A Factorial Analysis of the Yield Curve in the Extended Version of Nelson-Siegel with Underlying Macroeconomic Factors

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Contents

1 Introduction

1.1 Objective

The primary objective of this study is to perform a factorial analysis of the yield curve by extending the Nelson-Siegel model with underlying macroeconomic factors. Traditional yield curve models, such as the standard Nelson-Siegel model, capture yield curve dynamics using three main components—Level, Slope, and Curvature. While these components help explain the general structure of interest rates across maturities, they do not account for external macroeconomic influences that can impact the shape and movement of the yield curve over time. This study seeks to incorporate macroeconomic factors like GDP growth, CPI inflation, and Unemployment into the Nelson-Siegel framework to provide a more comprehensive view of the yield curve dynamics.

1.2 Importance

Understanding the yield curve is crucial for financial professionals and economists because it provides insights into future interest rate expectations, economic growth, and inflation. The yield curve, which plots yields against maturities, is often used as a predictor of economic cycles. For example, an inverted yield curve has historically been a reliable indicator of recessions. By extending the Nelson-Siegel model with macroeconomic factors, this study aims to enhance our understanding of how economic conditions influence interest rates across different maturities. This can be useful for bond pricing, portfolio management, and risk assessment in financial markets.

1.3 Literature Review

The Nelson-Siegel model was originally introduced by Nelson and Siegel (1987) and later extended by Diebold and Li (2006), who demonstrated that the Level, Slope, and Curvature factors can effectively capture the term structure of interest rates. Recent studies have explored further extensions of the Nelson-Siegel model, incorporating it into a state-space framework and adding macroeconomic factors to explain the yield curve's behavior better. Diebold, Rudebusch, and Aruoba (2006) provided a foundational study that integrated macroeconomic variables into the yield curve model, establishing a link between the yield curve and economic fundamentals such as inflation and output growth. This study builds on that foundation by incorporating updated macroeconomic factors and adapting the model for modern datasets.

2 Theoretical Background

2.1 Nelson-Siegel Model

The *Nelson-Siegel (NS) model* is a widely-used approach for modeling the term structure of interest rates. It approximates the yield curve using three factors: **Level**, **Slope**, and **Curvature**, which correspond to different maturity-related characteristics of the curve. The model assumes that the instantaneous forward rate at time t and maturity τ can be represented by a function with the following form:

$$f(t,\tau) = \beta_1 + \beta_2 \left(\frac{1 - e^{-\tau/\lambda}}{\tau/\lambda} \right) + \beta_3 \left(\frac{1 - e^{-\tau/\lambda}}{\tau/\lambda} - e^{-\tau/\lambda} \right) \tag{1}$$

where:

- β_1 represents the **Level** factor, capturing the long-term rate component.
- β_2 represents the **Slope** factor, describing the short- to long-term rate spread.
- β_3 represents the **Curvature** factor, modeling the mid-term hump in the yield curve.
- λ is a decay parameter that controls the exponential decay of the Slope and Curvature effects over maturity.

In this model:

- **Level** β_1 represents the overall interest rate level and remains relatively constant across maturities.
- **Slope** β_2 affects the short end of the yield curve, influencing the steepness between short and long maturities.
- **Curvature** β_3 is more prominent at intermediate maturities, capturing the shape of the yield curve at the mid-section.

The NS model is known for its simplicity and flexibility, providing a robust framework to model yield curves across different economic regimes.

2.2 Extension to a State-Space Model

The state-space model formulation of the Nelson-Siegel approach enables the incorporation of time-varying dynamics into the Level, Slope, and Curvature factors. In a state-space framework, the factors are treated as latent (unobserved) variables that evolve over time following certain stochastic processes. This formulation allows for the following components:

- State Equation: Describes the time evolution of the latent factors (Level, Slope, Curvature) as autoregressive (AR(1)) processes.
- Observation Equation: Connects the observed yields to the latent factors through a design matrix derived from the NS model equation.

$$y_t = Z\alpha_t + \varepsilon_t \tag{2}$$

$$\alpha_{t+1} = T\alpha_t + R\eta_t \tag{3}$$

where:

- y_t represents the observed yields at time t for different maturities.
- α_t is a vector of latent states (Level, Slope, Curvature) at time t
- Z is the design matrix derived from the NS yield equation
- T is the transition matrix that captures the dynamics of the latent states
- ε_t and η_t are error terms representing observation and state disturbances, respectively

he state-space model allows for dynamic estimation of yield curve factors, which is particularly useful for studying their behavior in response to changes in macroeconomic conditions.

2.3 Introduction of Macroeconomic Factors

Adding macroeconomic factors to the Nelson-Siegel framework allows for a more comprehensive understanding of yield curve dynamics, as it enables the model to incorporate economic fundamentals that drive interest rates. In this extended model, we include:

- **GDP Growth**: A measure of economic activity, which typically influences the slope of the yield curve. Higher GDP growth is often associated with higher interest rates at the short end, as central banks may tighten monetary policy.
- **CPI Inflation**: This factor captures inflationary pressures, which generally impact the Level factor. When inflation rises, long-term interest rates tend to increase as investors demand higher yields to offset anticipated inflation.
- **Unemployment**: Often viewed as a lagging indicator of economic cycles, unemployment can influence both slope and curvature, as it reflects underlying labor market conditions and consumer confidence.

These macroeconomic factors are incorporated into the state-space model as exogenous variables that impact the latent states over time, providing a more detailed view of how economic conditions affect the yield curve.

3 Data Collection and Preparation

3.1 Data Sources and Variables

For this analysis, we utilized four distinct datasets focused on yield curve data and macroeconomic variables from quarterly records. Each dataset was imported, with relevant details on its source and composition listed in the initial sheets of each file. **All data are about France, from 2004 to 2023.** Here's a summary of the datasets and their variables:

- **Yield Curve Data**:
  - **Source:** Based on **Banque de France** statistics, this dataset includes yields across maturities
  - **Variables:** the following maturities: **1** year, **2** year, **3** year, **5** year, **7** year, **10** year, **15** year, **20** year, **25** year, **30** year. The data is recorded quarterly.
- **GDP Data**:
  - **Source:** Collected from the *comptes nationaux trimestriels* of the **Insee**, detailing growth rates each quarter.
  - **Variables:** Quarterly GDP growth rate, indicating economic activity levels over time.
- **CPI Data**:
  - **Source:** Derived from consumer price index records from *Federal Reserve Economic Data* of **Federal Reserve Bank of St. Louis** , reflecting inflation trends.
  - **Variables:** The original CPI values were transformed into log-differentiated rates to represent the quarter-over-quarter inflation rate.
- **Unemployment Data**:
  - **Source:** Unemployment statistics collected at quarterly intervals from the *enquêtes Emploi* of the **Insee**'.
  - **Variables:** The original CPI values were transformed into log-differentiated rates to represent the quarter-over-quarter inflation rate.

3.2 Data Transformation and Preparation

Each dataset required careful transformation to align with the objectives of the study:

- **Quarterly Index Conversion:** The DATE column in each dataset was converted into a quarterly period index to ensure consistency. This was crucial for time series alignment during the merging process.

- **Log Transformation for CPI:** The CPI data was transformed using a log difference approach:

$$\text{CPI\_QoQ\_LogChange} = \log\left(\frac{\text{CPI}_t}{\text{CPI}_{t-1}}\right) \tag{4}$$

  This transformation captures the quarterly change in inflation while smoothing out extreme values, adhering to best practices in macroeconomic analysiserging Datasets**: All datasets were merged on the DATE index to produce a single, unified dataframe that aligns yield curve data with macroeconomic factors. Rows with missing values were dropped to ensure continuity in the dataset.

