Advanced Discretization Methods (WS 19/20)
Homework 8

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Deadline for submission (theory):	December 17th, 2019, 12:15
Deadline for submission (programming):	December 17th, 2019, 12:15

Remark: When you apply theorems, whether they were found in Knabner/Angermann or another source, make sure to **cite them**!

Brouwer's Fixed Point theorem
Let $\emptyset \neq B \subset \mathbb{R}^N$ be convex, compact and

$$\phi : B \to B$$

continuous. Then ϕ has at least one fixed point in B.

Exercise 17: A FV scheme for a nonlinear equation (1+3+3+3)

One of the perks of FV methods is that they are flexible to handle both linear and nonlinear problems. In the following we want to prove the existence of a solution of such a scheme. Consider the problem: Find $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$, s.t.

$$\begin{aligned} -u''(x) &= f(x, u(x)) \qquad \text{in } \Omega = (0,1) \\ u(0) &= u(1) = 0 \end{aligned}$$

with a right hand-side $f : (0,1) \times \mathbb{R} \to \mathbb{R}$ which should at the very least satisfy:

- $f(x,s)$ is Lebesgue-measurable w.r.t. $x \in (0,1)$ for all $s \in \mathbb{R}$
- $f(x,s)$ is continuous w.r.t. $s \in \mathbb{R}$ for a.e. $x \in (0,1)$
- $f \in L^\infty((0,1) \times \mathbb{R})$

We consider a mesh defined via

$$0 = x_0 = x_{\frac{1}{2}} < x_1 < x_{\frac{3}{2}} < ... < x_{i-\frac{1}{2}} < x_i < x_{i+\frac{1}{2}} < ...x_N < x_{N+\frac{1}{2}} = x_{N+1} = 1$$

And we set

$$\Omega_i := (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}), \qquad i = 1, ..., N$$

as our control volumes in Ω. The discretized FV scheme reads: Find $U = (U_0, ..., U_{N+1})^T \in \mathbb{R}^{N+2}$, s.t.

$$\begin{aligned} \mathcal{F}_{i+\frac{1}{2}} - \mathcal{F}_{i-\frac{1}{2}} &= h_i f_i(U_i), \qquad i = 1, ..., N \\ U_0 = U_{N+1} &= 0 \end{aligned} \tag{A}$$

where we used

$$\begin{aligned} \mathcal{F}_{i+\frac{1}{2}} &:= -\frac{U_{i+1} - U_i}{h_{i+\frac{1}{2}}}, \qquad i = 0, ..., N \\ f_i(U_i) &:= \frac{1}{h_i} \int_{\Omega_i} f(x, U_i) \, \mathrm{d}x, \qquad i = 1, ..., N \\ h_i &:= x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}, \qquad i = 1, ..., N \\ h_{i+\frac{1}{2}} &:= x_{i+1} - x_i, \qquad i = 0, ..., N \end{aligned}$$

a) Let $V \in \mathbb{R}^N$ be given. We linearize the discrete problem (A) by replacing $f_i(U_i)$ in the right-hand side with $f_i(V_i)$. Briefly comment on why this defines a linear, continuous functional

$$\phi : \mathbb{R}^N \to \mathbb{R}^N, \qquad V \mapsto U := \phi(V)$$

where U solves (A) if and only if

$$\phi(U) = U$$

b) Show that $U = \phi(V)$ obtained in a), given arbitrary input $V \in \mathbb{R}^N$ satisfies

$$\sum_{i=0}^{N} \frac{(U_{i+1} - U_i)^2}{h_{i+\frac{1}{2}}} \leq M \sum_{i=1}^{N} h_i |U_i|$$

where $M := \|f\|_{L^\infty((0,1)\times\mathbb{R})}$.
Hint: Multiply each discrete equation with U_i and sum over all of them.

c) Use the Cauchy-Schwarz inequality to prove

$$|U_i| \leq \left(\sum_{j=0}^{N} \frac{(U_{j+1} - U_j)^2}{h_{j+\frac{1}{2}}} \right)^{\frac{1}{2}}, \qquad i = 1, ..., N$$

Use this to find $C = C(f) > 0$, s.t.

$$\sum_{i=0}^{N} \frac{(U_{i+1} - U_i)^2}{h_{i+\frac{1}{2}}} \leq C$$

d) Use Brouwer's Fixed Point Theorem to conclude that (A) has at least one solution.

Hint: Consider $B = B_M(0)$ and $\|V\| := \left(\sum_{j=0}^{N} \frac{(V_{j+1}-V_j)^2}{h_{j+\frac{1}{2}}} \right)^{\frac{1}{2}}$ as a norm on $\mathbb{R}^N$ with $V_0, V_{N+1} := 0$ (prove that it is a norm!).

Programming exercise 7: Raviart-Thomas elements (20)

We will now pick up again where we left off in Exercise 12 (Homework 5) by finishing an implementation for the RT_0-$\mathcal{L}_0^0$ elements in MATLAB applied to Poisson's problem. Recall that the discrete problem reads: Find

$$\sigma_h \in V_h := \mathrm{RT}_0(\mathcal{T}_h), \quad u_h \in W_h := \mathcal{L}_0^0(\mathcal{T}_h)$$

such that

$$\begin{aligned} (\sigma_h, \tau_h) + (u_h, \operatorname{div} \tau_h) &= \int_{\partial\Omega} u_D \tau_h \cdot \nu \, d\sigma & \forall \tau_h \in V_h, \\ (v_h, \operatorname{div} \sigma_h) &= -(f, v_h) & \forall v_h \in W_h. \end{aligned}$$

A template on which your implementation should be based on can be downloaded through StudOn. The template for the main file is called `main.template.m`, mesh-related things can be found in `RTMesh.m`, it should not be necessary to change this file. Study those two files to understand the employed data structures. You can find premade triangle meshes in the directory `meshes/`.

a) Implement the assembly of the stiffness matrix using the results from Exercise 12. Implement the assembly of the right-hand side by evaluating the integral using the midpoint quadrature rule. Use pre-implemented methods when possible. Test your implementation on the square $\Omega := (-1,1)^2$ (file: `meshes/square-1.msh`) by choosing f such that

$$u(x,y) := \sin(\pi x)\sin(\pi y) \in H_0^{1,2}(\Omega)$$

is the analytic solution to $-\Delta u = f$.

b) Compute the order of convergence for the problem a) in the L^2-norm for different refinement levels (`meshes/square-1.msh`, ..., `meshes/square-5.msh`). Calculate the norm by evaluating the error integral using the midpoint rule on each triangle and plot the error vs. mesh size.

c) Implement Dirichlet boundary values, evaluating the boundary integral using the midpoint rule. Plot the solution u_h for $f = 0$ and

$$u_D(x,y) = \begin{cases} 4x(1-x) & \text{if } y = -1, \\ 0 & \text{else.} \end{cases}$$

on the L-shaped domain $\Omega = (0,1) \times (-1,1) \cup (0,2) \times (0,1)$ (`meshes/lshaped-3.msh`).

Note: Older Octave versions may have problems with constructor class methods - to prevent this ensure you use the latest version.