

Quantum Systems

(Lecture 3: Quantum states and computation)

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The principles

Quantum computation explores the laws of quantum theory as computational resources.

Thus, the principles of the former are directly derived from the postulates of the latter.

- The state **space** postulate
- The state **evolution** postulate
- The state **composition** postulate
- The state **measurement** postulate

The underlying maths is that of Hilbert spaces.

The underlying maths: Hilbert spaces

Complex, inner-product vector space

A complex vector space with **inner product** which measures how much two vectors **overlap**:

$$(-, -) : H \times H \longrightarrow \mathbb{C}$$

usually abbreviated as

$$\langle v|w \rangle \hat{=} (|v\rangle, |w\rangle)$$

such that

$$(1) \quad \langle v | \sum_i \lambda_i \cdot |w_i\rangle \rangle = \sum_i \lambda_i \langle v | w_i \rangle$$

$$(2) \quad \langle v | w \rangle = \overline{\langle w | v \rangle}$$

$$(3) \quad \langle v | v \rangle \geq 0 \text{ (with equality iff } |v\rangle = 0)$$

Note: $\langle - | - \rangle$ is **conjugate linear** in the first argument:

$$\langle \sum_i \lambda_i \cdot |w_i\rangle | v \rangle = \sum_i \overline{\lambda_i} \langle w_i | v \rangle$$

Dirac's notation

Dirac's bra/ket notation is a handy way to represent elements and constructions on an Hilbert space

- $|u\rangle$ A **ket** stands for a vector in an Hilbert space H . In \mathbb{C}^n , it is a column vector of complex entries. Note that the identity for $+$ (the **zero** vector) is just written 0.
- $\langle u|$ A **bra** is a vector in the **dual** space H^* , i.e. scalar-valued linear maps in H . In $(\mathbb{C}^n)^*$ it is the **adjoint**, i.e. the conjugate transpose, of the corresponding **ket**, therefore a row vector.

There is a bijective correspondence between $|u\rangle$ and $\langle u|$

$$|u\rangle = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \Leftrightarrow [\bar{u}_1 \cdots \bar{u}_n] = \langle u|$$

Inner product: examples

In \mathbb{C}

$$\langle a + bi | c + di \rangle = (a - bi)(c + di) = ac + adi - bci + bd$$

In \mathbb{C}^n : The dot product

Amost useful example of a **inner product** is the **dot product**

$$\langle u | v \rangle = \underbrace{[\overline{u_1} \quad \overline{u_2} \quad \cdots \quad \overline{u_n}]}_{\langle u |} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \sum_{i=1}^n \overline{u_i} v_i$$

where $\overline{c} = a - ib$ is the complex conjugate of $c = a + ib$

.

Old friends: The dual space

H^*

If H is a Hilbert space, H^* is the space of **linear maps** from H to \mathbb{C} .

Elements of H^* are denoted by

$$\langle u| : H \longrightarrow \mathbb{C} \text{ and defined as } \langle u|(|v\rangle) = \langle u|v\rangle$$

In a matricial representation $\langle u|$ is obtained as the **Hermitian conjugate** (i.e. the **transpose** of the vector composed by the **complex conjugate** of each element) of $|u\rangle$, therefore the dot product of $|u\rangle$ and $|v\rangle$.

Old friends: Norms and orthogonality

- The inner product measures the *degree of overlapping*: $|v\rangle$ and $|w\rangle$ are **orthogonal** if $\langle v|w\rangle = 0$
- The "length" of a vector uses the measure of its overlap with itself to yield the (Euclidean) **norm**:

$$\| |v\rangle \| = \sqrt{\langle v|v\rangle}$$

(generalizing the distance between two points)

- $|v\rangle$ is a **unit vector** if $\| |v\rangle \| = 1$
- **normalization**: $\frac{|v\rangle}{\| |v\rangle \|}$
- A set of vectors $\{|i\rangle, |j\rangle, \dots\}$ is **orthonormal** if each $|i\rangle$ is a unit vector and

$$\langle i|j\rangle = \delta_{i,j} = \begin{cases} i=j & \Rightarrow 1 \\ \text{otherwise} & \Rightarrow 0 \end{cases}$$

Old friends: Bases

Orthonormal basis

A orthonormal basis for a Hilbert space H of dimension n is a set $B = \{|i\rangle \mid i \in n - 1\}$ of n linearly independent elements of H st

- $\langle i|j\rangle = \delta_{i,j}$ for all $|i\rangle, |j\rangle \in B$
- and B **spans** H , i.e. every $|v\rangle$ in H can be written as

$$|v\rangle = \sum_i \alpha_i |i\rangle \quad \text{for some } \alpha_i \in \mathbb{C}$$

Note that the **amplitude** or **coefficient** of $|v\rangle$ wrt $|i\rangle$ satisfies

$$\alpha_i = \langle i|v\rangle$$

Why?

Bases

$\alpha_i = \langle i | v \rangle$ because

$$\begin{aligned}\langle i | v \rangle &= \langle i | \sum_j \alpha_j | j \rangle \\ &= \sum_j \alpha_j \langle i | j \rangle \\ &= \sum_j \alpha_j \delta_{i,j} \\ &= \alpha_i\end{aligned}$$

Note

If $|v\rangle$ is expressed wrt an orthonormal basis $\{|i\rangle \mid i \in n\}$, i.e.

