so.

If $g(x)$ has a finite number of jump discontinuities at $x_1, x_2, \ldots, x_j$ and is piecewise smooth elsewhere we define $h(x)$ by

$$h(x) = \begin{cases} g(x) - \frac{1}{\pi}\sum_j \big(g(x_j^+) - g(x_j^-)\big) f(x - x_j) & \text{if} \quad x \neq x_j, \\ \dfrac{g(x_j^+) + g(x_j^-)}{2} & \text{if} \quad x = x_j. \end{cases}$$

Then by the same argument as above $g(x)$ exhibits the Gibbs phenomenon near $x_1, x_2, \ldots, x_j$.

4.4 Removing the Gibbs Effect Using Positive Kernels

Recall a family of positive kernels introduced in Definition 2.24. The following theorem shows that convolution with positive kernels eliminates the Gibbs effect [19], [22].

Theorem 4.8. *Let $\{K_n\}_{n=1}^{\infty}$ be a a family of positive kernels and $m \leq f(x) \leq M$ for $x \in (a, b)$. Then for any $\varepsilon > 0$ and $0 < \delta < \dfrac{b-a}{2}$ there is an N such that $\sigma_n f(x) = (K_n * f)(x)$ satisfies $m - \varepsilon \leq \sigma_n f(x) \leq M + \varepsilon$ for $x \in (a + \delta, b - \delta)$ and $n > N$.*

Proof. Using the definition of convolutions we have

$$\begin{aligned} \sigma_n f(x) &= (K_n * f)(x) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-\theta)K_n(\theta)d\theta \\ &= \frac{1}{2\pi}\int_{-\delta}^{\delta} f(x-\theta)K_n(\theta)d\theta + \frac{1}{2\pi}\int_{\delta \leq |\theta| \leq \pi} f(x-\theta)K_n(\theta)d\theta \\ &= G_1 + G_2. \end{aligned}$$

Now, we show that G_2 converges to 0.

$$\begin{aligned}|G_2| &= \left|\frac{1}{2\pi}\int_{\delta\le|\theta|\le\pi} f(x-\theta)K_n(\theta)d\theta\right|\\ &\le \frac{1}{2\pi}\int_{\delta\le|\theta|\le\pi} |f(x-\theta)K_n(\theta)|d\theta\\ &\le \frac{1}{2\pi}\max_{\delta\le|\theta|\le\pi}|K_n(\theta)|\int_{\delta\le|\theta|\le\pi}|f(x-\theta)|d\theta.\end{aligned}$$

Since K_n is a positive kernel by property (iii′) of positive kernels, expression (2.8.5), we have

$$\max_{\delta\le|\theta|\le\pi}|K_n(\theta)| \to 0 \quad \text{as } n\to\infty.$$

Therefore, for fixed $\delta > 0$ given $\varepsilon > 0$ there exists $N_1 > 0$ such that $|G_1| < \varepsilon$ for $n > N_1$. Now, $x \in (a+\delta, b-\delta)$ implies that $x-\theta \in (a,b)$ for $|\theta| < \delta$. Therefore, $f(x-\theta) \le M$ for $|\theta| < \delta$. Thus, G_2 becomes

$$\begin{aligned}G_2 &= \frac{1}{2\pi}\int_{-\delta}^{\delta} f(x-\theta)K_n(\theta)d\theta\\ &\le M\frac{1}{2\pi}\int_{-\delta}^{\delta}K_n(\theta)d\theta\\ &\le M\frac{1}{2\pi}\int_{-\pi}^{\pi}K_n(\theta)d\theta\\ &\le M \quad \text{by (2.8.1), property (i) of good kernels.}\end{aligned}$$

So $|G_1+G_2| \le |G_1| + |G_2| \le M+\varepsilon$. Hence, $\sigma_n f(x) \le M+\varepsilon$. On the other hand,

$$\begin{aligned}G_1+G_2 &= \frac{1}{2\pi}\int_{-\delta}^{\delta} f(x-\theta)K_n(\theta)d\theta + \frac{1}{2\pi}\int_{\delta\le|\theta|\le\pi} f(x-\theta)K_n(\theta)d\theta\\ &\ge m\frac{1}{2\pi}\int_{-\delta}^{\delta}K_n(\theta)d\theta - \max_{\delta\le|\theta|\le\pi}|K_n(\theta)|\cdot||f(x-\theta)||_1\\ &\ge m-\varepsilon \quad \text{for } n\ge N_2.\end{aligned}$$

Let $n = \max\{N_1, N_2\}$. Then for $n \ge N$ we have

$$m-\varepsilon \le \sigma_n f(x) \le M+\varepsilon$$

□

Now, it is clear why the Gibbs phenomenon is absent in Figure 2.9.1 where the Cesàro means are used to approximate the function defined by (2.9.1).

Chapter 5

The Gibbs Phenomenon in Wavelets

In this chapter we give a brief introduction to wavelets and multiresolution analysis [3]. We give a condition under which the wavelet expansion of a function shows the Gibbs phenomenon near a jump discontinuity [8], [17]. This condition involves what is called a reproducing kernel. Next, we show that the Haar wavelets do not exhibit the Gibbs phenomenon whereas the Shannon wavelets do [7]. We revisit the concept of a family of good kernels reincarnated as what are called positive delta sequences and prove that wavelet approximations of functions using kernels that are positive delta sequences eliminates the Gibbs phenomenon [19]. We also revisit that the case of the Haar wavelets and show that the corresponding reproducing kernel is a positive delta sequence hence the absence of the the Gibbs phenomenon in the Haar wavelet expansion of a function with a jump discontinuity [20].

5.1 Wavelets and Multiresolution Analysis

Definition 5.1. Let $\psi \in L^2(\mathbb{R})$. Define the functions $\psi_{j,k}$ by $\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)$. Then by a wavelet system we mean a complete orthonormal set in $L^2(\mathbb{R})$ of the form $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$. In other words, $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is a an orthonormal basis for $L^2(\mathbb{R})$. The functions $\psi_{j,k}$ are called wavelets and the function ψ is called the mother wavelet.

Remark 5.2. If $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is a wavelet system then every function f in $L^2(\mathbb{R})$ can be written as

$$f = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \langle f, \psi_{j,k}\rangle \psi_{j,k}.$$

Definition 5.3. A multiresolution analysis, or MRA for short, is a sequence $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R})$ such that

(i) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$,

(ii) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$,

(iii) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$,

(iv) For each $j \in \mathbb{Z}$, $f(x) \in V_0$ if and only if $f(2^j x) \in V_j$,

(v) For each $k \in \mathbb{Z}$, $f(x) \in V_0$ if and only if $f(x-k) \in V_0$,

(vi) There exists a function $\varphi \in V_0$, called a scaling function, or father wavelet, such that $\{\varphi(x-k)\}_{k\in\mathbb{Z}}$ is an orthonormal basis for V_0.

Remark 5.4. For each $j \in \mathbb{Z}$, $\{\varphi_{j,k}\}_{k\in\mathbb{Z}}$ is an orthonormal basis for V_j, where $\varphi_{j,k}$ is defined by

$$\varphi_{j,k}(x) = 2^{j/2}\varphi(2^j x - k).$$

However, the functions $\varphi_{j,k}$ are not necessarily orthogonal at different levels j.

As a consequence of Remark 5.4, for $f \in L^2(\mathbb{R})$ the orthogonal projection of f onto V_j is

$$P_j f = \sum_{k\in\mathbb{Z}} \langle f, \varphi_{j,k}\rangle \varphi_{j,k}. \tag{5.1.1}$$

Moreover, the function $P_j f$ is the best approximation to f in the subspace V_j. The approximations $P_j f$ improve as j increases. In fact,

$$||P_j f - f||_{L^2(\mathbb{R})} \to 0 \quad \text{as} \quad j \to \infty.$$

We call the subspaces $\{V_j\}_{j\in\mathbb{Z}}$ approximation subspaces.

Remark 5.5. Since $\varphi \in V_0 \subseteq V_1$, by Remark 5.4 we have

$$\varphi(x) = \sum_{k\in\mathbb{Z}} u_k \varphi_{1,k}(x) = \sum_{k\in\mathbb{Z}} u_k \sqrt{2}\varphi(2x-k), \tag{5.1.2}$$

where the sequence $\{u_k\}_{k\in\mathbb{Z}}$ is in $l^2(\mathbb{Z})$. This sequence is called the scaling sequence and equation (5.1.2) is called the scaling equation. A property of the scaling sequence is that its even integer translates, $\{\tau_{2k}u\}_{k\in\mathbb{Z}}$, form an orthonormal set in $l^2(\mathbb{Z})$.

The following theorem of Stéphane Mallat shows how to obtain a wavelet system once we have a MRA. You can find a proof of this theorem in [3] page 391.

Theorem 5.6. *Let $\{V_j\}_{j\in\mathbb{Z}}$ be a MRA with scaling function φ and scaling sequence $\{u_k\}_{k\in\mathbb{Z}} \in l^1(\mathbb{Z})$. Define $v \in l^1(\mathbb{Z})$ by $v_k = (-1)^{k-1}\overline{u_{1-k}}$ for all $k \in \mathbb{Z}$. Also, define ψ by the following equation*

$$\psi(x) = \sum_{k\in\mathbb{Z}} v_k \varphi_{1,k}(x) = \sum_{k\in\mathbb{Z}} v_k \sqrt{2}\varphi(2x-k).$$

Then $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is a wavelet system in $L^2(\mathbb{R})$.

