Advanced Discretization Techniques
Homework 6

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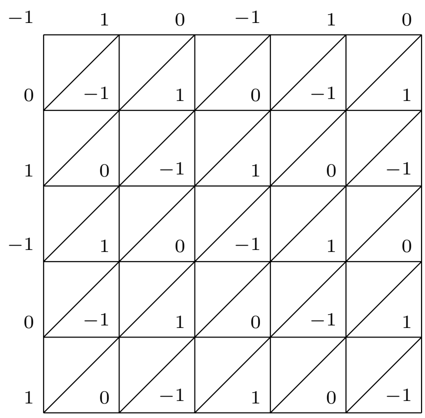
Exercise 13: Instability of $(\mathcal{P}_1)^2$-$(\mathcal{P}_1)$ for Stokes

a) One way to construct such a function $p_h \in S_0^1$ is as follows:



For any $T \in \mathcal{T}_h$, it's clear from the picture that the values of p_h at the three vertices a_1, a_2, a_3 of T are always a permutation of $\{0, 1, -1\}$. Without loss of generality, we can assume that $p_h(a_1) = 0, p_h(a_2) = 1, p_h(a_3) = -1$. Since p_h is an affine function when restricted to T, we get that

$$p_h(\lambda a_i + (1-\lambda)a_j) = \lambda p_h(a_i) + (1-\lambda)p_h(a_j)$$

holds for all $i, j \in \{1, 2, 3\}$ and all $\lambda \in [0, 1]$. Then, by the midpoint rule (which is exact for affine functions), it follows that

$$\int_T p_h(x)\,dx = \frac{|T|}{3}\left(p_h\left(\frac{1}{2}a_1 + \frac{1}{2}a_2\right) + p_h\left(\frac{1}{2}a_2 + \frac{1}{2}a_3\right) + p_h\left(\frac{1}{2}a_3 + \frac{1}{2}a_1\right)\right)$$

$$= \frac{|T|}{3}\left(\frac{1}{2}p_h(a_1) + \frac{1}{2}p_h(a_2) + \frac{1}{2}p_h(a_2) + \frac{1}{2}p_h(a_3) + \frac{1}{2}p_h(a_3) + \frac{1}{2}p_h(a_1)\right)$$

$$= \frac{|T|}{3}\left(\frac{0}{2} + \frac{1}{2} + \frac{1}{2} + \frac{-1}{2} + \frac{-1}{2} + \frac{0}{2}\right) = \frac{|T|}{3}\left(\frac{2-2}{2}\right) = 0.$$

This proves that $p_h \in L_0^2$ (the integral over Ω can be split as the sum of the integrals over $T \in \mathcal{T}_h$, and square-integrability comes from the fact that $\bar{\Omega}$ is a compact set). By construction, $p_h \in S_0^1$. Hence, $p_h \in X_h$, as requested.

b) For the Stokes problem, we have that $b_h \colon M_h \times X_h \to \mathbb{R}$ is defined as

$$b_h(\mathbf{v}_h, p_h) = -\int_\Omega p_h \operatorname{div}(\mathbf{v}_h)\, dx\,.$$

As usual, we then define the operator $B_h \colon M_h \to X_h^*$ so that

$$b_h(\mathbf{v}_h, p_h) = \langle B_h \mathbf{v}_h, p_h\rangle = \left\langle \mathbf{v}_h, B_h^T p_h\right\rangle.$$

As explained in Knabner-Angermann (8.42, p. 399,400), $\ker(B_h^T) \neq \{0\}$ is enough to ensure that the inf-sup conditions do not hold. We will now prove that $p_h \in \ker(B_h^T)$ using the fact that all functions in $(\mathcal{P}_1)^2$ have elementwise constant divergence (that's what we get when we derive a polynomial of degree at most 1). For any $\mathbf{v}_h \in M_h$,

$$\begin{aligned}\left\langle \mathbf{v}_h, B_h^T p_h\right\rangle &= -\int_\Omega p_h \operatorname{div}(\mathbf{v}_h)\, dx \;= -\sum_{T\in\mathcal{T}_h}\int_T p_h \operatorname{div}(\mathbf{v}_h)\, dx\\ &= -\sum_{T\in\mathcal{T}_h} \operatorname{div}(\mathbf{v}_h)\big|_T \int_T p_h\, dx \overset{a)}{=} -\sum_{T\in\mathcal{T}_h} \operatorname{div}(\mathbf{v}_h)\big|_T\, 0\, dx \;= 0.\end{aligned}$$