$|v\rangle = \sum_i \alpha_i |i\rangle$, then

$$\| |v\rangle \| = \sum_i \| \alpha_i \|^2$$

Example: The Hadamard basis

One of the infinitely many orthonormal bases for a space of dimension 2:

$$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

$$|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

Check, e. g.

$$\langle + | - \rangle = \frac{1}{2}(\langle 0| + \langle 1|, |0\rangle - |1\rangle) = \frac{1}{2} \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right) = \frac{1}{2} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0$$

$$\| |+\rangle \| = \sqrt{\langle + | + \rangle} = \sqrt{\frac{1}{2}(\langle 0| + \langle 1|, |0\rangle + |1\rangle)} = \sqrt{\frac{1}{2} \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)} = 1$$

Bases

A basis for H^*

If $\{|i\rangle \mid i \in n\}$ is an orthonormal basis for H , then

$$\{\langle i| \mid i \in n\}$$

is an orthonormal basis for H^* .

Hilbert spaces

The complete picture

An **Hilbert space** is an inner-product space H st the metric defined by its norm turns H into a **complete metric space**, i.e. any Cauchy sequence

$$|v_1\rangle, |v_2\rangle, \dots$$

$$\forall \epsilon > 0 \exists N \forall m, n > N \quad \| |v_m\rangle - |v_n\rangle \| \leq \epsilon$$

converges

(i.e. there exists an element $|s\rangle$ in H st $\forall \epsilon > 0 \exists N \forall n > N \quad \| |s\rangle - |v_n\rangle \| \leq \epsilon$)

The completeness condition is trivial in **finite dimensional** vector spaces

The state space postulate

Postulate 1

The state space of a quantum system is described by a unit vector in a Hilbert space

- In practice, with finite resources, one cannot distinguish between a **continuous** state space from a **discrete** one with arbitrarily small minimum spacing between adjacent locations.
- One may, then, restrict to **finite-dimensional** (complex) Hilbert spaces.

The state space postulate

A quantum (binary) state is represented as a **superposition**, i.e. a linear combination of vectors $|0\rangle$ and $|1\rangle$ with **complex** coefficients:

$$|\phi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

When state $|\phi\rangle$ is **measured** (i.e. **observed**) one of the two basic states $|0\rangle, |1\rangle$ is returned with probability

$$\|\alpha\|^2 \quad \text{and} \quad \|\beta\|^2$$

respectively.

Being probabilities, the norm squared of coefficients must satisfy

$$\|\alpha\|^2 + \|\beta\|^2 = 1$$

which enforces quantum states to be represented by **unit** vectors.

The state space of a qubit

Global phase

Unit vectors equivalent up to multiplication by a complex number of modulus one, i.e. a **phase factor** $e^{i\theta}$, represent the **same** state.

Let

$$|v\rangle = \alpha|u\rangle + \beta|u'\rangle$$

$$\|e^{i\theta}\alpha\|^2 = (\overline{e^{i\theta}\alpha})(e^{i\theta}\alpha) = (e^{-i\theta}\overline{\alpha})(e^{i\theta}\alpha) = \overline{\alpha}\alpha = \|\alpha\|^2$$

and similarly for β .

As the probabilities $\|\alpha\|^2$ and $\|\beta\|^2$ are the **only** measurable quantities, **global phase has no physical meaning**.

Representation redundancy

qubit state space \neq complex vector space used for representation

The state space of a qubit

Relative phase

It is a measure of the angle between the two complex numbers.
Thus, it cannot be discarded!

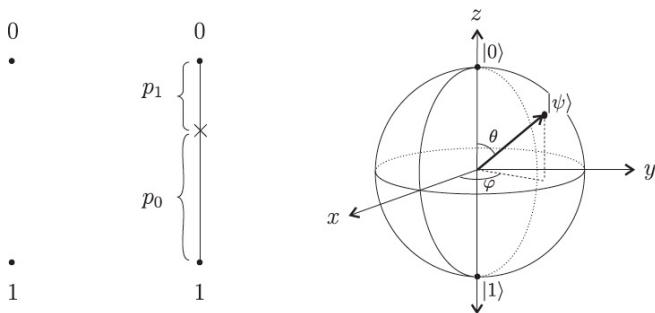
Those are different states

$$\frac{1}{\sqrt{2}}(|u\rangle + |u'\rangle) \quad \frac{1}{\sqrt{2}}(|u\rangle - |u'\rangle) \quad \frac{1}{\sqrt{2}}(e^{i\theta}|u\rangle + |u'\rangle)$$

...

The Bloch sphere

Deterministic, probabilistic and quantum bits



(from [Kaeys et al, 2007])

The Bloch sphere: Representing $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

- Express $|\psi\rangle$ in **polar** form

$$|\psi\rangle = \rho_1 e^{i\varphi_1} |0\rangle + \rho_2 e^{i\varphi_2} |1\rangle$$

- Eliminate one of the four real parameters multiplying by $e^{-i\varphi_1}$

$$|\psi\rangle = \rho_1 |0\rangle + \rho_2 e^{i(\varphi_2 - \varphi_1)} |1\rangle = \rho_1 |0\rangle + \rho_2 e^{i\varphi} |1\rangle$$

making $\varphi = \varphi_2 - \varphi_1$,

which is possible because **global phase factors** are **physically meaningless**.

The Bloch sphere: Representing $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

- Switching back the coefficient of $|1\rangle$ to Cartesian coordinates

$$|\psi\rangle = \rho_1|0\rangle + (a + bi)|1\rangle$$

the normalization constraint

$$\|\rho_1\|^2 + \|a+ib\|^2 = \|\rho_1\|^2 + (a-ib)(a+ib) = \boxed{\|\rho_1\|^2 + a^2 + b^2 = 1}$$

yields the [equation of a unit sphere](#) in the real tridimensional space with Cartesian coordinates: (a, b, ρ_1) .

