

Canonical Coin Systems for Change-Making Problems

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Abstract. The Change-Making Problem is to represent a given value with the fewest coins under a given coin system. As a variation of the knapsack problem, it is known to be NP-hard. Nevertheless, in most real money systems, the greedy algorithm can always yield optimal solutions. In this paper, we study what type of coin systems that can guarantee the optimality of the greedy algorithm. We provide novel proofs for the sufficient and necessary condition of the so-called *canonical* coin systems with four or five types of coins, and a sufficient condition of non-canonical coin systems, respectively. Moreover, we present an $O(m^2)$ algorithm that decides whether a tight coin system is canonical.

1 Introduction

The Change-Making Problem comes from the following scenario: in a shopping mall, the cashier needs to make change for many values of money based on some *coin system*¹ $\$ = \langle c_1, c_2, \dots, c_m \rangle$ with $1 = c_1 < c_2 < \dots < c_m$, where c_i denotes the value of the i -th type of coin in $\$$. For example, cent, nickel, dime and quarter are four types of US coins, and the corresponding coin system is $\$ = \langle 1, 5, 10, 25 \rangle$. Since the reserved coins are limited in reality, the cashier has to handle every exchange with as few coins as possible.

Formally, the Change-Making Problem is to solve the following integer programming with respect to a given value x .

$$\begin{aligned} & \min \sum_{i=1}^m \alpha_i \\ & \text{s.t. } \sum_{i=1}^m c_i \alpha_i = x, \quad c_i \geq 0 \end{aligned}$$

As usual, we call a feasible solution $(\alpha_1, \alpha_2, \dots, \alpha_m)$ of the integer programming above a *representation* of x under $\$$. If this representation satisfies $\sum_{j=1}^{i-1} \alpha_j c_j < c_i$ for $2 \leq i \leq m$, then it is the *greedy representation* of x , denoted by $\text{GRD}_\$(x)$, and $|\text{GRD}_\$(x)| = \sum_{i=1}^m \alpha_i$ is its size. Similarly, we also call the optimal solution $(\beta_1, \beta_2, \dots, \beta_m)$ the *optimal representation* of x , denoted by $\text{OPT}_\$(x)$, and $|\text{OPT}_\$(x)| = \sum_{i=1}^m \beta_i$ is its size.

¹ In this paper, all the variables range over the set \mathbb{N} of natural numbers.

1.1 Problem Statement

The Change-Making Problem is NP-hard [7, 3, 8] by a polynomial reduction from the knapsack problem. There are a large number of pseudo-polynomial exact algorithms [6, 9] solving this problem, including the one using dynamic programming [12]. However, the greedy algorithm, as a simpler approach, can produce the optimal solutions for many practical instances, especially canonical coin systems.

Definition 1. A coin system $\$$ is canonical if $|\text{GRD}_\$(x)| = |\text{OPT}_\$(x)|$ for all x .

For example, the coin system $\$ = \langle 1, 5, 10, 25 \rangle$ is canonical. Accordingly, the cashier can easily realize the optimality by repeatedly taking the biggest coin whose value is no larger than the remaining amount.

Definition 2. A coin system $\$$ is non-canonical if there is an x with $|\text{GRD}_\$(x)| > |\text{OPT}_\$(x)|$, and such x is called a counterexample of $\$$.

Definition 3. A coin system $\$ = \langle 1, c_2, \dots, c_m \rangle$ is tight if it has no counterexample smaller than c_m .

For example, both $\$_1 = \langle 1, 7, 10, 11 \rangle$ and $\$_2 = \langle 1, 7, 10, 50 \rangle$ are non-canonical, and $\$_1$ is tight but $\$_2$ is not. It is easy to verify that 14 is the counterexample of them, i.e., $\text{GRD}_{\$_1}(14) = \text{GRD}_{\$_2}(14) = (4, 0, 1, 0)$ and $\text{OPT}_{\$_1}(14) = \text{OPT}_{\$_2}(14) = (0, 2, 0, 0)$.

Nowadays, canonical coin systems have found numerous applications in many fields, e.g., finance [9], management [2] and computer networks [4]. It is desirable to give a full characterization of them.

1.2 Related Work

Chang and Gill [1] were the first ones to study canonical coin systems. They showed that there must be a counterexample x of the non-canonical $\$ = \langle 1, c_2, \dots, c_m \rangle$ such that $c_3 \leq x < \frac{c_m(c_m c_{m-1} + c_m - 3c_{m-1})}{c_m - c_{m-1}}$.

Concerning the smallest counterexamples of non-canonical coin systems, Tien and Hu established the following two important results in [11].

Theorem 1. Let x be the smallest counterexample of the non-canonical coin system $\$ = \langle 1, c_2, \dots, c_m \rangle$. Then $\alpha_i \beta_i = 0$ for all $i \in [1, m]$ such that $\text{OPT}_\$(x) = (\alpha_1, \alpha_2, \dots, \alpha_m)$ and $\text{GRD}_\$(x) = (\beta_1, \beta_2, \dots, \beta_m)$.

Theorem 2. Let $\$_1 = \langle 1, c_2, \dots, c_m \rangle$ and $\$_2 = \langle 1, c_2, \dots, c_m, c_{m+1} \rangle$ be two coin systems such that $\$_1$ is canonical but $\$_2$ is not. Then there is some k such that $k \cdot c_m < c_{m+1} < (k+1) \cdot c_m$ and $(k+1) \cdot c_m$ is a counterexample of $\$_2$.

These two results not only imply that many coin systems are canonical such as positive integer arithmetic sequences, geometric sequences and the Fibonacci sequence but also are the starting point of a lot of subsequent work.

Building on Theorem 1, Kozen and Zaks [5] gave a tight range of smallest counterexamples of non-canonical coin systems:

Theorem 3. *Let $\$ = \langle 1, c_2, \dots, c_m \rangle$ be a coin system. If $\$$ is not canonical, then the smallest counterexample lies in the range*

$$c_3 + 1 < x < c_{m-1} + c_m.$$

Furthermore, these bounds are tight.

