Worksheet 1

CA

x_J, x_D, x_p: Weight of Jamie, Daniel, Philipp

1. $$\boxed{x_J, x_D, x_p} \underset{\text{(ill-posed) inv. prob.}}{\overset{\text{weighing process (forward)}}{\rightleftarrows}} \boxed{x_J + x_D,\ x_D + x_p}$$

unknown paramet. — data

2. ill-posed, because no unique solution:

For example consider $x_J + x_D = 160$,
$x_D + x_p = 160$,

maybe $x_J = x_D = x_p = 80$, but

$\begin{cases} x_J = 100 \\ x_p = 60 \\ x_D = 100 \end{cases}$ is also a possible solution

3. new data: $x_J + x_p$

Now the inverse problem is well-posed:

Assume $y_1 = x_J + x_D$, $y_2 = x_D + x_p$, $y_3 = x_J + x_p$

Then $$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}}_{A} \begin{pmatrix} x_J \\ x_D \\ x_p \end{pmatrix}$$

and we can invert A to obtain x_J, x_D, x_p

2)

PA 2a) $\|u\|_{L^1} \leq \|u\|_{L^2}$, as

$$\|u\|_1 = \int_0^1 |u(x)|\,dx = \int_0^1 1 \cdot |u(x)|\,dx$$

$$\overset{\text{Hölder}}{\underset{\text{inequality}}{=}} \sqrt{\int_0^1 1^2} \cdot \sqrt{\int_0^1 |u(x)|^2 dx} = 1 \cdot \|u\|_{L^2}$$

But there is no constant C such that $\forall u \in C[0,1]$

$$\|u\|_{L^2} \leq C \cdot \|u\|_{L^1}:$$

Set $u_n(x) = (n+1) \cdot x^n$

$$\Rightarrow \|u_n\|_{L^1} = \int_0^1 |(n+1)x^n|\,dx = \int_0^1 (n+1)x^n dx = x^{n+1}\Big|_0^1 = 1$$

But $\|u_n\|_{L^2} = \sqrt{\int_0^1 (n+1)^2 x^{2n}\,dx} = \sqrt{\frac{(n+1)^2}{2n+1} x^{2n+1}\Big|_0^1}$

$$= \frac{n+1}{\sqrt{2n+1}} \simeq O(\sqrt{n}), \text{ i.e. } \frac{\|u_n\|_{L^2}}{\|u_n\|_{L^1}} \xrightarrow{n\to\infty} \infty$$

b) ~~For~~ example: $L: (\ell^2, \|\cdot\|_2) \to \mathbb{R}$

$$x \mapsto \sum x_k$$

is unbounded: Define $x^{(n)} = (1, \frac{1}{2}, \dots, \frac{1}{n}, 0, 0, \dots)$

$\|x^{(n)}\|_2 \leq \sqrt{\sum \frac{1}{k^2}} < \infty \Rightarrow x^{(n)}$ bounded in ℓ^2

But $Lx^{(n)} = \sum_{k=1}^{n} \frac{1}{k} \xrightarrow{n\to\infty} \infty \Rightarrow L$ is unbounded op.

L is indeed not continuous at 0:

Define $y^{(n)} := \dfrac{x^{(n)}}{\sum_{k=1}^{n} \frac{1}{k}} = \left(\dfrac{1}{\sum_{k=1}^{n} \frac{1}{k}}, \dfrac{1}{2 \cdot \sum_{k=1}^{n} \frac{1}{k}}, \dots, \dfrac{1}{n \cdot \sum_{k=1}^{n} \frac{1}{k}}, 0 \right.$

$\Rightarrow \|y^{(n)}\|_2 = \dfrac{\|x^{(n)}\|_2}{\sum_{k=1}^{n} \frac{1}{k}} \xrightarrow{n\to\infty} 0$ as $\|x^{(n)}\| < C$ and $\sum_{k=1}^{n} \frac{1}{k} \xrightarrow{n\to\infty} \infty$

$\Rightarrow y^{(n)} \to 0$ in ℓ^2

But $Ly^{(n)} = \dfrac{\sum_{k=1}^{n} \frac{1}{k}}{\sum_{k=1}^{n} \frac{1}{k}} = 1 \quad \forall n$

$\Rightarrow Ly^{(n)} \not\to 0$ in ℓ^2

$\Rightarrow L$ is not continuous at 0.

2c) • $L_1(x) = \sum_{k=1}^{\infty} \frac{x_k}{k}$. L_1 is linear (clear)

L_1 is bounded, because

$$|L_1(x)| = \left|\sum_{k=1}^{\infty} \frac{x_k}{k}\right| \overset{\text{Hölder}}{\leq} \sqrt{\sum_{k=1}^{\infty} x_k^2}\ \underbrace{\sqrt{\sum_{k=1}^{\infty} \frac{1}{k^2}}}_{=:C<\infty} = C \cdot \|x\|_{\ell^2}$$

$\Rightarrow L_1$ is bounded

$x_0 := (1, \frac{1}{2}, \frac{1}{3}, \dots) \in \ell^2$

And $L_1(x) = \langle x_0, x \rangle = \sum_{k=1}^{\infty} \frac{x_k}{k}$

• L_2 is unbounded (show similar to 2b)

Candidate for $x_0 = (1, -1, 1, -1, 1, \dots)$,
because (formally) $\langle x_0, x \rangle = L_1(x)$, but $x_0 \notin \ell^2$

④ 2d) If X is finite-dimensional, then I is compact, because (Bolzano-Weierstraß) bounded sets in $\mathbb{R}^d$ are compact.

The same does not hold, i.e. there are (in ∞-dim) sequences in bounded sets which do not have a converging subsequence (see 2e)), i.e. I is not compact.

2e) $x^{(n)} \not\rightarrow 0$ because $\|x^{(n)}\|_{\ell^2} = 1 \;\forall n$

but: let $f \in (\ell^2)^* \simeq \ell^2$:

$$\langle f, x^{(n)} \rangle = \sum_{k=1}^{\infty} f_k \cdot x_k^{(n)} \underset{\text{of } x^{(n)}}{\overset{\text{def.}}{=}} f_n$$

Now $f_n \xrightarrow{n\to\infty} 0$ because $f \in \ell^2$

(Assume that $\exists C : \forall N \; \exists n \geq N : |f_n| > C$

$$\Rightarrow \sum_{k=1}^{\infty} f_k^2 = \underbrace{\dots + C + \dots + C + \dots + C + \dots}_{(\infty\text{-often})}$$

$$= \infty \quad \text{↯})$$