Remark 5.7. If we define the space W_0 by

$$W_0 = \left\{ \sum_{k\in\mathbb{Z}} z_k \psi_{0,k} : \{z_k\}_{k\in\mathbb{Z}} \in l^2(\mathbb{Z}) \right\}$$

then $V_1 = V_0 \oplus W_0$. In other words, W_0 is the orthogonal complement of V_0 in V_1. This means that $V_0 \subseteq V_1$, $W_0 \subseteq V_1$, $V_0 \perp W_0$, and for all $f \in V_1$ there exist unique $g \in V_0, h \in W_0$ such that $f = g + h$. Similarly, we define W_j by

$$W_j = \left\{ \sum_{k \in \mathbb{Z}} z_k \psi_{j,k} : \{z_k\}_{k \in \mathbb{Z}} \in l^2(\mathbb{Z}) \right\}.$$

Then $V_{j+1} = V_j \oplus W_j$ for all $j \in \mathbb{Z}$.

It turns out that the subspaces $\{W_j\}_{j \in \mathbb{Z}}$ have the same dilation property as the the subspaces $\{V_j\}_{j \in \mathbb{Z}}$. In other words, $f(x) \in W_0$ if and only if $f(2^j x) \in W_j$ for all $j \in \mathbb{Z}$. Moreover, $\{\psi(x-k)\}_{k \in \mathbb{Z}}$ is an orthomormal basis for W_0, and in general, the functions $\{\psi_{j,k}\}_{k \in \mathbb{Z}}$ form an orthonormal basis for W_j. This implies that the orthogonal projection Q_j of $f \in L^2(\mathbb{R})$ onto the space W_j can be obtained by

$$Q_j f = \sum_{k \in \mathbb{Z}} \langle f, \psi_{j,k} \rangle \psi_{j,k}.$$

5.2 Construction of a Multiresolution Analysis

Theorem 5.6 of Mallat tells us how to construct the father wavelet ψ using a MRA and to obtain from it a wavelet system. However, it does not give a recipe to construct a MRA in the first place. The following two theorems show this construction. The proofs can be found in [3] pages 420, and 421.

Theorem 5.8. *Let $m_\circ : \mathbb{R} \to \mathbb{C}$ be a 2π-periodic function satisfying the following* *conditions:*

(i) $|m_\circ(\xi)|^2 + |m_\circ(\xi+\pi)|^2 = 1$ *for all* $\xi \in \mathbb{R}$,

(ii) $m_\circ(0) = 1$,

(iii) $m_\circ$ *satisfies a Lipschitz condition of order* $\delta > 0$ *at* 0, *i.e, there exist* $\delta > 0$ *and* $C > 0$ *such that* $|m_\circ(\xi) - m_\circ(0)| \leq C|\xi|^\delta$ *for all* $\xi \in \mathbb{R}$,

(iv) $\inf_{|\xi|\leq\pi/2} |m_\circ(\xi)| > 0$.

Let $\{u_k\}_{k\in\mathbb{Z}}$ *be such that*

$$m_\circ(\xi) = \frac{1}{\sqrt{2}} \sum_{k\in\mathbb{Z}} u_k e^{-ik\xi},$$

or alternatively, define u_k *by*

$$u_k = \frac{\sqrt{2}}{2\pi} \int_{-\pi}^{\pi} m_\circ(\xi) e^{ik\xi} d\xi = \sqrt{2}\check{m}_\circ(-k),$$

and suppose that $\{u_k\}_{k\in\mathbb{Z}} \in l^1(\mathbb{Z})$. *Then the infinite product* $\prod_{j=1}^{\infty} m_\circ(\xi/2^j)$ *converges uniformly on bounded sets to a function* $\hat{\varphi} \in L^2(\mathbb{R})$. *Let* $\varphi = (\hat{\varphi})\check{}$. *Then* φ *satisfies the scaling equation*

$$\varphi(x) = \sum_{k\in\mathbb{Z}} u_k \varphi_{1,k}(x) = \sum_{k\in\mathbb{Z}} u_k \sqrt{2}\varphi(2x-k),$$

and $\{\varphi_{0,k}\}_{k\in\mathbb{Z}}$ *is an orthonormal set in* $L^2(\mathbb{R})$. *For* $j \in \mathbb{Z}$ *define the spaces* V_j *by*

$$V_j = \left\{ \sum_{k\in\mathbb{Z}} z_k \varphi_{j,k} : \{z_k\}_{k\in\mathbb{Z}} \in l^2(\mathbb{Z}) \right\}.$$

Then $\{V_j\}_{j\in\mathbb{Z}}$ *is a MRA with scaling function* φ *and scaling sequence* $\{u_j\}_{j\in\mathbb{Z}}$.

Remark 5.9. Conditions (i), (ii), and (iii) ensure the convergence of the infinite product, and that φ satisfies the scaling equation. Condition (iv) is needed to show the orthonormality of $\{\varphi_{0,k}\}_{k\in\mathbb{Z}}$.

The next theorem details the construction in terms of the scaling sequence $\{u_j\}_{j\in\mathbb{Z}}$.

Theorem 5.10. *Let $\{u_j\}_{j\in\mathbb{Z}}$ be a sequence satisfying the following condtions*

$$(i)\ \sum_{k\in\mathbb{Z}} |k|^{\varepsilon}|u_k| < \infty \quad \text{for some } \varepsilon > 0,$$

$$(ii)\ \sum_{k\in\mathbb{Z}} u_k = \sqrt{2},$$

$$(iii)\ \{\tau_{2k}u\}_{k\in\mathbb{Z}} \text{ is an orthonormal set in } l^2(\mathbb{Z}),$$

$$(iv)\ \inf_{|\xi|\leq\pi/2} |m_\circ(\xi)| > 0, \text{ for } m_\circ(\xi) = \frac{1}{\sqrt{2}}\sum_{k\in\mathbb{Z}} u_k e^{-ik\xi}.$$

Then the infinite product $\prod_{j=1}^{\infty} m_\circ(\xi/2^j)$ converges uniformly on bounded subsets of $\mathbb{R}$ to a function $\hat{\varphi} \in L^2(\mathbb{R})$. Let $\varphi = (\hat{\varphi})\check{}$. For every $j \in \mathbb{Z}$ define V_j by

$$V_j = \left\{ \sum_{k\in\mathbb{Z}} z_k\varphi_{j,k} : \{z_k\}_{k\in\mathbb{Z}} \in l^2(\mathbb{Z}) \right\}.$$

Then $\{V_j\}_{j\in\mathbb{Z}}$ is a MRA with scaling function φ and scaling sequence $\{u_j\}_{j\in\mathbb{Z}}$.

So once we have a sequence $\{u_j\}_{j\in\mathbb{Z}}$ satisfying the conditions of Theorem 5.10 then by Theorem 5.6 of Mallat we are guaranteed to have a wavelet system.

5.3 Existence of the Gibbs Phenomenon for some Wavelets

In this section, we introduce the Schwartz space $\mathcal{S}(\mathbb{R})$ and give conditions under which wavelet expansions of functions with a jump discontinuity show the Gibbs phenomenon near the jump if the scaling function φ satisfies a certain decay condition [8], [17].

Definition 5.11. The Schwartz space $\mathcal{S}(\mathbb{R})$, or the space of rapidly decreasing C^∞ functions, is defined by

$$\mathcal{S}(\mathbb{R}) = \left\{ f \in C^\infty : |f^l(x)| \leq C_{l,k}(1+|x|)^{-k} \text{ for all integers } k, l \geq 0 \right\}.$$

Definition 5.12. We define the space of rapidly decreasing C^r functions $\mathcal{S}_r(\mathbb{R})$ by

$$\mathcal{S}_r(\mathbb{R}) = \Big\{ f \in C^r : |f^l(x)| \le C_{l,k}(1+|x|)^{-k} \text{ for integers } 0 \le l \le r,\ k \ge 0 \Big\}.$$

We define the Gibbs phenomenon for the wavelet approximation to the function $f : \mathbb{R} \to \mathbb{R}$, $f \in L^2(\mathbb{R})$ which is bounded and has a jump discontinuity at the origin similar to Definition 3.1 as follows:

Definition 5.13. Suppose a function $f(x)$ has a jump discontinuity at $x = 0$, i.e, $f(0^+) = \lim_{x\to 0^+} f(x) < \infty$, $f(0^-) = \lim_{x\to 0^-} f(x) < \infty$, and $f(0^+) \ne f(0^-)$. We say that the wavelet expansion of f exhibits the Gibbs phenomenon at the right hand side of $x = 0$ if there is sequence $x_j > 0$ converging to 0 such that $\lim_{j\to\infty} P_j f(x_j) > f(0^+)$ if $f(0^+) > f(0^-)$, or $\lim_{j\to\infty} P_j f(x_j) < f(0^+)$ if $f(0^+) < f(0^-)$. Similarly, we say that the wavelet expansion of f exhibits the Gibbs phenomenon at the left hand side of $x = 0$ if there is sequence $x_j < 0$ converging to 0 such that $\lim_{j\to\infty} P_j f(x_j) < f(0^-)$ if $f(0^+) > f(0^-)$, or $\lim_{j\to\infty} P_j f(x_j) > f(0^-)$ if $f(0^+) < f(0^-)$.

