Double auctions for cross-blockchain resource allocation*

Bruno Mazorra†[1] and Matheus V. X. Ferreira‡[2]

[1]Universitat Pompeu Fabra & Composable Foundation
[2]Harvard University

July 8, 2024

Abstract

The current design of rollup sequencing and L1 does not allow solvers and searchers to express combinatorial preferences over different domains or efficiently extract cross-domain MEV opportunities, nor does it allow efficient routing of transactions or intents through liquidity aggregation across different domains. Shared sequencing marketplaces have emerged as a potential solution to this problem by allowing third parties to sequence L2 batches concurrently. In this paper, we formalize the shared sequencing mechanism design and show that just-in-time shared sequencing has a revenue and welfare price of stability of $\Omega(\sqrt[3]{m})$ with m sellers, even when buyers' valuations are fully known. Moreover, we demonstrate that natural mechanisms such as Shapley payments and proportional payment rules for sharing revenue among rollups lead to a counterintuitive result akin to Braess's paradox. More formally, these mechanisms, in equilibrium, can result in less revenue and welfare than simultaneous first-price auctions. Finally, we present examples showing that slot auctions could address these issues, enhancing the efficiency of the shared sequencing marketplace.

Keywords: Mechanism design, Cross-Domain MEV

1 Introduction

Imagine a scenario where two siblings inherit parts of a substantial estate from a wealthy family. The estate is divided into two distinct sections: the first comprises a sprawling forest, a well-tended garden, and a quaint country house; the second includes a cozy guest house and a luxurious swimming pool. This division presents the siblings with several options: they could each utilize their portion for personal enjoyment, opt to sell their shares independently on the market, selling the entire estate as one entity or through a combinatorial auction.

The dilemma intensifies as it becomes clear that the market assigns a higher value to the estate when it is sold as a whole. This increased valuation could be attributed to various

*To whom it may concern, this paper is unfinished and a work in progress. We rushed it for presentation at the workshop. Any help and suggestions for future directions are welcome. We would like to thank the Composable Foundation for funding this research.

†Email: brunomazorra@gmail.com
‡Email: matheusventuryne@gmail.com

factors, such as the estate's historical significance or the potential for development projects like hotel construction. The central question that emerges from this scenario is how the siblings should proceed with the sale to optimize the estate's value. Furthermore, in the event that they decide to sell the estate as a single unit, another critical consideration is how they should divide the resulting profits.

This real estate example serves as an analog to a similar challenge found in the realm of blockchain technology, specifically within the concept of cross-domain Miner Extractable Value (MEV) between different blockchains (L1s) and roll-ups (L2s). In the blockchain scenario, m block proposals are poised to construct the next block in a chain. As introduced in [Oba+21], these blocks often contain synergies, such as the ability for third-party agents to leverage price discrepancies across Decentralized Exchanges (DEXs) in different domains. The question then arises: How can these block proposals be optimally sold to third parties and which mechanism ensures efficient blockspace allocation? In acknowledging the fragmentation of liquidity and applications across L2s within blockchain technology, the absence of protocols facilitating synchronous composability underscores a critical motivation. As the ecosystem evolves, the imperative for such protocols becomes increasingly apparent. Enabling seamless interaction across L2 environments, while harnessing their scalability benefits, stands as a pivotal step towards realizing efficient execution akin to Layer 1s. This motivation drives the ongoing search for innovative solutions that connect fragmented Layer 2 environments, fostering efficient execution similar to Layer 1. Different companies, such as Espresso[1], Astria[2], and NodeKit[3], are building a marketplace for buying and selling the right to sequence bundles of blocks on Ethereum and its L2s such as Optimisms[4] and Arbitrium[5]. These marketplaces are known as shared sequencing marketplaces. To the date, no formal analysis on shared sequencing marketplaces has been made.

At [Col23] conference, Ben Fisch introduced the challenge of allocating resources when dealing with complementary goods produced by different entities. He illustrated this with an example involving two companies: Company A makes right shoes, and Company B makes left shoes. Individually, a right or left shoe has value, but together, as a pair, their value significantly increases, suggesting they should be sold as a bundle. The dilemma arises in determining how much each company should be paid when the shoes are sold as a pair since the counterfactual revenue of selling the shoes separately is unknown. Moreover, there's a risk that the companies may misreport their separate selling potential to inflate their share of the bundle's revenue.

To address this, Ben Fisch discussed a combinatorial first-price auction, where buyers can bid on individual items (right shoe A, left shoe B) or the bundle (both A and B). If the bundle's highest bid exceeds the sum of the highest bids for the individual items, the bundle is sold; otherwise, the items are sold separately. A key issue is that, when selling the bundle, the payment to each company is based on the highest bid for their item, incentivizing them to place shill bids to overvalue their product and thus secure a larger share of the revenue. This strategy, however, undermines the fairness and efficiency of the auction. To remove the incentives from shill bidding, Fisch proposes the following auction. Agents report their bids for items A and B and for the bundle. With probability p the auction is a simultaneous first-price auction for both items (the two items are sold separately with two separate first-price auctions), and with probability $1-p$, the items are sold through a "grand bundle auction" (both items are sold as a bundle through a first-price auction). Given v_A, v_B and R be the

[1] https://www.espressosys.com/
[2] https://www.astria.org/
[3] https://www.nodekit.xyz/
[4] https://optimism.io/
[5] https://arbitrum.io/

highest bids of the first item, second item, and bundle auction respectively. The auction pays the sellers of A, the amount $(1-\beta)v_A$ in case that the item is sold through the separate auction and $(R+v_A-v_B)/2$ if the items are sold through the grand bundle auction. In case the block proposals of both chains have zero valuation of their own block, then a simple analysis shows that both agents do not have incentives to Shill bid as long as $\beta > \frac{1-p}{2p}$. Another example of Shill-Proof mechanism is selling both items in simultaneous first-price auctions or selling the items through two simultaneous Dutch auctions.

The auctioneer's challenge extends beyond ensuring individual rationality and incentive compatibility. They must also design an auction mechanism that remains competitive against each agent's potential to sell their items through separate auctions. In other words, the mechanism should, on average, generate more revenue than the sellers would expect to earn by auctioning their items individually. This paper explores this issue under the assumption that the seller's expected value, r_i, is observed either in a simultaneous auction or extracted by the seller through sequencing the block themselves. We investigate this question in the contexts of just-in-time auctions, where the auction happens at the moment that both buyers and sellers observe their value over the blocks, and slot auctions, where sellers sell the right to construct the block before the time when both buyers and sellers realize their value of the block, as introduced in [Ma23]. In this paper, we formalize shared sequencing marketplaces through mechanism design in the context of combinatorial just-in-time auctions and slot-auctions.

1.1 Organization of the paper

In section 2 we introduce the model of the Shared sequencing marketplace and discuss important background concepts such as mechanism design, superadditive valuations, and Bayesian price of anarchy. In section 2, we explore Braess's paradox within the context of shared sequencing, showing how allowing buyers to express preferences over bundles can paradoxically reduce overall welfare. In section 4.1, we analyze and compute bounds for the Price of Anarchy of some natural just-in-time combinatorial auctions, highlighting the inefficiencies introduced by non-shill-proof mechanisms. In section 4.1.2, we prove a negative result on just-in-time auctions when agents have single-minded valuations over the items and sellers have reserve values over their own items, demonstrating the inefficiencies and challenges in achieving optimal outcomes. Finally, in section 4.2, we present how slot auctions, where sellers commit to future sales before realizing the value, can circumvent these negative results, offering a promising approach to improve the overall outcome.

1.2 Related Work

Miner Extractable Value (MEV) is a concept that has gained significant attention in the blockchain community, primarily due to its implications on the stability and fairness of decentralized exchanges (DEXs). The seminal paper [Dai+20] laid the foundation for understanding MEV. This work explores how miners can exploit transaction ordering to extract additional value, beyond the standard block rewards and gas fees, by including, excluding, or reordering transactions.

Expanding on this foundational work, [Oba+21] explores the complexities of cross-domain MEV. This work formalizes the extraction of value across multiple blockchain platforms, such as Ethereum and Polygon, as well as centralized domains like CEXs. By providing a comprehensive framework for understanding and analyzing cross-domain arbitrage opportunities, the authors significantly broaden the scope of MEV research. Related to our work, [Oba+21] introduces the cost of collusion by block proposers of different domains. More specifically, cross-domain MEV is the maximum value that can be extracted when a unique agent can synchronize all the chains and optimally censor, reorder, or include transactions that maximize its payoff. However, in general, block proposers of different domains are different identities. This introduces some friction among the sequencers to extract the cross-domain MEV since they have to spin some infrastructure that will run mechanisms allowing block proposers to coordinate on which block to build. This can lead to operational costs, such as the cost of running the necessary infrastructure like the MEV-relay in the MEV-Boost. Moreover, in this paper, we argue that there is an additional cost — the cost of agents being strategic, known as the Price of Anarchy [KP99]. This refers to the ability of agents to be strategic to maximize their payoff, and its consequence to the overall welfare, such as block proposers delaying their blocks to increase profits, also known as timing games.

Other studies have focused on quantifying cross-domain MEV and the specific arbitrage opportunities between centralized exchanges (CEXs) and decentralized exchanges (DEXs) [SMC23; MR24]. The papers provides an empirical analysis of arbitrage opportunities by extracting and analyzing Uniswap data across multiple domains. The study offers insights into the prevalence and profitability of cross-domain arbitrage, shedding light on the practical aspects of MEV exploitation and its impact on market efficiency.