The Bloch sphere: Representing $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

- The **polar** coordinates (ρ, θ, φ) of a point in the surface of a sphere relate to Cartesian ones through the correspondence

$$x = \rho \sin \theta \cos \varphi$$

$$y = \rho \sin \theta \sin \varphi$$

$$z = \rho \cos \theta$$

- Recalling $\rho = 1$ (cf unit vector),

$$\begin{aligned} |\psi\rangle &= \rho_1|0\rangle + (a + ib)|1\rangle \\ &= \cos \theta|0\rangle + \sin \theta(\cos \varphi + i \sin \varphi)|1\rangle \\ &= \cos \theta|0\rangle + e^{i\varphi} \sin \theta|1\rangle \end{aligned}$$

which, with **two parameters**, defines a **point** in the sphere's surface.

The Bloch sphere

Actually, one may just focus on the **upper hemisphere** ($0 \leq \theta' \leq \frac{\pi}{2}$) as opposite points in the lower one differ only by a phase factor of -1 , as suggested by

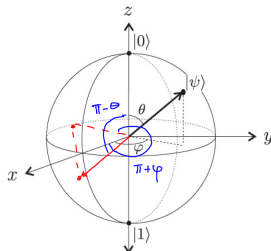
$$\theta' = 0 \Rightarrow |\psi\rangle = \cos 0|0\rangle + e^{i\varphi} \sin 0|1\rangle = |0\rangle$$

$$\theta' = \frac{\pi}{2} \Rightarrow |\psi\rangle = \cos \frac{\pi}{2}|0\rangle + e^{i\varphi} \sin \frac{\pi}{2}|1\rangle = e^{i\varphi}|1\rangle = |1\rangle$$

Note that **longitude** (φ) is irrelevant in a pole!

The Bloch sphere

Indeed, let $|\psi'\rangle$ be the opposite point on the sphere with polar coordinates $(1, \pi - \theta, \varphi + \pi)$:



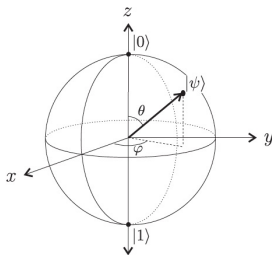
$$\begin{aligned}
 |\psi'\rangle &= \cos(\pi - \theta)|0\rangle + e^{i(\varphi + \pi)} \sin(\pi - \theta)|1\rangle \\
 &= -\cos\theta|0\rangle + e^{i\varphi} e^{i\pi} \sin\theta|1\rangle \\
 &= -\cos\theta|0\rangle + e^{i\varphi} \sin\theta|1\rangle \\
 &= -|\psi\rangle
 \end{aligned}$$

The Bloch sphere

which leads to

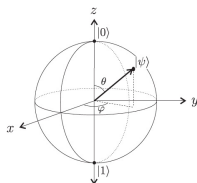
$$|\psi\rangle = \cos \frac{\theta}{2} |0\rangle + e^{i\varphi} \sin \frac{\theta}{2} |1\rangle$$

where $0 \leq \theta \leq \pi$, $0 \leq \varphi \leq 2\pi$



The map $\frac{\theta}{2} \mapsto \theta$ is **one-to-one** at any point but at $\frac{\theta}{2}$:
all points on the equator are mapped into a single point: the south pole.

The Bloch sphere



- The poles represent the classical bits. In general, **orthogonal states correspond to antipodal points** and every **diameter** to a **basis** for the single-qubit state space.
- Once measured a qubit collapses to one of the two poles. Which pole depends exactly on the arrow direction: The angle θ measures that **probability**: If the arrow points at the equator, there is 50-50 chance to collapse to any of the two poles.
- Rotating a vector wrt the z -axis results into a **phase change** (φ), and does not affect which state the arrow will collapse to, when measured.

The state evolution postulate

If a quantum state is a **ray** (i.e. a unit vector in a Hilbert space H up to a global phase), its evolution is specified as a certain kind of **linear** maps $U : H \longrightarrow H$.

Linearity

$$U \left(\sum_j \alpha_j |v_j\rangle \right) = \sum_j \alpha_j U(|v_j\rangle)$$

just by itself has an important consequence:

quantum states cannot be cloned

The no-cloning theorem

Linearity implies that quantum states cannot be cloned

Let $U(|a\rangle|0\rangle) = |a\rangle|a\rangle$ be a 2-qubit operator and $|c\rangle = \frac{1}{\sqrt{2}}(|a\rangle + |b\rangle)$ for $|a\rangle, |b\rangle$ orthogonal. Then,

$$\begin{aligned}U(|c\rangle|0\rangle) &= \frac{1}{\sqrt{2}}(U(|a\rangle|0\rangle) + U(|b\rangle|0\rangle)) \\&= \frac{1}{\sqrt{2}}(|a\rangle|a\rangle + |b\rangle|b\rangle) \\&\neq \frac{1}{\sqrt{2}}(|a\rangle|a\rangle + |a\rangle|b\rangle + |b\rangle|a\rangle + |b\rangle|b\rangle) \\&= |c\rangle|c\rangle \\&= U(|c\rangle|0\rangle)\end{aligned}$$

(Recall: $|x\rangle|y\rangle = |xy\rangle = |x\rangle \otimes |y\rangle$)

But, linearity is not enough ...

