Advanced Discretization Methods (WS 19/20)
Homework 9
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Deadline for submission (theory): January 7th, 2019, 12:15
Deadline for submission (programming): January 7th, 2019, 12:15

Remark: When you apply theorems, whether they were found in Knabner/Angermann or another source, make sure to **cite them**!

Kolmogorov's compactness lemma
Let Ω be an open, bounded set of $\mathbb{R}^d$, $d \geq 1$, $1 \leq q < \infty$ and $A \subset L^q(\Omega)$. Then, A is relatively compact in $L^q(\Omega)$ if and only if there exists $\{p(u) \mid u \in A\} \subset L^q(\mathbb{R}^d)$ such that

i) $p(u) = u$ a.e. on Ω, for all $u \in A$

ii) $\{p(u) \mid u \in A\}$ is bounded in $L^q(\mathbb{R}^d)$

iii) $\|p(u)(\cdot + \eta) - p(u)\|_{L^q(\mathbb{R}^d)} \to 0$ as $\eta \to 0$, uniformly w.r.t. $u \in A$.

Exercise 18: A FV scheme for a nonlinear equation (Part II) (2+2+2+2+3+2+2)

On the previous exercise sheet we proved the existence of a solution to the discrete problem on a given mesh. We now want to expand on that result by showing that a sequence of discrete solutions converges towards a solution of the original variational problem: Find $u \in H_0^{1,2}((0,1))$, s.t.

$$\int_0^1 u'(x)\varphi'(x)\,\mathrm{d}x = \int_0^1 f(x,u(x))\varphi(x)\,\mathrm{d}x \qquad \forall \varphi \in C_0^\infty((0,1))$$

We again consider an admissible mesh $\mathcal{T} = \{\Omega_i \mid i = 1,...,N\}$ (admissible meaning it is defined as in Exercise 17, all notations shall also remain the same). Like before, $(U_1,...,U_N)^T \in \mathbb{R}^N$ satisfies the estimate

$$\sum_{i=0}^{N} \frac{(U_{i+1}-U_i)^2}{h_{i+\frac{1}{2}}} \leq C$$

with $U_0 = U_{N+1} = 0$ and a constant $C = C(f)$. We further define

$$u_\mathcal{T}(x) := U_i \text{ for } x \in \Omega_i,\ i = 1,...,N$$

a) Show that $\{u_\mathcal{T} \mid \mathcal{T} \text{ an admissible mesh of } (0,1)\}$ is a bounded set in $L^2(\mathbb{R})$ if you extend $u_\mathcal{T}$ by 0 in $\mathbb{R}\backslash[0,1]$

b) Let $0 < \eta < 1$ and

$$\chi_{i+\frac{1}{2}}(x) = \begin{cases} 1 & x_{i+\frac{1}{2}} \in [x, x+\eta] \\ 0 & \text{else} \end{cases}$$

Show that for all $x \in \mathbb{R}$:

$$(u_\mathcal{T}(x+\eta) - u_\mathcal{T}(x))^2 \leq \left(\sum_{i=0}^{N} \frac{(U_{i+1}-U_i)^2}{h_{i+\frac{1}{2}}}\chi_{i+\frac{1}{2}}(x)\right)\left(\sum_{i=0}^{N}\chi_{i+\frac{1}{2}}(x)h_{i+\frac{1}{2}}\right)$$

c) Use b) to conclude that

$$\|u_\mathcal{T}(\cdot+\eta) - u_\mathcal{T}\|_{0,\mathbb{R}}^2 \leq C\eta(\eta + 2h)$$

with $h = \text{ size}(\mathcal{T}) = \max_{i=1,...,N} |\Omega_i|$.

Since $\eta + 2h \leq 3$, by the Kolmogorov compactness theorem (see remark), this yields the relative compactness of $A := \{u_\mathcal{T} \mid \mathcal{T} \text{ admissible mesh of } (0,1)\}$ in $L^2((0,1))$ (and $L^2(\mathbb{R})$ for that matter). What this means is that any sequence in A has a subsequence that converges in $L^2((0,1))$ (or $L^2(\mathbb{R})$).

d) Consider now such an L^2-convergent subsequence $(u_{\mathcal{T}_n})_{n\in\mathbb{N}}$ solving the respective discrete problem with size$(\mathcal{T}_n) \overset{n\to\infty}{\longrightarrow} 0$ and limit u in $L^2(\mathbb{R})$. Show that the difference quotient $\Delta_\eta u := \frac{u(\cdot+\eta)-u(\cdot)}{\eta}$ remains bounded in $L^2(\mathbb{R})$ for $\eta \to 0$.