In the broader context of MEV research, [McM23] provides a comprehensive survey of existing studies, categorizing various types of MEV and summarizing the current understanding of its implications on blockchain security and efficiency. This paper serves as a critical reference point for researchers, consolidating diverse strands of MEV research and highlighting areas requiring further exploration.

Related to our work, the study on shared sequencing [MS23] examines the economic implications of shared sequencing for cross-domain arbitrage. Their game-theoretic model compares the performance of shared sequencing versus separate sequencing, revealing that while shared sequencing can enhance the realization of arbitrage opportunities, it also introduces more wasteful latency competition. The findings indicate that the benefits of shared sequencing in terms of revenue and efficiency depend significantly on the transaction ordering rules applied and the potential arbitrage values realized.

In this paper, we will focus the design of such transaction or block production rules through the lens of auction theory and mechanism design. Auctions have long been utilized as an efficient mechanism for trading goods, particularly when buyers' and sellers' valuations are private, and prices need to be discovered. In single-item auctions, such as those detailed in *Auction Theory* by Vijay Krishna [Kri09], these mechanisms offer the advantage of quickly reaching equilibrium prices and improving overall market efficiency. The Federal Communications Commission (FCC) spectrum auctions serve as a notable example of large-scale auction applications that raised substantial revenue and allocated licenses efficiently. Peter Cramton's study on the FCC auctions demonstrates how the simultaneous ascending auction format allowed bidders to flexibly adjust their strategies, leading to efficient license allocations and substantial revenue generation for the U.S. Treasury [Cra98].

Recent studies have examined the Bayesian Price of Anarchy (BPoA) in the context of simultaneous first-price auctions, particularly when agents have subadditive valuations. In [ST13; RST17; Fel+20] the smooth framework provides a comprehensive analysis, showing that such auctions can achieve good efficiency bounds under certain conditions [ST13]. More specifically, they show that when m items are sold through simultaneous first-price auction to agents with subadditive valuations, the Bayesian price of anarchy is upper bounded by $\frac{e}{e-1}$. However, negative results have also been highlighted [Fei+15], especially when agents have complementarities. These studies underscore the challenges inherent in auction design, where the presence of complementarities among items can significantly impact the efficiency and feasibility of achieving optimal outcomes. Specifically, they show that the price of anarchy of selling m items through simulatenous m first-price auctions with general valuations has Bayesian price of anarchy $\Theta(m)$.

When moving beyond simultaneous single-item auctions, combinatorial auctions become essential, particularly for goods with complementarities or substitutabilities. In this context, combinatorial auctions allow bidders to place bids on bundles of items, which can significantly enhance the overall efficiency and social welfare. As discussed in the lecture on combinatorial auctions from the *Algorithmic Game Theory* book [Rou10], these auctions address the complexities of expressing preferences for combinations of items rather than individual ones. However, despite these positive aspects, there are significant challenges in achieving optimal outcomes in combinatorial auctions, especially when considering general valuations. The optimal allocation problem in combinatorial auctions is notably NP-hard, especially when considering single-minded bidders [Rou10]. More specifically, in [Rou10] it is shown that for single-minded valuations, there are no polynomial time algorithms that can asymptotically achieve better than a $\sqrt{m}$-approximation. This negative result indicates the difficulty of finding efficient solutions in the most general cases of combinatorial auctions. Nevertheless, in certain restricted cases where valuations belong to specific spaces, mechanisms can perform quite well, achieving much better approximations and sometimes even optimal solutions. This highlights the importance of understanding the structure of valuations.

In [LB10] greedy algorithms have also been studied in the context of combinatorial auctions. These algorithms offer practical approaches to bounding the Bayesian Price of Anarchy (BPoA) in settings where agents have general valuations. In the paper, the authors show that under mild conditions, an allocation algorithm that is $c-$approximate has Bayesian price of anarchy upper bounded by $c/(1-e^{-c})$. In particular, if the set of valuations are restricted to a subspace where the allocation problem is solvable in polynomial time, then the mechanism has Bayesian price of anarchy $\frac{e}{e-1}$.

Finally, combinatorial auctions in two-sided markets represent another layer of complexity where both buyers and sellers submit bids for bundles of items. This setup is common in business-to-business (B2B) marketplaces and requires sophisticated mechanisms to ensure efficient matchings. The study by [XSW05] on solving the combinatorial double auction problem provides valuable insights and methods for addressing these challenges in terms of computational complexity. Two other interesting works on agents buying (full) complementary goods are [HKL24; CP24]. [HKL24] focus on characterizes mechanism for consumer surplus maximization when a buyer is buying from a group perfect complementary goods. [CP24] on the other hand, explores a trading problem where a buyer needs to acquire an aggregate resource that is fragmented among many owners, with each fragment being a perfect complement to the others.

1.3 Our contributions

In this paper, we make the following contributions:

1. **Formalization of mechanisms for shared sequencing**: We define the structure of a market where participants both buy and sell the right sequencing blocks.

2. **Comparing Auction Types**: We prove that under some conditions on the valuation distributions on buyers, simultaneous first-price auction can have more revenue and welfare than combinatorial auctions with proportional payment rule and Shapley payment rule.

3. **Impossibility result**: We establish that if m sequencers possess non-zero reserve valuations for items that are not publicly known, the Bayesian price of anarchy—in terms of both welfare and revenue—will be at least proportional to the number of items, $\Omega(m^{1/3})$, for mechanisms that are ex-interim individually rational and maintain weak budget-balance (even when restricting to very small number of buyers when the allocation problem is not NP).

4. **Improving Auction Efficiency**: Finally, we explore how this negative result could be circumvented by removing ex-interim individually rational to ex-ante individually rationality by using slot auctions.

2 Model

There are m sellers, each with a distinct item $i \in [m] = \{1, 2, \ldots, m\}$ with a valuation r_i which can be either public or private. There n buyers, each with a valuation $v_i(A) \in \mathbb{R}_{\geq 0}$, $i \in [n]$ for the bundle $A \subseteq [m]$. Items present complementaries which is modeled by assuming buyers valuations are superadditive:

Definition 3 (Superadditive)**.** *A Buyer valuation v is superadditive if for any disjoint sets $A \subseteq [m]$ and $B \subseteq [m]$*

$$v(A \cup B) \geq v(A) + v(B)$$

A single-minded valuation is a type of valuation function that models goods that are perfect complements. Formally, let $M = [m]$ be the set of items. A bidder i with a single-minded valuation is interested in only one specific bundle $S_i \in 2^M$ and has a value v_i for that bundle. The valuation function $v_i : 2^M \to \mathbb{R}_{\geq 0}$ is defined as follows:

$$v_i(T) = \begin{cases} v_i & \text{if } S_i \subseteq T, \\ 0 & \text{otherwise.} \end{cases}$$

In other words, bidder i values any bundle $T \subseteq M$ at v_i if and only if $S_i \subseteq T$; otherwise, the value is 0.

An example of cross-domain MEV is as follows: Suppose an asset is listed on two DEXs, such as Radyium (in Solana) and Osmosis (in Cosmos), and these DEXs quote different prices. This allows an agent to buy on one exchange and sell on the other, thereby making a net profit. In this scenario, the agent places no value on having access to just one chain, since it cannot liquidate its position and realize profits. However, the value of being able to operate on both chains is equivalent to the value of the arbitrage opportunity.

An *auction mechanism* to allocate the items and extract (and give) payments induces an *allocation rule* x, a *payment rule* p and a transfer rule t that determine the allocation of the set of items $[m]$, a payment map p that determinates how much each buyer must pay to the auctioneer and a transfer map t that consist of positive transfers of money to each seller from the auctioneer. More formally, given a set of reported valuations functions $\mathbf{b} = (b_1, ..., b_n)$ and reserve prices $\mathbf{r} = (r_1, ..., r_m)$, $x_{j,S}(\mathbf{b}, \mathbf{r})$ is the probability that the bundle S is allocated to the player j, $p_j(\mathbf{b}, \mathbf{r})$ is the expected payment that agent j has to make to the auctioneer and $t_i(\mathbf{b}, \mathbf{r})$ are the transfers made from the auctioneer to seller i. In general, we will denote by $\mathbf{b} = (b_1, ..., b_n)$.

Examples of mechanisms

1. **Simultaneous first-price auctions with direct transfers** is defined as selling the independent item $i = 1, ..., m$ simultaneously via a sealed-bid first-price auction and transfer the highest bid b_j made in the auction j to the seller j. In the blockchain context, this will translate to, for each domain, running a simultaneous proposal-builder separation instance.

2. **Combinatorial first-price auction with proportional transfer rule** is defined by first running a combinatorial first-price auction and, when allocating a bundle of more than one item, pay the agent the proportion the bundles revenue taking into account the highest bid observe for his item isolated. More formally, the combinatorial first-price auction consists of the allocation $\mathcal{A}$ that comes from solving the allocation problem with m items and reported valuations $v_1, ..., v_n$ and $r_1, ..., r_m$. Then, if a bundle S is sold, pay to each $i \in S$ the amount $t_i = \frac{\max\{\max v_j(\{i\}), r_i\}}{\sum_{k=1}^{m} \max\{\max v_j(\{k\}), r_k\}} \max v_j(S)$.

3. **Combinatorial first-price auction with Shapley transfer rule** Consists of running a combinatorial first-price auction for every set S and paying each seller the marginal contribution of its item $\phi_i(S)$ where ϕ is the Shapley value.