Moreover, they gave a necessary and sufficient condition of the canonical coin system with three types of coins in [5].

Theorem 4. *The coin system $\$ = \langle 1, c_2, c_3 \rangle$ is non-canonical if and only if $0 < r < c_2 - q$ where $c_3 = qc_2 + r$ and $r \in [0, c_2 - 1]$.*

Here, we provide a proof of this theorem which will be used later.

Proof (of Theorem 4).

- (\Leftarrow) Consider the integer $x = c_2 + c_3 - 1$, $\text{GRD}_\$(x) = (c_2 - 1, 0, 1)$. Since $(r - 1, q + 1, 0)$ is a representation of x , we have $|\text{OPT}_\$(x)| \leq r + q$. By the precondition $0 < r < c_2 - q$, it is easy to see $|\text{OPT}_\$(x)| < |\text{GRD}_\$(x)|$. Thus, $c_3 + c_2 - 1$ is a counterexample of $\$$ and it is non-canonical.
- (\Rightarrow) Since $\$$ is non-canonical, let x be the smallest counterexample. By Theorem 3, $x \in [c_3 + 2, c_2 + c_3 - 1]$. Without loss of generality, let $\text{GRD}_\$(x) = (e, 0, 1)$ and $\text{OPT}_\$(x) = (0, k, 0)$ with $e \in [1, c_2 - 1]$. Then, we have $x = c_3 + e = kc_2$, i.e., $q = k - 1 < e$ and $r = c_2 - e \geq 1$. Thus, $0 < r < c_2 - q$. \square

Pearson [10] proved the following theorem that characterizes the smallest counterexample of the non-canonical coin system.

Theorem 5. *Let x be the smallest counterexample of the non-canonical coin system $\$ = \langle 1, c_2, \dots, c_m \rangle$. If $\text{OPT}_\$(x) = (\underbrace{0, \dots, 0}_0, \beta_l, \dots, \beta_r, \underbrace{0, \dots, 0}_0)$ with $\beta_l, \beta_r > 0$, then $\text{GRD}_\$(c_{r+1} - 1) = (\alpha_1, \dots, \alpha_{l-1}, \beta_l - 1, \beta_{l+1}, \dots, \beta_r, \underbrace{0, \dots, 0}_0)$.*

Based on this theorem, he has given an $O(m^3)$ algorithm that decides whether the coin system $\$ = \langle 1, c_2, \dots, c_m \rangle$ is canonical.

Recently, Niewiarowska and Adamaszek [13] investigate the structure of canonical coin systems and present a series of sufficient conditions of non-canonical coin systems.

1.3 Our Results

In this paper, we study canonical coin systems for the Change-Making Problem and obtain the following results.

- We give an easy proof for the sufficient and necessary condition of canonical coin systems with four or five types of coins.
- We provide a novel proof for natural sufficient condition of non-canonical coin systems.
- We present an $O(m^2)$ algorithm that decides whether a tight coin system is canonical.

The rest of this paper is organized as follows. In Section 2, we study canonical coin systems with four types of coins. In Section 3, we extend the study to canonical coin systems with five types of coins. Section 4 introduces tight canonical coin systems and Section 5 presents an $O(m^2)$ algorithm that decides whether a tight coin system is canonical. Finally, in Section 6, we address some of the questions left open and discuss future work.

2 Coin System with Four Types of Coins

In this section, we study canonical coin systems with four types of coins and give a full characterization of them based on Theorem 4.

Theorem 6. *A coin system $\$ = \langle 1, c_2, c_3, c_4 \rangle$ is non-canonical if and only if $\$$ satisfies exact one of the following conditions:*

1. $\langle 1, c_2, c_3 \rangle$ is non-canonical.
2. $|\text{GRD}_\$(k+1) \cdot c_3| > k+1$ with $k \cdot c_3 < c_4 < (k+1) \cdot c_3$.

The whole proof is based on an analysis of the coin system $\langle 1, c_2, c_3 \rangle$. If $\langle 1, c_2, c_3 \rangle$ is canonical, we can decide whether $\$$ is canonical by Theorem 2. Otherwise, the following lemma covers the rest case.

Lemma 1. *Let $\$ = \langle 1, c_2, c_3 \rangle$ be a coin system with $c_3 = qc_2 + r$. If $\$$ is non-canonical, then the coin system $\$' = \langle 1, c_2, c_3, c_4 \rangle$ is also non-canonical.*

Proof. Since $\$$ is non-canonical, we can find the smallest counterexample $x \in [c_3 + 2, c_2 + c_3 - 1]$ by Theorem 3. Assume that there is some $c_4 > c_3$ such that $\$' = \langle 1, c_2, c_3, c_4 \rangle$ is canonical. We will deduce a contradiction based on the analysis of x .

- If $x < c_4$, then x is a counterexample of $\$'$, a contradiction.
- Otherwise, $x \geq c_4$. It is easy to see $(0, 1, 1, 0)$ is a representation of $c_2 + c_3$, and $|\text{OPT}_{\$'}(c_2 + c_3)| = 2$ for $c_4 \leq x \leq c_2 + c_3 - 1$. By the above assumption, we know $\delta = c_2 + c_3 - c_4$ must be a coin, and here $\delta = 1$. Hence, $x = c_4 =$

$c_2 + c_3 - 1$. By the proof of Theorem 4, we have $x = kc_2 = c_2 + c_3 - 1$, i.e., $r = 1$ and $c_3 = qc_2 + 1$ and $c_4 = qc_2 + c_2$. Thus,

$$\$' = \langle 1, c_2, qc_2 + 1, qc_2 + c_2 \rangle.$$

For the integer $2qc_2 + c_2 - 1$, it is easy to see that $(c_2 - 3, 0, 2, 0)$ is a representation under $\$'$, and $\text{GRD}_{\$'}(2qc_2 + c_2 - 1) = (c_2 - 1, q - 1, 0, 1)$. Hence, we have $|(\underline{c}_2 - 3, 0, 2, 0)| < |\text{GRD}_{\$'}(2qc_2 + c_2 - 1)|$, that is, $2qc_2 + c_2 - 1$ is a counterexample of $\$'$, a contradiction.

