Advanced Discretization Techniques
Homework 7

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Exercise 15: Uzawa algorithm - a proof of convergence

a) The saddle point problem can be written in the following operator form:

$$\begin{aligned} Au + B^*p &= f \\ Bu &= g. \end{aligned}$$

A is invertible, because it's positive definite (by assumption). Hence, we can multiply the first equation by BA^{-1} and substitute g for Bu to get that

$$\begin{aligned} Bu + BA^{-1}B^*p &= BA^{-1}f \\ g + BA^{-1}B^*p &= BA^{-1}f \\ Sp &= BA^{-1}f - g. \end{aligned}$$

b) Given a linear system $Lx = r$, the ω-relaxated Richardson method is defined as the following iterative scheme:

$$x^{k+1} = x^k + \omega(r - Lx^k).$$

Uzawa's algorithm can be written in this form by substituting line 4 into line 6 (and eliminating u^{k+1} in the process):

$$\begin{aligned} p^{k+1} - p^k &= \omega(Bu^{k+1} - g) \\ &= \omega(B(A^{-1}f - A^{-1}B^*p^k) - g) \\ &= \omega(BA^{-1}f - g - BA^{-1}B^*p^k) \\ &= \omega(BA^{-1}f - g - Sp^k), \end{aligned}$$

As required, $L = S$ and $r = BA^{-1}f - g$.

c) By the spectral theorem, $A^{1/2}$ is a well-defined, real, s.p.d. operator. Then

$$\|v\|_A = \langle Av, v\rangle_{H^1}^{1/2} = \left\langle A^{1/2}A^{1/2}v, v\right\rangle^{1/2} = \left\langle A^{1/2}v, A^{1/2}v\right\rangle^{1/2} = \left\|A^{1/2}v\right\|_1$$

$$\|p\|_S = \langle Sp, p\rangle_{L^2}^{1/2} = \langle BA^{-1}B^*p, p\rangle_{L^2}^{1/2} = \left\langle A^{-1/2}B^*p, A^{-1/2}B^*p\right\rangle_{H^1}^{1/2} = \left\|A^{-1/2}B^*p\right\|_1$$

$$\langle Bv, p\rangle_{L^2} = \left\langle BA^{-1/2}A^{1/2}v, p\right\rangle_{L^2} = \left\langle A^{1/2}v, A^{-1/2}B^*p\right\rangle_{H^1}.$$

By the Cauchy-Schwarz inequality, it follows that

$$\sup_{v\in M_h} \frac{\langle Bv, p\rangle}{\|v\|_A} = \sup_{v\in M_h} \frac{\langle A^{1/2}v, A^{-1/2}B^*p\rangle}{\left\|A^{1/2}v\right\|_1} \leq \left\|A^{-1/2}B^*p\right\|_1 = \|p\|_S\,,$$

and this inequality can be turned into an equality by choosing $v \in M_h$ in the sup as $A^{-1}B^*p$, so that $A^{1/2}v = A^{-1/2}B^*p$.

d) In the Stokes case,

$$a(u, v) = \int_\Omega Du : Dv\,dx \qquad b(v, q) = -\int_\Omega q\operatorname{div}(v)\,dx\,.$$

Let $R(q)$ be the Rayleigh quotient associated with S:

$$R(q) = \frac{\langle Sq, q\rangle}{\langle q, q\rangle}.$$

Since S is s.p.d., it follows from the min-max theorem that

$$\lambda_{\min}(S) = \inf_{0\neq q\in X_h} R(q) \quad \text{and} \quad \lambda_{\max}(S) = \sup_{0\neq q\in X_h} R(q).$$

If we now combine the result from point c) with the inf-sup condition, we get that

$$\lambda_{\min}(S) = \inf_{0\neq q\in X_h} \frac{\langle Sq, q\rangle}{\langle q, q\rangle} = \inf_{0\neq q\in X_h} \frac{\|q\|_S^2}{\|q\|_0^2} = \inf_q \sup_v \frac{\langle Bv, q\rangle^2}{\|v\|_A^2\,\|q\|_0^2}$$

$$= \inf_q \sup_v \frac{\langle Bv, q\rangle^2}{a(v, v)\,\|q\|_0^2} = \inf_q \sup_v \frac{\langle Bv, q\rangle^2}{|v|_1^2\,\|q\|_0^2} = \beta^2,$$

