Advanced Discretization Methods (WS 19/20)
Homework 6

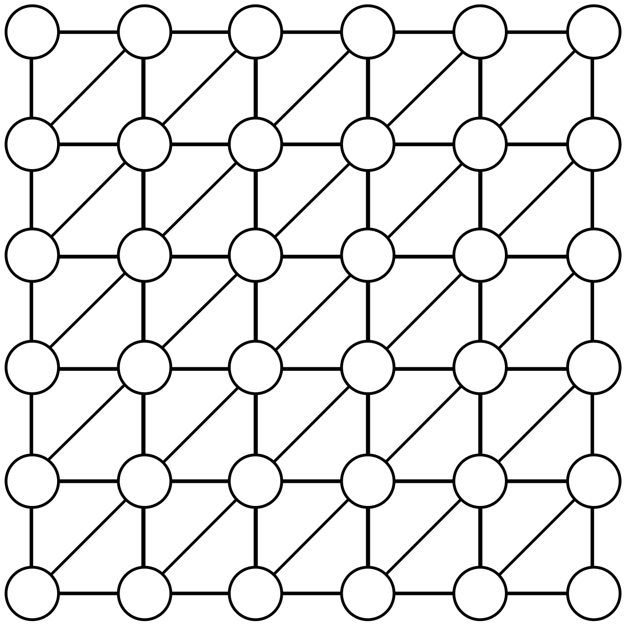
(P. Knabner, L. Wester)

Deadline for submission (theory):	December 3rd, 2019, 12:15
Deadline for submission (programming):	(c.f. Homework 5)

Remark: When you apply theorems, whether they were found in Knabner/Angermann or another source, make sure to **cite them**!

Exercise 13: Instability of $(\mathcal{P}_1)^2$-$(\mathcal{P}_1)$ for Stokes (10)

The goal of this exercise is to show that the pair of spaces $(\mathcal{P}_1)^2 - \mathcal{P}_1$ is not inf-sup stable for Stokes' problem (hence the need for a velocity ansatz space with more degrees of freedom, such as $(\mathcal{P}_2)^2$). Let $\Omega = (0,1)^2$ and consider the following triangulation $\mathcal{T}_h$ (the circles indicate degrees of freedom):



The finite element spaces for velocity and pressure are

$$M_h := (S^1_{0,0})^2 := \left\{ v_h \in C^0(\overline{\Omega}) \cap H^{1,2}_0(\Omega) \mid v_h|_T \in \mathcal{P}_1(T)\ \forall T \in \mathcal{T}_h \right\}^2,$$
$$X_h := S^1_0 \cap L^2_0(\Omega) := \{ q_h \in C^0(\overline{\Omega}) \cap L^2_0(\Omega) \mid v_h|_T \in \mathcal{P}_1(T)\ \forall T \in \mathcal{T}_h \}.$$

a) Construct a function $0 \neq p_h \in X_h$ such that $\int_T p_h = 0$ for all $T \in \mathcal{T}_h$. You may specify p_h by writing its nodal values into the circles of the figure above. Argue why $\int_T p_h = 0$ for all $T \in \mathcal{T}_h$ and conclude that $p_h \in X_h$.

b) Prove that the existence of a function p_h as constructed in a) is sufficient for M_h-X_h to violate the inf-sup condition.

Hint: Use the midpoint rule to evaluate integrals over linear functions.

Exercise 14: A stable element - Fortin's trick (2+1+2+1+1+3+3+2)

Note: Exercises marked with ☠ are a bit more difficult than others.

