# Advanced Discretization Techniques
# Homework 5

Bruno Degli Esposti, Xingyu Xu

November 19th - November 26th, 2019

## Exercise 11: A trace estimate using the Raviart-Thomas element $\mathrm{RT}_0$

**a)** For any $i \in \{1,2,3\}$ and any $x \in e_j \neq e_i$, we know that the vector $x - x_i$ is parallel to the side $e_j$, so

$$\frac{|e_i|}{2\,|K|}(x - x_i) \cdot \nu(x) = 0$$

by the definition of normal $\nu(x)$. Otherwise, if $x \in e_i$, we get that $(x - x_i) \cdot \nu(x) = h_i$, where $h_i$ is the height of the triangle $K$ with respect to the base $e_i$. It then follows by the formula for the area of a triangle that

$$\frac{|e_i|}{2\,|K|}(x - x_i) \cdot \nu(x) = \frac{1}{|K|}\,\frac{|e_i|\,h_i}{2} = 1,$$

as was required. This proves that $\tau_i(x) \cdot \nu(x) = \chi_{e_i}(x)$ for all $x \in \partial K$. Moving on to the second part of the proof, $\dim(\mathrm{RT}_0(K)) = 1(1+2) = 3$, so we only need to prove that the functions $\tau_i$ are linearly independent. For this purpose, let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ be such that $\alpha_1\tau_1(x) + \alpha_2\tau_2(x) + \alpha_3\tau_3(x) = 0$. If we now take the scalar product with respect to $\nu(x)$, we get that

$$\begin{aligned} 0 &= \alpha_1\tau_1(x) \cdot \nu(x) + \alpha_2\tau_2(x) \cdot \nu(x) + \alpha_3\tau_3(x) \cdot \nu(x) \\ &= \alpha_1\chi_{e_1}(x) + \alpha_2\chi_{e_2}(x) + \alpha_3\chi_{e_3}(x). \end{aligned}$$

By evaluating this identity in $x \in e_i$, we can conclude that $\alpha_i = 0$ for all $i \in \{1,2,3\}$. Hence $\{\tau_1, \tau_2, \tau_3\}$ forms a basis of $\mathrm{RT}_0(K)$.

**b)** We can get the required estimate by using the formula $\tau_i(x) \cdot \nu(x) = \chi_{e_i}(x)$ of point a), the divergence theorem and the Cauchy-Schwarz inequality:

$$\begin{aligned} \|u\|_{0,e_i}^2 &= \int_{e_i} u(x)^2\,d\sigma = \int_{\partial K} u(x)^2\,\chi_{e_i}(x)\,d\sigma \\ &= \int_{\partial K} u(x)^2\tau_i(x) \cdot \nu(x)\,d\sigma = \int_K \nabla\cdot\left(u(x)^2\tau_i(x)\right)dx \\ &= \int_K 2u(x)\nabla u(x) \cdot \tau_i(x)\,dx + \int_K u(x)^2\nabla\cdot\tau_i(x)\,dx \\ &\leq 2\|u\|_{0,K}\|\nabla u\|_{0,K}\|\tau_i\|_{L^\infty} + \|\nabla\cdot\tau_i\|_{L^\infty(K)}\|u\|_{0,K}^2. \end{aligned}$$

**c)** The constant $h$ is equal to the diameter of $K$. Therefore

$$\|\tau_i\|_{L^\infty(K)} = \operatorname*{ess\,sup}_{x\in\partial K}\left\{\frac{|e_i|}{2|K|}\|x-x_i\|\right\} \leq \frac{h}{2\,|K|}h \leq \frac{c}{2}.$$

Since $K$ is a subset of $\mathbb{R}^2$, we have that $\nabla\cdot x = 2$. Hence

$$\|\nabla\cdot\tau_i\|_{L^\infty(K)} = \operatorname*{ess\,sup}_{x\in\partial K}\left\{\frac{|e_i|}{2|K|}|\nabla\cdot(x-x_i)|\right\} = \frac{|e_i|}{2\,|K|}2 \leq \frac{h}{|K|} \leq \frac{c}{h}.$$

**d)** By the inequalities we have proved in points b) and c), we have that

$$\|u\|^2_{0,e_i} \leq \frac{c}{h}\|u\|^2_{0,K} + c\,\|u\|_{0,K}\,\|\nabla u\|_{0,K}\,.$$

Then, since $c$ and $h$ are both positive constants,

$$\begin{aligned}\|u\|^2_{0,e_i} &\leq \frac{c}{h}\|u\|^2_{0,K} + 2c\,\|u\|_{0,K}\,\|\nabla u\|_{0,K} + ch\,\|\nabla u\|^2_{0,K}\\ &= \left((c^{1/2}h^{-1/2}\,\|u\|_{0,K} + c^{1/2}h^{1/2}\,\|\nabla u\|_{0,K}\right)^2.\end{aligned}$$

The result follows by taking the square root on both sides and making the choice $C = c^{1/2}$. □

# Exercise 12: Poisson's problem with $\mathrm{RT}_0$-$\mathcal{L}^0_0$

Let $\lambda_1(x), \lambda_2(x), \lambda_3(x)$ be the barycentric coordinate maps for the triangle $K$, so that for each $x \in \mathbb{R}^2$ we have that

$$x = \sum_{i=1}^{3}\lambda_i(x)x_i \quad \text{and} \quad \sum_{i=1}^{3}\lambda_i(x) = 1.$$