In this paper, we will assume that both buyers and sellers are risk-neutral agents and have quasi-linear utilities. That is, if agents report $(\mathbf{b}, \mathbf{r})$ a buyer j with valuation v_j has utility $u_j(b_j, \mathbf{b}_{-j}, \mathbf{r}) = \sum_{S \subseteq [m]} x_{j,S}(\mathbf{b}, \mathbf{r}) v_j(S) - p_j(\mathbf{b}, \mathbf{r})$ and a seller i with private value r_i has utility $u_i(r_i, \mathbf{b}, \mathbf{r}_{-i}) = x_{i,\{i\}}(\mathbf{b}, \mathbf{r}) r_i + t_i(\mathbf{b}, \mathbf{r})$.

We will assume that buyers drawn their private valuations from a distributions $\mathbf{F} = \prod_{j=1}^{n} F_j$ and sellers drawn their valuations from a distribution $\mathbf{G} = \prod_{i=1}^{m} G_i$. In our analysis, we will focus on a set of mechanism that will hold some desiderata constraints, such as ex-interim individually rationality, no-deficit and incentive-compatibility.

A mechanism is *ex-interim individually rational* if all agents are at least as good as not participating in the mechanism conditioned to knowing their private valuation. More formally, a mechanism is *ex-interim individually rational* for sellers if: for every seller i with private valuation r_i, then $u_i(r_i, \mathbf{b}, \mathbf{r}_{-i}) \geq r_i$ for all reports $\mathbf{b}, \mathbf{r}_{-i}$. And is *ex-interim individually rational for buyers* if: for every buyer j with private valuation function v_j holds $u_j(v_j, \mathbf{b}_{-j}, \mathbf{r}) \geq 0$ for all reports $\mathbf{b}_{-j}, \mathbf{r}$.

A mechanism is *weak budget-balance* if, for every type vector profile $(\mathbf{b}, \mathbf{r})$, in expectation over the randomization of the mechanism, the auctioneer does not lose money. More formally,

$$\sum_{j=1}^{n} p_j(\mathbf{b}, \mathbf{r}) \geq \sum_{i=1}^{m} t_i(\mathbf{b}, \mathbf{r}).$$

In other words, the mechanism is no-deficit if the auctioneer does not have to provide extra cash to run the mechanism. This property is fundamental since it is not sustainable for auctioneers to incur losses in equilibrium by subsidizing trades, even though this is the current state of MEV-Boost.

In a permissionless set up it is not possible in general to identify if two different reported identities are held by the same agent. For this reason, some agents can exploit this weakness on the combinatorial auctions to increase their payoff unilaterally.

We say that the mechanism is *Shill-Proof* if no seller has incentives to shill bid ex-post in order to increase its payoff. More formally, for every bid vector profiles $\mathbf{b}$ with n components and bid vector profile $\tilde{r}$ with k components it holds

$$t_i(\mathbf{b}, \tilde{r}) - \sum_{i=n+1}^{n+k} p_i(\mathbf{b}, \tilde{r}) \leq t_i(\mathbf{b}).$$

For example, sealed-bid first-price auctions and dutch auctions are Shill-proof.

A **strategy** for a participant in a mechanism is the action that specifies the participant's moves or decisions at every possible point in the game, based on their available information. Formally, in our context, a strategy consists of a (potentially randomized) map

$$s_i : T_i \to A$$

where A is the set of all possible actions that can be realized in the game, such as reporting valuations or reserves, shill bidding or Sybil bidding.

A **Bayesian Nash Equilibrium (BNE)** extends the concept of Nash Equilibrium to games where players have incomplete information about each other's types such as private valuations. In a BNE, each player's strategy must be a best response to their beliefs about the other players' strategies. Formally, a strategy profile $(s_1, s_2, \dots, s_n)$ is a BNE if for every player i, given the strategies s_{-i} of other players, the strategy s_i maximizes player i's expected utility, based on their belief about the other players' valuation conditioned on their own valuation:

$$s_i \in \arg\max_{s_i'} \mathbb{E}_{t_{-i}|t_i} \left[u_i(s_i'(t_i), s_{-i}(t_{-i}), t_i) \right]$$

where t_i and t_{-i} represent the types of player i and all other players respectively, u_i is the utility function of player i, and the expectation is taken over the types of other players conditioned on player i's type t_i.

This equilibrium concept is fundamental in mechanism design, especially for analyzing and designing mechanisms that lead to desirable outcomes even when participants act strategically based on their private information.

Mechanism Design Objectives: In mechanism design literature, in general, its considered the problem of maximizing two primary objectives in expectation at BNE: utilitarian welfare and revenue. The utilitarian welfare is given by the overall utility of the outcomes, including the payments made to or by the auctioneer.

Formally, given a distribution of valuations $\mathbf{V}$, and agents playing strategies $s = (s_1, ..., s_n)$ the welfare is defined as

$$\mathcal{W}(M, s, \mathbf{F}) = \underbrace{\mathbb{E}_{(v,r)} \left[\sum_{j=1}^{n} u_i^b(s(v,r)) \right]}_{= \text{Buyers' utility}} + \underbrace{\mathbb{E}_{(v,r)} \left[\sum_{j=1}^{n} u_i^s(s(v,r)) \right]}_{= \text{Sellers' utility}} + \underbrace{\mathbb{E}_{(v,r)} \left[\sum_{j=1}^{n} p_j(s(v,r)) - \sum_{j=1}^{m} t_i(s(v,r)) \right]}_{= \text{Auctioneer utility}}$$

where (v, r) are drawn from $\mathbf{F} \times \mathbf{G}$. In our context, we consider the total revenue of the mechanism M as the total sellers' utility. Formally, the revenue of a mechanism M under s and $\mathbf{V}$ as

$$\text{Rev}(M, s, \mathbf{V}) = \mathbb{E}_{(v,r)} \left[\sum_{j=1}^{n} u_i^s(s(v,r)) \right].$$

For both welfare and revenue, we will suppress $\mathbf{F}$ and s when context makes the distributions of bids and values clear.

Our welfare benchmark is the outcome that gives that allocation that maximizes welfare. This can be implemented via the Vickrey-Clarke-Groves (VCG) mechanism. And so, we seek to approximate optimal feasible allocation. To measure the efficiency and the optimality of mechanisms we will use the Price of anarchy and the Price of stability for welfare and revenue. The *Bayesian Price of anarchy*, PoA for short, over a set feasible value distributions $\mathcal{F}$ is defined as

$$\text{PoA}_{\mathcal{W}}(M, \mathcal{F}) := \max_{\mathbf{V} \in \mathcal{F}} \max_{s \in \text{BNE}(M, \mathbf{V})} \frac{\text{OPT}(\mathbf{V})}{\mathcal{W}(M, s, \mathbf{V})},$$

where $\mathrm{BNE}(M, \mathbf{V})$ is the set of BNE for M under value distribution $\mathbf{V}$.

Similarly, the *Bayesian Price of Stability*, PoS for short, over a set feasible value distributions $\mathcal{F}$ is defined as

$$\mathrm{PoS}_{\mathcal{W}}(M, \mathcal{F}) := \max_{\mathbf{V} \in \mathcal{F}} \min_{s \in \mathrm{BNE}(M, \mathbf{V})} \frac{\mathrm{OPT}(\mathbf{V})}{\mathcal{W}(M, s, \mathbf{V})}.$$

Now, $\mathrm{OPT}_R(\mathbf{F})$ be the revenue-optimal mechanism for any valuation distribution $\mathbf{F}$. For revenue, we will measure the performance, in terms of revenue, of a mechanism by its worst-case approximation ratio:

$$\mathrm{PoA}_R(M, \mathcal{F}) = \max_{\mathbf{V} \in \mathcal{F}} \max_{s \in \mathrm{BNE}(M, \mathbf{V})} \frac{\mathrm{Rev}(\mathrm{OPT}_R(\mathbf{V}))}{\mathrm{Rev}(M, s, \mathbf{V})}.$$

3.1 Assumptions

In this section, we lay out a set of fundamental assumption that we will use through the paper.

Assumption 1. *Agents have private independent valuations. The distributions from which agents' valuations are drawn are common knowledge. All agents have quasi-linear utilities and are risk-neutral.*

These assumptions, while standard in auction theory [Kri09], serves as a practical framework to understand the behaviors of agents within the agents bidding in blocks. Despite its departure from absolute realism, it offers a necessary starting point for theoretical investigations. Although prior-free mechanisms might be preferred in ideal settings, such mechanisms face inherent limitations, as evidenced by impossibility results even in simple bilateral trading scenarios. Moreover, continuous repetition of auctions over time enables one to infer of information regarding agents' valuations, mitigating the constraints imposed by common knowledge assumptions.

Assumption 2. *We differentiate between two type of sellers, active and passive. Active sellers have valuations over their own blocks, optimized through sequencing. Passive sellers, if they choose, can sell their blocks via separate auctions and observe the resultant revenue but do not have an intrinsic valuation over they own block.*

This assumption clarifies the roles and strategic options available to sellers operating within the blockchain market. Active sellers, equipped with information and expertise in identifying and capitalizing on Miner Extractable Value opportunities, optimize the sequencing of their blocks. They may also choose to sell their blocks through separate auctions, leveraging their knowledge and capabilities to maximize returns. In contrast, passive sellers lack the ability to extract additional value from their blocks and can only sell them through separate auctions. Monitoring the revenue generated from these auctions offers passive sellers valuable insights into the market's valuation of their blocks. This assumption is plausible, especially considering the potential for L2 sequencers to implement analogous mechanisms akin to MEV-Boost for their own roll-ups.