... we need to enforce that the norm squared of the new amplitudes still represent a probability distribution

$$\text{If } \sum_j \alpha_j U(|v_j\rangle) = \sum_j \alpha'_j |v_j\rangle \text{ then } \sum_j \|\alpha'_j\|^2 = 1$$

This is achieved by making U **unitary**, i.e. such that

$$U^{-1} = U^\dagger$$

What is U^\dagger ? The adjoint map

Given a linear map $U : H \longrightarrow H'$, its **adjoint** $U^\dagger : H' \longrightarrow H$ is the **unique** linear map such that

$$(\textcolor{red}{U}^\dagger|a\rangle, |b\rangle) = (|a\rangle, \textcolor{red}{U}|b\rangle)$$

which, in Dirac notation, boils down to

$$\langle a|\textcolor{red}{U}|b\rangle$$

Note that $(\textcolor{blue}{UV})^\dagger = \textcolor{blue}{V}^\dagger \textcolor{blue}{U}^\dagger$ and $\textcolor{blue}{U}^{\dagger\dagger} = \textcolor{blue}{U}$ because

$$((\textcolor{blue}{UV})^\dagger|a\rangle, |b\rangle) = (|a\rangle, \textcolor{blue}{UV}|b\rangle) = (\textcolor{blue}{U}^\dagger|a\rangle, \textcolor{blue}{V}|b\rangle) = (\textcolor{blue}{V}^\dagger \textcolor{blue}{U}^\dagger|a\rangle, |b\rangle)$$

and

$$(\textcolor{blue}{U}^{\dagger\dagger}|a\rangle, |b\rangle) = (|a\rangle, \textcolor{blue}{U}^\dagger|b\rangle) = (\textcolor{blue}{U}|a\rangle, |b\rangle)$$

The state evolution postulate

Postulate 2

The evolution over time of the state of a closed quantum system is described by a unitary map.

The evolution is **linear**

$$U\left(\sum_j \alpha_j |v_j\rangle\right) = \sum_j \alpha_j U(|v_j\rangle)$$

and preserves the **normalization constraint**

$$\text{If } \sum_j \alpha_j U(|v_j\rangle) = \sum_j \alpha'_j |v_j\rangle \text{ then } \sum_j \|\alpha'_j\|^2 = 1$$

The state evolution postulate

Preservation of the **normalization constraint** means that unit length vectors (and thus orthogonal subspaces) are mapped by U to unit length vectors (and thus to orthogonal subspaces).

This entails a condition on valid quantum operators: they must **preserve** the inner product, i.e.

$$(U|a\rangle, U|b\rangle) = (U^\dagger U|a\rangle, |b\rangle) = (U^\dagger U|a\rangle, |b\rangle) = (|a\rangle, |b\rangle)$$

i.e.

$$\langle a|U^\dagger U|b\rangle = \langle a|b\rangle$$

which is only the case iff U is **unitary**, i.e. U^\dagger is the **inverse** of U :

$$U^\dagger U = UU^\dagger = I$$

Unitary maps

- Preserving the inner product means that a unitary operator maps **orthonormal bases** to **orthonormal bases**.
- Conversely, any operator with this property is unitary.
- If given in matrix form, being unitary means that the set of columns of its matrix representation are orthonormal (because the j th column is the image of $U|j\rangle$). Equivalently, rows are orthonormal (why?)

Unitary maps

Unitarity is the **only** constraint on quantum operators: Any unitary matrix specifies a valid quantum operator.

This means that there are many non-trivial operators on a single qubit (in contrast with the **classical** case where the only non-trivial operation on a bit is **complement**).

Finally, because the **inverse** of a unitary matrix is also a unitary matrix, a quantum operator can always be inverted by another quantum operator

Unitary transformations are **reversible**

Representing linear maps

A linear map $U : H \longrightarrow H'$ is fully characterized by specifying how it acts on a basis of H . If H is **finite** this leads to a natural representation of U as **matrix**.

Let $\{|j\rangle \mid j \in n-1\}$ be a basis for a n -dimensional Hilbert space H , and similarly $\{|i\rangle \mid i \in m-1\}$ for a m -dimensional H' . Then the $m \times n$ matrix corresponding to U is defined as

$$[U|0\rangle \quad U|1\rangle \quad \dots \quad U|n-1\rangle]$$

i.e. its j^{th} -column corresponds to m -dimensional vector $U|j\rangle$.

The Dirac notations provides a handy, alternative description of matrices via **outer products**.

Representing linear maps

Outer product

... is computed straightforwardly by matrix multiplication, e.g.

$$\begin{aligned} |0\rangle\langle 0| &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\ |1\rangle\langle 0| &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \end{aligned}$$

In general, for vectors $|i\rangle, |j\rangle$ in an orthonormal basis, $|i\rangle\langle j|$ is a square matrix with 1 in position (i, j) and 0 elsewhere. As an operator, $|i\rangle\langle j|$ maps $|j\rangle$ into $|i\rangle$ because

$$|i\rangle\langle j|j\rangle = |i\rangle\langle j|j\rangle = |i\rangle$$

A linear map $U: H \longrightarrow H'$ can be represented as a matrix

$$\sum_{i \in m-1, j \in n-1} U_{i,j} |i\rangle\langle j|$$

Representing linear maps

Decomposition of the identity (for an orthonormal basis)

$$I_H = \sum_{i \in n-1} |i\rangle\langle i|$$

Thus,

$$\begin{aligned} U &= I_H U I_H = \sum_{i \in m-1} |i\rangle\langle i| U \sum_{j \in n-1} |j\rangle\langle j| \\ &= \sum_{i \in m-1, j \in n-1} |i\rangle\langle i| U |j\rangle\langle j| \\ &= \sum_{i \in m-1, j \in n-1} \langle i|U|j\rangle |i\rangle\langle j| \end{aligned}$$