3.3 Exploratory Data Analysis

Before conducting the main analysis, we reviewed summary statistics for each variable and plotted initial trends over time. Key observations included:

- **GDP Growth**: Displayed cyclical trends aligned with economic expansions and contractions.
- **CPI Inflation**: Showed periodic fluctuations, with spikes during periods of increased economic stress or policy changes.
- **Unemployment**: Inversely correlated with GDP growth, peaking during economic downturns and dropping during expansions.

	GDP Growth	CPI Inflation (Log Change)	Unemployment
count	66.000000	66.000000	66.000000
mean	0.265023	0.003779	8.913636
std	2.625700	0.005021	1.087536
min	-12.318700	-0.005028	7.100000
25%	0.037536	-0.000289	7.925000
50%	0.274305	0.002990	9.200000
75%	0.619378	0.006985	9.850000
max	15.662327	0.015492	10.500000

3.4 Visualizing Macroeconomic Trends

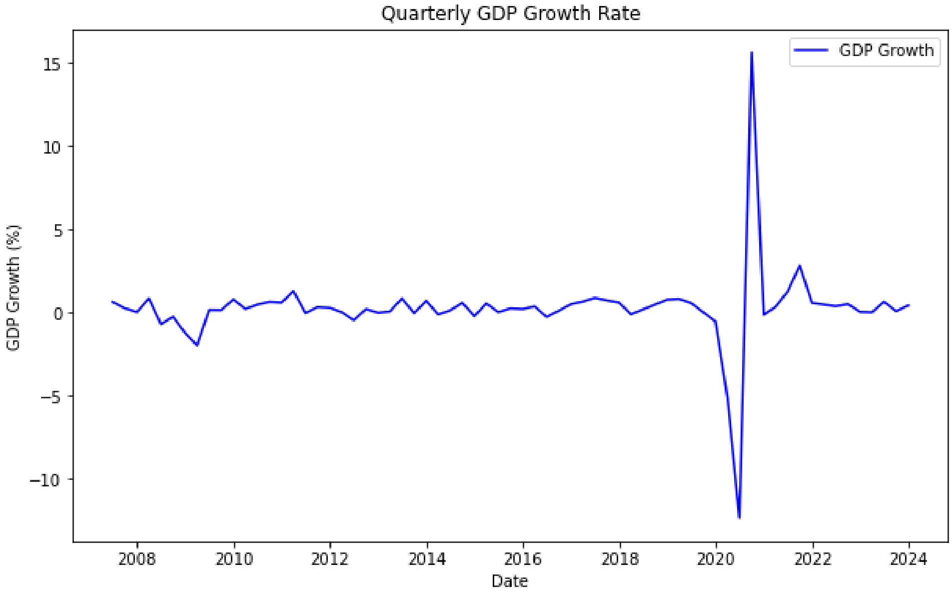

Figure 1: Quarterly GDP Growth Rate

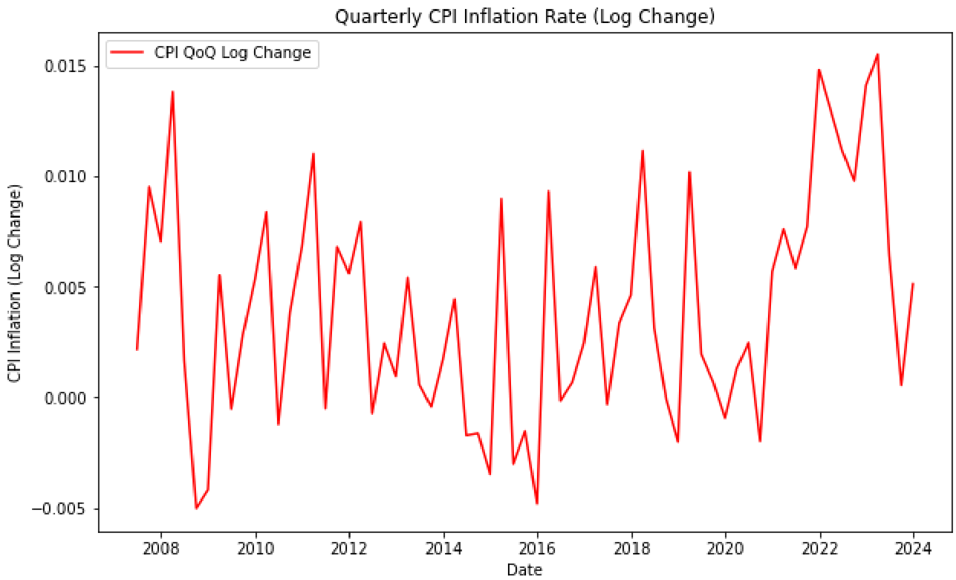

Figure 2: Quarterly CPI Inflation Rate (Log Change)

3.5 Analysis

Observations from the Macroeconomic Plots:

Based on Figures 1, 2, and 3, we observe the following:

- **Quarterly GDP Growth Rate:** The GDP growth rate shows significant volatility, particularly around 2020, with a sharp drop followed by a rapid rebound. This trend aligns with the economic disruptions caused by the COVID-19 pandemic, indicating a sudden contraction and a quick recovery. Prior to 2020, GDP growth remains relatively stable with minor fluctuations indicative of normal economic cycles.

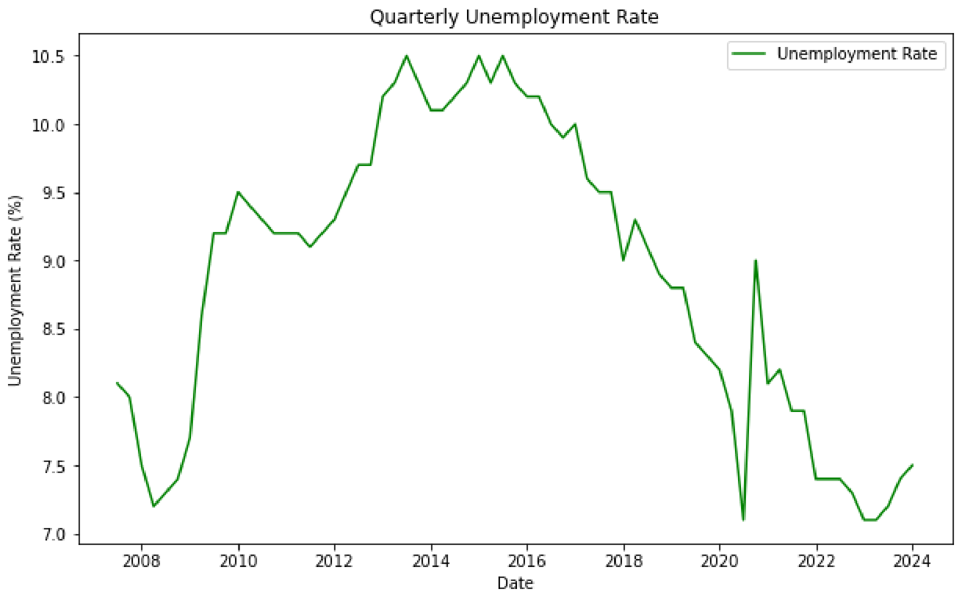

Figure 3: Quarterly Unemployment Rate

- **Quarterly CPI Inflation Rate (Log Change):** The CPI inflation rate displays regular fluctuations with periodic spikes, suggesting inflationary pressures, especially in the recent years. The rise in inflation around 2021-2022 is likely attributed to economic policies and supply chain disruptions following COVID-19, which impacted price stability worldwide.