Assumption 3. *Agents realize their values at the moment they build their blocks. Before that time, they know the expected value they can make.*

In general, arbitrage opportunities and other MEV opportunities, such as liquidations, depend fundamentally on the current state of the blockchain and the transactions in the mempool. That is the main reason we assume that agents realize their value at the moment the block must be built. While certain nuances, such as private order flow, may exist, for the sake of simplicity, this assumption focuses on the moment of block construction as the critical juncture for value realization. The second assumption here might be quite strong because agents don't have a fixed expected value for their earnings. Instead, they rely on signals indicating how much MEV they might get in the future. However, these signals might not always match reality perfectly. It's similar to how in oil auctions, predictions about the value of oil depend on what other buyers signals. So, a model that takes into account these interdependencies might be more appropriate here i.e. a model that takes into account interdependent valuations of the buyers and sellers. But exploring this is out of the scope of this paper.

Assumption 4. *There exist credible commitments that sellers must adhere to. More specifically, the validators and sequencers of a future blocks are known in advance an can commit to delegate the right to construct blocks to third parties.*

Smart contracts facilitate the establishment of credible commitments within the blockchain environment, ensuring the enforceability of agreements between parties. For instance, block proposers may commit to future block production rights, with mechanisms in place to deter contract breaches, either through slashing conditions or protocol-level enforcement.

Assumption 5. *Buyers and sellers do not form coalitions; each agent acts strategically and independently.*

Although the absence of coalitions may appear idealized, it simplifies the analysis by focusing on individual agent behavior. This is a standard assumption in mechanism design. In addition, it is important to note that the formation of coalitions in anonymous environments can be quite expensive [MDP23].

4 Cross domain PBS

Proposal Builder Separation (PBS) refers to the mechanism or protocol that allows block proposers (or sequencers) to delegate the right to construct the next block to specialized third parties called builders. The current implementation of this mechanism is known as MEV-Boost and involve selling the right to construct the block through an English auction, facilitated by an intermediary called Relay, which acts as an auctioneer. However, blocks from different domains may exhibit synergies, prompting the possibility of designing mechanisms that allow builders to bid for the right to construct bundles of blocks from various domains. In this section, we will first examine two simple mechanisms and their inefficiencies. Subsequently, we will demonstrate that these inefficiencies are, in fact, general across mechanisms that are ex-interim individually rational and weak budget-balanced. To circumvent these negative outcomes, in the final section, we explore the potential of altering the ex-interim individually rationality constraint to ex-ante individually rationality.

4.1 Just-in-time auctions

4.1.1 Braess's paradox for just-in-time auctions

Combinatorial just-in-time auctions consists of the set of mechanisms that allow different sellers to outsource the right to build the block through third parties at the same time of the block execution. We will formalize the notion of Braess's paradox for combinatorial just-in-time auctions. In summary, the paradox consists as even all parties have greater ability to express their preference on buying combination of blocks, or expressing reserve prices, the revenue, and the welfare can decrease compared to simple auctions such as simultaneous first-price auctions. In this section, we will assume that all block proposals are passive, and we will normalize the block reward to $r_1 = ... = r_m = 0$. First, let us analyze some simple mechanisms.

A first mechanism would be to run m simultaneous instances of proposal-builder-separation. More formally, each block is sold through a simultaneous auction (here we will assume sealed bid first-price auction). In [Fei+15], study the price of anarchy of simultaneous first-price auctions with buyers with different degrees of complementarities. From their results, the following is deduced.

Proposition 4.1. *([Fei+15], Appendix H) The Bayesian Price of anarchy of selling all blocks simultaneously through first-price auctions* $\Theta(m)$. *Moreover the Bayesian price of anarchy for* $m = 2$ *is at most* $1/(1 - \log 2) \approx 3.25$.

However, no lower bound is provided in [Fei+15] when there are exactly two items $m = 2$. A first non-trivial lower bound can be deduced by considering two family of single-minded bidders just valuing the items isolated with the distributions provided in [JL23]. This would imply that the price of anarchy is at least $1/(1 - 1/e^2) \approx 1.15$. We can slightly improve this lower bound.

Proposition 4.2. *Simultaneous first-price auctions have a Bayesian price of anarchy of at least* ≈ 1.36.[6]

Using a previous construction, the following result follows.

Corollary 4.3. *The revenue price of anarchy of simultaneous auctions with two items is at least* 1.36.

Two natural questions arise from previous results:

- Are there mechanisms that are better in terms of welfare?
- A mechanism adding the option to sell the items as bundles increases the revenue in equilibrium?

The first intuition tells that allowing buyers to express their preferences over bundles increases the expected revenue of sellers and the overall welfare, since buyers do not have to price in the risk of not buying one of the items, allowing them to bid more over the bundles. However, we claim that in some scenerios if the mechanism is not (ex-ante) Shill-proof, the welfare and the revenue in equilibrium can decrease. This have a similar idea to Braess's Paradox.

Braess's Paradox is a counterintuitive scenario in road networks where adding extra capacity to the network, such as an additional road, can lead to increased overall congestion rather than alleviating it. This phenomenon occurs under certain conditions when drivers choose their routes selfishly, aiming to minimize their own travel time without regard to the overall effect on the network.

The paradox unfolds as follows: In a road network, each driver selects the quickest route from their starting point to their destination, considering current traffic. When a new road

[6] For this proof we used a computer for computing some intregrals.

is introduced, it might initially appear to offer a quicker route. However, if enough drivers switch to the new road thinking it will be faster, it can become congested. Surprisingly, this shift can increase the overall congestion on the network, resulting in longer travel times for everyone compared to the scenario without the new road.

The fundamental insight of Braess's Paradox is that individual optimization does not necessarily lead to global optimization; personal benefits can sometimes lead to worse outcomes, even when we allow individual users to increase their preference expression.

We will see that a similar principle applies to the proportional and Shapley payment rules.

If both items were sold by the same identity, i.e. the block proposal of next block was the same identity on both chains, we could sell both items through a combinatorial first-price auction. By [LB10], is known that the price of anarchy is at most $1/(1-e) \approx 1.58$. However, in general the sellers are different rational agents who want to maximize their revenue, therefore, we need a way of distributing the revenue made in the auction. Therefore, the current design problem is how much do we pay to each seller? A first proposal made by Ben [Col23] is the following payment rule (the proportional payment rule introduced in 2): Let v_A, v_B and R be the highest bids on the item first, second and the bundle respectively. Then, in case that $R > v_A + v_B$, one option is to pay each seller $\frac{v_A}{v_A+v_B}R$ and $\frac{v_B}{v_A+v_B}R$, and otherwise pay v_A and v_B. A second proposal made in [Col23] is the Shapley value payment rule. Where if the bundle with total revenue R, the sellers' payments are $(R+v_A-v_B)/2$ and $(R-v_A+v_B)/2$ respectively. However, these mechanisms are not shill-proof since one agent can unilaterally shill bid and increase its payoff since its payoff is increasing over the highest bid of the independent item. For example, assume that we are selling the items through a combinatorial auction with proportional payment rule and with bids R, v_A, and v_B.

Then the seller of first item could report $\tilde{v}_A > v_A$ such that $R \geq \tilde{v}_A + v_B$, and increase its revenue. The fact that the mechanism is not shill-proof is not bad per se, since the welfare in equilibrium could be close to optimal, however we will see that this is not true for every buyers distribution $\mathbf{F}$. Actually, we will see that in the worst-case welfare in equilibrium is arbitrarily bad compared to the optimal one.

Proposition 4.4. *The Bayesian price of anarchy of the combinatorial first-price auction with proportional payments is $+\infty$.*

Using similar arguments as the previous lemma, we can prove that the Price of anarchy of the Shapley value payment rule mechanism is $+\infty$.

A similar thing happens with the revenue. For example, suppose that there are two single-minded bidders $i = 1, 2$ with valuations over the bundle drawn i.i.d. from a distribution with cdf F with support in $[1, M]$ and density function f. Let $R(r)$ be the total revenue in symmetric Nash equilibrium of selling both blocks through a bundle auction with reserve price r. And let $\beta(v, r, n)$ is the strategy of the unique Nash equilibrium with two players of the bundle auction with reserve price r.

Lemma 4.5. *The set of strategies $(\lambda\beta(v, 0, 2), (1-\lambda)\beta(v, 0, 2))$ with $\lambda \in [0, 1]$ are pure Nash equilibrium of the simultaneous auctions. In particular, the sum of expected revenues is $R(1)$.*

Proposition 4.6. *If $rR(r)$ is increasing close to 1 (i.e. $R(1) > -R'(1)$), the pure Nash equilibrium with $v_A + v_B \leq M$ are of the form $(q/2, q/2, \beta(\cdot, q, 2), \beta(\cdot, q, 2))$ with $q \in (1, M)$.*

Proof. First, observe that single-minded bidders do not have incentives to bid in the items separately and so bid in the bundle. If sellers shill bid v_A and v_B, then buyers interpret the auctions as a first-price auction with reserve price $r = v_A + v_B$, and so, the unique Nash equilibrium is given by $\beta(\cdot, r, 2)$. In this case, the total expected revenue is:

$$R(r) = 2rF(r)[1-F(r)] + 2\int_r^M (1-F(y))yf(y)dy$$

and $R'(r) = 2F(r)[1 - F(r) - rf(r)]$. If agent A and B report shill bids v_A and v_B, their expected payoffs are $\frac{v_A}{v_A+v_B}R(v_A+v_B)$ and $\frac{v_B}{v_A+v_B}R(v_A+v_B)$ respectively. And so, the sellers side interprets this game as pro-rata game. The symmetry on the sellers side of the pure Nash equilibrium follows similarly to the argument on concave pro-rata games [Joh+23] and with similar arguments, can be shown that are of the form $(q^\star/2, q^\star/2, \beta(\cdot, q^\star/2, 2), \beta(\cdot, q^\star/2, 2))$ with $q^\star \in \operatorname{argmax}_{q\geq 0} qR(q)$. Clearly, $qR(q)$ is increasing for $q \in [0,1)$ since $F(x) = 0$ for $x \in [0,1)$, $MR(M) = 0$ since $f(M) = 0$ and by assumption $qR(q)$ is locally increasing in 1, and so, $1, M \notin \operatorname{argmax}_{q\geq 0} qR(q)$. $\square$

Theorem 4.7. *The expected revenue of the proportional payment rule is not necessarily higher than the expected revenue of the simultaneous first-price auctions in equilibrium.*

As an example, consider the regular distribution $F(x) = x - 1$ with $x \in [0,1]$. The total revenue with reserve price r is

$$R(r) = \begin{cases} -\frac{4}{3}r^3 + 4r^2 - 4r + \frac{8}{3}, \text{ if } r \in (1,2), \\ 4/3 \text{ if } r \in [0,1], \\ 0 \text{ otherwise.} \end{cases} \tag{1}$$

The argmax of $qR(q)$ is given in $q \approx 1.4554$, and so the total revenue in equilibrium is $R(1.4554) = 1.2074 < 4/3 = R(1)$.