Clearly,

$$U_{i,j} = \langle i|U|j\rangle$$

Representing linear maps

because

$$\begin{aligned}\langle i|U|j\rangle &= \langle i|\left(\sum_{i'\in m-1, j'\in n-1} U_{i',j'} |i'\rangle\langle j'|\right)|j\rangle \\ &= \sum_{i'\in m-1, j'\in n-1} U_{i',j'} \langle i|i'\rangle\langle j|j'\rangle \\ &= \sum_{i'\in m-1, j'\in n-1} U_{i',j'} \delta_{ii'}\delta_{jj'} = U_{i,j}\end{aligned}$$

Representing linear maps

Any orthonormal provides a decomposition of the identity.

Is there a standard way to provide a decomposition for an arbitrary operator U over a Hilbert H ?

Yes, if U is **normal** operator, i.e. $UU^\dagger = U^\dagger U$, because of the

Spectral theorem

Any normal operator on a finite, n -dimensional Hilbert space H provides a basis for H consisting of its **eigenvectors**. Thus,

$$U = \sum_{i \in n-1} \lambda_i |\lambda_i\rangle \langle \lambda_i|$$

where each $(\lambda_i, |\lambda_i\rangle)$ is a eigenvalue / eigenvector pair.

Typical quantum gates on 1 qubit

The $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = |0\rangle\langle 1| + |1\rangle\langle 0|$ gate



$$X|0\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle$$

As $X|+\rangle = |+\rangle$ and $X|-\rangle = -|-\rangle$, its **spectral decomposition** yields

$$X = |+\rangle\langle +| - |-\rangle\langle -|$$

Typical quantum gates on 1 qubit

Acts as

$$Z|0\rangle = |0\rangle \text{ and } Z|1\rangle = -|1\rangle$$

i.e. leaves $|0\rangle$ invariant, but injects a phase $e^{i\pi} = -1$ to $|1\rangle$, corresponding to a rotation of π radians around the Z axis.

Clearly, its spectral decomposition yields:

$$Z = |0\rangle\langle 0| - |1\rangle\langle 1|$$

Typical quantum gates on 1 qubit

The phase shift gate

$$P_{\phi} = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{bmatrix}$$

i.e. $P_{\phi} |0\rangle = |0\rangle$ and $P_{\phi} |1\rangle = e^{i\phi} |1\rangle$.

The probability of measuring a $|0\rangle$ or $|1\rangle$ remains unchanged, but it modifies the phase of the quantum state.

This corresponds to a rotation of ϕ radians around the Z axis (i.e. along a line of latitude on the Bloch sphere) by ϕ radians.

Typical quantum gates on 1 qubit

Examples

- $Z = P_\pi$
- $S = P_{\frac{\pi}{2}} = \sqrt{Z} = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$
- $T = P_{\frac{\pi}{4}} = \sqrt{S}$ (also called the $\frac{\pi}{8}$ gate)

$$T = P_{\frac{\pi}{4}} = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{bmatrix}$$

which, up to a global phase factor $e^{i\frac{\pi}{8}}$, is equivalent to

$$\begin{bmatrix} e^{-i\frac{\pi}{8}} & 0 \\ 0 & e^{i\frac{\pi}{8}} \end{bmatrix}$$

Typical quantum gates on 1 qubit

Pauli gates

X, Y, Z specify a rotation by π radians around the corresponding axes on the Bloch sphere.

$$I = |0\rangle\langle 0| + |1\rangle\langle 1| = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$X = |1\rangle\langle 0| + |0\rangle\langle 1| = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$Z = |0\rangle\langle 0| - |1\rangle\langle 1| = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$Y = i(-|1\rangle\langle 0| + |0\rangle\langle 1|) = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

Typical quantum gates on 1 qubit

Rotation gates

Correspond to arbitrary rotations around the three axes of the Bloch sphere

$$R_e(\theta) \hat{=} e^{\frac{-i\theta E}{2}} = \cos\left(\frac{\theta}{2}\right)I - i\sin\frac{\theta}{2}E$$

where $e \hat{=} x, y, z$ and $E \hat{=} X, Y, Z$.

because, for any real number θ and matrix R st $R^2 = I$, which is the case for X , Y , and Z ,

$$e^{i\theta R} = \cos(\theta)I + i\sin(\theta)R$$

Typical quantum gates on 1 qubit

Rotation gates as matrices in the computational basis

$$R_x(\theta) = \begin{bmatrix} \cos\left(\frac{\theta}{2}\right) & -i \sin\left(\frac{\theta}{2}\right) \\ -i \sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} \cos\left(\frac{\theta}{2}\right) & -\sin\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) \end{bmatrix}$$

$$R_z(\theta) = \begin{bmatrix} e^{-i\frac{\theta}{2}} & 0 \\ 0 & e^{i\frac{\theta}{2}} \end{bmatrix}$$