This implies that $B_h^T p_h = 0$ and $p_h \in \ker(B_h^T)$, as required. □

Exercise 14: A stable element - Fortin's trick

a) The subspace $S_{0,0}^1$ of $H_0^1(\Omega)$ is still a Hilbert space with respect to the same scalar product, because it's finite dimensional. The bilinear form a_h is equal to said scalar product, so it's continuous by Cauchy-Schwarz's inequality, and coercive by definition ($\alpha = 1$). The inclusion $S_{0,0}^1 \subset H_0^1(\Omega)$ induces the opposite inclusion for the dual spaces, so v (as an element of $(H_0^1(\Omega))^*$, by Riesz's isometry) is also an element of $(S_{0,0}^1)^*$. Then, by Lax-Milgram's lemma,

$$\exists!\; \pi_h^0 v \in S_{0,0}^1 \;\text{ such that }\; a_h(\pi_h^0 v, w_h) = \langle v, w_h\rangle_1 \;\text{ for all } w_h \in S_{0,0}^1$$

and we get the stability estimate

$$\left\|\pi_h^0 v\right\|_1 \le \frac{\|v\|_{(S_{0,0}^1)^*}}{\alpha} \le \frac{\|v\|_{(H_0^1(\Omega))^*}}{\alpha} = \frac{\|v\|_1}{\alpha} = \|v\|_1\,.$$

This is exactly what we wanted to prove.

b) By equation (A) we get that $a_h(w_h, \pi_h^0 v - v) = 0$, and by equation (B) we get that $a_h(u, \pi_h^0 v - v) = (f, \pi_h^0 v - v)_0$. Since we have assumed here that $f = \pi_h^0 v - v$, it is now clear by linearity that $a_h(u - w_h, \pi_h^0 v - v) = \left\|\pi_h^0 v - v\right\|_0^2$.

c) Since $\Omega \subset \mathbb{R}^2$, H^2-regularity of u is enough for $I_h u$ to be well-defined (I_h is the Lagrange interpolation operator), with an interpolation error estimate of $\|u - I_h u\|_1 \leq C_I h \|u\|_2$. If we now choose $w_h = I_h u$ in the result of point b), we get that

$$\begin{aligned}
\left\|\pi_h^0 v - v\right\|_0^2 &= a_h(u - I_h u, \pi_h^0 v - v) = \left\langle u - I_h u, \pi_h^0 v - v\right\rangle_1 \\
&\leq \|u - I_h u\|_1 \left\|\pi_h^0 v - v\right\|_1 \leq C_I h \|u\|_2 \left\|\pi_h^0 v - v\right\|_1 \\
&\leq C_I C h \|f\|_0 \left\|\pi_h^0 v - v\right\|_1 = C_I C h \left\|\pi_h^0 v - v\right\|_0 \left\|\pi_h^0 v - v\right\|_1 .
\end{aligned}$$

All that's left to do now is to divide by $\left\|\pi_h^0 v - v\right\|_0$.

d) By the triangle inequality and the results of points a) and c), we get that

$$\left\|\pi_h^0 v - v\right\|_0 \leq Ch \left\|\pi_h^0 v - v\right\|_1 \leq Ch \left\|\pi_h^0 v\right\|_1 + Ch \|v\|_1 \leq 2Ch \|v\|_1 .$$

e) By the definition of B_3, we need to find a constant C_T for each $T \in \mathcal{T}_h$ such that

$$\int_T v \, dx = \int_T \pi_h^1 v \, dx = \int_T C_T \, \varphi_{T,1}\varphi_{T,2}\varphi_{T,3} \, dx .$$

Solving for C_T gives

$$C_T = \left(\int_T v \, dx\right)\left(\int_T \varphi_{T,1}\varphi_{T,2}\varphi_{T,3} \, dx\right)^{-1} .$$

First of all, we need to make sure that we are not dividing by 0. Let $\hat{T}$ be the reference element in $\mathbb{R}^2$. Then, by the change of variables formula,

$$\begin{aligned}
&\int_T \varphi_{T,1}(\mathbf{x})\varphi_{T,2}(\mathbf{x})\varphi_{T,3}(\mathbf{x}) \, d\mathbf{x} = \int_{\hat{T}} xy(1 - x - y) \, 2\,|T| \, dx dy \\
&= 2\,|T| \int_0^1 \int_0^{1-x} xy(1 - x - y) \, dy \, dx = \frac{2\,|T|}{120} = \frac{|T|}{60} \neq 0.
\end{aligned}$$

Next, in order to conclude that π_h^1 is well-defined, we need to prove that a function $\pi_h^1 v$ defined on every element T as

$$\frac{60}{|T|}\left(\int_T v \, dx\right) \varphi_{T,1}\varphi_{T,2}\varphi_{T,3}$$

belongs to $H_0^1(\Omega)$. This means that we can't have discontinuities on any edge of the triangulation. For any edge e, let T_1 and T_2 be the two elements in $\mathcal{T}_h$ such that $e = \partial T_1 \cap \partial T_2$. Let a_i, a_j be the vertices in T_1, T_2 opposite to e (i and j are in local numbering notation). Then, for each $x \in e$,

$$\begin{aligned}
&\frac{60}{|T_1|}\left(\int_{T_1} v \, dx\right) \varphi_{T_1,1}(x)\varphi_{T_1,2}(x)\varphi_{T_1,3}(x) = 0 \\
&\frac{60}{|T_2|}\left(\int_{T_2} v \, dx\right) \varphi_{T_2,1}(x)\varphi_{T_2,2}(x)\varphi_{T_2,3}(x) = 0,
\end{aligned}$$

because $\varphi_{T_1,i}(x) = 0 = \varphi_{T_2,j}(x)$. Hence, $\pi_h^1 v$ vanishes on the boundary of every element. This not only ensures continuity (so that $\pi_h^1 v \in H^1(\Omega)$), but also that $\pi_h^1 v$ has 0 trace on $\partial\Omega$, as required.

