Advanced Discretization Methods (WS 19/20)
Homework 7

(P. Knabner, L. Wester)

Deadline for submission (theory): December 10th, 2019, 12:15
Deadline for submission (programming): December 10th, 2019, 12:15

Remark: When you apply theorems, whether they were found in Knabner/Angermann or another source, make sure to **cite them**!

Exercise 15: Uzawa algorithm - a proof of convergence (1+1+3+2+3)

Given the saddle point problem: Find $u \in M_h, p \in X_h$, s.t.

$$\begin{aligned} a(u,v) + b(v,p) &= f(v) \qquad \forall v \in M_h \\ b(u,q) &= g(q) \qquad \forall q \in X_h \end{aligned}$$

denote as always the operators $A : M_h \to M_h'$ and $B : M_h \to X_h'$ (along with its adjoint $B_h^* : X \to M_h'$) by:

$$\langle Au, v \rangle := a(u,v), \qquad \langle Bu, q \rangle = \langle u, B^* q \rangle := b(u,q) \qquad \forall u,v \in M_h,\ q \in X_h$$

You may assume A to be s.p.d..

We can then write Uzawa's algorithm as follows:

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Let ω > 0, TOL > 0, p^0 ∈ X_h be given
for k ≥ 0 do
    Find u^{k+1} ∈ M_h, s.t.
        A u^{k+1} = f − B* p^k
    Find p^{k+1} ∈ X_h, s.t.
        p^{k+1} − p^k = −ω (g − B u^{k+1})
    if ||p^{k+1} − p^k||_X < TOL then
        STOP
    end if
end for
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a) Show that p solves the following problem:

$$Sp = BA^{-1}f - g \qquad (*)$$

where $S := BA^{-1}B^*$ denotes the Schur complement of the saddle point problem.

b) Show that Uzawa's algorithm can be interpreted as a relaxated Richardson method applied to $(*)$.

c) We define the S-energy norm via $\|p\|_S := (p,p)_S^{1/2} := \langle Sp, p \rangle^{1/2}$. Show that it satisfies:

$$\|p\|_S = \sup_{v \in M_h} \frac{\langle Bv, p \rangle}{\|v\|_A}$$

d) Beyond this point we consider only the Stokes case (in particular $M_h \subset H_0^{1,2}(\Omega), X_h \subset L_0^2(\Omega)$). Assume the inf-sup-condition to hold for B with constant $\beta > 0$, i.e.:

$$\inf_{q \in X_h} \sup_{v \in M_h} \frac{\langle Bv, p \rangle}{|v|_1 \|q\|_0} = \beta > 0$$

Show that S fulfills:

$$\lambda_{\min}(S) = \beta^2, \qquad \lambda_{\max}(S) \le 1$$

for its smallest and largest eigenvalue.

e) Conclude the following error estimates for Uzawa's algorithm given the choice of $\omega = 1$:

$$\begin{aligned} \|p - p^k\|_0 &\le (1-\beta^2)^k \|p - p^0\|_0 \\ |u - u^{k+1}|_1 &\le (1-\beta^2)^k \|p - p^0\|_0 \end{aligned}$$

Hint: You may use that for Richardson's method, the error can be written as

$$p - p^k = (Id - \omega S)(p - p^{k-1})$$

with the corresponding error estimate

$$\|p - p^k\| \le \rho(Id - \omega S)\|p - p^{k-1}\|$$

Exercise 16: Voronoi diagrams (4+3+3)

Let $\mathcal{A} := \{a_i\}_{i \in \overline{\Lambda}} \subset \mathbb{R}^2$ be a set of points with corresponding Voronoi polygons

$$\widetilde{\Omega}_i := \{x \in \mathbb{R}^2 \mid |x - a_i| < |x - a_j| \ \forall j \neq i\}$$

a) Show that $\widetilde{\Omega}_i$ is convex and polygonally bounded.

b) Let $p \in \mathbb{R}^2$ and $B(p)$ be the largest open ball with center p that fulfills $B(p) \cap \mathcal{A} = \emptyset$. Prove:

i) $p \in \mathbb{R}^2$ is a Voronoi vertex, if and only if $\#(\partial B(p) \cap \mathcal{A}) \geq 3$.

ii) $p \in \mathbb{R}^2$ is on an edge of the diagram, if and only if $\#(\partial B(p) \cap \mathcal{A}) = 2$.

Programming exercise 6: Uzawa's algorithm (10)

Until now we have always assembled the full system and simply solved it via the backslash-operator. While in MatLab this may generally be an acceptable (and usually the fastest) way to solve the linear system, the question arises how to treat saddle point problems when such a handy solution is not at hand. The indefinite matrix system characteristic of saddle point problems tends to be much more involved than the simple elliptic problems we treated in Programming exercise 1, since we lose a large class of efficient solvers (e.g. CG). A common algorithm used for such systems shall be implemented here.

Expand Programming exercise 5 by solving the linear system with the Uzawa algorithm. Note one significant difference compared to the pseudocode written in Exercise 15: Line 6, in matrix form, is actually

$$M(p^{k+1} - p^k) = -\omega(g - Bu^{k+1})$$

with the mass matrix M (why might that be?).

Also note that you do not need to pose any constraints on the pressure, an initial value suffices. Test your program for fixed g and $u_{\max}$, but various mesh sizes and values for $\rho \in (1, 2)$. Write down your observations.