A non-linear specification of the dynamics, allowing for Smooth Transition Regression (STR) Models

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1 Introduction

In economic and financial modeling, traditional linear frameworks often fall short in capturing the complexities and inherent nonlinearities present in many real-world systems. An increasing interest has thus developed around non-linear models, particularly those allowing for smooth transition dynamics that reflect gradual changes in regimes rather than abrupt shifts. The Smooth Transition Regression (STR) model emerges as a robust approach for modeling such gradual and smooth changes, thus enhancing the accuracy and interpretability of econometric models in dynamic environments.

The STR framework is particularly beneficial for financial markets, where the impact of shocks, policy changes, or shifts in investor sentiment typically manifests gradually over time rather than instantaneously. For instance, in assessing the influence of macroeconomic variables on asset prices, one may observe that the effects of a policy shift (such as a change in interest rates or monetary policy) do not appear immediately but rather transition smoothly over time. This transition can reflect investor adaptation, market inertia, or delayed adjustment of prices and returns.

From a methodological perspective, the STR model offers a way to capture these dynamic transitions through a flexible non-linear specification. By introducing a transition function, typically modeled as a logistic or exponential function, STR models allow for varying parameter estimates depending on the position within the regime. For instance, the transition function may be specified as:

$$G(q_t;\gamma,c) = \frac{1}{1+\exp(-\gamma(q_t-c))}$$

where q_t represents a transition variable, γ denotes the smoothness parameter, and c is the threshold parameter. The function $G(q_t;\gamma,c)$ smoothly varies between 0 and 1 as q_t changes, thus allowing for different model parameters across regimes based on the position of q_t. As a result, STR models provide a flexible framework that accommodates gradual shifts in economic relationships and captures the underlying structural dynamics.

In recent literature, STR models have seen applications across various domains of finance and economics. For instance, Teräsvirta (1994) explored the applicability of STR models in capturing cyclical dynamics in exchange rates, while González et al. (2005) demonstrated their utility in modeling nonlinear relationships in monetary policy transmission. The use of STR models has been further extended to portfolio risk modeling, term structure modeling, and even the analysis of economic growth patterns, making it a versatile tool for understanding the smooth transitions that characterize many economic phenomena.

1.1 Part 1: Theoretical Background and Applications in Finance

The theoretical foundation of Smooth Transition Regression (STR) models is rooted in non-linear time series analysis, a field that addresses the limitations of linear models in capturing complex, dynamic behavior in economic and financial data. Unlike traditional linear models that assume constant parameters across time, STR models enable the parameters to vary in a smooth, continuous manner as a function of an underlying transition variable. This flexibility allows the STR model to represent different regimes or states within the data while avoiding the abrupt regime shifts typical of Markov-Switching models.

- 1.1 The Structure of the STR Model The standard form of an STR model can be represented as:

  $$y_t = \alpha_0 + \alpha_1 x_t + (\beta_0 + \beta_1 x_t)\, G(q_t;\gamma,c) + \varepsilon_t$$

  where:

  - y_t is the dependent variable,
  - x_t represents the independent variables,
  - $G(q_t;\gamma,c)$ is the transition function that varies smoothly between 0 and 1,
  - $\alpha_0, \alpha_1, \beta_0, \beta_1$ are the parameters to be estimated,
  - $\varepsilon_t \sim N(0,\sigma^2)$ is an error term.

  The transition function $G(q_t;\gamma,c)$, often defined as logistic or exponential, controls the pace and extent of the transition between different regimes. For instance, with a logistic transition function:

  $$G(q_t;\gamma,c) = \frac{1}{1+\exp(-\gamma(q_t - c))}$$

  the parameter γ controls the smoothness of the transition, with higher values of γ indicating a sharper transition between regimes, while c denotes the threshold level of q_t at which the transition occurs.

1.2 1.2 Applications in Finance

Smooth transition models (STR) are widely applied in finance due to their ability to capture gradual adjustments in financial dynamics. In interest rate modeling, STR models effectively illustrate how rates transition in response to economic changes, mirroring central bank policies' impact over time. For asset pricing and portfolio management, these models capture varying market conditions, such as shifts in volatility based on investor sentiment, allowing gradual transitions in asset prices. In exchange rate dynamics, STR models handle currency value adjustments due to trade balances and interest rate differences, better reflecting non-linear currency behaviors. Additionally, STR models enhance Value-at-Risk calculations by accommodating smooth changes in risk exposure linked to macroeconomic factors, thereby improving risk estimates. Finally, STR models contribute to business cycle analysis, capturing the transition from recession to expansion, aiding policymakers in monitoring economic shifts.

This report analyzes key economic indicators and their trends over recent years, setting the groundwork for examining non-linear dynamics and potential threshold effects in Smooth Transition Regression (STR) models. We explore the SP 500, CPI, Industrial Production Index, 10-Year Treasury vs. 2-Year Spread, VIX, Unemployment Rate, and Federal Funds Rate.

2 Data Preparation and Loading

Our dataset consists of multiple economic and financial indicators:

- **S&P 500 (SP500):** Represents overall stock market performance. It is taken from S&P 500®' website.
- **Consumer Price Index (CPIAUCSL):** Measures inflation trends over time.U.S. It is downloaded from FRED ST LOUIS.FED website, which is based on the data from US Bureau of Labor Statistics.
- **Industrial Production Index (INDPRO):** Indicates economic activity in industrial sectors. It is downloaded from FRED ST LOUIS.FED website, which is based on the data from Board of Governors of the Federal Reserve System (US).
- **10-Year Treasury vs. 2-Year Spread (T10Y2Y):** A yield spread often used as a recession indicator. It is downloaded from FRED ST LOUIS.FED website, based on their own data.
- **Volatility Index (VIX):** Captures market volatility and investor sentiment. It is downloaded from FRED ST LOUIS.FED website, Chicago Board Options Exchange which is based on the data from Chicago Board Options Exchange.
- **Unemployment Rate (UNRATE):** Reflects labor market conditions. It is downloaded from FRED ST LOUIS.FED website, Chicago Board Options Exchange which is based on the data from U.S. Bureau of Labor Statistics.
- **Federal Funds Rate (FEDFUNDS):** Indicates central bank policy rate. It is downloaded from FRED ST LOUIS.FED website, Chicago Board Options Exchange which is based on the data from Board of Governors of the Federal Reserve System (US).

The data was resampled to a monthly frequency, merged, and cleaned to ensure consistent analysis across all indicators.