f) By Jensen's inequality and the change of variables formula we get that

$$\begin{aligned}
\left\|\pi_h^1 v\right\|_{0,T}^2 &= 60^2 \left(\frac{1}{|T|}\int_T v\,dx\right)^2 \int_T \big(\varphi_{T,1}(x)\varphi_{T,2}(x)\varphi_{T,3}(x)\big)^2\,dx \\
&\le \frac{60^2}{|T|}\left(\int_T v^2\,dx\right)\int_T \big(\varphi_{T,1}(x)\varphi_{T,2}(x)\varphi_{T,3}(x)\big)^2\,dx \\
&= \frac{60^2}{|T|}\left(\int_T v^2\,dx\right) 2\,|T| \int_{\hat{T}} \big(\varphi_{T,1}(\hat{x})\varphi_{T,2}(\hat{x})\varphi_{T,3}(\hat{x})\big)^2\,d\hat{x} \\
&= \frac{60^2}{|T|}\left(\int_T v^2\,dx\right)\frac{2\,|T|}{5040} = \frac{10}{7}\,\|v\|_{0,T}^2 = C\,\|v\|_{0,T}^2\,.
\end{aligned}$$

The rest of the proof is straightforward:

$$\left\|\pi_h^1 v\right\|_0^2 = \sum_{T\in\mathcal{T}_h}\left\|\pi_h^1 v\right\|_{0,T}^2 \le \sum_{T\in\mathcal{T}_h} C\,\|v\|_{0,T}^2 = C\sum_{T\in\mathcal{T}_h}\|v\|_{0,T}^2 = C\,\|v\|_0^2$$

and then we take the square root on both sides.

g) We begin with integration by parts:

$$\begin{aligned}
b(v-\Pi_h v, q_h) &= -\int_\Omega q_h \operatorname{div}(v - \pi_h^0 v - \pi_h^1(v-\pi_h^0 v))\,dx \\
&= -\int_{\partial\Omega} q_h (v - \pi_h^0 v - \pi_h^1(v-\pi_h^0 v))\cdot n\,d\sigma \\
&\quad + \int_\Omega \nabla q_h \cdot (v - \pi_h^0 v - \pi_h^1(v-\pi_h^0 v))\,dx\,.
\end{aligned}$$

The boundary term vanishes, because the functions v, $\pi_h^0 v$ and $\pi_h^1(v-\pi_h^0 v)$ are all in $H_0^1(\Omega)^2$. As for the other term, we will use point e) and the fact that ∇q_h is elementwise constant (it's the gradient of a polynomial of degree at most 1) to show that it vanishes, too:

$$\begin{aligned}
&\int_\Omega \nabla q_h \cdot (v - \pi_h^0 v - \pi_h^1(v-\pi_h^0 v))\,dx = \sum_{T\in\mathcal{T}_h}\int_T \nabla q_h \cdot (v - \pi_h^0 v - \pi_h^1(v-\pi_h^0 v))\,dx \\
&= \sum_{T\in\mathcal{T}_h}\nabla q_h \cdot \int_T (v-\pi_h^0 v) - \pi_h^1(v-\pi_h^0 v)\,dx \overset{e)}{=} \sum_{T\in\mathcal{T}_h}\nabla q_h\cdot(0,0)^T\,dx = 0.
\end{aligned}$$

h) By Lemma 4.5.3 in Brenner and Scott's *The Mathematical Theory of FEM*, the following global inverse estimate holds:

$$\left\|\pi_h^1(v-\pi_h^0 v)\right\|_1 \le C_{\mathrm{inv}} h^{-1}\left\|\pi_h^1(v-\pi_h^0 v)\right\|_0\,.$$

Now we can combine all the results from the previous points to get that

$$\begin{aligned}
\|\Pi_h v\|_1 &= \left\|\pi_h^0 v + \pi_h^1(v-\pi_h^0 v)\right\|_1 \le \left\|\pi_h^0 v\right\|_1 + \left\|\pi_h^1(v-\pi_h^0 v)\right\|_1 \\
&\overset{a)}{\le} \|v\|_1 + \left\|\pi_h^1(v-\pi_h^0 v)\right\|_1 \le \|v\|_1 + C_{\mathrm{inv}}h^{-1}\left\|\pi_h^1(v-\pi_h^0 v)\right\|_0 \\
&\overset{f)}{\le} \|v\|_1 + C_{\mathrm{inv}}h^{-1}C_{\mathrm{f}}\left\|v-\pi_h^0 v\right\|_0 \overset{d)}{\le} \|v\|_1 + C_{\mathrm{inv}}h^{-1}C_{\mathrm{f}}C_{\mathrm{d}}h\,\|v\|_1 \\
&= (1 + C_{\mathrm{inv}}C_{\mathrm{f}}C_{\mathrm{d}})\,\|v\|_1 = C\,\|v\|_1\,. \quad \square
\end{aligned}$$