Advanced Discretization Techniques
Homework 11

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Exercise 20: Stability of the one-step θ method

The one-step θ method is defined as the iterative scheme

$$\xi^0 = \xi_0, \quad \frac{\xi^{n+1} - \xi^n}{\Delta t} + \lambda(\theta \xi^{n+1} + (1-\theta)\xi^n) = 0,$$

which can be reformulated as

$$\xi^0 = \xi_0, \quad \xi^{n+1} = \frac{1 - \lambda \Delta t (1-\theta)}{1 + \lambda \Delta t \theta} \xi^n.$$

Therefore, the complex function R_θ is defined as

$$R_\theta(z) = \frac{1 + (1-\theta)z}{1 - \theta z},$$

so that $\xi^{n+1} = R_\theta(-\lambda \Delta t)\xi^n$. If $\theta \in \{0, \frac{1}{2}, 1\}$, then we have

$$R_0(z) = z + 1, \qquad R_{1/2}(z) = \frac{1 + z/2}{1 - z/2}, \qquad R_1(z) = \frac{1}{1-z}.$$

Now we can determine the three domains of stability:

- S_{R_0} is a circle with radius 1 and center $(-1, 0)$:

$$\begin{aligned} S_{R_0} &= \{z \in \mathbb{C} \mid |z+1| < 1\} \\ &= \{z \in \mathbb{C} \mid (z+1)\overline{(z+1)} < 1\} \\ &= \{x + yi \in \mathbb{C} \mid (x + yi + 1)(x - yi + 1) < 1\} \\ &= \{x + yi \in \mathbb{C} \mid (x+1)^2 + y^2 < 1\}. \end{aligned}$$

- $S_{R_{1/2}}$ is the left half-plane:

$$\begin{aligned}
S_{R_{1/2}} &= \left\{ z \in \mathbb{C} \mid \left| \frac{1+z/2}{1-z/2} \right| < 1 \right\} \\
&= \left\{ z \in \mathbb{C} \mid \left(\frac{1+z/2}{1-z/2} \right) \left(\frac{1+\bar{z}/2}{1-\bar{z}/2} \right) < 1 \right\} \\
&= \left\{ z \in \mathbb{C} \mid \frac{1+\mathrm{Re}(z)+z\bar{z}/4}{1-\mathrm{Re}(z)+z\bar{z}/4} < 1 \right\} \\
&= \{ z \in \mathbb{C} \mid 1+\mathrm{Re}(z)+z\bar{z}/4 < 1-\mathrm{Re}(z)+z\bar{z}/4 \} \\
&= \{ z \in \mathbb{C} \mid 2\,\mathrm{Re}(z) < 0 \}.
\end{aligned}$$

- S_{R_1} is the whole complex plane, minus the unit circle centered at $(1,0)$:

$$\begin{aligned}
S_{R_1} &= \{ z \in \mathbb{C} \mid \left| \frac{1}{1-z} \right| < 1 \} \\
&= \{ z \in \mathbb{C} \mid |1-z| > 1 \} \\
&= \{ z \in \mathbb{C} \mid (1-z)\overline{(1-z)} > 1 \} \\
&= \{ x+yi \in \mathbb{C} \mid (1-x-yi)(1-x+yi) > 1 \} \\
&= \{ x+yi \in \mathbb{C} \mid (1-x)^2+y^2 > 1 \}.
\end{aligned}$$

Moving on to L-stability, we have that

$$\begin{aligned}
\lim_{Re(z)\to-\infty} R_\theta(z) &= \lim_{Re(z)\to-\infty} \frac{1+(1-\theta)z}{1-\theta z} \\
&= \lim_{Re(z)\to-\infty} \frac{1/z+(1-\theta)}{1/z-\theta} = \frac{1-\theta}{-\theta} = 0
\end{aligned}$$

if and only if $\theta = 1$ (if $\theta = 0$, the limit doesn't even exist). □

Exercise 21: Fully discrete estimate

The implicit Euler method is defined as the iterative scheme

$$U^0 = u_0, \quad M\frac{U^{n+1}-U^n}{\tau_n} + AU^{n+1} = F^{n+1}.$$

We begin the proof by testing the last equation with U^{n+1}:

$$(U^{n+1})^T M\frac{U^{n+1}-U^n}{\tau_n} + (U^{n+1})^T AU^{n+1} = (U^{n+1})^T F^{n+1}.$$

At the continuous level, this identity corresponds to

$$\int_\Omega u^{n+1}\left(\frac{u^{n+1}-u^n}{\tau_n}\right)dx + \int_\Omega \nabla u^{n+1}\cdot\nabla u^{n+1}\,dx = \int_\Omega f^{n+1}u^{n+1}\,dx \qquad (1)$$

Using (1) and Cauchy-Schwarz's inequality (twice), we can now prove that the following estimate holds for each timestep τ_n:

$$\frac{\|u^{n+1}\|_0 - \|u^n\|_0}{\tau_n} = \frac{\|u^{n+1}\|_0^2 - \|u^n\|_0\|u^{n+1}\|_0}{\tau_n\|u^{n+1}\|_0} \leq \frac{\|u^{n+1}\|_0^2 - \langle u^n, u^{n+1}\rangle_0}{\tau_n\|u^{n+1}\|_0}$$

$$= \frac{1}{\|u^{n+1}\|_0} \int_\Omega u^{n+1} \left(\frac{u^{n+1} - u^n}{\tau_n}\right) dx$$

$$\stackrel{(1)}{=} \frac{-\left|u^{n+1}\right|_1 + \langle f^{n+1}, u^{n+1}\rangle_0}{\|u^{n+1}\|_0}$$

$$\leq \frac{\|f^{n+1}\|_0\|u^{n+1}\|_0}{\|u^{n+1}\|_0} = \|f^{n+1}\|_0,$$

which is equivalent to

$$\left\|u^{n+1}\right\|_0 - \|u^n\|_0 \leq \tau_n \left\|f^{n+1}\right\|_0.$$

Taking the sum on both sides as n goes from 0 to $s-1$, we finally obtain

$$\sum_{n=0}^{s-1} \left(\left\|u^{n+1}\right\|_0 - \|u^n\|_0\right) = \|u^s\|_0 - \left\|u^0\right\|_0 \leq \sum_{n=0}^{s-1} \tau_n \left\|f^{n+1}\right\|_0,$$

which is a fully-discrete equivalent to the estimate proven in Exercise 19a). $\square$