Date	SP500	CPIAUCSL	INDPRO	T10Y2Y	VIXCLS	UNRATE	FEDFUNDS
2014-11-30	2067.56	236.983	103.5978	1.71	13.33	5.8	0.09
2014-12-31	2058.90	236.252	103.6151	1.50	19.20	5.6	0.12
2015-01-31	1994.99	234.747	102.7923	1.21	20.97	5.7	0.11
2015-02-28	2104.50	235.342	102.1366	1.37	13.34	5.5	0.11
2015-03-31	2067.89	235.976	101.7869	1.38	15.29	5.4	0.11
2015-04-30	2085.51	236.222	101.2255	1.47	14.55	5.4	0.12
2015-05-31	2107.39	237.001	100.7675	1.51	13.84	5.6	0.12
2015-06-30	2063.11	237.657	100.4588	1.71	18.23	5.3	0.13
2015-07-31	2103.84	238.034	101.0893	1.53	12.12	5.2	0.13

3 Indicator Analysis and Visualizations

3.1 S&P 500 (SP500)

The S&P 500 index shows a general upward trend, with a significant drop in 2020 due to the COVID-19 pandemic and a rapid recovery. The increase post-2020 highlights market resilience and response to monetary stimulus.

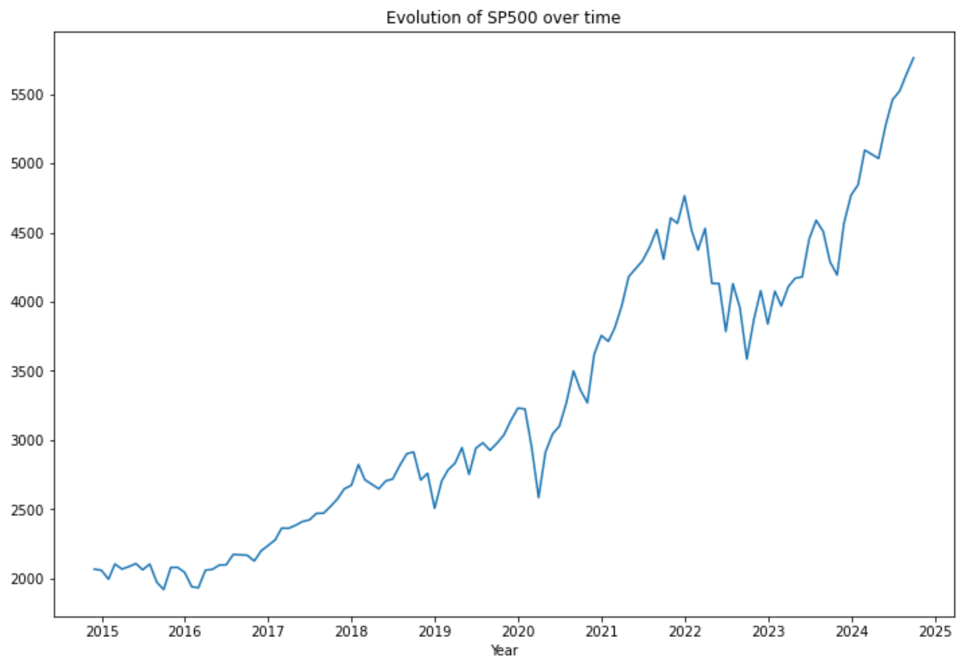

Figure 1: Evolution of S&P 500 over time

Economic Insight: The sharp dip in 2020 corresponds with the pandemic-induced market crash, followed by rapid recovery fueled by monetary stimulus and investor optimism.

3.2 Consumer Price Index (CPIAUCSL)

The CPI shows a steady increase, with a notable rise starting in 2021, indicative of inflationary pressures likely driven by supply chain disruptions and increased post-pandemic demand.

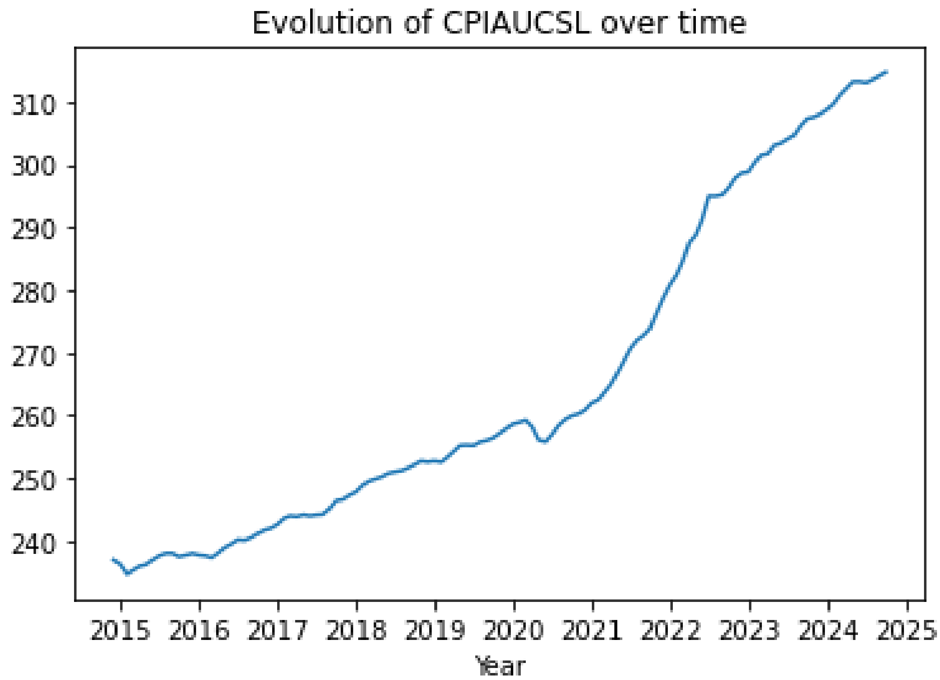

Figure 2: Evolution of CPIAUCSL over time

Economic Insight: The steep rise in CPI post-2021 suggests significant inflation, potentially linked to expansive fiscal and monetary policies and supply chain disruptions.

3.3 Industrial Production Index (INDPRO)

Industrial production experienced stability with a significant drop in 2020, followed by recovery, reflecting the impact of economic lockdowns and the sector's resilience.

Economic Insight: The dip and subsequent recovery in 2020 highlights the impact of COVID-19 lockdowns and the resilience of industrial activity in bouncing back to pre-pandemic levels.

3.4 10-Year Treasury vs. 2-Year Spread (T10Y2Y)

The T10Y2Y spread trends downwards post-2018, turning negative, signaling potential recessionary pressures as investors favor long-term over short-term bonds.

Economic Insight: Negative spreads, particularly around 2019-2020 and recent periods, often signal an upcoming recession, reflecting investor preference for long-term stability.

