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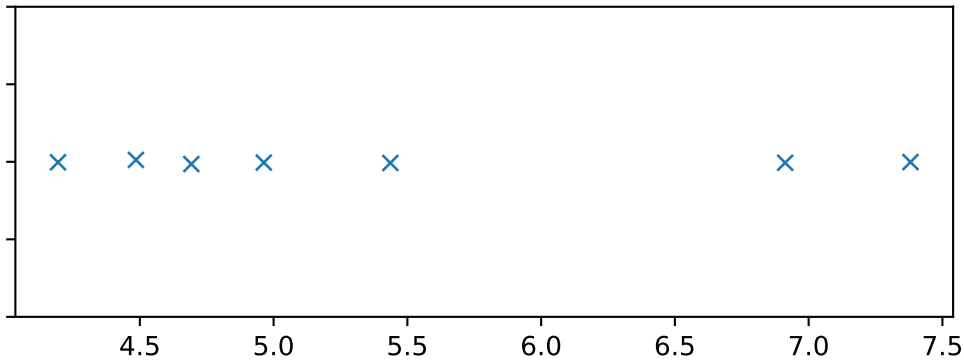
Inverse Problems - Worksheet 9 Winter term 2019/2020

1 Consolidation assignment

1. **Complete the survey about the project phase on StudOn.**

2. "Seven scientists". Assume that you are trying to infer the numerical value of some natural constant, μ, which is unknown to you. You ask seven (or N) scientists to use their measurement devices in order to give you an estimate of μ. We assume that the ith scientist's device returns the value $y_i = \mu + \epsilon_i$, where each ϵ_i is a measurement error distributed according to a Gaussian random variable $N(0, \sigma_i)$.

   Consider measurements `y = [4.96334778, 5.43644897, 7.38067265, 4.69244271, 6.91227063, 4.48530689, 4.19409183]`. (see figure below)



   Write down explicitly the expression for the *likelihood* $\mathbb{P}(y|\mu, \{\sigma_i\})$, i.e. the probability density of a measurement y given a fixed set of parameters μ and $\{\sigma_i\}$.

   a) First assume that all devices have the same measurement uncertainty $\sigma_i = 1$. What is the maximum likelihood estimator for μ (i.e. the value $\hat{\mu}_{ML} = \text{argmax}_\mu \ \mathbb{P}(y|\mu, \{\sigma_i\})$)?

   b) Now we assume that we don't know each scientist's measuring uncertainty σ_i. What is the joint maximum likelihood estimator for μ and all σ_i, i.e. which values of μ and σ_i, $i = 1, \dots, N$ minimize the likelihood $\mathbb{P}(y|\mu, \{\sigma_i\})$? What goes wrong?

   c) Elaborate on a possible solution for the problem in b).

3. Let $X \sim N(\mu, \sigma^2)$, $\varepsilon \sim N(0, \gamma^2)$ and $Y = a \cdot X + \varepsilon$. In class we derived the following:

$$Y|X \sim N(a \cdot X, \gamma^2)$$

   For simplicity we assume that $\mu = 0$. This means that

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \exp(-\frac{x^2}{2\sigma^2})$$
$$f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi\gamma^2}} \cdot \exp(-\frac{(y-ax)^2}{2\gamma^2}).$$

Calculate $f_{X,Y}(x,y) = f_{Y|X}(y|x) \cdot f_X(x)$ and thus derive the distribution of (X,Y). Hint: The density for a 2-d multivariate Gaussian distribution $(W,Z) \sim N(\vec{m}, C)$ is

$$f_{W,Z}(w,z) = \frac{1}{\sqrt{2\pi \det C}} \exp\left(-\frac{\left(\begin{pmatrix} w \\ z \end{pmatrix} - \vec{m}\right)^T \cdot C^{-1} \cdot \left(\begin{pmatrix} w \\ z \end{pmatrix} - \vec{m}\right)}{2} \right)$$

4. (optional) If you're interested in Simpson's paradox, you can read more about it on `https://pwacker.com/simpson.html`

2 Preparation assignment

We consider the setting of linear regression with a polynomial regression function: Given data $z \in \mathbb{R}^{2\times N}$ (i.e. each of the N data points z_i has two coordinates which we abbreviate by $z_i = (x_i, y_i)$).

We assume that the data was generated by noisy evaluation of a polynomial with n unknown coefficients, i.e.

$$y_i = a_0 + a_1 \cdot x_i + a_2 \cdot x_i^2 + \cdots + a_{n-1} \cdot x_i^{n-1} + \varepsilon_i$$

where $\varepsilon_i \sim N(0, \sigma^2)$ are independent noise terms (with known variance σ^2.

a) Write Matlab/Octave code that can do the following: Given parameters $\vec{a} = (a_0, \ldots, a_{n-1})$, measurements positions x_i and noise parameter σ, generate data y as above.

b) Write the measurement process in vectorial form

$$\vec{y} = M \cdot \vec{a} + \vec{\varepsilon}.$$

3 Notes / Insights from class