Let $\varphi \in \mathcal{S}_r(\mathbb{R})$ and define the function $f \in L^2(\mathbb{R})$ as

$$f(x) = \begin{cases} -1 - x, & -1 \le x < 0; \\ 1 - x, & 0 < x \le 1; \\ 0, & \text{otherwise}. \end{cases} \tag{5.3.1}$$

Recall the orthogonal projection of f onto V_j given by (5.1.1).

$$\begin{aligned} P_j f(x) &= \sum_{k\in\mathbb{Z}} \langle f, \varphi_{j,k}\rangle \varphi_{j,k}(x) \\ &= \sum_{k\in\mathbb{Z}} \left(\int_{-\infty}^{\infty} f(y)\overline{\varphi_{j,k}}(y)dy \right) \varphi_{j,k}(x) \\ &= \sum_{k\in\mathbb{Z}} \left(\int_{-\infty}^{\infty} f(y)\varphi_{j,k}(y)dy \right) \varphi_{j,k}(x) \\ &= \int_{-\infty}^{\infty} f(y) \left(\sum_{k\in\mathbb{Z}} \varphi_{j,k}(y)\varphi_{j,k}(x) \right) dy \\ &= \int_{-\infty}^{\infty} f(y) K_j(x,y) dy, \end{aligned} \tag{5.3.2}$$

where $K_j(x,y) = \sum_{k\in\mathbb{Z}} \varphi_{j,k}(x)\varphi_{j,k}(y)$. We call $K_j(x,y)$ the reproducing kernel of V_j. We can express K_j in terms of K_0, the reproducing kernel of V_0 as follows

$$\begin{aligned} K_j(x,y) &= \sum_{k\in\mathbb{Z}} \varphi_{j,k}(x)\varphi_{j,k}(y) \\ &= \sum_{k\in\mathbb{Z}} 2^{j/2}\varphi(2^j x - k) 2^{j/2}\varphi(2^j y - k) \\ &= 2^j \sum_{k\in\mathbb{Z}} \varphi(2^j x - k)\varphi(2^j y - k) \\ &= 2^j \sum_{k\in\mathbb{Z}} \varphi_{0,k}(2^j x)\varphi_{0,k}(2^j y) \\ &= 2^j K_0(2^j x, 2^j y), \end{aligned} \tag{5.3.3}$$

where $K_0(x,y) = \sum_{k\in\mathbb{Z}} \varphi(x-k)\varphi(y-k)$.

We need the following result given in [17].

Lemma 5.14. *Let K_0 be defined as above and $\varphi \in \mathcal{S}_r(\mathbb{R})$. Then*

(i) $|K_0(x,y)| \le \dfrac{C_\beta}{(1+|x-y|)^\beta}, \quad \beta \in \mathbb{N},$

(ii) $\displaystyle\int_{-\infty}^{\infty} K_0(x,y)dy = 1,$ *for all* $x \in \mathbb{R}$.

Proof. (i) Since $K_0(x,y) = K_0(y,x) = K_0(x+1, y+1)$ we can assume without loss of generality that $|x+y| \le 1$ and $x \ge y$. Now, if $k \ge 0$ then

$$\begin{aligned}
|x-k| &= \left|\frac{x+y}{2} + \frac{x-y}{2} - k\right| \\
&= \left|\left(\frac{x-y}{2} - k\right) - \left(-\frac{x+y}{2}\right)\right| \\
&\ge \left|\frac{x-y}{2} - k\right| - \left|\frac{x+y}{2}\right| \\
&\ge \left|\frac{x-y}{2} - k\right| - \frac{1}{2},
\end{aligned} \tag{5.3.4}$$

and

$$\begin{aligned}
|y-k| &= \left|\frac{x+y}{2} - \frac{x-y}{2} - k\right| \\
&= \left|-\left(\frac{x-y}{2} + k\right) - \left(-\frac{x+y}{2}\right)\right| \\
&\ge \left|\frac{x-y}{2} + k\right| - \left|\frac{x+y}{2}\right| \\
&\ge \frac{x-y}{2} + k - \frac{1}{2} \\
&\ge \frac{x-y}{2} - \frac{1}{2}.
\end{aligned} \tag{5.3.5}$$

Now, by (5.3.4) and (5.3.5) we obtain

$$\begin{aligned}
(1+|x-k|)(1+|y-k|) &\ge \left(\left|\frac{x-y}{2} - k\right| + \frac{1}{2}\right)\left(\frac{x-y}{2} + \frac{1}{2}\right) \\
&= \frac{1}{4}\big(|x-y-2k| + 1\big)(x-y+1).
\end{aligned} \tag{5.3.6}$$

Similarly, if $k < 0$ we obtain

$$(1+|x-k|)(1+|y-k|) \ge \frac{1}{4}\big(|x-y+2k| + 1\big)(x-y+1). \tag{5.3.7}$$

The assumption $\varphi \in \mathcal{S}_r(\mathbb{R})$ implies that there exist a constant K and $\beta > 1$ such that

$$|\varphi(x)| \le \frac{K}{\big(1+|x|\big)^{\beta}}.$$

This in turn implies that

$$|\varphi(x-k)| \le \frac{K}{\big(1+|x-k|\big)^{\beta}},$$
$$|\varphi(y-k)| \le \frac{K}{\big(1+|y-k|\big)^{\beta}}.$$

Thus

$$\begin{aligned}
|K_0(x,y)| &\le \sum_{k\in\mathbb{Z}} |\varphi(x-k)||\varphi(y-k)| \\
&\le \sum_{k\in\mathbb{Z}} \frac{K}{\big(1+|x-k|\big)^{\beta}} \frac{K}{\big(1+|y-k|\big)^{\beta}} \\
&= K^2\Bigg(\sum_{k>0} \frac{1}{\big(1+|x-k|\big)^{\beta}} \frac{1}{\big(1+|y-k|\big)^{\beta}} \\
&\quad + \sum_{k<0} \frac{1}{\big(1+|x-k|\big)^{\beta}} \frac{1}{\big(1+|y-k|\big)^{\beta}}\Bigg).
\end{aligned}$$

Now, by inequalities (5.3.6), and (5.3.7) we obtain

$$\begin{aligned}
|K_0(x,y)| &\le K^2\Bigg(\sum_{k>0} \frac{4^{\beta}}{\big(1+|x-y-2k|\big)^{\beta}(x-y+1)^{\beta}} \\
&\quad + \sum_{k<0} \frac{4^{\beta}}{\big(1+|x-y+2k|\big)^{\beta}(x-y+1)^{\beta}}\Bigg) \\
&= \frac{4^{\beta}K^2}{(x-y+1)^{\beta}} 2\sum_{k>0} \frac{1}{\big(1+|t-2k|\big)^{\beta}},\ t\in\mathbb{R}.
\end{aligned}$$

Now, $\big(1+|t-2k|\big)^{\beta} \ge |t-2k|^{\beta}$. So $\dfrac{1}{\big(1+|t-2k|\big)^{\beta}} \le \dfrac{1}{|t-2k|^{\beta}}$. For fixed $t\in\mathbb{R}$, we can find $N\in\mathbb{N}$ such that $|t-2k|\ge k$ for $k\ge N$. Therefore, $\dfrac{1}{|t-2k|^{\beta}} \le \dfrac{1}{k^{\beta}}$. Now, by the comparison test we have

$$\sum_{k>0} \frac{1}{\big(1+|t-2k|\big)^{\beta}} \le \sum_{k>0} \frac{1}{|t-2k|^{\beta}} \le \sum_{k>0} \frac{1}{k^{\beta}} < \infty \quad \text{since } \beta>1.$$

Thus,

$$K_0(x,y) \le \frac{C_{\beta}}{\big(1+|x-y|\big)^{\beta}}.$$

(ii) Let $d = \dfrac{m}{2^n}, m \in \mathbb{Z}, n \in \mathbb{N}$, be a dyadic number, and $j \in \mathbb{Z}$ such that $j \geq n$. We know that $V_{-j} = \overline{span\{\varphi_{-j,k}\}_{k\in\mathbb{Z}}}$. This implies that $\varphi_{-j,0}(x) = 2^{-j/2}\varphi(2^{-j}x) \in V_{-j}$. Therefore, $h(x) := \varphi(2^{-j}x) \in V_{-j}$. Thus, $h(x) \in V_0$ since $V_{-j} \subseteq V_0$. Therefore, $\varphi(2^{-j}x + d) \in V_0$ since $\varphi(2^{-j}x + d) = \varphi(2^{-j}(x + 2^{j-n}m))$. Now, by the definition of the projection onto V_0 we have

$$P_0 g(x) = \int_{-\infty}^{\infty} K_0(x,y)g(y)dy \quad \text{for all } g \in L^2(\mathbb{R}).$$

So, in particular, for $g(x) = \varphi(2^{-j}x + d) \in V_0$, we have $g(x) = P_0 g(x)$, so

$$\varphi(2^{-j}x + d) = g(x) = \int_{-\infty}^{\infty} K_0(x,y)\varphi(2^{-j}y + d)dy.$$

By part (ii), $K_0(x,y)$ is integrable, and $\varphi(x)$ is bounded. Thus by the Lebesgue Dominated Convergence Theorem as $j \to \infty$ we have

$$\varphi(d) = \int_{-\infty}^{\infty} K_0(x,y)\varphi(d)dy = \varphi(d)\int_{-\infty}^{\infty} K_0(x,y).$$

Now we show that there exists a dyadic number d such that $\varphi(d) \neq 0$. Assume that $\varphi(d) = 0$ for all dyadic numbers d. Let a be a real number such that $\varphi(a) \neq 0$. Because of the density of the dyadic numbers in the reals we can find a sequence of dyadic numbers $\{d_n\}_{n=1}^{\infty}$ such that $d_n \to a$ as $n \to \infty$. Now, by the continuity of φ it follows that $\varphi(d_n)$ converges to $\varphi(a)$ which is a contradiction. Therefore we have

$$\int_{-\infty}^{\infty} K_0(x,y)dy = 1.$$

$\square$

Now, we can prove Kelly's main theorem given in [8].