- **Quarterly Unemployment Rate:** The unemployment rate generally follows an inverse pattern to GDP growth, with higher unemployment rates occurring during economic downturns (e.g., 2020). There is a visible decline in unemployment in recent years, indicating a recovery in labor market conditions after 2020.

Summary of Findings:

- **Cyclical Trends:** The GDP growth and unemployment plots reflect typical cyclical economic behavior, with GDP and unemployment moving inversely.

- **Inflation Fluctuations:** The CPI plot shows inflationary pressures that correspond to economic disruptions, particularly post-2020.

- **Interrelations:** The unemployment rate typically rises when GDP declines, supporting the expected relationship between economic output and labor market health.

These observations highlight the impact of macroeconomic factors on the yield curve, which provides context for understanding how economic fundamentals influence interest rate dynamics across maturities.

4 Methodology

4.1 Nelson-Siegel Model for Yield Curve Fitting

The Nelson-Siegel model provides a powerful framework for representing the term structure of interest rates. For each quarter in the dataset, we estimated the Nelson-Siegel parameters—Level β_1, Slope β_2, and Curvature β_3 to capture the yield curve's structure over time :

$$y(t,\tau) = \beta_1(t) + \beta_2(t)\frac{1-e^{-\tau/\lambda}}{\tau/\lambda} + \beta_3(t)\left(\frac{1-e^{-\tau/\lambda}}{\tau/\lambda} - e^{-\tau/\lambda}\right) \tag{5}$$

where:

- $\beta_1(t)$ is the **Level** factor, representing the long-term interest rate.
- $\beta_2(t)$ is the **Slope** factor, which represents the spread between short-term and long-term yields.
- $\beta_3(t)$ is the **Curvature** factor, capturing the medium-term shape of the yield curve.
- λ is a decay parameter that controls the rate at which the Slope and Curvature effects diminish with maturity.
- **Parameter Estimation**: For each time point in the yield data, we estimated these parameters by minimizing the sum of squared errors between observed and fitted yields across maturities. The optimization process aims to find the best fit of the model to the observed data, enabling us to capture the dynamic behavior of the yield curve over time.
- **Interpretation of Parameters**:
  - **Level** (β_1): Reflects the general interest rate level across all maturities. Higher values suggest a high-yield environment.
  - **Slope** (β_2): Measures the difference between short-term and long-term yields. Positive values indicate a steep curve, typically associated with economic expansion.
  - **Curvature** (β_3): Captures the mid-term hump of the yield curve, showing market sentiment toward medium-term risks and policy.

4.2 Objective Function for Parameter Estimation

To fit the Nelson-Siegel model to observed yields at different maturities, we define an objective function that minimizes the sum of squared errors between the observed yields and the fitted yields. Given the parameter vector $\theta = (\beta_1, \beta_2, \beta_3, \lambda)$, the objective function is:

$$\text{Objective Function} = \sum_{i=1}^{N} \left(y_{\text{observed}}(t,\tau_i) - y_{\text{fitted}}(t,\tau_i;\theta)\right)^2 \tag{6}$$

where:

- $y_{\text{observed}}(t, \tau_i)$ is the observed yield for maturity τ_i at time t.
- $y_{\text{fitted}}(t, \tau_i; \theta)$ is the yield fitted by the Nelson-Siegel model for maturity τ_i at time t using parameters θ.

This optimization is performed for each time point to obtain time-varying estimates of β_1, β_2, β_3, and λ.

4.3 Principal Component Analysis (PCA) on Yield Curve Changes

Principal Component Analysis (PCA) is a dimensionality reduction technique that helps validate the structure of the yield curve by identifying the principal components that explain the most variance in the yields. This complements the Nelson-Siegel model by providing a statistical basis for the Level, Slope, and Curvature factors.To analyze the principal components of yield curve changes, we perform PCA on the differenced yield data. This transformation focuses on changes in yields and helps us identify the main components driving yield variations.

1. **Data Preparation**: The yield data is differenced to focus on yield changes, producing a stationary series:

$$\Delta y(t, \tau) = y(t, \tau) - y(t-1, \tau) \tag{7}$$

2. **PCA Formulation**: Given a matrix Y of yield changes, we perform PCA to decompose Y into principal components. The decomposition is:

$$Y = W \Sigma V^{\top} \tag{8}$$

where:

- W is the matrix of principal component scores.
- Σ is the diagonal matrix of singular values.
- V is the matrix of principal component loadings.

3. **Interpretation of Principal Components**: The first few principal components explain most of the variance in yield changes:

- **PC1 (Level)**: Explains the overall level shift in the yield curve.
- **PC2 (Slope)**: Reflects the steepness of the yield curve, with positive loadings at shorter maturities and negative at longer maturities.
- **PC3 (Curvature)**: Captures the curvature or hump in the middle of the yield curve.

4.4 State-Space Model with Macroeconomic Factors

To extend the Nelson-Siegel model, we incorporate macroeconomic factors into a state-space model. This approach allows the Level, Slope, and Curvature factors to evolve dynamically over time, influenced by macroeconomic variables and so to capture the dynamic nature of the yield curve and its dependence on economic conditions, we extended the Nelson-Siegel model into a state-space model with macroeconomic factors

1. **State Equation**: The latent factors $\alpha_t = [\beta_1(t), \beta_2(t), \beta_3(t)]^\top$ evolve according to an autoregressive process, incorporating macroeconomic variables as external inputs:

$$\alpha_{t+1} = T\alpha_t + BX_t + \eta_t \tag{9}$$

where:

- T is the transition matrix, defining the AR(1) dynamics of each factor.
- B is the matrix of coefficients linking macroeconomic variables X_t (such as GDP growth, CPI inflation, and unemployment) to the latent factors.
- η_t is the state disturbance with covariance matrix Q.

2. **Observation Equation**: The observed yields at different maturities are modeled as linear functions of the latent factors:

$$y_t = Z\alpha_t + \varepsilon_t \tag{10}$$

where:

- Z is the design matrix derived from the Nelson-Siegel equation, mapping the latent factors to observed yields.
- ε_t is the observation noise with covariance matrix R.

3. **Kalman Filter for Estimation**: The Kalman filter is used to estimate the latent factors α_t over time, providing an optimal estimate of the state given the observed data. The filter computes the conditional mean and covariance of α_t recursively as follows:

$$\begin{aligned}
\text{Prediction Step:} \quad & \hat{\alpha}_{t|t-1} = T\hat{\alpha}_{t-1|t-1} + BX_t && (11)\\
& P_{t|t-1} = TP_{t-1|t-1}T^\top + Q && (12)\\
\text{Update Step:} \quad & K_t = P_{t|t-1}Z^\top(ZP_{t|t-1}Z^\top + R)^{-1} && (13)\\
& \hat{\alpha}_{t|t} = \hat{\alpha}_{t|t-1} + K_t(y_t - Z\hat{\alpha}_{t|t-1}) && (14)\\
& P_{t|t} = (I - K_tZ)P_{t|t-1} && (15)
\end{aligned}$$

where:

- K_t is the Kalman gain.

- $\hat{\alpha}_{t|t}$ and $P_{t|t}$ are the updated state estimate and covariance after incorporating the observation y_t.

- $\hat{\alpha}_{t|t-1}$ and $P_{t|t-1}$ are the predicted state estimate and covariance prior to the observation.

Interpretation of State-Space Parameters:

- **AR(1) Transition Parameters**: Reflect the persistence of Level, Slope, and Curvature over time, indicating how long-lasting these factors are.

- **Macroeconomic Effects**: External factors contribute to state transitions, providing insights into how economic shocks influence the yield curve dynamics.

This methodology captures the dynamic evolution of the yield curve, allowing for the integration of macroeconomic factors into the Nelson-Siegel framework, enhancing the model's ability to describe yield curve movements in response to economic conditions.