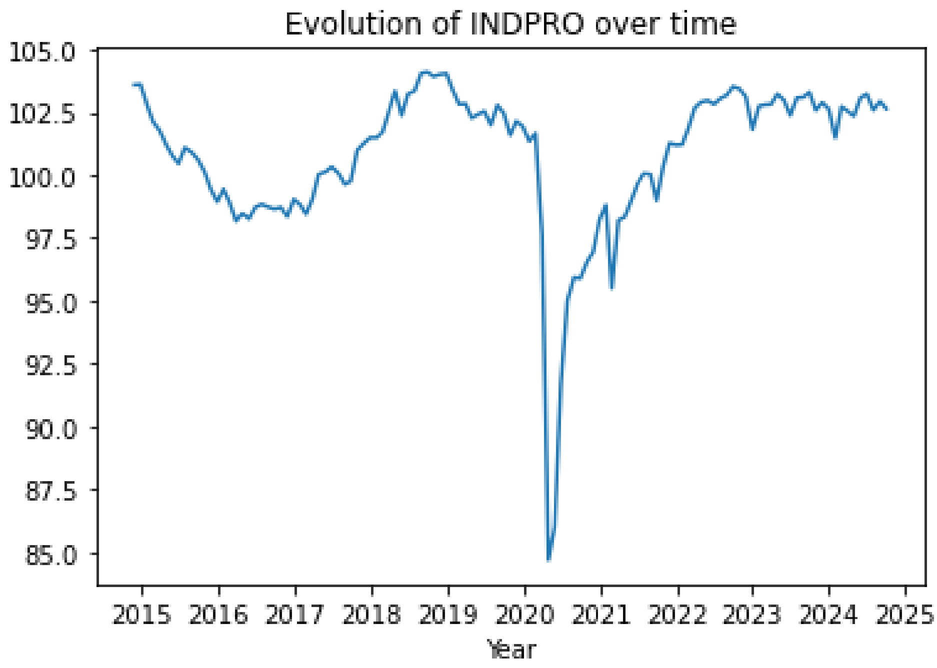

Figure 3: Evolution of Industrial Production Index (INDPRO) over time

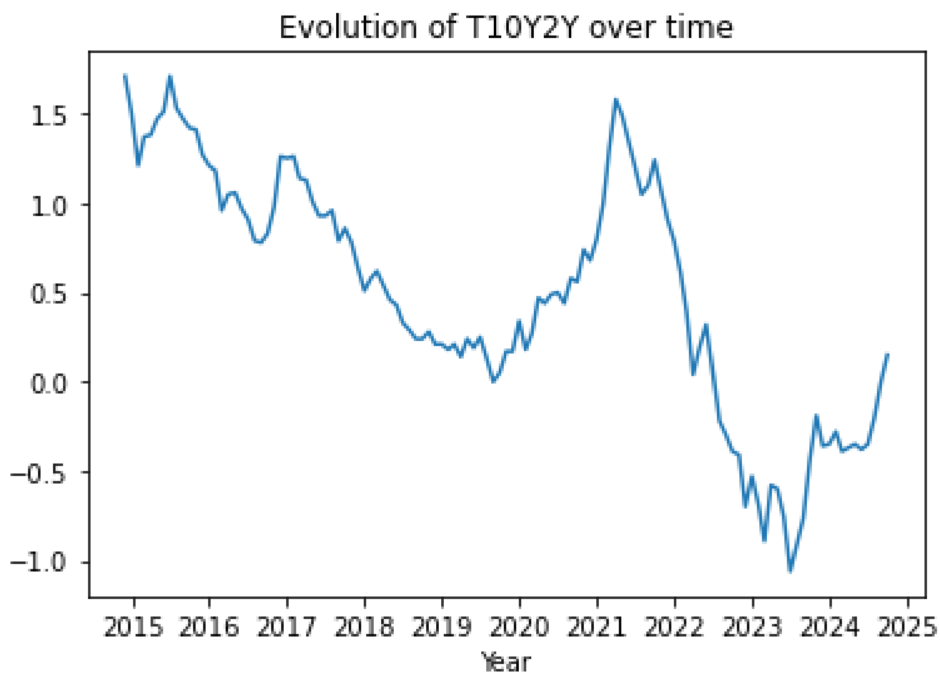

Figure 4: Evolution of 10-Year Treasury vs. 2-Year Spread (T10Y2Y) over time

3.5 Volatility Index (VIX)

The VIX spikes in 2020, signaling heightened market uncertainty during the early stages of the pandemic, followed by moderate fluctuations.

Economic Insight: Elevated VIX levels during significant economic disruptions reflect heightened market fear and risk aversion.

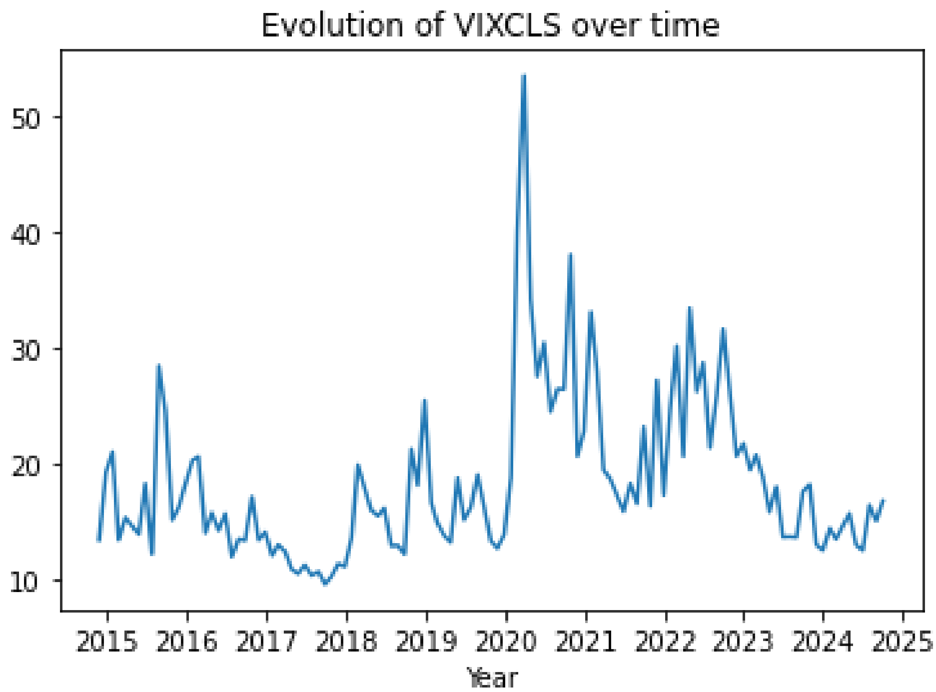

Figure 5: Evolution of VIX (Volatility Index) over time

3.6 Unemployment Rate (UNRATE)

The unemployment rate spiked dramatically in 2020 due to pandemic-induced job losses, followed by a sharp decline as the labor market rebounded.

