Advanced Discretization Methods (WS 19/20)
Homework 5

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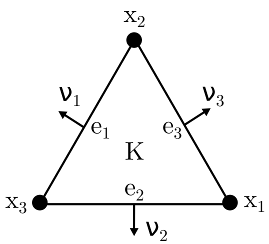
Deadline for submission (theory):	November 26th, 2019, 12:15
Deadline for submission (programming):	December 3rd, 2019, 12:15

Exercise 11: A trace estimate using the Raviart-Thomas element RT_0 (3+2+2+3)



Let K be a triangle with vertices $x_1, x_2, x_3 \in \mathbb{R}^2$. Denote by e_i the edge opposite to x_i and by ν_i its normal, see the figure above. Define functions $\tau_i \in \mathrm{RT}_0(K)$ by

$$\tau_i(x) := \frac{|e_i|}{2|K|}(x - x_i) \quad \text{for } i = 1, 2, 3.$$

Let $h := \max\{|e_1|, |e_2|, |e_3|\}$ be the maximal edge length and $c > 0$ be chosen such that $h^2 \leq c|K|$.

a) Prove that $\tau_i(x) \cdot \nu(x) = \chi_{e_i}(x)$ for $x \in \partial K$ a.e., where χ denotes the characteristic function. Use (8.15) to conclude that $\{\tau_1, \tau_2, \tau_3\}$ is a basis of $\mathrm{RT}_0(K)$.

b) Prove that for any $u \in H^{1,2}(K)$ and $i \in \{1, 2, 3\}$ there holds:

$$\|u\|_{0,e_i}^2 \leq \|\nabla \cdot \tau_i\|_{L^\infty(K)} \|u\|_{0,K}^2 + 2\|\tau_i\|_{L^\infty(K)} \|u\|_{0,K} \|\nabla u\|_{0,K}.$$

Hint: Use a) and apply the divergence theorem: $\int_{\partial\Omega} f \cdot \nu \, d\sigma = \int_\Omega \nabla \cdot f \, \mathrm{d}x$.

c) Prove that for any $i \in \{1, 2, 3\}$ there holds:

$$\|\tau_i\|_{L^\infty(K)} \leq \frac{c}{2} \quad \text{and} \quad \|\nabla \cdot \tau_i\|_{L^\infty(K)} \leq \frac{c}{h}.$$

d) Finally, prove the following trace estimate for $u \in H^{1,2}(K)$:

$$\|u\|_{0,e_i} \leq C \left(h^{-1/2} \|u\|_{0,K} + h^{1/2} \|\nabla u\|_{0,K} \right)$$

with $C > 0$ depending on c.

Exercise 12: Poisson's problem with RT_0-$\mathcal{L}_0^0$ (10)

We want to study the saddle point formulation of the non-homogeneous Poisson problem

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = u_D \quad \text{on } \partial\Omega$$

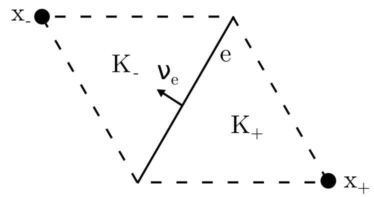
with RT_0-$\mathcal{L}_0^0$ elements. The discrete problem reads: Find

$$\sigma_h \in V_h := \mathrm{RT}_0(\mathcal{T}_h), \quad u_h \in W_h := \mathcal{L}_0^0(\mathcal{T}_h)$$

such that

$$\begin{aligned} (\sigma_h, \tau_h) + (u_h, \nabla \cdot \tau_h) &= \int_{\partial\Omega} u_D \tau_h \cdot \nu \, d\sigma \qquad && \forall \tau_h \in V_h, \\ (v_h, \nabla \cdot \sigma_h) &= -(f, v_h) && \forall v_h \in W_h. \end{aligned}$$

We fix the orientation of interior edges e by fixing a normal vector ν_e for each such edge. For given interior edge e we denote K_+ and K_- the adjacent triangles, such that ν_e points from K_+ to K_-. We denote by $x_\pm$ the vertices opposite to e in $K_\pm$, see the figure below for all notations. We extend the notation to exterior edges, denoting by ν_e the outward normal.