We completes the proof. \square

Moreover, we prove the following Theorem 7 in which the coin system with three types of coins plays a somewhat surprising role.

Theorem 7. *If a coin system $\$_1 = \langle 1, c_2, c_3 \rangle$ is non-canonical, then the coin system $\$_2 = \langle 1, c_2, c_3, \dots, c_m \rangle$ is also non-canonical for $m \geq 4$.*

Actually, we can prove the following stronger result on the counterexamples of non-canonical systems with more than three types of coins.

Theorem 8. *If the coin system $\$_1 = \langle 1, c_2, c_3 \rangle$ is non-canonical, then the coin system $\$_2 = \langle 1, c_2, c_3, \dots, c_m \rangle$ is non-canonical and there exists some counterexample $x < c_m + c_3$ for $m \geq 4$.*

The proof is based on an induction on m and an exhaustive case-by-case analysis of $c_{k+1} < c_k + c_3$. The long proof is placed in the Appendix.

3 Coin System with Five Types of Coins

In this section, we give a full characterization of canonical coin systems with five types of coins.

Theorem 9. *A coin system $\$ = \langle 1, c_2, c_3, c_4, c_5 \rangle$ is non-canonical if and only if $\$$ satisfies exact one of the following conditions:*

1. $\langle 1, c_2, c_3 \rangle$ is non-canonical.
2. $\$ \neq \langle 1, 2, c_3, c_3 + 1, 2c_3 \rangle$.
3. $|\text{GRD}_{\$}((k+1) \cdot c_4)| > k+1$ with $k \cdot c_4 < c_5 < (k+1) \cdot c_4$.

The proof of this theorem is similar to that of Theorem 6 except for the second item “ $\$ \neq \langle 1, 2, c_3, c_3 + 1, 2c_3 \rangle$ ”. We actually need to prove the following theorem.

Theorem 10. *The coin system $\$_1 = \langle 1, c_2, c_3, c_4 \rangle$ is non-canonical and the coin system $\$_2 = \langle 1, c_2, c_3, c_4, c_5 \rangle$ is canonical if and only if $c_3 > 3$ and $\$_2 = \langle 1, 2, c_3, c_3 + 1, 2c_3 \rangle$.*

The proof is based on an exhaustive case-by-case analysis of the smallest counterexamples of some coin systems. The proof can be found in the Appendix.

4 Tight Coin System

For a coin system $\$ = \langle 1, c_2, \dots, c_m, c_{m+1} \rangle$, once there is an untight subsystem $\langle 1, c_2, \dots, c_i \rangle$ with $i \leq m+1$, $\$$ is clearly non-canonical. Therefore, it is necessary to decide whether a tight coin system is canonical.

Theorem 11. *Let $\$_1 = \langle 1, c_2, c_3 \rangle$, $\$_2 = \langle 1, c_2, c_3, \dots, c_m \rangle$ and $\$_3 = \langle 1, c_2, c_3, \dots, c_m, c_{m+1} \rangle$ be three tight coin systems such that $\$_1$ is canonical but $\$_2$ is not. If $\$_3$ is non-canonical, then there is a counterexample $x = c_i + c_j > c_{m+1}$ of $\$_3$ with $1 < c_i \leq c_j \leq c_m$.*

To establish Theorem 11, we first prove Lemma 2. Here, we define $c_0 = 0$ and $d_i := c_i - c_{i-1}$ with $1 \leq i \leq m+1$.

Lemma 2. *Let $\$_1 = \langle 1, c_2, c_3 \rangle$, $\$_2 = \langle 1, c_2, c_3, \dots, c_m \rangle$ and $\$_3 = \langle 1, c_2, c_3, \dots, c_m, c_{m+1} \rangle$ be three tight coin systems. $\$_1$ is canonical but both $\$_2$ and $\$_3$ are not. If any $c_m + c_i > c_{m+1}$ is not the counterexample of $\$_3$ with $1 < c_i \leq c_m$, then $d_{m+1} = \max\{d_i \mid 1 \leq i \leq m+1\}$.*

Proof. Assume that there is some $d_{j+1} > d_{m+1}$ with $0 < j < m$. Without loss of generality, let $d_{j+1} = d_{m+1} + \varepsilon$ with $0 < \varepsilon < d_{j+1}$.

For $c_{j+1} + c_m$, we have $c_{j+1} + c_m = c_j + d_{j+1} + c_m = c_j + \varepsilon + c_{m+1}$. Since $c_j + \varepsilon \in (c_j, c_{j+1})$, we have that $c_{j+1} + c_m$ is a counterexample of $\$_3$. This is a contradiction. \square

Proof (of Theorem 11). Assume that any $x = c_i + c_j > c_{m+1}$ with $1 < c_i \leq c_j \leq c_m$ is not the counterexample of $\$_3$. By Lemma 2 and the assumption, $d_{m+1} = \max\{d_i \mid 1 \leq i \leq m+1\}$. For simplicity, we introduce some notations.

- “ $x = c_{m+1} + \delta$ ” is the smallest counterexample of $\$_3$.
- “ c_s ” is the biggest coin used in all the optimal representations of x with $s \leq m$.
- “ c_l ” and “ c_h ” are the smallest coin and the biggest coin used in the optimal representation of $x - c_s$ respectively.

Thus, we have $c_l > x - c_{s+1}$ by definition of c_s .

