Advanced Discretization Techniques
Homework 9

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Exercise 18: A FV scheme for a nonlinear equation (Part II)

a) Since $u_{\mathcal{T}}(x)$ is a piecewise constant function, $\|u_{\mathcal{T}}(x)\|_{L^2(\mathbb{R})}$ can be computed in the following way:

$$\|u_{\mathcal{T}}(x)\|^2_{L^2(\mathbb{R})} = U_1^2 h_1 + U_2^2 h_2 + \cdots + U_N^2 h_n.$$

By Ex.17 c),

$$\|u_{\mathcal{T}}(x)\|^2_{L^2(\mathbb{R})} \leq C(h_1 + h_2 + \cdots + h_n) = C.$$

And the boundedness of $u_{\mathcal{T}}$ is thus proved.

b) Since $u_{\mathcal{T}}(x)$ is a piecewise constant function, we may assume that $u_{\mathcal{T}}(x+\eta) = U_{i+j}$ and $u_{\mathcal{T}}(x) = U_i$, where $1 \leq i, i+j \leq N$.

$$\begin{aligned}
(u_{\mathcal{T}}(x+\eta) - u_{\mathcal{T}}(x))^2 &= (U_{i+j} - U_i)^2 \\
&\leq (|U_{i+j} - U_{i+j-1}| + |U_{i+j-1} - U_{i+j-2}| + \cdots + |U_{i+1} - U_i|)^2 \\
&\leq \left(\sum_{i=0}^{N} |U_{i+1} - U_i| \right)^2 \\
&\leq ??? \left(\sum_{i=0}^{N} |U_{i+1} - U_i| \chi_{i+\frac{1}{2}}(x) \right)^2 \\
&\leq \left(\sum_{i=0}^{N} \frac{(U_{i+1} - U_i)^2}{h_{i+\frac{1}{2}}} \chi_{i+\frac{1}{2}}(x) \right) \left(\sum_{i=0}^{N} \chi_{i+\frac{1}{2}}(x) h_{i+\frac{1}{2}} \right)
\end{aligned}$$

c) $\sum\limits_{i=0}^{N} \chi_{i+\frac{1}{2}}(x) h_{i+\frac{1}{2}} \leq \eta + 2h???$

d) By c) and the condition $\eta + 2h \leq 3$, we have

$$\| \frac{u_{\mathcal{T}_n}(\cdot + \eta) - u_{\mathcal{T}_n}}{\eta} \|^2_{0,\mathbb{R}} \leq 3C.$$

Since $\lim_{n\to\infty} u_{\mathcal{T}_n} = u \in L^2(\mathbb{R})$, and noticing that the right hand side of the above inequality is a constant, we have

$$\|\frac{u(\cdot+\eta)-u}{\eta}\|_{0,\mathbb{R}}^2 \le 3C,$$

and furthermore

$$\|\lim_{\eta\to 0}\frac{u(\cdot+\eta)-u}{\eta}\|_{0,\mathbb{R}}^2 \le 3C.$$

The boundedness of $\Delta_\eta u$ is thus proved.

e) Multiplying the equation (A) in Ex.17 of the discretized FV scheme by φ_i, we get

$$(\mathcal{F}_{i+\frac{1}{2}} - \mathcal{F}_{i-\frac{1}{2}})\varphi_i = h_i f_i(U_i)\varphi_i, \quad i = 1,\cdots,N$$

By the conditions $\mathcal{F}_{i+\frac{1}{2}} = -\frac{U_{i+1}-U_i}{h_{i+\frac{1}{2}}}, \quad i = 1,\cdots,N$ and $U_0 = U_{N+1} = 0$, we have further that:

$$\sum_{i=1}^{N}\left(-\frac{U_{i+1}-U_i}{h_{i+\frac{1}{2}}} - \left(-\frac{U_i - U_{i-1}}{h_{i-\frac{1}{2}}}\right)\right)\varphi_i = \sum_{i=1}^{N} f_i\varphi_i h_i$$

Noticing that the l.h.s. can be rewritten as

$$\sum_{i=1}^{N}\left(-\frac{U_{i+1}-U_i}{h_{i+\frac{1}{2}}} - \left(-\frac{U_i - U_{i-1}}{h_{i-\frac{1}{2}}}\right)\right)\varphi_i = \sum_{i=1}^{N}\left(\frac{\varphi_i-\varphi_{i-1}}{h_{i-\frac{1}{2}}} - \frac{\varphi_{i+1}-\varphi_i}{h_{i+\frac{1}{2}}}\right)u_i$$

$$= \sum_{i=1}^{N}\frac{1}{h_i}\left(\frac{\varphi_i-\varphi_{i-1}}{h_{i-\frac{1}{2}}} - \frac{\varphi_{i+1}-\varphi_i}{h_{i+\frac{1}{2}}}\right)u_i h_i$$

Then we get

$$\int_0^1 u_{\mathcal{T}}\psi_{\mathcal{T}}dx = \sum_{i=1}^{N}\frac{1}{h_i}\left(\frac{\varphi_i-\varphi_{i-1}}{h_{i-\frac{1}{2}}} - \frac{\varphi_{i+1}-\varphi_i}{h_{i+\frac{1}{2}}}\right)u_i h_i = \sum_{i=1}^{N} f_i\varphi_i h_i = \int_0^1 f_{\mathcal{T}}\varphi_{\mathcal{T}}dx$$

f) Since φ is by definition a smooth function, we have

$$\frac{\varphi_{i+1}-\varphi_i}{h_{i+\frac{1}{2}}} = \varphi'(x_{i+\frac{1}{2}}) + R_{i+\frac{1}{2}},$$

where $R_{i+\frac{1}{2}}$ is a remainder term. Then

$$\int_0^1 u_{\mathcal{T}}(x)\psi_{\mathcal{T}}(x)dx = \sum_{i=1}^{N}\int_{\Omega_i}\frac{U_i}{h_i}(\varphi'(x_{i-\frac{1}{2}}) - \varphi'(x_{i+\frac{1}{2}}))dx + \sum_{i=1}^{N} U_i(R_{i-\frac{1}{2}} - R_{i+\frac{1}{2}})$$

$$= \int_0^1 -u_{\mathcal{T}}(x)\theta_{\mathcal{T}}(x)dx + \sum_{i=0}^{N} R_{i+\frac{1}{2}}(U_{i+1}-U_i).$$

g) By the regularity of φ again, we have

$$\frac{\varphi'(x_{i+\frac{1}{2}}) - \varphi'(x_{i-\frac{1}{2}})}{h_i} = \varphi''(x_i) + R_i'.$$

Define $\varphi''(x) := \sum_{i=1}^{N} \varphi''(x_i)\chi_{\Omega_i}(x)$, then

$$\int_0^1 -u_{\mathcal{T}}(x)\theta_{\mathcal{T}}(x)dx = \int_0^1 -u_{\mathcal{T}}(x)\varphi''(x)dx + \sum_{i=1}^{N} U_i R_i'. \tag{1}$$

If $h \to 0$, i.e. $n \to \infty$, then $R_i', R_{i-\frac{1}{2}} \to 0$ for all $i \in \mathbb{N}$ and $u_{\mathcal{T}_n} \to u$, $f_{\mathcal{T}_n} \to f$, moreover by the conclusions of f) and (1), we get the limiting case of conclusion e):

$$-\int_0^1 u(x)\varphi''(x)dx = \int_0^1 f(x, u(x))\varphi(x)dx.$$