Theorem 5.15. *Let f be defined as in* (5.3.1)*, and a be a real number. Then*

$$\lim_{j\to\infty} P_j f(2^{-j}a) = 2\int_0^{\infty} K_0(a,u)du - 1.$$

Proof. Using (5.3.2), we can write $P_j f(x)$ as

$$\begin{aligned} P_j f(x) &= \int_{-\infty}^{\infty} f(y) K_j(x,y) dy \\ &= \int_{-1}^{0} (-1-y) K_j(x,y) dy + \int_{0}^{1} (1-y) K_j(x,y) dy. \end{aligned}$$

Now, with the change of variables $t = -y$ on the first integral, we obtain

$$\begin{aligned} P_j f(x) &= -\int_{1}^{0} (-1+t) K_j(x,-t) dt + \int_{0}^{1} (1-t) K_j(x,t) dt \\ &= \int_{0}^{1} (-1+t) K_j(x,-t) dt + \int_{0}^{1} (1-t) K_j(x,t) dt \\ &= -\int_{0}^{1} (1-t) K_j(x,-t) dt + \int_{0}^{1} (1-t) K_j(x,t) dt \\ &= \int_{0}^{1} (1-t)[K_j(x,t) - K_j(x,-t)] dt \\ &= 2^j \int_{0}^{1} (1-t)[K_0(2^j x, 2^j t) - K_0(2^j x, -2^j t)] dt, \quad \text{by (5.3.3).} \end{aligned}$$

Now, let $u = 2^j t$. Then $du = 2^j dt$ and the integral becomes

$$\begin{aligned} P_j f(x) &= \int_{0}^{2^j} (1 - 2^{-j} u)[K_0(2^j x, u) - K_0(2^j x, -u)] du \\ &= \int_{0}^{\infty} \chi_{[0,2^j]}(u)(1 - 2^{-j} u)[K_0(2^j x, u) - K_0(2^j x, -u)] du. \end{aligned}$$

Letting $x = 2^{-j} a$, where $a \in \mathbb{R}$, we obtain

$$P_j f(2^{-j} a) = \int_{0}^{\infty} \chi_{[0,2^j]}(u)(1 - 2^{-j} u)[K_0(a,u) - K_0(a,-u)] du.$$

Now, let $h_j(u) = \chi_{[0,2^j]}(u)(1 - 2^{-j} u)[K_0(a,u) - K_0(a,-u)]$, and $g(u) = |K_0(a,u)| + |K_0(a,-u)|$. The functions $h_j(u)$ are bounded by $g(u)$, i.e. $|h_j(u)| \le g(u)$, and $g \in L^1(\mathbb{R})$ since $\varphi \in \mathcal{S}_r(\mathbb{R})$. Thus, by the Lebesgue Dominated Convergence Theorem we have

$$\begin{aligned}
\lim_{j\to\infty} P_j f(2^{-j}a) &= \lim_{j\to\infty} \int_0^\infty h_j(u)du \\
&= \int_0^\infty \lim_{j\to\infty} h_j(u)du \\
&= \int_0^\infty \lim_{j\to\infty} \chi_{[0,2^j]}(u)(1-2^{-j}u)[K_0(a,u)-K_0(a,-u)]du \\
&= \int_0^\infty \chi_{[0,\infty]}(u)[K_0(a,u)-K_0(a,-u)]du \\
&= \int_0^\infty K_0(a,u)du - \int_0^\infty K_0(a,-u)du
\end{aligned}$$

The change variable $t = -u$ in the second integral will yield

$$\begin{aligned}
\lim_{j\to\infty} P_j f(2^{-j}a) &= \int_0^\infty K_0(a,u)du + \int_0^{-\infty} K_0(a,t)dt \\
&= \int_0^\infty K_0(a,u)du - \int_{-\infty}^0 K_0(a,t)dt \\
&= \int_0^\infty K_0(a,u)du - \int_{-\infty}^0 K_0(a,u)du \qquad (5.3.8) \\
&= \int_0^\infty K_0(a,u)du + \int_0^\infty K_0(a,u)du - \int_{-\infty}^\infty K_0(a,u)du \\
&= 2\int_0^\infty K_0(a,u)du - \int_{-\infty}^\infty K_0(a,u)du \\
&= 2\int_0^\infty K_0(a,u)du - 1, \quad \text{by Lemma 5.14.} \qquad (5.3.9)
\end{aligned}$$

□

Now, the following corollary gives a criterion for the existence of the Gibbs phenomenon.

Corollary 5.16. *Let f be defined as in* (5.3.1)*, and let $\varphi \in \mathcal{S}_r(\mathbb{R})$. Then the wavelet expansion of f exhibits the Gibbs phenomenon near $x = 0$ if there exists a real number*

$a > 0$ such that

$$\int_0^\infty K_0(a,u)du > 1,$$

and/or if there exists a real number $a < 0$ such that

$$\int_0^\infty K_0(a,u)du < 0.$$

Proof. Let $x_j = 2^{-j}a$, $a \in \mathbb{R}$. Then

$$\lim_{j\to\infty} f(x_j) = 1 \text{ if } a > 0$$
$$\lim_{j\to\infty} f(x_j) = -1 \text{ if } a < 0.$$

Thus by Theorem 5.15,

$$\begin{aligned} \lim_{j\to\infty} P_j f(x_j) &> \lim_{j\to\infty} f(x_j) \quad \text{if} \quad \int_0^\infty K_0(a,u)du > 1 \text{ for some } a > 0, \text{ or} \\ \lim_{j\to\infty} P_j f(x_j) &< \lim_{j\to\infty} f(x_j) \quad \text{if} \quad \int_0^\infty K_0(a,u)du < 0 \text{ for some } a < 0. \end{aligned}$$

□

The next theorem of Shim and Volkmer relaxes the condition that the scaling function φ be in $\mathcal{S}_r(\mathbb{R})$. To prove it we need the following remarks and lemmas.

Remark 5.17. Let the function $h(x)$ be defined by

$$h(x) = \begin{cases} 1, & \text{if } \; x \geq 0; \\ -1, & \text{if } \; x < 0. \end{cases}$$

Then by (5.3.8) we have

$$\lim_{j\to\infty} P_j f(2^{-j}x) = \int_{-\infty}^\infty K_0(x,y)h(y)dy.$$

This limit is sometimes called the Gibbs function. Let us also define the function $r(x)$ as

$$r(x) = h(x) - \int_{-\infty}^{\infty} K_0(x,y)h(y)dy. \tag{5.3.10}$$

Using (5.3.9) we can write the Gibbs function as

$$\int_{-\infty}^{\infty} K_0(x,y)h(y)dy = 2\int_0^{\infty} K_0(x,y)dy - 1.$$

Thus we can rewrite $r(x)$ as

$$\begin{aligned} r(x) &= h(x) - \left(2\int_0^{\infty} K_0(a,u)du - 1\right) \qquad (5.3.11)\\ &= \begin{cases} 2 - 2\int_0^{\infty} K_0(x,y)dy & x \geq 0; \\ -1 - 2\int_0^{\infty} K_0(x,y)dy + 1 & x < 0. \end{cases} \\ &= \begin{cases} 2 - 2\Big(\int_{-\infty}^{\infty} K_0(x,y)dy - \int_{-\infty}^{0} K_0(x,y)dy\Big) & x \geq 0; \\ -2\int_0^{\infty} K_0(x,y)dy & x < 0. \end{cases} \\ &= \begin{cases} 2\int_{-\infty}^{0} K_0(x,y)dy & x \geq 0; \\ -2\int_0^{\infty} K_0(x,y)dy & x < 0. \end{cases} \qquad (5.3.12) \end{aligned}$$

Notice that if $\varphi(x)$ is continuous then $r(x) - h(x) = -2\int_0^{\infty} K_0(a,u)du + 1$ is also continuous. In the next lemma we will make use of the representation of $r(x)$ given in (5.3.12).

Lemma 5.18. *Suppose that $\varphi(x)$ is a continuous scaling function satisfying*

$$|\varphi(x)| \leq \frac{C}{(1+|x|)^{\beta}} \quad \text{for } x \in \mathbb{R},\ C > 0, \text{ and } \beta > 1. \tag{5.3.13}$$

Then there exists $M > 0$ such that

$$|r(x)| \leq \frac{M}{(1+|x|)^{\beta-1}} \quad \text{for all } x \in \mathbb{R},$$

where $r(x)$ is the function defined in (5.3.12). Moreover, if $\beta > \frac{3}{2}$ then $r \in L^2(\mathbb{R})$ and it is orthogonal to V_0.