- (1) If $s \leq m-1$, then $c_l > x - c_m = \delta + d_{m+1}$.
 - If $c_s + c_l < x$, then $\text{OPT}_{\$_3}(x)$ uses one coin c_h besides a coin c_l and a coin c_s . Since $|\text{GRD}_{\$_3}(x - c_h)| = |\text{OPT}_{\$_3}(x - c_h)|$ and $x - c_h \geq c_s + c_l > c_{s+1}$, we have c_{s+1} appears in $\text{OPT}_{\$_3}(x)$, a contradiction.
 - Otherwise, $c_s + c_l = x > c_{m+1}$. By the assumption, there is $c_{m+1} + c_i = c_s + c_l$, a contradiction.
- (2) Otherwise, $s = m$.
 - If $c_h + c_m > c_{m+1}$, there exists $c_{m+1} + c_i = c_h + c_s$ by the assumption. Thus, we can get a new representation of x replacing c_h and c_m with c_{m+1} and c_i in $\text{OPT}_{\$_3}(x)$. It is easy to see this new representation uses the coin c_{m+1} and remains optimality, a contradiction.

- If $c_h + c_m < c_{m+1}$, then $\text{OPT}_{\$_3}(x)$ uses one coin c_l besides a coin c_h and a coin c_m . Since $s = m$, we have $c_l > x - c_{m+1}$, i.e., $\delta < c_l \leq c_h < d_{m+1}$. Thus, $c_m + \delta \in (c_m, c_{m+1})$. It is easy to see $|\text{GRD}_{\$_3}(x)| > |\text{OPT}_{\$_3}(x)| = |\text{GRD}_{\$_2}(x)|$. By the assumption, there is $c_{m+1} + c_i = 2c_m$. Then, we have

$$|\text{GRD}_{\$_3}(c_m + \delta)| = |\text{GRD}_{\$_3}(x)| > 1 + |\text{GRD}_{\$_3}(c_m + \delta - c_i)|$$

It implies $c_m + \delta$ is a counterexample of $\$_3$, a contradiction.

- If $c_h + c_m = c_{m+1}$, then we can get a new representation of x replacing c_h and c_m with c_{m+1} in $\text{OPT}_{\$_3}(x)$. It is easy to see this new representation has the smaller size, a contradiction.

This completes the proof. \square

5 The Algorithm

In this section, we present an $O(m^2)$ algorithm that decides whether a tight coin system $\$ = \langle 1, c_2, \dots, c_m, c_{m+1} \rangle$ with $m \geq 5$ is canonical. By Theorem 11, we have if $\$$ is non-canonical, then there is a counterexample that is the sum of two coins.

Algorithm 1: IsCanonical

Require: a tight coin system $\$ = \langle 1, c_2, \dots, c_m, c_{m+1} \rangle$ with $m \geq 5$

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1: if  $0 < r < c_2 - q$  with  $c_3 = qc_2 + r$  then
2:   return  $\$$  is non-canonical
3: else
4:   for  $i = m$  downto 1 do
5:     for  $j = i$  downto 1 do
6:       if  $c_i + c_j > c_{m+1}$  and  $|\text{GRD}_{\$}(c_i + c_j)| > 2$  then
7:         return  $\$$  is non-canonical
8:       end if
9:     end for
10:   end for
11:   return  $\$$  is canonical
12: end if
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In addition, we observe this algorithm can deal with most of tight coin systems in $2m$ steps, except a small number of non-canonical coin systems, and they are almost arithmetic sequences, for example,

$$\$ = \langle 1, 2, \dots, 11, 12, 14, 15, \dots, 20, 21, 24, 25, 26, 28, 29, 30, 39 \rangle$$

Moreover, we can characterize the smallest counterexample of such non-canonical coin system.

Lemma 3. Let $\$1 = \langle 1, c_2, c_3 \rangle$, $\$2 = \langle 1, c_2, c_3, \dots, c_m \rangle$ and $\$3 = \langle 1, c_2, c_3, \dots, c_m, c_{m+1} \rangle$ be three tight coin systems such that $\$1$ is canonical but $\$2$ and $\$3$ are not. If $\$3$ has no counterexample x such that $x = c_m + c_i > c_{m+1}$ or $x = c_{m-1} + c_j > c_{m+1}$ with $1 < c_i, c_j \leq c_m$, then the smallest counterexample of $\$3$ is the sum of two coins.

Proof. Modifying the proof of Theorem 11 slightly, it is easy to get this proof. \square

6 Conclusion

In this paper, we have given an $O(m^2)$ algorithm that decides whether a tight coin system is canonical. Although our algorithm can only handle tight coin systems, it is more efficient than Pearson's algorithm. As some future work, we expect to obtain an $O(m)$ algorithm for tight canonical coin systems and an $o(m^3)$ general algorithm.

We have shown the sufficient and necessary condition of canonical coin systems with four or five types of coins by a novel method. Meantime, we have also obtained the sufficient condition of non-canonical coin systems. Many algorithm including Pearson's can benefit from it. However, it is still left open to give full characterizations of canonical coin systems with more than five types of coins. It is a challenge to explore the corresponding necessary condition in the future.

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Appendix

Proof (of Theorem 8). The proof bases on the induction on m and the exhaustive case-by-case analysis of $c_{m+1} < c_m + c_3$.

The result is trivial for $m = 4$ by Lemma 1 and Theorem 3. Now assume that $\$'_2 = \langle 1, c_2, c_3, \dots, c_k \rangle$ is non-canonical and there exists some counterexample $x < c_k + c_3$.

Then, we will check $\$'_2 = \langle 1, c_2, c_3, \dots, c_k, c_{k+1} \rangle$ based on c_{k+1} in detail. Here, we only consider the non-trivial cases as follows:

- “ $x > c_k$ ”. Otherwise, x is also a counterexample of $\$'_2$ and $x < c_{k+1} + c_3$.
- “ $c_i - c_{i-1} < c_3$ for $i \in [4, k+1]$ ”. Otherwise, there must be some integer y such that y is a counterexample of $\$'_2$ and $y < c_{k+1} + c_3$ by the previous assumption.
- “Once $c_i + c_j \in (c_s, c_{s+1})$ and $c_i + c_j < c_{k+1} + c_3$ with $i \leq j < s \leq k+1$, we have $c_i + c_j - c_s \in \{1, c_2, c_3, \dots, c_m\}$ ”. Otherwise, $c_i + c_j$ is a counterexample of $\$'_2$ and $c_i + c_j < c_{k+1} + c_3$.