This can be further generalized when there are m sellers.

Theorem 4.8. Let $\mathcal{F}$ be the set of distribution that that are composed by single-minded buyers that only value the grand-bundle. Then $\text{PoA}_{\mathcal{W}}(PT, \{\mathbf{F}\}) = +\infty$. In case we restrict to the set $\mathcal{F}_S \subseteq \mathcal{F}$ where agents have i.i.d valuations and the number of buyers is at least $n \geq 2$, then

$$\text{PoA}_R(PT, \mathcal{F}_S) = O(1/m)\text{PoS}_R(SS, \mathcal{F}_S).$$

In other words, when sellers are strategic and sell their blocks just-in-time, the ability of buyers to express their preference over bundles does not necessarily increase the revenue and the welfare of the auction. In next section, we will see the limits of mechanisms when sellers are strategic. On the other hand, what can we say when the block proposals are passive and do not have access to side-auctions? For example, consider a combinatorial first-price auction and then transfer uniformly randomly independently of the bids observed in the mechanism. Moreover, in case we know the identity of the buyers, we could even transfer to each seller its marginal contribution that their item makes to the revenue of the auction.

Proposition 4.9. *When sellers are passive and there is no side-auctions, or all the sellers are the same identity and sellers can not shill bid, then running a combinatorial first-price auction with random transfers is shill-proof and has Bayesian price of anarchy $e/(e-1)$.*

This results follow by the main theorem of [LB10] and the assumptions on the sellers. Currently, it is realistic to assume that validators are not sophisticated enough to reorder and insert transactions to increase their payoff. However, it is not realistic to assume that there are no competing auctions. That is, the auction mentioned enforces the agents to sell their item through the combinatorial auction, not allowing the block proposal to sell its own item by its own means. So the refined question is, what can we say when all block proposals are passive and have access to a side-auction? We will first start by analysing simple mechanisms such as the simultaneous first-price auctions (SF), the combinatorial first-price auction with proportional transfer rule and Shapley transfer rule. We will see that, in general, they are not better than simultaneous first-price auctions.

4.1.2 Imposibility result

We've seen that when considering the shapley and the proportional transfer rules, the mechanism does not necessarily have better welfare or revenue than the simultaneous first-price auction mechanism. In this section, we prove that when block proposals are active, then every individually rational and weak budget-balance mechanism has a high cost of coordination in terms of welfare and revenue. More formally, the revenue and welfare are upper bounded by $O(1/m)$OPT. In other words, the negative results seen in previous sections generalize for any other mechanisms holding these desiderata properties.

We are able to prove it even for the following simplified scenario: There is unique buyer single-minded valuation $v([m]) = 1$ and $v(S) = 0$ for all $S \subsetneq [m]$ and m sellers with valuations drawn from the joint distribution $\mathbf{G} = \prod_i G_i$. This scenario is similar to the one proposed in [HKL24]. However, in this case, we not impose the mechanism to pay the same to each seller share. Moreover, our objective is not to maximize the buyer's utility but rather, maximize the welfare or the seller's utility.

Since the buyers' valuation is public knowledge, we will remove it from the notation of the mechanism. In this section, focus on mechanisms that 1) are ex-interim individually rational for both the buyer and sellers, i.e. the agents have the ability drop out if they want to and 2) in expectancy, the mechanism is weak budget-balance, i.e., in expectancy the auctioneer does not incur losses.

Let's formalize this. Let (x, p, t) be a shared sequencing mechanism. If a seller have valuation r_i over its item, then there exists an action a_i such that $u_i(a_i, a_{-i}) \geq r_i$ for every action profile a_{-i}.

Now, for a given Bayes-Nash equilibrium of the mechanism $\mathcal{M} = (x, p, t)$, by the revelation principle, there is a mechanism $\mathcal{M}'$ such that is individually rational and incentive compatible, and when all agents reveal their valuation truthfully, all agents have the same expected utility as the Bayes-Nash equilibrium. Moreover, since the underlying mechanism is weak budget-balance over all type vector profiles, the mechanism $\mathcal{M}'$ is also weak budget-balance. So, from now on let us assume that the agent report their valuation truthfully to the mechanism.

Let's denote by $X_i(r_i) = \mathbb{E}_{r_{-i}}[x_i(r_i, r_{-i})]$ the probability that the item i is allocated to the seller i, when reports r_i to the mechanism and $T_i(r_i) = \mathbb{E}[t_i(r_i, r_{-i})]$ the expected transfer received when reporting r_i. If r_i is drawn from G_i, we can think of X_i as a random variable $\mathbb{E}_{r_{-i}}[x_i(G_i, r_{-i}) \mid G_i]$ and analogously with T_i. Then if an agent has value r_i, its expected utility is $X_i(r_i)r_i + T_i(r_i)$. Let Y be the event that all items are allocated to the buyer.

In this context, a mechanism is ex-interim individually rational for the buyer if

$$\Pr[Y]v([m]) - \mathbb{E}_{\mathbf{r}\sim\mathbf{G}}[p_1(\mathbf{r})] \geq 0,$$

and is individually rational for the seller with valuation r_i if $X_i(r_i)r_i + T_i(r_i) \geq r_i$. The mechanism is incentive compatible for the sellers if

$$X_i(r_i)r_i + T_i(r_i) \geq X_i(r_i')r_i + T_i(r_i') \text{ for all } r_i, r_i' \in \operatorname{supp} G_i.$$

Since the mechanism is weak budget-balance, $\mathbb{E}_{\mathbf{r}\sim\mathbf{G}}[p_1(\mathbf{r})] \geq \mathbb{E}_{\mathbf{r}\sim\mathbf{G}}[\sum_{i=1}^m t_i(\mathbf{r})]$, and so $\Pr[Y]v([m]) - \mathbb{E}_{\mathbf{r}\sim\mathbf{G}}[\sum_{i=1}^m t_i(\mathbf{r})] \geq 0$.

Theorem 4.10. There exists sellers valuations distribution $\mathbf{G} = \prod \mathbf{G}_i$ such that all Bayesian Mechanism M that are weak budget-balance, ex-interim individually rational, and incentive compatible have $\mathcal{W}(M, \mathbf{G}) \leq O(1/m^{1/3})$OPT and REV$(M, \mathbf{G}) \leq O(1/m^{1/3})OPT_R$.

Proof. Let's consider the distribution of valuations drawn by i.i.d. valuations $V_1, ..., V_n$ with $\Pr[V_i = 0] = 1 - \frac{1}{m^\gamma}$ and $\Pr[V_i = 1/m^\gamma] = \frac{1}{m^\gamma}$ with $\gamma \in (0, 1)$. Observe that $\mathbb{E}[V_i] = \frac{1}{m^{2\gamma}}$. Now, given a Bayesian mechanism, we can consider the mechanism that consists of first picking a random permutation of S_m with probability $1/m!$ and then computing the outcome of the mechanism with indexes of the agents order with respect to the random permutation. Since the random variables V_i are i.i.d. this mechanism has the same expected social-welfare and is symmetric. So, we will assume that the underlying mechanism is symmetric.

Let Y denote that random variable, such that $Y = 1$ if all items are allocated to the buyer and $Y = 0$ otherwise and let p be the probability that $Y = 1$. So, $p = \mathbb{E}[Y]$. By the feasibility constraint, for a given event w (takes into account the agents reported valuations and the internal randomization of the mechanism) if $Y(w) = 1$, then $x_i(w) = 0$, and so $Y(w) \leq 1 - \max x_i(w)$. In particular, $\mathbb{E}[Y] \leq 1 - \max\{\mathbb{E}[X_i]\}$. First, observe that $p < 1$. To show it, first observe that by the non-deficit condition, we have that $\Pr[Y] \geq \mathbb{E}_{\mathbf{v}}[t_i(\mathbf{v})]$ and by incentive compatibility, we can deduce that $T_i(0) \geq T_i(1/m^\gamma)$. And so, $T_i(1/m^\gamma) \leq 1/m$. By individually rationality, $X_i(1/m^\gamma)/m^\gamma + T_i(1/m^\gamma) \geq 1/m^\gamma$. But since $Y = 1$ a.s., $X_i = 0$ a.s. and so we would deduce that $1/m \geq 1/m^\gamma$ and this contradicts the fact that $\gamma \in (0, 1)$.

