Advanced Discretization Techniques
Homework 3

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Exercise 7: Non-Consistency by Quadrature

a) We can assume without loss of generality that $h < 1$ for the whole exercise, so that $h^\alpha \leq h^\beta$ whenever $\alpha > \beta$. As per the hint, we will make use of the following inequality, called a *local inverse estimate*:

$$\|v_h\|_{k;T} \leq C_{\text{inv}} h^{-k+1} \|v_h\|_{1;T} \quad \text{for all } v_h \in X_h,\, T \in \mathcal{T}_h.$$

By the assumption on the quadrature rule, the Cauchy-Schwarz inequality and the inverse estimate, we have that

$$\begin{aligned}
|(f, v_h) - Q(f, v_h)| &\leq Ch^r \sum_{T \in \mathcal{T}_h} \|f\|_{r;T} \|v_h\|_{r;T} \\
&\leq Ch^r \left(\sum_{T \in \mathcal{T}_h} \|f\|_{r;T}^2 \right)^{1/2} \left(\sum_{T \in \mathcal{T}_h} \|v_h\|_{r;T}^2 \right)^{1/2} \\
&= Ch^r \|f\|_r \left(\sum_{T \in \mathcal{T}_h} \|v_h\|_{k;T}^2 \right)^{1/2} \\
&\leq Ch^r \|f\|_r C_{\text{inv}} h^{-k+1} \left(\sum_{T \in \mathcal{T}_h} \|v_h\|_{1;T}^2 \right)^{1/2} \\
&= C_a h^{r-k+1} \|f\|_r \|v_h\|_1 .
\end{aligned}$$

We have also used the fact that $k \leq r$ and that $D^{k+1} v_h$ vanishes, because v_h is a polynomial of degree at most k on each T.

b) Similar arguments as in a) prove that

$$\begin{aligned}
|a(v_h, w_h) - a_h(v_h, w_h)| &= |(A\nabla v_h, \nabla w_h) - Q(A\nabla v_h, \nabla w_h)| \\
&\le Ch^r \sum_{T\in\mathcal{T}_h} \|A\nabla v_h\|_{r;T} \|\nabla w_h\|_{r;T} \\
&\le Ch^r \|A\|_{r,\infty} \sum_{T\in\mathcal{T}_h} \|v_h\|_{r;T} \|w_h\|_{r;T} \\
&= Ch^r \|A\|_{r,\infty} \sum_{T\in\mathcal{T}_h} \|v_h\|_{k;T} \|w_h\|_{k;T} \\
&\le Ch^r \|A\|_{r,\infty} \left(\sum_{T\in\mathcal{T}_h} \|v_h\|_{k;T}^2\right)^{1/2} \left(\sum_{T\in\mathcal{T}_h} \|w_h\|_{k;T}^2\right)^{1/2}.
\end{aligned}$$

c) We rely on the coercivity of a, given by the uniform ellipticity of A. For each $v_h \in X_h$, we have that

$$|a_h(v_h, v_h)| \ge |a(v_h, v_h)| - |a(v_h, v_h) - a_h(v_h, v_h)|$$

$$\ge \alpha \|v_h\|_1^2 - Ch^r \|A\|_{r,\infty} \left(\sum_{T\in\mathcal{T}_h} \|v_h\|_{k;T}^2\right)^{1/2} \left(\sum_{T\in\mathcal{T}_h} \|v_h\|_{k;T}^2\right)^{1/2}$$

$$\ge \alpha \|v_h\|_1^2 - Ch^r \|A\|_{r,\infty} \sum_{T\in\mathcal{T}_h} \|v_h\|_{k;T}^2$$

$$\ge \alpha \|v_h\|_1^2 - Ch^r \|A\|_{r,\infty} C_{\text{inv}}^2 h^{-2k+2} \sum_{T\in\mathcal{T}_h} \|v_h\|_{1;T}^2$$

$$= \alpha \|v_h\|_1^2 - \tilde{C}h^{r-2k+2} \|v_h\|_1^2$$

By taking the limit as $h \to 0$, the coefficient $\tilde{C}h^{r-2k+2}$ vanishes, since by assumption $r - 2k + 2 > 0$. Thus a_h is uniformly coercive, for small enough h.

d) Strang's 1st lemma asserts that, if a_h is uniformly coercive (which we have just proven), the following estimate holds:

$$\|u - u_h\|_1 \le C_s \left(\inf_{v_h\in X_h} \left\{ \|u - v_h\|_1 + \sup_{w_h\in X_h} \frac{|a(v_h, w_h) - a_h(v_h, w_h)|}{\|w_h\|_1} \right\} + \sup_{w_h\in X_h} \frac{|(f, w_h) - Q(f, w_h)|}{\|w_h\|_1} \right)$$

We can remove the $\inf\{\dots\}$ by choosing $v_h = I_h u$, and we can remove the two $\sup\{\dots\}$ by substituting the results from points a) and b):

$$\|u - u_h\|_1 \le C_s \left(\|u - I_h u\|_1 + \sup_{w_h\in X_h} \frac{Ch^r \|A\|_{r,\infty} \|I_h u\|_k \|w_h\|_k}{\|w_h\|_1} + C_a h^{r-k+1} \|f\|_r \right)$$

$$\|u - u_h\|_1 \le C_s \left(\|u - I_h u\|_1 + Ch^r \|A\|_{r,\infty} \|I_h u\|_k C_{\text{inv}} h^{-k+1} + C_a h^{r-k+1} \|f\|_r \right)$$

$$\|u - u_h\|_1 \le C_s \left(C_i h^k \|u\|_{k+1} + Ch^r \|A\|_{r,\infty} C_I \|u\|_k C_{\text{inv}} h^{-k+1} + C_a h^{r-k+1} \|f\|_r \right)$$