Next, we will analyze $c_{k+1} < c_k + c_3$ exhaustively, and either find a counterexample of $\$'_2$ or exclude a case for a contradiction.

- (1) If $c_{k+1} = c_k + 1$, then $c_k + c_2$ is a counterexample of $\$'_2$ and $c_k + c_2 < c_{k+1} + c_3$.
- (2) If $c_{k+1} = c_k + \varepsilon$ with $\varepsilon \in (1, c_2)$, then $c_k + c_2 \in (c_{k+1}, c_{k+1} + c_2 - 1)$ and $c_k + c_3 \in (c_{k+1}, c_{k+1} + c_2 - 1)$. Thus, we have

$$\begin{aligned} c_k + c_2 - c_{k+1} &= c_2 - \varepsilon \in \{1\} \\ c_k + c_3 - c_{k+1} &= c_3 - \varepsilon \in \{1, c_2\} \end{aligned}$$

It is easy to see $2c_2 = c_3 + 1$ for $c_3 - \varepsilon > c_2 - \varepsilon$. Thus, $\$'_1$ is canonical by Theorem 4, a contradiction.

- (3) If $c_{k+1} = c_k + c_3 - \varepsilon$ with $\varepsilon \in (0, c_3 - c_2]$, then $c_k + c_3 \in (c_{k+1}, c_{k+1} + c_3 - c_2]$ and $c_k + c_4 \in (c_{k+1}, c_{k+1} + c_4 - c_2]$. Thus, we have

$$\begin{aligned} c_k + c_3 - c_{k+1} &= \varepsilon \in \{1, c_2\} \\ c_k + c_4 - c_{k+1} &= c_4 - c_3 + \varepsilon \in \{1, c_2, c_3\} \end{aligned}$$

If $\varepsilon = c_2$, then it is easy to see $2c_2 \leq c_3$ and $c_4 = 2c_3 - c_2 \geq c_2 + c_3$. By Theorem 3, the smallest counterexample y of $\$'_1$ is also the counterexample of $\$'_2$ and $y < c_{k+1} + c_3$.

If $\varepsilon = 1$, then $c_{k+1} = c_k + c_3 - 1$ and $c_4 - c_3 + 1 \in \{c_2, c_3\}$.

(a) If $c_4 - c_3 + 1 = c_3$, then $c_4 = 2c_3 - 1$. Since $\$1$ is non-canonical, by the proof of Theorem 4, we have the smallest counterexample $y = (q+1)c_2$ of $\$1$ with $0 < r < c_2 - q$. Replacing c_3 with $qc_2 + r$, it is easy to see $c_4 = 2qc_2 + 2r - 1 > y$. Thus, y is a counterexample of $\$'_2$ and $y < c_{k+1} + c_3$.

(b) If $c_4 - c_3 + 1 = c_2$, then $c_4 = (q+1)c_2 + r - 1$ with $c_3 = qc_2 + r$. Similarly, we have the smallest counterexample $y = (q+1)c_2$ of $\$1$ with $0 < r < c_2 - q$.

If $r > 1$, then y is a counterexample of $\$'_2$ and $y < c_{k+1} + c_3$.

If $r = 1$, then $c_3 = qc_2 + 1$ and $c_4 = (q+1)c_2$. Since $c_{k-1} > c_k - c_3$ and $c_{k+1} - c_k = c_3 - 1$, we have $c_{k-1} + c_4 \in (c_k + c_2 - 1, c_k + c_4)$ and $c_k + c_4 > c_{k+1}$.

- If $c_{k-1} + c_4 \in (c_k + c_2 - 1, c_{k+1})$, then $c_{k-1} + c_4 - c_k \in \{c_2\}$, i.e., $c_k - c_{k-1} = c_{k+1} - c_k = c_4 - c_2 = qc_2$. Since $c_{k-1} < c_{k-2} + c_3$, we have $c_{k-2} + c_3 \in (c_{k-1}, c_k]$.

If $c_{k-2} + c_3 = c_k$, then $c_{k-1} - c_{k-2} = 1$. It is easy to see $c_{k-2} + c_2 \in (c_{k-1}, c_k)$ is a counterexample of $\$'_2$.

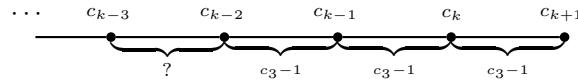
If $c_{k-2} + c_3 < c_k$, then $c_{k-2} + c_3 - c_{k-1} \in \{1, c_2\}$.

- If $c_{k-2} + c_3 - c_{k-1} = c_2$, then $c_{k-1} - c_{k-2} = c_3 - c_2$.

If $c_3 - c_2 > c_2$, then $2c_3 = c_4 + c_2$. Since $c_4 = (q+1)c_2$, we have $c_2 = \frac{2}{2-q}$ and $q \geq 1$, i.e., $c_2 = 2$. However, $\$1 = \langle 1, 2, 3 \rangle$ is canonical, a contradiction.

If $c_3 - c_2 < c_2$, then $q = 1$ and $c_3 = c_2 + 1$ and $c_4 = 2c_2$. Thus, $c_{k-2} + c_2 \in (c_{k-1}, c_k)$ is a counterexample of $\$'_2$.

- If $c_{k-2} + c_3 - c_{k-1} = 1$, then $c_{k-1} - c_{k-2} = c_3 - 1$.



If $c_{k-2} - c_{k-3} < c_3 - 1$, then there is a counterexample z of $\langle 1, c_2, \dots, c_{k-3} \rangle$ such that $z < c_{k-3} + c_3 \leq c_{k-1}$ by the previous assumption.