Proof. By Lemma 5.14 we know that

$$|K_0(x,y)| \leq \frac{C_\beta}{(1+|x-y|)^\beta}.$$

Therefore we can bound $r(x)$ as follows:

Case 1: $x \geq 0$

$$\begin{aligned}
|r(x)| &= \left|2\int_{-\infty}^{0} K_0(x,y)dy\right| \\
&\leq 2\int_{-\infty}^{0} |K_0(x,y)|dy \\
&\leq 2C_\beta \int_{-\infty}^{0} \frac{1}{(1+|x-y|)^\beta}dy
\end{aligned}$$

Using the substitution $u = x - y$ we obtain

$$\begin{aligned}
\int_{-\infty}^{0} \frac{1}{(1+|x-y|)^\beta}dy &= -\int_{\infty}^{x} \frac{1}{(1+|u|)^\beta}du \\
&= \int_{x}^{\infty} \frac{1}{(1+|u|)^\beta}du \\
&= \int_{x}^{\infty} \frac{1}{(1+u)^\beta}du \quad \text{since } u \geq 0.
\end{aligned}$$

Now, let $w = 1 + u$ to obtain

$$\begin{aligned}
\int_{x}^{\infty} \frac{1}{(1+u)^\beta}du &= \int_{1+x}^{\infty} \frac{1}{(w)^\beta}dw \\
&= \lim_{c\to\infty} \left.\frac{w^{-\beta+1}}{-\beta+1}\right|_{1+x}^{c} \\
&= \frac{1}{1-\beta}\left(\lim_{c\to\infty}\frac{1}{c^{\beta-1}} - \frac{1}{(1+x)^{\beta-1}}\right) \\
&= \frac{1}{\beta-1}\frac{1}{(1+|x|)^{\beta-1}} \quad \text{since } x \geq 0 \text{ and } \beta > 1.
\end{aligned}$$

Thus

$$\begin{aligned}
|r(x)| &\leq 2C_\beta \frac{1}{\beta-1}\frac{1}{(1+|x|)^{\beta-1}} \\
&\leq \frac{M}{(1+|x|)^{\beta-1}} \quad \text{for } x \geq 0.
\end{aligned}$$

Case 2: $x < 0$

$$\begin{aligned}
|r(x)| &= \left| -2\int_0^\infty K_0(x,y)dy \right| \\
&\leq 2\int_0^\infty |K_0(x,y)|dy \\
&\leq 2C_\beta \int_0^\infty \frac{1}{(1+|x-y|)^\beta}dy
\end{aligned}$$

Let $u = x - y$. Then we obtain

$$\begin{aligned}
\int_0^\infty \frac{1}{(1+|x-y|)^\beta}dy &= -\int_x^{-\infty} \frac{1}{(1+|u|)^\beta}du \\
&= \int_{-\infty}^x \frac{1}{(1+|u|)^\beta}du \\
&= \int_{-\infty}^x \frac{1}{(1-u)^\beta}du \quad \text{since } u < 0.
\end{aligned}$$

Now, let $w = 1 - u$ to obtain

$$\begin{aligned}
\int_{-\infty}^x \frac{1}{(1-u)^\beta}du &= -\int_\infty^{1-x} \frac{1}{(w)^\beta}dw \\
&= \int_{1-x}^\infty \frac{1}{(w)^\beta}dw \\
&= \lim_{c\to\infty} \left.\frac{w^{-\beta+1}}{-\beta+1}\right|_{1-x}^c \\
&= \frac{1}{1-\beta}\left(\lim_{c\to\infty}\frac{1}{c^{\beta-1}} - \frac{1}{(1-x)^{\beta-1}}\right) \\
&= \frac{1}{\beta-1}\frac{1}{(1+|x|)^{\beta-1}} \quad \text{since } x < 0 \text{ and } \beta > 1.
\end{aligned}$$

Thus

$$\begin{aligned}
|r(x)| &\leq 2C_\beta \frac{1}{\beta-1}\frac{1}{(1+|x|)^{\beta-1}} \\
&\leq \frac{M}{(1+|x|)^{\beta-1}} \quad \text{for } x < 0.
\end{aligned}$$

Therefore, we have the required bound for all x.

To show that $r \in L^2(\mathbb{R})$ we use the bound we just found as follows

$$\begin{aligned}\int_{-\infty}^{\infty} |r(x)|^2 dx &\leq \int_{-\infty}^{\infty} \left(\frac{M}{(1+|x|)^{\beta-1}}\right)^2 dx \\ &= \int_{-\infty}^{\infty} \frac{M^2}{(1+|x|)^{2(\beta-1)}} dx \\ &< \infty \quad \text{since } 2(\beta-1) > 1 \text{ as } \beta > 3/2.\end{aligned}$$

Hence $r(x) \in L^2(\mathbb{R})$. Now, suppose that $f \in V_0$ is integrable. Multiplying both sides of (5.3.10) by $f(x)$ and integrating with respect to x will yield

$$\int_{-\infty}^{\infty} r(x)f(x)dx = \int_{-\infty}^{\infty} f(x)h(x)dx - \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} K_0(x,y)f(x)h(y)dy\right)dx.$$

Since $f \in V_0$ we have

$$f(y) = P_j f(y) = \int_{-\infty}^{\infty} K_0(x,y)f(x)dx. \tag{5.3.14}$$

Using Fubini's theorem and (5.3.14) we obtain

$$\begin{aligned}\int_{-\infty}^{\infty} r(x)f(x)dx &= \int_{-\infty}^{\infty} f(x)h(x)dx - \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} K_0(x,y)f(x)dx\right)h(y)dy \\ &= \int_{-\infty}^{\infty} f(x)h(x)dx - \int_{-\infty}^{\infty} f(y)h(y)dy \\ &= 0.\end{aligned}$$

Hence $r(x)$ is orthogonal to V_0. □

Lemma 5.19. *Suppose that $\varphi(x)$ is a continuous and bounded scaling function such that it is differentiable at a dyadic number d with $\varphi'(d) \neq 0$. Let $g \in L^2(\mathbb{R})$ be orthogonal to V_0, and $xg(x)$ be in $L^1(\mathbb{R})$. Then $\int_{-\infty}^{\infty} xg(x)dx = 0$.*

Proof. First we prove that $g \in L^1(\mathbb{R})$.

$$\begin{aligned}\int_{\mathbb{R}} |g(x)|dx &= \int_{|x|<1} |g(x)|dx + \int_{|x|>1} |g(x)|dx \\ &= I_1 + I_2.\end{aligned}$$

Now, we show that both I_1 and I_2 are bounded.

$$\begin{aligned} I_1 &= \int_{\mathbb{R}} |g(x)|\chi_{|x|<1}(x)dx \\ &\leq ||g(x)||_2||\chi_{|x|<1}(x)||_2 \quad \text{by Hölder's inequality} \\ &< \infty. \end{aligned}$$

As for I_2, we rewrite the integral

$$\begin{aligned} I_2 &= \int_{|x|>1} |x^{-1}||xg(x)|dx \\ &\leq \max_{|x|>1} \frac{1}{|x|} \int_{|x|>1} |xg(x)|dx \\ &\leq 1 \cdot \int_{|x|>1} |xg(x)|dx \\ &< \infty \quad \text{by the assumption that } xg(x) \in L^1(\mathbb{R}). \end{aligned}$$

So we conclude that $\int_{\mathbb{R}} |g(x)|dx < \infty$. Now, let $c = \dfrac{m}{2^n}, m \in \mathbb{Z}, n \in \mathbb{N}$, be a dyadic number. Then $\varphi(2^{-j}x + c) \in V_0$ for all $j \geq n$ as proved in Lemma 5.14. Therefore, by the assumption that $g \perp V_0$ we have

$$\int_{-\infty}^{\infty} \varphi(2^{-j}x + c)g(x)dx = 0 \quad \text{for all } j \geq n. \tag{5.3.15}$$

Now, as $j \to \infty$ by the Lebesgue Dominated Convergence Theorem we obtain

$$\varphi(c) \int_{-\infty}^{\infty} g(x)dx = 0.$$

Since $\varphi(c)$ is continuous there exists a dyadic number c such that $\varphi(c) \neq 0$. Hence

$$\int_{-\infty}^{\infty} g(x)dx = 0.$$

This implies that

$$\varphi(d) \int_{-\infty}^{\infty} g(x)dx = 0.$$

Now, using (5.3.15) with $c = d$ we obtain

$$\begin{aligned}\int_{-\infty}^{\infty} \varphi(2^{-j}x + d)g(x)dx - \int_{-\infty}^{\infty} \varphi(d)g(x)dx &= 0 \\ \int_{-\infty}^{\infty} \frac{\varphi(2^{-j}x + d) - \varphi(d)}{x} xg(x)dx &= 0 \\ \int_{-\infty}^{\infty} \frac{\varphi(2^{-j}x + d) - \varphi(d)}{2^{-j}x} xg(x)dx &= 0.\end{aligned}$$

Letting $j \to \infty$ and applying the Lebesgue Dominated Convergence Theorem one more time we obtain

$$\int_{-\infty}^{\infty} \varphi'(d)xg(x)dx = 0.$$

Since by assumption $\varphi'(d) \neq 0$ we conclude that

$$\int_{-\infty}^{\infty} xg(x)dx = 0.$$

□

Remark 5.20. First, notice that if $\varphi(x)$ satisfies the condition (5.3.13) then $\varphi \in \mathcal{S}_r(\mathbb{R})$ and we can use Corollary 5.16. Now, if there is $a > 0$ such that $r(x) < 0$ then (5.3.11) implies that $2\int_0^\infty K_0(a,u)du - 1 > 1$ which in turn implies that $\int_0^\infty K_0(a,u)du > 1$. Now, by Corollary 5.16 there is a Gibbs phenomenon at the right of $x = 0$. Similarly, if there is $a < 0$ such that $r(x) > 0$ then (5.3.11) implies that $2\int_0^\infty K_0(a,u)du - 1 < -1$. This implies that $\int_0^\infty K_0(a,u)du < 0$, and again by Corollary 5.16 we conclude that there is a Gibbs phenomenon at the left of $x = 0$.

Now, we are ready to prove Shim and Volkmer's theorem [17].

Theorem 5.21. *Suppose φ is a continuous scaling function such that $\varphi'(d) \neq 0$ for some dyadic rational number d and that*

$$|\varphi(x)| \leq \frac{C}{(1+|x|)^\beta} \quad \text{for } x \in \mathbb{R},\ C > 0, \text{ and } \beta > 3.$$

Then the corresponding wavelet expansion shows Gibbs phenomenon on the right or the left at 0.