We have also used the Lagrange interpolation inequality and the boundedness of I_h. We can conclude the proof using the assumption $r \geq 2k-1$, which implies that $h^{r-k+1} \leq h^k$:

$$\|u-u_h\|_1 \leq C_s \left(C_i \|u\|_{k+1} + C \|A\|_{r,\infty} C_I \|u\|_{k+1} C_{\text{inv}} + C_a \|f\|_r \right) h^k = Ch^k.$$

Exercise 8: Poisson with Neumann boundaries as a saddle point problem

a) For any $v \in C_0^\infty(\Omega) \subset H^1(\Omega)$, we have that

$$\int_\Omega fv \, dx \, + 0 = \int_\Omega \nabla u \cdot \nabla v \, dx \, = \int_\Omega (-\Delta u) v \, dx \, ,$$

so

$$(\Delta u + f, v)_{L^2} = 0.$$

Since $C_0^\infty(\Omega)$ is a dense subset of $L^2(\Omega)$, this proves that $-\Delta u$ and f are equal as L^2 functions. For the second part of the proof, we have that

$$\begin{aligned} 0 &= \int_\Omega \nabla u \cdot \nabla v \, dx \, - \int_\Omega fv \, dx \, - \int_{\partial\Omega} gv \, d\sigma \\ &= \int_{\partial\Omega} (\partial_n u) v \, d\sigma - \int_\Omega (\Delta u) v \, dx \, - \int_\Omega fv \, dx \, - \int_{\partial\Omega} gv \, d\sigma \\ &= -(\Delta u + f, v)_{L^2} + \int_{\partial\Omega} (\partial_n u - g) v \, d\sigma = \int_{\partial\Omega} (\partial_n u - g) v \, d\sigma. \end{aligned}$$

b) Just pick $v = 1$ in (1). This choice is allowed by the boundedness of Ω.

c) As per the hint, the proof relies on the Poincaré-Wirtinger inequality, which asserts that there exists a constant $C > 0$ such that

$$\|u - \bar{u}\|_0 \leq C \|\nabla u\|_0$$

for any $u \in H^1(\Omega)$ with integral mean $\bar{u}$. Let $\gamma \in (0,1)$ be such that

$$\gamma \leq C |\Omega| \, .$$

Then $\tilde{a}$ is coercive, with coercivity constant $\alpha = \min\{(1-\gamma), \gamma C^{-1}\}$:

$$\begin{aligned} \tilde{a}(u,u) &= \int_\Omega \nabla u \cdot \nabla u \, dx \, + (\bar{u} |\Omega|)^2 \\ &= (1-\gamma) \|\nabla u\|_0^2 + \gamma \|\nabla u\|_0^2 + \bar{u}^2 |\Omega|^2 \\ &\geq (1-\gamma) \|\nabla u\|_0^2 + \gamma C^{-1} (u - \bar{u}, u - \bar{u})_{L^2} + \bar{u}^2 |\Omega|^2 \\ &= (1-\gamma) \|\nabla u\|_0^2 + \gamma C^{-1} \|u\|_0^2 - 2\gamma C^{-1} (u, \bar{u})_{L^2} + \gamma C^{-1} (\bar{u}, \bar{u})_{L^2} + \bar{u}^2 |\Omega|^2 \\ &= (1-\gamma) \|\nabla u\|_0^2 + \gamma C^{-1} \|u\|_0^2 - \gamma C^{-1} \bar{u}^2 |\Omega| + \bar{u}^2 |\Omega|^2 \\ &= (1-\gamma) \|\nabla u\|_0^2 + \gamma C^{-1} \|u\|_0^2 + (-\gamma C^{-1} + |\Omega|) \bar{u}^2 |\Omega| \\ &\geq (1-\gamma) \|\nabla u\|_0^2 + \gamma C^{-1} \|u\|_0^2 \geq \alpha \|u\|_1^2 \, . \end{aligned}$$

d) Once again, we can just pick $v = 1$.

e) We begin by introducing some notation:

$$M = H^1(\Omega), \quad X = \mathbb{R}, \quad a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx \,,$$

$$b(u,\mu) = \mu \int_\Omega u \, dx \,, \quad \ell_1(v) = \int_\Omega f v \, dx + \int_{\partial\Omega} g v \, d\sigma, \quad \ell_2(\mu) = 0.$$

The required formulation of a saddle problem is to find $(u, \lambda) \in M \times X$ such that

$$\begin{aligned} a(u,v) + b(v,\lambda) &= \ell_1(v) \quad \text{for all } v \in M \\ b(u,\mu) &= \ell_2(\mu) \quad \text{for all } \mu \in X. \end{aligned}$$

f) If we choose $v = 1$ in the first equation of the saddle point problem, we get

$$0 + \lambda \, |\Omega| = \int_\Omega f \, dx + \int_{\partial\Omega} g \, d\sigma.$$

If (2) holds, this means that $\lambda \, |\Omega| = 0$, so $\lambda = 0$ and $b(v, \lambda)$ vanishes for every $v \in M$. Thefore the first equation of the saddle point problem (satisfied by u) becomes (1).