Since $p < 1$, we can consider $a = -(p/(1-p))^{1/2}$ and $b = ((1-p)/p)^{1/2}$ and the random variable

$$f_i = \begin{cases} a, & \text{if } Y \neq 1 \\ b, & \text{otherwise} \end{cases} \tag{2}$$

Lets define $Y_i = \mathbb{E}[Y \mid V_i]$. Then,

$$\begin{aligned} p = \Pr[Y] &\geq \mathbb{E}_{\mathbf{v}}\left[\sum_{i=1}^{m} T_i(\mathbf{v})\right] \text{ (By no-deficit)} \\ &\geq m \sum_{i=1}^{m} T_i(1/m^\gamma) \text{ (By IC)} \\ &\geq m^{1-\gamma}(1 - X_i(1/m^\gamma)) \text{ (By IR)} \\ &\geq m^{1-\gamma} Y_i(1/m^\gamma). \end{aligned}$$

Now, observe that $\mathbb{E}[f] = 0$ and $\mathrm{Var}(f) = 1$. Now, for $i = 1, ..., m$ consider $f_i = \mathbb{E}[f \mid V_i]$.

Observe that $\mathbb{E}[f_i] = \mathbb{E}[f] = 0$, $\mathbb{E}[Y_i] = p$ and the f_i i.i.d since V_i are i.i.d. In particular, $\mathrm{Var}(f_i) = \mathrm{Var}(f_j)$ for all i, j. By Bessel's inequality, $\mathrm{Var}(f) \geq \sum_{i=1}^m \mathrm{Var}(f_i)$ and so $\mathrm{Var}(f_i) \leq 1/m$. Observe that $Y_i = \frac{f_i - a}{b-a}$, so $\mathrm{Var}(Y_i) = \mathrm{Var}(f_i)/(b-a)^2$. Using that $(b-a)^{-1} = \sqrt{p(1-p)}$, we have that $\mathrm{Var}(Y_i) \leq p(1-p)/m$. And so, we deduce that $Y_i(1/m^\gamma) - p \geq -\sqrt{p(1-p)}m^{\frac{\gamma-1}{2}}$. Putting all together, we deduce that $p \geq [p - \sqrt{p(1-p)}m^{\frac{\gamma-1}{2}}]m^{1-\gamma}$, and rearranging, we obtain $p \leq \frac{m^{1-\gamma}}{m^{2(\gamma-1)} - m^{1-\gamma} + 1} \leq \frac{4}{3m^{1-\gamma}}$.

Now, the expected utility of the buyer is $Y - \mathbb{E}[p(\mathbf{v})]$ and so the expected welfare is $\mathcal{W} = \mathbb{E}_{\mathbf{v}}[Y - p(\mathbf{v}) + \sum_{i=1}^m x_i(\mathbf{v})v_i + t_i(v_i)]$. Since the mechanism is ex-ante non-deficit, we have that $\mathbb{E}[p(\mathbf{v})] \geq \mathbb{E}[\sum_{i=1}^m t_i(v)]$. And so $\mathcal{W} \leq \mathbb{E}[Y] + \sum_{i=1}^m x_i(\mathbf{v})v_i \leq \Pr[Y] + \mathbb{E}[\sum_{i=1}^m V_i] \leq \frac{4}{3m^{1-\gamma}} + \frac{1}{m^{2\gamma-1}}$. For $\gamma = 2/3$, we obtain $\mathcal{W} \leq \frac{7}{3m^{1/3}}$. On the other hand, the optimal welfare is, at least 1, and so, we obtain the result. □

Corollary 4.11. *The revenue and welfare price of stability of any mechanism with n buyers and m non-pasive block-proposals is $\Omega(m^{1/3})$.*

In summary, we have shown that when block proposals are active, the worst-case welfare and revenue is far from optimal. By [Fei+15] the lower bound is not that far from being tight.

Proposition 4.12. *As long as sellers do not overbid for their own item, the simultaneous first-price auctions with active sellers have a Bayesian price of anarchy of $\Theta(m)$.*

In other words, when sellers are rational, can shill-bid, have access to side-auctions, and the blocks are sold just-in-time in the worst-case scenario is not that different to sell the blocks through simultaneous first-price auction and any other type of mechanism. On the other hand, if the sequencer of all blocks is the same one, he/she can sell the blocks close to optimally via combinatorial auction by [LB10] (without taking into account the computational complexity [Rou10]).

4.2 Slot auctions and ex-ante individually rationality

Previous leaves a bad flavour on the possibility of creating an efficient marketplace for shared sequencing. In this section, we explore mechanisms that exploit the ability of block proposal to sell their future block through a slot auction. More specifically, we will see an example of a mechanism that using such commitments, remove the inefficiencies of the counter-example provided in section 4.1.2.

Imagine a scenario where sellers can commit to selling their block before anyone knows its value. In other words, both the buyers and the sellers are unaware of the block's value at the time of the sale. Because of this setup, we only need to consider the "ex-ante" individual rationality (the rational decision based on expectations before the actual outcome is known) rather than "ex-interim" individual rationality (the rational decision based on known outcomes at some intermediate point).

To simplify, let's start by examining a basic scenario where there is a unique seller ($m = 1$, or validator, and is passive. This means the value they assign to building the block is zero ($v = 0$).

Now, the seller has the option, but not the obligation, to sell a contract. This contract would require them to outsource their future block to the holder of the contract at a predetermined as long as they pay the price of the contract π. At this point, buyers don't yet know their exact valuations because they don't know the future state of the blockchain or the transactions that will be available. Essentially, they don't know how valuable the future block will be.

Assume there are n potential buyers whose valuations are i.i.d. according to a cumulative distribution function F. When the time comes to build the block, the contract holder (or the seller, if they still hold the contract) can participate in a just-in-time auction.

A key question arises: what is the clearing price of the contract in equilibrium? In simpler terms, how much is this contract worth under balanced market conditions?

The value of the contract to sellers is the expected revenue they would earn from the just-in-time auction. In an auction where buyers' valuations are i.i.d., the revenue can be determined by looking at the expected value of the second-highest bid (in a second-price auction) or the highest bid (in a first-price auction). By the revenue equivalence theorem, this expected revenue corresponds to the $(n-1)$-th order statistic of n samples from the distribution F. Analogously, any risk-neutral agent that has no value over the blocks will be willing to pay the expected revenue of the just-in-time auction. A natural question is, how much the set of buyers' value the contract?

To do this, let's imagine that one of the buyers, buyer n, holds the contract. When the just-in-time auction starts, buyer n discovers their private valuation of the block, which we'll call v_n.

Now, suppose buyer n decides to sell the block using a second-price auction, setting a reserve price equal to their own valuation, v_n. This means that the block will only be sold if someone else bids at least v_n. In a second-price auction, the highest bidder wins but pays the second-highest bid. Then its expected payoff of the just-in-time auction is

$$\mathbb{E}_{\mathbf{v}_{-n}\sim F^{n-1}}\left[\max_{(2)}\{v_n, v_1, ..., v_{n-1}\}\mathbb{1}_{\max\{v_1,..,v_{n-1}\}>v_n} + v_n\mathbb{1}_{\max\{v_1,..,v_{n-1}\}\leq v_n}\right].$$

where $\mathbf{v}_{-n} = (v_1, ..., v_{n-1})$. And so, before in the time of the slot auction, its value over the contract is

$$\begin{aligned}
\pi &= \mathbb{E}_{v_n\sim F}\left[\mathbb{E}_{\mathbf{v}_{-n}\sim F^{n-1}}\left[\max_{(2)}\{v_n, v_1, ..., v_{n-1}\}\mathbb{1}_{\max\{v_1,..,v_{n-1}\}>v_n} + v_n\mathbb{1}_{\max\{v_1,..,v_{n-1}\}\leq v_n}\right]\right]\\
&= \mathbb{E}_{\mathbf{v}\sim F^n}\left[\max_{(2)}\{v_n, v_1, ..., v_{n-1}\}\mathbb{1}_{\max\{v_1,..,v_{n-1}\}>v_n} + v_n\mathbb{1}_{\max\{v_1,..,v_{n-1}\}\leq v_n}\right]\\
&= \mathbb{E}_{\mathbf{v}\sim F^n}\left[\max_{(2)}\{v_n, v_1, ..., v_{n-1}\}\mathbb{1}_{\max\{v_1,..,v_{n-1}\}>v_n} + \max\{v_1, ..., v_n\}\mathbb{1}_{\max\{v_1,..,v_{n-1}\}\leq v_n}\right]\\
&\geq \mathbb{E}_{\mathbf{v}\sim F^n}\left[\max_{(2)}\{v_n, v_1, ..., v_{n-1}\}\mathbb{1}_{\max\{v_1,..,v_{n-1}\}>v_n} + \max_{(2)}\{v_1, ..., v_n\}\mathbb{1}_{\max\{v_1,..,v_{n-1}\}\leq v_n}\right]\\
&= \mathbb{E}_{\mathbf{v}\sim F^n}[\max_{(2)}\{v_1, ..., v_n\}]
\end{aligned}$$

Putting all together, the value that a payoff of the seller selling its block before time is higher than selling it directly in the just-in-time auction. In case that F is not a mass points distribution, the inequality is strict, and so the value of the contract is strictly larger than the expected value that the seller will make in the just-in-time auction. In other words, the revenue of the slot auction is strictly higher than the revenue of the just-in-time auction, when the buyers have the option to resell its item in the just-in-time auction.

By the revenue equivalence theorem, this also holds when the just-in-time auction is a first-price sealed bid auction.

