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Inverse Problems - Worksheet 3 Winter term 2019/2020

1 Consolidation assignment

Implement the inverse problem of obtaining f from F from $F(x) = \int_0^x f(x)\,\mathrm{d}x$. Orient yourself along the following steps:

1. Prepare a ground truth function $f : [0,1] \to \mathbb{R}$ which you want to recover. As a suggestion, take $f(x) = x^2$. Sample this function on a fine grid (at least 200 points). We will call this the *quasi-continuous* grid.

2. We discretize the operator A in the equation $F = Af$ as follows: $\tilde{F}(x_k) = \Delta x \cdot \sum_{i=1}^k f(x_i)$. Write down the operator A in matrix form.

3. Apply the matrix on the ground truth to obtain a *quasicontinuous* primitive integral $\tilde{F}$.

4. In order to avoid an inverse crime, we now have to downsample the data on a coarser grid[1]. Prepare a coarser grid (which should not overlap with the fine grid, i.e. take for example a power of 2 if you chose a multiple of 100 for your quasicontinuous grid), and interpolate the quasicontinuous data $\tilde{F}$ on this coarser grid. This will be the data F we will work with. Also prepare a noisy version F_{noisy} with additive normal random noise.

5. Construct the operator A again, this time compatible with the coarser grid. This means that we "pretend that F was the result of this operator" although it was actually prepared via interpolation on a finer grid. This way we do not commit an inverse crime.

6. Reconstruct f naively by inverting the matrix A and applying it to both the perfect data and the noisy data. The reconstruction from perfect data should work reasonably well, but the reconstruction from noisy data should amplify the noise to the point that the original function is not well recovered all.

This assignment is continued in the preparation assignments.

2 Preparation assignment

1) **Peruse the pdf titled "compact integral operators" in StudOn**.

2) Concerning theorem 3.4: The form of $\mathscr{A}f$ includes the following two applications as examples. Write down explicitly to what choice of Hilbert–Schmidt kernel $K(x,y)$ they correspond:

[1] see the code for reading assignment 2 on how to do this

(a) Convolution/blurring of a function f with a point spread function ψ (see the reading assignment of worksheet 2).

(b) Computing the primitive of a function, i.e. $(Af)(x) = F(x)$ with $F'(x) = f(x)$ (and $F(0) = 0$ in order to fix the constants).

In the light of theorem 3.2, what does this tell you about the corresponding inverse problems, i.e. *deconvolution* and *computing the derivative*?

3) Concerning section “regularized inversion”: We continue with the consolidation assignment and we try to recover the parameter more smoothly. For this, we define a regularized inverse problem of the form

$$F = Af + \alpha \cdot f.$$

Show that this corresponds to a regularization strategy in the sense of the reading assignment above. Write down the operator $\mathscr{R}_\alpha$ explicitly. Continue with your code from the consolidation assignment and implement this regularized inversion. Comment on the quality of the reconstruction for various levels for α and describe what happens from $\alpha \to 0$ and $\alpha \to \infty$

4) Get psyched about Anki, e.g. by reading `http://augmentingcognition.com/ltm.html`. If you want, you can download the desktop app and/or the free Android app under `https://apps.ankiweb.net/`

3 Notes / Insights from class