Advanced Discretization Techniques
Homework 4

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Exercise 9: Saddle point problems (revisited)

a) As per the hint, we just need to check that the hypothesis of Theorem 6.12 [K,A] are satisfied. The bilinear form a is continuous by the Cauchy-Schwarz inequality. The bilinear form b is continuous by Hölder's inequality and the assumption that Ω has finite measure. The functional l_1 in the right-hand side is continuous by the Cauchy-Schwarz inequality and the boundedness of the trace operator. The operator B is defined so that the identity

$$b(u, \mu) = \langle Bu, \mu \rangle_X = \langle Bu, \mu \rangle_{\mathbb{R}} = \mu \cdot Bu$$

holds for any $(u, \mu) \in M \times X = H^1(\Omega) \times \mathbb{R}$, so it's clear that

$$Bu = \int_\Omega u \, dx \quad \text{and that} \quad N = \ker(B) = \left\{ u \in H^1(\Omega) \mid \int_\Omega v \, dx = 0 \right\}.$$

Now we need to check that a satisfies conditions (NB1) and (NB2) (defined in Theorem 6.1 [K,A]) on $N \times N$. As per Remark 6.2.3, it's actually enough to show that a is H^1-coercive. This follows from the Poincaré-Wirtinger inequality:

$$\begin{aligned} a(u, u) = \frac{1}{2} |u|_1^2 + \frac{1}{2} |u|_1^2 \geq \frac{c}{2} \|u - \bar{u}\|_0^2 + \frac{1}{2} |u|_1^2 \\ = \frac{c}{2} \|u - 0\|_0^2 + \frac{1}{2} |u|_1^2 \geq c' \|u\|_1^2 . \end{aligned}$$

We have used the fact that $\bar{u}$, the integral mean of u, vanishes (by definition) for any u in N. To conclude the proof, we will check that $\tilde{b}$ satisfies (NB2) on $\mathbb{R} \times N^\perp$:

$$\inf_{\substack{\mu \in \mathbb{R} \\ \mu \neq 0}} \sup_{\substack{u \in N^\perp \\ u \neq 0}} \frac{\tilde{b}(\mu, u)}{\|\mu\|_{\mathbb{R}} \|u\|_1} \geq \inf_\mu \frac{\mu \int_\Omega \operatorname{sgn}(\mu) \, dx}{\|\mu\|_{\mathbb{R}} \|\operatorname{sgn}(\mu)\|_1} = \inf_\mu \frac{\int_\Omega 1 \, dx}{\|\operatorname{sgn}(\mu)\|_0} = 1 > 0.$$

The function $u \in N^\perp$ was chosen to be constantly equal to the sign of μ.

b) This exercise can be enormously simplified by choosing a different scalar product in $H_0^1(\Omega)$:

$$\langle u, v \rangle = \int_\Omega \nabla u \cdot \nabla v \, dx .$$

By Poincaré's inequality, this scalar product is well-defined and induces a norm equivalent to the usual one. The continuity of a and l_2 is trivial. The bilinear form b is equal to this new scalar product, so it's automatically continuous. The operator B is the identity on $H_0^1(\Omega)$, hence $\ker(B) = \{0\}$. Since N is trivial, a satisfies conditions (NB1) and (NB2) on $N \times N$ without any effort. The bilinear form $\tilde{b}$, as all scalar products, is clearly H_0^1-coercive with coercivity constant 1. $\square$

Exercise 10: The nature of saddle point problems

In order to show the equivalence, J will be the following Lagrange functional:

$$J(v,q) = \frac{1}{2} v^T A v - f^T v + q^T (Bv - g).$$

$\Leftarrow$) Let $(u,p) \in \mathbb{R}^n \times \mathbb{R}^m$ be such that

$$J(u,p) = \max_{q \in \mathbb{R}^m} \min_{v \in \mathbb{R}^n} J(v,q).$$

We want to prove that (u,p) is a solution to the linear system. Let

$$I(q) = \min_{v \in \mathbb{R}^n} J(v,q).$$

For a given q, the function J is quadratic in v through A, a symmetric positive definite matrix, so the minimization problem $\min_v J(v,q)$ has a unique solution and $I(q)$ is well defined. Now we want to show that the maximization problem $\max_q I(q)$ also has a unique solution. This can be done by explicitly calculating an expression for $I(q)$:

$$I(q) = \min_{v \in \mathbb{R}^n} J(v,q) = J\left(\operatorname*{argmin}_{v \in \mathbb{R}^n} J(v,q), q\right)$$

$$\partial_v J(v,q) = Av - f + B^T q = 0 \quad \Leftrightarrow \quad v = A^{-1}(f - B^T q)$$

$$\operatorname*{argmin}_{v \in \mathbb{R}^n} J(v,q) = A^{-1}(f - B^T q) \quad \Rightarrow \quad I(q) = J(A^{-1}(f - B^T q), q)$$

Expanding the last term, we get

$$\begin{aligned} I(q) &= \frac{1}{2}(A^{-1}(f - B^T q))^T A A^{-1}(f - B^T q) - f^T A^{-1}(f - B^T q) \\ &\quad + q^T (B A^{-1}(f - B^T q) - g) \\ &= \frac{1}{2}(f^T - q^T B) A^{-1}(f - B^T q) - f^T A^{-1} f + f^T A^{-1} B^T q \\ &\quad + q^T B A^{-1} f - q^T B A^{-1} B^T q - q^T g \\ &= -\frac{1}{2} f^T A^{-1} f - \frac{1}{2} q^T B A^{-1} B^T q + f^T A^{-1} B^T q - q^T g. \end{aligned}$$

The existence and uniqueness of the maximum of $I(q)$ is given by the quadratic term in q through the matrix $BA^{-1}B^T$, which is positive definite thanks to the rank condition on B:

$$q \neq 0 \Rightarrow B^T q \neq 0 \Rightarrow (B^T q)^T A^{-1} B^T q > 0 \Rightarrow q^T B A^{-1} B^T q > 0.$$

Back to the main argument, what we have shown so far proves that (u, p) is the unique solution to the max min problem for J, and that

$$\begin{cases} u = \operatorname{argmin}_v J(v, p) \\ p = \operatorname{argmax}_q I(q). \end{cases} \tag{1}$$

On the one hand, equation (1u) implies that

$$0 = \partial_v J(u, p) = Au - f + B^T p, \tag{2}$$

and this shows that (u, p) solves the first equation of the linear system. On the other hand, equation (1p) implies that

$$\begin{aligned} 0 = \partial_q I(p) &= -BA^{-1}B^T p + BA^{-1} f - g \\ &= BA^{-1}(-B^T p + f) - g \overset{(2)}{=} BA^{-1}Au - g = Bu - g. \end{aligned}$$

This proves that (u, p) also solves the second equation of the linear system.

$\Rightarrow$) Let (u, p) be a solution to the linear system. We want to prove that

$$J(u, p) = \max_{q \in \mathbb{R}^m} \min_{v \in \mathbb{R}^n} J(v, q).$$

In the last point we have shown that the max min problem for J always admits a solution, say $(\bar{v}, \bar{q})$, and that $(\bar{v}, \bar{q})$ automatically solves the linear system. All that's left to do is to show that the linear system can't have two different solutions. We will do this by showing that the kernel of the block matrix is trivial. Let (w, r) be such that

$$\begin{cases} Aw + B^T r = 0 \\ Bw = 0. \end{cases}$$

Then $w = -A^{-1}B^T r$, which means that $0 = Bw = -BA^{-1}B^T r$. Since the matrix $BA^{-1}B^T$ is positive definite, hence invertible, it follows that $r = 0$, $Aw = 0$, and finally $w = 0$. Now, $(u, p) = (\bar{v}, \bar{q})$ and therefore

$$J(u, p) = J(\bar{v}, \bar{q}) = \max_{q \in \mathbb{R}^m} \min_{v \in \mathbb{R}^n} J(v, q). \quad \square$$