- * If $z < c_{k-2}$, then $z < c_{k+1} + c_3$ is a counterexample of $\$'_2$.
- * If $c_{k-2} \leq z < c_{k-3} + c_3 - 1$, then $c_{k-2} + z - c_{k-3} \in (c_{k-2}, c_{k-1})$, which is a counterexample of $\$'_2$.
- * If $z = c_{k-3} + c_3 - 1$, then $c_{k+1} + c_3 - 1$ is a counterexample of $\$'_2$.

Otherwise, $c_{k-2} - c_{k-3} = c_3 - 1$. Similarly, we either find a counterexample of $\$'_2$ or obtain

$$c_{k-3} - c_{k-4} = \dots = c_4 - c_3 = c_3 - 1 = qc_2.$$

Thus, $c_2 = \frac{1}{1-q}$, a contradiction.

- If $c_{k-1} + c_4 = c_{k+1}$, then $c_k - c_{k-1} = c_2$. Since $c_{k-1} + c_3 = c_k + c_3 - c_2 \in (c_k, c_{k+1})$, we have $c_3 - c_2 \in \{1, c_2\}$.
 - If $c_3 - c_2 = c_2$, then $c_3 = 2c_2$, a contradiction.
 - If $c_3 - c_2 = 1$, then $c_3 = c_2 + 1$ and $c_4 = 2c_2$. Since $c_{k-2} + c_3 \in (c_{k-1}, c_k)$, we have $c_{k-2} + c_3 - c_{k-1} \in \{1\}$, i.e., $c_{k-1} - c_{k-2} = c_2$. Similar to the above proof, we either find a counterexample of $\$'_2$ or obtain

$$c_{k-2} - c_{k-3} = \cdots = c_4 - c_3 = c_2.$$

It is a contradiction.

- If $c_{k-1} + c_4 \in (c_{k+1}, c_{k+1} + c_2)$, then $c_{k-1} + c_4 - c_{k+1} \in \{1\}$. Since $c_{k+1} = c_k + c_3 - 1$, we have $c_k - c_{k-1} = c_2 - 1$, and $c_{k-1} + c_3 = c_k + c_3 + 1 - c_2$. Since $c_3 + 1 - c_2 \in \{c_2\}$, we have $c_3 = 2c_2 - 1$, a contradiction.

The proof is complete. \square

Proof (of Theorem 10). To show the above theorem, we first need to prove the following lemma.

Lemma 4. *Let $\$_1 = \langle 1, c_2, c_3, c_2 + c_3 - 1 \rangle$ and $\$_2 = \langle 1, c_2, c_3, c_2 + c_3 - 1, c_2 + 2c_3 - 2 \rangle$ be two coin systems such that $\$_2$ is canonical but $\$_1$ is not. Then the coin system $\$_3 = \langle 1, c_2, c_3 \rangle$ is canonical.*

Proof. Assume that $\$_3$ is non-canonical. By Theorem 3, there exists the smallest counterexample $y \in [c_3 + 2, c_2 + c_3 - 1]$ of $\$_3$. Next, we will deduce a contradiction by analyzing y in detail.

- (1) If $y \in [c_3 + 2, c_2 + c_3 - 2]$, then y is also a counterexample of $\$_2$, a contradiction.
- (2) Otherwise, $y = c_2 + c_3 - 1 = kc_2$ with $1 < k < c_2$ by the proof of Theorem 4. Thus, we have

$$\$_2 = \langle 1, c_2, kc_2 - c_2 + 1, kc_2, 2kc_2 - c_2 \rangle$$

Since $\$_1$ is non-canonical, we have the smallest counterexample $x \in [c_3 + 2, c_2 + 2c_3 - 2]$ by Theorem 3.

- If $x \in [c_3 + 2, c_2 + 2c_3 - 3]$, then x is also a counterexample of $\$_2$, a contradiction.
- Otherwise, $x = c_2 + 2c_3 - 2 = 2kc_2 - c_2$. By Theorem 1, we have

$$\begin{aligned} \text{GRD}_{\$_1}(x) &= (0, \quad k-1, \quad 0, \quad 1) \\ \text{OPT}_{\$_1}(x) &= (\beta_1, \quad 0, \quad \beta_3, 0) \text{ with } \beta_3 \leq 2 \end{aligned}$$

- If $\beta_3 \leq 1$, then $\beta_1 > k-1$. It is easy to see $|\text{OPT}_{\$_1}(x)| \geq |\text{GRD}_{\$_1}(x)|$, a contradiction.
- If $\beta_3 = 2$, then $\beta_1 = c_2 - 2 > k-2$. Thus, we have $|\text{OPT}_{\$_1}(x)| > |\text{GRD}_{\$_1}(x)|$, a contradiction.

Therefore, $\$_3 = \langle 1, c_2, c_3 \rangle$ is canonical. \square

(\Leftarrow) First, we show that $\$1 = \langle 1, 2, c_3, c_3 + 1 \rangle$ is non-canonical with $c_3 > 3$. It is easy to see $\text{OPT}_{\$1}(2c_3) = (0, 0, 2, 0)$. By Theorem 1, we have $\text{GRD}_{\$1}(2c_3) = (\alpha_1, \alpha_2, 0, 1)$. Since $c_3 > 3$, we have $|\text{GRD}_{\$1}(c_3 - 1)| > 1$, i.e., $|\text{GRD}_{\$1}(2c_3)| > 2$. Thus, $\$1$ is non-canonical.

Secondly, we show that $\$2 = \langle 1, 2, c_3, c_3 + 1, 2c_3 \rangle$ is canonical. Assume that $\$2$ is non-canonical. By Theorem 3, there is the smallest counterexample x of $\$2$ such that $x \in [c_3 + 2, 3c_3]$ and $x \neq 2c_3$. Next, we will deduce a contradiction by analyzing x in detail.

(1) If $x \in [c_3 + 2, 2c_3]$, then it is also the smallest counterexample of $\$1$. For simplicity, let $x = c_3 + \varepsilon$ with $\varepsilon \in [2, c_3]$.