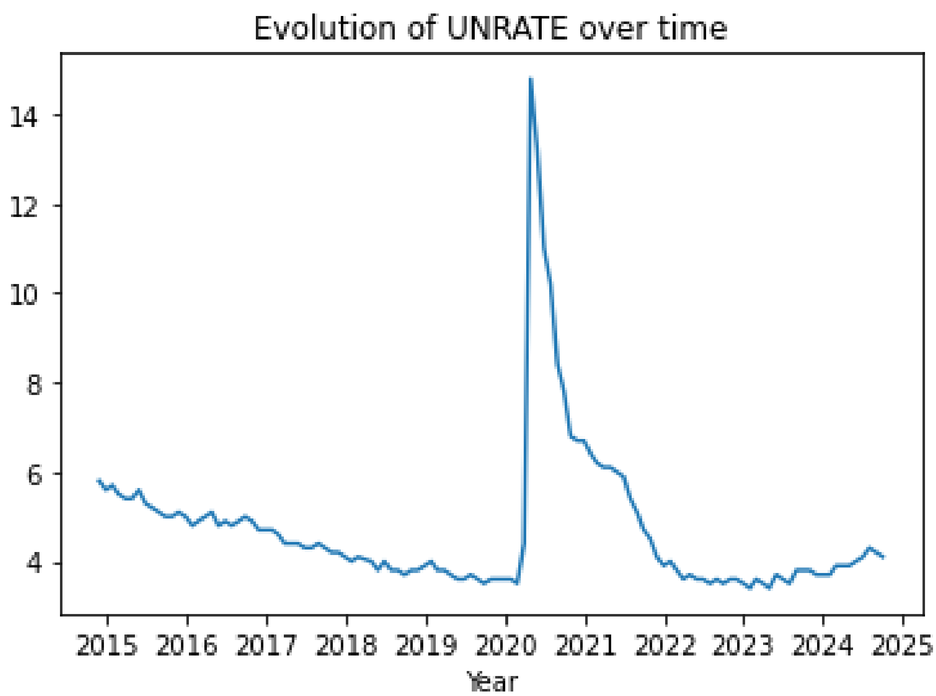

Figure 6: Evolution of Unemployment Rate (UNRATE) over time

Economic Insight: The rapid increase and subsequent decrease in unemployment highlight the labor market's response to the pandemic and the gradual recovery as economies reopened.

3.7 Federal Funds Rate (FEDFUNDS)

The Federal Funds Rate remained near zero until 2016, rose modestly, then dropped again in 2020 as a response to the pandemic, with recent hikes in response to rising inflation.

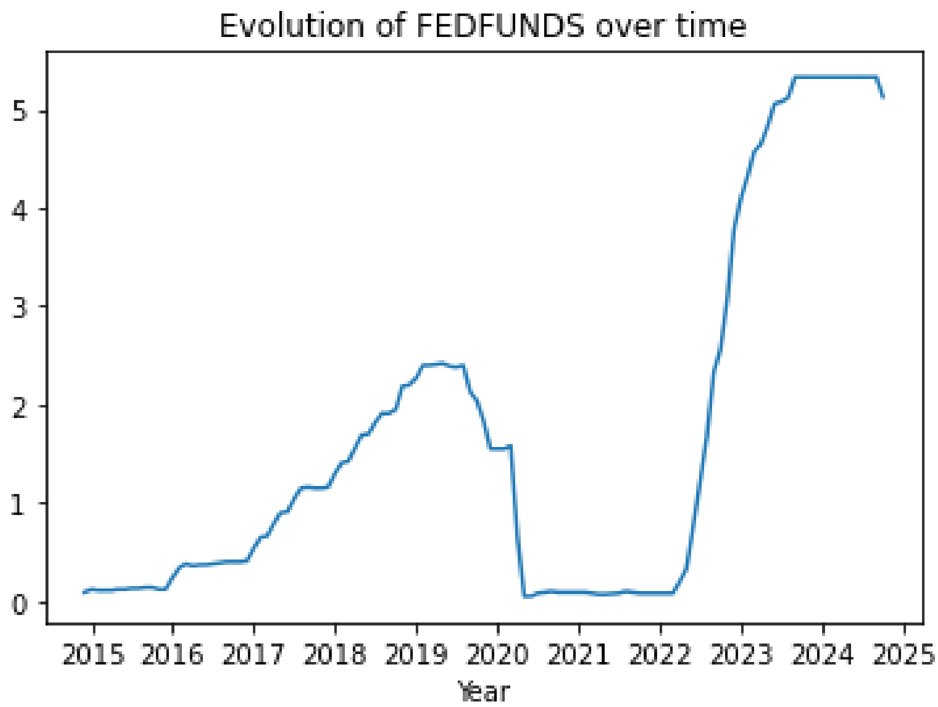

Figure 7: Evolution of Federal Funds Rate (FEDFUNDS) over time

Economic Insight: The recent rate increases reflect the Federal Reserve's tightening measures to combat inflation, indicating a shift in monetary policy.

4 Data Manipulation

In order to stabilize variance and remove trends, we applied log transformations to selected variables such as SP500, CPIAUCSL, and INDPRO, generating new variables (e.g., ln_SP500, ln_CPI, ln_INDPRO). For the Federal Funds Rate (FEDFUNDS), we opted for differencing directly due to the potential issues with zero values in rate data.

The log transformations and first differencing were computed as follows: Logarithmic transformations are applied to CPI, Industrial Production, and S&P 500 indices, represented as $\ln(\text{SP500}), \ln(\text{CPI}), \text{and} \ln(\text{INDPRO})$. First differences are then taken to compute growth rates, exemplified by $D\ln(\text{SP500}) = \ln(\text{SP500}_t) - \ln(\text{SP500}_{t-1})$. Direct differencing is further used for variables such as T10Y2Y, UNRATE, and FEDFUNDS to analyze their changes over time, producing variables D_T10Y2Y, D_UNRATE, and $D\_ln\_FEDFUNDS$.

5 Stationarity Analysis with the Augmented Dickey-Fuller Test

To confirm stationarity, we conducted the Augmented Dickey-Fuller (ADF) test on each series. Stationarity is essential in time series analysis to avoid spurious results. The ADF test checks for the presence of a unit root, where a low p-value (typically less than 0.05) indicates that the series is stationary.

$$H_0 : \text{Series has a unit root (non-stationary)} \quad \text{vs.} \quad H_1 : \text{Series is stationary}$$

Test statistic: The ADF statistic is calculated, with the corresponding p-value evaluated against a significance level (e.g., 0.05).

Critical Values: Compared to the ADF statistic to determine stationarity.

Table 2: ADF Test Results for Economic Indicators

Variable	ADF Statistic	p-value	Stationary?
SP500	0.855	0.992	No
CPIAUCSL	0.664	0.989	No
INDPRO	-2.611	0.091	No
T10Y2Y	-1.822	0.370	No
VIXCLS	-4.568	0.000	Yes
UNRATE	-3.249	0.017	Yes
FEDFUNDS	-2.102	0.244	No
ln_SP500	0.059	0.963	No
ln_CPI	0.602	0.988	No
ln_INDPRO	-2.694	0.075	No
ln_FEDFUNDS	-2.845	0.052	No
D_ln_SP500	-11.978	3.76e-22	Yes
D_ln_CPI	-4.170	2.17e-17	Yes
D_ln_INDPRO	-9.099	3.65e-15	Yes
D_T10Y2Y	-8.488	1.34e-13	Yes
D_UNRATE	-10.312	3.17e-18	Yes
D_ln_FEDFUNDS	-8.003	2.31e-12	Yes

Interpretation: The level series, such as SP500, CPIAUCSL, and FEDFUNDS, are non-stationary, as indicated by their high p-values. However, the differenced series, like D_ln_SP500 and D_T10Y2Y, exhibit stationarity, confirmed by very low p-values in the ADF test results.

