Theorem 2.1.4
Let $u, v, w \in \Sigma^*$. Then $(uv)w = u(vw)$.

Proof The proof is by induction on the length of the string w. The string w was chosen for compatibility with the recursive definition of strings, which builds on the right-hand side of an existing string.

Basis: $length(w) = 0$. Then $w = \lambda$, and $(uv)w = uv$ by the definition of concatenation. On the other hand, $u(vw) = u(v) = uv$.

Inductive Hypothesis: Assume that $(uv)w = u(vw)$ for all strings w of length n or less.

Inductive Step: We need to prove that $(uv)w = u(vw)$ for all strings w of length $n + 1$. Let w be such a string. Then $w = xa$ for some string x of length n and $a \in \Sigma$ and

$$\begin{aligned}
(uv)w &= (uv)(xa) && \text{(substitution, } w = xa\text{)}\\
&= ((uv)x)a && \text{(definition of concatenation)}\\
&= (u(vx))a && \text{(inductive hypothesis)}\\
&= u((vx)a) && \text{(definition of concatenation)}\\
&= u(v(xa)) && \text{(definition of concatenation)}\\
&= u(vw). && \text{(substitution, } xa = w\text{)}
\end{aligned}$$

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Since associativity guarantees the same result regardless of the order of the operations, parentheses are omitted from a sequence of applications of concatenation. Exponents are used to abbreviate the concatenation of a string with itself. Thus uu may be written u^2, uuu may be written u^3, etc. u^0, which represents concatenating u with itself zero times, is defined to be the null string. The operation of concatenation is not commutative. For strings $u = ab$ and $v = ba$, $uv = abba$ and $vu = baab$. Note that $u^2 = abab$ and not $aabb = a^2b^2$.

Substrings can be defined using the operation of concatenation. Intuitively, u is a substring of v if u "occurs inside of" v. Formally, u is a **substring** of v if there are strings x and y such that $v = xuy$. A **prefix** of v is a substring u in which x is the null string in the decomposition of v. That is, $v = uy$. Similarly, u is a **suffix** of v if $v = xu$.

The reversal of a string is the string written backward. The reversal of $abbc$ is $cbba$. Like concatenation, this unary operation is also defined recursively on the length of the string. Removing an element from the right-hand side of a string constructs a smaller string that can then be used in the recursive step of the definition. Theorem 2.1.6 establishes the relationship between the operations of concatenation and reversal.

Definition 2.1.5
Let u be a string in Σ^*. The **reversal** of u, denoted u^R, is defined as follows:

i) Basis: $length(u) = 0$. Then $u = \lambda$ and $\lambda^R = \lambda$.
ii) Recursive step: If $length(u) = n > 0$, then $u = wa$ for some string w with length $n - 1$ and some $a \in \Sigma$, and $u^R = aw^R$.