– If $\varepsilon = 2\ell + 1$, then $1 \leq \ell \leq \frac{c_3 - 1}{2} - 1$. By Theorem 1, we have

$$\text{GRD}_{\$1}(x) = (0, \ell, 0, 1) \quad \text{OPT}_{\$1}(x) = (\beta_1, \beta_2, 1, 0)$$

Thus, $\beta_1 + 2\beta_2 = 2\ell + 1$ and $\beta_1 \in \{0, 1\}$. It is easy to see $|\text{GRD}_{\$1}(x)| < |\text{OPT}_{\$1}(x)|$, a contradiction.

– If $\varepsilon = 2\ell$, then $2 \leq \ell \leq \frac{c_3}{2} - 1$. Similarly, we have

$$\text{GRD}_{\$1}(x) = (1, \ell - 1, 0, 1) \quad \text{OPT}_{\$1}(x) = (0, \ell, 1, 0)$$

Obviously, $|\text{OPT}_{\$1}(x)| = |\text{GRD}_{\$1}(x)|$, a contradiction.

(2) Otherwise, $x \in (2c_3, 3c_3)$. Let $x = 2c_3 + \varepsilon$ with $\varepsilon \in [2, c_3]$.

– If $\varepsilon = 2\ell + 1$, then $1 \leq \ell \leq \frac{c_3 - 1}{2} - 1$. By Theorem 1, we have

$$\text{GRD}_{\$2}(x) = (1, \ell, 0, 0, 1) \quad \text{OPT}_{\$2}(x) = (0, \beta_2, \beta_3, \beta_4, 0)$$

If $\beta_3 + \beta_4 \geq 3$, then $2\beta_2 + \beta_3 c_3 + (c_3 + 1)\beta_4 \geq 3c_3 > x$, a contradiction.

And for $\beta_3 + \beta_4 = 0$, $\beta_3 + \beta_4 = 1$ and $\beta_3 + \beta_4 = 2$, it is easy to get a contradiction similarly.

– If $\varepsilon = 2\ell$, then we can also deduce a contradiction similar to the above case.

Thus, $\$2 = \langle 1, 2, c_3, c_3 + 1, 2c_3 \rangle$ is canonical.

(\Rightarrow) For the integer $c_3 + c_4$, it is easy to see $\text{GRD}_{\$2}(c_3 + c_4) = (\alpha_1, \alpha_2, 0, 0, 1)$ and $\text{OPT}_{\$2}(c_3 + c_4) = (0, 0, 1, 1, 0)$. Since $\$2$ is canonical, we have either $\alpha_1 = 1, \alpha_2 = 0$ or $\alpha_1 = 0, \alpha_2 = 1$, i.e.,

$$\text{either } c_5 = c_3 + c_4 - 1 \text{ or } c_5 = c_3 + c_4 - c_2. \tag{A}$$

Similar to the integer $2c_4$, we have

$$\text{either } c_5 = 2c_4 - 1 \text{ or } c_5 = 2c_4 - c_2 \text{ or } c_5 = 2c_4 - c_3 \tag{B}$$

Correlating (A) with (B), we have 3 feasible equations as follows:

$$\begin{array}{l} \textcircled{1} \left\{ \begin{array}{l} c_5 = c_3 + c_4 - c_2 \\ c_5 = 2c_4 - c_3 \end{array} \right. \textcircled{2} \left\{ \begin{array}{l} c_5 = c_3 + c_4 - 1 \\ c_5 = 2c_4 - c_3 \end{array} \right. \textcircled{3} \left\{ \begin{array}{l} c_5 = c_3 + c_4 - 1 \\ c_5 = 2c_4 - c_2 \end{array} \right. \end{array}$$

Next, we will deduce a contradiction from ① and ② respectively.

(1) Solving ①, we have $c_4 = 2c_3 - c_2$ and $c_5 = 3c_3 - 2c_2$. Thus,

$$\$2 = \langle 1, c_2, c_3, 2c_3 - c_2, 3c_3 - 2c_2 \rangle$$

Since $\$1$ is non-canonical, there is the smallest counterexample $x \in [c_3 + 2, 3c_3 - c_2 - 1]$ of $\$1$.

If $x \in [c_3 + 2, 3c_3 - c_2 - 1]$, then x is also a counterexample of $\$2$, contradiction.

Otherwise, $x \in [3c_3 - 2c_2, 3c_3 - c_2 - 1]$. By Theorem 1, we have

$$\begin{aligned} \text{GRD}_{\$1}(x) &= (\alpha_1, \alpha_2, 0, 1) \\ \text{OPT}_{\$1}(x) &= (\beta_1, \beta_2, \beta_3, 0) \text{ with } \beta_3 \leq 2 \end{aligned}$$

- If $c_3 \geq 2c_2$, then $\text{GRD}_{\$1}(x) = (\alpha_1, \alpha_2, 0, 1)$ and $\text{OPT}_{\$1}(x) = (\beta_1, 0, \beta_3, 0)$ where $\beta_3 \in \{0, 1, 2\}$ and $\alpha_2 > 0$. Since $x \in [3c_3 - 2c_2, 3c_3 - c_2 - 1]$, we have $|\text{GRD}_{\$1}(x)| \leq c_3 - 2c_2 + 2$.

For $\beta_3 = 0$, $\beta_3 = 1$ and $\beta_3 = 2$, it is easy to deduce a contradiction respectively.