6 Implications for Smooth Transition Regression (STR) Modeling

With the differenced and stationary series, we have transformed our indicators into suitable forms for further modeling. Stationary series like D_ln_SP500 and D_T10Y2Y are critical for analyzing regime shifts, particularly in non-linear models like Smooth Transition Regression (STR). By selecting a variable such as D_T10Y2Y or D_ln_FEDFUNDS as a transition variable, we can investigate how economic regimes may shift in response to monetary policy or economic cycles.

7 Smooth Transition Regression (STR) Model Specification

The STR model separates linear and nonlinear components, utilizing a logistic transition function to model changes in the regime.

Logistic Transition Function

The logistic transition function is defined as:

$$G(s_t;\gamma,c) = \frac{1}{1+\exp(-\gamma(s_t - c))}$$

where s_t (e.g., $D\ln(\text{INDPRO})$) is the transition variable, γ is the transition speed parameter, and c is the threshold.

Regression Model

The STR model is specified as:

$$Y_t = X_t\beta_0 + G(s_t;\gamma,c)X_t\beta_1 + \varepsilon_t$$

where:

- Y_t: Dependent variable (e.g., $D\ln(\text{SP500})$)
- X_t: Regressors (e.g., VIXCLS, D_{T10Y2Y}, $D\ln(\text{FEDFUNDS})$, DUNRATE)
- $G(s_t;\gamma,c)$: Transition function based on s_t
- β_0 and β_1: Coefficients for the linear and nonlinear parts, respectively

Objective Function

The objective function is minimized to fit parameters β_0, β_1, γ, and c:

$$\text{Objective} = \sum_t (Y_t - (X_t\beta_0 + G(s_t;\gamma,c)X_t\beta_1))^2$$

8 Smooth Transition Regression (STR) Model Results

The STR model estimated two sets of coefficients:

- **Linear Regime Coefficients** (β_0): These coefficients represent the influence of the independent variables (VIXCLS, D_T10Y2Y, D_ln_FEDFUNDS, D_UNRATE) when the transition function $G(s_t;\gamma,c)$ is close to 0, indicating the "linear regime."
- **Nonlinear Regime Coefficients** (β_1): These coefficients represent the influence of the same variables in the "nonlinear regime," which takes effect as the transition function approaches 1.

The estimated parameters are as follows:

- **Linear Coefficients (β_0)**: [0.045, 0.034, -0.024, 0.040, 0.577]
- **Nonlinear Coefficients (β_1)**: [0.054, -0.074, -0.008, -0.016, -1.208]
- **Transition Speed (γ)**: 1.032
- **Threshold (c)**: -0.032

8.1 Interpretation of the Results

Linear and Nonlinear Influence: The STR model suggests that the relationship between stock returns (D_ln_SP500) and each economic variable (e.g., VIXCLS, D_T10Y2Y, etc.) changes based on the state of the economy, as indicated by D_ln_INDPRO. For instance, the effect of VIXCLS is positive in the linear regime ($\beta_0 = 0.034$) but turns negative in the nonlinear regime ($\beta_1 = -0.074$). This change implies that during different economic conditions, the impact of market volatility (VIX) on stock returns can vary, potentially due to shifting investor sentiment or economic stress.

Transition Speed (γ): The estimated γ value of 1.032 controls the smoothness of the transition. A higher γ would indicate an abrupt switch between regimes, while a lower γ implies a more gradual shift. Here, $\gamma = 1.032$ suggests a moderate transition speed, meaning that as D_ln_INDPRO approaches the threshold, the model shifts smoothly between the linear and nonlinear effects.

Threshold Parameter (c): The threshold value of $c = -0.032$ indicates the point in the transition variable D_ln_INDPRO where the model starts to shift between the regimes. This threshold suggests that around a slight contraction in industrial production growth, the model transitions between linear and nonlinear dynamics.

8.2 Economic Interpretation

Nonlinear Influence of Economic Variables:

- This model suggests that the impact of variables such as market volatility (VIX), yield spread (D_T10Y2Y), interest rates (D_ln_FEDFUNDS), and unemployment rate (D_UNRATE) on stock market returns (D_ln_SP500) changes depending on economic conditions, as proxied by industrial production growth.
- During periods of low or contracting industrial production (i.e., when D_ln_INDPRO is below the threshold), the linear part β_0 dominates. When industrial production growth improves and crosses the threshold, the nonlinear coefficients β_1 take a more significant role.

Application to Economic Cycles:

- This STR model captures the idea that economic relationships can differ across business cycles. For instance, in times of economic contraction, volatility might have a smaller or even positive effect on stock returns, but in periods of expansion, the effect might turn negative as captured by the change in β_1 coefficients.

Summary

- This This model provides insights into how financial and economic indicators influence stock returns under different economic conditions. The estimated threshold (c) and transition speed (gamma) highlight a smooth but responsive shift in the impact of these variables based on industrial production growth, reflecting the cyclical nature of financial markets.

Observations from the Plot

1. **Overall Fit**:
    - The fitted values (orange dashed line with ”x” markers) generally follow the pattern of the actual values (solid blue line with ”o” markers), indicating that the STR model is capturing some of the dynamics of D_ln_SP500.

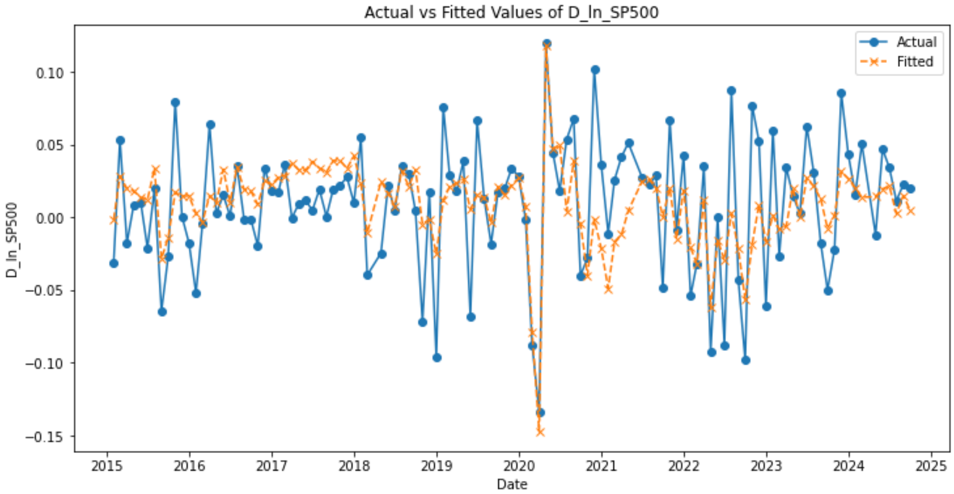

Figure 8: Evolution of Federal Funds Rate (FEDFUNDS) over time

- However, there are periods where the model diverges slightly from the actual values, which may indicate limitations in capturing extreme volatility or other unmodeled dynamics.

