Advanced Discretization Techniques
Homework 12

Bruno Degli Esposti, Xingyu Xu

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Exercise 22: Fully discrete error estimate

a) For any fixed time t, consider the following elliptic boundary value problem in weak form:

Find $w \in V$ such that $a(w, v) = a(u(t), v)$ for each $v \in V$.

Of course, $w = u(t)$ is a solution, and it's actually unique because of Lax-Milgram's lemma. On the other hand, the finite element solution of the problem will be some $w_h \in V_h$ such that $a(w_h, v_h) = a(u(t), v_h)$ for each $v_h \in V_h$, and we must then have $w_h = R_h u(t)$ by the definition of R_h. Now, the Aubin/Nitsche theorem (3.37 in the book by Knabner and Angermann, inequality (3)) gives us exactly what we need to prove:

$$\|u(t) - R_h u(t)\|_0 \leq Ch^2 \, |u(t)|_2$$

However, we still need to check that the hypothesis of the theorem hold. In particular, we need to check that the adjoint boundary value problem is regular. As $a(\cdot, \cdot)$ is symmetric, the adjoint boundary value problem is the same as the original one. Regularity follows from Lax-Milgram's lemma and the elliptic regularity estimate

$$|u(t)|_2 \leq C \, \|a(u(t), \cdot)\| \, .$$

The proof of this estimate can be found in Evans' book on partial differential equations (see theorem 4, chapter 6.3, page 317).

b) By point a) and the fundamental theorem of calculus for Bochner integrals, it follows that

$$\|u(t_n) - R_h u(t_n)\|_0 \leq Ch^2 \, |u(t_n)|_2 = Ch^2 \left| u_0 + \int_0^{t_n} \partial_t u(t) \, dt \right|_2 .$$

Then, by the triangle inequality (once for sums, once for integrals),

$$\begin{aligned}
Ch^2 \left| u_0 + \int_0^{t_n} \partial_t u(t)\, dt \right|_2 &\le Ch^2 \left(|u_0|_2 + \left| \int_0^{t_n} \partial_t u(t)\, dt \right|_2 \right) \\
&\le Ch^2 \left(|u_0|_2 + \int_0^{t_n} |\partial_t u(t)|_2 \, dt \right) \\
&= Ch^2 \left(\|\partial_{xx} u_0\|_0 + \int_0^{t_n} \|\partial_{t,xx} u(t)\|_0 \, dt \right).
\end{aligned}$$

c) By the definitions of θ_n, implicit Euler scheme and Ritz-projection, we can prove the following chain of equalities:

$$\begin{aligned}
&\left(\frac{\theta_n - \theta_{n-1}}{\tau_{n-1}}, v_h \right)_0 + a(\theta_n, v_h) \\
&= \left(\frac{R_h u(t_n) - U^n - R_h u(t_{n-1}) + U^{n-1}}{\tau_{n-1}}, v_h \right)_0 + a(R_h u(t_n) - U^n, v_h) \\
&= \left(\frac{R_h u(t_n) - R_h u(t_{n-1})}{\tau_{n-1}}, v_h \right)_0 + a(R_h u(t_n), v_h) \\
&\qquad\qquad - \left(\frac{U^n - U^{n-1}}{\tau_{n-1}}, v_h \right)_0 - a(U^n, v_h) \\
&= \left(\frac{R_h u(t_n) - R_h u(t_{n-1})}{\tau_{n-1}}, v_h \right)_0 + a(R_h u(t_n), v_h) - (f^n, v_h)_0 \\
&= \left(\frac{R_h u(t_n) - R_h u(t_{n-1})}{\tau_{n-1}}, v_h \right)_0 + a(u(t_n), v_h) - (f^n, v_h)_0.
\end{aligned}$$

Since $u(t)$ solves the weak formulation of the problem, we can choose v_h as a test function and get that $(\partial_t u(t_n), v_h)_0 + a(u(t_n), v_h) = (f^n, v_h)_0$. Then,

$$\begin{aligned}
&\left(\frac{R_h u(t_n) - R_h u(t_{n-1})}{\tau_{n-1}}, v_h \right)_0 + a(u(t_n), v_h) - (f^n, v_h)_0 \\
&= \left(\frac{R_h u(t_n) - R_h u(t_{n-1})}{\tau_{n-1}}, v_h \right)_0 - (\partial_t u(t_n), v_h)_0.
\end{aligned}$$

d) In point c), we've essentially proved that θ_i satisfies an implicit Euler scheme with right-hand side $-\omega_i^1 - \omega_i^2$:

$$\begin{aligned}
&\frac{R_h u(t_i) - R_h u(t_{i-1})}{\tau_{i-1}} - \partial_t u(t_i) \\
&= \frac{R_h u(t_i) - u(t_i) + u(t_i) - u(t_{i-1}) + u(t_{i-1}) - R_h u(t_{i-1})}{\tau_{i-1}} - \partial_t u(t_i) \\
&= - \left(\partial_t u(t_i) - \frac{u(t_i) - u(t_{i-1})}{\tau_{i-1}} \right) - \left(\frac{u(t_i) - R_h u(t_i)}{\tau_{i-1}} - \frac{u(t_{i-1}) - R_h u(t_{i-1})}{\tau_{i-1}} \right).
\end{aligned}$$