Proposition 4.13. *Suppose there are n buyers with i.i.d. valuation and a seller with no valuation over their item. If everyone can sell once all buyers have realized their value, then the seller makes more revenue by selling the item before buyers realize their valuations.*

Now lets move to the case where there are buyers with combinatorial valuations. We have seen that in just-in-time cross-domain auctions, when buyers have single-minded valuations over all items, the ability of sellers to veto power in the negotiation lead to a large amount of inefficiency.

One of the fundamental constraints in the mechanism that imply a high inefficiency is the ex-interim individually rationality and the weak budget-balance. Since in equilibrium no third party will subsidize the demand, we can just weaken the first assumption. A weaker assumption would be to just impose ex-ante individually rationality for both buyers and sellers. That means, we are just imposing that each agent has incentive to participate in the mechanism before realizing its own valuation.

Let's consider a similar particular case as the that lead to the impossibility result of section 4.1.2. Suppose we have a buyer with valuation v drawn from F over all the items and a set of m sellers with valuations r_i drawn from G_i.

Consider the following mechanism:

- If $\mathbb{E}[v] \leq \mathbb{E}\left[\sum_{i=1}^{m} r_i\right]$, do nothing.
- Otherwise, the buyer makes a take-it-or-leave-it offer to each seller i the amount $\mathbb{E}[r_i]$ before both agents realizing their valuation (that is before the time of the concurrent blocks).

Proposition 4.14. *The mechanism is ex-ante IR for both buyers and sellers, budget-balance and is* $2-$*approximate.*

Proof. In case that $\mathbb{E}[v] < \mathbb{E}\left[\sum_{i=1}^{m} r_i\right]$. Otherwise the expected welfare is $\mathbb{E}[\sum_{i=1}^{m} r_i]$. The optimal welfare is $\mathbb{E}[\max\{v, \sum r_i\}]$. In either case, the ratio of the optimal welfare and the expected welfare of the mechanism is

$$\frac{\mathbb{E}[\max\{v, \sum r_i\}]}{\max\{\mathbb{E}[v], \mathbb{E}[\sum r_i]\}} \leq \frac{\mathbb{E}[v + \sum r_i]}{\max\{\mathbb{E}[v], \mathbb{E}[\sum r_i]\}} \leq 2.$$

□

Therefore, when we can sell the item before sellers realize their value, in worst-case scenario, the expected outcome is more efficient in terms of both welfare and revenue. We leave for future work to analyse how to make these mechanisms prior-independent with general buyers' valuations.

5 Conclusions and future work

In this paper, we explored the design and implications of double auctions for cross-blockchain resource allocation, focusing on optimizing Miner Extractable Value (MEV) across different blockchain domains. We formalized shared sequencing marketplaces and compared various auction mechanisms, finding that simultaneous first-price auctions suffer from inefficiencies due to strategic seller behavior, while combinatorial auctions, though potentially more efficient, are not inherently shill-proof. Also, we proved that no just-in-time auction suffers of an inefficiency of order of the number of items m. Slot auctions, where sellers commit to selling future blocks, emerged as a promising solution, enhancing efficiency by reducing strategic manipulation. Future directions include studying slot auctions with arbitrary buyers' valuations, exploring mechanisms for agents with interdependent valuations, and conducting simulations to evaluate the performance of different auction implementations. Additionally, integrating these auction mechanisms with decentralized finance platforms and addressing the computational complexity of combinatorial auctions will be crucial for practical deployment. Empirical validation using real-world blockchain data will further refine these models and enhance their applicability in achieving efficient and fair resource allocation in decentralized networks.

References

[Cra98] Peter Cramton. "The efficiency of the FCC spectrum auctions". In: *The Journal of Law and Economics* 41.S2 (1998), pp. 727–736.

[KP99] Elias Koutsoupias and Christos Papadimitriou. "Worst-case equilibria". In: *Annual symposium on theoretical aspects of computer science*. Springer. 1999, pp. 404–413.

[XSW05] Mu Xia, Jan Stallaert, and Andrew B Whinston. "Solving the combinatorial double auction problem". In: *European Journal of Operational Research* 164.1 (2005), pp. 239–251.

[Kri09] Vijay Krishna. *Auction theory*. Academic press, 2009.

[LB10] Brendan Lucier and Allan Borodin. “Price of anarchy for greedy auctions”. In: *Proceedings of the twenty-first annual ACM-SIAM symposium on Discrete Algorithms*. SIAM. 2010, pp. 537–553.

[Rou10] Tim Roughgarden. “Algorithmic game theory”. In: *Communications of the ACM* 53.7 (2010), pp. 78–86.

[ST13] Vasilis Syrgkanis and Eva Tardos. “Composable and efficient mechanisms”. In: *Proceedings of the forty-fifth annual ACM symposium on Theory of computing*. 2013, pp. 211–220.

[Fei+15] Uriel Feige et al. “A unifying hierarchy of valuations with complements and substitutes”. In: *Proceedings of the AAAI Conference on Artificial Intelligence*. Vol. 29. 1. 2015.

[RST17] Tim Roughgarden, Vasilis Syrgkanis, and Eva Tardos. “The price of anarchy in auctions”. In: *Journal of Artificial Intelligence Research* 59 (2017), pp. 59–101.

[Dai+19] Philip Daian et al. “Flash boys 2.0: Frontrunning, transaction reordering, and consensus instability in decentralized exchanges”. In: *arXiv preprint arXiv:1904.05234* (2019).

[Dai+20] Philip Daian et al. “Flash boys 2.0: Frontrunning in decentralized exchanges, miner extractable value, and consensus instability”. In: *2020 IEEE symposium on security and privacy (SP)*. IEEE. 2020, pp. 910–927.

[Fel+20] Michal Feldman et al. “Simultaneous auctions without complements are (almost) efficient”. In: *Games and Economic Behavior* 123 (2020), pp. 327–341.

[Oba+21] Alexandre Obadia et al. “Unity is strength: A formalization of cross-domain maximal extractable value”. In: *arXiv preprint arXiv:2112.01472* (2021).

[MRD22] Bruno Mazorra, Michael Reynolds, and Vanesa Daza. “Price of mev: towards a game theoretical approach to mev”. In: *Proceedings of the 2022 ACM CCS Workshop on Decentralized Finance and Security*. 2022, pp. 15–22.

[Bab+23] Kushal Babel et al. “Clockwork finance: Automated analysis of economic security in smart contracts”. In: *2023 IEEE Symposium on Security and Privacy (SP)*. IEEE. 2023, pp. 2499–2516.

[Col23] Columbia Engineering. *Columbia CryptoEconomics Workshop 2023*. Online video. Available at: `https://www.youtube.com/watch?v=GHPFWMQOwCk&list=PLpktWkixc1gX-L5NT-vuDP54kW--3gDnK&ab_channel=ColumbiaEngineering`. 2023.

[JL23] Yaonan Jin and Pinyan Lu. “First Price Auction is 1-1/e 2 Efficient”. In: *Journal of the ACM* 70.5 (2023), pp. 1–86.

[Joh+23] Nicholas AG Johnson et al. “Concave pro-rata games”. In: *International Conference on Financial Cryptography and Data Security*. Springer. 2023, pp. 266–285.

[Ma23] Julian Ma. *Block vs. Slot Auction PBS*. `https://mirror.xyz/0x03c29504CEcCa30B93FF5774 CPYI91s98cp9zKFkanKs_qotYzw09kWvouaAa9GXBrQ`. Accessed: 2024-06-06. 2023.

[MS23] Akaki Mamageishvili and Jan Christoph Schlegel. “Shared Sequencing and Latency Competition as a Noisy Contest”. In: *arXiv preprint arXiv:2310.02390* (2023).

[MDP23] Bruno Mazorra and Nicolás Della Penna. "The cost of sybils, credible commitments, and false-name proof mechanisms". In: *arXiv preprint arXiv:2301.12813* (2023).

[McM23] Conor McMenamin. "Sok: Cross-domain mev". In: *arXiv preprint arXiv:2308.04159* (2023).

[SMC23] Johan Hagelskjar Sjursen, Weizhi Meng, and Wei-Yang Chiu. "A closer look at cross-domain maximal extractable value for blockchain decentralisation". In: *2023 IEEE International Conference on Blockchain and Cryptocurrency (ICBC)*. IEEE. 2023, pp. 1–3.

[CP24] Rakesh Chaturvedi and Ashish Kumar Pandey. "Double auction for trading perfect complements". In: *Journal of Public Economic Theory* 26.1 (2024), e12672.

[HKL24] Nima Haghpanah, Aditya Kuvalekar, and Elliot Lipnowski. "Buying from a Group". In: (2024). URL: https://elliotlipnowski.com/wp-content/uploads/buying-from-a-group.pdf.

[MR24] Ori Mazor and Ori Rottenstreich. "An Empirical Study of Cross-chain Arbitrage in Decentralized Exchanges". In: *2024 16th International Conference on COMmunication Systems & NETworkS (COMSNETS)*. IEEE. 2024, pp. 488–496.

A A note on Miner extractable value vs Maximal extractable value

Miner Extractable Value (MEV) and Maximal Extractable Value have evolved significantly in their definitions and implications within the blockchain ecosystem. The term "miner extractable value" originally referred to the profit miners could extract by ordering, censoring, and including transactions in a block, as highlighted in the [Dai+19]. In essence, MEV was defined as the maximum profit that an agent could make by exploiting its monopolistic rights over transaction inclusion.

In the follow-up paper [Bab+23] the authors expanded on this concept by formalizing the notion of *extractable value* as an optimization problem. This problem considers the current state of the blockchain, the transactions visible in the mempool, and the potential profit from strategically constructing the next block. The maximal miner extractable value was defined as the solution to this optimization problem.