2. **Periods of Divergence**:

   - Around 2020, there's a sharp drop and rapid recovery in D_ln_SP500, likely corresponding to the COVID-19 market crash. The model approximates this movement but doesn't fully capture the extreme values, which suggests that sudden shocks may be difficult for the STR model to handle without further adjustments.
   - In other periods, such as around 2021-2022, the model appears to track D_ln_SP500 reasonably well but shows some lag or mismatch in amplitude, which is common in models that are sensitive to smooth transitions rather than abrupt regime shifts.

3. **Transition Dynamics**:

   - The use of the logistic transition function, as represented by G_t allows the model to adapt based on the value of s_t D_ln_INDPRO, This allows it to capture shifts in dynamics as the economy moves through different regimes, providing a more flexible fit compared to a purely linear model.
   - The model may perform particularly well in periods when D_ln_INDPRO, hanges gradually, allowing the logistic transition function to modulate smoothly. Abrupt changes in D_ln_INDPRO however, may still present challenges.

Implications for Model Performance

- **Nonlinear Flexibility**: The STR model's ability to capture nonlinear relationships based on D_ln_INDPRO as a transition variable adds flexibility, which helps in tracking D_ln_SP500

better than a linear model might.

- **Potential for Improvement**:
  - If extreme events (like the 2020 crash) are critical, additional modifications—such as including more robust terms for volatility or switching to a different transition variable—may improve model accuracy during turbulent times
- **Threshold Effects**: The differences between the actual and fitted values in certain periods could indicate that the threshold level or transition speed γ could be refined to capture regime changes more effectively, especially during volatile periods

This analysis provides a solid basis for understanding the strengths and limitations of the STR model, along with potential areas for refinement to better align with real-world data, particularly during high-volatility periods.

9 Residual Analysis of the STR Model

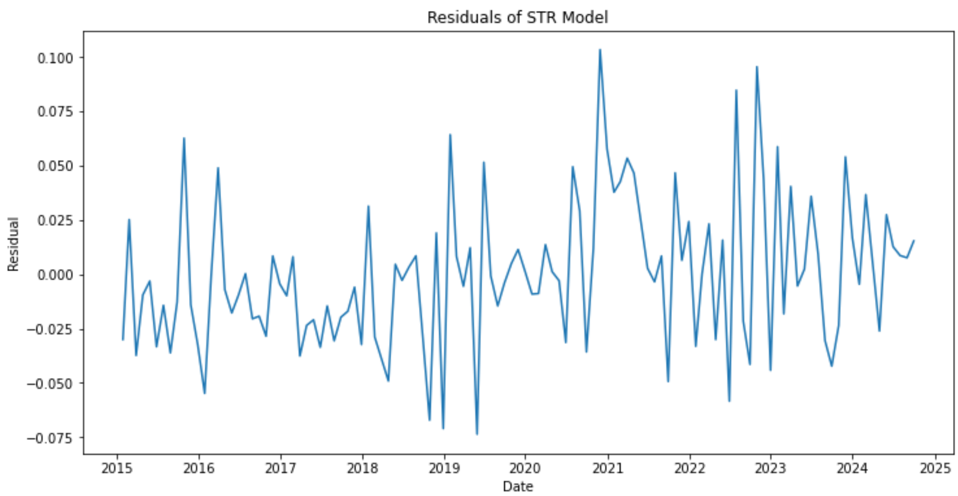

Figure 9: Evolution Residual of the STR Model over time

9.1 Residuals Over Time

The first plot shows the residuals ($\varepsilon_t = Y_t - Y_{t,\text{fitted}}$ differences between the actual and fitted values of $D\_\ln\_SP500$) over the time period from 2015 to 2025.

- The residuals oscillate around zero, which is expected if the model is unbiased.
- There are some periods of larger residuals, particularly around 2020 and 2021, indicating that the model struggled to capture certain extreme movements in the data. This aligns

with the earlier observation regarding the COVID-19 shock, where the model may have been less effective at capturing sudden shifts.

- There does not appear to be a clear pattern or trend in the residuals, which is a good indication that the model captures most of the systematic dynamics of the data.

Implications:

- If the residuals are truly random (white noise), this suggests the model has effectively captured the underlying structure of $D\_\ln\_SP500$.

- However, the presence of any autocorrelation in the residuals (which would require further testing) might indicate model misspecification or omitted variables.

9.2 Histogram of Residuals

The histogram provides a view of the distribution of residuals.

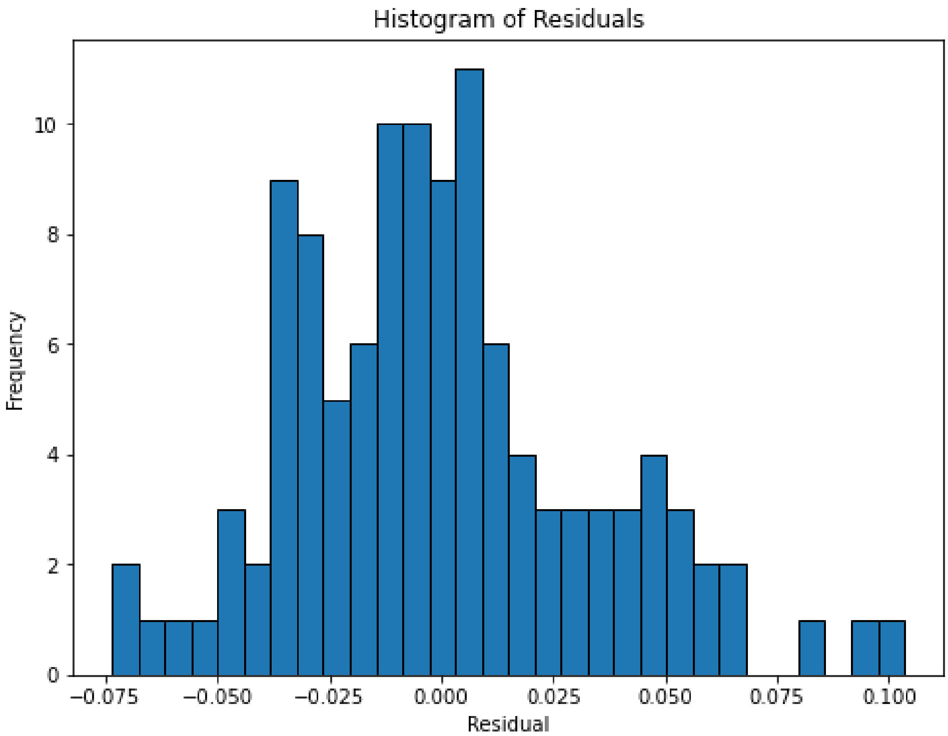

Figure 10: Histogram Analysis of the Residuals from the STR Model

- The residuals are roughly centered around zero, which is a sign of a well-calibrated model.

- The distribution appears slightly skewed, with a few larger positive and negative residuals.

- The shape of the histogram is somewhat bell-shaped but has heavier tails, which could imply that the residuals may not be perfectly normally distributed.

