BASICS OF LINEAR ALGEBRA

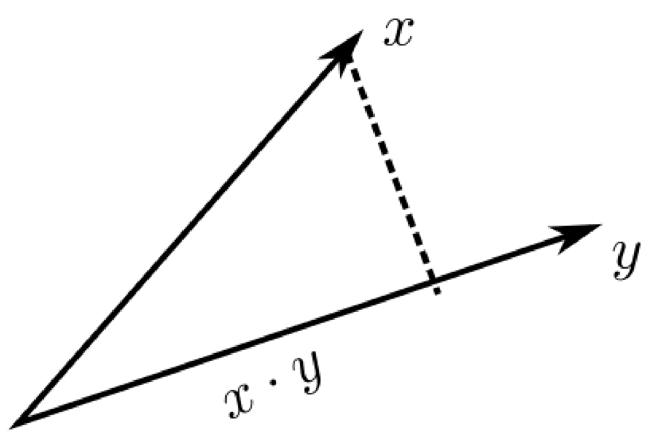
Vectors and inner products

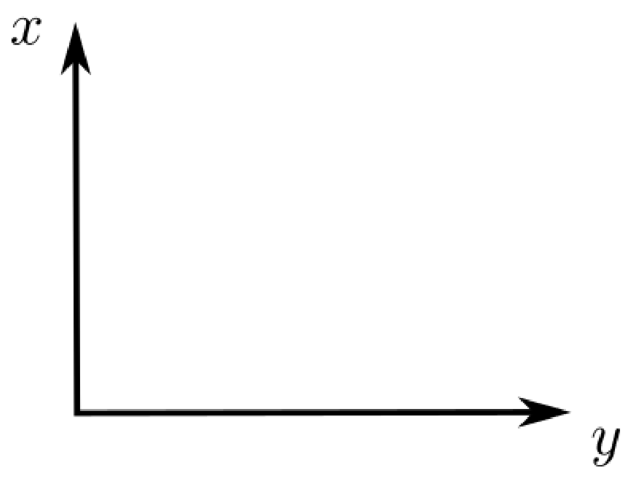
A d-dimensional vector **x** is just a collection of d numbers [x_1, x_2 … x_d]

The **inner/scalar/dot product** of two d-dimensional vectors **x** and **y** is written as $x \cdot y$, $x^T y$ or $\langle x, y \rangle$ is given by the formula $\sum_{i=1}^{d} x_i y_i$

This is a single number, measuring how similar **x** and **y** are to each other

- The inner product of a vector with itself is the square of its length
- If one of the vectors (say **y**) has length 1, then $x \cdot y$ gives the length of the projection of x on y
- Two vectors whose inner product is 0 are **orthogonal** to each other





Matrices

An $m \times n$ matrix is a collection of numbers organized in m rows and n columns

If **A** is an $m \times n$ matrix, and **B** is an $n \times p$ matrix, then **C** = **AB** is an $m \times p$ matrix whose (j,k)th element (in row j and column k) is the inner product of the jth row of **A** and kth column of **B**

- A common special case is when **B** is a column vector (that is an $n \times 1$ matrix). Then, the result of multiplying **A** and **B** is an $m \times 1$ matrix

An identity matrix **I** is a square matrix with 1 in the diagonal and 0 elsewhere

The inverse $\mathbf{A}^{-1}$ of an $m \times m$ matrix **A** is another $m \times m$ matrix satisfying $\mathbf{A}\,\mathbf{A}^{-1} = \mathbf{I}$

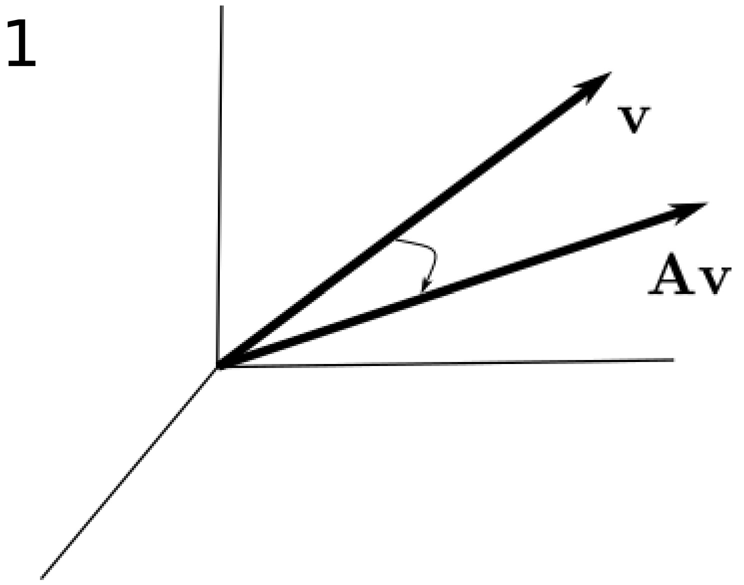
Orthogonal/rotation matrices

The transpose of $\mathbf{A}^T$ of an $m \times n$ matrix **A** is an $n \times m$ matrix obtained by swapping row and column indices

- That is, element (j,k) of **A** equals element (k,j) of $\mathbf{A}^T$

A matrix is said to be orthonormal if it satisfies $\mathbf{A}\mathbf{A}^T = \mathbf{I}$ (its inverse equals its transpose)

For such a matrix all columns are orthogonal to each other and have length 1



Such a matrix is also called a rotation matrix:

- For a vector **v** and a rotation matrix **A**, **v** and **Av** have the same length

Eigenvalue decomposition

A matrix **A** is symmetric if it is equal to its transpose: $\mathbf{A} = \mathbf{A}^T$

- A common example is a distance or covariance matrix, with element (j,k) giving the distance/covariance between datapoints j and k

Eigendecomposition (EVD): A symmetric matrix **A** can be decomposed as $\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{U}^T$

- Here **D** is a diagonal matrix, and **U** is an orthogonal matrix ($\mathbf{U}\mathbf{U}^T = \mathbf{I}$)
- U and D are usual chosen to that the diagonal of **D** is decreasing in magnitude

The i^{th} diagonal element of **D** is called the i^{th} eigenvalue
The i^{th} row of **U** is called the i^{th} eigenvector

Each (eigenvalue,eigenvector) pair (d_i,$\mathbf{u}_i$) satisfies: $\mathbf{A}\mathbf{u}_i = d_i\mathbf{u}_i$

Also useful is the more general singular value decomposition (SVD)