Proof. Our assumptions satisfy those of Lemma 5.18. Therefore, there exists an $M > 0$ such that

$$|r(x)| \leq \frac{M}{(1+|x)^{\beta-1}} \quad \text{for all } x \in \mathbb{R}.$$

Recall that $r(x)$ is the function defined in (5.3.12). Also, since $\beta > 3$ the second part of Lemma 5.18 implies that $r(x) \in L^2(\mathbb{R})$ and it is orthogonal to V_0. Now, $xr(x) \in L^1(\mathbb{R})$ because

$$\begin{aligned} \int_{\mathbb{R}} |xr(x)| dx &\leq \int_{\mathbb{R}} \frac{xM}{(1+|x|)^{\beta-1}} dx \\ &< \infty \quad \text{since } \beta > 3. \end{aligned}$$

Thus, the assumptions of Lemma 5.19 are satisfied with $g(x) = r(x)$. Therefore, we have

$$\int_{-\infty}^{\infty} xr(x) dx = 0. \tag{5.3.16}$$

Now, suppose that $r(x) \geq 0$ for all $x > 0$ and $r(x) \leq 0$ for all $x < 0$. Then (5.3.16) implies that $r(x) = 0$ almost everywhere. But this is a contradiction since by Remark 5.17, $r(x) - h(x)$ is continuous whereas $h(x)$ has a jump discontinuity at $x = 0$. Therefore, there exists an $x > 0$ such that $r(x) < 0$, or there exists an $x < 0$ such that $r(x) > 0$. Now, Remark 5.20 implies the existence of the Gibbs phenomenon at the right or left hand side of the origin. □

5.4 A Wavelet with no Gibbs Effect

In [7], [8], and [16], Corollary 5.16 is used to show that the Haar system does not exhibit the Gibbs phenomenon. They proceed as follows:

The scaling function for the Haar MRA is given by $\varphi(x) = \chi_{[0,1)}(x)$. Then

$$\begin{aligned}\int_0^\infty K_0(a,u)du &= \int_0^\infty \sum_{k\in\mathbb{Z}} \varphi(a-k)\varphi(u-k)du \\ &= \int_0^\infty \sum_{k\in\mathbb{Z}} \chi_{[0,1)}(a-k)\chi_{[0,1)}(u-k)du \\ &= \sum_{k\in\mathbb{Z}} \chi_{[0,1)}(a-k)\int_0^\infty \chi_{[0,1)}(u-k)du.\end{aligned}$$

By the change of variables $t = u - k$ we obtain

$$\begin{aligned}\int_0^\infty K_0(a,u)du &= \sum_{k\in\mathbb{Z}} \chi_{[0,1)}(a-k)\int_{-k}^\infty \chi_{[0,1)}(t)dt \\ &= \int_{-[a]}^\infty \chi_{[0,1)}(t)dt \\ &= \begin{cases} 1, & a \geq 0; \\ 0, & a < 0, \end{cases}\end{aligned}$$

which implies the absence of the Gibbs phenomenon. However, in proving Corollary 5.16 it was assumed that the scaling function $\varphi(x)$ belongs to the class $\mathcal{S}_r(\mathbb{R})$ and in case of the Haar system $\varphi(x) = \chi_{[0,1)}(x)$ is clearly not in $\mathcal{S}_r(\mathbb{R})$. In Section 5.6, we see how one can get around this problem.

5.5 The Shannon Wavelet and its Gibbs Phenomenon

In this section, we define the Shannon system and show the existence of the Gibbs phenomenon for an approximation using the Shannon wavelet to a function with a jump discontinuity at $x = 0$, see [7]. We will use the following theorem known as the Shannon Sampling Theorem. For a proof of this theorem see [11] page 44.

Theorem 5.22. *Suppose that the support of $\hat{f}$ is in $[-\pi/T, \pi/T]$. Then*

$$f(t) = \sum_{k=-\infty}^{\infty} f(kT)h_T(t-kT),$$

where

$$h_T(t) = \frac{\sin(\pi t/T)}{\pi t/T}.$$

Definition 5.23. We define the scaling function for the Shannon wavelets on the Fourier side by

$$\hat{\varphi}(\xi) = \begin{cases} 1, & -\pi \leq \xi < \pi; \\ 0, & \text{otherwise.} \end{cases}$$

Then, taking the inverse Fourier transform we obtain

$$\begin{aligned} \varphi(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\varphi}(\xi) e^{i\xi x} d\xi \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \chi_{[-\pi,\pi)}(\xi) e^{i\xi x} d\xi \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\xi x} d\xi \\ &= \frac{1}{2\pi}\frac{e^{i\xi x}}{ix}\bigg|_{-\pi}^{\pi} \\ &= \frac{1}{\pi x}\frac{e^{i\pi x} - e^{-i\pi x}}{2i} \\ &= \frac{\sin(\pi x)}{\pi x}. \end{aligned} \tag{5.5.1}$$

Let the function f be defined as follows:

$$f(x) = \begin{cases} -1, & -1 \leq x < 0; \\ 1, & 0 < x \leq 1; \\ 0, & \text{otherwise.} \end{cases} \tag{5.5.2}$$

Recall that by (5.3.2) $P_j f$, the orthogonal projection of f onto V_j, can be written as

$$P_j f(x) = \int_{-\infty}^{\infty} f(y) K_j(x,y) dy,$$

where $K_j(x,y) = \sum_{k\in\mathbb{Z}} \varphi_{j,k}(x)\varphi_{j,k}(y)$. If we have a closed form formula for $K_0(x,y) = \sum_{k\in\mathbb{Z}} \varphi(x-k)\varphi(y-k)$ then using (5.3.3) we can find an expression for K_j. If we take $T = 1$ in Theorem 5.22 we obtain

$$f(t) = \sum_{k\in\mathbb{Z}} f(k)\frac{\sin(\pi(t-k))}{\pi(t-k)}. \tag{5.5.3}$$

By (5.5.1) the scaling function for the Shannon wavelet is given by $\varphi(x) = \dfrac{\sin(\pi x)}{\pi x}$ and the support of $\hat{\varphi}$ is in $[-\pi, \pi]$ as defined in (5.5.1). So we can apply (5.5.3) to $g(x) = \varphi(x-y)$ to obtain

$$\begin{aligned}
\varphi(x-y) &= \sum_{k\in\mathbb{Z}} \varphi(k-y)\frac{\sin(\pi(x-k))}{\pi(x-k)} \\
&= \sum_{k\in\mathbb{Z}} \frac{\sin(\pi(k-y))}{\pi(k-y)}\frac{\sin(\pi(x-k))}{\pi(x-k)} \\
&= \sum_{k\in\mathbb{Z}} \frac{\sin(\pi(x-k))}{\pi(x-k)}\frac{\sin(\pi(y-k))}{\pi(y-k)} \\
&= \sum_{k\in\mathbb{Z}} \varphi(x-k)\varphi(y-k) \\
&= K_0(x,y).
\end{aligned}$$

Then by (5.3.3) we obtain

$$\begin{aligned}
K_j(x,y) &= 2^j K_0(2^j x, 2^j y) \\
&= 2^j \frac{\sin \pi(2^j x - 2^j y)}{\pi(2^j x - 2^j y)} \\
&= \frac{\sin(2^j \pi(x-y))}{\pi(x-y)}.
\end{aligned}$$

Therefore we can write $P_j f(x)$ as

$$\begin{aligned}
P_j f(x) &= \int_{-\infty}^{\infty} \frac{\sin(2^j\pi(x-y))}{\pi(x-y)} f(y)dy \\
&= \int_{-1}^{1} \frac{\sin(2^j\pi(x-y))}{\pi(x-y)} f(y)dy \\
&= -\int_{-1}^{0} \frac{\sin(2^j\pi(x-y))}{\pi(x-y)} dy + \int_{0}^{1} \frac{\sin(2^j\pi(x-y))}{\pi(x-y)} dy.
\end{aligned}$$

We want to examine the behavior of these projections as $j \to \infty$ near the origin. So we let $x = 2^{-j}a$ where a is a positive real number. Then we have

$$\begin{aligned} P_j f(2^{-j}a) &= -\int_{-1}^{0} \frac{\sin(2^j\pi(2^{-j}a - y))}{\pi(2^{-j}a - y)} dy + \int_{0}^{1} \frac{\sin(2^j\pi(2^{-j}a - y))}{\pi(2^{-j}a - y)} dy \\ &= -\int_{-1}^{0} \frac{\sin(\pi(a - 2^j y))}{\pi(2^{-j}a - y)} dy + \int_{0}^{1} \frac{\sin(\pi(a - 2^j y))}{\pi(2^{-j}a - y)} dy. \end{aligned}$$

Now, let $u = a - 2^j y$. Then $y = 2^{-j}(a - u)$, $du = -2^j dy$, and $dy = -2^{-j} du$. With this change of variables we obtain

$$\begin{aligned} P_j f(2^{-j}a) &= -\int_{a+2j}^{a} \frac{\sin(\pi u)}{\pi(2^{-j}a - 2^{-j}(a-u))}(-2^{-j} du) \\ &\quad + \int_{a}^{a-2^j} \frac{\sin(\pi u)}{\pi(2^{-j}a - 2^{-j}(a-u))}(-2^{-j} du) \\ &= \int_{a+2^j}^{a} \frac{\sin(\pi u)}{2^{-j}\pi u} 2^{-j} du - \int_{a}^{a-2^j} \frac{\sin(\pi u)}{2^{-j}\pi u} 2^{-j} du \\ &= -\int_{a}^{a+2^j} \frac{\sin(\pi u)}{\pi u} du + \int_{a-2^j}^{a} \frac{\sin(\pi u)}{\pi u} du \\ &= \int_{a-2^j}^{a} \frac{\sin(\pi u)}{\pi u} du - \int_{a}^{a+2^j} \frac{\sin(\pi u)}{\pi u} du. \end{aligned}$$