Implications:

- If the residuals were normally distributed, the histogram would have a more symmetric, bell-shaped appearance. The heavier tails observed here might suggest that the model doesn't fully capture extreme values, which could lead to potential outliers.

- If normality is a requirement for further analysis, applying transformations or using a different distributional assumption may improve the fit.

10 Diagnostic Tests on STR Model Residuals

10.1 Ljung-Box Test for Autocorrelation

The Ljung-Box test checks for autocorrelation in the residuals up to lag h. The null hypothesis is:

$$H_0 : \text{No autocorrelation in residuals up to lag } h$$

The test statistic $Q(h)$ is computed as:

$$Q(h) = T(T+2)\sum_{k=1}^{h}\frac{\hat{\rho}^2(k)}{T-k}$$

Significance: This test assesses whether residuals exhibit serial correlation, which could indicate potential model issues.

The results are as follows:

- **Test Statistic (lb_stat)**: Approximately 17.57

- **p-value (lb_pvalue)**: 0.1295

Interpretation:

- The p-value is greater than 0.05, so we fail to reject the null hypothesis that the residuals are uncorrelated. This suggests there is no significant autocorrelation in the residuals up to the 12th lag.

- The absence of autocorrelation is a good indication that the model has effectively captured the temporal structure in the data and that there are no lingering patterns in the residuals.

10.2 Jarque-Bera Test for Normality

The Jarque-Bera test evaluates the normality of residuals. The null hypothesis is:

$$H_0 : \text{Residuals are normally distributed}$$

The test statistic is based on skewness S and kurtosis K, calculated as:

$$\mathrm{JB} = \frac{T}{6}\left(S^2 + \frac{(K-3)^2}{4}\right)$$

Interpretation: The Jarque-Bera test determines if the residuals are normally distributed; non-normality may suggest a need for alternative model specifications.

The results are:

- **JB Statistic**: Approximately 5.08
- **p-value**: 0.079

Interpretation:

- With a p-value slightly above 0.05, we cannot reject the null hypothesis of normality at the 5% significance level, but it's close to the threshold. This result suggests that the residuals are roughly normal, though there might be some deviation (as suggested by the histogram with slightly heavy tails).
- This suggests that the residuals are approximately normally distributed. However, the slight deviation from perfect normality (as seen in the histogram with slightly heavy tails) indicates that there may be some outliers or skewness in the data.

10.3 Summary and Implications

- **No Significant Autocorrelation:** The lack of significant autocorrelation indicates that the STR model captures the dynamics of $D\_\ln\_SP500$ effectively, as there are no remaining systematic patterns in the residuals.
- **Approximate Normality:** The residuals are approximately normal, which supports the validity of the model. However, the near-threshold p-value suggests it may be beneficial to monitor for potential outliers or skewness.

11 Analysis of Time-varying Coefficients in STR Model

11.1 Interpretation of the Time-Varying Coefficients

The plot of the time-varying coefficients illustrates the evolution of each variable's impact on $D\_\ln\_SP500$ over time, as governed by the transition function $G(s_t;\gamma,c)$. Here is an analysis of

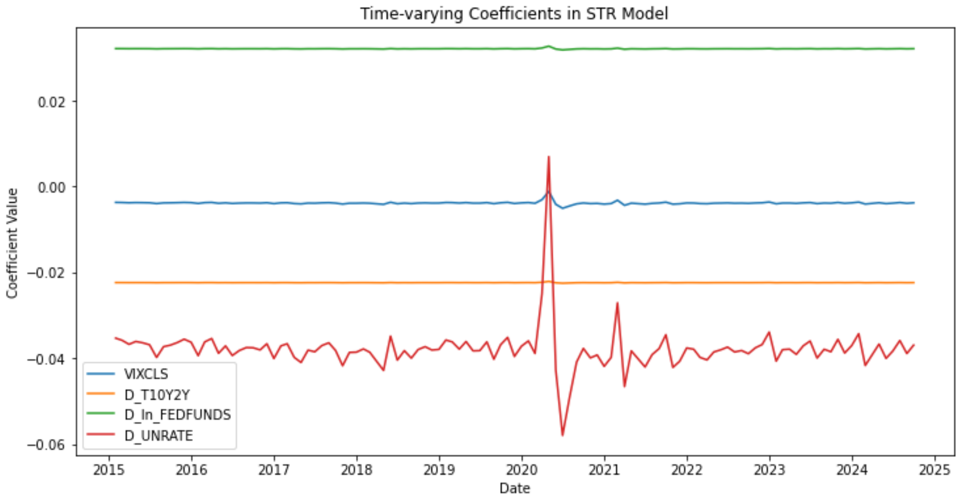

Figure 11: Evolution Residual of the STR Model over time

the observed behavior:

11.1.1 Overall Structure

- Each line in the plot represents the combined effect of the linear (β_0) and nonlinear (β_1) coefficients, modulated by the logistic transition function G_t.
- As G_t varies over time with changes in the transition variable $D\_\ln\_INDPRO$, the impact of each explanatory variable on $D\_\ln\_SP500$ shifts between the linear and nonlinear regimes.

11.1.2 Variable-Specific Observations

- **VIXCLS:** The coefficient for VIXCLS shows a mostly stable effect with minimal fluctuations over time, suggesting that the influence of market volatility on stock returns remains relatively constant, with minor adjustments between regimes.
- **D_T10Y2Y:** Similar to VIXCLS, the coefficient for D_T10Y2Y remains fairly stable over the period. This could indicate that the model captures the term spread's consistent impact on stock returns without substantial regime-dependent variation.
- **D_ln_FEDFUNDS:** The coefficient for the Federal Funds rate exhibits stability, implying that changes in interest rates exert a relatively constant effect on stock returns, regardless of the economic cycle.
- **D_UNRATE:** This coefficient shows more pronounced fluctuations, especially around 2020. The sharp spikes and dips indicate that the influence of the unemployment rate on stock returns varies more significantly with economic conditions, potentially reflecting

the labor market's heightened sensitivity to economic shocks like the COVID-19 pandemic.

11.1.3 Transition Dynamics

- The spikes around 2020 align with the COVID-19 period, suggesting that certain variables especially D_UNRATE, experience larger shifts in influence during times of economic stress.
- The STR model's flexibility allows it to capture these shifts, with the coefficient values adapting based on the economic regime as indicated by $D\_\ln\_INDPRO$.

11.2 Economic Insights

- **Constant vs. Regime-Sensitive Variables:** Variables like VIXCLS and D_T10Y2Y appear to have more stable effects, indicating that they impact $D\_\ln\_SP500$ consistently across economic cycles. In contrast, D_UNRATE displays greater variability, showing that labor market conditions influence stock returns differently depending on the economic environment.
- **Economic Cycles and Transition Effects:** The fluctuations in coefficients, particularly during periods of economic shocks, highlight the importance of models that allow for non-linear dynamics. This behavior underscores how certain variables may have asymmetric effects on stock returns during expansions versus contractions.

11.3 Summary

The time-varying coefficients demonstrate the STR model's capability to account for shifts in economic impact based on the business cycle. The observed variations, especially in D_UNRATE, emphasize the model's sensitivity to structural changes in the economy.