5 Estimation and Results

5.1 Nelson-Siegel Parameter Estimation

For each quarter in the dataset, we estimated the parameters of the Nelson-Siegel model: **Level** (β_1), **Slope** (β_2), **Curvature** (β_3), and the decay factor λ. The parameters were fitted by minimizing the sum of squared errors between the observed and modeled yields across selected maturities.

The following table shows the estimated parameters for a sample of quarters from 2007 to 2011, capturing varying economic conditions over time:

Table 1: Nelson-Siegel Parameters from 2007 to 2011

Date	**Level** (β_1)	**Slope** (β_2)	**Curvature** (β_3)	**Decay Factor** (λ)
2007-03-31	4.41	-0.28	-0.85	2.99
2007-06-30	4.83	-0.41	-0.24	3.93
2008-09-30	5.01	-1.35	-2.06	1.88
2009-03-31	4.62	-3.83	-4.44	1.22
2009-12-31	4.71	-4.26	-3.69	1.45
2010-03-31	4.51	-3.50	-6.28	1.16
2011-03-31	4.49	-3.53	-2.84	1.30

5.2 Data Summaries and Trends

- **Analysis of Yield Curve Trends**
  - **Yield Curve Across Selected Maturities Over Time**

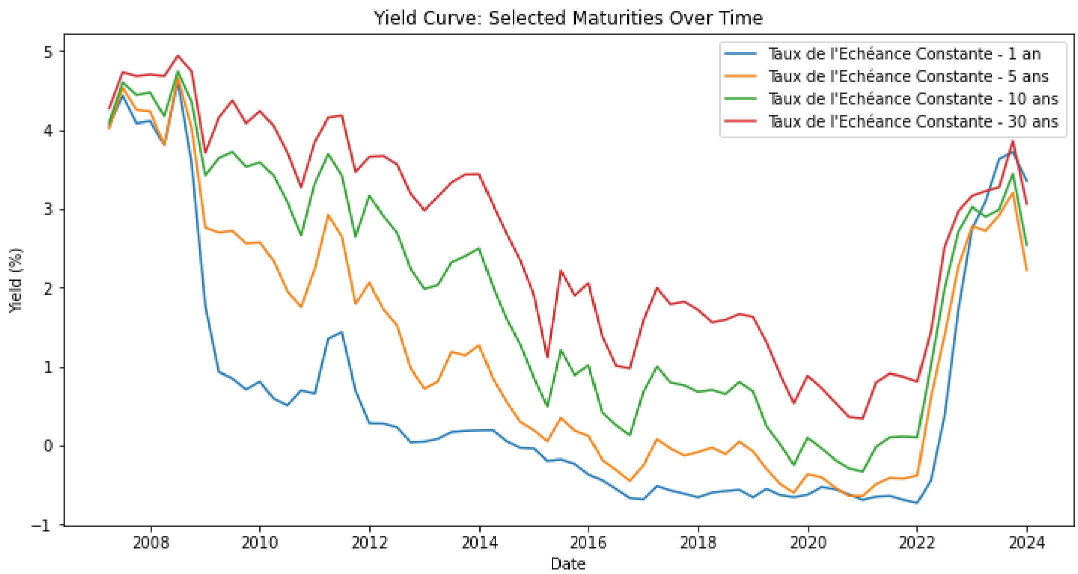

Figure 4: Yield Curve Across Selected Maturities Over Time

In Figure 4, we observe the yield for four key maturities (1-year, 5-year, 10-year, and 30-year) from 2007 to 2024.

- **General Downward Trend:** All maturities exhibit a general downward trend from 2008 through around 2021, reflecting the broader trend of declining interest rates globally. This prolonged decrease likely corresponds to various factors, including accommodative monetary policies by central banks post-2008 financial crisis and ongoing low inflation pressures.
- **COVID-19 Spike and Recovery:** In 2020, a sharp drop occurs across all maturities, followed by a sudden increase as economies began recovering from the COVID-19 pandemic. The rate hikes by central banks to counter inflationary pressures in the post-pandemic period are evident in this rapid increase around 2022-2023.
- **Flattening at Long Maturities:** Longer maturities, such as the 30-year yield, display a flatter trajectory post-2015, indicating a reduced difference between short-term and long-term yields, often observed during periods of economic uncertainty or low inflation expectations.

- **Yield Curve at Different Quarters Over Time**

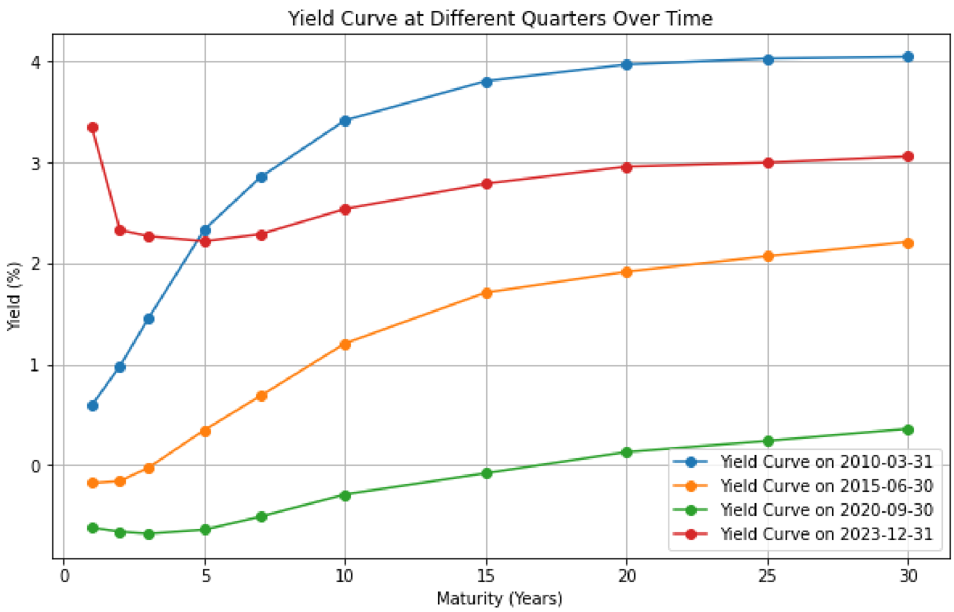

Figure 5: Yield Curve at Different Quarters Over Time

Figure 5 shows the entire yield curve for selected quarters (2010-03-31, 2015-06-30, 2020-09-30, 2023-12-31), allowing us to compare the curve's shape and steepness across different economic conditions.

- **Steep Curve in 2010:** In 2010, we see a steep yield curve, with shorter maturities (1–5 years) yielding much less than longer maturities (20–30 years). This shape is typical in economic recovery phases, where short-term rates remain low due to central bank policies, while long-term rates rise on expectations of future growth and inflation.

- **Flatter Curve in 2015:** By mid-2015, the curve has flattened, especially at the long end, as seen in the reduced yield gap between the 5-year and 30-year maturities. This flattening may reflect lower long-term growth expectations or persistent low inflation, leading to subdued long-term yields.

- **Inverted Curve in 2020:** The yield curve on 2020-09-30 (shown in green) is almost inverted at the short end, particularly between the 1-year and 3-year maturities. This inversion is often viewed as a recession indicator, as investors expect future economic conditions to worsen, prompting central banks to lower short-term rates.

- **Upward Shift in 2023:** By 2023, the entire curve shifts upwards, with relatively higher yields across all maturities. This rise reflects the increased inflationary pressures post-COVID and subsequent rate hikes by central banks. The yield curve shape remains upward-sloping but less steep, suggesting that while economic conditions have improved, expectations for long-term growth remain moderate.

