Advanced Discretization Techniques
Homework 10

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Exercise 19: Estimates for parabolic equations

a) We first prove a stronger, differential version of the inequality. If $\|u(t)\|_0 = 0$, then we're already done. Otherwise, by the usual theorem for differentiation under the (Lebesgue) integral sign,

$$\frac{d}{dt}\|u(t)\|_0 = \frac{d}{dt}\langle u(t), u(t)\rangle^{1/2} = \frac{2\langle u'(t), u(t)\rangle_0}{2\|u(t)\|_0} = \frac{\langle u'(t), u(t)\rangle_0}{\|u(t)\|_0}.$$

For $v = u(t)$, the last numerator is equal to the first integral in the weak formulation of the boundary value problem, so we get

$$\frac{\langle u'(t), u(t)\rangle_0}{\|u(t)\|_0} = \frac{\int_\Omega u'(t)u(t)\,dx}{\|u(t)\|_0} = \frac{\int_\Omega f(t)u(t)\,dx\, - a(u(t), u(t))}{\|u(t)\|_0}.$$

By the positive definiteness of the bilinear form $a(\cdot,\cdot)$ and Cauchy-Schwarz's inequality, we can conclude that

$$\frac{\int_\Omega f(t)u(t)\,dx\, - a(u(t), u(t))}{\|u(t)\|_0} \le \frac{\int_\Omega f(t)u(t)\,dx}{\|u(t)\|_0} \le \frac{\|f(t)\|_0\,\|u(t)\|_0}{\|u(t)\|_0} = \|f(t)\|_0\,.$$

Now we can integrate the differential version of the inequality over time:

$$\int_0^s \frac{d}{dt}\|u(t)\|_0\,dt \le \int_0^s \|f(t)\|_0\,dt$$
$$\|u(s)\|_0 \le \|u(0)\|_0 + \int_0^s \|f(t)\|_0\,dt\,.$$

b) Again, we first prove a stronger, differential version of the inequality. By Poincaré's inequality in $H_0^1(\Omega)$, there exists a constant $C_1 > 0$ independent of t such that

$$\|u(t)\|_0^2 \le C_1^2\,|u(t)|_1^2$$

holds for all $u(t) \in H^1(0,T;V)$. Then, taking $v = 2u(t)$, it follows that

$$\begin{aligned}
\frac{d}{dt}\|u(t)\|_0^2 &= \langle u'(t), 2u(t)\rangle_0 \\
&= \langle f(t), 2u(t)\rangle_0 - a(u(t), 2u(t)) \\
&= \int_\Omega 2f(t)u(t)\,dx - |u(t)|_1^2 - |u(t)|_1^2 \\
&\le \int_\Omega 2f(t)u(t)\,dx - C_1^{-2}\|u(t)\|_0^2 - |u(t)|_1^2 \\
&= \int_\Omega C_1^2 f(t)^2\,dx - \int_\Omega \left(C_1 f(t) - C_1^{-1}u(t)\right)^2 dx - |u(t)|_1^2 \\
&\le C_1^2\|f(t)\|_0^2 - |u(t)|_1^2 .
\end{aligned}$$

In the last steps, we've completed the square and discarded a negative term. Now we can integrate the differential version of the inequality over time:

$$\begin{aligned}
\int_0^s \frac{d}{dt}\|u(t)\|_0^2\,dt &\le \int_0^s C_1^2\|f(t)\|_0^2\,dt - \int_0^s |u(t)|_1^2\,dt \\
\|u(s)\|_0^2 - \|u(0)\|_0^2 &\le C_1^2\int_0^s \|f(t)\|_0^2\,dt - \int_0^s |u(t)|_1^2\,dt \\
\|u(s)\|_0^2 + \int_0^s |u(t)|_1^2\,dt &\le \|u(0)\|_0^2 + C_1^2\int_0^s \|f(t)\|_0^2\,dt
\end{aligned}$$

c) The proof is very similar to the one of point b). The only differences are that this time we choose $v = 2u'(t)$ and use the symmetry of $a(\cdot,\cdot)$:

$$\begin{aligned}
\frac{d}{dt}|u(t)|_1^2 &= \frac{d}{dt}a(u(t), u(t)) = a(u(t), 2u'(t)) \\
&= \int_\Omega 2f(t)u'(t)\,dx - \int_\Omega u'(t)^2\,dx - \int_\Omega u'(t)^2\,dx \\
&= \int_\Omega f(t)^2\,dx - \int_\Omega (f(t) - u'(t))^2\,dx - \int_\Omega u'(t)^2\,dx \\
&\le \int_\Omega f(t)^2\,dx - \int_\Omega u'(t)^2\,dx = \|f(t)\|_0^2 - \|u'(t)\|_0^2
\end{aligned}$$

Now we can integrate the differential version of the inequality over time:

$$\begin{aligned}
\int_0^s \frac{d}{dt}|u(t)|_1^2\,dt &\le \int_0^s \|f(t)\|_0^2\,dt - \int_0^s \|u'(t)\|_0^2\,dt \\
|u(s)|_1^2 - |u(0)|_1^2 &\le \int_0^s \|f(t)\|_0^2\,dt - \int_0^s \|u'(t)\|_0^2\,dt \\
|u(s)|_1^2 + \int_0^s \|u'(t)\|_0^2\,dt &\le |u(0)|_1^2 + \int_0^s \|f(t)\|_0^2\,dt
\end{aligned}$$

d) By Poincaré's inequality in $H_0^1(\Omega)$, there exists a constant $C_2 > 0$ independent of t such that

$$\|u(t)\|_1^2 \le C_2^2\,|u(t)|_1^2$$

holds for all $u(t) \in H^1(0,T;V)$. Let $v \in V$ be the weak solution to the given Poisson problem. This v can also be seen as a constant function in $H^1(0,T;V)$. Then, by taking the difference of the weak formulations, we immediately get that $u(t) - v$ is a weak solution to the corresponding homogenous parabolic equation ($f(t) \equiv 0$), and so the three previous estimates also apply to $u(t) - v$. In particular, if we sum the differential versions of inequalities b) and c), we get that

$$\begin{aligned}\frac{d}{dt} \|u(t) - v\|_1^2 &\leq - |u(t) - v|_1^2 - \|\partial_t (u(t) - v)\|_0^2 \\ &\leq - |u(t) - v|_1^2 \leq -C_2^{-2} \|u(t) - v\|_1^2 .\end{aligned}$$

By Grönwall's lemma (in differential form), it follows that

$$\|u(t) - v\|_1^2 \leq \|u(0) - v\|_1^2 e^{\int_0^t -C_2^{-2}\, ds} = \|u(0) - v\|_1^2 e^{-tC_2^{-2}} ,$$

and this implies

$$\lim_{t \to \infty} \|u(t) - v\|_1 = 0,$$

as required. □