Advanced Discretization Techniques
Homework 2

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Exercise 5: A different kind of element

a) Let $\{e_1, \dots, e_d\}$ be the canonical basis of $\mathbb{R}^d$. We will first prove the statement for the case $T = \hat{T}$, where $\hat{T}$ is the convex hull of the points $\{0, e_1, \dots, e_d\}$. The volume of $\hat{T}$ is $1/d!$. The functions

$$v \mapsto \int_T v(y)\, dy \qquad \text{and} \qquad v \mapsto \frac{1}{d!} v\left(\frac{1}{d+1}, \dots, \frac{1}{d+1}\right)$$

belong to $\mathbb{P}_1^*$, so in order to show that they are equal it suffices to check that they agree on a basis of $\mathbb{P}_1$. In this proof, we will choose the monomial basis. The case $v(y) = 1$ is trivial. If $v(y) = y_1$, then

$$\int_T v(y)\, dy = \int_T y_1\, dy_1 \dots dy_d = \int_0^1 \left(\int_{L(y_1)} y_1\, dy_2 \dots dy_d \right) dy_1 = \int_0^1 y_1 \left| L(y_1) \right| \, dy_1$$

by Fubini's theorem, where

$$L(y_1) = \{y_1\} \times \{(y_2, \dots, y_d) \in \mathbb{R}_+^{d-1} \mid y_2 + \dots + y_d \le 1 - y_1\}.$$

Now,

$$|L(y_1)| = |(1 - y_1)L(0)| = (1 - y_1)^{d-1} |L(0)| = \frac{(1 - y_1)^{d-1}}{(d-1)!},$$

hence

$$\int_0^1 y_1 \left| L(y_1) \right| \, dy_1 = \frac{1}{(d-1)!} \int_0^1 y_1 (1 - y_1)^{d-1}\, dy_1$$

$$= 0 - \frac{1}{(d-1)!} \int_0^1 -\frac{(1 - y_1)^d}{d}\, dy_1 = \frac{1}{(d-1)!} \left[-\frac{(1 - y_1)^{d+1}}{d(d+1)} \right]_0^1$$

$$= \frac{1}{(d+1)!} = \frac{1}{d!} v\left(\frac{1}{d+1}, \dots, \frac{1}{d+1}\right),$$

as was requested. The cases $v(y) = y_2, \dots, v(y) = y_d$ can be proven in the same way. Now we prove the theorem for a generic T. Let $m \colon \hat{T} \mapsto T$ be the affine mapping sending $\hat{T}$ to T. Then

$$\int_T v(y)\,dy = \int_{\hat{T}} v(m(y))\,|\det(J_m)|\,dy = |\det(J_m)| \int_{\hat{T}} v(m(y))\,dy$$

$$= d!\,|T|\,\frac{1}{d!}\,v\left(m\left(\frac{1}{d+1},\dots,\frac{1}{d+1}\right)\right) = |T|\,v\left(\frac{1}{d+1}\sum_{i=0}^{d} x_i\right)$$

b) The only nontrivial condition to check is that the functions $\{\sigma_0,\dots,\sigma_d\}$ are linearly independent. Let $a_0,\dots,a_d \in \mathbb{R}$ be such that

$$a_0\sigma_0 + \cdots + a_d\sigma_d = 0.$$

We want to show that all a_i are 0. As per the hint, we can evaluate the linear combination in $\theta_i(y) = 1 - d\lambda_i(y) \in \mathbb{P}_1$:

$$0 = \sum_{j=0}^{d} a_j\sigma_j(\theta_i) = \sum_{j=0}^{d} \frac{a_j}{|e_j|}\int_{e_j} \theta_i(y)\,dy \overset{a)}{=} \sum_{j=0}^{d} \frac{a_j}{|e_j|}\,|e_j|\,\theta_i\left(\frac{1}{d}\sum_{k=0,k\neq j}^{d} x_k\right)$$

$$= \sum_{j=0}^{d} a_j\left(1 - d\lambda_i\left(\frac{1}{d}\sum_{k=0,k\neq j}^{d} x_k\right)\right) = \sum_{j=0}^{d} a_j\delta_{ij} = a_i$$

for all $i = 0,\dots,d$, as required for linear independence.

c) Let p_1 and p_2 be the restrictions of v_h to T_1 and T_2, two elements sharing the face e. Let e_c be the centroid of e. Since v_h is piecewise affine, the size of the jump discontinuity $[\![v_h]\!]$ across e is given by $p_1 - p_2$, and v_h is continuous in e_c if and only if $p_1(e_c) = p_2(e_c)$. Let's prove this equality:

$$0 = \int_e [\![v_h]\!]\,dy = \int_e p_1(y) - p_2(y)\,dy$$

$$\int_e p_1(y)\,dy = \int_e p_2(y)\,dy \overset{a)}{\Rightarrow} |e|\,p_1(e_c) = |e|\,p_2(e_c) \Rightarrow p_1(e_c) = p_2(e_c). \quad \square$$

Exercise 6: A different kind of elliptic equation

a) We multiply the equation by a test function $\varphi \in X = H_0^2(\Omega)$, then integrate over Ω, integrate by parts twice and use the fact that both φ and $\nabla\varphi$ vanish on the boundary of the domain:

$$\int_\Omega \Delta^2 u\,\varphi\,dx = \int_\Omega f\varphi\,dx$$

$$\int_{\partial\Omega} (\nabla(\Delta u)\cdot\mathbf{n})\,\varphi\,d\sigma - \int_\Omega \nabla(\Delta u)\cdot\nabla\varphi\,dx = \int_\Omega f\varphi\,dx$$

$$-\int_{\partial\Omega} \Delta u\,(\nabla\varphi\cdot\mathbf{n})\,d\sigma + \int_\Omega \Delta u\,\Delta\varphi\,dx = \int_\Omega f\varphi\,dx$$

$$\int_\Omega \Delta u\,\Delta\varphi\,dx = \int_\Omega f\varphi\,dx\,.$$

Let

$$a(u,\varphi) = \int_\Omega \Delta u\, \Delta\varphi\, dx\,, \quad f(\varphi) = \int_\Omega f\varphi\, dx\,.$$

Then the weak form of the equation is $a(u,\varphi) = f(\varphi)$. In the continuous setting, the problem is:

Find $u \in X$ such that $a(u,\varphi) = f(\varphi)$ holds for every $\varphi \in X$.