Adding and subtracting $\int_{-a+2^j}^{a} \frac{\sin(\pi u)}{\pi u} du$ we obtain

$$\begin{aligned} P_j f(2^{-j}a) &= \int_{a-2^j}^{a} \frac{\sin(\pi u)}{\pi u} du - \int_{a}^{a+2^j} \frac{\sin(\pi u)}{\pi u} du \\ &\quad - \int_{-a+2^j}^{a} \frac{\sin(\pi u)}{\pi u} du + \int_{-a+2^j}^{a} \frac{\sin(\pi u)}{\pi u} du. \end{aligned}$$

Combining the second and the third integral we have

$$\begin{aligned} P_j f(2^{-j}a) &= \int_{a-2^j}^{a} \frac{\sin(\pi u)}{\pi u} du + \int_{-a+2^j}^{a} \frac{\sin(\pi u)}{\pi u} du - \int_{-a+2^j}^{a+2^j} \frac{\sin(\pi u)}{\pi u} du \\ &= A + B + C. \end{aligned}$$

Now, if $a - 2^j < -a$ we can break A into two integrals as follows

$$\begin{aligned} P_j f(2^{-j}a) &= \int_{-a}^{a} \frac{\sin(\pi u)}{\pi u} du + \int_{a-2^j}^{-a} \frac{\sin(\pi u)}{\pi u} du + \int_{-a+2^j}^{a} \frac{\sin(\pi u)}{\pi u} du \\ &\quad - \int_{-a+2^j}^{a+2^j} \frac{\sin(\pi u)}{\pi u} du \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

Since $\dfrac{\sin(\pi u)}{\pi u}$ is and even function we can write I_3 as $-\int_{a-2^j}^{-a} \frac{\sin(\pi u)}{\pi u} du$. Therefore, $I_2 + I_3 = 0$ and we obtain

$$P_j f(2^{-j}a) = \int_{-a}^{a} \frac{\sin(\pi u)}{\pi u} du - \int_{-a+2^j}^{a+2^j} \frac{\sin(\pi u)}{\pi u} du.$$

Similarly, if $-a < a - 2^j$ we can break B into two integrals as

$$\begin{aligned} P_j f(2^{-j}a) &= \int_{a-2^j}^{a} \frac{\sin(\pi u)}{\pi u} du + \int_{-a+2^j}^{-a} \frac{\sin(\pi u)}{\pi u} du + \int_{-a}^{a} \frac{\sin(\pi u)}{\pi u} du \\ &\quad - \int_{-a+2^j}^{a+2^j} \frac{\sin(\pi u)}{\pi u} du \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

With a similar argument as before $I_1 + I_2 = 0$ and we obtain again

$$P_j f(2^{-j}a) = \int_{-a}^{a} \frac{\sin(\pi u)}{\pi u} du - \int_{-a+2^j}^{a+2^j} \frac{\sin(\pi u)}{\pi u} du.$$

Now, $\int_{-a+2^j}^{a+2^j} \frac{\sin(\pi u)}{\pi u} du \to 0$ as $j \to \infty$. Thus we have

$$\begin{aligned} \lim_{j\to\infty} P_j f(2^{-j}a) &= \int_{-a}^{a} \frac{\sin(\pi u)}{\pi u} du \\ &= 2\int_{0}^{a} \frac{\sin(\pi u)}{\pi u} du \quad \text{since the integrand is an even function.} \end{aligned}$$

The change of variables $\theta = \pi u$ will yield

$$\begin{aligned} \lim_{j\to\infty} P_j f(2^{-j}a) &= \frac{2}{\pi} \int_{0}^{\pi a} \frac{\sin(\theta)}{\theta} d\theta \\ &= \frac{2}{\pi} \mathrm{Si}(\pi a). \end{aligned}$$

If we take $a = 1$ then

$$\begin{aligned}\lim_{j\to\infty} P_j f(2^{-j}) &= \frac{2}{\pi}\int_0^{\pi}\frac{\sin(\theta)}{\theta}d\theta \\ &= \frac{2}{\pi}\mathrm{Si}(\pi) \\ &= \frac{2}{\pi}\frac{\pi}{2}(1.17898) \\ &= 1.17898 > \lim_{x\to 0^+} f(x)\end{aligned}$$

Figures 5.5.1, and 5.5.2 show the overshoot and the undershoot near 0.

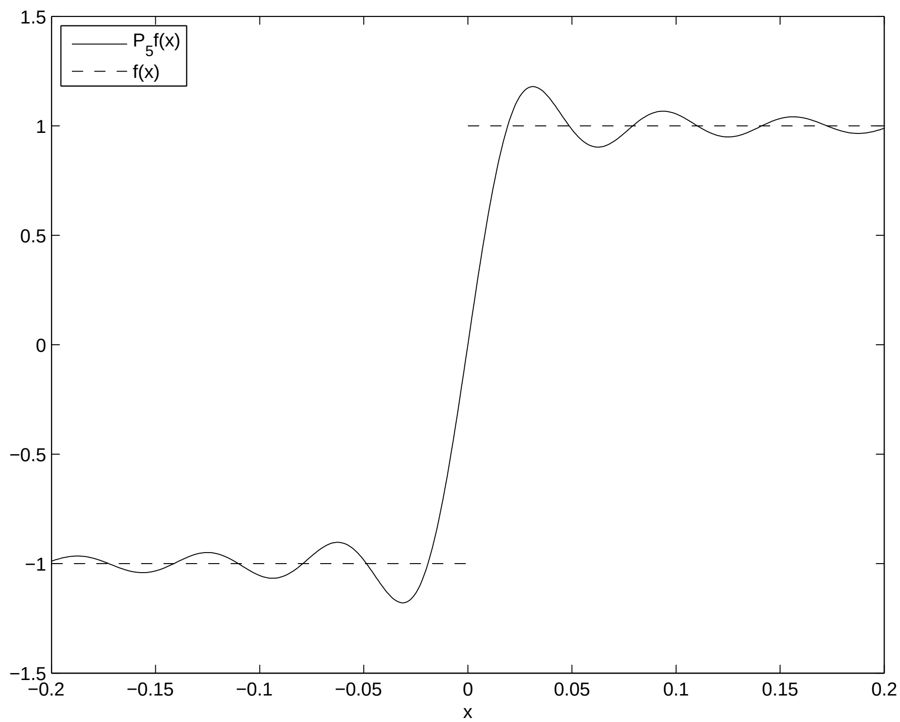

Figure 5.5.1: Graph of f(x) and its Shannon approximation for j=5

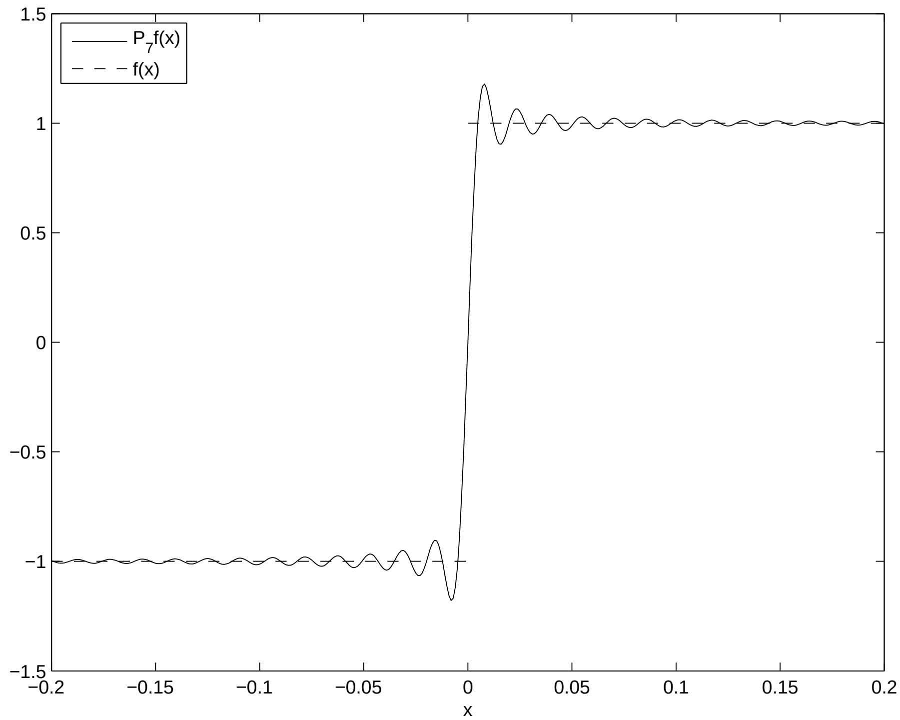

Figure 5.5.2: Graph of f(x) and its Shannon approximation for j=7

5.6 Quasi-positive and Positive Delta Sequences

In this section we introduce Quasi-positive and positive delta sequences and show that similar to the good kernels of Chapter 3 approximations using positive delta sequences do not show the Gibbs phenomenon [20], [19].