For any $K \in \mathcal{T}_h$ and adjacent edge e we set

$$\tau_{h,e}(x) := \begin{cases} \pm\frac{|e|}{2|K_\pm|}(x - x_\pm) & \text{if } x \in K_\pm, \\ 0 & \text{else,} \end{cases} \qquad v_{h,K}(x) := \begin{cases} 1 & \text{if } x \in K, \\ 0 & \text{else} \end{cases}$$

and $\{\tau_{h,e}; e \text{ edge of } \mathcal{T}_h\}$ forms a basis of V_h (cf. Ex. 11a), $\{v_{h,K}; K \in \mathcal{T}_h\}$ a basis of W_h.

With the notation from Exercise 11a) for a single triangle $K \in \mathcal{T}_h$, find explicit expressions for $(\tau_{h,e_i}, \tau_{h,e_j})_K$ and $(v_{h,K}, \nabla \cdot \tau_{h,e_i})$ for $i, j = 1, 2, 3$. The sign of $\tau_{h,e}$ may be represented as $\sigma_i := \nu_i \cdot \nu_{e_i}$, where ν_i is the normal relative to K, as in Ex. 11a, and ν_{e_i} the fixed normal.

Hint: Show that for barycentric coordinates λ_i, λ_j in K it holds

$$\int_K \lambda_i \lambda_j \,\mathrm{d}x = \frac{|K|}{12}(1+\delta_{ij})$$

Programming exercise 5: Stokes with Taylor-Hood elements (25)

Now we finally want to solve a Stokes problem: Given a domain $\Omega \subset \mathbb{R}^2$ find $(\boldsymbol{u}, p) \in (H_0^{1,2}(\Omega))^2 \times L_0^2(\Omega)$, s.t.

$$\begin{aligned} a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) &= (\boldsymbol{f} + \boldsymbol{f}_D, \boldsymbol{v}) \qquad \forall \boldsymbol{v} \in (H_0^{1,2}(\Omega))^2 \\ b(\boldsymbol{u}, q) &= (g_D, q) \qquad \forall q \in L_0^2(\Omega) \end{aligned}$$

where $L_0^2(\Omega) := \{q \in L^2(\Omega) \mid \int_\Omega q \,\mathrm{d}x = 0\}$ and

$$a(\boldsymbol{u}, \boldsymbol{v}) := \int_\Omega \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{v} \,\mathrm{d}x, \qquad b(\boldsymbol{v}, p) := -\int_\Omega \operatorname{div} \boldsymbol{v} p \,\mathrm{d}x, \qquad \boldsymbol{f} \equiv [0, -g]^T$$

for $\boldsymbol{u}, \boldsymbol{v} \in (H_0^{1,2}(\Omega))^2$, $p \in L_0^2(\Omega)$ and $g \geq 0$ is a constant representing gravity.
Reminder: in continuous form, this problem is given as: Find $\boldsymbol{u} \in C^2(\Omega)^2 \cap C^0(\overline{\Omega})^2$, $p \in C^1(\Omega)$, s.t.

$$\begin{aligned} -\Delta \boldsymbol{u} + \nabla p &= \boldsymbol{f} && \text{in } \Omega \\ \operatorname{div} \boldsymbol{u} &= 0 && \text{in } \Omega \\ \boldsymbol{u} &= \boldsymbol{u}_D && \text{on } \partial\Omega \end{aligned}$$

