Exercises

1. Give a recursive definition of the length of a string over Σ. Use the primitive operation from the definition of string.
2. Using induction on i, prove that $(w^R)^i = (w^i)^R$ for any string w and all $i \geq 0$.
3. Prove, using induction on the length of the string, that $(w^R)^R = w$ for all strings $w \in \Sigma^*$.
4. Give a recursive definition of the set of strings over $\{a, b\}$ that contains all and only those strings with an equal number of a's and b's. Use concatenation as the operator.
5. Give a recursive definition of the set $\{a^i b^j \mid 0 < i < j\}$.
6. Prove that every string in the language defined in Example 2.2.1 has even length. The proof is by induction on the recursive generation of the strings.
7. Prove that every string in the language defined in Example 2.2.2 has at least as many a's as b's. Let $n_a(u)$ and $n_b(u)$ be the number of a's and b's in a string u. The inductive proof should establish the inequality $n_a(u) \geq n_b(u)$.
8. Let L be the language over $\{a, b\}$ generated by the recursive definition
   i) Basis: $\lambda \in$ L.
   ii) Recursive step: If $u \in$ L then $aaub \in$ L.
   iii) Closure: A string w is in L only if it can be obtained from the basis by a finite number of applications of the recursive step.

   a) Give the sets L_0, L_1, and L_2 generated by the recursive definition.
   b) Give an implicit definition of the set of strings defined by the recursive definition.
   c) Prove, by mathematical induction, that for every string u in L the number of a's in u is twice the number b's in u. Let $n_a(u)$ denote the number of a's in a string u and $n_b(u)$ denote the number of b's in u.
9. A **palindrome** over an alphabet Σ is a string in Σ^* that is spelled the same forward and backward. The set of palindromes over Σ can be defined recursively as follows:
   i) Basis: λ and a, for all $a \in \Sigma$, are palindromes.
   ii) Recursive step: If w is a palindrome and $a \in \Sigma$, then awa is a palindrome.
   iii) Closure: w is a palindrome only if it can be obtained from the basis elements by a finite number of applications of the recursive step.

   The set of palindromes can also be defined by $\{w \mid w = w^R\}$. Prove that these two definitions generate the same set.
10. Let X $= \{aa, bb\}$ and Y $= \{\lambda, b, ab\}$.
    a) List the strings in the set XY.
    b) List the strings of the set Y* of length three or less.
    c) How many strings of length 6 are there in X*?