so the first half of the proof is complete. For the second half, we first need to show that $\|\mathrm{div}(v)\|_0^2 \le \|v\|_A^2$:

$$
\begin{aligned}
\|\mathrm{div}(v)\|_0^2 &= \int_\Omega \left(\sum_{i=1}^d \partial_i v_i(x) \right)^2 dx \;= \int_\Omega \sum_{i,j=1}^d \partial_i v_i(x) \partial_j v_j(x)\, dx \\
&= 0 + 0 + \int_\Omega \sum_{i,j=1}^d \partial_j v_i(x) \partial_i v_j(x)\, dx \\
&= \int_\Omega \sum_{i=1}^d (\partial_i v_i(x))^2\, dx \;+ \int_\Omega \sum_{1\le i<j\le d} 2\partial_j v_i(x) \partial_i v_j(x)\, dx \\
&\le \int_\Omega \sum_{i=1}^d (\partial_i v_i(x))^2\, dx \;+ \int_\Omega \sum_{1\le i<j\le d} (\partial_j v_i(x))^2 + (\partial_i v_j(x))^2\, dx \\
&= \int_\Omega \sum_{i,j=1}^d (\partial_i v_j(x))^2\, dx \;= a(v,v) = \|v\|_A^2 .
\end{aligned}
$$

We have used integration by parts twice (and, implicity, a density argument with $C_0^\infty(\Omega)$ functions), symmetry of second derivatives and the inequality $2ab \le a^2 + b^2$. We can now conclude the second part of the proof:

$$
\begin{aligned}
\lambda_{\max}(S) &= \sup_{0\ne q\in X_h} \frac{\langle Sq, q\rangle}{\langle q, q\rangle} = \sup_q \sup_v \frac{b(v,q)^2}{\|v\|_A^2 \|q\|_0^2} \\
&= \sup_q \sup_v \frac{\left(\int_\Omega q\,\mathrm{div}(v)\, dx\right)^2}{\|v\|_A^2 \|q\|_0^2} \le \sup_q \sup_v \frac{\left(\int_\Omega q^2\, dx\right)\left(\int_\Omega \mathrm{div}(v)^2\, dx\right)}{\|v\|_A^2 \|q\|_0^2} \\
&= \sup_q \sup_v \frac{\|q\|_0^2 \|\mathrm{div}(v)\|_0^2}{\|v\|_A^2 \|q\|_0^2} \le \sup_{0\ne v\in M_h} \frac{\|v\|_A^2}{\|v\|_A^2} = 1.
\end{aligned}
$$

e) By induction on the error estimate for Richardson's method, we get that

$$
\left\|p - p^k\right\|_0 \le (\rho(I-S))^k \left\|p - p^0\right\|_0 .
$$

In point 4) we have shown that $\sigma(S) \subseteq [\beta^2, 1]$, so $\sigma(I-S) \subseteq [0, 1-\beta^2]$ and $\rho(I-S) \le 1-\beta^2$. This is enough to prove the first error estimate for Uzawa's algorithm. To prove the second one, consider the first equation of the saddle point problem and line 4 in the algorithm:

$$
Au + B^* p = f \qquad Au^{k+1} + B^* p^k = f.
$$

Subtracting one equation from the other gives

$$
A(u - u^{k+1}) + B^*(p - p^k) = 0,
$$

from which we can deduce that

$$
\begin{aligned}
\left|u-u^{k+1}\right|_1^2 &= \left\|u-u^{k+1}\right\|_A^2 = \left\langle A(u-u^{k+1}), u-u^{k+1}\right\rangle_{H^1} \\
&= \left\langle -B^*(p-p^k), -A^{-1}B^*(p-p^k)\right\rangle_{H^1} \\
&= \left\langle S(p-p^k), p-p^k\right\rangle_{L^2} \le \left\|S(p-p^k)\right\|_0 \left\|p-p^k\right\|_0 \\
&\le \|S\|_2 \left\|p-p^k\right\|_0^2 \le \rho(S) \left\|p-p^k\right\|_0^2 \le (1-\beta^2)^{2k} \left\|p-p^0\right\|_0^2 .
\end{aligned}
$$