Typical quantum gates on 1 qubit

Compute $R_z(\theta)|\psi\rangle$ for $|\psi\rangle = \cos\left(\frac{\sigma}{2}\right)|0\rangle + e^{i\gamma}\sin\left(\frac{\sigma}{2}\right)|1\rangle$

$$\begin{aligned}\begin{bmatrix} e^{-i\frac{\theta}{2}} & 0 \\ 0 & e^{i\frac{\theta}{2}} \end{bmatrix} \begin{bmatrix} \cos\left(\frac{\sigma}{2}\right) \\ e^{i\gamma}\sin\left(\frac{\sigma}{2}\right) \end{bmatrix} &= \begin{bmatrix} e^{-i\frac{\theta}{2}} \cos\left(\frac{\sigma}{2}\right) \\ e^{i\frac{\theta}{2}} e^{i\gamma} \sin\left(\frac{\sigma}{2}\right) \end{bmatrix} \\ &= e^{-i\frac{\theta}{2}} \begin{bmatrix} \cos\left(\frac{\sigma}{2}\right) \\ e^{i\theta} e^{i\gamma} \sin\left(\frac{\sigma}{2}\right) \end{bmatrix} \\ &= e^{-i\frac{\theta}{2}} \left(\cos\left(\frac{\sigma}{2}\right)|0\rangle + e^{i(\gamma+\theta)} \sin\left(\frac{\sigma}{2}\right)|1\rangle \right)\end{aligned}$$

As global phase is insignificant, the angle mapping $\gamma \mapsto \gamma + \theta$ is a rotation of θ around the z-axis of the Bloch sphere.

Typical quantum gates on 1 qubit

Theorem

Let U be a 1-gate, and v, w any two non-parallel axes of the Bloch sphere. Then there exist real numbers $\alpha, \beta, \gamma, \delta$ st

$$U = e^{i\alpha} R_v(\beta) R_w(\gamma) R_v(\delta)$$

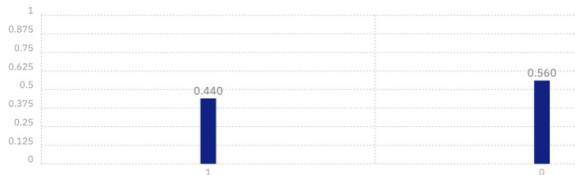
which means that any 1-gate can be expressed as a sequence of **two rotations about an axis** and **one rotation about another non parallel axis**, multiplied by a suitable **phase factor**.

proof hint: Recall U is unitary and unfold the definition of rotation gate.

Typical quantum gates on 1 qubit

The Hadamard gate creates superpositions

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$



$$\begin{aligned} H|0\rangle &= |+\rangle = \overbrace{\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)}^{\text{superposition}} \\ H|1\rangle &= |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \end{aligned}$$

Building larger states from smaller

Operator U in the no-cloning theorem acts on a 2-dimensional state, i.e. over the composition of two qubits.

What does composition mean?

Postulate 3

The state space of a combined quantum system is the tensor product $V \otimes W$ of the state spaces V and W of its components.

Composing quantum states

State spaces in a **quantum** system combine through **tensor**: \otimes

n m -dimensional vectors \rightsquigarrow a vector in m^n -dimensional space

i.e. the state space of a quantum system grows exponentially with the number of particles: cf, Feynman's original motivation

Example

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \otimes \begin{bmatrix} d \\ e \\ f \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \begin{bmatrix} d \\ e \\ f \end{bmatrix} = \begin{bmatrix} ad \\ ae \\ af \\ bd \\ be \\ bf \\ cd \\ ce \\ cf \end{bmatrix}$$

Composing quantum states

Tensor $V \otimes W$

- $B_{V \otimes W}$ is a set of elements of the form $|v_i\rangle \otimes |w_j\rangle$, for each $|v_i\rangle \in B_V$, $|w_j\rangle \in B_W$ and $\dim(V \otimes W) = \dim(V) \times \dim(W)$
- $(|u_1\rangle + |u_2\rangle) \otimes |z\rangle = |u_1\rangle \otimes |z\rangle + |u_2\rangle \otimes |z\rangle$
- $|z\rangle \otimes (|u_1\rangle + |u_2\rangle) = |z\rangle \otimes |u_1\rangle + |z\rangle \otimes |u_2\rangle$
- $(\alpha|u\rangle) \otimes |z\rangle = |u\rangle \otimes (\alpha|z\rangle) = \alpha(|u\rangle \otimes |z\rangle)$
- $\langle (|u_2\rangle \otimes |z_2\rangle) | (|u_1\rangle \otimes |z_1\rangle) \rangle = \langle u_2 | u_1 \rangle \langle z_2 | z_1 \rangle$

Composing quantum states

Clearly, every element of $V \otimes W$ can be written as

$$\alpha_1(|v_1\rangle \otimes |w_1\rangle) + \alpha_2(|v_2\rangle \otimes |w_1\rangle) + \cdots + \alpha_{nm}(|v_n\rangle \otimes |w_m\rangle)$$

Example

The basis of $V \otimes W$, for V, W qubits with the computational basis is

$$\{|0\rangle \otimes |0\rangle, |0\rangle \otimes |1\rangle, |1\rangle \otimes |0\rangle, |1\rangle \otimes |1\rangle\}$$

Thus, the tensor of $\alpha_1|0\rangle + \alpha_2|1\rangle$ and $\beta_1|0\rangle + \beta_2|1\rangle$ is

$$\alpha_1\beta_1|0\rangle \otimes |0\rangle + \alpha_1\beta_2|0\rangle \otimes |1\rangle + \alpha_2\beta_1|1\rangle \otimes |0\rangle + \alpha_2\beta_2|1\rangle \otimes |1\rangle$$

i.e., in a simplified notation,

$$\alpha_1\beta_1|00\rangle + \alpha_1\beta_2|01\rangle + \alpha_2\beta_1|10\rangle + \alpha_2\beta_2|11\rangle$$