In Exercise 13 we have seen that the discrete inf-sup-condition for Stokes' problem is not fulfilled for the case in which the discrete velocity space has the same number of degrees of freedom as the discrete pressure space. This inspires the definition of the MINI element as a way of enriching the velocity ansatz space:

$$M_h = (S^1_{0,0} \oplus B_3)^2, \qquad X_h := S^1_0 \cap L^2_0(\Omega)$$

where

$$B_3 := \{ v \in H^{1,2}_0(\Omega) \mid \exists C \in \mathbb{R} : v(x)|_T = C\varphi_{T,1}(x)\varphi_{T,2}(x)\varphi_{T,3}(x)\ \forall T \in \mathcal{T}_h \}$$

where $\varphi_{T,i}$ are the Lagrange $\mathcal{P}_1$-shape functions on T. Over the course of this exercise we will show that this suffices to imply inf-sup-stability.

a) For given $v \in H^{1,2}_0(\Omega)$ on a sufficiently smooth domain Ω, define $\pi^0_h : H^{1,2}_0(\Omega) \to S^1_{0,0}$ as the solution operator to the problem: Find $\pi^0_h v \in S^1_{0,0}$ s.t.

$$a_h(\pi^0_h v, w_h) := (\pi^0_h v, w_h)_0 + (\nabla \pi^0_h v, \nabla w_h)_0 = (v, w_h)_0 + (\nabla v, \nabla w_h) \qquad \forall w_h \in S^1_{0,0} \tag{A}$$

Show that this problem has a unique solution which satisfies

$$\|\pi^0_h v\|_1 \leq \|v\|_1$$

b) Consider now the problem: Given $f \in L^2(\Omega)$ find $u \in H^{1,2}_0(\Omega)$, s.t.

$$a_h(u, w) = (f, w)_0 \qquad \forall w \in H^{1,2}_0(\Omega) \tag{B}$$

Show that the unique solution for $f = \pi_h^0 v - v$ given $v \in H_0^{1,2}(\Omega)$ and $w_h \in S_{0,0}^1$, fulfills:

$$a_h(u - w_h, \pi_h^0 v - v) = \|\pi_h^0 v - v\|_0^2$$

c) Use b) and assume H^2-regularity of (B), i.e. the solution of (B) satisfies $u \in H^2(\Omega)$ and $\|u\|_2 \leq C\|f\|_0$, to show that

$$\|\pi_h^0 v - v\|_0 \leq Ch\|\pi_h^0 v - v\|_1$$

d) Conclude with a) that

$$\|\pi_h^0 v - v\|_0 \leq Ch\|v\|_1$$

e) We now define a second projector $\pi_h^1 : L^2(\Omega) \to B_3$ via: (with $v \in L^2(\Omega)$)

$$\int_T (\pi_h^1 v - v)\,\mathrm{d}x = 0 \qquad \forall T \in \mathcal{T}_h$$

Show that the operator is well-defined and find an explicit representation of $\pi_h^1 v$.

f) ☠ Use the representation found in e) and show that it satisfies

$$\|\pi_h^1 v\|_{0,T}^2 \leq C\|v\|_{0,T}^2 \qquad \forall v \in L^2(\Omega)$$

with a constant C independent of T. Conclude that

$$\|\pi_h^1 v\|_0 \leq C\|v\|_0$$

g) ☠ We now define the Fortin operator via

$$\Pi_h \boldsymbol{v} := \pi_h^0 \boldsymbol{v} + \pi_h^1(\boldsymbol{v} - \pi_h^0 \boldsymbol{v})$$

(to be understood entry-wise for $\boldsymbol{v} \in (H_0^{1,2}(\Omega))^2$)
Let further for $q \in L_0^2(\Omega)$

$$b(\boldsymbol{v}, q) := -\int_\Omega \operatorname{div} \boldsymbol{v} q\,\mathrm{d}x$$

be the bilinear form of Stokes' problem. Show that the Fortin operator satisfies

$$b(\boldsymbol{v} - \Pi_h \boldsymbol{v}, q_h) = 0 \qquad \forall \boldsymbol{v} \in (H_0^{1,2}(\Omega))^2,\ q_h \in X_h$$

Hint: Proceed element-wise, perform integration by parts and use previous knowledge about π_h^0 and π_h^1.

h) Use an inv. estimate, a), f), and d) to show that

$$\|\Pi_h \boldsymbol{v}\|_1 \leq C\|\boldsymbol{v}\|_1 \qquad \forall \boldsymbol{v} \in (H_0^{1,2}(\Omega))^2$$