Advanced Discretization Methods (WS 19/20)
Homework 3
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Deadline for submission (theory):	November 12th, 2019, 12:15
Deadline for submission (programming):	November 12th, 2019, 12:15

Remark: When you apply theorems, whether they were found in Knabner/Angermann or another source, make sure to **cite them**!

Exercise 7: Non-Consistency by Quadrature (2+2+3+3)

Let $\Omega \subset \mathbb{R}^d$ be a bounded polytope, $f\colon \Omega \to \mathbb{R}^d$ and $A\colon \Omega \to \mathbb{R}^{d\times d}$ be symmetric and uniformly elliptic, i.e. there exists $\alpha > 0$ such that

$$A(x) = A(x)^T \quad \forall x \in \Omega, \qquad \xi^T A(x)\xi \geq \alpha|\xi|^2 \quad \forall x \in \Omega,\ \xi \in \mathbb{R}^d \setminus \{0\}.$$

Consider the problem of finding $u\colon \Omega \to \mathbb{R}^d$ s.t.

$$-\nabla \cdot (A(x)\nabla u(x)) = f(x) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

The weak version of this problem reads: Find $u \in X := H_0^{1,2}(\Omega)^d$ such that

$$a(u, v) := (A\nabla u, \nabla v) = (f, v) \quad \forall v \in X.$$

The well-definedness of this problem follows by our assumptions on A. Let $(\mathcal{T}_h)_{h\to 0}$ denote a shape-regular, quasi-uniform family of triangulations of Ω and $X_h \subset X$ be the Lagrange finite element of order $k \in \mathbb{N}$ on $\mathcal{T}_h$. For simplicity, we make the following regularity assumptions: Let $A \in H^{r,\infty}(\Omega)$, $f \in H^{r,2}(\Omega)$ and $r > 2k - 2$ where k is large enough for $H^{k,2}(\Omega) \hookrightarrow C(\overline{\Omega})$. We use quadrature for both the evaluation of the bilinear form and right-hand side. Let $Q(v, w)$ for $v, w \in L^2(\Omega)$ with $v|_T, w|_T \in H^{r,2}(\Omega)$ for all $T \in \mathcal{T}_h$ denote a quadrature rule of order $r \in \mathbb{N}$ on a triangulation $\mathcal{T}_h$ of Ω in the sense of

$$|(v, w) - Q(v, w)| \leq Ch^r \sum_{T\in\mathcal{T}_h} \|v\|_{r,2;T}\|w\|_{r,2;T}.$$

The (inconsistent) discrete problem then reads: Find $u_h \in X_h$ such that

$$a_h(u_h, v_h) := Q(A\nabla u_h, \nabla v_h) = Q(f, v_h) \quad \forall v_h \in X_h.$$

a) Prove that for all $v_h \in X_h$ there holds

$$|(f, v_h) - Q(f, v_h)| \leq Ch^{r-k+1}\|f\|_r\|v_h\|_1.$$

Hint: Use inverse estimates.

b) For $v_h, w_h \in X_h$ prove that

$$|a(v_h, w_h) - a_h(v_h, w_h)| \leq Ch^r\|A\|_{r,\infty}\Big(\sum_{T\in\mathcal{T}_h}\|v_h\|^2_{k,2;T}\Big)^{\frac{1}{2}}\Big(\sum_{T\in\mathcal{T}_h}\|w_h\|^2_{k,2;T}\Big)^{\frac{1}{2}}.$$

c) Using b), prove that a_h is coercive for sufficiently small h.

Hint: Use inverse estimates to show that $\|a - a_h\| \to 0$ for $h \to 0$.

d) Use Strang's 1st Lemma and a) – c) to prove

$$\|u - u_h\|_1 \leq Ch^k.$$

with a constant depending on $\|f\|_r$, $\|A\|_{r,\infty}$ and $\|u\|_{k+1}$. You can use without proof for the Lagrange interpolation operator I_h that $\|I_h u\|_{k,2;T} \leq C\|u\|_{k,2;T}$.

Exercise 8: Poisson with Neumann boundaries as a saddle point problem (2+1+2+2+2+1)

Let Ω be a sufficiently regular domain and $f \in L^2(\Omega)$, $g \in L^2(\partial\Omega)$. Let $u \in H^{1,2}(\Omega)$ satisfy

$$\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x = \int_\Omega fv \,\mathrm{d}x + \int_{\partial\Omega} gv \, d\sigma \quad \forall\, v \in H^{1,2}(\Omega). \tag{1}$$

a) Let $u \in H^{2,2}(\Omega)$. Show that $-\Delta u = f$ and that $\partial_n u = g$ in the sense of

$$\int_{\partial\Omega} (\partial_n u - g)v \, d\sigma = 0 \quad \forall\, v \in H^{1,2}(\Omega).$$

b) Show that the compatibility condition

$$\int_\Omega f \, dx + \int_{\partial\Omega} g \, d\sigma = 0 \tag{2}$$

is a necessary condition for a solution u of (1) to exist.

Note that $u + c$ is a solution to (1) for any $c \in \mathbb{R}$. This motivates us to consider the following bilinear form:

$$\widetilde{a}(u, v) := \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x + \Big(\int_\Omega u \,\mathrm{d}x\Big)\Big(\int_\Omega v \,\mathrm{d}x\Big).$$

c) Show that $\widetilde{a}$ is coercive in $H^{1,2}(\Omega)$.
Hint: Recall Poincaré's inequality in $H^{1,2}(\Omega)$.

d) Let $\widetilde{u} \in H^{2,2}(\Omega)$ solve (1) with left-hand side $\widetilde{a}$. Show that

$$|\Omega| \int_{\Omega} \widetilde{u}\, dx = \int_{\Omega} f\, dx + \int_{\partial\Omega} g\, d\sigma\,.$$

In particular, if (2) is satisfied, then $\widetilde{u}$ solves (1) with $\int_\Omega \widetilde{u}\,\mathrm{d}x = 0$.

e) Formulate a saddle point problem corresponding to (1) with the side constraint $\int_\Omega u\,\mathrm{d}x = 0$.

f) Prove that a solution $(u,\lambda) \in H^{1,2}(\Omega) \times \mathbb{R}$ of the saddle point problem also solves (1), if (2) is satisfied.

Programming exercise 2: Time stepping schemes (10)

We now want to expand on the code from Programming Exercise 1 and introduce simple time stepping schemes. Note that a solution to the first exercise has been made available on StudOn and can be used if you're not sure about your own code so far.

Write a function

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uh = TimeStepping(...,DT,T,theta)
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which solves the following system

$$\begin{aligned} \partial_t u - \nabla \cdot (a(x)\nabla u) + r(x)u &= f \quad \text{in } (0,T)\times\Omega \\ u &= u_D \quad \text{on } (0,T)\times\partial\Omega \\ u(0,x) &= u_0(x) \end{aligned}$$

using one of the following time stepping schemes for fixed time step size Δt:

- $\theta = 0$ - Explicit Euler
- $\theta = 1$ - Implicit Euler
- $\theta = \frac{1}{2}$ - Crank-Nicolson

`uh` should store the solution at every time step between 0 and T.
Note that while $a = a(x), r = r(x)$, it may now be the case that f and u_D are time-dependent. Test your code for $\Omega = [0,1]^2$, $u(t,x,y) = \exp(-t)\sin(\pi x)\sin(\pi y)$, $a \equiv c_1, r \equiv c_2$ for appropriate constants c_1, c_2.