Bases

The computational basis for a vector space

$$\underbrace{V \otimes V \otimes \dots \otimes V}_n$$

corresponding to the composition of n qubits (each living in V) is the set

$$\begin{aligned} & \{ \underbrace{|0\rangle \dots |0\rangle}_n |0\rangle, \underbrace{|0\rangle \dots |0\rangle}_n |1\rangle, \underbrace{|0\rangle \dots |1\rangle}_n |0\rangle, \dots, \underbrace{|1\rangle \dots |1\rangle}_n |1\rangle \} \\ \stackrel{\text{abv}}{=} & \{ \underbrace{|0 \dots 00\rangle}_n, \underbrace{|0 \dots 01\rangle}_n, \underbrace{|0 \dots 10\rangle}_n, \dots, \underbrace{|1 \dots 11\rangle}_n \} \end{aligned}$$

which may be written in a compressed (decimal) way as

$$\{|0\rangle, |1\rangle, |2\rangle, |3\rangle, \dots, |2^n - 1\rangle\}$$

Bases

The **computational basis** for a two qubit system would be

$$\{|0\rangle, |1\rangle, |2\rangle, |3\rangle\}$$

with

$$|0\rangle = |00\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad |1\rangle = |01\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad |2\rangle = |10\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad |3\rangle = |11\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Bases

There are of course other bases ... besides the **standard** one, e.g.

The Bell basis

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

Compare with the Hadamard basis for the single qubit systems

Representing multi-qubit states

Any unit vector in a 2^n Hilbert space represents a possible n -qubit state, but for

... a certain level of redundancy

- As before, vectors that differ only in a **global phase** represent the **same** quantum state
- but also the **same phase factor in different qubits** of a tensor product represent the **same** state:

$$|u\rangle \otimes (e^{i\phi}|z\rangle) = e^{i\phi}(|u\rangle \otimes |z\rangle) = (e^{i\phi}|u\rangle) \otimes |z\rangle$$

Actually, phase factors in qubits of a single term of a superposition can always be factored out into a coefficient for that term, i.e. **phase factors distribute over tensors**

Representing multi-qubit states

Representation

- Relative phases still matter (of course!)

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \text{ differs from } \frac{1}{\sqrt{2}}(e^{i\phi}|00\rangle + |11\rangle)$$

even if

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(e^{i\phi}|00\rangle + e^{i\phi}|11\rangle) = \frac{e^{i\phi}}{\sqrt{2}}(|00\rangle + |11\rangle)$$

- The complex **projective space** of dimension 1 (depicted in the **Block sphere**) generalises to higher dimensions, although in practice linearity makes Hilbert spaces easier to use.

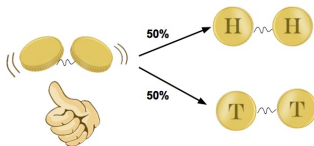
Entanglement

Most states in $V \otimes W$ cannot be written as $|u\rangle \otimes |z\rangle$

For example, the **Bell state**

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

is **entangled**



Entanglement

Actually, to make $|\Phi^+\rangle$ equal to

$$(\alpha_1|0\rangle + \beta_1|1\rangle) \otimes (\alpha_2|0\rangle + \beta_2|1\rangle) = \alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle + \beta_1\alpha_2|10\rangle + \beta_1\beta_2|11\rangle$$

would require that $\alpha_1\beta_2 = \beta_1\alpha_2 = 0$ which implies that either

$$\alpha_1\alpha_2 = 0 \text{ or } \beta_1\beta_2 = 0$$

Note

Entanglement can also be observed in simpler structures, e.g. **relations**:

$$\{(a, a), (b, b)\} \subseteq A \times A$$

cannot be **separated**, i.e. written as a Cartesian product of subsets of A .

2-gates: *CNOT*

Acts on the standard basis for a 2-qubit system, flipping the second bit if the first bit is 1 and leaving it unchanged otherwise.

$$\begin{aligned} CNOT &= |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X \\ &= |0\rangle\langle 0| \otimes (|0\rangle\langle 0| + |1\rangle\langle 1|) + |1\rangle\langle 1| \otimes (|1\rangle\langle 0| + |0\rangle\langle 1|) \\ &= |00\rangle\langle 00| + |01\rangle\langle 01| + |11\rangle\langle 10| + |10\rangle\langle 11| \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{aligned}$$

CNOT is unitary and is its own inverse, and **cannot be decomposed into a tensor product of two 1-qubit transformations**

2-gates: *CNOT*

The importance of *CNOT* is its ability to **change the entanglement** between two qubits, e.g.

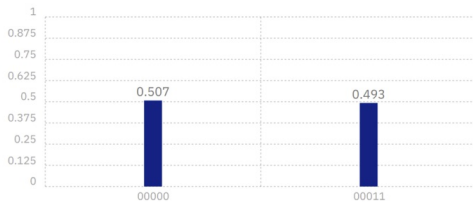
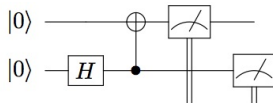
$$\begin{aligned} \text{CNOT} \left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle \right) &= \text{CNOT} \left(\frac{1}{\sqrt{2}}(|00\rangle + |10\rangle) \right) \\ &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \end{aligned}$$

Being its own inverse, also takes an entangled state to an unentangled one.