- **Summary of Observations**
  - **Economic Recovery and Recession Indicators:** The steepness or inversion of the yield curve at different times reflects investor sentiment regarding future economic conditions. A steep curve (2010) often signifies recovery, while a flatter or inverted curve (2020) signals concerns over economic slowdown.
  - **Impact of Monetary Policy:** Central banks' responses to economic crises (e.g., post-2008 and post-2020) are evident in these yield curve movements, where low short-term rates and a flat long end indicate prolonged low-interest policies.
  - **Inflation and Yield Curve Shifts:** The 2023 upward shift demonstrates the curve's sensitivity to inflation expectations, showing a broad rise across maturities as inflationary pressures necessitated interest rate hikes.

These yield curve patterns align with economic theory, illustrating how interest rates across maturities reflect macroeconomic expectations and central bank policies over time.

5.2.1 Interpretation of Nelson-Siegel Parameters

The estimated Nelson-Siegel parameters (β_1, β_2, and β_3) reveal significant insights into the yield curve's behavior over time, as depicted in Figure 6. The results illustrate notable trends in each parameter over time, particularly in response to macroeconomic conditions:

- **Level Factor (β_1)**: This factor, representing the general yield level, remains relatively stable but shows a slight increase during periods of economic stress (e.g., 2008-2009). Higher values for β_1 suggest an environment with generally elevated interest rates, which can be attributed to uncertainty and higher risk premiums demanded by investors. The 1 factor, which represents the long-term interest rate level, demonstrates a downward trend from 2007 to 2020, reflecting a prolonged period of low interest rates following the 2008 financial crisis. This trend reverses post-2020, with 1 showing an increase, likely in response to inflationary pressures and central bank rate hikes aimed at countering recent economic instability.

- **Slope Factor (β_2)**: The Slope factor, which measures the spread between short- and long-term yields, becomes increasingly negative during periods of economic downturn, such as the 2008 financial crisis. This indicates a steepening yield curve, where short-term rates are suppressed in response to monetary easing, while long-term rates remain relatively elevated. The 2 factor, indicating the slope of the yield curve, remains largely negative from 2008 onwards, a period marked by economic stagnation and accommodative monetary policies. The recent sharp rise in 2 signals a steepening yield curve, suggesting market expectations for economic recovery and potentially higher short-term rates as a response to inflation.

- **Curvature Factor (β_3)**: The Curvature factor fluctuates significantly, with sharp declines and increased negative values around times of economic stress. This behavior suggests an emphasis on medium-term maturities, likely as a response to shifting expectations regarding monetary policy and economic recovery. The 3 factor, capturing the curvature of the yield curve, shows heightened volatility during economic crises, especially around 2008 and 2020. Peaks in 3 indicate a pronounced hump in the mid-term maturities, reflecting uncertainty over medium-term economic conditions. The recent decrease in 3 suggests a more stable medium-term outlook as the market adjusts post-COVID

- **Decay Factor (λ)**: The decay factor λ generally ranges from around 1.2 to 3.9. Lower values of λ imply that the Slope and Curvature effects diminish more quickly with maturity, which is often observed during volatile periods. For instance, during the 2008-2009 crisis, λ decreases, showing that investors' focus was more on short-term dynamics rather than long-term stability.

5.2.2 Summary of Findings

The Nelson-Siegel parameter estimates effectively capture the shifts in the yield curve's structure over time:

- **Response to Economic Conditions**: The parameter estimates from 2008 to 2009 show an immediate response to the financial crisis, with the Slope factor becoming increasingly negative and the Curvature factor showing higher volatility. This suggests a short-term focus by investors, who expect lower rates to remain due to monetary interventions.

- **Yield Curve Shape**: The behavior of the Level, Slope, and Curvature factors allows us to observe the shape of the yield curve over time, transitioning from a relatively steep, upward-sloping curve during stable periods to a flatter or even inverted curve in times of economic stress.

These parameters form the basis for analyzing yield curve changes in response to macroeconomic variables, which we explore further in the subsequent sections.

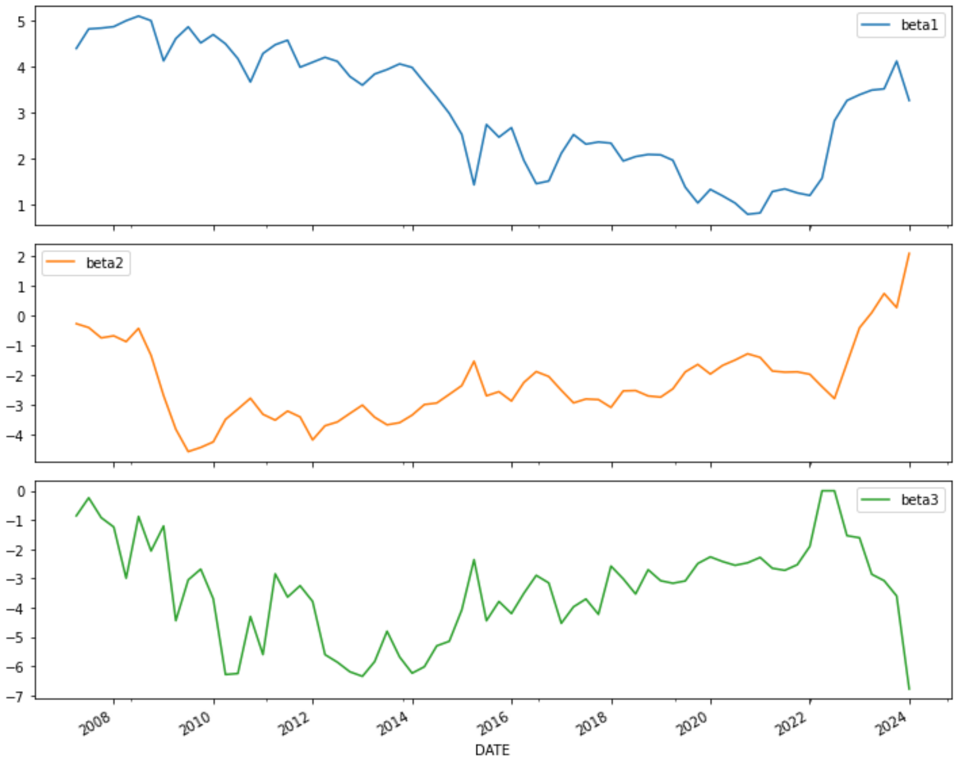

Figure 6: Time Series of Nelson-Siegel Beta Parameters (β_1, β_2, β_3)

5.3 Principal Component Analysis (PCA) of Yield Curve Changes

To further understand the yield curve structure, we performed a Principal Component Analysis (PCA) on the changes in the yield curve. The PCA identifies the main components that explain the variance in the yield curve dynamics, confirming the presence of Level, Slope, and Curvature factors. Principal Component Analysis (PCA) is performed on the yield curve data to identify the main com- ponents explaining the variance in yields across maturities. This approach helps capture the essential structure of the yield curve, primarily focusing on the Level, Slope, and Curvature factors. By analyzing both the original yield levels and the yield changes over time, we gain insights into both the static shape and dynamic shifts of the yield curve.

5.3.1 PCA on Original Yield Curve Data

We first apply PCA on the original yield curve data, represented by the matrix Y where each row corresponds to a time point and each column corresponds to a specific maturity (e.g., 1 year, 2 years, ..., 30 years). The decomposition is as follows:

$$Y = W\Sigma V^{\top} \tag{16}$$

where:

- W is the matrix of principal component scores,
- Σ is the diagonal matrix of singular values,
- V is the matrix of principal component loadings.

