Advanced Discretization Methods (WS 19/20)
Homework 1

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Deadline for submission (theory):	October 29th, 2019, 12:15
Deadline for submission (programming):	(this sheet: no programming)

Exercise 1: Minimization property (10)

Let V be a Hilbert space, a be a symmetric, continuous, and coercive bilinear form, $f \in V'$, $J(u) := \frac{1}{2}a(u,u) - f(u)$, and $U \subset V$ be closed and convex. Show that u is a solution of

$$J(u) = \inf_{v \in U} J(v)$$

if and only if

$$a(u, v-u) \geq f(v-u) \qquad \forall v \in U.$$

Exercise 2: Chain rule for weak derivatives (5+5)

a) Let $f \in C^1(\mathbb{R}), f' \in L^\infty(\mathbb{R}), u \in L^1_{\text{loc}}(\Omega)$ with weak derivatives of order 1 (i.e. there exists $D^\alpha u \in L^1_{\text{loc}}(\Omega)$ for all multi-indices $\alpha \in \mathbb{N}^d$ with $|\alpha| = 1$). Prove that a version of the chain rule also holds in the weak sense, i.e. $f \circ u$ is weakly differentiable and

$$D(f \circ u) = f'(u)Du$$

Hint: You can use the following theorem without proof:
Let u and v be locally integrable in Ω. Then $v = D^\alpha u$ if and only if there exists a sequence of $C^\infty(\Omega)$ functions $(u_m)_{m\in\mathbb{N}}$ converging in $L^1_{loc}(\Omega)$ whose derivatives $D^\alpha u_m$ converge to v in $L^1_{loc}(\Omega)$.

b) Let $u \in H^1(\Omega)$, $u^+ := \max\{u, 0\}$, $u^- := \min\{u, 0\}$.
Show that u^+, u^-, $|u| \in H^1(\Omega)$ and

$$Du^+ = \begin{cases} Du & u > 0 \\ 0 & u \leq 0 \end{cases}$$

$$Du^- = \begin{cases} 0 & u \geq 0 \\ Du & u < 0 \end{cases}$$

$$D|u| = \begin{cases} Du & u > 0 \\ 0 & u = 0 \\ -Du & u < 0 \end{cases}$$

Hint: Approximate u^+ with

$$f_\varepsilon(u) := \begin{cases} (u^2 + \varepsilon^2)^{1/2} - \varepsilon & u > 0 \\ 0 & u \leq 0 \end{cases}$$

Exercise 3: H^1-Estimate for the Poisson Problem (3+2+5)

Let $\Omega \subset \mathbb{R}^d$ be a sufficiently smooth domain. For $f \in L^2(\Omega)$ we consider the Poisson problem with Dirichlet boundary values:

$$-\triangle u = f \quad \text{in } \Omega, \qquad\qquad u|_{\partial\Omega} = 0.$$

Let $X_h \subset X := H_0^{1,2}(\Omega)$ be the Lagrange finite element space of order $k \in \mathbb{N}$.

a) Derive the weak form of this equation, both for the continuous problem with $u \in X$ and the discrete problem with $U_h \in X_h$.

b) Prove for the discrete solution $U_h \in X_h$ the following error identity:

$$\int_\Omega \nabla(U_h - u)\nabla\phi_h \; dx = 0 \qquad \forall \phi_h \in X_h.$$

c) Use Céa's Lemma and the Lagrange interpolation operator I_h to derive the following error estimate if $u \in H^{k+1,2}(\Omega)$ and $d < 2k+2$:

$$\|u - U_h\|_X \leq C \; h^k \|u\|_{k+1,2}.$$

Explain the necessity of the condition on d.

Exercise 4: L^2-Estimate for the Poisson Problem (2+3+3+2)

Let $\Omega \subset \mathbb{R}^d$ be a sufficiently smooth domain. We use a duality argument to derive an L^2-estimate for the Poisson problem from Exercise 3. For this, consider the dual problem: Find $w \in X$ such that

$$\int_\Omega \nabla w \nabla \phi \; dx = \int_\Omega (U_h - u)\phi \; dx \qquad \forall \phi \in X.$$

We assume H^2-regularity of our problem, i.e. for $U_h - u \in L^2(\Omega)$ there holds $w \in H^{2,2}(\Omega) \cap H_0^{1,2}(\Omega)$ with bound $\|w\|_2 \leq C \; \|U_h - u\|$.

a) Let $W_h \in X_h$ be the finite element solution of the dual problem. Using the Galerkin orthogonality, prove that

$$\|U_h - u\|^2 = \int_\Omega \nabla(U_h - u)\nabla(w - W_h) \; dx.$$

b) Use the preceeding identity to prove

$$\|U_h - u\|^2 \leq C\|\nabla(U_h - u)\| \inf_{V_h \in X_h} \|\nabla(w - V_h)\|.$$

c) Use Lagrange interpolation estimates and the assumed H^2-regularity to prove for $d \leq 3$:

$$\inf_{V_h \in X_h} \|\nabla(w - V_h)\| \leq C\ h\|U_h - u\|.$$

d) If $u \in H^{k+1,2}(\Omega)$ and $d < 2k + 2$ conclude with Exercise 3:

$$\|u - U_h\| \leq C\, h^{k+1}\|u\|_{k+1,2}.$$