Analysis of ACF and PACF of Residuals

The Autocorrelation Function (ACF) and Partial Autocorrelation Function (PACF) plots of the residuals provide insights into the properties of the residuals from the Smooth Transition Regression (STR) model. Below is an analysis of each plot.

1. Autocorrelation Function (ACF)

The ACF plot shows the degree of correlation between the residuals and their lagged values over 24 lags.

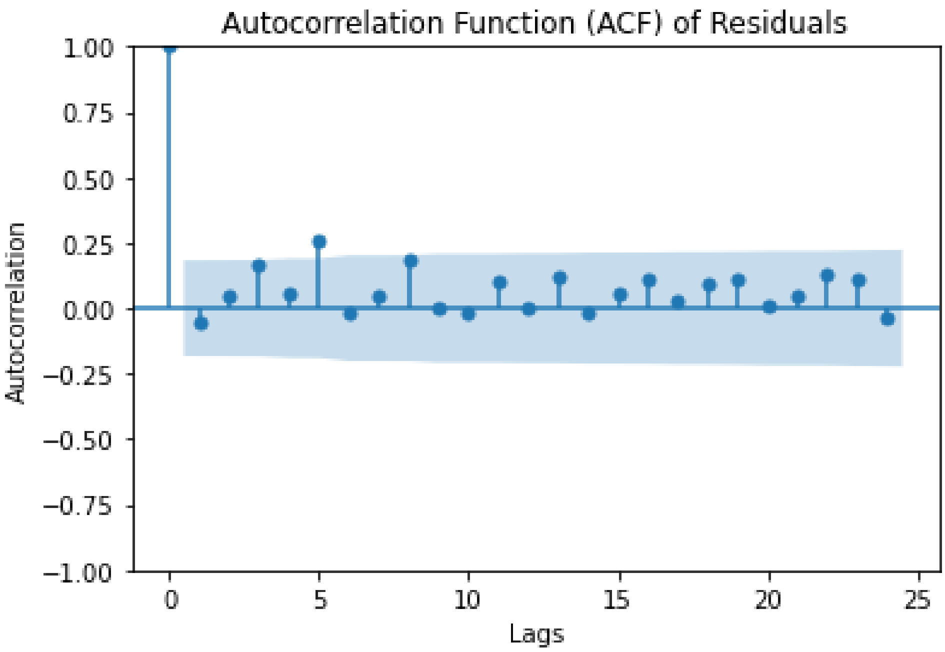

Figure 12: Evolution Residual of the STR Model over time

- **Key Observations:**
  - The autocorrelations for most lags are within the 95% confidence bands (the shaded area), indicating that the residuals do not exhibit significant autocorrelation at most lags.
  - There is a slight autocorrelation at lag 1, but it falls close to the confidence band, suggesting it's not strongly significant.
- **Implications:**
  - The lack of significant autocorrelation supports the idea that the residuals are approximately white noise, meaning the model has captured the main patterns in the data.
  - Minimal autocorrelation also confirms the results of the Ljung-Box test, which indicated no significant autocorrelation up to lag 12.

2. Partial Autocorrelation Function (PACF)

The PACF plot shows the partial correlation between the residuals and their lagged values, controlling for the values of the residuals at shorter lags.

- **Key Observations:**
  - Similar to the ACF plot, most of the partial autocorrelations fall within the confidence bands, indicating no significant partial autocorrelation at most lags. There is

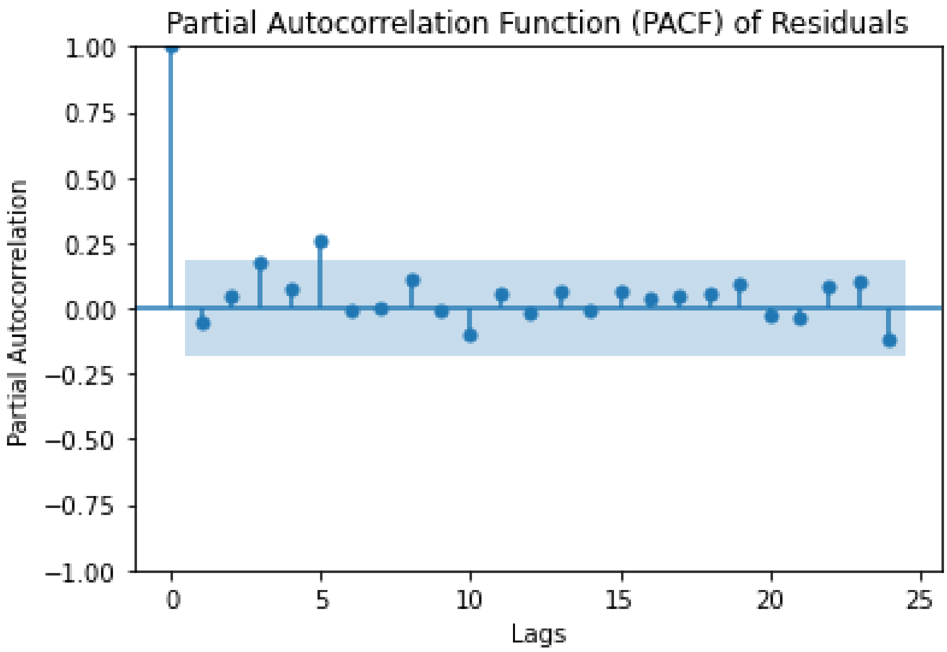

Figure 13: Evolution Residual of the STR Model over time

a small spike at lag 1, but it is relatively minor and doesn't suggest a strong need for model adjustment.

- **Implications:**
  - The absence of significant partial autocorrelation further suggests that the model's residuals behave like white noise.
  - This supports the effectiveness of the STR model in capturing the temporal structure in D_ln_SP500, with no remaining systematic patterns in the residuals.

Overall Summary

The ACF and PACF plots both indicate that the residuals from the STR model are approximately uncorrelated, suggesting that the model has effectively captured the dynamics in the data. This is consistent with the Ljung-Box test results and implies that the model does not exhibit serial correlation issues.

Conclusion

The Smooth Transition Regression (STR) model applied in this analysis demonstrates the flexibility and capability of capturing the nonlinear relationships among key economic indicators and their impacts on stock returns. The model successfully highlights the regime-dependent nature of financial and economic variables, such as the VIX, yield spread, and unemployment rate, showcasing how their influence on returns fluctuates across different economic cycles. Furthermore, diagnostic tests and residual analyses confirm the model's robustness in capturing time-varying dynamics while maintaining approximate normality and low autocorrelation in residuals.

Despite these strengths, the model shows some limitations in extreme volatility periods, as observed during the COVID-19 pandemic. This suggests that while STR models are effective in representing gradual transitions, additional modifications may be necessary to better capture sudden economic shifts. Overall, this analysis provides a strong foundation for understanding nonlinear economic impacts in financial modeling and highlights areas for potential model refinement, especially for extreme market conditions.