The principal components obtained from this analysis explain most of the variance in yield levels:

$$\begin{aligned} \text{Explained variance by component} &= [0.936, 0.060, 0.003] \\ \text{Cumulative explained variance} &= [0.936, 0.996, 0.999] \end{aligned}$$

Interpretation of Principal Components:

- **PC1 (Level)**: The first principal component has roughly equal loadings across all maturities, capturing the general level of the yield curve.
- **PC2 (Slope)**: The second principal component exhibits positive loadings at short maturities and negative loadings at long maturities, reflecting the slope of the yield curve.
- **PC3 (Curvature)**: The third principal component shows high loadings at intermediate maturities, representing the curvature or hump in the middle of the yield curve.

- **Explained Variance by Components**: The first three principal components (PCs) explain the majority of the variance in the yield curve changes. This result validates that the yield curve dynamics are primarily driven by three main factors, aligning with the Level, Slope, and Curvature factors in the Nelson-Siegel model.

$$\text{Explained Variance} = \sum_{i=1}^{k} \frac{\sigma_i^2}{\sum_{j=1}^{N} \sigma_j^2} \tag{17}$$

where σ_i^2 represents the variance explained by the i-th component, and k is the number of components chosen to explain the yield curve.

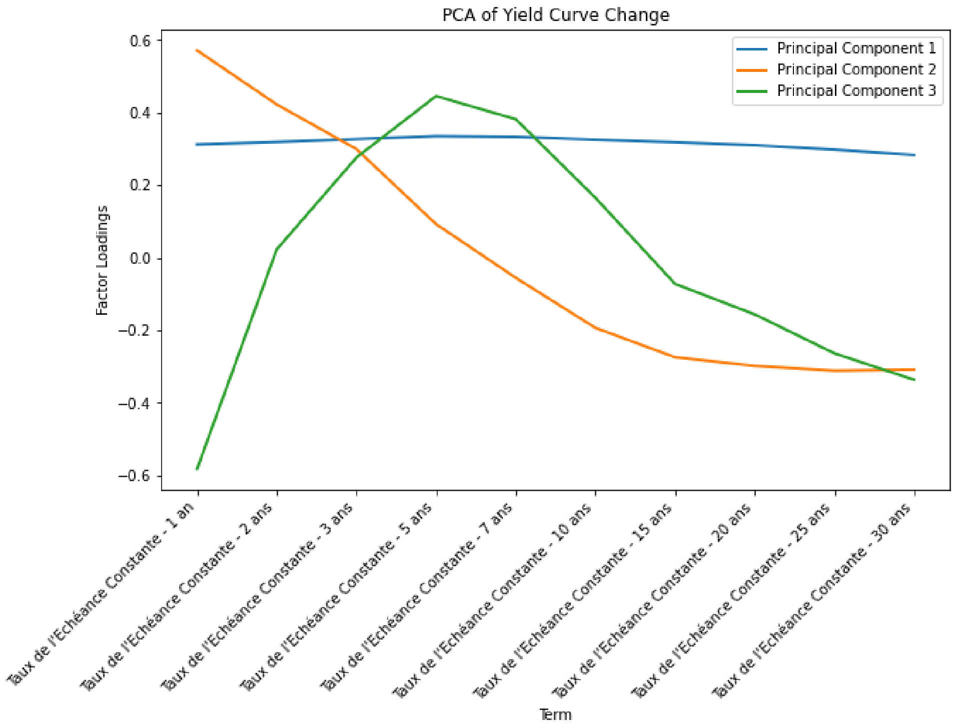

Figure 7: Factor Loadings of Principal Components on Original Yield Curve Data

- **PC1 (Level)**: Explains 93.6% of the variance, representing the general level shift across all maturities.
- **PC2 (Slope)**: Accounts for 6.04% of the variance, capturing the slope or steepness of the yield curve between short- and long-term maturities.
- **PC3 (Curvature)**: Contributes 0.3% of the variance, reflecting the curvature or hump shape at mid-term maturities.

The cumulative explained variance of these three components is 99.95%, validating that the yield curve dynamics are primarily driven by these three factors.

	Explained Variance	**Cumulative Variance**
PC1 (Level)	0.9361	0.9361
PC2 (Slope)	0.0604	0.9964
PC3 (Curvature)	0.0031	0.9995

Summary of Findings: The PCA results confirm that the Level, Slope, and Curvature factors are the primary drivers of yield curve changes, aligning closely with the Nelson-Siegel framework. PC1 captures the general level shifts across maturities, PC2 reflects the steepness of the curve, and PC3 represents the curvature, especially in the mid-maturity range. This decomposition provides a statistical basis for the importance of these three factors in explaining yield curve dynamics.

5.3.2 PCA on Differenced Yield Curve Data

To analyze the dynamics of yield curve changes over time, we perform PCA on the differenced yield curve data, denoted as ΔY, where:

$$\Delta y(t,\tau) = y(t,\tau) - y(t-1,\tau) \tag{18}$$

This approach captures the shifts in yields at each maturity from one period to the next, providing insights into the dynamics of yield curve evolution.

The PCA decomposition on the differenced data is given by:

$$\Delta Y = W_\Delta \Sigma_\Delta V_\Delta^\top \tag{19}$$

where the interpretation of W_Δ, Σ_Δ, and V_Δ is analogous to the decomposition on the original data.

The principal components for the differenced data show the following explained variance:

$$\begin{aligned}\text{Explained variance by component} &= [0.839, 0.133, 0.021]\\ \text{Cumulative explained variance} &= [0.839, 0.973, 0.994]\end{aligned}$$

Interpretation of Principal Components on Differenced Data:

- **PC1 (Level Change)**: Reflects overall shifts in the yield curve, indicating a change in the general interest rate level across maturities.
- **PC2 (Slope Change)**: Describes the steepening or flattening of the yield curve, with positive loadings on shorter maturities and negative on longer ones.
- **PC3 (Curvature Change)**: Captures dynamic changes in the curvature, especially for intermediate maturities.

Summary: Performing PCA on both the original and differenced yield curve data allows us to understand the structural factors driving the yield curve as well as the dynamic changes over time. The PCA on original data focuses on the shape of the yield curve, while PCA on differenced data highlights how this shape evolves in response to economic events.

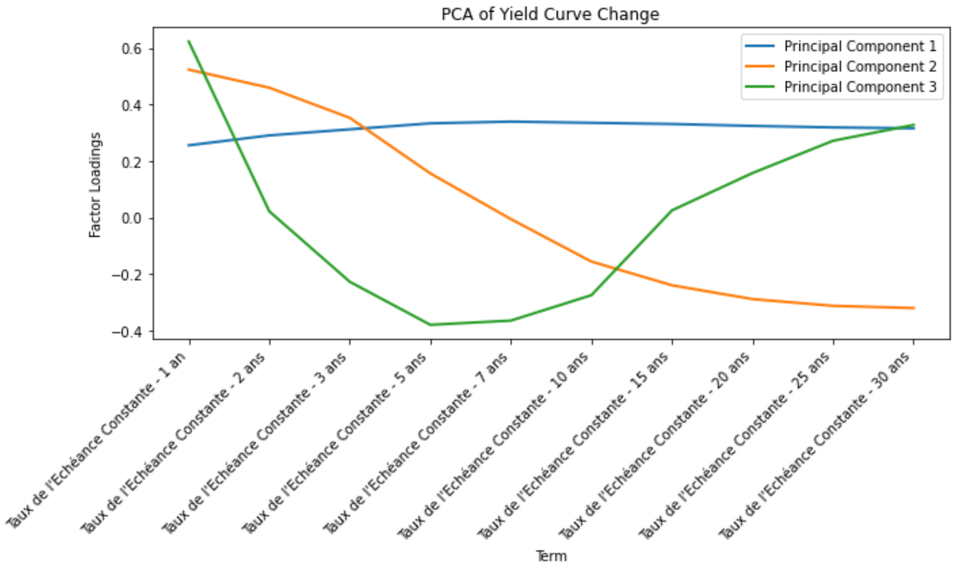

Figure 8: Factor Loadings of Principal Components on Differenced Yield Curve Data

5.4 State-Space Model Results with Macroeconomic Factors

To capture the dynamic nature of the yield curve in response to economic conditions, we extend the Nelson-Siegel model with a state-space framework that incorporates macroeconomic factors, such as GDP growth, CPI inflation, and unemployment. The estimated time series for the Level, Slope, and Curvature factors allow us to interpret the yield curve's reaction to these economic indicators.

