Martin Burger, Janic Föcke, Daniel Tenbrinck, Philipp Wacker

Inverse Problems - Worksheet 4 Winter term 2019/2020

1 Consolidation assignment

Finish the Matlab/Octave code for inversion of the heat equation using a Fourier series decomposition from the last session. Try an own regularized inversion method by neglecting all high frequency coefficients $a_k(t)$.
Follow these steps to complete the assignment:

1. Download the code `Incomplete Matlab code for Inverse Heat Equation using Fourier Decomposition` from StudOn by navigating to session on November 5th.

2. Complete the missing code passages for:

   (a) computing the Fourier coefficients $a_k(t)$

   (b) generating noisy measurements by perturbing the Fourier coefficients as $m_k(t) = a_k(t) + \epsilon$, where $\epsilon \sim \mathcal{N}(0, \sigma^2)$

   (c) solving the inverse heat equation by starting with the noisy coefficients m and changing the sign of the time t in the equation of the coefficients $a_k(t)$

   (d) realizing a regularized inversion by introducing a threshold $K \in \mathbb{N}$, such that $a_k(t) = 0$ for all $k > K$.

3. Play around with the noise parameter $\sigma^2 > 0$ and the threshold K and write down your observations.

2 Preparation assignment

1) **Peruse** the pdf titled "The SVD for matrices" in StudOn.

2) **Peruse** the web page titled "Wikipedia article on Singular Value Decomposition" in StudOn.

3) Compute the singular value decomposition (SVD) $A = U\Sigma V^T$ for a diagonal matrix $\Sigma \in \mathbb{R}^{2\times 3}$ containing the singular values σ_i of A and two unitary matrices $U \in \mathbb{R}^{2\times 2}$ and $V \in \mathbb{R}^{3\times 3}$.

$$A = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix}$$

   Follow these steps to compute the SVD:

   (a) Compute the eigenvalues of $AA^T \in \mathbb{R}^{2\times 2}$ by computing the roots of the characteristic polynomial $\det(AA^T - \lambda I_2)$. Set up the matrix Σ containing the square roots of the eigenvalues of AA^T on the main diagonal.

   (b) Determine the eigenvectors of the matrix $A^TA \in \mathbb{R}^{3\times 3}$ by computing unit-length vectors $v \in \mathbb{R}^3$ in the kernel of the matrix $A^TA - \lambda I_3$ for the computed eigenvalues λ above. Note that 0 is also an eigenvalue of A^TA. Explain why.

(c) Set up the unitary matrix $V \in \mathbb{R}^{3\times 3}$ by writing the computed eigenvectors column-wise into V.

(d) Compute the unitary matrix $U \in \mathbb{R}^{2\times 2}$ by the formula $Av_i = \sigma u_i$ for the two non-trivial eigenvectors v_i. Remember to normalize the vectors. Explain this formula.

(e) Check your SVD in Matlab or Octave for correctness. You can compare your result to the SVD computed by the Matlab/Octave function `svd(A)`. Explain any differences between your manual computation and the software solution.

4) Create at least 5 Anki flashcards on the content learned in this course so far.
Export your card decks and send them to `wacker@math.fau.de`
We will collect all flash cards, curate them, and upload a collective deck on StudOn for everyone to learn.

3 Notes / Insights from class