The system is not overconstrained because the weak gradient of every function in H_0^2 vanishes on the boundary (has trace 0), so the boundary condition $Du = 0$ is quite natural. Otherwise, by just asking for $u = 0$ we would have to work in $H_0^1(\Omega) \cap H^2(\Omega)$, but then we can't integrate by parts a second time without introducing an unwanted extra term.

b) The first half of the proof boils down to Schwarz's theorem and integration by parts. Indeed, for each $\varphi \in C_0^\infty(\Omega)$, we have that

$$\begin{aligned}
\int_\Omega (\Delta\varphi(x))^2\, dx &= \int_\Omega \Big(\sum_{i=1}^d \partial_{x_i x_i}\varphi(x)\Big)^2 dx = \int_\Omega \sum_{i,j=1,\dots,d} \partial_{x_i x_i}\varphi(x)\, \partial_{x_j x_j}\varphi(x)\, dx \\
&= \int_\Omega \sum_{i,j=1,\dots,d} \partial_{x_i}\varphi(x)\, \partial_{x_j x_j x_i}\varphi(x)\, dx = \int_\Omega \sum_{i,j=1,\dots,d} \partial_{x_i}\varphi(x)\, \partial_{x_i x_j x_j}\varphi(x)\, dx \\
&= \int_\Omega \sum_{i,j=1,\dots,d} \partial_{x_i x_j}\varphi(x)\, \partial_{x_i x_j}\varphi(x)\, dx = \int_\Omega \sum_{|\alpha|=2} \big(\partial_{x^\alpha}\varphi(x)\big)^2 dx = |\varphi|_2^2\,.
\end{aligned}$$

The second half of the proof follows by the density of $\varphi \in C_0^\infty(\Omega)$ in $H_0^2(\Omega)$ and the continuity of the two functionals, which is readily checked:

$$\begin{aligned}
\int_\Omega (\Delta\varphi(x))^2\, dx &= \int_\Omega \Big(\sum_{i=1}^d \partial_{x_i x_i}\varphi(x)\Big)^2 dx \\
&\le d \int_\Omega \sum_{i=1}^d \big(\partial_{x_i x_i}\varphi(x)\big)^2 dx \le d\, |\varphi|_2^2 \le d\, \|\varphi\|_2^2\,.
\end{aligned}$$

The first inequality is just the usual AM-QM inequality.

c) We derive the following inequality, which will be useful in point d):

$$\exists C > 0 \text{ such that } |u|_2^2 \ge C\, \|u\|_2^2 \text{ for each } u \in H_0^2(\Omega).$$

The proof relies on the fact that weak derivatives of functions in $H_0^2(\Omega)$ belong to $H_0^1(\Omega)$, so the basic version of Poincaré's inequality still applies to them: there exist constants $c, c_1, \dots, c_d > 0$ such that

$$\|u\|_{L^2}^2 \le c\, |u|_1^2 \text{ and } \|\partial_{x_i} u\|_{L^2}^2 \le c_i\, |\partial_{x_i} u|_1^2 \text{ for each } i = 1, \dots, d.$$

From now on, it's just algebra:

$$\begin{aligned}
\|u\|_{L^2}^2 &\le c\, |u|_1^2 = c\, \|\nabla u\|_{L^2}^2 = c \sum_{i=1}^d \|\partial_{x_i} u\|_{L^2}^2 \le c \sum_{i=1}^d c_i\, |\partial_{x_i} u|_1^2 \\
&= c \sum_{i=1}^d c_i\, \|\nabla \partial_{x_i} u\|_{L^2}^2 \le c' \sum_{i,j=1}^d \big\|\partial_{x_i x_j} u\big\|_{L^2}^2 = c'\, |u|_2^2\,.
\end{aligned}$$

Comparing the first, second and last terms we get that

$$\|u\|_2^2 = \|u\|_{L^2}^2 + |u|_1^2 + |u|_2^2 \le \left(c' + \frac{c'}{c} + 1\right) |u|_2^2 = c'' \, |u|_2^2 \, ,$$

so we can just choose $C = 1/c''$.

d) Existence and uniqueness of a weak solution $u \in X$ is given by the Lax-Milgram theorem applied to the equation $a(u,v) = f(v)$. Let's check that the hypothesis are satisfied. The continuity of f is given by $f \in X^*$. The continuity of $a(\cdot,\cdot)$ follows from the Cauchy-Schwarz inequality and point b):

$$\begin{aligned} a(u,v) = \int_\Omega \Delta u \, \Delta v \, dx &\le \left(\int_\Omega (\Delta u)^2 \, dx\right)^{1/2} \left(\int_\Omega (\Delta v)^2 \, dx\right)^{1/2} \\ &= |u|_2 \, |v|_2 \le \|u\|_2 \, \|v\|_2 \, . \end{aligned}$$

The coercivity of $a(\cdot,\cdot)$ follows from points b) and c):

$$a(u,u) = \int_\Omega (\Delta u)^2 \, dx \; = |u|_2^2 \ge C \, \|u\|_2^2 \, . \quad \square$$