We begin by proving the identity given as a hint. Let $\hat{K}$ be the reference triangle in $\mathbb{R}^2$, as defined in Chapter 2 of [K,A]. For any $i,j \in \{1,2,3\}$, let $F\colon \hat{K} \to K$ be an affine and bijective mapping such that $F(1,0) = x_i$ and, if $j \neq i$, $F(0,1) = x_j$. By the change of variables formula, it follows that

$$\int_K \lambda_i(\mathbf{x})\lambda_j(\mathbf{x})\,d\mathbf{x} = \int_{\hat{K}}\lambda_i(F(x,y))\lambda_j(F(x,y))\,|\det(dF(x,y))|\,dx dy\,.$$

By the choice of $F$, and the fact that $\lambda_i \circ F$ is still an affine map, it's easy to see that $\lambda_i(F(x,y)) = x$ and, if $j \neq i$, $\lambda_j(F(x,y)) = y$. Moreover, $dF$ is a constant matrix such that $|\det(dF)| = 2\,|K|$. If $i = j$, we get that

$$\begin{aligned}\int_K \lambda_i(\mathbf{x})\lambda_j(\mathbf{x})\,d\mathbf{x} &= 2\,|K|\int_{\hat{K}} x^2\,dx dy = 2\,|K|\int_0^1\int_0^{1-x} x^2\,dy\,dx\\ &= 2\,|K|\int_0^1 x^2 - x^3\,dx = 2\,|K|\left(\frac{1}{3}-\frac{1}{4}\right) = \frac{|K|}{12}(1+\delta_{ij}),\end{aligned}$$

as was required. If $i \neq j$, we still get that

$$\begin{aligned}
\int_K \lambda_i(\mathbf{x})\lambda_j(\mathbf{x})\,d\mathbf{x} &= 2\,|K| \int_{\hat{K}} xy\,dx dy \;= 2\,|K| \int_0^1 \int_0^{1-x} xy\,dy\;dx \\
&= 2\,|K| \int_0^1 \frac{1}{2}x(1-x)^2\,dx \;= |K| \int_0^1 x - 2x^2 + x^3\,dx \\
&= |K| \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) = |K| \left(\frac{6-8+3}{12}\right) = \frac{|K|}{12}(1+\delta_{ij}),
\end{aligned}$$

so the identity in the hint has been proved for all $i, j \in \{1, 2, 3\}$. Moving on to the central part of the proof,

$$\begin{aligned}
(\tau_{h,e_i}, \tau_{h,e_j})_K &= \int_K \left\langle \tau_{h,e_i}(x), \tau_{h,e_j}(x) \right\rangle\,dx \\
&= \int_K \sigma_i\sigma_j \frac{|e_i|\,|e_j|}{4\,|K|^2} \left\langle x - x_i, x - x_j \right\rangle\,dx\,.
\end{aligned}$$

Let $c$ be the constant in the last integral. We can introduce barycentric coordinates to get that

$$\begin{aligned}
\int_K c \left\langle x - x_i, x - x_j \right\rangle\,dx &= c \int_K \left\langle \sum_{s=1}^{3} \lambda_s(x)(x_s - x_i), \sum_{r=1}^{3} \lambda_r(x)(x_r - x_j) \right\rangle\,dx \\
&= c \sum_{s,r=1}^{3} \left\langle x_s - x_i, x_r - x_j \right\rangle \int_K \lambda_s(x)\lambda_r(x)\,dx \\
&= c \sum_{s,r=1}^{3} \left\langle x_s - x_i, x_r - x_j \right\rangle \frac{|K|}{12}(1+\delta_{sr}).
\end{aligned}$$

Let $x_c$ be the barycenter of $K$. Then we can further simplify the sum:

$$\begin{aligned}
(\tau_{h,e_i}, \tau_{h,e_j})_K &= \frac{c\,|K|}{12} \left( \sum_{s,r=1}^{3} \left\langle x_s - x_i, x_r - x_j \right\rangle + \sum_{s=1}^{3} \left\langle x_s - x_i, x_s - x_j \right\rangle \right) \\
&= \frac{c\,|K|}{12} \left( \left\langle \sum_{s=1}^{3}(x_s - x_i), \sum_{r=1}^{3}(x_r - x_j) \right\rangle + \sum_{s=1}^{3} \left\langle x_s - x_i, x_s - x_j \right\rangle \right) \\
&= \frac{c\,|K|}{12} \left( 9 \left\langle x_c - x_i, x_c - x_j \right\rangle + \sum_{s=1}^{3} \left\langle x_s - x_i, x_s - x_j \right\rangle \right) \\
&= \sigma_i\sigma_j \frac{|e_i|\,|e_j|}{48\,|K|} \left( 9 \left\langle x_c - x_i, x_c - x_j \right\rangle + \sum_{s=1}^{3} \left\langle x_s - x_i, x_s - x_j \right\rangle \right).
\end{aligned}$$

As for the last part of the proof, we compute that

$$\begin{aligned}
(v_{h,K}, \nabla\cdot\tau_{h,e_i})_K &= \int_K v_{h,K}(x)\,\nabla\cdot\tau_{h,e_i}(x)\,dx \;= \int_K \sigma_i \frac{|e_i|}{2\,|K|} \nabla\cdot(x - x_i)\,dx \\
&= \int_K \sigma_i \frac{|e_i|}{2\,|K|} 2\,dx \;= \sigma_i \frac{|e_i|}{|K|} \int_K 1\,dx \;= \sigma_i\,|e_i|\,. \quad \square
\end{aligned}$$