1. **Estimated Latent Factors**: The Kalman filter is applied to estimate the time series for the latent factors:

    - **Level Factor (Lt)**: Represents the overall interest rate level, affected by the macroeconomic environment and central bank policies.
    - **Slope Factor (St)**: Reflects changes in short-term vs. long-term yields, which may respond to shifts in economic growth expectations.
    - **Curvature Factor (Ct)**: Shows the hump in the yield curve, often responding to medium-term economic outlooks.

2. **Macroeconomic Influence on Latent Factors**: The effect of macroeconomic variables on each latent factor is examined by analyzing the state transition matrix and coefficients. The following relationships are observed:

$$\alpha_{t+1} = T\alpha_t + BX_t + \eta_t \tag{20}$$

    where:

    - T is the transition matrix for the AR(1) dynamics.

- B captures the sensitivity of the latent factors to the macroeconomic factors X_t (GDP growth, CPI, unemployment).
- η_t is the state disturbance term.

3. **Correlation Analysis with Macroeconomic Variables**: Finally, we calculate the correlations between the estimated latent factors (Level, Slope, and Curvature) and the macroeconomic variables:

$$\rho(F_i, M_j) = \frac{\mathrm{Cov}(F_i, M_j)}{\sigma_{F_i}\sigma_{M_j}} \tag{21}$$

where F_i represents the i-th latent factor, and M_j is the j-th macroeconomic variable. Significant correlations indicate that certain macroeconomic factors have a measurable influence on the yield curve dynamics, helping to explain the observed movements in the Level, Slope, and Curvature factors.

5.5 Estimation Methodology

To ensure parameter stability, initial values for the autoregressive (AR) and variance terms were constrained:

- **AR terms**: Bounded between 0.7 and 0.95 to allow persistence without destabilizing the estimates.
- **Variance terms**: Ranged from 0.01 to 0.5 for state factors and 0.01 to 1.0 for observation variance, balancing model flexibility and overfitting prevention.

Optimization was performed using a maximum likelihood approach, maximizing the log-likelihood function within these bounds.

5.6 Results

The estimation produced the following optimal values for the Level, Slope, and Curvature factors over the observation period:

5.7 Level Factor

The Level factor displays a clear downward trend, as seen in Figure 1, starting above 2.5 and gradually decreasing toward zero over time. This trend suggests a persistent shift toward a low-interest-rate environment, potentially driven by central bank policies, low inflation, or subdued economic growth.

$$\begin{aligned}\text{Level Factor} = [&2.2359, 2.4564, 2.5247, 2.5172, 2.5404, 2.5086, 2.4035, 2.3295, 2.2689, 2.1985,\\ &2.1354, 2.0672, 1.9917, 1.9092, 1.8445, 1.7876, 1.7300, 1.6624, 1.6011, 1.5407, \ldots]\end{aligned}$$

5.8 Slope Factor

The Slope factor initially starts positive, indicating an upward-sloping yield curve (typical in growth phases) but declines rapidly toward zero and eventually turns slightly negative, as shown in Figure 2. This flattening of the curve indicates potential economic slowdown or recession expectations, as a flattening or inverted yield curve often signals reduced growth prospects.

$$\text{Slope Factor} = [1.5503, 1.5731, 1.5119, 1.3906, 1.3534, 1.2489, 1.0926, 0.9287, 0.7890, 0.6726, 0.5727, 0.4826, 0.4051, 0.3457, 0.2923, 0.2559, 0.2215, 0.1872, 0.1554, 0.1252, \ldots]$$

5.9 Curvature Factor

The Curvature factor rises initially, reaching a peak around 1.0 before declining steadily, as seen in Figure 3. This behavior suggests an initial increase in demand for medium-term securities, resulting in a pronounced hump in the yield curve. However, as the factor decreases, this demand appears to stabilize, potentially reflecting adjustments in monetary policy expectations.

$$\text{Curvature Factor} = [0.4636, 0.5790, 0.6505, 0.7101, 0.7401, 0.7786, 0.8089, 0.8644, 0.9130, 0.9421, 0.9646, 0.9786, 0.9807, 0.9669, 0.9598, 0.9452, 0.9284, 0.9074, 0.8892, 0.8704, \ldots]$$

5.10 Visual Analysis

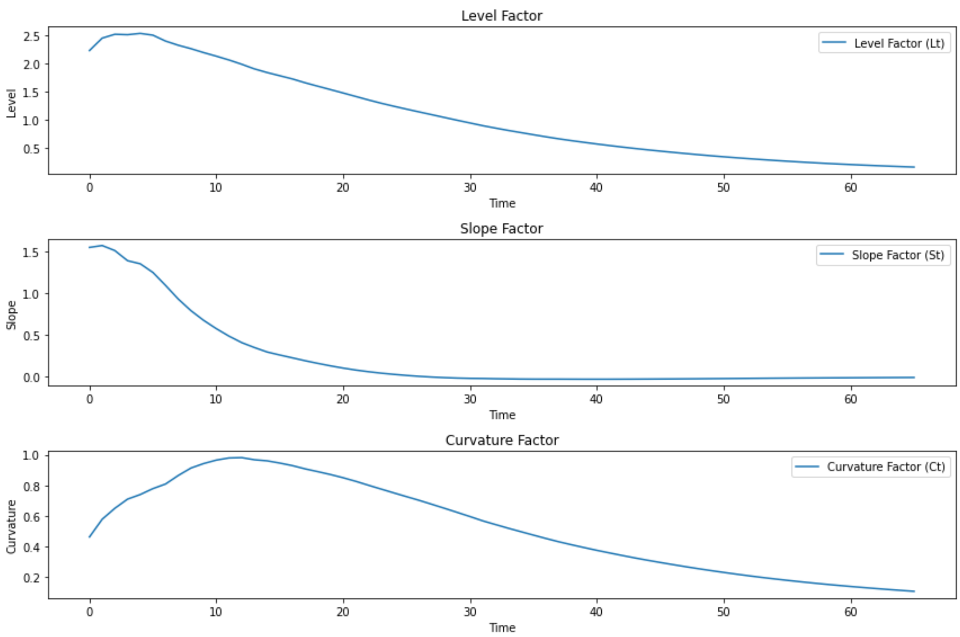

Figure 9: Estimated Level, Slope, and Curvature Factors over Time

The trends observed in Figure 9 support the analysis of the factor estimates:

- **Level Factor**: The gradual decline in the Level factor indicates a persistent low-yield environment.
- **Slope Factor**: The flattening and eventual slight inversion of the Slope factor suggest an economic slowdown.
- **Curvature Factor**: The initial increase followed by a decrease in the Curvature factor shows a shift in medium-term bond demand, potentially due to adjusted yield expectations.