Note that **entanglement** is not a local property in the sense that transformations that act separately on two or more subsystems cannot affect the entanglement between those subsystems:

$$(U \otimes V) |v\rangle \text{ is entangled iff } |v\rangle \text{ is}$$

2-gates: *CNOT*

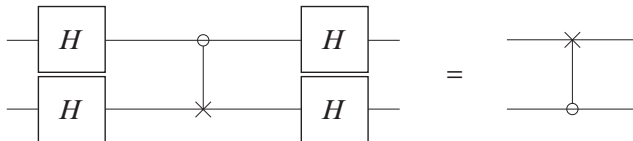


2-gates: *CNOT*

The notions of control/target bit in *CNOT* are **arbitrary**: they depend on what basis is considered. The standard behaviour is obtained in the computational basis. However, roles are interchanged in the Hadamard basis in which the effect of *CNOT* is

$$|++\rangle \mapsto |++\rangle \quad |+-\rangle \mapsto |--\rangle \quad |-+\rangle \mapsto |-+\rangle \quad |--\rangle \mapsto |+-\rangle$$

Exercise



The proof

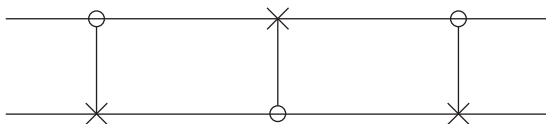
$$\begin{aligned}
 \text{LHS} &= \frac{1}{2} \begin{bmatrix} H & H \\ H & -H \end{bmatrix} \overbrace{\begin{bmatrix} I & 0 \\ 0 & X \end{bmatrix}}^{\text{CNOT}} \begin{bmatrix} H & H \\ H & -H \end{bmatrix} \\
 &= \frac{1}{2} \begin{bmatrix} H & HX \\ H & -HX \end{bmatrix} \begin{bmatrix} H & H \\ H & -H \end{bmatrix} \\
 &= \frac{1}{2} \begin{bmatrix} I + HXH & I - HXH \\ I - HXH & I + HXH \end{bmatrix} = \frac{1}{2} \begin{bmatrix} I + Z & I - Z \\ I - Z & I + Z \end{bmatrix} \\
 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \\
 &= I \otimes |0\rangle\langle 0| + X \otimes |1\rangle\langle 1| = \text{RHS}
 \end{aligned}$$

noting that

$$H \otimes H = (I \otimes H)(H \otimes I) = \frac{1}{\sqrt{2}} \begin{bmatrix} H & 0 \\ 0 & H \end{bmatrix} \begin{bmatrix} I & I \\ I & -I \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} H & H \\ H & -H \end{bmatrix}$$

Exercise

Discuss



Controlled Q-gates

From



to



$$C_Q|0\rangle|\varphi\rangle = |0\rangle|\varphi\rangle$$

$$C_Q|1\rangle|\varphi\rangle = |1\rangle Q|\varphi\rangle$$

$$C_Q = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes Q$$

corresponding to the following matrix in the standard basis:

$$C_Q = \begin{bmatrix} 1 & 0 \\ 0 & Q \end{bmatrix}$$

Controlled phase shift gate

$$C_{e^{i\theta}} = |00\rangle\langle 00| + |01\rangle\langle 01| + e^{i\theta}|10\rangle\langle 10| + e^{i\theta}|11\rangle\langle 11|$$

$$C_{e^{i\theta}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{i\theta} & 0 \\ 0 & 0 & 0 & e^{i\theta} \end{bmatrix}$$

Transforming a global into a local phase

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \longrightarrow \frac{1}{\sqrt{2}}(|00\rangle + e^{i\theta}|11\rangle)$$

Actually, a unitary transformation is completely determined by its action on a basis, but **not** by specifying what states the states corresponding to basis states are sent to.

Example: $e^{i\theta}$ takes the four quantum states to themselves (because e.g. $|10\rangle$ and $e^{i\theta}|10\rangle$ represent the same state), but a global phase can be transformed into a local one, as above

CCNOT or Toffoli gate

A 3-bit gate corresponding to **controlled *CNOT***. If the first two bits are in the state $|1\rangle$ applies X the third bit, else it does nothing:

$$|q_1 q_2 q_3\rangle \mapsto |q_1 q_2, q_3 \oplus (q_1 \wedge q_2)\rangle$$

In matrix form,

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Universal set of gates?

Is there a universal set of quantum gates?

In general **no**: there are uncountably many quantum transformations, and a finite set of generators can only generate countably many elements. However, it is possible for **finite sets of gates** to generate **arbitrarily close approximations to all unitary transformations**.

Definitions

- The **error** in approximating U by V is

$$Er(U, V) = \max_{|\phi\rangle} \|(U - V)|\phi\rangle\|$$

- An operator U can be **approximated to arbitrary accuracy** if for any positive ϵ there exists another unitary transformation V st $Er(U, V) \leq \epsilon$.
- A set of gates is **universal** if for any integer $n \geq 1$, any n -qubit unitary operator can be approximated to arbitrary accuracy by a quantum circuit using only gates from that set.

Universal set of gates?

Some examples

- The set $\{H, T\}$ is universal for 1-gates.
- The set $\{H, T, CNOT\}$ is a universal set of gates.

How efficient is an approximation?

To approximate an unitary transformation encoding some specific computation, one would expect to use a number of gates from the universal set which is **polynomial** in the number of qubits and the inverse of the quality factor ϵ .

Main result: theorem of **Solovay-Kitaev**