(i.e. the right-hand side $\boldsymbol{f}_D$ and g_D in the variational formulation are not additional forcing terms, but only the result of shifting the Dirichlet-b.c. In other words: $(\boldsymbol{f}_D, \boldsymbol{v}) = -a(\boldsymbol{u}_D, \boldsymbol{v})$, $(g_D, q) = -b(\boldsymbol{u}_D, q)$)
To solve this problem, we use $\mathcal{P}_2$ Lagrange elements for the velocity components u_1, u_2, but $\mathcal{P}_1$-elements for the pressure p. Let $\varphi_1, ..., \varphi_{N_2}$ be the $\mathcal{P}_2$ shape functions and $\psi_1, ..., \psi_{N_1}$ the $\mathcal{P}_1$ shape functions. Then the above problem leads to a discretized system of the form

$$\begin{bmatrix} A & 0 & B_1^T \\ 0 & A & B_2^T \\ B_1 & B_2 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ p \end{bmatrix} = \begin{bmatrix} f_{D_1} \\ -g + f_{D_2} \\ g_D \end{bmatrix}$$

where $A \in \mathbb{R}^{N_2 \times N_2}$, $B_1, B_2 \in \mathbb{R}^{N_1 \times N_2}$ are given as follows:

$$\begin{aligned} A_{ij} &= a(\varphi_j, \varphi_i) && i, j = 1, ..., N_2 \\ (B_1)_{ij} &= \int_\Omega -\partial_{x_1} \varphi_j \psi_i \,\mathrm{d}x && i = 1, ..., N_1;\ j = 1, ..., N_2 \\ (B_2)_{ij} &= \int_\Omega -\partial_{x_2} \varphi_j \psi_i \,\mathrm{d}x && i = 1, ..., N_1;\ j = 1, ..., N_2 \end{aligned}$$

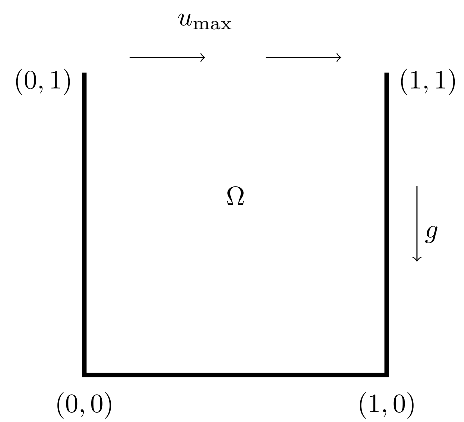
As mentioned before, f_{D_1}, f_{D_2}, g_D are an artifact of introducing Dirichlet-b.c. into the system.
Write a function

```
B = AssembleMixedMatrix(coord,elemNodeTable)
```

which assembles `B`$= [B_1\ B_2]$. (Note: After Programming Exercise 03, your `coord` should be ordered so that you can extract the $\mathcal{P}_1$ mesh from the first 3 columns of `elemNodeTable`. If this is not the case, we recommend generating a mesh using `gen_mesh_rectangle.m` from the solution of PE03 on StudOn).
Use `AssembleMatrices.m` to generate A, `AssembleRHS.m` to generate the right-hand side and test your function by solving the **lid-driven cavity** problem for different values of $g \geq 0, u_{\max} > 0$:

- $\Omega = (0,1)^2$
- $\boldsymbol{u}|_{[0,1]\times\{1\}} = \begin{bmatrix} u_{\max} \\ 0 \end{bmatrix}$
- $\boldsymbol{u}|_{\partial\Omega\setminus([0,1]\times\{1\})} = \boldsymbol{0}$

You may assemble the full block system and simply solve it using the backslash-operator.
Plot $\boldsymbol{u}$ using `quiver` and p with a method of choice (e.g. `trisurf`, `contour`). Comment on your results.
Note: The constraint for the pressure ($\int_\Omega p \,\mathrm{d}x = 0$) is not straightforward to implement. Instead, you can simply assume the Dirichlet-b.c. $p(\boldsymbol{x}_0) = 0$ for a single, but fixed, $\boldsymbol{x}_0 \in \partial\Omega$.



2nd Note: The programming exercise is due in **2 weeks**, on **Dec. 3rd**.