The boundedness of the difference quotients actually suffices to yield $u \in H^{1,2}(\mathbb{R})$ (c.f. Gilbarg/Trudinger, 2001). Since $u = 0$ on $\mathbb{R}\backslash[0,1]$ we can further conclude $u \in H_0^{1,2}((0,1))$. All we need to show now is that u solves the original variational problem.

e) For fixed mesh $\mathcal{T}$ let $\varphi \in C_0^\infty((0,1))$, $\varphi_i := \varphi(x_i)$, $i = 1,...,N$ and $\varphi_0 := \varphi_{N+1} := 0$. Use the discrete problem formulation to show that

$$\int_0^1 u_\mathcal{T}\psi_\mathcal{T}\,\mathrm{d}x = \int_0^1 f_\mathcal{T}\varphi_\mathcal{T}\,\mathrm{d}x$$

where for $x \in \Omega_i$:

$$\psi_\mathcal{T}(x) := \frac{1}{h_i}\left(\frac{\varphi_i - \varphi_{i-1}}{h_{i-\frac{1}{2}}} - \frac{\varphi_{i+1}-\varphi_i}{h_{i+\frac{1}{2}}}\right)$$

$$f_\mathcal{T}(x) = f(x, U_i)$$

$$\varphi_\mathcal{T}(x) = \varphi_i$$

f) Use the regularity of φ and e) to show that

$$\int_0^1 u_{\mathcal{T}}(x)\psi_{\mathcal{T}}(x)\,\mathrm{d}x = -\int_0^1 u_{\mathcal{T}}(x)\theta_{\mathcal{T}}(x)\,\mathrm{d}x + \sum_{i=0}^{N} R_{i+\frac{1}{2}}(U_{i+1} - U_i)$$

where $R_{i+\frac{1}{2}}$ is a remainder term that satisfies $|R_{i+\frac{1}{2}}| \lesssim h$ and

$$\theta_{\mathcal{T}} = \sum_{i=1}^{N} \frac{\varphi'(x_{i+\frac{1}{2}}) - \varphi'(x_{i-\frac{1}{2}})}{h_i}\chi_{\Omega_i}(x)$$

g) Conclude that in the limit of a convergent subsequence we obtain

$$-\int_0^1 u(x)\varphi''(x)\,\mathrm{d}x = \int_0^1 f(x, u(x))\varphi(x)\,\mathrm{d}x$$

(and so integration by parts yields the desired result)

Programming exercise 8: A convection-dominated problem with FVM (15)

We implement a time-dependent transport equation using FVM in MatLab. In its entirety, we consider the following problem:
Find $u : [0, T] \times [0, 1] \to \mathbb{R}$, s.t.

$$\begin{aligned} \partial_t u(t,x) - \partial_x(k\partial_x u(t,x)) + c\partial_x u(t,x) + ru(t,x) &= f(t,x) \text{ in } (0,T)\times(0,1) \\ u(t,x^*) &= g(t,x^*) \text{ for } t \in (0,T) \\ u(0,x) &= 0 \text{ for } x \in (0,1) \end{aligned}$$

where $k \geq 0$, $c \neq 0$, $r \geq 0$ are constants. We will also fix $T = 5$.
Note that the Dirichlet condition varies depending on the type of equation: In the non-diffusive case ($k = 0$), we can only pose a Dirichlet condition at the inflow boundary, that is

$$x^* = \begin{cases} 0 & c > 0 \\ 1 & c < 0 \end{cases}$$

In the case where $k > 0$, we need to pose Dirichlet b.c. on both sides.
Let further $0 = x_0 < x_1 < ... < x_N < x_{N+1} = 1$ be given, then we define:

$$x_{i+\frac{1}{2}} := \frac{x_{i+1} + x_i}{2},\ i = 0, ..., N, \qquad h_i := x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}},\ i = 1, ..., N$$

and our control volumes:

$$\Omega_i = (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}),\ i = 1, ..., N, \qquad \Omega_0 = (x_0, x_{\frac{1}{2}}),\ \Omega_{N+1} = (x_{N-\frac{1}{2}}, x_N)$$

On StudOn, you can find a template that your implementation should be based on. It comes equipped with a basic structure and plotting routines for each part of this exercise (you should not have to change these).
Discretize the problem with the Finite Volume Method to arrive at a formulation similar to (7.9) in the script. Use the Implicit Euler scheme to discretize the problem in time. When approximating the integrals, you can treat the time difference quotient similarly as you would treat the reactive term. You can further assemble the matrices directly, no element-wise assembly is needed. Proceed as follows:

a) First implement the pure convection equation ($k = 0, r = 0, f = 0$). Use a full-upwinding scheme to discretize the convective term. Test your code for $g = 1$ and $c = \pm 0.2$ and varying mesh & timestep size. Comment on the results.

b) Add the reactive term to a) and plot the solution for $r = 1$. Again, comment on your results.

c) Add the diffusive term and the right-hand side to your implementation. Make sure you introduce your Dirichlet values in a way that can handle both diffusive and non-diffusive problems. Test your implementation for varying k, c and r using $u(t,x) = \sin(t)\sin(2\pi x)$ as the analytical solution. Your code should also be able to handle unstructured meshes, for that you can generate a mesh e.g. using

```
x = [0;sort(rand(N,1));1]
```

d) Choose $c = 0.2$, $r = 0.1$ together with inflow values $g(t,0) = t((1.05t - \lfloor 1.05t \rfloor)^3 + 0.2)$ and right-hand side $f = 0$. Solve on a grid with $N = 5000$ using 1000 timesteps and plot the solution with the given plot script. $\lfloor \cdot \rfloor$ in Matlab/Octave is given via `floor`. Should your code be correct, happy holidays :)

We wish you all joyful holidays and a happy New Year!