The term "miner" started to lose its relevance for several reasons, primarily due to the transition from proof-of-work (PoW) to proof-of-stake (PoS). In PoS, the agents responsible for the security of the blockchain validate rather than mine blocks, rendering the notion of a "miner" obsolete. Moreover, the agents who strategically reordered and included transactions were often not miners or validators but third-party entities known as searchers, who could identify and act on opportunities in the mempool.

In [MRD22] the authors discuss the negative externalities of the transaction ordering mechanism when agents extract this value. To address these issues, Flashbots introduced MEV-geth, a centralized auction that allowed searchers to express complex preferences through bundles in a combinatorial auction format. With Ethereum's transition to PoS, this auction system was decentralized via MEV-Boost, enabling any party to propose their block by bidding in an English auction. Validators, in this context, no longer reorder transactions themselves but outsource this right to sophisticated parties known as builders. Validators thus exploit their monopolistic power over their slot to maximize revenue.

The term "maximal extractable value" becomes ambiguous in this new transaction supply chain. In this context, users and searchers bid in combinatorial auctions held by builders, who create blocks and bid for the right to construct them, sending their bids to MEV-Boost relays (trusted entities ensuring fair exchanges between builders and validators). Validators then select the block with the highest bid. This raises the question: What constitutes the maximal value extracted? Is it the revenue the validator extracts from the auction, which doesn't account for the profits of builders and searchers or the benefits to users executing their transactions?

When referring to maximal extractable value, it's crucial to consider the total value extracted by all participants in the supply chain. However, there is often tension between the maximal value a validator can extract and the overall value accrued by the system. For instance, a validator could impose a monopolistic reserve price on block building, maximizing their expected payoff but reducing the overall system utility since builders, searchers, and users might not access the block if the total revenue is less than the reserve price.

For example, lets recall our definition of welfare, i.e. total value of all users:

$$\mathcal{W}(M, s, \mathbf{F}) = \underbrace{\mathbb{E}_{(v,r)}\left[\sum_{j=1}^{n} u_i^b(s(v,r))\right]}_{=\text{ Buyers' utility}} + \underbrace{\mathbb{E}_{(v,r)}\left[\sum_{j=1}^{n} u_i^s(s(v,r))\right]}_{=\text{ Sellers' utility}} + \underbrace{\mathbb{E}_{(v,r)}\left[\sum_{j=1}^{n} p_j(s(v,r)) - \sum_{j=1}^{m} t_i(s(v,r))\right]}_{=\text{ Auctioneer utility}}$$

In this equation, the natural definition of miner extractable value implies the mechanism that maximizes the total utility of the seller. Thus, the term "maximal extractable value" can be seen as vacuous, as it is subject to interpretation. In mechanism design terms, when maximal extractable value refers to the total value extracted in the supply chain, it corresponds to social welfare—the maximum value all participants can achieve by engaging in the mechanism. In terms of the equation, implies maximizing the welfare W. Conversely, if it only considers validator profits, it pertains to the revenue made. These two concepts are often in tension, as welfare-maximizing mechanisms do not necessarily maximize revenue and vice versa.

Philosophically, the value extracted by validators should neither be minimized nor maximized. Instead, we should design incentive-compatible mechanisms that maximize welfare by allocating block space to users who value it the most, regardless of whether this increases or decreases payments to intermediaries such as builders, searchers, or validators.

B Proofs

Proof. Consider a cumulative distribution function F such that his $(n-1)-$order statistic is $G(x) = 2x$ for $x \in [0, 1/2]$. Equivalently,

$$nF(x)^{n-1}(1 - F(x)) + F(x)^n = 2x.$$

Now assume that we have n agents N_1 with valuation for the first item drawn i.i.d. from F and analogously n more agents N_2 with valuation for the second item drawn i.i.d. from F. Finally, assume that the last agent has a single-minded valuation $v = 1$ on the bundle. Let's call this agent the global bidder. We claim that agents N_1 and N_2 playing the unique symmetric equilibrium with symmetric i.i.d valuations and the global bidder not bidding in the auctions (or bidding 0 in both auctions) is a Nash equilibrium. Clearly, if global bidders does not bid, the other agents will not deviate since their are playing the symmetric Nash equilibrium for a single item first price auction. By revenue equivalence theorem, we know that the highest observed bid follows the second-order statistic over the distribution F, and so, follows the distribution $2x$. Now, the global bidder can not rationally deviate from not bidding. To prove so, assume that he bids b_1 and b_2 in the first and the second item respectively. Then, $b_1 + b_2 \leq 1$, and his expected payoff is

$$\text{Expected payoff bidding } b_1 \text{ and } b_2 = \begin{cases} (1-b_1-b_2)2b_1 2b_2 - b_1(1-2b_1) - b_2(1-2b_2), & \text{if } b_1, b_2 \in [0,1/2], \\ (1-b_1-b_2)2b_2 - b_2(1-2b_2), & \text{if } b_1 \in [1/2,1] \text{ and } b_2 \in [0,1/2], \\ (1-b_1-b_2)2b_1 - b_1(1-2b_1), & \text{if } b_1 \in [0,1/2] \text{ and } b_2 \in [1/2,1] \end{cases}$$

When $b_1, b_2 \in [0, 1/2]$ the expected payoff is equivalent to $-(2b_1-1)(2b_2-1)(b_1+b_2)$, and since $b_1, b_2 \geq 0$, and so the expression is bounded above by zero. In case $b_1 \in [1/2, 1]$ and $b_2 \in [0, 1/2]$ that $(1-b_1-b_2)2b_2 - b_2(1-2b_2) \leq (1-1/2-b_2)2b_2 - b_2(1-2b_2) = 0$. Analogously for the remaining case. Therefore, the expected payoff of bidding b_1, b_2 is upper bounded by 0, implying that the global bidder bidding zero on both auctions is a Nash equilibrium. Now, the expected welfare of this Nash equilibrium is the $2\mathbb{E}[X^{(n)}]$, where X is a random variable following F. Taking $n = 1000000$ and with help of a computer, we approximate it $2\mathbb{E}[X^{(n)}] = 2 \times 0.365$. Since the optimal allocation is allocating the item to the global bidder, it follows that the Bayesian price of anarchy is at least ≈ 1.35. □

Proof. 4.4 Suppose that there is a unique buyer that values the bundle v drawn from a random variable V defined by $\Pr[V \leq v] = 1 - 1/v$ for $v \in [1, H]$ and $\Pr[V = H] = 1/H$. Then the following is a Bayes Nash equilibrium. The buyer bids his own value v to the mechanism, and both sellers report a Shill bid $r_1 = r_2 = H/2$. Clearly, the buyer has no proftible deviation. Now, if first seller places a shill bid x and the second seller places a shill bid y, then their expected payoffs are

$$\begin{aligned} u_1(x,y) &= \frac{x}{x+y}\mathbb{E}[V\mathbb{1}_{V\geq x+y}] = \frac{x}{x+y}(\log H - \log(x+y) + 1) \\ u_2(x,y) &= \frac{y}{x+y}\mathbb{E}[V\mathbb{1}_{V\geq x+y}] = \frac{y}{x+y}(\log H - \log(x+y) + 1) \end{aligned}$$

for $x + y \leq H$. Taking derivatives, one can easily show that $x = H/2$ is a global maximum of $u_1(x, y)$ with $x + y \leq H$, proving that it is a Nash equilibrium. Now, the expected welfare of the Nash equilibrium is 1. However, the optimal outcome always allocates the item to the buyer. This outcome has a expected social welfare $\mathbb{E}[V] = \log H$. Taking $H \to +\infty$, we obtain the result. □

Proof. 4.7 Consider a distribution with cdf F with support in $[1, M]$ and density function f such that $R(r)$ is decreasing for $r \geq 1$ and $rR(r)$ is locally increasing. Then, by lemma 4.5, there are Nash equilibrium such that the total expected revenue is $R(1)$. By proposition 4.6, there are Nash equilibrium of the form $(q/2, q/2, \beta(\cdot, q, 2), \beta(\cdot, q, 2))$ with $q \in (1, M)$. The total expected revenue in this case is $R(q)$ with $q > 1$. Since R is strictly decreasing, we have that the revenue of the first equilibrium is strictly bigger than the second one, leading to the result. □

Proof. 4.8 The first statements is already shown for case $m = 2$ and follows analogously for other m. Now lets prove the second statement. Suppose that we have $n = 2$ buyers with single-minded valuations over the set of all items m drawn i.i.d. from a distribution F. Consider a distribution F such that $R(r)$ is strictly concave on a closed interval $[0, w]$ with $R(w) = 0$. For example, consider the regular distribution $F(x) = x - 1$ with $x \in [0, 1]$. Then, if agents report a shill bid $r_1, ..., r_m$, then buyers are bidding in an auction with reserve price

$r = \sum_{k=1}^{m} r_k$. And so, in equilibrium, they will bid $\beta(v_i, r, n)$. In this case, the expected utility of a seller is i:

$$u_i(r_i, r_{-i}) = \frac{r_i}{\sum_{k=1}^{m} r_k} R\left(\sum_{k=1}^{m} r_k\right)$$

This game has a pure Nash equilibrium where sellers shill bid with bids q/m where $q \in \operatorname{argmax}_{q \geq 0} q^{m-1} R(q)$, and buyers follow the strategy $\beta(\cdot, q, n)$. The total revenue of the sellers is $R(q)$ that holds the equation

$$R(q) = -\frac{qR'(q)}{m-1}.$$

And by similar arguments showed in [Joh+23], $R(q) = O(1/m)$. □