5.11 Correlation Analysis

The estimated factors were combined with macroeconomic data (GDP growth, CPI inflation, and unemployment) to analyze their correlations. Table 2 presents the correlation matrix:

Table 2: Correlation Matrix of Factors and Macroeconomic Variables

	Level	Slope	Curvature	GDP_Growth	CPI_Inflation	Unemployment
Level	1.000	0.835	0.860	-0.054	-0.108	0.093
Slope	0.835	1.000	0.436	-0.050	0.028	-0.342
Curvature	0.860	0.436	1.000	-0.041	-0.201	0.472
GDP_Growth	-0.054	-0.050	-0.041	1.000	-0.007	0.160
CPI_Inflation	-0.108	0.028	-0.201	-0.007	1.000	-0.393
Unemployment	0.093	-0.342	0.472	0.160	-0.393	1.000

5.12 Impulse Response Analysis

Figure 10 shows the impulse response functions of the Level, Slope, and Curvature factors following shocks to GDP growth, CPI inflation, and unemployment:

- **Level Factor**: A shock to GDP growth leads to a gradual decline in the Level factor, while shocks to CPI inflation and unemployment yield a similar downward trend. This response implies that economic growth and inflation shocks may reinforce low yield levels.
- **Slope Factor**: The Slope factor declines after shocks to GDP growth, CPI inflation, and unemployment. This reaction suggests that macroeconomic shocks contribute to a flattening of the yield curve, potentially signaling recessionary pressures.
- **Curvature Factor**: The Curvature factor shows an initial rise in response to GDP growth and inflation shocks, followed by a decline. This response suggests that economic shocks can initially increase the convexity of the yield curve, with effects tapering off over time.

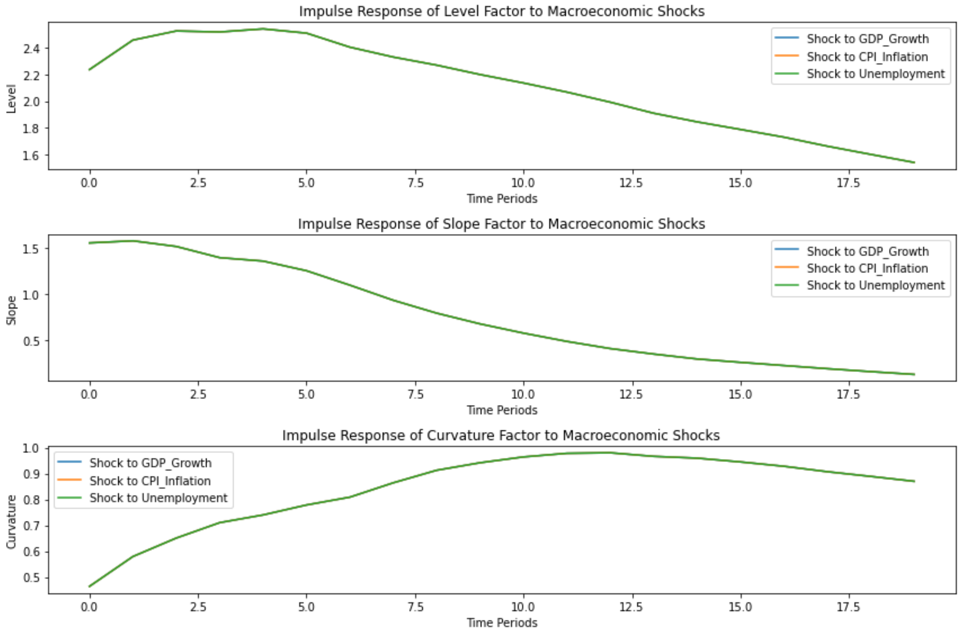

Figure 10: Impulse Response of Level, Slope, and Curvature Factors to Macroeconomic Shocks

6 Conclusion

This study provides a comprehensive analysis of the yield curve by extending the traditional Nelson-Siegel model to incorporate macroeconomic factors, including GDP growth, CPI inflation, and unemployment. By augmenting the Level, Slope, and Curvature factors within a state-space framework, the model captures dynamic yield curve behavior in response to evolving economic conditions.

The results reveal several key insights:

- **Yield Curve Dynamics**: The Level, Slope, and Curvature factors effectively describe the yield curve's general structure, steepness, and mid-term hump. These parameters respond to changing economic conditions, as demonstrated by significant shifts during periods of financial stress, such as the 2008 financial crisis and the COVID-19 pandemic.

- **Impact of Macroeconomic Conditions**: The extended model illustrates how macroeconomic variables influence the yield curve. For instance, GDP growth shocks often result in a decline in the Level factor, while shocks to inflation and unemployment show a consistent downward pressure on the Slope and Level factors. These findings align with economic theory, suggesting that investors adjust yield expectations in response to anticipated monetary policy and economic health.

- **Principal Component Validation**: The Principal Component Analysis (PCA) confirms that yield curve variations are predominantly driven by Level, Slope, and Curvature, with cumulative explained variance exceeding 99%. This statistical validation reinforces the relevance of these three factors in explaining yield curve dynamics.

- **Impulse Response Insights**: Impulse response analysis further demonstrates how macroeconomic shocks propagate through the yield curve. For example, inflation shocks initially increase the Curvature factor, implying greater convexity at mid-term maturities, which then stabilizes over time. This response highlights the yield curve's sensitivity to short-term economic disruptions, which often result in temporary adjustments in yield expectations.

In conclusion, this study demonstrates the effectiveness of the Nelson-Siegel Extended Model with macroeconomic integration for understanding and forecasting yield curve behavior. The model's capacity to dynamically capture yield curve shifts in response to economic changes offers valuable insights for bond pricing, risk assessment, and economic forecasting. Future research could expand on these findings by applying this framework to different regions and incorporating additional macroeconomic variables, potentially enhancing the model's predictive power for various economic scenarios.

6.1 Consideration of Innovation Parts Only

While it might seem sufficient to consider only the innovation parts of GDP, CPI, and Unemployment in the model, doing so would overlook critical aspects of the broader economic context that impact the yield curve dynamics. Innovations, or unexpected changes in macroeconomic variables, certainly play a significant role in explaining sudden shifts or adjustments in yields. However, the full economic conditions — including both the predictable (systematic) and unexpected (innovative) changes in these variables — provide a more comprehensive understanding of interest rate dynamics.

The yield curve responds not only to shocks but also to expectations and trends within the economy. For instance:

- **Systematic Trends and Expectations**: The long-term expectations of inflation, GDP growth, and labor market health drive investors' demand for long-term bonds. If we consider only the innovation parts, we ignore the impact of sustained economic trends, such as prolonged low inflation or persistent economic growth, which influence interest rate levels and shape over time.

- **Policy Reactions to Baseline Levels**: Central banks react to both the current levels and deviations (innovations) in macroeconomic variables. For example, a consistently high inflation rate might prompt a different policy response than a sudden inflation spike. Capturing the entire trajectory of GDP, CPI, and Unemployment is essential to reflect how monetary policy and market expectations evolve in response to both shocks and ongoing economic conditions.

- **Yield Curve Stability and Long-term Signals**: The yield curve is affected by both cyclical innovations and structural trends. Systematic components of macroeconomic variables often shape long-term yield expectations, while innovations contribute to short-term volatility. Ignoring the systematic parts would lead to an incomplete model, primarily capturing immediate fluctuations rather than stable trends that define the yield curve's long-term shape.

Therefore, incorporating both systematic components and innovations of macroeconomic variables into the model provides a fuller, more accurate representation of yield curve dynamics. This approach enables a nuanced analysis that accounts for both short-term shocks and long-term economic expectations, aligning with the complexity of real-world financial markets and investor behavior.