Now we just have to take the square root of both sides. □

Exercise 16: Voronoi diagrams

a) Let $S_{ij} = \{x \in \mathbb{R}^2 \mid |x-a_i| < |x-a_j|\}$, so that $\tilde{\Omega}_i = \bigcap_{j \neq i} S_{ij}$. Geometric intuition suggests that the sets S_{ij} are open half-planes, and that their boundaries ∂S_{ij} are perpendicular bisectors of the line segments $a_i a_j$. To show that this is really the case, consider the following:

$$
\begin{gathered}
x \in S_{ij} \iff |x-a_i| < |x-a_j| \iff |x-a_j|^2 - |x-a_i|^2 > 0 \\
\iff \langle x-a_j, x-a_j\rangle + \langle x-a_i, a_i-x\rangle > 0 \\
\iff \langle x-a_j, x-a_j\rangle + \langle x-a_i, a_i-x\rangle + \langle x-a_j, a_i-x\rangle + \langle x-a_i, x-a_j\rangle > 0 \\
\iff \langle x-a_j, x-a_j\rangle + \langle x-a_j, a_i-x\rangle + \langle x-a_i, a_i-x\rangle + \langle x-a_i, x-a_j\rangle > 0 \\
\iff \langle x-a_j, x-a_j+a_i-x\rangle + \langle x-a_i, a_i-x+x-a_j\rangle > 0 \\
\iff \langle x-a_j+x-a_i, a_i-a_j\rangle > 0 \iff \left\langle x - \frac{a_i+a_j}{2}, a_i-a_j\right\rangle > 0.
\end{gathered}
$$

We can now recognize the definition of open half-plane, the midpoint $(a_i+a_j)/2$ and the direction $a_i - a_j$ of the line segment $a_i a_j$. This proves that $\tilde{\Omega}_i$ is convex, because we've expressed $\tilde{\Omega}_i$ as the intersection of convex sets (the open semiplanes S_{ij}). Moreover, the intersection of open half-planes is a (topologically) open polygonal region, so its boundary must be a polygonal chain (allowing half-lines as edges, if $\tilde{\Omega}_i$ is unbounded).

b) Let $n = \#(\partial B(p) \cap \mathcal{A})$. To prove i), let p be a Voronoi vertex. This means that p is a vertex of the boundary of some Voronoi polygon $\tilde{\Omega}_i$, so there exist $j, k \in \bar{\Lambda}$ such that i, j, k are all distinct and $p \in \partial S_{ij} \cap \partial S_{ik}$. The point p is therefore the circumcenter of the triangle $a_i a_j a_k$, since we have proved in point a) that ∂S_{ij} and ∂S_{ik} are the perpendicular bisectors of the sides $a_i a_j$ and $a_i a_k$. As Euclid could confirm, this in turn implies that p has equal distance from all the vertices a_i, a_j, a_k of the triangle, so $n \geq 3$ because we know that $\{a_i, a_j, a_k\} \subseteq \partial B(p) \cap \mathcal{A}$. To prove the converse, we can follow the same line of reasoning. Let $n \geq 3$. Then there exist distinct $\{a_i, a_j, a_k\}$ in $\partial B(p) \cap \mathcal{A}$, and once again p is the circumcenter of the triangle $a_i a_j a_k$. This means that $p \in \partial S_{ij} \cap \partial S_{ik}$, so p is a Voronoi vertex.

To prove ii), let p be a point on the edge of a Voronoi polygon, say $\tilde{\Omega}_i$. We want to prove that $n = 2$. On the one hand, we know that n can't be greater than 2, otherwise p would be a vertex by i) (we implicitly assume that p is not on the boundary of the edge). On the other hand, we know that $p \in \partial S_{ij}$ for some $j \neq i$, so p must be the midpoint of $a_i a_j$. This means that, if $\partial B(p) \cap \mathcal{A}$ contains a_i (which it does, by the choice of i), then it must also contain a_j. So n can only be 2. To prove the converse, let $p \in \mathbb{R}^2$ such that $n = 2$. For the sake of contradiction, suppose that p belongs to any of the open sets $\tilde{\Omega}_i$. Then $|p - a_i| < |p - a_j|$ for every $j \neq i$, so $\partial B(p)$ can only contain one point from $\mathcal{A}$, namely a_i. This contradicts the assumption $n = 2$, so p must be on the boundary of some Voronoi polygon. But p cannot be a vertex, otherwise we would have $n \geq 3$ by point i). Therefore p can only be on an edge, as required. $\square$