- If $c_3 < 2c_2$, then $\text{GRD}_{\$1}(x) = (\alpha_1, 0, 0, 1)$ and $\text{OPT}_{\$1}(x) = (0, \beta_2, \beta_3, 0)$ where $\beta_3 \in \{0, 1, 2\}$ and $\beta_2 + \beta_3 < 1 + \alpha_1$.
 - If $\beta_3 = 2$, then $\beta_2 = 0$ and $\alpha_1 = c_2$, a contradiction.
 - If $\beta_3 = 1$, then $c_3 = (\beta_2 + 1)c_2 - \alpha_1$. Since $c_3 \in (c_2, 2c_2)$, we have $c_3 = 2c_2 - \alpha_1$. Thus,

$$\$2 = \langle 1, c_2, 2c_2 - \alpha_1, 3c_2 - 2\alpha_1, 4c_2 - 3\alpha_1 \rangle$$

If $\alpha_1 = 1$, then $\$2 = \langle 1, c_2, 2c_2 - 1, 3c_2 - 2, 4c_2 - 3 \rangle$. It is easy to see $\langle 1, c_2, 2c_2 - 1, 3c_2 - 2 \rangle$ is canonical, a contradiction.

If $\alpha_1 > 1$, then $4c_2 - 2\alpha_1$ is a counterexample of $\$2$, a contradiction.

- If $\beta_3 = 0$, then $c_3 = \frac{\beta_2 - 1}{2}c_2 + (c_2 - \frac{\alpha_1}{2}) = qc_2 + r$. Since $c_3 < 2c_2$, we have $\beta_2 = 3$ and $x = 3c_2$ and $q = 1, r = c_2 - \frac{1}{2}\alpha_1$. Thus,

$$\$2 = \langle 1, c_2, c_2 + r, c_2 + 2r, c_2 + 3r \rangle$$

Since $\beta_2 + \beta_3 < 1 + \alpha_1$, we have $\alpha_1 > 2$. By Theorem 4, $\$3 = \langle 1, c_2, c_2 + r \rangle$ is non-canonical. It is easy to see $2c_2 < c_2 + 2r$ is the smallest counterexample of $\$2$, a contradiction.

(2) Solving ②, we have $c_4 = 2c_3 - 1$ and $c_5 = 3c_3 - 2$. Thus,

$$\$2 = \langle 1, c_2, c_3, 2c_3 - 2, 3c_3 - 2 \rangle$$

It is easy to see $c_3 + c_4 - 1 < c_5$, i.e., the smallest counterexample of $\$1$ is also a counterexample of $\$2$, a contradiction.

(3) Solving ③, we have $c_4 = c_3 + c_2 - 1$ and $c_5 = 2c_3 + c_2 - 2$. Thus,

$$\$2 = \langle 1, c_2, c_3, c_2 + c_3 - 1, c_2 + 2c_3 - 2 \rangle$$

By Lemma 4, $\langle 1, c_2, c_3 \rangle$ is canonical. Since $\$1$ is non-canonical, $2c_3$ is a counterexample of $\$1$ by Theorem 2. And we claim $2c_3 \geq c_2 + 2c_3 - 2$. Otherwise, $2c_3$ is a counterexample of $\$2$, a contradiction. Thus, $c_2 \leq 2$, i.e.,

$$\$2 = \langle 1, 2, c_3, c_3 + 1, 2c_3 \rangle$$

This completes the proof. \square

Proof (of Lemma 3). First, we introduce some denotations as follows:

- “ $x = c_{m+1} + \delta$ ” is the smallest counterexample of $\$3$.
- “ c_s ” is the biggest coin used in all the optimal representations of x with $s \leq m$.
- “ c_l ” and “ c_h ” are the smallest coin and the biggest coin used in the optimal representation of $x - c_s$ respectively.

Thus, we have $c_l > x - c_{s+1}$ by definition of c_s . Since x is the smallest counterexample of $\$3$, we have

$$\begin{aligned} |\text{GRD}_{\$3}(x - c_s)| &= |\text{OPT}_{\$3}(x - c_s)| \\ |\text{GRD}_{\$3}(x - c_l)| &= |\text{OPT}_{\$3}(x - c_l)| \\ |\text{GRD}_{\$3}(x - c_h)| &= |\text{OPT}_{\$3}(x - c_h)| \end{aligned}$$

Assume that x is not the sum of two coin, i.e., $c_s + c_l < x$. Thus, we can find one coin c_h besides a coin c_l and a coin c_s in the optimal representation of x .

- (1) If $s \leq m - 1$, then $c_l > x - c_m = \delta + d_{m+1}$. By Lemma 2, it is easy to see $x - c_h \geq c_s + c_l > c_{s+1}$. Thus, c_{s+1} should appear in the optimal representation of x , a contradiction.
- (2) Otherwise, $s = m$.
 - If $c_h + c_m > c_{m+1}$, there exists $c_{m+1} + c_i = c_h + c_s$ by the hypothesis. We can replace c_h and c_m with c_{m+1} and c_i , a contradiction.
 - If $c_h + c_m < c_{m+1}$, then $\text{OPT}_{\$3}(x)$ uses one coin c_l besides a coin c_h and a coin c_m . Since $s = m$, we have $c_l > x - c_{m+1}$, i.e., $\delta < c_l \leq c_h < d_{m+1}$. By the hypothesis, we have $c_{m+1} + c_i = 2c_m$. Consider $c_m + \delta < c_{m+1}$.

$$\begin{aligned} |\text{GRD}_{\$3}(c_m + \delta)| &= 1 + |\text{GRD}_{\$3}(\delta)| > 1 + |\text{GRD}_{\$3}(\delta + d_{m+1})| \\ &= 1 + |\text{GRD}_{\$3}(c_m + \delta - c_i)| \end{aligned}$$

Therefore, $c_m + \delta$ is a counterexample of $\$3$, a contradiction.

- If $c_h + c_m = c_{m+1}$, then $\text{OPT}_{\$3}(x)$ uses one coin c_l besides a coin c_h and a coin c_m . Thus, $\delta < c_l \leq c_h = d_{m+1}$.

$$|\text{GRD}_{\$3}(x)| = 1 + |\text{GRD}_{\$3}(\delta)| < 2 + |\text{GRD}_{\$3}(\delta)| = |\text{OPT}_{\$3}(x)|$$

This is a contraction.

We complete the proof. \square