Definition 5.24. We call the sequence $\{\delta_m(x, y)\}$ of functions in $L^1(\mathbb{R})$ with parameter $x \in \mathbb{R}$ a quasi-positive delta sequence if the following conditions are satisfied:
(i) There exists a $C > 0$ such that

$$\int_{-\infty}^{\infty} |\delta_m(x, y)| dy \leq C, \text{ for all } x \in \mathbb{R},\ m \in \mathbb{N}. \tag{5.6.1}$$

(ii) There exists a $c > 0$ such that

$$\int_{x-c}^{x+c} \delta_m(x,y)dy \to 1 \text{ as } m \to \infty \tag{5.6.2}$$

uniformly on compact sets of $\mathbb{R}$.

(iii) For each $\gamma > 0$,

$$\int_{|x-y|\geq\gamma} |\delta_m(x,y)|dy \to 0 \text{ as } m \to \infty. \tag{5.6.3}$$

Definition 5.25. Let $\{\delta_m(x,y)\}$ be as in Definition 5.24. If we replace condition (i) with

(i′) For all $x, y \in \mathbb{R}$

$$\delta_m(x,y) \geq 0, \tag{5.6.4}$$

then $\{\delta_m(x,y)\}$ is called a positive delta sequence.

Remark 5.26. Every positive delta sequence is also a quasi-positive delta sequence since conditions (i′) and (iii) imply condition (i).

Theorem 5.27. *Suppose $\{\delta_m(x,y)\}$ is a positive delta sequence. Let f be a piecewise continuous function with compact support on the real line. Let $I = [a,b]$ be a finite interval with $0 < \mu < b - a/2$, and let $\varepsilon > 0$. Then*

$$m - \varepsilon \leq f_m(x) = \int \delta_m(x,y)f(y)dy \leq M + \varepsilon \text{ for } x \in [a+\mu, b-\mu],$$

where $m = \inf_{x\in I} f$, $M = \sup_{x\in I} f$.

Proof. First, notice that properties (i) and (ii) in Definition 5.24 imply that

$$\int_{\mathbb{R}} \delta_m(x,y)dy \to 1 \quad \text{uniformly for } x \in I. \tag{5.6.5}$$

Now, suppose that $suppf(x) \subseteq I$. Then we have

$$\inf_{t\in I} f(t) \int_{\mathbb{R}} \delta_m(x,y)dy \leq \int_{\mathbb{R}} \delta_m(x,y)f(y)dy,$$
$$\int_{\mathbb{R}} \delta_m(x,y)f(y)dy \leq \sup_{t\in I} f(t) \int_{\mathbb{R}} \delta_m(x,y)dy.$$

Therefore, using (5.6.5) we obtain

$$\inf_{t\in I} f(t)\int_{\mathbb{R}} \delta_m(x,y)dy \to m,$$
$$\sup_{t\in I} f(t)\int_{\mathbb{R}} \delta_m(x,y)dy \to M,$$

and the required result follows. Now, suppose that $supp f(x) \nsubseteq I$. Let $f_I(x) = f(x)\chi_I(x)$. Thus

$$\begin{aligned}\int_{\mathbb{R}} \delta_m(x,y)f(y)dy &= \int_{\mathbb{R}} \delta_m(x,y)(f(y)-f_I(y))dy + \int_{\mathbb{R}} \delta_m(x,y)f_I(y)dy\\ &= A+B.\end{aligned}$$

Now, we can write A as

$$\begin{aligned}A &= \int_{\mathbb{R}} \delta_m(x,y)f(y)dy - \int_{\mathbb{R}} \delta_m(x,y)f_I(y)dy\\ &= \int_{\mathbb{R}} \delta_m(x,y)f(y)dy - \int_{\mathbb{R}} \delta_m(x,y)f(y)\chi_I(y)dy\\ &= \int_{\mathbb{R}} \delta_m(x,y)f(y)dy - \int_a^b \delta_m(x,y)f(y)dy\\ &= \int_{-\infty}^a \delta_m(x,y)f(y)dy + \int_b^\infty \delta_m(x,y)f(y)dy. \qquad (5.6.6)\end{aligned}$$

Now, for $x \in [a+\mu, b-\mu]$ we have $b-x \geq \mu$ and $x-a \geq \mu$. Thus, by property (ii) of positive delta sequences (5.6.6) converges to zero uniformly in I. Now, we choose $N \in \mathbb{N}$ so that (5.6.6) is less then $\varepsilon/2$ and that

$$\left|\int_I \delta_m(x,y)dy - 1\right| < \frac{\varepsilon}{2M}.$$

Then we obtain

$$
\begin{aligned}
A + B &\leq \varepsilon/2 + \int_{\mathbb{R}} \delta_m(x,y) f_I(y) dy \\
&= \varepsilon/2 + \int_{\mathbb{R}} \delta_m(x,y) f(y) \chi_I(y) dy \\
&= \varepsilon/2 + \int_a^b \delta_m(x,y) f(y) dy \\
&\leq \varepsilon/2 + \sup_{t\in[a,b]} f(t) \int_a^b \delta_m(x,y) dy \\
&\leq \varepsilon/2 + M(\frac{\varepsilon}{2M} + 1) \\
&= M + \varepsilon.
\end{aligned}
$$

The other inequality can be proved in a similar way. □

5.7 Revisiting the Haar MRA

To prove the absence of the Gibbs phenomenon in the Haar wavelet we can show that $K_j(x,y)$, the reproducing kernel associated to the Haar scaling function, is a positive delta sequence [16], and then apply Theorem 5.27. The Haar scaling function is given by $\varphi(x) = \chi_{[0,1)}(x)$. Then $K_0(x,y)$, the reproducing kernel of V_0 is

$$
\begin{aligned}
K_0(x,y) &= \sum_{k\in\mathbb{Z}} \varphi(x-k)\varphi(y-k) \\
&= \varphi(x - [y]) \\
&= K_0(y,x) \\
&= \varphi(y - [x]),
\end{aligned}
$$

where $[x]$ is the largest integer less than or equal to x. Then by (5.3.3) we have

$$
\begin{aligned}
K_j(x,y) &= 2^j K_0(2^j x, 2^j y) \\
&= 2^j \varphi(2^j y - [2^j x]).
\end{aligned}
$$

It is clear that $K_j(x,y) \geq 0$. Now, we need to show (5.6.2) and (5.6.3). For property (ii), observe that

$$\int_{-\infty}^{\infty} K_j(x,y)dy = \int_{-\infty}^{\infty} 2^j \varphi(2^j y - [2^j x])dy.$$

By the change of variables $u = 2^j y - [2^j x]$ we obtain

$$\begin{aligned}\int_{-\infty}^{\infty} K_j(x,y)dy &= \int_{-\infty}^{\infty} \varphi(u)du \\ &= \int_0^1 \varphi(u)du \\ &= 1.\end{aligned}$$

As for property (iii), for $\gamma > 0$ we have

$$\int_{|x-y|\geq\gamma} |K_j(x,y)| = \int_{|x-y|\geq\gamma} \varphi(2^j y - [2^j x])dy.$$

Now, by the change of variable $u = 2^j y$ we obtain

$$\begin{aligned}\int_{|x-y|\geq\gamma} \varphi(2^j y - [2^j x])dy &= \int_{|u/2^j - x|\geq\gamma} \varphi(u - [2^j x])du \\ &= \int_{|u-2^j x|\geq 2^j\gamma} \varphi(u - [2^j x])du.\end{aligned}$$

This last integral converges to zero as $j \to \infty$ since for 2^j sufficiently large we can make $2^j\gamma > 1$ which results in $\varphi(u - [2^j x]) = 0$.

Therefore, $K_j(x,y)$ is a positive delta sequence. Hence, by Theorem 5.27 the Haar system does not exhibit the Gibbs phenomenon.

Chapter 6

Conclusion

In this thesis, we showed the existence of the Gibbs phenomenon for two simple functions both having a jump discontinuity at zero. We also calculated the size of the overshoot near the jump for both functions using two different approaches. Using a convergence result and one of these functions, we showed that any piecewise smooth function with a jump discontinuity at zero exhibits the Gibbs phenomenon. After that, we did the generalization to a jump discontinuity at a general point $x = x_\circ$, and to a finite number of jump discontinuities. In other words, we showed that any piecewise smooth function with a finite number of jump discontinuities exhibits the Gibbs phenomenon at each of those discontinuities. Next, we showed that if in approximating a discontinuous function f, where the discontinuity is a jump one, we use a summation method which is the convolution of f with a positive kernel then we will not see the Gibbs phenomenon. Then, we gave a criterion for the existence of the Gibbs phenomenon in wavelet approximations to a function with a jump discontinuity at 0. This criterion can be used for those scaling functions that are C^r and rapidly decreasing and involves the reproducing kernel $K_j(x, y)$. After that, we showed the existence of the Gibbs phenomenon near zero for a certain class of wavelets. More precisely, we showed that if the scaling function $\varphi(x)$ corresponding to our wavelet system is continuous, has a non-vanishing derivative at a dyadic number, and

$$|\varphi(x)| \leq \frac{C}{(1+|x|)^{\beta}} \quad \text{for } x \in \mathbb{R},\, C > 0, \text{ and } \beta > 3,$$

then the corresponding wavelet approximation to f shows the Gibbs phenomenon on the right and/or left of 0. Next, we examined the Shannon wavelets and showed, by direct calculation, that they exhibit the Gibbs phenomenon, as seen on the graphs provided. Finally, we proved a theorem which shows that those wavelet approximations involving positive delta sequences do not show the Gibbs phenomenon, and used this result to show the absence of the phenomenon in Haar wavelets.

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