Therefore, the fully discrete estimate from Exercise 21 tells us that

$$\|\theta_n\|_0 - \|\theta_0\|_0 \le \sum_{i=1}^{n} \tau_{i-1} \| - \omega_i^1 - \omega_i^2 \|_0,$$

and then it follows by the triangle inequality and the definition of τ that

$$\|\theta_n\|_0 \le \|\theta_0\|_0 + \tau \sum_{i=1}^{n} \left\|\omega_i^1 + \omega_i^2\right\|_0 \le \|\theta_0\|_0 + \tau \sum_{i=1}^{n} \left\|\omega_i^1\right\|_0 + \tau \sum_{i=1}^{n} \left\|\omega_i^2\right\|_0 .$$

e) Taylor's formula with remainder in integral form gives us that

$$u(t_{i-1}) = u(t_i - \tau_{i-1}) = u(t_i) + \partial_t u(t_i)(-\tau_{i-1}) + \int_{t_{i-1}}^{t_i} \partial_{tt} u(t)(t - t_{i-1})\, dt ,$$

therefore

$$\begin{aligned}
\left\|\omega_i^1\right\|_0 &= \left\| \partial_t u(t_i) - \frac{u(t_i) - u(t_{i-1})}{\tau_{i-1}} \right\|_0 \\
&= \left\| \int_{t_{i-1}}^{t_i} \partial_{tt} u(t) \frac{t - t_{i-1}}{\tau_{i-1}}\, dt \right\|_0 \le \int_{t_{i-1}}^{t_i} \left\| \partial_{tt} u(t) \frac{t - t_{i-1}}{\tau_{i-1}} \right\|_0 dt \\
&\le \int_{t_{i-1}}^{t_i} \|\partial_{tt} u(t)\|_0 \frac{t_i - t_{i-1}}{\tau_{i-1}}\, dt = \int_{t_{i-1}}^{t_i} \|\partial_{tt} u(t)\|_0 \, dt .
\end{aligned}$$

$$\tau \sum_{i=1}^{n} \left\|\omega_i^1\right\|_0 = \tau \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \|\partial_{tt} u(t)\|_0 \, dt = \tau \int_0^{t_n} \|\partial_{tt} u(t)\|_0 \, dt .$$

f) By the fundamental theorem of calculus and point a) (which can be proved for $\partial_t u(t)$ as well, since $\partial_t u(t) \in H^2(0,1)$), we have that

$$\begin{aligned}
\|\omega_i^2\|_0 &= \left\| \int_{t_{i-1}}^{t_i} \partial_t \left(\frac{u(t) - R_h(u(t))}{\tau_{i-1}} \right) dt \right\|_0 \\
&= \left\| \int_{t_{i-1}}^{t_i} \left(\frac{\partial_t u(t) - R_h(\partial_t u(t))}{\tau_{i-1}} \right) dt \right\|_0 \\
&\le \int_{t_{i-1}}^{t_i} \frac{\|\partial_t u(t) - R_h(\partial_t u(t))\|_0}{\tau_{i-1}}\, dt \\
&\le \int_{t_{i-1}}^{t_i} \frac{Ch^2 |\partial_t u(t)|_2}{\tau_{i-1}}\, dt = \int_{t_{i-1}}^{t_i} \frac{Ch^2}{\tau_{i-1}} \|\partial_{t,xx} u(t)\|_0 \, dt .
\end{aligned}$$

If we now sum both sides for $i = 1, \ldots, n$ and multiply them by τ, we end up with

$$\tau \sum_{i=1}^{n} \|\omega_i^2\|_0 \le \frac{\tau}{\min_{i=0,\ldots,N-1} \tau_i} Ch^2 \int_0^{t_n} \|\partial_{t,xx} u(t)\|_0 \, dt \le \tilde{C} h^2 \int_0^{t_n} \|\partial_{t,xx} u(t)\|_0 \, dt$$

We've made the assumption that the time discretization is quasi-uniform.

g) By the triangle inequality and the definition of θ_n,

$$\|u(t_n) - U^n\|_0 = \|u(t_n) - R_h u(t_n) + \theta_n\|_0 \le \|u(t_n) - R_h u(t_n)\|_0 + \|\theta_n\|_0 .$$

Then, by points b) and d), it follows that

$$\|u(t_n) - R_h u(t_n)\|_0 + \|\theta_n\|_0 \leq Ch^2 \left(\|\partial_{xx} u_0\|_0 + \int_0^{t_n} \|\partial_{t,xx} u(t)\|_0 \, dt \right) + \|\theta_0\|_0 + \tau \sum_{i=1}^{n} \|\omega_i^1\|_0 + \tau \sum_{i=1}^{n} \|\omega_i^2\|_0 .$$

The term $\|\theta_0\|_0$ vanishes because $R_h u(0) - U^0 = 0$ by assumption, and finally we can conclude the proof using the results from points e) and f):

$$\begin{aligned}
&\|u(t_n) - U^n\|_0 \\
&\leq Ch^2 \left(\|\partial_{xx} u_0\|_0 + 2 \int_0^{t_n} \|\partial_{t,xx} u(t)\|_0 dt \right) + \tau \int_0^{t_n} \|\partial_{tt} u(t)\|_0 \, dt \\
&\leq 2Ch^2 \left(\|\partial_{xx} u_0\|_0 + \int_0^{t_n} \|\partial_{t,xx} u(t)\|_0 dt \right) + \tau \int_0^{t_n} \|\partial_{tt} u(t)\|_0 \, dt . \quad \square
\end{aligned}$$